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PARTIAL DIFFERENTIAL EQUATIONS

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Chapter 1

The Compactness Method

The compactness method is one of the most effective and widely used tools in the theory of Partial Differential Equations, primarily for establishing the existence of solutions for initial or boundary value problems. When employing this method, the core strategy involves two fundamental steps:

1. Construction of an approximate solution (e.g., using the Faedo-Galerkin method).

2. Obtaining a priori estimates for the sequence of approximate solutions.

The combination of the sequence of approximate solutions being bounded and the compactness properties of the Sobolev spaces (or spaces of distributions) allows the extraction of a subsequence that converges weakly or strongly to an element that, due to the closure properties of the functional spaces and the a priori estimates, is identified as the weak solution to the original problem.

1.1 The Linear Equation $u_{tt} - \Delta u = f$ (Weak Case)

In this section, we analyze the existence of solutions for the linear problem defined in the cylinder $\Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set, with boundary $\partial\Omega$ of class C^2. The problem is given by:

$$u_{tt} - \Delta u = f \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

with boundary and initial conditions:

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.2}$$
$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \tag{1.3}$$
$$u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \tag{1.4}$$

where $T > 0$ is a given final time.

We denote by V the space $H_0^1(\Omega)$, equipped with the inner product

$$((u, v)) = \int_\Omega \nabla u \cdot \nabla v \, dx, \tag{1.5}$$

and the norm $\|u\| = \sqrt{((u,u))} = \|\nabla u\|_{L^2(\Omega)}$. H is the space $L^2(\Omega)$, endowed with the inner product and norm:

$$(u,v) = \int_\Omega uv\,dx, \quad \|u\| = \|u\|_{L^2(\Omega)}. \tag{1.6}$$

Note that the norms defined on V and H are equivalent, respectively, to the norms of $H^1(\Omega)$ and $L^2(\Omega)$. Since Ω is bounded, the embedding $V \hookrightarrow H$ is compact.

1.1.1 Weak Formulation

A function u is a weak solution to problem (1.1)-(1.4) if u satisfies the following identity:

$$(u_{tt}(t),v) + ((\nabla u(t),\nabla v)) = (f(t),v) \quad \text{in } \mathcal{D}'(0,T), \tag{1.7}$$

for every test function $v \in V$, where $u(t) = u(\cdot,t)$ and $f(t) = f(\cdot,t)$.

Remark 1.1. *The identity (1.7) means that, for any $\varphi \in \mathcal{D}(0,T)$, we have:*

$$\int_0^T \{(u_{tt}(t),v) + ((\nabla u(t),\nabla v))\}\,\varphi(t)\,dt = \int_0^T (f(t),v)\varphi(t)\,dt. \tag{1.8}$$

Remark 1.2. *The initial conditions must be satisfied in the sense of functional spaces:*

$$u(\cdot,0) = u_0 \quad \textit{in } V, \tag{1.9}$$

$$u_t(\cdot,0) = u_1 \quad \textit{in } H. \tag{1.10}$$

The space for weak solutions is usually taken as:

$$u \in L^\infty(0,T;V) \quad \textit{with } u_t \in L^\infty(0,T;H) \quad \textit{and } u_{tt} \in L^\infty(0,T;V'). \tag{1.11}$$

We adopt the same notation used in Lions [9]. Ω represents a bounded open set in $\mathbb{R}^n$ with a smooth boundary Γ, $T > 0$ is an arbitrary but fixed positive real number, and Q is the cylinder $\Omega \times\,]0,T[$ whose lateral boundary Σ is given by $\Gamma \times\,]0,T[$. We denote, respectively, by $(\cdot,\cdot)$ and $|\cdot|$, the inner product and the norm in $L^2(\Omega)$. Similarly, we denote, respectively, by $((\cdot,\cdot))$ and $||\cdot||$, the inner product and the norm in $H_0^1(\Omega)$.

The linear problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f & \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(0) = u_0(x); \ \dfrac{\partial u}{\partial t}(0) = u_1(x); \ x \in \Omega \end{cases} \tag{1.12}$$

where

$$u_0 \in H_0^1(\Omega); \quad u_1 \in L^2(\Omega) \quad \text{and} \quad f \in L^1(0,T;L^2(\Omega)) \tag{1.13}$$

admits a unique weak solution $u\colon Q \to \mathbb{R}$, in the class

$$u \in L^\infty(0,T;H_0^1(\Omega)); \quad u' = \frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)). \tag{1.14}$$

More precisely, we have

$$\frac{d}{dt}(u'(t), v) + ((u(t), v)) = (f(t), v) \tag{1.15}$$

in $\mathcal{D}'(0,T)$ for every $v \in H_0^1(\Omega)$.

$$u(0) = u_0\,; \qquad u'(0) = u_1\,. \tag{1.16}$$

Proof:
Step 1: Approximate Problem

Let $\{\omega_\nu\}_{\nu\in\mathbb{N}}$ be a basis of $H_0^1(\Omega)$, orthonormal in $L^2(\Omega)$ (such a property can be achieved by applying the Gram-Schmidt orthogonalization process to the basis of $H_0^1(\Omega)$). We define

$$V_m = [\omega_1, \cdots, \omega_m]\,,$$

the subspace spanned by the first m elements of the basis, and consider the approximate problem in $[0,T]$:

$$\text{Determine } u_m(t) \in V_m \;\Leftrightarrow\; u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i, \tag{1.17}$$

$$(u_m''(t), \omega_j) + ((u_m(t), \omega_j)) = (f(t), \omega_j); \quad j = 1, 2, \ldots, m, \tag{1.18}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega), \tag{1.19}$$

$$u_m'(0) = u_{1m} \to u_1 \quad \text{in} \quad L^2(\Omega). \tag{1.20}$$

We have, by virtue of (1.17) that

$$u_{0m} = \sum_{i=1}^{m} \alpha_{im}\omega_i = u_m(0) = \sum_{i=1}^{m} g_{im}(0)\omega_i,$$

$$u_{1m} = \sum_{i=1}^{m} \beta_{im}\omega_i = u_m'(0) = \sum_{i=1}^{m} g_{im}'(0)\omega_i.$$

and, hence, from (1.17)-(1.20) we can write

$$\begin{cases} \displaystyle\sum_{i=1}^{m} g_{im}''(t)(\omega_i, \omega_j) + \sum_{i=1}^{m} g_{im}(t)((\omega_i, \omega_j)) = (f(t), \omega_j), \\ g_{jm}(0) = \alpha_{jm},\; g_{jm}'(0) = \beta_{jm}, \quad j = 1, \cdots, m. \end{cases} \tag{1.21}$$

or alternatively

$$\underbrace{\begin{bmatrix} (\omega_1,\omega_1)\cdots(\omega_1,\omega_m) \\ (\omega_2,\omega_1)\cdots(\omega_2,\omega_m) \\ \vdots \qquad\quad \vdots \\ (\omega_m,\omega_1)\cdots(\omega_m,\omega_m) \end{bmatrix}\begin{bmatrix} g_{1m}''(t) \\ g_{2m}''(t) \\ \vdots \\ g_{mm}''(t) \end{bmatrix}}_{=A} + \underbrace{\begin{bmatrix} ((\omega_1,\omega_1))\cdots((\omega_1,\omega_m)) \\ ((\omega_2,\omega_1))\cdots((\omega_2,\omega_m)) \\ \vdots \qquad\quad \vdots \\ ((\omega_m,\omega_1))\cdots((\omega_m,\omega_m)) \end{bmatrix}}_{=B}\begin{bmatrix} g_{1m}(t) \\ g_{2m}(t) \\ \vdots \\ g_{mm}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} (f(t),\omega_1) \\ (f(t),\omega_2) \\ \vdots \\ (f(t),\omega_m) \end{bmatrix}}_{=F(t)}. \tag{1.22}$$

We define

$$z(t)=\begin{bmatrix} g_{1m}(t)\\ g_{2m}(t)\\ \vdots\\ g_{mm}(t)\end{bmatrix} \Rightarrow z(0)=\begin{bmatrix} \alpha_{1m}\\ \alpha_{2m}\\ \vdots\\ \alpha_{mm}\end{bmatrix}=z_0 \text{ and } z'(0)=\begin{bmatrix} \beta_{1m}\\ \beta_{2m}\\ \vdots\\ \beta_{mm}\end{bmatrix}=z_1. \tag{1.23}$$

Observe that, since $\omega_{\nu\nu\in\mathbb{N}}$ is orthonormal in $L^2(\Omega)$, the matrix A is the identity matrix, and therefore from (1.22) and (1.23) it follows that

$$\begin{cases} z''(t)+Bz(t)=F(t)\\ z(0)=z_0, \quad z'(0)=z_1\end{cases} \tag{1.24}$$

Defining

$$Y_1(t)=z(t),\ Y_2(t)=z'(t) \text{ and } Y(t)=\begin{bmatrix} Y_1(t)\\ Y_2(t)\end{bmatrix},$$

from (1.24) it follows that

$$Y'(t)=\begin{bmatrix} Y_1'(t)\\ Y_2'(t)\end{bmatrix}=\begin{bmatrix} z'(t)\\ z''(t)\end{bmatrix}=\begin{bmatrix} z'(t)\\ F(t)-Bz(t)\end{bmatrix}=\begin{bmatrix} Y_2(t)\\ F(t)-BY_1(t)\end{bmatrix}, \tag{1.25}$$

that is,

$$Y'(t)=\begin{bmatrix} 0 & I\\ -B & 0\end{bmatrix}\begin{bmatrix} Y_1(t)\\ Y_2(t)\end{bmatrix}+\begin{bmatrix} 0\\ F(t)\end{bmatrix},$$

or further,

$$Y'(t)=\underbrace{\begin{bmatrix} 0 & I\\ -B & 0\end{bmatrix}}_{=D}Y(t)+\underbrace{\begin{bmatrix} 0\\ F(t)\end{bmatrix}}_{=G(t)}.$$

We thus obtain the following system:

$$\begin{cases} Y'(t)=DY(t)+G(t)\\ Y(0)=Y_0,\end{cases} \tag{1.26}$$

where $Y_0=\begin{bmatrix} z_0\\ z_1\end{bmatrix}$, which possesses a solution in $[0,T]$ given by

$$Y(t)=e^{tD}Y_0+e^{tD}\int_0^t e^{-sD}G(s)\,ds, \quad \text{remembering that } e^{tD}=\sum_{k=0}^{\infty}\frac{t^k}{k!}D^k.$$

Note that the integral representation above is well-defined because the components of $G(\cdot)$ belong to $L^1(0,T)$ since $f\in L^1(0,T;L^2(\Omega))$. Furthermore, $Y,Y'\in L^1(0,T)$, i.e., $Y\in W^{1,1}(0,T)$. Thus $Y(\cdot)$ identifies with an absolutely continuous representative in $[0,T]$. It follows that $z(t)$ and $z'(t)$ are absolutely continuous with $z''(t)$ existing almost everywhere in $(0,T)$, the same holding for $g_{jm}(t)$, for all $j=1,\cdots,m$. We conclude, then, that the Dini derivatives and the distributional derivatives of $g_{jm}(t)$ with respect to the variable t coincide up to the second order, for all $j = 1, \cdots, m$.

Step 2: A Priori Estimate

Multiplying (1.18) by $g'_{jm}(t)$ and summing in j from 1 to m, we obtain

$$(u''_m(t), u'_m(t)) + ((u_m(t), u'_m(t))) = (f(t), u'_m(t)). \tag{1.27}$$

Note that

$$\begin{aligned}(u''_m(t), u'_m(t)) &= \left(\sum_{i=1}^m g''_{im}(t)\omega_i, \sum_{j=1}^m g'_{jm}(t)\omega_j\right)\\ &= \sum_{i=1}^m g''_{im}(t)g'_{im}(t)|\omega_i|^2\\ &= \frac{1}{2}\frac{d}{dt}\sum_{i=1}^m (g'_{im}(t))^2|\omega_i|^2\\ &= \frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^m g'_{im}(t)\omega_i, \sum_{j=1}^m g'_{jm}(t)\omega_j\right)\\ &= \frac{1}{2}\frac{d}{dt}|u'_m(t)|^2.\end{aligned} \tag{1.28}$$

Similarly we have

$$((u_m(t), u'_m(t))) = \frac{1}{2}\frac{d}{dt}||u_m(t)||^2. \tag{1.29}$$

Combining (1.27), (1.28) and (1.29) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u'_m(t)|^2 + \frac{1}{2}\frac{d}{dt}||u_m(t)||^2 &= (f(t), u'_m(t))\\ &\le |f(t)||u'_m(t)|,\end{aligned} \tag{1.30}$$

where the last inequality follows from the Cauchy-Schwarz inequality. Integrating (1.30) from $(0, t)$ with $t \in [0, T]$ and using the inequality $2ab \le a^2 + b^2$, we obtain

$$\begin{aligned}|u'_m(t)|^2 + ||u_m(t)||^2 \le{}& |u'_m(0)|^2 + ||u_m(0)||^2\\ &+ 2\int_0^t |f(s)|\,|u'_m(s)|\,ds.\end{aligned} \tag{1.31}$$

We use the simplified inequality $|f(t)||u'_m(t)| \le \frac{1}{2}|f(t)|^2 + \frac{1}{2}|u'_m(t)|^2$. Integrating $2|f(s)||u'_m(s)|$ (from multiplying (1.30) by 2) from $(0, t)$, and using $2ab \le a^2 + b^2$, we adjust the integration result, obtaining:

$$\begin{aligned}|u'_m(t)|^2 + ||u_m(t)||^2 \le{}& |u'_m(0)|^2 + ||u_m(0)||^2\\ &+ \int_0^t |f(s)|^2\,ds + \int_0^t |u'_m(s)|^2\,ds,\end{aligned} \tag{1.32}$$

and consequently, from (1.19), (1.20) and (1.32), we have

$$\begin{aligned}|u'_m(t)|^2 + ||u_m(t)||^2 \le{}& |u_{1m}|^2 + ||u_{0m}||^2\\ &+ \int_0^T |f(s)|^2\,ds + \int_0^t \left[|u'_m(s)|^2 + ||u_m(s)||^2\right]\,ds,\end{aligned} \tag{1.33}$$

for all $t \in [0,T]$. Since $u_{0m} \to u_0$ in $H_0^1(\Omega)$, $u_{1m} \to u_1$ in $L^2(\Omega)$ and $f \in L^1(0,T;L^2(\Omega))$, then there exists a constant $K > 0$, independent of m such that

$$|u_{1m}|^2 + ||u_{0m}||^2 + \int_0^T |f(s)|^2 \, ds \leq K. \tag{1.34}$$

Combining (1.33) and (1.34) yields

$$|u'_m(t)|^2 + ||u_m(t)||^2 \leq K + \int_0^t \left[|u'_m(s)|^2 + ||u_m(s)||^2 \right] \, ds. \tag{1.35}$$

Employing Gronwall's inequality in (1.35) we obtain

$$|u'_m(t)|^2 + ||u_m(t)||^2 \leq \underbrace{Ke^T}_{=C}, \quad \text{for all } t \in [0,T] \text{ and for all } m \in \mathbb{N}. \tag{1.36}$$

The inequality in (1.36) tells us that

$$\begin{aligned} &\{u_m\} \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)) \equiv \left(L^1(0,T;H^{-1}(\Omega))\right)', \\ &\{u'_m\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \equiv \left(L^1(0,T;L^2(\Omega))\right)'. \end{aligned} \tag{1.37}$$

Consequently, from (1.37) there exists a subsequence $\{u_\mu\}_{\mu\in\mathbb{N}}$ of $\{u_\nu\}_{\nu\in\mathbb{N}}$ such that

$$\begin{aligned} &u_\mu \overset{*}{\rightharpoonup} u \text{ weak-star in } L^\infty(0,T;H_0^1(\Omega)) \equiv \left(L^1(0,T;H^{-1}(\Omega))\right)', \\ &u'_\mu \overset{*}{\rightharpoonup} v \text{ weak-star in } L^\infty(0,T;L^2(\Omega)) \equiv \left(L^1(0,T;L^2(\Omega))\right)'. \end{aligned} \tag{1.38}$$

We affirm that $v = u'$. Indeed, note that from the chain of injections

$$L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^2(Q) \hookrightarrow \mathcal{D}'(Q),$$

and since the differentiation operator is continuous in $\mathcal{D}'(Q)$ from (1.38) it follows that

$$\begin{aligned} &u_\mu \overset{*}{\rightharpoonup} u \text{ in } \mathcal{D}'(Q) \Rightarrow u'_\mu \overset{*}{\rightharpoonup} u' \text{ in } \mathcal{D}'(Q), \\ &u'_\mu \overset{*}{\rightharpoonup} v \text{ in } \mathcal{D}'(Q). \end{aligned} \tag{1.39}$$

From the uniqueness of the limit in $\mathcal{D}'(Q)$ and from (1.39) it follows that $u' = v$.

Step 3: Limit Process

From the convergence given in (1.38) we can write

$$\int_0^T \langle u_\mu(t), w(t)\rangle_{H_0^1,H^{-1}} \, dt \to \int_0^T \langle u(t), w(t)\rangle_{H_0^1,H^{-1}} \, dt, \text{ when } \mu \to +\infty, \tag{1.40}$$

for every $w \in L^1(0,T;H^{-1}(\Omega))$. Taking, in particular, $w = (-\Delta v)\theta$, $v \in H_0^1(\Omega)$ and $\theta \in \mathcal{D}(0,T)$, we have, noting that $\langle -\Delta u, v\rangle = ((u,v))$, for every $u, v \in H_0^1(\Omega)$, that

$$\int_0^T ((u_\mu(t), v))\theta(t) \, dt \to \int_0^T ((u(t), v))\theta(t) \, dt, \text{ when } \mu \to +\infty. \tag{1.41}$$

Similarly, from (1.38) we obtain

$$\int_0^T \langle u'_\mu(t), w(t)\rangle_{L^2,(L^2)'} \, dt \to \int_0^T \langle u'(t), w(t)\rangle_{L^2,(L^2)'} \, dt, \text{ when } \mu \to +\infty, \tag{1.42}$$

for every $w \in L^1(0,T;(L^2(\Omega)')$. Taking, in particular, $w = v\theta'$, $v \in H_0^1(\Omega)$ and $\theta \in \mathcal{D}(0,T)$, we have

$$\int_0^T (u'_\mu(t), v)\theta'(t)\,dt \to \int_0^T (u'(t), v)\theta'(t)\,dt, \text{ when } \mu \to +\infty. \tag{1.43}$$

Let $j \in \mathbb{N}$ be arbitrary but fixed, and consider $\mu > j$. Then from (1.18) it follows that

$$(u''_\mu(t), \omega_j) + ((u_\mu(t), \omega_j)) = (f(t), \omega_j). \tag{1.44}$$

Multiplying (1.44) by $\theta \in \mathcal{D}(0,T)$ and integrating from 0 to T, we obtain

$$\int_0^T (u''_\mu(t), \omega_j)\theta(t)\,dt + \int_0^T ((u_\mu(t), \omega_j))\theta(t)\,dt = \int_0^T (f(t), \omega_j)\theta(t)\,dt. \tag{1.45}$$

But, by integration by parts (since $\theta \in \mathcal{D}(0,T)$):

$$\int_0^T (u''_\mu(t), \omega_j)\theta(t)\,dt = \int_0^T \frac{d}{dt}(u'_\mu(t), \omega_j)\theta(t)\,dt = -\int_0^T (u'_\mu(t), \omega_j)\theta'(t)\,dt. \tag{1.46}$$

Combining (1.45) and (1.46) yields

$$-\int_0^T (u'_\mu(t), \omega_j)\theta'(t)\,dt + \int_0^T ((u_\mu(t), \omega_j))\theta(t)\,dt = \int_0^T (f(t), \omega_j)\theta(t)\,dt. \tag{1.47}$$

Taking the limit in (1.47) as $\mu \to +\infty$, and considering the convergences in (1.41) and (1.43), we infer

$$-\int_0^T (u'(t), \omega_j)\theta'(t)\,dt + \int_0^T ((u(t), \omega_j))\theta(t)\,dt = \int_0^T (f(t), \omega_j)\theta(t)\,dt, \quad \text{for all } j \in \mathbb{N}. \tag{1.48}$$

Consider, now, $v \in H_0^1(\Omega)$. Since the finite linear combinations of the basis elements $\{\omega_\nu\}_{\nu\in\mathbb{N}}$ are dense in $H_0^1(\Omega)$, there exists a sequence $\{z_k\}_{k\in\mathbb{N}}$, $z_k = \sum_{i=1}^{\nu(k)} \xi_{ik}\omega_{ik}$ such that $z_k \to v$ in $H_0^1(\Omega)$ when $k \to +\infty$.

Hence, from (1.48), for every $k \in \mathbb{N}$, we have

$$-\int_0^T (u'(t), z_k)\theta'(t)\,dt + \int_0^T ((u(t), z_k))\theta(t)\,dt = \int_0^T (f(t), z_k)\theta(t)\,dt, \quad \text{for all } k \in \mathbb{N}. \tag{1.49}$$

From the strong convergence $z_k \to v$ in $H_0^1(\Omega)$ and the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the following convergences result:

$$\begin{cases} ((z_k, \xi)) \to ((v, \xi)), & \text{when } k \to \infty \text{ for all } \xi \in H_0^1(\Omega), \\ (z_k, \eta) \to (v, \eta), & \text{when } k \to \infty \text{ for all } \eta \in L^2(\Omega). \end{cases} \tag{1.50}$$

From (1.49) and (1.50) we obtain

$$-\int_0^T (u'(t), v)\theta'(t)\,dt + \int_0^T ((u(t), v))\theta(t)\,dt = \int_0^T (f(t), v)\theta(t)\,dt, \quad \text{for all } v \in H_0^1(\Omega) \tag{1.51}$$

and for every $\theta \in \mathcal{D}(0,T)$, or alternatively, in the sense of distributions:

$$\left\langle \frac{d}{dt}(u'(t),v),\theta \right\rangle + \langle ((u(t),v)),\theta\rangle = \langle (f(t),v),\theta\rangle \quad \text{for all } v \in H_0^1(\Omega), \tag{1.52}$$

and for every $\theta \in \mathcal{D}(0,T)$.

Step 4: Initial Conditions and Uniqueness

The initial conditions are verified through the continuity of the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and by the strong convergence (which will be established later) of the subsequence $\{u_\mu\}$ in $L^2(Q)$. The full proof of existence requires establishing strong convergence to rigorously verify the initial data, but assuming strong convergence as a final step, we obtain:

$$u(\cdot,0) = u_0 \quad \text{in } V, \tag{1.12}$$
$$u_t(\cdot,0) = u_1 \quad \text{in } H. \tag{1.13}$$

Step 4: Localization of the 2nd Derivative

We observe, initially, that the operator $-\Delta$ defined by the triplet $\{H_0^1(\Omega), L^2(\Omega), ((\cdot,\cdot))\}$, satisfies the condition:

$$(-\Delta u, v) = ((u,v)), \quad \text{for all } u \in D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega) \text{ and for all } v \in H_0^1(\Omega),$$

since Ω is a bounded open set with a smooth boundary. Moreover, the operator $-\Delta$ admits a unique continuous extension, indeed an isometry from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. Thus, $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric bijection, i.e.,

$$||\Delta u||_{H^{-1}(\Omega)} = ||u||_{H_0^1(\Omega)}, \text{ for all } u \in H_0^1(\Omega). \tag{1.53}$$

Furthermore, such an extension verifies the identity

$$\langle -\Delta u, v\rangle_{H^{-1},H_0^1} = ((u,v)), \text{ for all } u,v \in H_0^1(\Omega). \tag{1.54}$$

Making the above considerations, from (1.52) and (1.54) we can write

$$-\int_0^T (u'(t),v)\theta'(t)\,dt = \int_0^T \langle \Delta u(t),v\rangle\,\theta(t)\,dt + \int_0^T (f(t),v)\theta(t)\,dt, \quad \text{for all } v \in H_0^1(\Omega) \tag{1.55}$$

and for every $\theta \in \mathcal{D}(0,T)$, or alternatively,

$$\Bigg(\underbrace{-\int_0^T u'(t)\theta'(t)\,dt}_{\in L^2(\Omega)}, v\Bigg) = \Bigg\langle \underbrace{\int_0^T \Delta u(t)\theta(t)\,dt}_{\in H^{-1}(\Omega)}, v\Bigg\rangle + \Bigg(\underbrace{\int_0^T f(t)\theta(t)\,dt}_{\in L^2(\Omega)}, v\Bigg),$$

for all $v \in H_0^1(\Omega)$ and for every $\theta \in \mathcal{D}(0,T)$. Identifying $L^2(\Omega)$ with its dual, via the Riesz Theorem, from the last identity it follows that

$$\left\langle -\int_0^T u'(t)\theta'(t)\,dt, v\right\rangle = \left\langle \int_0^T \Delta u(t)\theta(t)\,dt, v\right\rangle + \left\langle \int_0^T f(t)\theta(t)\,dt, v\right\rangle. \tag{1.56}$$

for all $v \in H_0^1(\Omega)$ and for every $\theta \in \mathcal{D}(0,T)$.

Defining

$$g(t) = \Delta u(t) + f(t),$$

from (1.56) it results that

$$-\int_0^T u'(t)\theta'(t)\,dt = \int_0^T g(t)\theta(t)\,dt \text{ in } H^{-1}(\Omega). \tag{1.57}$$

Furthermore, since $f \in L^1(0,T;H^{-1}(\Omega))$, from (1.53) we obtain

$$\int_0^T ||\Delta u(t)||_{H^{-1}(\Omega)}\,dt = \int_0^T ||u(t)||_{H_0^1(\Omega)}\,dt < +\infty,$$

which implies that $g \in L^1(0,T;H^{-1}(\Omega))$.

Let us define:

$$v(t) = u'(t) - \int_0^t g(s)\,ds \in H^{-1}(\Omega).$$

Since $v \in L^1(0,T;H^{-1}(\Omega))$, v' defines a vector distribution, and moreover,

$$\langle v', \theta\rangle = -\langle v, \theta'\rangle = -\langle u', \theta'\rangle + \left\langle \int_0^t g(s)\,ds, \theta'\right\rangle.$$

But,

$$\left\langle \int_0^t g(s)\,ds, \theta'\right\rangle = \int_0^T \underbrace{\int_0^t g(s)\,ds}_{=h(t)}\theta'(t)\,dt$$

$$= \underbrace{h(t)\theta(t)|_0^T}_{=0} - \int_0^T g(t)\theta(t)\,dt = -\int_0^T g(t)\theta(t)\,dt,$$

which implies

$$\langle v', \theta\rangle = -\langle u', \theta'\rangle \; - \int_0^T g(t)\theta(t)\,dt. \tag{1.58}$$

From (1.57) and (1.58) it follows that

$$\langle v', \theta\rangle = 0, \text{ for all } \theta \in \mathcal{D}(0,T),$$

and consequently, $v' = 0$. Hence, $v(t) = \xi_x =$ constant with respect to t.

Thus,

$$u'(t) = \xi_x + \int_0^t g(s)\,ds \Rightarrow u''(t) = g(t),$$

which leads us to

$$u'' \in L^1(0,T;H^{-1}(\Omega)). \tag{1.59}$$

Step 5: Initial Conditions

Let us note initially that due to

$$u \in L^\infty(0,T;H_0^1(\Omega)), u' \in L^\infty(0,T;L^2(\Omega)), u'' \in L^\infty(0,T;H^{-1}(\Omega)),$$

it results, by virtue of Lemma 1.2 in Lions [10] and Lemma 8.1 in Lions and Magenes [9], that

$$\begin{cases} u \in C([0,T];L^2(\Omega)), \ u \in C_s(0,T;H_0^1(\Omega)), \\ u' \in C([0,T];H^{-1}(\Omega)), \ u' \in C_s(0,T), L^2(\Omega). \end{cases}$$

Thus, it makes sense to refer to $u(t)$ and $u'(t)$ for any $t \in [0,T]$.

We will prove that:

(i) $u(0) = u_0$

Indeed, we have previously established that

$$u'_\nu \overset{*}{\rightharpoonup} u' \ \text{ in } L^\infty(0,T;L^2(\Omega)) \ \text{ when } \nu \to \infty.$$

Let $\theta \in C^1([0,T])$ and $v \in L^2(\Omega)$. Then, identifying $L^2(\Omega)$ with its dual, from the last convergence it follows that

$$\int_0^T (u'_\nu(t),v)\theta(t)\,dt \ \to \int_0^T (u'(t),v)\theta(t)\,dt, \quad \text{when } \nu \to \infty,$$

that is,

$$\int_0^T \frac{d}{dt}(u_\nu(t),v)\theta(t)\,dt \ \to \int_0^T \frac{d}{dt}(u(t),v)\theta(t)\,dt, \quad \text{when } \nu \to \infty,$$

or alternatively,

$$(u_\nu(t),v)\theta(t)|_0^T - \int_0^T (u_\nu(t),v)\theta'(t)\,dt \to (u(t),v)\theta(t)|_0^T - \int_0^T (u(t),v)\theta'(t)\,dt,$$

when $\nu \to \infty$.

Choosing θ such that $\theta(T) = 0$ and $\theta(0) = 1$, and observing that

$$\int_0^T (u_\nu(t),v)\theta'(t)\,dt \to \int_0^T (u(t),v)\theta'(t)\,dt, \quad \text{when } \nu \to \infty,$$

we obtain

$$(u_\nu(0),v) \to (u(0),v), \quad \text{when } \nu \to \infty, \quad \text{for all } v \in L^2(\Omega).$$

Hence,

$$u_\nu(0) = u_{0\nu} \rightharpoonup u(0), \quad \text{weakly in } L^2(\Omega).$$

On the other hand, $u_{0\nu} \to u_0$ strongly in $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. By the uniqueness of the weak limit in $L^2(\Omega)$ we conclude that $u(0) = u_0$, as was to be proved.

We will prove next that

(ii) $u'(0) = u_1$.

In fact, let $\theta \in C^1([0,T])$ such that $\theta(0) = 1$ and $\theta(T) = 0$. Returning to the approximate problem we can write

$$\int_0^T (u''_\nu(t),\omega_j)\theta(t)\,dt + \int_0^T ((u_\nu,\omega_j))\theta(t)\,dt = \int_0^T (f(t),\omega_j)\theta(t)\,dt, \ \nu \geq j(\text{fixed}).$$

Integrating the above identity by parts we arrive at

$$(u'_\nu(t),\omega_j)\theta(t)|_0^T - \int_0^T (u'_\nu(t),\omega_j)\theta'(t)\,dt + \int_0^T ((u_\nu(t),\omega_j))\theta(t)\,dt = \int_0^T (f(t),\omega_j)\theta(t)\,dt,$$

or alternatively, based on the characteristics of θ,

$$-(u'_\nu(0),\omega_j) - \int_0^T (u'_\nu(t),\omega_j)\theta'(t)\,dt + \int_0^T ((u_\nu(t),\omega_j))\theta(t)\,dt = \int_0^T (f(t),\omega_j)\theta(t)\,dt.$$

Taking the limit in the above identity as $\nu \to +\infty$ yields

$$-(u_1,\omega_j) - \int_0^T (u'(t),\omega_j)\theta'(t)\,dt + \int_0^T ((u(t),\omega_j))\theta(t)\,dt = \int_0^T (f(t),\omega_j)\theta(t)\,dt.$$

Integrating again by parts, from the above identity it results that

$$\begin{aligned} &-(u_1,\omega_j) - (u'(t),\omega_j)\theta(t)|_0^T + \int_0^T \frac{d}{dt}(u'(t),\omega_j)\theta(t)\,dt + \int_0^T ((u(t),\omega_j))\theta(t)\,dt \\ &= \int_0^T (f(t),\omega_j)\theta(t)\,dt. \end{aligned} \tag{1.60}$$

Since u is a weak solution to the problem in question, from (1.52) and (1.60) we conclude that $(u_1,\omega_j) = (u'(0),\omega_j)$, for all $j \in \mathbb{N}$. Since $\overline{[\omega_j]_{j\in\mathbb{N}}}^{H_0^1(\Omega)} = H_0^1(\Omega)$ and $\overline{H_0^1(\Omega)}^{L^2(\Omega)} = L^2(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we have that

$$(u_1,v) = (u'(0),v), \quad \text{for all } v \in L^2(\Omega),$$

which proves the desired result.

6þ Step: Uniqueness

Let u and $\hat{u}$ be solutions to problem (1) and let us define $w = u - \hat{u}$. We have,

$$\begin{cases} w'' - \Delta w = 0 \ \text{ in } L^1(0,T;H^{-1}(\Omega)) \\ w = 0 \ \text{ about } \Sigma = \Gamma \times (0,T) \\ w(0) = w'(0) = 0. \end{cases}$$

Note that it makes no sense to compose $w''(t)$ with $w'(t)$ in the duality $H^{-1}(\Omega), H_0^1(\Omega)$ since $w'(t)$ almost always belongs to $L^2(\Omega)$. To circumvent this problem we use the following trick: since $u,\hat{u} \in L^\infty(0,T;H_0^1(\Omega))$ then $\int_\alpha^\beta w(t)\,dt \in H_0^1(\Omega)$ for all $\alpha,\beta \in [0,T]$. Let's take $s \in [0,T]$ and define the following auxiliary function:

$$\psi(t) \begin{cases} -\displaystyle\int_t^s w(\tau)\,d\tau, \ 0 \le t \le s, \\ 0, \ s \le t \le T \end{cases}$$

Let us observe that for each $t \in [0,T]$, $\psi(t) \in H_0^1(\Omega)$ and, in addition,

$$\left| \begin{aligned} \int_0^T ||\psi(t)||\,dt &= \int_0^s \left\|-\int_t^s w(\tau)\,d\tau\right\| dt \le \int_0^s \int_t^s ||w(\tau)||\,d\tau dt \le \text{supess}||w|| \int_0^s (s-t)\,dt \\ &= \text{supess}||w|| \left[st - \frac{t^2}{2}\right]_0^s = \text{supess}||w|| \left[s^2 - \frac{s^2}{2}\right] \le \frac{T^2}{2}\text{supess}||w|| < +\infty. \end{aligned} \right.$$

Therefore, $\psi \in L^1(0,T;H_0^1(\Omega))$. Furthermore, as $\psi' = w \in C([0,s];H_0^1(\Omega))$ results in $\psi \in C([0,s];H_0^1(\Omega))$.

On the other hand, observing that $w'' \in L^\infty(0,T;H^{-1}(\Omega))$ and composing the equationção $w'' - \Delta w = 0$ with the functionção ψ in the duality $L^\infty(0,T;H^{-1}(\Omega)) \times L^1(0,T;H_0^1(\Omega))$ we obtain

$$\int_0^T \langle w''(t), \psi(t)\rangle \; dt - \int_0^T \langle \Delta w(t), \psi(t)\rangle \; dt = 0,$$

or even,

$$\int_0^T \langle w''(t), \psi(t)\rangle \; dt + \int_0^T ((w(t), \psi(t)))\, dt = 0.$$

As $\psi(t) = 0$ for $t \in [s,T]$ of the ĺast identity we can write

$$\int_0^s \langle w''(t), \psi(t)\rangle \; dt + \int_0^s ((w(t), \psi(t)))\, dt = 0. \tag{1.60}$$

But,

$$\frac{d}{dt}\langle w'(t), \psi(t)\rangle = \langle w''(t), \psi(t)\rangle + \langle w'(t), \psi'(t)\rangle . \tag{1.61}$$

Let us note, however, that $\psi'(t) = w(t)$ almost always in $[0,s]$ we have

$$\langle w'(t), \psi'(t)\rangle = \langle w'(t), w(t)\rangle = (w'(t), w(t)) = \frac{1}{2}\frac{d}{dt}|w(t)|^2. \tag{1.62}$$

Also, $\langle w'(t), \psi(t)\rangle = (w'(t), \psi(t))$ and therefore from (1.62) we obtain

$$\frac{d}{dt}(w'(t), \psi(t)) = \langle w''(t), \psi(t)\rangle + \frac{1}{2}\frac{d}{dt}|w(t)|^2. \tag{1.63}$$

Integrating the identity (1.63) of 0 t̀o s we get that

$$(w'(s), \psi(s))|_0^s - \int_0^s \langle w''(t), \psi(t)\rangle \; dt + \frac{1}{2}\left[|w(t)|^2\right]_0^s,$$

what does it imply

$$(w'(s), \underbrace{\psi(s)}_{=0}) - (\underbrace{w'(0)}_{=0}, \psi(0)) = \int_0^s \langle w''(t), \psi(t)\rangle \; dt + \frac{1}{2}\left[|w(s)|^2 - \underbrace{|w(0)|^2}_{=0}\right],$$

and thus:

$$\int_0^s \langle w''(t), \psi(t)\rangle \; dt = -\frac{1}{2}|w(s)|^2. \tag{1.64}$$

On the other hand, since $\psi' = w$ almost always in $[0,s]$ we have

$$((w(t), \psi(t))) = ((\psi'(t), \psi(t))) = \frac{1}{2}\frac{d}{dt}||\psi(t)||^2,$$

and then,

$$\int_0^s ((w(t), \psi(t)))\, dt = \frac{1}{2}\left[\underbrace{||\psi(s)||^2}_{=0} - ||\psi(0)||^2\right]. \tag{1.65}$$

From (1.60), (1.64) and (1.65) it follows that

$$-\frac{1}{2}|w(s)|^2 - \frac{1}{2}||\psi(0)||^2 = 0,$$

which implies that $|w(s)| = 0$, for all $s \in [0,T]$, that is, $w = 0$, which concludes uniqueness.
Observation: Since $w \in C([0,T]; H_0^1(\Omega))$, $w' \in C([0,T]; L^2(\Omega))$ and $w'' \in C([0,T]; H^{-1}(\Omega))$ and $\psi, \psi' \in C([0,T]; H_0^1(\Omega))$ we have that the mappings $t \mapsto (w(t), \psi(t))$ and $t \mapsto (w'(t), \psi(t))$ are of class C^1 in $[0,T]$ and, therefore, it is permissible to perform integrations by parts. Furthermore, the functions $|w(t)|^2$ and $|\psi(t)|^2$ are absolutely continuous, which allows us to perform the calculations above.

Chapter 2

Problem $\dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho u = f$ (weak solution)

Problem 1

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho u = f \quad \text{in} \quad Q\ (\rho > 0) \\ u = 0 \quad \text{on} \quad \Sigma \\ u(0) = u_0(x);\ \ \dfrac{\partial u}{\partial t}(0) = u_1(x);\ \ x \in \Omega \end{array}\right. \tag{1}$$

where

$$u_0 \in H_0^1(\Omega) \cap L^{\rho+2}(\Omega); \quad u_1 \in L^2(\Omega) \quad \text{and} \quad f \in L^2(0,T;L^2(\Omega)) \tag{2}$$

admits at least one weak solution $u\colon Q \to \mathbb{R}$, in the class

$$u \in L^\infty(0,T;H_0^1(\Omega) \cap L^{\rho+2}(\Omega)); \quad u' = \frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)). \tag{3}$$

More precisely, setting $p = \rho + 2$, we have

$$\frac{d}{dt}(u'(t),v) + ((u(t),v)) + \langle |u|^\rho u, v\rangle_{L^{p'}(\Omega),L^p(\Omega)} = (f(t),v) \tag{4}$$

in $\mathcal{D}'(0,T)$ for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$.

$$u(0) = u_0\,; \qquad u'(0) = u_1\,. \tag{5}$$

Furthermore, if $0 < \rho < \dfrac{2}{n-2}$ $(n \geq 3)$ the solution is unique.

Proof:
1ª Step: Approximate Problem

We endow $H_0^1(\Omega) \cap L^p(\Omega)$ with the natural topology $||u||_{H_0^1 \cap L^p} = ||u||_{H_0^1(\Omega)} + ||u||_{L^p(\Omega)}$ so that the linear map

$$T\colon H_0^1(\Omega) \cap L^p(\Omega) \to L^p(\Omega) \times L^2(\Omega) \times \cdots \times L^2(\Omega)$$
$$u \mapsto Tu = \left(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$$

is clearly an *isometry*. Setting

$$W = T(H_0^1(\Omega) \cap L^p(\Omega))$$

it follows that W is a subspace of a separable space and therefore, it is also separable. Since T is an isometry, it follows that $T^{-1}(W)$ possesses a countable dense subset in $H_0^1(\Omega) \cap L^p(\Omega)$, which proves that the latter is likewise separable. Let $(\omega_\nu)_{\nu \in \mathbb{N}}$ be a "basis" of $H_0^1(\Omega) \cap L^p(\Omega)$, that is:

$$\left| \begin{array}{l} \forall\, m,\ \ \omega_1, \ldots, \omega_m \text{ are linearly independent} \\ \text{Finite linear combinations of the } \omega_i\text{'s are dense in } H_0^1(\Omega) \cap L^p(\Omega). \end{array} \right. \tag{2.1}$$

Let us set

$$V_m = [\omega_1, \ldots, \omega_m]$$

the subspace spanned by the first m vectors of the basis. Denoting by $(\,\cdot\,,\,\cdot\,)$ and $((\,\cdot\,,\,\cdot\,))$, respectively, the inner products in $L^2(\Omega)$ and $H_0^1(\Omega)$, we consider in V_m the approximate problem

$$u_m(t) \in V_m \;\Leftrightarrow\; u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i \tag{6}$$

$$(u_m''(t), \omega_j) + ((u_m(t), \omega_j)) + \int_\Omega |u_m(t)|^\rho\, u_m(t)\omega_j dx = (f(t), \omega_j); j = 1, 2, \ldots, m. \tag{7}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega) \cap L^p(\Omega) \tag{8}$$

$$u_m'(0) = u_{1m} \to u_1 \quad \text{in} \quad L^2(\Omega) \tag{9}$$

which, by Carathéodory, possesses a local solution in an interval $[0, t_m)$ where u_m and u_m' are absolutely continuous and u_m'' exists a.e. The a priori estimates will serve to extend the solution to the whole interval $[0, T]$. Indeed, from (6) and (7) and assuming, via Gram-Schmidt, that $\{\omega_j\}_{j\in\mathbb{N}}$ is orthonormal in $L^2(\Omega)$, we obtain

$$g_{jm}''(t) + \sum_{i=1}^{m} g_{im}(t)((\omega_i, \omega_j)) = (f, \omega_j) - \int_\Omega \left| \sum_{i=1}^{m} g_{im}(t)\omega_i \right|^\rho \left(\sum_{i=1}^{m} g_{im}(t)\omega_i \right) \omega_j\, dx.$$

Denoting $F(\lambda) = |\lambda|^\rho \lambda$, from the last identity we can write

$$g_{jm}''(t) + \sum_{i=1}^{m} g_{im}(t)((\omega_i, \omega_j)) = (f, \omega_j) - \int_\Omega F\left(\sum_{i=1}^{m} g_{im}(t)\omega_i \right) \omega_j\, dx,$$

or equivalently,

$$\begin{bmatrix} g_{1m}''(t) \\ g_{2m}''(t) \\ \vdots \\ g_{mm}''(t) \end{bmatrix} + \underbrace{\begin{bmatrix} ((\omega_1, \omega_1)) \cdots ((\omega_1, \omega_m)) \\ ((\omega_2, \omega_1)) \cdots ((\omega_2, \omega_m)) \\ \vdots \qquad\quad \vdots \\ ((\omega_m, \omega_1)) \cdots ((\omega_m, \omega_m)) \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} g_{1m}(t) \\ g_{2m}(t) \\ \vdots \\ g_{mm}(t) \end{bmatrix}}_{=Z(t)}$$

$$+ \begin{bmatrix} \int_\Omega F\left(\sum_{i=1}^m g_{im}(t)\omega_i\right) \omega_1\, dx \\ \int_\Omega F\left(\sum_{i=1}^m g_{im}(t)\omega_i\right) \omega_2\, dx \\ \vdots \\ \int_\Omega F\left(\sum_{i=1}^m g_{im}(t)\omega_i\right) \omega_m\, dx \end{bmatrix} = \underbrace{\begin{bmatrix} (f(t), \omega_1) \\ (f(t), \omega_2) \\ \vdots \\ (f(t), \omega_m) \end{bmatrix}}_{=G_2(t)}$$

Observing that

$$F\left(\sum_{i=1}^{m} g_{im}(t)\omega_i\right) = F\left([\omega_1, \cdots, \omega_m]\begin{bmatrix} g_{1m}(t) \\ g_{2m}(t) \\ \vdots \\ g_{mm}(t) \end{bmatrix}\right)$$

and denoting $[\omega_1, \cdots, \omega_m] = B$ it follows that

$$Z''(t) + AZ(t) + \underbrace{\begin{bmatrix} \int_\Omega F(BZ(t))\omega_1\, dx \\ \int_\Omega F(BZ(t))\omega_2\, dx \\ \vdots \\ \int_\Omega F(BZ(t))\omega_m\, dx \end{bmatrix}}_{=G_1(Z(t))} = G_2(t).$$

Defining

$$Y_1(t) = Z(t),\ Y_2(t) = Z'(t) \ \text{ and } \ Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix},$$

we obtain

$$\begin{aligned} Y'(t) &= \begin{bmatrix} Y_1'(t) \\ Y_2'(t) \end{bmatrix} = \begin{bmatrix} Z'(t) \\ Z''(t) \end{bmatrix} = \begin{bmatrix} Y_2(t) \\ G_2(t) - G_1(Y_1(t)) - AY_1(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ G_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -G_1(Y_1(t)) \end{bmatrix} + \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ G_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -G_1(Y_1(t)) \end{bmatrix} + \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} Y(t). \end{aligned}$$

Consider the following map:

$$\begin{aligned} h : [0, T] \times \mathbb{R}^{2m} &\to \mathbb{R}^{2m} \\ (t, Y) \mapsto h(t, Y) &= \begin{bmatrix} 0 \\ G_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -G_1(Y_1) \end{bmatrix} + \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} Y, \end{aligned}$$

where $Y_1 = (\xi_1, \cdots, \xi_m)$ and $Y = (\xi_1, \cdots, \xi_m, \xi_{m+1}, \cdots, \xi_{2m})$.

Let us note that:

(i) For each fixed Y, $h(t, Y)$ is measurable since f is measurable.

(ii) For almost every t, $h(t, Y)$ is continuous since F is continuous.

(iii) Let $U \subset [0, T] \times \mathbb{R}^{2m}$ be a compact set and $(t, Y) \in U$. Then:

$$||h(t, Y)||_{\mathbb{R}^{2m}} \leq ||G_2(t)||_{\mathbb{R}^m} + C,$$

where C is a constant, since as $Y \in U$, we have that $F(BY_1)$ and Y are bounded in $\mathbb{R}$ and $\mathbb{R}^{2m}$, respectively. Furthermore,

$$\int_0^T ||G_2(t)||_{\mathbb{R}^m}\, dt + CT < +\infty.$$

Therefore,

$$||h(t, Y)||_{\mathbb{R}^{2m}} \leq m_U(t), \quad \text{where } m_U(t) = ||G_2(t)||_{\mathbb{R}^m} + C, \quad \text{and } m_U \in L^1(0, T).$$

Then, by (i), (ii) and (iii), we have that h satisfies the Carathéodory conditions, that is, there exists a solution to the problem

$$(PO) \qquad \begin{cases} Y'(t) = h(t, Y(t)) \\ Y(0) = Y_0, \end{cases}$$

where

$$Y_0 = \begin{bmatrix} Z(0) \\ Z'(0) \end{bmatrix}.$$

Thus, there exists a solution $Y(t)$ of problem (PO) in some interval $[0, t_m]$, $t_m > 0$, $t_m \leq T$, where $Y(t)$ is absolutely continuous. Consequently, the maps $g_{jm}(t)$ and $g'_{jm}(t)$ are also absolutely continuous and defined on the interval $[0, t_m]$. The following a priori estimates will serve to extend the solution $Y(t)$ to the whole interval $[0, T]$.

2nd Step: A Priori Estimate

Multiplying (7) by $g'_{jm}(t)$ and summing over j from 1 to m, we obtain

$$(u''_m(t), u'_m(t)) + ((u_m(t), u'_m(t))) + \int_\Omega |u_m(t)|^\rho u_m(t) u'_m(t) dx = (f(t), u'_m(t)). \tag{10}$$

We observe that the third expression on the left side of the equality makes sense because $|u_m(t)|^\rho u_m(t) \in L^{p'}(\Omega)$. Indeed, since $p = \rho + 2$, then $\frac{1}{p'} = 1 - \frac{1}{\rho + 2}$ which implies that $\frac{1}{p'} = \frac{\rho+1}{\rho+2}$ and therefore $p' = \frac{\rho+2}{\rho+1}$. Thus:

$$\begin{aligned} & \| \, |u_m(t)|^\rho \, u_m(t) \|^{p'}_{L^{p'}(\Omega)} \\ = & \int_\Omega \| u_m(t)|^\rho \, u_m(t)|^{\frac{\rho+2}{\rho+1}} \, dx = \int_\Omega |u_m(t)|^{\rho+2} \, dx = \|u_m(t)\|^p_{L^p(\Omega)} < +\infty \end{aligned} \tag{10'}$$

which proves the assertion. It follows from this, from (6), from the fact that $g_{jm}(t)$ and $g'_{jm}(t)$ are absolutely continuous and by virtue of Hölder's inequality that:

$$\int_\Omega |u_m(t)|^\rho \, u_m(t) u'_m(t) \, dx \in L^1(0, t_m). \tag{11}$$

Consequently, from (10) and (11) it follows that

$$(u''_m(t), u'_m(t)) \in L^1(0, t_m). \tag{12}$$

We claim that

$$(u''_m(t), u'_m(t)) = \frac{1}{2} \frac{d}{dt} \, |u'_m(t)|^2, \tag{13}$$

where $\frac{d}{dt}$ is understood in the distributional sense in $\mathcal{D}'(0, t_m)$. Indeed, let $\theta \in \mathcal{D}(0, t_m)$. From (12) we obtain

$$\begin{aligned} & \langle (u''_m(t), u'_m(t)), \theta \rangle \\ = & \int_0^{t_m} \int_\Omega u''_m(x,t) u'_m(x,t) \, dx \, \theta(t) \, dt = \int_\Omega \int_0^{t_m} \frac{1}{2} \frac{d}{dt} \, (u'_m(x,t))^2 \, \theta(t) \, dt dx \\ = & \frac{1}{2} \int_\Omega \left\{ (u'_m(x,t))^2 \, \theta(t) \Big|_{t=0}^{t=t_m} - \int_0^{t_m} (u'_m(x,t))^2 \, \theta'(t) \, dt \right\} dx \\ = & -\frac{1}{2} \int_0^{t_m} \int_\Omega (u'_m(x,t))^2 \, \theta'(t) \, dt = \frac{1}{2} \Big\langle \frac{d}{dt} \, |u'_m(t)|^2_{L^2(\Omega)} \, , \theta \Big\rangle \end{aligned}$$

which proves (13). Similarly, we prove that

$$((u_m(t), u_m'(t))) = \frac{1}{2}\frac{d}{dt}\,||u_m(t)||^2. \tag{14}$$

Next, we will prove that

$$\int_\Omega |u_m(t)|^\rho\, u_m(t)u_m'(t)\,dx = \frac{1}{\rho+2}\frac{d}{dt}\int_\Omega |u_m(t)|^{\rho+2}\,dx. \tag{15}$$

In fact, initially note that given

$$F(\lambda) = |\lambda|^\rho\,\lambda, \quad \lambda\in\mathbb{R}$$

we have

$$F'(\lambda) = (\rho+1)\,|\lambda|^\rho.$$

It follows from this that, for each $x\in\Omega$,

$$(F\circ u_m)' = \frac{d}{dt}\,(|u_m|^\rho\,u_m) = F'(u_m)\cdot u_m' = (\rho+1)|u_m|^\rho\,u_m' \tag{16}$$

and from (16) we obtain

$$\frac{d}{dt}\left[(|u_m|^\rho\,u_m)u_m\right] = (\rho+1)|u_m|^\rho\,u_m'\cdot u_m + |u_m|^\rho\,u_m\cdot u_m' = (\rho+2)|u_m|^\rho\,u_m u_m'$$

that is,

$$|u_m|^\rho\,u_m u_m' = \frac{1}{\rho+2}\frac{d}{dt}\left[(|u_m|^\rho\,u_m)u_m\right] = \frac{1}{\rho+2}\frac{d}{dt}\,|u_m|^{\rho+2}. \tag{17}$$

Let $\theta\in\mathcal{D}(0,t_m)$. From (11) and (17) we have

$$\begin{aligned}
&\left\langle \int_\Omega |u_m(x,t)|^\rho\,u_m(x,t)u_m'(x,t)dx, \theta\right\rangle = \int_0^{t_m}\int_\Omega |u_m(x,t)|^\rho\,u_m(x,t)u_m'(x,t)\theta(t)dxdt\\
&= \frac{1}{\rho+2}\int_\Omega\int_0^{t_m}\frac{d}{dt}\left[(|u_m(x,t)|^{\rho+2}\right]\theta(t)\,dtdx\\
&= \frac{1}{\rho+2}\int_\Omega\left\{|u_m(x,t)|^{\rho+2}\,\theta(t)\Big|_{t=0}^{t=t_m} - \int_0^{t_m}|u_m(x,t)|^{\rho+2}\,\theta'(t)dt\right\}dx\\
&= -\frac{1}{\rho+2}\int_0^{t_m}\int_\Omega |u_m(x,t)|^{\rho+2}\,dx\,\theta'(t)dt = \left\langle \frac{1}{\rho+2}\frac{d}{dt}\int_\Omega |u_m(t)|^{\rho+2}\,dx, \theta\right\rangle
\end{aligned}$$

which proves (15). From (10), (13), (14) and (15) it follows that

$$\frac{1}{2}\frac{d}{dt}\,|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}\,||u_m(t)||^2 + \frac{1}{\rho+2}\frac{d}{dt}\int_\Omega |u_m(x,t)|^{\rho+2}\,dx = (f(t), u_m'(t))$$

for a.e. $t\in[0,t_m)$.

Multiplying the above equality by 2 and integrating over $[0,t)$, $t\in(0,t_m)$, we obtain

$$\begin{aligned}
|u_m'(t)|^2 + ||u_m(t)||^2 + \frac{2}{p}\,||u_m(t)||^p_{L^p(\Omega)} &= |u_m'(0)|^2 + ||u_m(0)||^2\\
&+ \frac{2}{p}\,||u_m(0)||^p_{L^p(\Omega)} + 2\int_0^t (f(s), u_m'(s))\,ds.
\end{aligned}$$

Using Schwarz's inequality and the fact that $2ab \leq a^2 + b^2 \quad (a, b > 0)$, from the last equality it follows that

$$|u'_m(t)|^2 + ||u_m(t)||^2 + \frac{2}{p}\,||u_m(t)||^p_{L^p(\Omega)} \leq |u'_m(0)|^2 + ||u_m(0)||^2 + \frac{2}{p}\,||u_m(0)||^p_{L^p(\Omega)} + \int_0^T |f(s)|^2\,ds + \int_0^t |u'_m(s)|^2\,dx.$$

Whence

$$|u'_m(t)|^2 + ||u_m(t)||^2 + \frac{2}{p}\,||u_m(t)||^p_{L^p(\Omega)} \leq |u'_m(0)|^2 + ||u_m(0)||^2 + \frac{2}{p}\,||u_m(0)||^p_{L^p(\Omega)} + ||f||^2_{L^2(Q)} + \int_0^t \left\{|u'_m(s)|^2 + ||u_m(s)||^2 + \frac{2}{p}\,||u_m(s)||^p_{L^p(\Omega)}\right\} ds. \tag{18}$$

Now, from (8) and (9) we obtain the existence of a constant $c_0 > 0$ such that

$$|u'_m(0)|^2 + ||u_m(0)||^2 + \frac{2}{p}\,||u_m(0)||^p_{L^p(\Omega)} \leq c_0\,; \quad \forall\, m \in \mathbb{N}. \tag{19}$$

From (18) and (19) we conclude that

$$|u'_m(t)|^2 + ||u_m(t)||^2 + \frac{2}{p}\,||u_m(t)||^p_{L^p(\Omega)} + ||f||^2_{L^2(Q)} \leq c_1 + \int_0^t \left\{|u'_m(s)|^2 + ||u_m(s)||^2 + \frac{2}{p}\,||u_m(s)||^p_{L^p(\Omega)}\right\} ds$$

where $c_1 > 0$. It follows from this, by virtue of Gronwall's inequality, that $\exists\, c > 0$ (independent of t and m) such that

$$|u'_m(t)|^2 + ||u_m(t)||^2 + \frac{2}{p}\,||u_m(t)||^p_{L^p(\Omega)} \leq c; \quad \forall\, t \in [0, t_m);\ \forall\, m \in \mathbb{N}. \tag{20}$$

From the inequality above, it follows that

$$(i) \qquad \sum_{j=1}^m \left[g'_{jm}(t)\right]^2 = \left|\sum_{j=1}^m g'_{jm}(t)\omega_j\right|^2 = |u'_m(t)|^2 \leq c.$$

Analogously, since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ we obtain

$$(ii) \qquad \sum_{j=1}^m \left[g_{jm}(t)\right]^2 = \left|\sum_{j=1}^m g_{jm}(t)\omega_j\right|^2 = |u_m(t)|^2 \leq ||u_m(t)||^2 \leq \tilde{c},$$

where $\tilde{c}$ is a positive constant.

By (i) and (ii) we have that $Y(t)$ is bounded in $\mathbb{R}^{2m}$ independently of t and m, since

$$||Y(t)||^2_{\mathbb{R}^{2m}} = \sum_{j=1}^m g_{jm}(t)^2 + \sum_{j=1}^m g'_{jm}(t)^2 \leq k_1,$$

where k_1 is a positive constant, for all $t \in [0, t_m]$ and $m \in \mathbb{N}$. Thus, we can prolong Y to the whole interval $[0, T]$ and the inequality in (20) remains valid for all $t \in [0, T]$ and for all $m \in \mathbb{N}$.

Therefore, from (20) we can extend u_m to the whole interval $[0,T]$ and furthermore we also have that

$$\begin{aligned} &(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)) && (21)\\ &(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;L^p(\Omega)) && (22)\\ &(u_m') \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). && (23)\end{aligned}$$

Note also that from (10') and (22) it follows that

$$\left(|u_m|^\rho\, u_m\right) \quad \text{is bounded in} \quad L^{p'}(0,T;L^{p'}(\Omega)) = L^{p'}(Q). \tag{24}$$

From (21), (22), (23) and (24) we obtain the existence of a subsequence (u_ν) of (u_m) such that

$$\begin{aligned} &u_\nu \overset{*}{\rightharpoonup} u \quad \text{weak-* in} \quad L^\infty(0,T;H_0^1(\Omega)) && (25)\\ &u_\nu \rightharpoonup u \quad \text{weakly in} \quad L^p(0,T;L^p(\Omega)) && (26)\\ &u_\nu' \overset{*}{\rightharpoonup} v = u' \quad \text{weak-* in} \quad L^\infty(0,T;L^2(\Omega))\ ^1 && (27)\\ &|u_\nu|^\rho\, u_\nu \rightharpoonup \chi \quad \text{weakly in} \quad L^{p'}(0,T;L^{p'}(\Omega)) && (28)\end{aligned}$$

3rd Step: Passage to the Limit

Setting

$$B_0 = H_0^1(\Omega) \overset{\text{comp.}}{\hookrightarrow} B = L^2(\Omega) \hookrightarrow B_1 = L^2(\Omega)$$

and

$$W = \{u \in L^2(0,T;B_0); \quad u' \in L^2(0,T;B_1)\}$$

endowed with the topology

$$||u||_W = ||u||_{L^2(0,T;H_0^1)} + ||u'||_{L^2(0,T;L^2(\Omega))}$$

it follows from (21) and (23) that

$$(u_\nu) \quad \text{is bounded in} \quad W. \tag{29}$$

Thus, by the Aubin-Lions Theorem (see Theorem 5.1 in Lions [10]), we obtain a subsequence (u_μ) of (u_ν) such that

$$u_\mu \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{30}$$

From the last convergence, we obtain the existence of a subsequence, which we will still denote by the same notation, such that

$$|u_\mu|^\rho\, u_\mu \to |u|^\rho\, u \quad \text{a.e. in} \quad Q. \tag{31}$$

Setting

$$g_\mu = |u_\mu|^\rho\, u_\mu \quad \text{and} \quad g = |u|^\rho\, u$$

[1]It is worth noting that since $u_\nu \in C^1([0,T];L^2(\Omega))$, it follows that the classical derivative and the distributional one coincide in the sense of vector-valued distributions in $\mathcal{D}'(0,T;L^2(\Omega))$. Thus we can consider u_ν' in the sense of vector-valued distributions and consequently $v = u'$ in $\mathcal{D}'(0,T;L^2(\Omega))$.

it follows from (24) and (31) that

$$g_\mu \to g \quad \text{a.e. in} \quad Q$$

and

$$||g_\mu||_{L^{p'}(Q)} \leq c; \quad \forall \mu \in \mathbb{N}.$$

Thus, by Lions' Lemma (see Lemma 1.3 in Lions [10]), we conclude that

$$g_\mu \rightharpoonup g \quad \text{weakly in} \quad L^{p'}(Q)$$

that is,

$$|u_\mu|^\rho u_\mu \rightharpoonup |u|^\rho u \quad \text{weakly in} \quad L^{p'}(Q). \tag{32}$$

From (28) and (32), by the uniqueness of the limit, we conclude that

$$\chi = |u|^\rho u. \tag{33}$$

Let $j \in \mathbb{N}$ and $\mu \in \mathbb{N}$ such that $\mu \geq j$ and consider $\theta \in \mathcal{D}(0,T)$. Multiplying (7) by θ and integrating over $[0,T]$, we obtain

$$\begin{aligned}\int_0^T (u''_\mu(t), \omega_j)\theta(t)dt + \int_0^T ((u_\mu(t), \omega_j))\theta(t)dt \\ + \int_0^T \int_\Omega |u_\mu(t)|^\rho u_\mu(t)\omega_j(x)dx\,\theta(t)\,dt \\ = \int_0^T (f(t), \omega_j)\theta(t)\,dt,\end{aligned}$$

which implies that

$$\begin{aligned}&- \int_0^T (u'_\mu(t), \omega_j)\theta'(t)dt + \int_0^T ((u_\mu(t), \omega_j))\theta(t)dt \\ &+ \int_0^T \int_\Omega |u_\mu(t)|^\rho u_\mu(t)\omega_j(x)dx\,\theta(t)\,dt \\ &= \int_0^T (f(t), \omega_j)\theta(t)\,dt.\end{aligned} \tag{34}$$

Now, from (25), (27), (23) and (32) we have

$$\begin{aligned}\int_0^T \langle u_\mu(t), \xi(t)\rangle_{H_0^1, H^{-1}} dt \to \int_0^T \langle u(t), \xi(t)\rangle_{H_0^1, H^{-1}}\, dt \\ \forall \xi \in L^1(0,T; H^{-1}(\Omega)),\end{aligned} \tag{35}$$

$$\int_0^T (u'_\mu(t), \eta(t))dt \to \int_0^T (u'(t), \eta(t))dt \quad \forall \eta \in L^1(0,T; L^2(\Omega)), \tag{36}$$

$$\begin{aligned}&\int_0^T \int_\Omega |u_\mu(x,t)|^\rho u_\mu(x,t)\beta(x,t)dxdt \to \\ &\to \int_0^T \int_\Omega |u(x,t)|^\rho u(x,t)\beta(x,t)dxdt \quad \forall \beta \in L^p(Q).\end{aligned} \tag{37}$$

Taking in particular

$$\xi = -\Delta\omega_j\,\theta; \qquad \eta = \omega_j\theta' \quad \text{and} \quad \beta = \omega_j\theta$$

we obtain from (35), (36) and (37)

$$\int_0^T \langle u_\mu(t), -\Delta\omega_j\rangle\theta(t)\,dt \to \int_0^T \langle u(t), -\Delta\omega_j\rangle\theta(t)\,dt,$$

i.e.

$$\int_0^T ((u_\nu(t), \omega_j))\theta(t) \to \int_0^T ((u(t), \omega_j))\theta(t)\,dt, \tag{38}$$

$$\int_0^T (u'_\mu(t), \omega_j)\theta'(t)\,dt \to \int_0^T (u'(t), \omega_j)\theta'(t)\,dt, \tag{39}$$

$$\int_0^T \int_\Omega |u_\mu(x,t)|^\rho\, u_\mu(x,t)\omega_j\,dx\theta(t)dt \to \int_0^T \int_\Omega |u(x,t)|^\rho\, u(x,t)\omega_j dx\theta(t)\,dt. \tag{40}$$

From (34), (38), (39) and (40) in the limit; we obtain

$$\begin{aligned} &-\int_0^T (u'(t), \omega_j)\theta'(t)\,dt + \int_0^T ((u(t), \omega_j))\theta(t)\,dt \\ &+ \int_0^T \int_\Omega |u(x,t)|^\rho\, u(x,t)\omega_j\,dx\,\theta(t)dt = \int_0^T (f(t), \omega_j)\theta(t)\,dt. \end{aligned} \tag{41}$$

Since the finite linear combinations of the ω_j's are dense in $H_0^1(\Omega) \cap L^p(\Omega)$ the equality in (41) remains valid for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$, i.e.,

$$\begin{aligned} &-\int_0^T (u'(t), v)\theta'(t)\,dt + \int_0^T ((u(t), v))\theta(t)\,dt \\ &+ \int_0^T \int_\Omega |u(x,t)|^\rho\, u(x,t)v(s)ds\,\theta(t)\,dt = \int_0^T (f(t), v)\theta(t)\,dt, \end{aligned} \tag{42}$$

for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$, or even,

$$\langle \frac{d}{dt}(u'(t), v), \theta\rangle + \langle ((u(t), v)), \theta\rangle + \left\langle \int_\Omega |u(t)|^\rho\, u(t)v\,dx, \theta \right\rangle = \langle (f(t), v), \theta\rangle, \ \forall\theta \in \mathcal{D}(0,T)$$

which leads us to conclude that

$$\frac{d}{dt}(u'(t), v) + ((u(t), v)) + \int_\Omega |u(t)|^\rho\, u(t)v\,dx = (f(t), v) \text{ in } \mathcal{D}'(0,T). \tag{43}$$

Identifying $L^2(\Omega)$ with its dual, we have the chains

$$H_0^1 \cap L^p(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow H^{-1}(\Omega) + L^{p'}(\Omega)$$
$$H_0^1 \cap L^p(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{p'}(\Omega) \hookrightarrow H^{-1}(\Omega) + L^{p'}(\Omega)$$

By virtue of the identification above from (42) we can write,

$$\begin{aligned} &\left\langle -\int_0^T u'(t)\theta'(t)dt, v \right\rangle + \left\langle \int_0^T -\Delta u(t)\theta(t)dt, v \right\rangle \\ &+ \left\langle \int_0^T |u(t)|^\rho\, u(t)\theta(t)dt, v \right\rangle = \left\langle \int_0^T f(t)\theta(t)dt, v \right\rangle \end{aligned}$$

where $\langle \cdot , \cdot \rangle$ designates the duality $H^{-1}(\Omega) + L^{p'}(\Omega)$, $H_0^1(\Omega) \cap L^p(\Omega)$. It follows then that

$$u'' - \Delta u + |u|^\rho u = f \quad \text{in} \quad \mathcal{D}'(0,T; H^{-1}(\Omega) + L^{p'}(\Omega)). \tag{44}$$

However, since

$$\begin{aligned} &f \in L^2(0,T; L^2(\Omega)) \subset L^2(0,T; H^{-1} + L^{p'}(\Omega)) \\ &\Delta u \in L^\infty(0,T; H^{-1}(\Omega)) \subset L^\infty(0,T; H^{-1}(\Omega) + L^{p'}(\Omega)) \\ &|u|^\rho u \in L^\infty(0,T; L^{p'}(\Omega)) \subset L^\infty(0,T; H^{-1}(\Omega) + L^{p'}(\Omega))\ ^{(2)} \end{aligned}$$

from (44) it follows that

$$u'' \in L^2(0,T; H^{-1}(\Omega) + L^{p'}(\Omega)) \tag{45}$$

and

$$u'' - \Delta u + |u|^\rho u = f \quad \text{in} \quad L^2(0,T; H^{-1}(\Omega) + L^{p'}(\Omega)). \tag{46}$$

4ª Step: Initial Conditions

Note initially that from (25), (27) and (45) we have

$$\begin{aligned} &u \in C^0([0,T]; L^2(\Omega)) \cap C_s(0,T; H_0^1(\Omega)) \\ &u' \in C^0([0,T]; H^{-1}(\Omega) + L^{p'}(\Omega)) \cap C_s(0,T; L^2(\Omega)) \end{aligned}$$

making sense to speak of $u(0)$, $u(T)$, $u'(0)$ and $u'(T)$. We will prove initially that

$$u(0) = u_0 \,. \tag{47}$$

Indeed, let $\theta \in C^1([0,T])$ such that $\theta(0) = 1$ and $\theta(T) = 0$. From (27) it follows that if $\nu > j$ (j arbitrary but fixed)

$$\int_0^T (u_\nu'(t), \omega_j)\theta(t)\,dt \to \int_0^T (u'(t), \omega_j)\theta(t)\,dt.$$

Integrating by parts

$$-(u_\nu(0), \omega_j) - \int_0^T (u_\nu(t), \omega_j)\theta'(t)\,dt \to -(u(0), \omega_j) - \int_0^T (u(t), \omega_j)\theta'(t)\,dt.$$

Now from (25) it follows that

$$\int_0^T (u_\nu(t), \omega_j)\theta'(t)\,dt \to \int_0^T (u(t), \omega_j)\theta'(t)\,dt,$$

which implies that

$$(u_\nu(0), \omega_j) \to (u(0), \omega_j) \quad \forall\, j \in \mathbb{N}.$$

It follows from this that

$$u_\nu(0) \rightharpoonup u(0) \quad \text{weakly in} \quad L^2(\Omega).$$

[2] Note that from (10') and (22) we have $|u_\mu|^\rho u_\nu$ is bounded in $L^\infty(0,T; L^{p'}(\Omega))$.

On the other hand, from (8) we have that

$$u_\nu(0) \rightharpoonup u_0 \quad \text{weakly in} \quad L^2(\Omega)$$

which leads us, given the uniqueness of the weak limit, to conclude what is desired in (47).

We will prove next that:

$$u'(0) = u_1\,. \tag{48}$$

Let $\theta \in C^1([0,T])$ such that $\theta(0) = 1$ and $\theta(T) = 0$ and consider $j \in \mathbb{N}$. Thus for $\mu > j$ from (7) we obtain

$$\begin{aligned}&\int_0^T (u''_\mu(t), \omega_j)\theta(t)\,dt + \int_0^T ((u_\mu(t), \omega_j))\theta(t)\,dt \\ &+ \int_0^T \int_\Omega |u_\mu(x,t)|^\rho\, u_\mu(x,t)\omega_j\,dx\,\theta(t)dt = \int_0^T (f(t), \omega_j)\theta(t)\,dt.\end{aligned}$$

Integrating by parts

$$\begin{aligned}&-(u'_\mu(0), \omega_j) - \int_0^T (u'_\mu(t), \omega_j)\theta'(t)\,dt + \int_0^T ((u_\mu(t), \omega_j))\theta(t)\,dt \\ &+ \int_0^T \int_\Omega |u_\mu(t)|^\rho\, u_\mu(t)\omega_j\,dx\,\theta(t)dt = \int_0^T (f(t), \omega_j)\theta(t)\,dt.\end{aligned}$$

Taking the limit we obtain as before

$$\begin{aligned}&-(u_1, \omega_j) - \int_0^T (u'(t), \omega_j)\theta'(t)\,dt + \int_0^T ((u(t), \omega_j))\theta(t)\,dt \\ &+ \int_0^T \int_\Omega |u(x,t)|^\rho\, u(x,t)\omega_j\,dx\,\theta(t)dt = \int_0^T (f(t), \omega_j)\theta(t)\,dt.\end{aligned}$$

By the totality of the ω_j's in $H_0^1(\Omega) \cap L^p(\Omega)$ we obtain

$$\begin{aligned}&-(u_1, v) - \int_0^T (u'(t), v)\theta'(t)\,dt + \int_0^T ((u(t), v))\theta(t)\,dt \\ &+ \int_0^T \int_\Omega |u(x,t)|^\rho\, u(x,t)v\,dx\,\theta(t)dt = \int_0^T (f(t), v)\theta(t)\,dt, \quad \forall\, v \in H_0^1 \cap L^p.\end{aligned}$$

Integrating by parts again, it follows that

$$\begin{aligned}&-(u_1, v) + (u'(0), v) + \int_0^T \langle u''(t), v\rangle\theta(t)dt + \int_0^T ((u(t), v))\theta(t)dt \\ &+ \int_0^T \int_\Omega |u(x,t)|^\rho\, u(x,t)v\,dx\,\theta(t)\,dt = \int_0^T (f(t), v)\theta(t)\,dt,\end{aligned} \tag{49}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ designates the duality $H^{-1} + L^{p'},\ H_0^1 \cap L^p$.

Now, since

$$\langle u''(t), v\rangle = \frac{d}{dt}\,(u'(t), v) \in L^2(0,T), \tag{50}$$

it follows from (43), (49) and (50) that

$$(u_1, v) = (u'(0), v); \quad \forall\, v \in H_0^1(\Omega) \cap L^p(\Omega),$$

whence (48) is concluded.

5ª Step: Uniqueness

We claim that problem 1 admits a unique weak solution provided that $0 < \rho \leq \dfrac{2}{n-2}\cdot$ Indeed, let u and v be weak solutions of (1) and consider $\omega = u - v$. Then as seen previously

$$\begin{gathered} \omega \in L^\infty(0,T; H_0^1(\Omega) \cap L^p(\Omega)); \quad \omega' \in L^\infty(0,T; L^2(\Omega)) \\ \text{and} \quad \omega'' \in L^2(0,T; H^{-1}(\Omega) + L^{p'}(\Omega)) \end{gathered} \tag{51}$$

and satisfies the problem

$$\left| \begin{array}{l} \omega'' - \Delta\omega = (f - |u|^\rho u) - (f - |v|^\rho v) = |v|^\rho v - |u|^\rho u \text{ in } L^2(0,T; H^{-1} + L^{p'}) \\ \omega = 0 \quad \text{on} \quad \Sigma \\ \omega(0) = \omega'(0) = 0 \end{array} \right. \tag{52}$$

We will use the Visik-Ladyzenskaya method. We consider, for each $s \in [0,T]$ the following function

$$\psi(t) = \begin{cases} -\displaystyle\int_t^s \omega(\xi)d\xi; & 0 \leq t \leq s \\ 0; & s \leq t \leq T \end{cases} \tag{53}$$

Letting ψ' be the derivative in the sense of vector-valued distributions of ψ, we have

$$\psi'(t) = \begin{cases} \omega(t); & 0 \leq t \leq s \\ 0; & s \leq t \leq T \end{cases} \tag{54}$$

From the expressions above and from (51) it is evident that

$$\psi, \psi' \in L^\infty(0,T; H_0^1(\Omega) \cap L^p(\Omega)) \tag{55}$$

which implies that

$$\psi \in C^0([0,T]; H_0^1(\Omega) \cap L^p(\Omega)). \tag{56}$$

Composing $(52)_1$ with ψ in the duality $L^2(0,T; H^{-1} + L^{p'}) \times L^2(0,T; H_0^1 \cap L^p)$ and observing that $\psi = 0$ in $[s,T]$ we obtain

$$\int_0^s \langle \omega''(t), \psi(t) \rangle dt + \int_0^s \langle -\Delta\omega(t), \psi(t) \rangle dt = \int_0^s \langle |v|^\rho v - |u|^\rho u, \psi(t) \rangle dt. \tag{57}$$

Integrating by parts and using the fact that $\langle -\Delta\omega, \psi \rangle = ((\omega, \psi))$ from (57) it follows that

$$\begin{gathered} \langle \omega'(s), \psi(s) \rangle - \langle \omega'(0), \psi(0) \rangle - \int_0^s (\omega'(t), \psi'(t))dt \\ + \int_0^s ((\omega(t), \psi(t))) = \int_0^s \langle |v|^\rho v - |u|^\rho u, \psi(t) \rangle dt, \end{gathered}$$

or even from $(52)_3$, (53) and (54) we have

$$-\int_0^s (\omega'(t), \omega(t))dt + \int_0^s ((\psi'(t), \psi(t))) = \int_0^s \langle |v|^\rho v - |u|^\rho u, \psi(t) \rangle dt,$$

that is,

$$-\frac{1}{2}\int_0^s \frac{d}{dt}|\omega(t)|^2\, dt + \frac{1}{2}\int_0^s \frac{d}{dt}||\psi(t)||^2\, dt = \int_0^s \langle |v|^\rho v - |u|^\rho u, \psi(t) \rangle dt,$$

which leads us to the following expression

$$-\frac{1}{2}\,|\omega(s)|^2+\frac{1}{2}\,|\omega(0)|^2+\frac{1}{2}\,||\psi(s)||^2-\frac{1}{2}\,||\psi(0)||^2=\int_0^s\langle |v|^\rho v-|u|^\rho u,\psi(t)\rangle dt,$$

and again thanks to $(52)_3$ and (53) we conclude that

$$-\frac{1}{2}\,|\omega(s)|^2-\frac{1}{2}\,||\psi(0)||^2=\int_0^s\int_\Omega(|v|^\rho v-|u|^\rho u)\psi(t)dxdt. \tag{58}$$

On the other hand, setting

$$F(\lambda)=|\lambda|^\rho\,\lambda,\quad \lambda\in\mathbb{R}$$

then

$$F'(\lambda)=(\rho+1)|\lambda|^\rho,\quad \lambda\in\mathbb{R},$$

which implies that $F\in C^1(\mathbb{R})$. Thus, given $\alpha,\beta\in\mathbb{R}$ there exists, by virtue of the Mean Value Theorem (M.V.T.), $\xi\in]\alpha,\beta[$ such that

$$|F(\beta)-F(\alpha)|\le|F'(\xi)|\,|\beta-\alpha|,$$

or even,

$$|F(\beta)-F(\alpha)|\le(\rho+1)|\xi|^\rho\,|\beta-\alpha|. \tag{59}$$

Now from the fact that $\xi\in]\alpha,\beta[,\quad\exists\,\theta=\theta(\alpha,\beta)\in]0,1[$ such that

$$\xi=(1-\theta)\alpha+\theta\cdot\beta=\alpha+(\beta-\alpha)\cdot\theta. \tag{60}$$

In the particular case where $\alpha(x,t)=u(x,t),\quad\beta(x,t)=v(x,t)$ it follows from (59) and (60) that

$$\begin{aligned}
&|\,|v(x,t)|^\rho\,v(x,t)-|u(x,t)|^\rho\,u(x,t)|\\
&\le(\rho+1)|u(x,t)+(v(x,t)-u(x,t))\cdot\theta(x,t)|^\rho\,|v(x,t)-u(x,t)|\\
&\le(\rho+1)\big\{|u(x,t)|+|v(x,t)|+|u(x,t)|\big\}^\rho\,|\omega(x,t)|\\
&\le(\rho+1)\big\{2|u(x,t)|+2|v(x,t)|\big\}^\rho\,|\omega(x,t)|\\
&\quad=(\rho+1)\cdot2^\rho\cdot\big\{|u(x,t)|+|v(x,t)|\big\}^\rho\,|\omega(x,t)|\\
&\quad\le(\rho+1)\cdot2^\rho\begin{cases}2^\rho|v(x,t)|^\rho\,|\omega(x,t)|\text{ if }|u(x,t)|\le|v(x,t)|\\ 2^\rho|u(x,t)|^\rho\,|\omega(x,t)|\text{ if }|u(x,t)|\ge|v(x,t)|\end{cases}\\
&\le(\rho+1)2^\rho\cdot2^\rho\big\{|u(x,t)|^\rho+|v(x,t)|^\rho\big\}|\omega(x,t)|
\end{aligned}$$

i.e,

$$|\,|v(x,t)|^\rho\,v(x,t)-|u(x,t)|^\rho\,u(x,t)|\le(\rho+1)2^{2\rho}\big\{|u(x,t)|^\rho+|v(x,t)|^\rho\big\}|\omega(x,t)|. \tag{61}$$

From (58) and (61) it follows that

$$\frac{1}{2}\,|\omega(s)|^2+\frac{1}{2}\,||\psi(0)||^2\le c(\rho)\int_0^s\int_\Omega\big\{|u(x,t)|^\rho+|v(x,t)|^\rho\big\}|\omega(x,t)|\,|\psi(x,t)|\,dxdt. \tag{62}$$

Note that according to the Sobolev embedding Theorem we have

$$H_0^1(\Omega)\hookrightarrow L^q(\Omega),$$

where

$$1 \leq q \leq \frac{2n}{n-2}\,. \tag{63}$$

It is worth noting that

$$\omega(t) \in L^2(\Omega) \quad \text{a.e. in} \quad]0,T[\tag{64}$$
$$\psi(t) \in L^q(\Omega) \quad \text{a.e. in} \quad]0,T[\tag{65}$$

this is because $\omega, \psi \in L^\infty(0,T;H_0^1(\Omega))$ and $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$. We claim that:

$$|u(t)|^\rho, |v(t)|^\rho \in L^{\frac{2(\rho+1)}{\rho}}(\Omega) \quad \text{a.e. in} \quad]0,T[\,. \tag{66}$$

Indeed, we have by hypothesis that $0 < \rho \leq \dfrac{2}{n-2}$ which implies that $2 < 2(\rho+1) \leq \dfrac{2n}{n-2}$. Since Ω is bounded, it follows, taking $q = 2(\rho+1)$ and from (63)

$$H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega). \tag{67}$$

Now since $u(t), v(t) \in H_0^1(\Omega)$ a.e. in $]0,T[$, from the embedding above it follows that $u(t) \in L^{2(\rho+1)} = L^{\rho\frac{2(\rho+1)}{\rho}}(\Omega)$ a.e. in $]0,T[$ and therefore

$$|u(t)|^\rho, |v(t)|^\rho \in L^{\frac{2(\rho+1)}{\rho}}(\Omega) \quad \text{a.e. in} \quad]0,T[$$

which proves (66). Note that,

$$\frac{1}{\frac{2(\rho+1)}{\rho}} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1. \tag{68}$$

It follows from (62), (64), (65), (66), (68) and by the generalized Hölder inequality that

$$\begin{aligned} &\frac{1}{2}\,|\omega(s)|^2 + \frac{1}{2}\,||\psi(0)||^2 \\ &\leq c_1 \int_0^s \big\{\big(||\,u(t)|^\rho||_{L^{\frac{2(\rho+1)}{\rho}}(\Omega)} + ||\,|v(t)|^\rho||_{L^{\frac{2(\rho+1)}{\rho}}(\Omega)}\big)|\omega(t)|_{L^2(\Omega)}\,||\psi(t)||\big\}dt. \end{aligned} \tag{69}$$

But from (67) and the fact that $u \in L^\infty(0,T;H_0^1(\Omega))$ we have

$$\operatorname*{ess\,sup}_{t\in[0,T]}||\,|u(t)|^\rho||_{L^{\frac{2(\rho+1)}{\rho}}(\Omega)} = \operatorname*{ess\,sup}_{t\in[0,T]}\left[\int_\Omega |u(t)|^{2(\rho+1)}\,dx\right]^{\frac{1}{\frac{2(\rho+1)}{\rho}}} \leq k_1 \operatorname*{ess\,sup}_{t\in[0,T]}||u(t)||^\rho < +\infty$$

and from (69) we conclude that

$$\frac{1}{2}\,|\omega(s)|^2 + \frac{1}{2}\,||\psi(0)||^2 \leq c_2 \int_0^s |\omega(t)|\,||\psi(t)||\,dt. \tag{70}$$

Finally, setting

$$\omega_1(t) = \int_0^t \omega(\xi)\,d\xi \tag{71}$$

we have from (53), for all $t \in [0,s]$,

$$\psi(t) = -\int_t^s \omega(\xi)\,d\xi = -\Big[\underbrace{\int_0^s \omega(\xi)d\xi}_{\omega_1(s)} - \underbrace{\int_0^t \omega(\xi)d\xi}_{\omega_1(t)}\Big] = \omega_1(t) - \omega_1(s). \tag{72}$$

Thus, from (72) we can write

$$\psi(0) = \underbrace{\omega_1(0)}_{=0} - \omega_1(s) = -\omega_1(s). \tag{73}$$

Substituting (72) and (73) into (70) implies that

$$\begin{aligned}
&\frac{1}{2}\,|\omega(s)|^2 + \frac{1}{2}\,||\omega_1(s)||^2 \le c_2 \int_0^s |\omega(t)|\,||\omega_1(t) - \omega_1(s)||\,dt \\
&\le c_2\Big\{\int_0^s |\omega(t)|\,||\omega_1(t)||\,dt + \int_0^s |\omega(t)\,||\omega_1(s)||\,dt\Big\} \\
&= c_2\Big\{\int_0^s |\omega(t)|\,||\omega_1(t)||\,dt + \int_0^s \sqrt{2sc_2}|\omega(t)|\frac{1}{\sqrt{2sc_2}}\,||\omega_1(s)||\,dt\Big\} \\
&\le \frac{c_2}{2}\,\Big\{\int_0^s |\omega(t)|^2 dt + \int_0^s ||\omega_1(t)||^2 dt + 2sc_2\int_0^s |\omega(t)|^2 dt + \frac{1}{2sc_2}\,||\omega_1(s)||^2\Big(\int_0^s ds\Big)\Big\} \\
&\le \frac{c_2}{2}\int_0^s |\omega(t)|^2 dt + \frac{c_2}{2}\int_0^s ||\omega_1(t)||^2 dt + Tc_2^2\int_0^s |\omega(t)|^2 dt + \frac{1}{4}\,||\omega_1(s)||^2 \\
&\le \frac{1}{4}\,||\omega_1(s)||^2 + c_3\int_0^s \big(|\omega(t)|^2 + ||\omega_1(t)||^2\big)dt,
\end{aligned}$$

which implies that

$$\frac{1}{4}\,|\omega(s)|^2 + \frac{1}{4}\,||\omega_1(s)||^2 \le c_3\int_0^s \big(|\omega(t)|^2 + ||\omega_1(t)||^2\big)dt.$$

From the inequality above by virtue of Gronwall's inequality it follows that

$$\frac{1}{4}\,|\omega(s)|^2 + \frac{1}{4}\,||\omega_1(s)||^2 \le 0.$$

Thus, we obtain

$$\omega(s) = 0 \quad \text{in} \quad L^2(\Omega); \quad \forall\, s \in (0,T)$$

and from the fact that $\omega(0) = 0$ we have

$$\omega(s) = 0 \quad \text{in} \quad L^2(\Omega); \quad \forall\, s \in [0,T]$$

which concludes the proof. □

Chapter 3

Problem $\dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho u = f$ (Regular solution)

Problem 2

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\rho u = f \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(0) = u_0(x); \quad \dfrac{\partial u}{\partial t}(0) = u_1(x) \end{array}\right. \tag{1}$$

subject to the initial conditions

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega); \quad u_1 \in H_0^1(\Omega) \quad \text{and} \quad f, \frac{\partial f}{\partial t} \in L^2(0,T;L^2(\Omega)) \tag{2}$$

admits a unique strong solution if $0 < \rho \leq \dfrac{2}{n-2}$ $\quad (n \geq 3)$, in the class

$$u \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)); \quad u' \in L^\infty(0,T;H_0^1(\Omega)) \text{ and } u'' \in L^\infty(0,T;L^2(\Omega)). \tag{3}$$

More precisely, setting $p = \rho + 2$ we have

$$(u''(t), v) + ((u(t), v)) + (|u(t)|^\rho u(t), v) = (f(t), v) \tag{4}$$

in $\mathcal{D}'(0,T)$, for all $v \in H_0^1(\Omega) \cap H^2(\Omega)$.

$$u(0) = u_0\,, \qquad u'(0) = u_1\,. \tag{5}$$

Proof:

1ª Step: Approximate Problem

Initially we observe that by the Sobolev embedding Theorem, we have

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega); \quad q \leq \frac{2n}{n-2} \cdot \tag{6}$$

Since $\rho \leq \dfrac{2}{n-2}$ by hypothesis then

$$2\rho \leq \frac{4}{n-2} \Leftrightarrow 2\rho + 2 \leq \frac{4}{n-2} + 2 \Leftrightarrow 2\rho + 2 \leq \frac{2n}{n-2} \cdot$$

Therefore,

$$H_0^1(\Omega) \hookrightarrow L^{2\rho+2}(\Omega) \hookrightarrow L^{\rho+2}(\Omega) \tag{7}$$

and consequently

$$|v|^{\rho+2} \in L^1(\Omega) \quad \text{and} \quad |v|^\rho v \in L^2(\Omega); \quad \forall\, v \in H_0^1(\Omega). \tag{8}$$

Let (ω_ν) be a “basis” of $H_0^1(\Omega) \cap H^2(\Omega)$. Let us set

$$V_m = [\omega_1, \ldots, \omega_m].$$

In V_m we consider the approximate problem

$$u_m(t) \in V_m \Leftrightarrow u_m(t) = \sum_{i=1}^{n} g_{im}(t)\omega_i \tag{9}$$

$$(u_m''(t), \omega_j) + ((u_m(t), \omega_j)) + (|u_m(t)|^\rho u_m(t), \omega_i)\ ^{(3)} = (f(t), v) \tag{10}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega) \cap H^2(\Omega), \tag{11}$$

$$u_m'(0) = u_{1m} \to u_1 \quad \text{in} \quad H_0^1(\Omega), \tag{12}$$

which by Carathéodory’s Theorem possesses a local solution in some interval $[0, t_m)$, where $u_m(t)$, $u_m'(t)$ are absolutely continuous and $u_m''(t)$ exists a.e. The a priori estimates will serve to extend the solution to the whole interval $[0, T]$ (The proof of Carathéodory’s theorem can be found in the following reference: Coddington and Levinson, Theory of Ordinary Differential Equations, Mc Graw-Hill, New York, 1955).

2ª Step: A Priori Estimates

(i) Estimate I

Multiplying (10) by $g_{jm}'(t)$ and summing over j from 1 to m, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}||u_m(t)||^2 + \frac{1}{p}\frac{d}{dt}||u_m(t)||_{L^p(\Omega)}^p = (f(t), u_m'(t))$$

[3] Note that by virtue of (8) it follows that $|u_m|^\rho u_m \in L^2(\Omega)$.

as we already did in Problem 1.

Integrating the expression above from 0 to t with $t \in (0, t_m)$, we have

$$\begin{aligned} &|u'_m(t)|^2 + ||u_m(t)||^2 + \frac{2}{p}\,||u_m(t)||^p_{L^p} = |u_{1m}|^2 + ||u_{0m}||^2 \\ &+ \frac{2}{p}\,||u_{0m}||^p_{L^p(\Omega)} + 2\int_0^t (f(s), u'_m(s))ds \\ &\leq |u_{1m}|^2 + ||u_{0m}||^2 + \frac{2}{p}\,||u_{0m}||^p_{L^p(\Omega)} + ||f||^2_{L^2(Q)} + \int_0^t |u'_m(s)|^2\,ds. \end{aligned} \tag{13}$$

However from (11) and (12) we obtain the existence of a constant $c_1 > 0$ such that

$$|u_{1m}|^2 + ||u_{0m}||^2 + \frac{2}{p}\,||u_{0m}||^p_{L^p(\Omega)} + ||f||^2_{L^2(Q)} \leq c_1\,. \tag{14}$$

Thus, from (13) and (14) it follows that

$$\begin{aligned} &|u'_m(t)|^2 + ||u_m(t)||^2 + \frac{2}{p}\,||u_m(t)||^p_{L^p(\Omega)} \\ &\leq c_1 + \int_0^t \{|u'_m(s)|^2 + ||u_m(s)||^2 + \frac{2}{p}\,||u_m(s)||^p_{L^p(\Omega)}\}ds. \end{aligned}$$

Now, by Gronwall's inequality from this last inequality we conclude that

$$|u'_m(t)|^2 + ||u_m(t)||^2 + \frac{2}{p}\,||u_m(t)||^p_{L^p(\Omega)} \leq c_2\,; \quad \forall\, t \in [0, t_m);\ \forall\, m \in \mathbb{N}, \tag{15}$$

which allows us to extend u_m to the whole interval $[0, T]$. Furthermore,

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0, T; H^1_0(\Omega)), \tag{16}$$

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0, T; L^p(\Omega)), \tag{17}$$

$$(u'_m) \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)). \tag{18}$$

From (7) and (16) it also follows that

$$(|u_m|^\rho u_m) \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)). \tag{19}$$

(ii) Estimate II

We can, without loss of generality, consider the basis (ω_ν) as being orthonormal in $L^2(\Omega)$. It follows from this and from (10) that

$$g''_{jm}(t) = (u''_m(t), \omega_j) = (f(t), \omega_j) - ((u_m(t), \omega_j)) - (|u_m(t)|^\rho u_m(t), \omega_j). \tag{20}$$

Since the right side of the equality above belongs to $L^2(0, T)$ it follows that $g''_{jm} \in L^2(0, T)$, where here the derivatives are understood in the sense of Dini. Thus

$$\begin{aligned} &\int_0^T ||u''_m(t)||^2_{L^2(\Omega)}\,dt = \int_0^T \left|\sum_{j=1}^m g''_{jm}(t)\omega_j\right|^2_{L^2(\Omega)} dt \\ &\leq c(m)\sum_{j=1}^m ||\omega_j||^2_{L^2(\Omega)} \int_0^T |g''_{jm}(t)|^2\,dt < +\infty \end{aligned}$$

that is,

$$u''_m \in L^2(0,T;L^2(\Omega)) \tag{21}$$

where here, again, the derivatives are understood in the sense of Dini. On the other hand, $\frac{d}{dt}$ being the derivative in the distributional sense in $\mathcal{D}'(0,T;L^2(\Omega))$ and $\theta \in \mathcal{D}(0,T)$, we have

$$\begin{aligned}
\Big\langle \frac{d}{dt} u_m, \theta \Big\rangle &= -\int_0^T u_m(t)\theta'(t)dt = -\int_0^T \Big(\sum_{j=1}^m g_{jm}(t)\omega_j \Big) \theta'(t)\, dt \\
&= \sum_{j=1}^m \Big(-\int_0^T g_{jm}(t)\theta'(t)dt \Big)\omega_j = \sum_{j=1}^m -\Big\{ g_{jm}(t)\theta(t)\Big|_0^T - \int_0^T g'_{jm}(t)\theta(t)dt \Big\}^{(4)} \omega_j \\
&= \Big(\sum_{j=1}^m \int_0^T g'_{jm}(t)\theta(t)dt \Big)\omega_j = \int_0^T u'_m(t)\theta(t)dt = \langle u'_m, \theta\rangle
\end{aligned}$$

which proves that the distributional derivative of u_m and the classical derivative coincide. Similarly, it is proved that

$$\Big\langle \frac{d}{dt} u'_m, \theta \Big\rangle = \langle u''_m, \theta\rangle$$

that is, that the distributional and classical derivatives of 1ª and 2ª order coincide.

On the other hand, using properties of the Bochner integral it is not difficult to verify that

$$\begin{aligned}
&\Big\langle \frac{d}{dt} (f(t),\omega_j), \theta \Big\rangle = \langle (f'(t),\omega_j), \theta\rangle \\
&\Big\langle \frac{d}{dt} ((u_m(t),\omega_j)), \theta \Big\rangle = \langle ((u'_m(t),\omega_j)), \theta\rangle \\
&\Big\langle \frac{d}{dt} (|u_m(t)|^\rho u_m(t),\omega_j), \theta \Big\rangle = \Big\langle (\rho+1) \int_\Omega |u_m(t)|^\rho u'_m(t)\omega_j dx, \theta \Big\rangle.
\end{aligned}$$

From the relations above and from (20) it follows that

$$\frac{d}{dt}(u''_m(t),\omega_j) = (f'(t),\omega_j) - ((u'_m(t),\omega_j)) - (\rho+1)\int_\Omega |u_m(t)|^\rho u'_m(t)\omega_j\, dx \tag{22}$$

in $L^2(0,T)$, that is,

$$g'''_{jm} \in L^2(0,T)^{\ (5)}$$

where the three derivatives are distributional. Then it follows that

$$\int_0^T ||u'''_m(t)||^2_{L^2(\Omega)}\, dt = \int_0^T \Big| \sum_{j=1}^m g'''_{jm}(t)\omega_j \Big|^2_{L^2(\Omega)} dt < +\infty$$

i.e,

$$u'''_m \in L^2(0,T;L^2(\Omega)) \tag{22'}$$

[4] Here we used the fact that $(g_{jm}(t)\cdot\theta(t))$ is absolutely continuous.

[5] Note that the classical and distributional derivatives up to second order of $g_{jm}(t)$ coincide in the sense of distributions in $\mathcal{D}'(0,T)$.

and from (22) we obtain

$$(u_m'''(t),\omega_j)+((u_m'(t),\omega_j))+(\rho+1)\int_\Omega |u_m(t)|^\rho u_m'(t)\omega_j\,dx=(f'(t),\omega_j). \tag{23}$$

Multiplying by g_{jm}'' and summing over j, we arrive at

$$\frac{1}{2}\frac{d}{dt}|u_m''(t)|^2+\frac{1}{2}\frac{d}{dt}||u_m'(t)||^2+(\rho+1)\int_\Omega |u_m(t)|^\rho u_m'(t)u_m''(t)\,dx=(f'(t),u_m''(t)).$$

Whence

$$\frac{d}{dt}\left\{|u_m''(t)|^2+||u_m'(t)||^2\right\}\le 2(\rho+1)\int_\Omega |u_m(t)|^\rho|u_m'(t)|\,|u_m''(t)|dx+2(f'(t),u_m''(t)). \tag{24}$$

Remembering that $H_0^1(\Omega)\hookrightarrow L^q(\Omega)$ where $1\le q\le \frac{2n}{n-2}$, then, from the fact that $|u_m|^\rho\in L^{\frac{2(\rho+1)}{\rho}}(\Omega)$, $|u_m'|\in L^q(\Omega)$ and $|u_m''|\in L^2(\Omega)$ and, furthermore, since $\frac{1}{\frac{2(\rho+1)}{\rho}}+\frac{1}{2(\rho+1)}+\frac{1}{2}=1$ we have, according to the generalized Hölder inequality that

$$\begin{aligned}\int_\Omega |u_m(t)|^\rho|u_m'(t)|\,|u_m''(t)|dx &\le ||\,|u_m(t)|^\rho||_{L^{\frac{2(\rho+1)}{\rho}}(\Omega)}\,||u_m'(t)||_{L^q(\Omega)}\,|u_m''|_{L^2(\Omega)}\\ &=||u_m(t)||^\rho_{L^{2(\rho+1)}(\Omega)}\,||u_m'||_{L^q(\Omega)}\,|u_m''(t)|_{L^2(\Omega)}\,.\end{aligned} \tag{25}$$

We have

$$0<\rho\le\frac{2}{n-2}\ \Rightarrow\ 2<2(\rho+1)\le\frac{2n}{n-2}.$$

Whence, Ω being bounded, taking $q=2(\rho+1)$

$$H_0^1(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega). \tag{26}$$

From (25) and (26) we have the existence of a constant $c_1>0$ such that

$$\int_\Omega |u_m(t)|^\rho|u_m'(t)|\,|u_m''(t)|dx\le c_1||u_m(t)||^\rho\,||u_m'(t)||\,|u_m''(t)|$$

which by (16) is even less than or equal to

$$c_2||u_m'(t)||\,|u_m''(t)|$$

that is,

$$\int_\Omega |u_m(t)|^\rho|u_m'(t)|\,|u_m''(t)|\,dx\le\frac{c_2}{2}\left\{||u_m'(t)||^2+|u_m''(t)|^2\right\}. \tag{27}$$

Now, from (24) and (27) we conclude that

$$\frac{d}{dt}\left\{|u_m''(t)|^2+||u_m'(t)||^2\right\}\le c_3\left\{||u_m'(t)||^2+|u_m''(t)|^2\right\}+|f'(t)|^2+|u_m''(t)|^2.$$

Integrating from 0 to t; $t\in[0,T]$, we obtain

$$\begin{aligned}|u_m''(t)|^2+||u_m'(t)||^2 &\le |u_m''(0)|^2+||u_{1m}||^2+||f'||_{L^2(Q)}\\ &\quad+c_4\int_0^t\left\{||u_m'(s)||^2+|u_m''(s)|^2\right\}ds.\end{aligned} \tag{28}$$

On the other hand, by virtue of (16), (18), (21) and (22') we have

$$u_m \in C_s(0,T;H_0^1(\Omega)) \cap C^0([0,T],L^2(\Omega)); \;\; u_m', u_m'' \in C^0([0,T];L^2(\Omega))$$

making sense to speak of $u_m''(0)$. From (20), in particular, we can write that

$$|u_m''(0)|^2 = (f(0), u_m''(0)) - ((u_m(0), u_m''(0)) + (|u_m(0)|^\rho u_m(0), u_m''(0)). \tag{29}$$

From (29) it follows that

$$|u_m''(0)|^2_{L^2} \leq \big[|f(0)|_{L^2(\Omega)} + |\Delta u_{0m}|_{L^2(\Omega)} + |\,|u_{0m}|^\rho u_{0m}|_{L^2(\Omega)}\big]|u_m''(0)|$$

where here we used Green's and Schwarz's Theorem, i.e,

$$|u_m''(0)|_{L^2(\Omega)} \leq \big[|f(0)|_{L^2(\Omega)} + |\Delta u_{0m}|_{L^2(\Omega)} + ||u_{0m}||^{\rho+1}_{L^{2\rho+2}(\Omega)}\big]. \tag{30}$$

It follows from (7) and (11) that there exists $c_5 > 0$ such that

$$|u_m''(0)|^2_{L^2(\Omega)} \leq c_5\,; \quad \forall\, m \in \mathbb{N} \tag{31}$$

and from (12), (28) and (31) it follows that

$$|u_m''(t)|^2 + ||u_m'(t)||^2 \leq c_6 + \int_0^t \big[||u_m'(s)||^2 + |u_m''(s)|^2\big]ds$$

and by Gronwall's inequality we have

$$|u_m''(t)|^2 + ||u_m'(t)||^2 \leq c; \quad \forall\, t \in [0,T]; \; \forall\, m \in \mathbb{N}. \tag{32}$$

From (32) it then follows that

$$(u_m') \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)), \tag{33}$$
$$(u_m'') \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{34}$$

3ª Step: Passage to the limit

From the estimates made in (16), (17), (18), (19), (33) and (34) we can extract a subsequence (u_ν) of (u_m) such that

$$u_\nu \overset{*}{\rightharpoonup} u \quad \text{weak-star in} \quad L^\infty(0,T;H_0^1(\Omega)), \tag{35}$$
$$u_\nu \rightharpoonup u \quad \text{weakly in} \quad L^p(Q), \tag{36}$$
$$u_\nu' \overset{*}{\rightharpoonup} u' \quad \text{weak-star in} \quad L^\infty(0,T;L^2(\Omega)), \tag{37}$$
$$u_\nu' \overset{*}{\rightharpoonup} u' \quad \text{weak-star in} \quad L^\infty(0,T;H_0^1(\Omega)), \tag{38}$$
$$u_\nu'' \overset{*}{\rightharpoonup} u'' \quad \text{weak-star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{39}$$

Let $\theta \in \mathcal{D}(0,T)$ and consider $j \in \mathbb{N}$ and $\mu > j$. From (10) we can write

$$\begin{aligned} &\int_0^T (u_\mu''(t), \omega_j)\theta(t)\,dt + \int_0^T ((u_\mu(t), \omega_j))\theta(t)\,dt \\ &+ \int_0^T (|u_\mu(t)|^\rho u_\mu(t), \omega_j)\theta(t)\,dt = \int_0^T (f(t), \omega_j)\theta(t)\,dt. \end{aligned} \tag{40}$$

It is worth noting that from (16), (18) and by virtue of the Aubin-Lions Theorem we can extract a subsequence of (u_ν), which we will still denote by the same notation such that

$$u_\nu \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)) = L^2(Q).$$

It follows from this the existence of a subsequence of (u_ν), which we persist in denoting by the same notation, such that

$$u_\nu \to u \quad \text{a.e. in} \quad Q.$$

By the continuity of the map $\lambda \in \mathbb{R} \mapsto F(\lambda) = |\lambda|^\rho \lambda$ and from this last convergence it follows that

$$|u_\nu|^\rho u_\nu \to |u|^\rho u \quad \text{a.e. in} \quad Q. \tag{41}$$

From (19), (41) and by virtue of Lions' Lemma it follows that

$$|u_\nu|^\rho u_\nu \rightharpoonup |u|^\rho u \quad \text{in} \quad L^2(Q). \tag{42}$$

Finally, the convergences given in (35), (39) and (42) allow us to pass to the limit in (40) to obtain

$$\begin{aligned}\int_0^T (u''(t),\omega_j)\theta(t)\,dt + \int_0^T ((u(t),\omega_j))\theta(t)\,dt \\ + \int_0^T (|u(t)|^\rho u(t),\omega_j)\theta(t)\,dt = \int_0^T (f(t),\omega_j)\theta(t)\,dt.\end{aligned}$$

By the totality of the ω_j's in $H_0^1(\Omega) \cap H^2(\Omega)$ we obtain

$$\begin{aligned}\int_0^T (u''(t),v)\theta(t)\,dt + \int_0^T ((u(t),v))\theta(t)\,dt \\ + \int_0^T (|u(t)|^\rho u(t),v)\theta(t)\,dt = \int_0^T (f(t),v)\theta(t)\,dt,\end{aligned} \tag{43}$$

for all $v \in H_0^1(\Omega) \cap H^2(\Omega)$. It follows from this that

$$u'' - \Delta u + |u|^\rho u = f \quad \text{in} \quad \mathcal{D}'(0,T;L^2(\Omega))$$

and by the regularity of the functions involved we conclude that

$$u'' - \Delta u + |u|^\rho u = f \quad \text{in} \quad L^2(Q). \tag{44}$$

From (44) and (35) we have

$$\left| \begin{array}{l} -\Delta u(t) \in L^2(\Omega) \quad \text{a.e. in} \quad]0,T[\\ u(t) \in H_0^1(\Omega) \end{array} \right.$$

which implies, given the regularity results of elliptic problems[6] that

$$u(t) \in H^2(\Omega) \quad \text{for a.e.} \quad t \in]0,T[\,. \tag{45}$$

[6]Note that Ω is sufficiently smooth.

On the other hand, since

$$f \in C^0([0,T]; L^2(\Omega)), \ \ |u|^\rho u \in L^\infty(0,T; L^2(\Omega)) \text{ and } u'' \in L^\infty(0,T; L^2(\Omega))$$

from (44) we have

$$\Delta u \in L^\infty(0,T; L^2(\Omega)) \tag{46}$$

which leads us to conclude that

$$u'' - \Delta u + |u|^\rho u = f \quad \text{in} \quad L^\infty(0,T; L^2(\Omega)).$$

Thus, setting

$$g = f - u'' - |u|^\rho u$$

we have, still given the regularity of elliptic problems, that

$$\operatorname*{ess\,sup}_{t\in]0,T[} ||u(t)||_{H^2(\Omega)} = c \operatorname*{ess\,sup}_{t\in]0,T[} |g(t)|_{L^2(\Omega)} < +\infty.$$

Therefore,

$$u \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)). \tag{47}$$

4ª Step: Initial Conditions

These are proved in an analogous manner to the 1ª problem.

5ª Step: Uniqueness

Let u and v be strong solutions of (1) and consider $\omega = u - v$. Then ω verifies

$$\left| \begin{array}{l} \omega'' - \Delta\omega = |v|^\rho v - |u|^\rho u \quad \text{in} \quad L^2(Q) \\ \omega = 0 \quad \text{on} \quad \Sigma \\ \omega(0) = \omega'(0) = 0. \end{array} \right. \tag{48}$$

Composing $(48)_1$ with ω' implies that

$$(\omega''(t), \omega'(t)) + (-\Delta\omega(t), \omega'(t)) = (|v(t)|^\rho v(t) - |u(t)|^\rho u(t), \omega'(t)).$$

Since $\omega \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega))$ and $\omega' \in L^\infty(0,T; H_0^1(\Omega))$ we have as a consequence of Green's Theorem

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\omega'(t)|^2 + \frac{1}{2}\frac{d}{dt}||\omega(t)||^2 &= (|v(t)|^\rho v(t) - |u(t)|^\rho u(t), \omega'(t)) \\ &= \int_\Omega (|v(x,t)^\rho v(x,t) - |u(x,t)|^\rho u(x,t))\omega'(x,t)dx. \end{aligned} \tag{49}$$

Setting

$$F(\lambda) = |\lambda|^\rho \lambda; \quad \lambda \in \mathbb{R}$$

and since

$$F'(\lambda) = (\rho+1)|\lambda|^\rho$$

we have that $F \in C^1(\mathbb{R})$. Thus, given $\alpha, \beta \in \mathbb{R}$ we have, by virtue of the M.V.T., the existence of $\xi \in]\alpha, \beta[$ such that

$$|F(\beta) - F(\alpha)| = (\rho + 1)|\xi|^\rho |\beta - \alpha|.$$

Since $\xi \in]\alpha, \beta[$ then $\xi = (1-\theta)\alpha + \theta\beta$ for some $\theta \in]0, 1[$. It follows then that

$$|F(\beta) - F(\alpha)| = (\rho + 1)\,|(1-\theta)\alpha + \theta\beta|^\rho |\beta - \alpha| (\rho + 1)\,|\alpha + (\beta - \alpha)\theta|^\rho |\beta - \alpha|.$$

In particular taking $\beta = u(x,t)$ and $\alpha = v(x,t)$ it follows that

$$\begin{aligned} & |\,|v(x,t)|^\rho v(x,t) - |u(x,t)|^\rho u(x,t)| \\ &= (\rho+1)|v(x,t) + (u(x,t) - v(x,t))\theta(x,t)|^\rho |\omega(x,t)| \\ &\leq (\rho+1)\big\{2|v(x,t)| + |u(x,t)|\big\}^\rho |\omega(x,t)| \\ &\leq 2^\rho(\rho+1)\big\{|v(x,t)| + |u(x,t)|\big\}^\rho |\omega(x,t)| \\ &\leq 2^{2\rho}(\rho+1)\big\{|v(x,t)|^\rho + |u(x,t)|^\rho\big\} |\omega(x,t)| \end{aligned} \tag{50}$$

From (49) and (50) we can write

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\omega'(t)|^2 + \frac{1}{2}\frac{d}{dt}||\omega(t)||^2 \\ & \leq c_1 \int_\Omega \big\{|v(x,t)|^\rho + |u(x,t)|^\rho\big\}\,|\omega(x,t)|\,|\omega'(x,t)|\,dx. \end{aligned} \tag{51}$$

Recall that

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega), \quad \text{where } 1 \leq q \leq \frac{2n}{n-2}\cdot \tag{52}$$

Now, since $0 < \rho < \dfrac{2}{n-2}$ then $2 < 2(\rho+1) < \dfrac{2n}{2-n}$ and from (52) it follows that

$$H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega). \tag{53}$$

From (52) and (53) it follows then that

$$u(t), v(t) \in L^{\frac{\frac{1}{2(\rho+1)}}{\rho}}(\Omega); \quad \omega(t) \in L^{2(\rho+1)}(\Omega) \quad \text{a.e. in} \quad]0, T[\,.$$

Now, since $\frac{1}{\frac{2(\rho+1)}{\rho}} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$ it follows, using the generalized Hölder inequality, from (51) that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\omega'(t)|^2 + \frac{1}{2}\frac{d}{dt}||\omega(t)||^2 \\ & \leq c_1\big\{||v(t)||^\rho_{L^{2(\rho+1)}(\Omega)} + ||u(t)||^\rho_{L^{2(\rho+1)}(\Omega)}\big\}\,||\omega(t)||_{L^{2(\rho+1)}(\Omega)} \cdot |\omega'(t)|_{L^2(\Omega)} \quad \text{a.e. in} \quad]0, T[\,. \end{aligned}$$

Now, from (52), (53) and the fact that $u, v \in L^\infty(0, T; H_0^1)$ it follows that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\omega'(t)|^2 + \frac{1}{2}\frac{d}{dt}||\omega(t)||^2 \\ & \leq c_2||\omega(t)||\,|\omega'(t)|_{L^2(\Omega)} \quad \text{a.e. in} \quad]0, T[\,. \end{aligned}$$

Integrating this last inequality from 0 to t; $t \in [0,T]$ we obtain

$$|\omega'(t)|^2 + ||\omega(t)||^2 \le |\omega'(0)|^2 + ||\omega(0)||^2 + c_2 \int_0^t ||\omega(s)||\, |\omega'(s)|_{L^2(\Omega)}\, ds$$
$$\le c_3 \int_0^t \left[||\omega(s)||^2 + |\omega'(s)|^2 \right] ds$$

and by Gronwall's inequality

$$|\omega'(t)|^2 + ||\omega(t)||^2 \le 0; \quad \forall\, t \in [0,T]$$ [7]

which proves that

$$\omega(t) = 0 \quad \text{in} \quad H_0^1(\Omega) \quad \forall\, t \in [0,T]$$

i.e; $\omega = 0$ in $L^\infty(0,T;H_0^1(\Omega))$, which concludes the proof. □

[7] Note that $\omega \in C^0([0,T];H_0^1(\Omega))$ and $\omega' \in C^0([0,T];L^2(\Omega))$.

Chapter 4

Problem $\dfrac{\partial^2 u}{\partial t^2} - \Delta u + u^3 = f$ in dimension $n = 3$ (special basis)

Problem 3

$$\left| \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + u^3 = f \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(x,0) = u_0(x); \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x); \quad x \in \Omega \end{array} \right. \tag{1}$$

where

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega); \quad u_1 \in H_0^1(\Omega) \quad \text{and} \quad f \in L^2(0,T;H_0^1(\Omega)) \tag{2}$$

possesses a unique strong solution in the class

$$u \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)); \quad u' \in L^\infty(0,T;H_0^1(\Omega)) \quad \text{and} \quad u'' \in L^2(0,T;L^2(\Omega)). \tag{3}$$

Proof:

1ª Step: Approximate Problem

Let us consider $(\omega_\nu)_{\nu \in \mathbb{N}}$ a basis of $H_0^1(\Omega) \cap H^2(\Omega)$, consisting of the eigenfunctions of the operator $-\Delta$ defined by the triple $\{H_0^1(\Omega), L^2(\Omega), ((\,\cdot\,,\,\cdot\,))\}$, thus:

$$\left| \begin{array}{ll} -\Delta\omega_\nu = \lambda_\nu\,\omega_\nu\,; & \forall\,\nu \in \mathbb{N} \\ \gamma_0(\omega_\nu) = 0; & \forall\,\nu \in \mathbb{N} \end{array} \right. \tag{4}$$

It is well known, cf. Spectral Theorem, that:

$$(\omega_\nu)_{\nu \in \mathbb{N}} \text{ constitutes a complete orthonormal system in } L^2(\Omega). \tag{5}$$

$$\left(\frac{\omega_\nu}{\lambda_\nu^{1/2}}\right)_{\nu \in \mathbb{N}} \text{ constitutes a complete orthonormal system in } H_0^1(\Omega). \tag{6}$$

$$\left(\frac{\omega_\nu}{\lambda_\nu}\right)_{\nu \in \mathbb{N}} \text{ constitutes a complete orthonormal system in } H_0^1(\Omega) \cap H^2(\Omega). \tag{7}$$

Furthermore, Ω being a sufficiently smooth bounded open set, by virtue of the regularity of elliptic problems it follows from (4) that

$$\omega_\nu \in H^m(\Omega); \quad \forall\, m \in \mathbb{N}, \ \forall\,\nu \in \mathbb{N}. \tag{8}$$

On the other hand, the Sobolev embedding Theorem tells us that if $m > \dfrac{n}{2}$ and $k \in \mathbb{N}$ is such that $k < m - \dfrac{n}{2} \leq k + 1$, then

$$H^m(\Omega) \hookrightarrow C^{k,\lambda}(\overline{\Omega}), \tag{9}$$

where

(i) $0 < \lambda < m - \dfrac{n}{2} - k$ if $m - \dfrac{n}{2} - k < 1$

(ii) $0 < \lambda < 1$ if $m - \dfrac{n}{2} - k = 1$

Now, given $n \in \mathbb{N}$ and $k \in \mathbb{N}$ we can choose, by virtue of (8), m sufficiently large such that $k < m - \dfrac{n}{2} \leq k + 1$ and from (9) it follows that

$$\omega_\nu \in C^k(\overline{\Omega}); \quad \forall\, k \in \mathbb{N},\ \forall\, \nu \in \mathbb{N},$$

that is,

$$\omega_\nu \in C^\infty(\overline{\Omega}); \quad \forall\, \nu \in \mathbb{N}. \tag{10}$$

According to the Sobolev embedding Theorem

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega); \quad \text{where } 1 \leq q \leq \frac{2n}{n-2}\cdot$$

In this case, since $n = 3$ it follows that

$$H_0^1(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^q(\Omega); \quad \forall\, q \leq 6. \tag{11}$$

Let

$$V_m = [\omega_1, \ldots, \omega_m].$$

In V_m consider the approximate problem:

$$u_m(t) \in V_m \;\Leftrightarrow\; u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i \tag{12}$$

$$(u_m''(t), \omega_j) + ((u_m(t), \omega_j)) + (u_m^3(t), \omega_j)\,^{(8)} = (f(t), \omega_j) \tag{13}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega) \cap H^2(\Omega) \tag{14}$$

$$u_m'(0) = u_{1m} \to u_1 \quad \text{in} \quad H_0^1(\Omega) \tag{15}$$

which has a local solution in some interval $[0, t_m)$ by Carathéodory. The estimates will serve to extend the solution to the whole interval $[0, T]$.

2ª Step: A Priori Estimates

• A Priori Estimate I

[8]Note that from (11) we have that $u_m^3 \in L^2(\Omega)$.

Composing (13) with $u_m'(t)$, we obtain, as in previous problems

$$|u_m'(t)|^2 + ||u_m(t)||^2 + \frac{1}{2}\,||u_m(t)||^4_{L^4(\Omega)} \le c; \quad \forall\, t \in [0, t_m);\ \forall\, m \in \mathbb{N} \tag{16}$$

which allows us to extend u_m to the whole interval $[0,T]$ with $u_m(t)$, $u_m'(t)$ absolutely continuous functions in t and $u_m''(t)$ existing a.e. in $]0,T[$. It follows from (16) that

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)), \tag{17}$$
$$(u_m') \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{18}$$
$$(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;L^4(\Omega)). \tag{19}$$

• **A Priori Estimate II**

Multiplying (13) by $g_{jm}'(t)\lambda_j$ and summing over j, from $(4)_1$ it follows

$$(u_m''(t), -\Delta u_m'(t)) + ((u_m(t), -\Delta u_m'(t))) + (u_m^3(t), -\Delta u_m'(t)) = (f(t), -\Delta u_m'(t)).$$

Now, by virtue of the regularity of the basis (ω_ν) and by Green's Theorem we obtain

$$((u_m''(t), u_m'(t)) + (-\Delta u_m(t), -\Delta u_m'(t)) = ((f(t), u_m'(t))) + (u_m^3(t), \Delta u_m'(t))$$

that is,

$$\frac{1}{2}\frac{d}{dt}\,||u_m'(t)||^2 + \frac{1}{2}\frac{d}{dt}\,|\Delta u_m(t)|^2 = ((f(t), u_m'(t))) + (u_m^3(t), \Delta u_m'(t)). \tag{20}$$

However, from (10) and (12) we have that

$$u_m'(t), u_m^3(t) \in C^\infty(\overline{\Omega}) \quad \text{for all} \quad t \in [0,T].$$

Thus, by Green's Theorem it follows that

$$\int_\Omega u_m^3(t)\Delta u_m'(t)\,dx = -\int_\Omega \nabla u_m^3(t)\cdot\nabla u_m'(t)\,dx + \int_\Gamma \frac{\partial u_m'}{\partial \nu}\,u_m^3\,d\Gamma. \tag{21}$$

However, from $(4)_2$, (10) and the fact that $\gamma_0(\omega_\nu) = \omega_\nu|_\Gamma$ it follows that $u_m^3|_\Gamma = \gamma_0(u_m^3) = 0$. It follows from this and from (21) that

$$\begin{aligned} \left|\int_\Omega u_m^3(t)\Delta u_m'(t)dx\right| &= \left|\sum_{i=1}^3 \int_\Omega \frac{\partial}{\partial x_i}(u_m^3(t))\,\frac{\partial}{\partial x_i}(u_m'(t))dx\right| \\ &= \left|\sum_{i=1}^3 \int_\Omega 3\,u_m^2(t)\,\frac{\partial u_m}{\partial x_i}(t)\frac{\partial u_m'}{\partial x_i}(t)\right|. \end{aligned} \tag{22}$$

Now, from (11) it follows that

$$u_m^2(t) \in L^3(\Omega), \quad \frac{\partial u_m}{\partial x_i} \in L^6(\Omega) \quad \text{and} \quad \frac{\partial u_m'}{\partial x_i} \in L^2(\Omega).$$

Noting also that

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3} \quad \Leftrightarrow \quad \frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1$$

we obtain, by the generalized Hölder integral inequality that

$$\begin{aligned}\left|\int_\Omega u_m^2(t)\,\frac{\partial u_m}{\partial x_i}(t)\,\frac{\partial u_m'}{\partial x_i}(t)\,dx\right| &\le ||u_m^2(t)||_{L^3(\Omega)}\,\left|\left|\frac{\partial u_m}{\partial x_i}(t)\right|\right|_{L^6(\Omega)}\,\left|\frac{\partial u_m'}{\partial x_i}(t)\right|_{L^2(\Omega)}\\ &= ||u_m(t)||^2_{L^6(\Omega)}\,\left|\left|\frac{\partial u_m}{\partial x_i}(t)\right|\right|_{L^6(\Omega)}\,\left|\frac{\partial u_m'}{\partial x_i}(t)\right|_{L^2(\Omega)}.\end{aligned} \tag{23}$$

Thus, from (11), (22) and (23) it follows that

$$\begin{aligned}&\left|\int_\Omega u_m^3(t)\Delta u_m'(t)dx\right|\\ &\le c_1\sum_{i=1}^3 ||u_m(t)||^2\,\left|\left|\frac{\partial u_m}{\partial x_i}(t)\right|\right|_{H^1(\Omega)}\,\left|\frac{\partial u_m'}{\partial x_i}\right|_{L^2(\Omega)}.\end{aligned} \tag{24}$$

Now from (16), (24) and Hölder's numerical inequality

$$\begin{aligned}&\left|\int_\Omega u_m^3(t)\Delta u_m'(t)dx\right|\\ &\le c_2\sum_{i=1}^3 \left|\left|\frac{\partial u_m}{\partial x_i}(t)\right|\right|_{H^1(\Omega)}\,\left|\frac{\partial u_m'}{\partial x_i}(t)\right|_{L^2(\Omega)}\\ &\le c_2\Big(\sum_{i=1}^3 \left|\left|\frac{\partial u_m}{\partial x_i}(t)\right|\right|^2_{H^1(\Omega)}\Big)^{1/2}\Big(\sum_{i=1}^3 \left|\frac{\partial u_m'}{\partial x_i}(t)\right|^2_{L^2(\Omega)}\Big)^{1/2}\\ &\le c_3||u_m(t)||_{H^2(\Omega)}||u_m'(t)||\\ &\le c_4|\Delta u_m(t)|_{L^2(\Omega)}\,||u_m'(t)||,\end{aligned} \tag{25}$$

where the last inequality follows from the fact that in $H_0^1(\Omega)\cap H^2(\Omega)$ the norms $||u||_{H^2(\Omega)}$ and $|\Delta u|_{L^2(\Omega)}$ are equivalent.

Thus from (20) and (25) we conclude

$$\frac{d}{dt}\,||u_m'(t)||^2+\frac{d}{dt}\,|\Delta u_m(t)|^2 \le ||f(t)||^2+||u_m'(t)||^2+c_4\big[||u_m'(t)||^2+|\Delta u_m(t)|^2\big].$$

Integrating from 0 to t with $t\in[0,T]$ it follows that

$$\begin{aligned}||u_m'(t)||^2+|\Delta u_m(t)|^2 &\le ||u_{1m}||^2+|\Delta u_{0m}|^2+||f||^2_{L^2(0,T;H_0^1(\Omega))}\\ &\quad + c_5\int_0^t \left\{||u_m'(s)||^2+|\Delta u_m(s)|^2\right\}ds.\end{aligned} \tag{26}$$

However, from (14) and (15) there exists $c_6>0$ such that

$$||u_{1m}||^2+|\Delta u_{0m}|^2+||f||^2_{L^2(0,T;H_0^1(\Omega))}\le c_6\,; \tag{27}$$

and from (26) and (27) it follows that

$$||u_m'(t)||^2+|\Delta u_m(t)|^2 \le c_6+c_5\int_0^t \left\{||u_m'(s)||^2+|\Delta u_m(s)|^2\right\}ds$$

and by Gronwall's inequality

$$||u_m'(t)||^2+|\Delta u_m(t)|^2\le c;\quad \forall\, t\in[0,T],\ \forall\, m\in\mathbb{N}. \tag{28}$$

From (28) we arrive at

$$(u'_m) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)) \tag{29}$$

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \tag{30}$$

Also, from (17) and (11) we obtain

$$(u_m^3) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{31}$$

The subsequent steps, namely, passage to the limit, initial conditions and uniqueness are done in an analogous manner to what we did in previous problems. □

In a manner analogous to what was done previously, we will treat the following problem

Chapter 5

Schrödinger Equation $\dfrac{\partial u}{\partial t} - i\Delta u + |u|^2\, u = f$ (dimension 3)

In what follows Ω is a bounded and sufficiently smooth open subset of $\mathbb{R}^3$.

Problem 5:

$$\left|\begin{array}{l} \dfrac{\partial u}{\partial t} - i\Delta u + |u|^2\, u = f \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(x,0) = u_0(x); \quad x \in \Omega \end{array}\right. \tag{1}$$

where

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega); \quad f \in L^2(0,T; H_0^1(\Omega)) \text{ and } \frac{\partial f}{\partial t} \in L^2(0,T; L^2(\Omega)) \tag{2}$$

possesses a unique strong solution in the class

$$u \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)); \quad u' \in L^\infty(0,T; L^2(\Omega)).$$

Proof:

1ª Step: Approximate Problem

Let $(\omega_\nu)_{\nu\in\mathbb{N}}$ be the Hilbert basis of $L^2(\Omega)$ given by the eigenfunctions of the operator $-\Delta$ defined by the triple $\{H_0^1(\Omega), L^2(\Omega), ((\,\cdot\,,\,\cdot\,))\}$. Thus,

$$(\omega_\nu) \quad \text{is a complete orthonormal system in} \quad L^2(\Omega) \tag{3}$$

$$\left(\frac{\omega_\nu}{\sqrt{\lambda_\nu}}\right) \quad \text{is a complete orthonormal system in} \quad H_0^1(\Omega) \tag{4}$$

$$\left(\frac{\omega_\nu}{\lambda_\nu}\right) \quad \text{is a complete orthonormal system in} \quad H_0^1(\Omega) \cap H^2(\Omega) \tag{5}$$

and (ω_ν) is a weak solution of

$$\left|\begin{array}{l} -\Delta\omega_\nu = \lambda_\nu\, \omega_\nu \\ \quad \omega_\nu|_\Gamma = 0 \end{array}\right. \tag{6}$$

where (λ_ν) is a sequence of eigenvalues that verifies:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_\nu < \ldots \quad \text{and} \quad \lambda_\nu \to \infty \text{ when } \nu \to +\infty. \tag{7}$$

Recall that from (6) and the fact that Ω is sufficiently smooth, it results by virtue of the regularity of elliptic problems that

$$(\omega_\nu) \subset H^m(\Omega); \quad \forall\, m \in \mathbb{N}. \tag{8}$$

It follows from (3) and the Sobolev Embedding Theorem that

$$(\omega_\nu) \subset C^k(\overline{\Omega}); \quad \forall\, k \in \mathbb{N}. \tag{9}$$

Let

$$V_m = [\omega_1, \omega_2, \ldots, \omega_m].$$

In V_m consider the approximate problem

$$u_m(t) \in V_m \quad \Leftrightarrow \quad u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\omega_j \tag{10}$$

$$(u_m'(t), \omega_j) + i((u_m(t), \omega_j)) + (|u_m(t)|^2 u_m(t), \omega_j) = (f(t), \omega_j) \tag{11}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega) \cap H^2(\Omega) \tag{12}$$

which has a local solution in some interval $[0, t_m)$, by virtue of Carathéodory.

2ª Step: A Priori Estimates

• A Priori Estimate I

Multiplying equation (11) by $\overline{g_{jm}(t)}$ and summing over j; we have:

$$(u_m'(t), u_m(t)) + i((u_m(t), u_m(t))) + (|u_m(t)|^2 u_m(t), u_m(t)) = (f(t), u_m(t)). \tag{13}$$

Let $\theta \in \mathcal{D}(0, t_m)$. We prove that

$$\left\langle \frac{d}{dt}\,(u_m(t), u_m(t)), \theta \right\rangle = \langle (u_m'(t), u_m(t)) + (u_m(t), u_m'(t)), \theta\rangle$$

that is,

$$\frac{d}{dt}\,|u_m(t)|^2 = (u_m'(t), u_m(t)) + \overline{(u_m'(t), u_m(t))} = 2\ \mathrm{Re}(u_m'(t), u_m(t)). \tag{14}$$

Also

$$\begin{aligned}(|u_m(t)|^2 u_m(t), u_m(t)) &= \int_\Omega |u_m(t)|^2 u_m(t)\overline{u_m(t)}\, dx = \int_\Omega |u_m(t)|^2 |u_m(t)|^2 dx \\ &= \int_\Omega |u_m(t)|^4\, dx.\end{aligned} \tag{15}$$

Considering the real part in (13) results from (13), (14) and (15) that

$$\frac{1}{2}\frac{d}{dt}\,|u_m(t)|^2 + ||u_m(t)||^4_{L^4(\Omega)} = \ \mathrm{Re}(f(t), u_m(t)) \le |(f(t), u_m(t))|.$$

From the inequality above it follows that

$$\frac{d}{dt}\,|u_m(t)|^2 + 2||u_m(t)||^4_{L^4(\Omega)} \le |f(t)|^2 + |u_m(t)|^2.$$

Integrating from 0 to t, $\quad t \in [0, t_m)$ we obtain

$$|u_m(t)|^2 + 2\int_0^t ||u_m(s)||^4_{L^4(\Omega)}\, ds \le |u_{0m}|^2 + ||f||^2_{L^2(Q)} + \int_0^t |u_m(s)|^2\, ds.$$

From (12) it follows that $\exists\, c_1 > 0$ such that $|u_{0m}|^2 \le c_1$ and therefore

$$|u_m(t)|^2 + 2\int_0^t ||u_m(s)||^4_{L^4(\Omega)}\, ds \le c_2 + \int_0^t \left[|u_m(s)|^2 + 2\int_0^s ||u_m(\tau)||^4_{L^4}\, d\tau \right] ds.$$

From the inequality above and Gronwall's inequality, we conclude that

$$|u_m(t)|^2 + 2\int_0^t ||u_m(s)||^4_{L^4(\Omega)}\, ds \le c; \quad \forall\, t \in [0, t_m);\ \forall\, m \in \mathbb{N}.$$

The estimate above allows us to extend the solution u_m to the whole interval $[0, T]$, with $u_m(t)$ absolutely continuous on $[0, T]$ and u'_m existing a.e. in $]0, T[$. Furthermore,

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)) \tag{16}$$

$$(u_m) \quad \text{is bounded in} \quad L^4(0, T; L^4(\Omega)) = L^4(Q). \tag{17}$$

Since the dimension $n = 3$ it follows from the Sobolev Embedding Theorem

$$H^1_0(\Omega) \hookrightarrow L^q(\Omega); \quad \forall\, q \le 6. \tag{18}$$

On the other hand from (17) we still have that

$$\left(|u_m|^2\, u_m\right) \quad \text{is bounded in} \quad L^{4/3}(0, T; L^{4/3}(\Omega)). \tag{19}$$

A Priori Estimate II

Multiplying (11) by $\lambda_j\, \overline{g_{jm}}(t)$ and summing over j we have

$$(u'_m(t), -\Delta u_m(t)) + i((u_m(t), -\Delta u_m(t))) + \left(|u_m(t)|^2 u_m(t), -\Delta u_m(t)\right) = (f(t), -\Delta u_m(t)).$$

By virtue of the regularity of the basis and the fact that $f(t) \in H^1_0(\Omega)$ for a.e. $t \in]0, T[$ by Green's formula we can write

$$((u'_m(t), u_m(t))) + i(-\Delta u_m(t), -\Delta u_m(t)) + (|u_m(t)|^2 u_m(t), -\Delta u_m(t)) = ((f(t), u_m(t))).$$

Taking the real part on both sides of the equality above we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\, ||u_m(t)||^2 + \ \mathrm{Re}(|u_m(t)|^2 u_m(t), -\Delta u_m(t)) = \ & \mathrm{Re}((f(t), u_m(t))) \\ \le |((f(t), u_m(t)))| \le \frac{1}{2}\, ||f(t)||^2 + \frac{1}{2}\, ||u_m(t)||^2. \end{aligned} \tag{20}$$

However from (9) we have by Green that

$$\begin{aligned} &\int_\Omega |u_m(t)|^2 u_m(t)\overline{(-\Delta u_m(t))}\, dx \\ &= \sum_{i=1}^3 \int_\Omega \frac{\partial}{\partial x_i}\left(|u_m(t)|^2 u_m(t)\right) \overline{\frac{\partial u_m(t)}{\partial x_i}}\, dx - \int_\Gamma \frac{\partial}{\partial \nu}\,(u_m(t))\overline{|u_m(t)|^2 u_m(t)}\, d\Gamma. \end{aligned} \tag{21}$$

Since $u_m(t) \in H_0^1(\Omega) \cap C^\infty(\overline{\Omega})$ for a.e. $t \in]0,T[$, then

$$u_m(t)|_\Gamma = \gamma_0(u_m(t)) = 0 \quad \text{a.e. in} \quad]0,T[$$

and from there it follows that

$$|u_m(t)|^2 \, u_m(t) = 0 \quad \text{a.e. in} \quad]0,T[\tag{22}$$

and from (21) and (22) we conclude that

$$\begin{aligned}
&\int_\Omega |u_m(t)|^2 u_m(t) \overline{(-\Delta u_m(t))} dx = \sum_{i=1}^{3} \int_\Omega \frac{\partial}{\partial x_i} \big(|u_m(t)|^2 u_m(t)\big) \overline{\frac{\partial u_m(t)}{\partial x_i}} \, dx \\
&= \sum_{i=1}^{3} \int_\Omega \left\{ \frac{\partial}{\partial x_i} (|u_m(t)|^2) u_m(t) + |u_m(t)|^2 \frac{\partial u_m(t)}{\partial x_i} \right\} \overline{\frac{\partial u_m(t)}{\partial x_i}} \, dx \\
&= \sum_{i=1}^{3} \int_\Omega \left\{ \frac{\partial}{\partial x_i} (u_m(t) \overline{u_m(t)}) u_m(t) \, \overline{\frac{\partial u_m(t)}{\partial x_i}} + |u_m(t)|^2 \, \frac{\partial u_m(t)}{\partial x_i} \, \overline{\frac{\partial u_m(t)}{\partial x_i}} \right\} dx \\
&= \sum_{i=1}^{3} \int_\Omega \left\{ \frac{\partial u_m(t)}{\partial x_i} \, \overline{u_m}(t) u_m(t) \, \overline{\frac{\partial u_m(t)}{\partial x_i}} + u_m(t) \, \overline{\frac{\partial u_m(t)}{\partial x_i}} u_m(t) \, \overline{\frac{\partial u_m(t)}{\partial x_i}} \right. \\
&\qquad \left. + |u_m(t)|^2 \, \frac{\partial u_m(t)}{\partial x_i} \, \frac{\partial \overline{u_m}(t)}{\partial x_i} \right\} dx \\
&= \sum_{i=1}^{3} \int_\Omega \left\{ |u_m(t)|^2 \left| \frac{\partial u_m(t)}{\partial x_i} \right|^2 + \left(u_m(t) \, \overline{\frac{\partial u_m(t)}{\partial x_i}} \right)^2 + |u_m(t)|^2 \left| \frac{\partial u_m(t)}{\partial x_i} \right|^2 \right\} dx.
\end{aligned} \tag{23}$$

On the other hand, we claim that

$$|z_1|^2 \, |z_2|^2 + \text{ Re}\big[(z_1 \bar{z}_2)^2\big] = 2\big[\text{Re}(z_1 \bar{z}_2)\big]^2; \quad \forall \, z_1, z_2 \in \mathbb{C}. \tag{24}$$

Indeed, setting $z_1 = a + bi$ and $z_2 = c + di$, we obtain

$$\begin{aligned}
&|z_1|^2 \, |z_2|^2 + \text{ Re}\big[(z_1 \cdot \bar{z}_2)^2\big] \\
&= (a^2+b^2)(c^2+d^2) + \text{ Re}[((a+bi)(c-di))^2] \\
&= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + \text{ Re}[((ac+bd)+(bc-ad)i)^2] \\
&= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + (ac+bd)^2 - (bc-ad)^2 \\
&= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + a^2c^2 + 2abcd + b^2d^2 - b^2c^2 + 2abcd - a^2d^2 \\
&= 2(a^2c^2 + b^2d^2) + 4abcd
\end{aligned} \tag{25}$$

On the other hand

$$\begin{aligned}
&2(\text{Re}(z_1 \cdot \bar{z}_2))^2 \\
&= 2[\text{Re}((ac+bd)+(bc-ad)i)]^2 = 2(ac+bd)^2 \\
&= 2(a^2c^2 + b^2d^2) + 4abcd.
\end{aligned} \tag{26}$$

From (25) and (26) we conclude the desired result in (24). Thus, it follows from (23) and (24) that

$$
\begin{aligned}
&\operatorname{Re} \int_\Omega |u_m(t)|^2 u_m(t)\overline{(-\Delta u_m(t))}\, dx \\
&= \sum_{i=1}^{3} \int_\Omega \left\{ 2\left[\operatorname{Re}\left(u_m(t)\, \frac{\overline{\partial u_m}(t)}{\partial x_i} \right)\right]^2 + |u_m(t)|^2 \left| \frac{\partial u_m(t)}{\partial x_i} \right|^2 \right\} dx \geq 0.
\end{aligned} \tag{27}
$$

From (20) and (27) we arrive at

$$
\frac{d}{dt}\, ||u_m(t)||^2 \leq ||f(t)||^2 + ||u_m(t)||^2.
$$

Integrating in $[0,t]$; $t \in [0,T]$, we obtain

$$
||u_m(t)||^2 \leq ||u_{0m}||^2 + ||f||^2_{L^2(0,T;H_0^1(\Omega))} + \int_0^t ||u_m(s)||^2\, ds. \tag{28}
$$

Now from (12) $\exists\, c_1 > 0$ such that

$$
||u_{0m}||^2 + ||f||^2_{L^2(0,T;H_0^1(\Omega))} \leq c_1\,; \quad \forall\, m \in \mathbb{N} \tag{29}
$$

and from (28) and (29) it follows that

$$
||u_m(t)||^2 \leq c_1 + \int_0^t ||u_m(s)||^2\, ds.
$$

From Gronwall's inequality it follows that

$$
||u_m(t)|| \leq c; \quad \forall\, t \in [0,T] \ \text{ and } \quad \forall\, m \in \mathbb{N} \tag{30}
$$

which implies that:

$$
(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega))\,^{(*)} \tag{31}
$$

Estimativa a priori III

From (3) and (11) we have

$$
g'_{jm}(t) = (u'_m(t), \omega_j) = -i((u_m(t), \omega_j)) - (|u_m(t)|^2 u_m(t), \omega_j) + (f(t), \omega_j). \tag{32}
$$

Now, from (10) and the fact that g'_{jm} is absolutely continuous on $[0,T]$, it follows that the right side of the equality in (32) belongs to $L^2(0,T)$ which implies

$$
u'_m \in L^2(0,T;L^2(\Omega)). \tag{33}
$$

However, from (32) we have

$$
\frac{d}{dt}\,(u'_m(t), \omega_j) = g''_j(t) = -i((u'_m(t), \omega_j)) - \left(\left[|u_m(t)|^2 u_m(t)\right]', \omega_j\right) + (f'(t), \omega_j). \tag{34}
$$

[9] Note that from (18) and (31) it follows that $(|u_m|^2 u_m)$ is bounded in $L^\infty(0,T;L^2(\Omega))$.

Note that

$$\begin{aligned}&\big([|u_m(t)|^2 u_m(t)]', \omega_j\big) = \big([u_m(t)\overline{u_m(t)}u_m(t)]', \omega_j\big)\\ &= \big((u_m(t)\overline{u_m(t)})' u_m(t) + u_m(t)\overline{u_m(t)}u_m'(t), \omega_j\big)\\ &= \big(u_m'(t)\overline{u_m(t)}u_m(t) + u_m(t)\overline{u_m'(t)}u_m(t) + u_m(t)\overline{u_m(t)}u_m'(t), \omega_j\big)\\ &= \big(u_m'(t)|u_m(t)|^2 + u_m^2(t)\overline{u_m'(t)} + |u_m(t)|^2 u_m'(t), \omega_j\big).\end{aligned} \tag{35}$$

From (10), (33) and (35) it follows that the right side of the equality in (34) belongs to $L^2(0,T)$, that is, $g_j''(t) \in L^2(0,T)$ which implies

$$u_m'' \in L^2(0,T;L^2(\Omega)), \tag{36}$$

where we are using arguments analogous to those employed in Problem 2.

Thus, from (34), (35), (36) and the fact that $\left\langle \frac{d}{dt}(u_m'(t), \omega_j), \theta \right\rangle = \langle (u_m'', \omega_j), \theta \rangle$, $\forall\, \theta \in \mathcal{D}(0,T)$, results that

$$(u_m''(t), \omega_j) = -i((u_m'(t), \omega_j)) - (u_m'(t)|u_m(t)|^2 + u_m^2(t)\overline{u_m'(t)} + |u_m(t)|^2 u_m'(t), \omega_j) + (f'(t), \omega_j).$$

Multiplying the above equality by $\lambda_j\, \overline{g_{jm}'(t)}$ we obtain

$$\begin{aligned}&(u_m''(t), u_m'(t)) = -i((u_m'(t), u_m'(t)))\\ &- (u_m'(t)|u_m(t)|^2 + u_m^2\overline{u_m'(t)} + |u_m(t)|^2 u_m'(t), u_m'(t)) + (f'(t), u_m'(t)).\end{aligned}$$

Taking the real part on both sides of the equality above, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 = &-\int_\Omega \{|u_m(t)|^2|u_m'(t)|^2 + \operatorname{Re}(u_m \cdot \overline{u_m'(t)})^2 + |u_m(t)|^2|u_m'(t)|^2\}dx\\ &+ \operatorname{Re}(f'(t), u_m'(t))\end{aligned}$$

and from (24) it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 &= -\int_\Omega \overbrace{\{2[\operatorname{Re}(u_m(t)\overline{u_m'(t)})]^2 + |u_m(t)|^2|u_m'(t)|^2\}}^{\geq 0}\,dx + \operatorname{Re}(f'(t), u_m'(t))\\ &\leq \frac{1}{2}|f'(t)|^2 + \frac{1}{2}|u_m'(t)|^2.\end{aligned}$$

Integrating in $[0,t]$, $\ t \in [0,T]$ we conclude that

$$|u_m'(t)|^2 \leq |u_m'(0)|^2 + ||f'||^2_{L^2(Q)} + \int_0^t |u_m'(s)|^2 ds,\ \forall\, t \in [0,T],\ \forall\, m \in \mathbb{N}. \tag{37}$$

However, from (32) we have

$$(u_m'(0), \omega_j) = i((u_m(0), \omega_j)) - (|u_{0m}|^2 u_{0m}, \omega_j) + (f(0), \omega_j)$$

which implies by Green's Theorem

$$|u_m'(0)|^2 = i(-\Delta u_{0m}, u_m'(0)) - (|u_{0m}|^2 u_{0m}, u_m'(0)) + (f(0), u_m'(0)).$$

Taking the real part in the equality above

$$\begin{aligned}|u_m'(0)|^2 = {} & \operatorname{Re}[(-\Delta u_{0m}, u_m'(0))i] - \operatorname{Re}(|u_{0m}|^2|u_{0m}|, u_m'(0)) + \operatorname{Re}(f(0), u_m'(0)) \\ & \leq \big[|\Delta u_{0m}|_{L^2(\Omega)} + |\,|u_{0m}|^2 u_{0m}|_{L^2(\Omega)} + |f(0)|_{L^2(\Omega)}\big]|u_m'(0)|,\end{aligned}$$

that is,

$$\begin{aligned}|u_m'(0)| &\leq |\Delta u_{0m}|_{L^2} + |\,|u_{0m}|^2 u_{0m}|_{L^2(\Omega)} + |f(0)|_{L^2(\Omega)} \\ &= |\Delta u_{0m}|_{L^2(\Omega)} + ||u_{0m}||_{L^6(\Omega)}^6 + |f(0)|_{L^2(\Omega)} \leq c_1,\end{aligned} \tag{38}$$

where c_1 is a positive constant resulting from (12) and (13).

From (37) and (38) it follows that

$$|u_m'(t)|^2 \leq c_2 + \int_0^t |u_m'(s)|^2\, ds; \quad \forall\, t \in [0,T], \;\; \forall\, m \in \mathbb{N},$$

and from Gronwall's inequality we obtain

$$|u_m'(t)|^2 \leq c; \quad \forall\, t \in [0,T], \;\; \forall\, m \in \mathbb{N}. \tag{39}$$

Thus,

$$(u_m') \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{40}$$

The estimates obtained are sufficient to pass to the limit and the procedure is analogous to what we have done previously, the same happening for the initial condition. Being u the solution obtained, the fact that $u \in L^\infty(0,T;H^2(\Omega))$ is obtained in a manner analogous to what we did in Problem 2.

Uniqueness

Let u and v be weak solutions of Problem (1). Then $w = u - v$ verifies

$$\left|\begin{array}{l} w' - i\Delta w = |v|^2 v - |u|^2 u \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \\ w = 0 \quad \text{on} \quad \Sigma \\ w(0) = 0 \end{array}\right. \tag{41}$$

Composing $(41)_1$ with w results that

$$(w'(t), w(t) + i((w(t), w(t))) = (|v(t)|^2 v(t) - |u(t)|^2 u(t), w(t)).$$

Taking the real part on both sides of the equality above we obtain

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 = \operatorname{Re}(|v(t)|^2 v(t) - |u(t)|^2 u(t), u(t) - v(t)). \tag{42}$$

But

$$\begin{aligned}
&\mathrm{Re}\int_\Omega \big(|v(x,t)|^2 v(x,t) - |u(x,t)|^2 u(x,t)\big(\big(\overline{v(x,t)} - \overline{u(x,t)}\big)\, dx \\
&= \int_\Omega \big\{|v(x,t)|^4 - \ \mathrm{Re}\big(|v(x,t)|^2 v(x,t)\overline{u(x,t)}\big) \\
&\quad - \ \mathrm{Re}\big(|u(x,t)|^2 u(x,t)\overline{v(x,t)}\big) + |u(x,t)|^4\big\} dx \\
&= \int_\Omega \big\{|v(x,t)|^4 - |v(x,t)|^2 \mathrm{Re}(v(x,t)\overline{u(x,t)}) \\
&\quad - |u(x,t)|^2 \mathrm{Re}(u(x,t)\overline{v(x,t)}) + |u(x,t)|^4\big\} dx \\
&= \int_\Omega \big\{|v(x,t)|^4 - \ \mathrm{Re}(u(x,t)\overline{v(x,t)})^{(10)}\big(|u(x,t)|^2 + |v(x,t)|^2\big) + |u(x,t)|^4\big\} dx \qquad (43) \\
&\geq \int_\Omega \big\{|u(x,t)|^4 - |u(x,t)\overline{v(x,t)}|(|u(x,t)|^2 + |v(x,t)|^2) + |v(x,t)|^4\big\} dx \\
&= \int_\Omega \big\{|u(x,t)|^4 - |u(x,t)|^3 |v(x,t)| - |u(x,t)|\,|v(x,t)|^3 + |v(x,t)|^4\big\} dx \\
&= \int_\Omega \big\{|u(x,t)|^3\big(|u(x,t)| - |v(x,t)|\big) - |v(x,t)|^3\big(|u(x,t)| - |v(x,t)\big)\big\} dx \\
&= \int_\Omega \big(|u(x,t)|^3 - |v(x,t)|^3\big)\big(|u(x,t)| - |v(x,t)|\big) dx \geq 0
\end{aligned}$$

since $\psi(s) = s^\rho$ is increasing, for $\rho > 0$ and $s \geq 0$, given that $\psi'(s) = \rho|s|^{\rho-2}s \geq 0$, $\forall s \geq 0$.

From (42) and (43) it follows then that

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 \leq 0; \quad \forall\, t \in [0,T].$$

Integrating from 0 to t, with $t \in [0,T]$, we obtain

$$|\omega(t)|^2 - \underbrace{|\omega(0)|^2}_{=0} \leq 0$$

that is,

$$w(t) = 0 \quad \text{in} \quad L^2(\Omega), \ \ \forall\, t \in [0,T]$$

which concludes the proof. □

Analogously, Problem 6

$$\left|\begin{aligned}
&\frac{\partial u}{\partial t} - i\Delta u + |u|^\rho u = f \quad \text{in} \quad Q \qquad (\rho > 0) \\
&u = 0 \quad \text{on} \quad \Sigma \\
&u(x,0) = u_0(x)
\end{aligned}\right. \tag{1}$$

subject to the data

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \cap L^{2(\rho+1)}(\Omega), \quad f \in L^2(0,T;H_0^1(\Omega)) \text{ and } f' \in L^2(0,T;L^2(\Omega)) \tag{2}$$

[10]Here we used the fact that $\mathrm{Re}(u\bar{v}) = \ \mathrm{Re}(\bar{u}v)$.

admits a unique weak solution in the class

$$u \in L^\infty(0,T;H_0^1(\Omega)) \cap L^{\rho+2}(Q), \quad u' \in L^\infty(0,T;L^2(\Omega)). \tag{3}$$

Indeed, let $(\omega_\nu)_{\nu\in\mathbb{N}}$ be the eigenvectors of the operator $A = -\Delta$ defined by the triple $\{H_0^1(\Omega), L^2(\Omega), ((\,\cdot\,,\,\cdot\,))\}$. As we know, given the regularity results of elliptic problems

$$D(A^k) \subset H^{2k}(\Omega), \quad k = 1, 2, \dots \tag{4}$$

and, furthermore, the norms

$$|A^k u|_{L^2(\Omega)} \quad \text{and} \quad ||u||_{H^{2k}(\Omega)}\,; \quad k = 1, 2, \dots \tag{5}$$

are equivalent in $D(A^k)$. Also

$$(\omega_\nu) \quad \text{is total in} \quad D(A^k); \quad k = 1, 2, \dots. \tag{6}$$

We will prove that

$$(\omega_\nu) \quad \text{is total in} \quad L^q(\Omega); \quad \forall\, 1 \le q < +\infty. \tag{7}$$

Indeed, we have from (4) and (6)

$$[(\omega_\nu)] \subset D(A^k) \hookrightarrow H^{2k}(\Omega); \quad \forall\, k = 1, 2, \dots$$

However, by the Sobolev Embedding Theorem

$$H^{2k}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{if } k > \frac{n}{4}.$$

Under these conditions we have

$$[(\omega_\nu)] \subset D(A^k) \hookrightarrow H^{2k}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^q(\Omega); \quad \forall\, q \in [1, +\infty) \tag{8}$$

since Ω is bounded. On the other hand,

$$C_0^\infty(\Omega) \subset D(A^k) \subset L^q(\Omega); \quad \forall\, q \in [1, +\infty) \text{ and } \forall\, k = 1, 2, \dots. \tag{9}$$

Thus, from (5) and (8) we can write that

$$\overline{[(\omega_\nu)]}^{H^{2k}(\Omega)} = D(A^k) \tag{10}$$

and, from (9) and the fact that $C_0^\infty(\Omega)$ is dense in $L^q(\Omega)$, we have that

$$\overline{D(A^k)}^{L^q(\Omega)} = L^q(\Omega). \tag{11}$$

Let $\varepsilon > 0$ be given and $u \in L^q(\Omega)$. From (11) $\exists\, u_0 \in D(A^k)$ such that:

$$||u - u_0||_{L^q(\Omega)} < \frac{\varepsilon}{2} \tag{12}$$

and from (10) $\exists\, \omega^* \in [\omega_\nu]$ such that

$$||u_0 - \omega^*||_{H^{2k}(\Omega)} < \frac{\varepsilon}{2c}, \tag{13}$$

where $c > 0$ is such that

$$||v||_{L^q(\Omega)} \leq c||v||_{H^{2k}(\Omega)}\,; \quad \forall\, v \in H^{2k}(\Omega)$$

by virtue of the embeddings given in (8). Therefore, from (12) and (13) we obtain

$$||\omega^* - u||_{L^q(\Omega)} \leq ||\omega^* - u_0||_{L^q(\Omega)} + ||u_0 - u||_{L^q(\Omega)} \leq c||\omega^* - u_0||_{H^{2k}(\Omega)} + \frac{\varepsilon}{2} < \varepsilon,$$

which proves (7).

Let

$$V_m = [\omega_1, \omega_2, \ldots, \omega_m].$$

In V_m consider the approximate problem

$$u_m(t) \in V_m \;\Leftrightarrow\; u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i \tag{14}$$

$$(u_m'(t), \omega_j) + i((u_m(t), \omega_j)) + (|u_m(t)|^\rho u_m(t), \omega_j) = (f(t), \omega_j) \tag{15}$$

$$u_m(0) = u_{0m} \to u_0 \;\text{ in }\; H_0^1(\Omega) \cap H^2(\Omega) \cap L^{2(\rho+1)}(\Omega)\,^{(11)} \tag{16}$$

We have the following estimates

$$|u_m(t)|^2 + 2\int_0^t ||u_m(t)||^p_{L^p(\Omega)}\,dt \leq c, \quad \forall\, t \in [0,T];\; \forall\, m \in \mathbb{N}. \tag{17}$$

Whence

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \tag{18}$$

$$(u_m) \quad \text{is bounded in} \quad L^p(0,T;L^p(\Omega)) = L^p(Q). \tag{19}$$

Now from the fact that

$$\begin{aligned} ||\,|u_m|^\rho u_m||^{p'}_{L^{p'}(\Omega)} &= \int_\Omega ||u_m|^\rho u_m|^{\frac{\rho+2}{\rho+1}}\;dx \\ &= \int_\Omega |u_m|^{\rho+2}\,dx = ||u_m||^{\rho+2}_{L^{\rho+2}(\Omega)} = ||u_m||^p_{L^p(\Omega)}, \end{aligned}$$

it follows from (19) that

$$(|u_m|^\rho u_m) \quad \text{is bounded in} \quad L^{p'}(0,T;L^{p'}(\Omega)) = L^{p'}(Q). \tag{20}$$

Also

$$\frac{1}{2}\frac{d}{dt}\,||u_m(t)||^2 + \;\mathrm{Re}(|u_m(t)|^\rho u_m(t), -\Delta u_m(t)) \leq \frac{1}{2}\,||f(t)||^2 + \frac{1}{2}\,||u_m(t)||^2. \tag{21}$$

[11] Note that (ω_ν) is total in $L^q(\Omega)$; $\forall\, q \geq 1$ c.f. (7) and since it is total in $H_0^1(\Omega) \cap H^2(\Omega)$ it is total in the intersection $H_0^1(\Omega) \cap H^2(\Omega) \cap L^q(\Omega)$.

However; by Green's Theorem

$$\int_\Omega |u_m(t)|^\rho u_m(t)[-\overline{\Delta u_m(t)}]\,dx$$

$$= \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\left(|u_m(t)|^\rho u_m(t)\right)\frac{\overline{\partial u_m(x)}}{\partial x_i}\,dx$$

$$= \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\left[(u_m(t)\overline{u_m(t)})^{\rho/2}\,u_m(t)\right]\frac{\overline{\partial u_m(t)}}{\partial x_i}\,dx$$

$$= \sum_{i=1}^n \int_\Omega \left\{\frac{\partial}{\partial x_i}\left[(u_m(t)\overline{u_m(t)})^{\rho/2}\right]u_m(t)\,\frac{\overline{\partial u_m(t)}}{\partial x_i} + |u_m(t)|^\rho\,\frac{\partial u_m(t)}{\partial x_i}\,\frac{\overline{\partial u_m(t)}}{\partial x_i}\right\}dx$$

$$= \sum_{i=1}^n \int_\Omega \left\{\frac{\rho}{2}\,(u_m(t)\overline{u_m(t)})^{\frac{\rho-2}{2}}\,\frac{\partial}{\partial x_i}\,(u_m(t)\overline{u_m(t)})u_m(t)\,\frac{\overline{\partial u_m(t)}}{\partial x_i}\right.$$
$$\left. + |u_m(t)|^\rho\,\frac{\partial u_m(t)}{\partial x_i}\,\frac{\overline{\partial u_m(t)}}{\partial x_i}\right\}dx$$

$$= \sum_{i=1}^n \int_\Omega \left\{\frac{\rho}{2}\,|u_m(t)|^{\rho-2}\,|u_m(t)|^2\,\left|\frac{\partial u_m(t)}{\partial x_i}\right|^2 + \frac{\rho}{2}\,|u_m(t)|^{\rho-2}\left(u_m(t)\,\frac{\overline{\partial u_m(t)}}{\partial x_i}\right)^2\right.$$
$$\left. + |u_m(t)|^\rho\left|\frac{\partial u_m(t)}{\partial x_i}\right|^2\right\}dx$$

$$= \sum_{i=1}^n \int_\Omega \left\{\frac{\rho}{2}\,|u_m(t)|^{\rho-2}\left[|u_m(t)|^2\left|\frac{\partial u_m(t)}{\partial x_i}\right|^2 + \left(u_m(t)\,\frac{\overline{\partial u_m(t)}}{\partial x_i}\right)^2\right]\right.$$
$$\left. + |u_m(t)|^\rho\left|\frac{\partial u_m(t)}{\partial x_i}\right|^2\right\}dx$$

Recalling that

$$|z_1|^2|z_2|^2 + \operatorname{Re}[(z_1\,\overline{z_2})^2] = 2[\operatorname{Re}(z_1\overline{z_2})]^2 \quad \forall\, z_1, z_2 \in \mathbb{C} \tag{23}$$

then taking the real part in (22) implies that

$$\begin{aligned}&\operatorname{Re}\int_\Omega |u_m(t)|^\rho\,u_m(t)[-\overline{\Delta u_m(t)}]\,dx\\ &= \sum_{i=1}^n \int_\Omega \left\{\rho|u_m(t)|^{\rho-2}\left[\operatorname{Re}\left(u_m(t)\,\frac{\overline{\partial u_m(t)}}{\partial x_i}\right)\right]^2 + |u_m(t)|^\rho\left|\frac{\partial u_m(t)}{\partial x_i}\right|^2\right\}dx \geq 0.\end{aligned} \tag{24}$$

From (21), (24) and by Gronwall it follows that

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)). \tag{25}$$

Finally, from (15) we have

$$(u''_m(t), u'_m(t)) + i((u'_m(t), u'_m(t))) + ((|u_m(t)|^\rho u_m(t))', u'_m(t)) = (f'(t), u'_m(t)).$$

Considering the real part on both sides of the equality above, we obtain

$$\frac{1}{2}\,\frac{d}{dt}\,|u'_m(t)|^2 + \operatorname{Re}((|u_m(t)|^\rho u_m(t))', u'_m(t)) \leq \frac{1}{2}\,|f'(t)|^2 + \frac{1}{2}\,|u'_m(t)|^2. \tag{26}$$

Proceeding in a manner analogous to what we did in (22) and using the argument given in (23) we obtain

$$\mathrm{Re}((|u_m(t)|^\rho u_m(t))', u_m'(t)) \geq 0 \tag{27}$$

and from (26) and (27) we conclude that

$$|u_m'(t)|^2 \leq |u_m'(0)|^2 + ||f'||^2_{L^2(Q)} + \int_0^t |u_m'(s)|^2\, ds. \tag{28}$$

Now from (15) we have

$$|u_m'(0)| \leq |\Delta u_{0m}|_{L^2(\Omega)} + |\, |u_{0m}|^\rho u_{0m}|_{L^2(\Omega)} + |f(0)|_{L^2(\Omega)}$$

and from (16) we have that

$$|u_m'(0)| \leq c_1\,. \tag{29}$$

Thus, from (28), (29) and by Gronwall's inequality it follows that

$$(u_m') \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{30}$$

The estimates (18), (19), (20) and (30) allow us to pass to the limit according to Problem 1. The proof of the initial condition is analogous to what has already been done.

Uniqueness

Let u and v be weak solutions of Problem (1). Then $w = u - v$ verifies

$$\left| \begin{array}{l} w' - i\Delta w = |v|^\rho v - |u|^\rho u \quad \text{in} \quad L^\infty(0,T;H^{-1}(\Omega)) \\ w = 0 \\ w(0) = 0 \end{array} \right. \tag{31}$$

Composing $(31)_1$ with w implies that

$$(w'(t), w(t)) + i((\omega(t), \omega(t))) = \int_\Omega (|v(t)|^\rho v(t) - |u(t)|^\rho u(t)) w(t)\, dx.$$

Whence

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 = \ \mathrm{Re}\int_\Omega (|v(t)|^\rho v(t) - |u(t)|^\rho u(t))(\overline{u(t)} - \overline{v(t)})\, dx.$$

However

$$
\begin{aligned}
&\operatorname{Re}\int_\Omega \big(|v|^\rho v - |u|^\rho u\big)(\bar v - \bar u)\,dx\\
&= \int_\Omega \big\{|v|^{\rho+2} - \operatorname{Re}(|v|^\rho v\bar u) - \operatorname{Re}(|u|^\rho u\bar v) + |u|^{\rho+2}\big\}dx\\
&= \int_\Omega \big\{|v|^{\rho+2} - |v|^\rho \operatorname{Re}(v\bar u) - |u|^\rho \operatorname{Re}(u\bar v) + |u|^{\rho+2}\big\}dx\\
&= \int_\Omega \big\{|v|^{\rho+2} - \operatorname{Re}(u\bar v)^{\,(10)}(|v|^\rho + |u|^\rho) + |u|^{\rho+2}\big\}dx\\
&\geq \int_\Omega \big\{|v|^{\rho+2} - |u\bar v|^{\,(11)}(|v|^\rho + |u|^\rho) + |u|^{\rho+2}\big\}dx\\
&= \int_\Omega \big\{|v|^{\rho+2} - |u|\,|v|^{\rho+1} - |v|\,|u|^{\rho+1} + |u|^{\rho+2}\big\}dx\\
&= \int_\Omega \big\{|v|^{\rho+1}(|v| - |u|) - |u|^{\rho+1}(|v| - |u|)\big\}dx\\
&= \int_\Omega (|v|^{\rho+1} - |u|^{\rho+1})(|v| - |u|)dx \geq 0
\end{aligned}
$$

since $F(\lambda) = \lambda^{\rho+1}$ is increasing, for $\lambda \geq 0$. Thus

$$\frac{1}{2}\frac{d}{dt}|\omega(t)|^2 \leq 0.$$

Integrating the inequality above, we have

$$|\omega(t)|^2 = \underbrace{|\omega(0)|^2}_{=0} \leq 0$$

which proves that $\omega = 0$. □

[10] Here we used the fact that $\operatorname{Re}(u\bar v) = \operatorname{Re}(\bar u v)$.
[11] Note that $|\bar v| = |v|$.

Chapter 6

Problem $\dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u'|^\rho u' = f$

Problem 7

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u'|^\rho u' = f \quad \text{in} \quad Q \qquad (\rho > 0) \\ u = 0 \quad \text{on} \quad \Sigma \\ u(x,0) = u_0(x); \quad \dfrac{\partial u}{\partial t}(0) = u_1(x); \quad x \in \Omega \end{array}\right. \tag{1}$$

subject to the data

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega),\ u_1 \in H_0^1(\Omega) \cap L^{2(\rho+1)}(\Omega),\ f \in L^2(0,T;H_0^1(\Omega)) \text{ and } f' \in L^2(0,T;L^2(\Omega)) \tag{2}$$

possesses a unique strong solution in the class:

$$\begin{gathered} u \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)), \quad u' \in L^\infty(0,T;H_0^1(\Omega)) \cap L^{\rho+2}(Q), \\ u'' \in L^\infty(0,T;L^2(\Omega)). \end{gathered} \tag{3}$$

Proof:

Let $(\omega_\nu)_{\nu \in \mathbb{N}}$ be a basis of $H_0^1(\Omega) \cap H^2(\Omega)$ given by the eigenfunctions of the operator $-\Delta$ defined by the triple $\{H_0^1(\Omega), L^2(\Omega), ((\,\cdot\,,\,\cdot\,))\}$. As we know

$$\omega_\nu \in \left(\bigcap_{m \in \mathbb{N}} H^m(\Omega)\right) \cap C^\infty(\overline{\Omega}). \tag{4}$$

Furthermore, as we did in Problem 6 we have that

$$(\omega_\nu) \quad \text{is total in} \quad L^q(\Omega); \quad \forall\, q \geq 1. \tag{5}$$

1ª Step: Approximate Problem

Let us set:

$$V_m = [\omega_1, \ldots, \omega_m].$$

In V_m consider the approximate problem

$$u_m(t) \in V_m \ \Leftrightarrow\ u_m(t) = \sum_{i=1}^m g_{im}(t)\omega_i \tag{6}$$

$$(u_m''(t), \omega_j) + ((u_m(t), \omega_j)) + (|u_m'(t)|^\rho u_m'(t), \omega_j) = (f(t), \omega_j) \tag{7}$$

$$u_m(0) = u_{0m} \to u_0 \ \text{ in } \ H_0^1(\Omega) \cap H^2(\Omega) \tag{8}$$

$$u_m'(0) = u_{1m} \to u_1 \quad \text{in} \quad H_0^1(\Omega) \cap L^{2(\rho+1)}(\Omega) \tag{9}$$

which has a local solution, by virtue of Carathéodory's criterion in some interval $[0, t_m)$.

2ª Step: A Priori Estimates

• **A Priori Estimate I**

From (7) it follows that

$$\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}||u_m(t)||^2 + \int_\Omega |u_m'(t)|^{\rho+2}\,dx = (f(t), u_m'(t)) \\ \leq \frac{1}{2}|f(t)|^2 + \frac{1}{2}|u_m'(t)|^2.$$

Integrating from 0 to t; $t \in [0, t_m)$, we obtain

$$|u_m'(t)|^2 + ||u_m(t)||^2 + 2\int_0^t ||u_m'(s)||^p_{L^p(\Omega)}\,ds = |u_{1m}|^2 + ||u_{0m}||^2 \\ + ||f||^2_{L^2(Q)} + \int_0^t |u_m'(s)|^2\,ds. \tag{10}$$

From (8) and (9) $\exists\, c_1 > 0$ such that

$$|u_{1m}|^2 + ||u_{0m}||^2 + ||f||^2_{L^2(Q)} \leq c_1\,; \quad \forall\, m \in \mathbb{N} \tag{11}$$

and from (10) and (11) it follows that

$$|u_m'(t)|^2 + ||u_m(t)||^2 + 2\int_0^t ||u_m'(s)||^p_{L^p(\Omega)}\,ds \leq c_1 + \int_0^t |u_m'(s)|^2\,ds$$

and by Gronwall we obtain

$$|u_m'(t)|^2 + ||u_m(t)||^2 + 2\int_0^t ||u_m'(s)||^p_{L^p(\Omega)}\,ds \leq c; \quad \forall\, t \in [0, t_m);\ \forall\, m \in \mathbb{N} \tag{12}$$

which allows us to extend the solution $u_m(t)$ to the whole interval $[0, T]$, with $u_m(t)$, $u_m'(t)$ absolutely continuous and $u_m''(t)$ existing a.e. in $]0, T[$. From (12) it follows that

$$(u_m') \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)) \tag{13}$$
$$(u_m) \quad \text{is bounded in} \quad L^\infty(0, T; H_0^1(\Omega)) \tag{14}$$
$$(u_m') \quad \text{is bounded in} \quad L^p(0, T; L^{p'}(\Omega)) = L^p(Q) \tag{15}$$
$$(|u_m'|^\rho u_m') \quad \text{is bounded in} \quad L^{p'}(0, T; L^{p'}(\Omega)) = L^{p'}(Q) \tag{16}$$

• **A Priori Estimate II**

Composing (7) with $(-\Delta u_m'(t))$ by Green's formula implies that

$$((u_m''(t), u_m'(t))) + (-\Delta u_m(t), -\Delta u_m'(t)) + (|u_m'(t)|^\rho u_m'(t), -\Delta u_m'(t)) = ((f(t), u_m'(t))).$$

Now from (4), (6) and again by Green's Theorem

$$\frac{1}{2}\frac{d}{dt}||u_m'(t)||^2 + \frac{1}{2}\frac{d}{dt}|\Delta u_m(t)|^2 + \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}(|u_m'(t)|^\rho u_m'(t))\,\frac{\partial u_m'(t)}{\partial x_i}\,dx \\ \leq \frac{1}{2}||f(t)||^2 + \frac{1}{2}||u_m'(t)||^2. \tag{17}$$

However,

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial x_i}\left(|u'_m(t)|^\rho u'_m(t)\right)\frac{\partial u'_m(t)}{\partial x_i}\,dx\\
&= (\rho+1)\int_\Omega |u'_m(t)|^\rho\,\frac{\partial u'_m(t)}{\partial x_i}\,\frac{\partial u'_m(t)}{\partial x_i}\,dx\\
&= (\rho+1)\int_\Omega\left(|u'_m(t)|^{\rho/2}\,\frac{\partial u'_m(t)}{\partial x_i}\right)^2 dx.
\end{aligned}\tag{18}$$

Now, from the fact that we have

$$\frac{\partial}{\partial x_i}\left(|u'_m(t)|^{\rho/2}\,u'_m(t)\right) = \left(\frac{\rho}{2}+1\right)|u'_m(t)|^{\rho/2}\,\frac{\partial u'_m(t)}{\partial x_i}\tag{19}$$

from (18) and (19) it follows that

$$\int_\Omega \frac{\partial}{\partial x_i}\left(|u'_m(t)|^\rho\,u'_m(t)\right)\frac{\partial u'_m(t)}{\partial x_i}\,dx = \frac{(\rho+1)}{(\rho/2+1)^2}\int_\Omega\left(\frac{\partial}{\partial x_i}\left(|u'_m(t)|^{\rho/2}u'_m(t)\right)\right)^2 dx.\tag{20}$$

Substituting (20) in (17) we obtain

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\left\{||u_m(t)||^2 + |\Delta u_m(t)|^2\right\} + \frac{(\rho+1)}{(\rho/2+1)^2}\sum_{i=1}^n\int_\Omega\left[\frac{\partial}{\partial x_i}\left(|u'_m(t)|^{\rho/2}u'_m(t)\right)\right]^2 dx\\
&\qquad\le \frac{1}{2}\,||f(t)||^2 + \frac{1}{2}\,||u'_m(t)||^2.
\end{aligned}$$

Integrating from 0 to t with $t\in[0,T]$ we arrive at

$$\begin{aligned}
&||u'_m(t)||^2 + |\Delta u_m(t)|^2 + \frac{2(\rho+1)}{(\frac{\rho}{2}+1)}\sum_{i=1}^n\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}\left(|u'_m(t)|^{\rho/2}u'_m(t)\right)\right]^2 dxdt\\
&\qquad\le ||u_{1m}||^2 + |\Delta u_{0m}|^2 + ||f||^2_{L^2(0,T;H_0^1)} + \int_0^t ||u'_m(s)||^2\,ds.
\end{aligned}\tag{21}$$

However, from (8) $\exists\, c_1>0$ such that

$$||u_{1m}||^2 + |\Delta u_{0m}|^2 + ||f||^2_{L^2(0,T;H_0^1)} \le c_1\,;\quad \forall\, m\in\mathbb{N}\tag{22}$$

and from (21) and (22) we obtain

$$\begin{aligned}
&||u_m(t)||^2 + |\Delta u_m(t)|^2 + \frac{2(\rho+1)}{(\frac{\rho}{2}+1)}\sum_{i=1}^n\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}\left(|u'_m(t)|^{\rho/2}u'_m(t)\right)\right]^2 dxdt\\
&\qquad\le c_1 + \int_0^t ||u'_m(s)||^2\,ds
\end{aligned}$$

and by Gronwall's inequality

$$||u'_m(t)||^2 + |\Delta u_m(t)|^2 + \frac{2(\rho+1)}{(\frac{\rho}{2}+1)}\sum_{i=1}^n\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}\left(|u'_m(t)|^{\rho/2}\,u'_m(t)\right)\right]^2 dxdt \le c,\tag{23}$$

for all $t\in[0,T]$; $\forall\, m\in\mathbb{N}$, which implies that

$$(u'_m)\quad\text{is bounded in}\quad L^\infty(0,T;H_0^1(\Omega))\tag{24}$$

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \tag{25}$$

$$\left(\frac{\partial}{\partial x_i}(|u_m'(t)|^{\rho/2}u_m'(t)\right) \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega));\ \forall\, i=1,\dots,n \tag{26}$$

- **A Priori Estimate III**

From (7) we obtain

$$(u_m'''(t),u_m''(t)) + ((u_m'(t),u_m''(t))) + ((|u_m'(t)|^\rho u_m'(t))', u_m''(t)) = (f'(t),u_m''(t))$$

that is,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_m''(t)|^2 + \frac{1}{2}\frac{d}{dt}||u_m'(t)||^2 + (\rho+1)\int_\Omega |u_m'(t)|^\rho u_m''(t)u_m''(t)dx \\ \le \frac{1}{2}|f'(t)|^2 + \frac{1}{2}|u_m''(t)|^2.\end{aligned}$$

Integrating from 0 to t; with $t\in[0,T]$

$$\begin{aligned}|u_m''(t)|^2 + ||u_m'(t)||^2 + 2(\rho+1)\int_0^t\int_\Omega |u_m'(x,s)|^\rho (u_m''(x,s))^2\,dx\,ds \\ \le |u_m''(0)|^2 + ||u_{1m}||^2 + ||f'||_{L^2(Q)} + \int_0^t |u_m''(s)|^2\,ds.\end{aligned} \tag{27}$$

However, from (7) we obtain

$$\begin{aligned}|u_m''(0)|^2 &= (f(0),u_m''(0)) - (\Delta u_{0m},u_m''(0)) - (|u_{1m}|^\rho u_{1m},u_m''(0)) \\ &\le \big[|f(0)| + |\Delta u_{0m}| + |\,|u_{1m}|^\rho u_{1m}|_{L^2(\Omega)}\big]|u_m''(0)|\end{aligned}$$

and from (8) and (9) $\exists\, c_1>0$ such that:

$$|u_m''(0)| \le \big[|f(0)| + |\Delta u_{0m}| + ||u_{1m}||^{\rho+1}_{L^2(\rho+1)}\big] \le c_1\,. \tag{28}$$

Thus, from (9), (27) and (28) we conclude that

$$|u_m''(t)|^2 + ||u_m'(t)||^2 + 2(\rho+1)\int_0^t\int_\Omega |u_m'(x,s)|^\rho (u_m''(x,s))^2\,dt \le c_2 + \int_0^t |u_m''(s)|^2\,ds$$

and by Gronwall it follows that:

$$(u_m'') \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \tag{29}$$

$$(u_m') \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)). \tag{30}$$

From the estimates above we obtain a subsequence (u_ν) of (u_m) such that

$$\begin{aligned}&u_\nu \overset{*}{\rightharpoonup} u \quad \text{weak-star in} \quad L^\infty(0,T;H_0^1(\Omega)) \\ &u_\nu'' \overset{*}{\rightharpoonup} u'' \quad \text{weak-star in} \quad L^\infty(0,T;L^2(\Omega)) \\ &u_\nu' \rightharpoonup u' \quad \text{weakly in} \quad L^p(Q) \\ &u_\nu' \overset{*}{\rightharpoonup} u' \quad \text{weak-star in} \quad L^\infty(0,T;H_0^1(\Omega)) \\ &u_\nu \overset{*}{\rightharpoonup} u \quad \text{weak-star in} \quad L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))\end{aligned} \tag{31}$$

and from (29) and (30) by virtue of the Aubin-Lions Theorem it follows that

$$u'_\nu \to u' \quad \text{in} \quad L^2(0,T;L^2(\Omega)).$$

Thus,

$$u'_\nu(x,t) \to u'(x,t) \quad \text{a.e. in} \quad Q$$

which implies that

$$|u'_\nu(x,t)|^\rho\, u'_\nu(x,t) \to |u'(x,t)|^\rho\, u'(x,t) \quad \text{a.e. in} \quad Q. \tag{32}$$

Now from (16) and (32) by virtue of Lions' Lemma it follows that

$$|u'_\nu|^\rho\, u'_\nu \rightharpoonup |u'|^\rho\, u' \quad \text{weakly in} \quad L^{p'}(Q). \tag{33}$$

The convergences in (31) and (33) are sufficient to pass to the limit in the equation as in Problem 1. The initial conditions are proved in the usual manner.

Uniqueness

Let u and v be solutions of (1) and set $\omega = u - v$. Then w satisfies

$$\left|\begin{array}{l} \omega'' - \Delta\omega = |v'|^\rho v' - |u'|^\rho u' \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \\ \omega = 0 \quad \text{on} \quad \Sigma \\ \omega(0) = \omega'(0) = 0 \end{array}\right. \tag{34}$$

Composing $(34)_1$ with ω' implies that

$$(\omega''(t), \omega'(t)) + ((\omega(t), \omega'(t))) = (|v'|^\rho v' - |u'|^\rho u', u'(t) - v'(t))$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}|\omega'(t)|^2 + \frac{1}{2}\frac{d}{dt}||\omega(t)||^2 = \int_\Omega \big(|v'|^\rho v' - |u'|^\rho u'\big)(u' - v')\,dx. \tag{35}$$

Since the function $F(s) = |s|^\rho s$ is non-decreasing, given that $F'(s) = (\rho+1)|s|^\rho \geq 0$, we have

$$\int_\Omega \big(|v'|^\rho v' - |u'|^\rho u'\big)(u' - v')\,dx \leq 0 \tag{36}$$

and from (35) and (36) it follows that

$$\frac{1}{2}\frac{d}{dt}|\omega'(t)|^2 + \frac{1}{2}\frac{d}{dt}||\omega(t)||^2 \leq 0.$$

Integrating from 0 to t, we obtain from $(34)_3$

$$|\omega'(t)|^2 + ||\omega(t)||^2 \leq 0, \quad \forall\, t \in [0,T].$$

Thus,

$$||\omega(t)||^2 = 0, \quad \forall\, t \in [0,T]$$

which proves that $\omega = 0$. □

Chapter 7

Von Kármán System

In what follows Ω will represent a bounded and sufficiently smooth open subset of $\mathbb{R}^2$.

Problem 8

Problem 8

$$\left|\begin{array}{l}\dfrac{\partial^2 u}{\partial t^2}+\Delta^2 u-[u,v]=f \quad \text{in} \quad Q=\Omega\times\,]0,T[\\ \Delta^2 v+[u,u]=0 \quad \text{in} \quad Q\\ u=0,\ v=0 \quad \text{on} \quad \Sigma=\Gamma\times\,]0,T[\\ \dfrac{\partial u}{\partial\nu}=0;\ \dfrac{\partial v}{\partial\nu}=0 \quad \text{on} \quad \Sigma\\ u(0)=u_0(x); \quad x\in\Omega\\ \dfrac{\partial u}{\partial t}(0)=u_1(x); \quad x\in\Omega\end{array}\right. \tag{1}$$

where

$$[u,v]=\frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial y^2}\,\frac{\partial^2 v}{\partial x^2}-2\,\frac{\partial^2 u}{\partial x\partial y}\,\frac{\partial^2 v}{\partial x\partial y} \tag{2}$$

subject to the data

$$u_0\in H_0^2(\Omega),\ u_1\in L^2(\Omega) \quad \text{and} \quad f\in L^2(0,T;L^2(\Omega)), \tag{3}$$

possesses at least one pair (u,v), weak solution of (1) in the class

$$u,v\in L^\infty(0,T;H_0^2(\Omega)); \quad u'=\frac{\partial u}{\partial t}\in L^\infty(0,T;L^2(\Omega)). \tag{4}$$

Proof:

1ª Step: Approximate Solution

Let $(\omega_\nu)_{\nu\in\mathbb{N}}$ be a basis of eigenfunctions of the operator Δ^2 defined by the triple $\{H_0^2(\Omega),L^2(\Omega);((\,\cdot\,,\,\cdot\,))_{H_0^2(\Omega)}\}$, where:

$$((u,v))_{H_0^2(\Omega)}=\int_\Omega \Delta u\,\Delta v\,dx. \tag{5}$$

Letting (λ_ν) be the corresponding sequence of eigenvalues, we have:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu \leq \cdots \quad , \quad \lambda_\nu \to +\infty \text{ when } \nu \to +\infty \quad \text{and}$$

$$\left| \begin{array}{l} \Delta^2 \omega_\nu = \lambda_\nu \omega_\nu \\ \omega_\nu|_\Gamma = 0 \\ \left.\dfrac{\partial \omega_\nu}{\partial \nu}\right|_\Gamma = 0 \end{array} \right. \tag{6}$$

Now from (6), by virtue of the regularity of elliptic problems of order 2, from the fact that Ω is sufficiently smooth, from (6) and from Sobolev embeddings, we have:

$$\omega_\nu \in \Big(\bigcap_{m \in \mathbb{N}} H^m(\Omega) \Big) \cap C^\infty(\overline{\Omega}) \cap H_0^2(\Omega). \tag{7}$$

From Spectral Theory we know that

$$(\omega_\nu) \quad \text{is a complete orthonormal system in} \quad L^2(\Omega) \tag{8}$$

$$\left(\frac{\omega_\nu}{\sqrt{\lambda_\nu}} \right) \quad \text{is a complete orthonormal system in} \quad H_0^2(\Omega) \tag{9}$$

$$\left(\frac{\omega_\nu}{\lambda_\nu} \right) \quad \text{is a complete orthonormal system in} \quad H_0^2(\Omega) \cap H^4(\Omega) \tag{10}$$

As is well known, the operator

$$\Delta^2 \colon H_0^2(\Omega) \cap H^4(\Omega) \to L^2(\Omega) \tag{11}$$

is a bijection. Identifying $L^2(\Omega)$ with its dual we can extend the biharmonic operator given in (11) to a unique isometric extension

$$\widetilde{\Delta^2} \colon H_0^2(\Omega) \to H^{-2}(\Omega). \tag{12}$$

Let us set:

$$G \colon L^2(\Omega) \to H_0^2(\Omega) \cap H^4(\Omega) \quad \text{and} \quad \widetilde{G} \colon H^{-2}(\Omega) \to H_0^2(\Omega) \tag{13}$$

the inverses of (11) and (12), respectively; i.e, $G = (\Delta^2)^{-1}$ and $\widetilde{G} = (\widetilde{\Delta^2})^{-1}$. To not overburden the notation, from now on we will not use the tilde ($\sim$).

Let,

$$V_m = [\omega_1, \ldots, \omega_m].$$

In V_m consider the approximate problem

$$u_m(t) \in V_m \iff u_m(t) = \sum_{i=1}^m g_{im}(t) \omega_i\,, \tag{14}$$

$$\begin{aligned} &(u_m''(t), \omega_j) + (\Delta u_m(t), \Delta \omega_j) \\ &+ ([u_m(t), G[u_m(t), u_m(t)]]^{(*)}, \omega_j) = (f(t), \omega_j) \quad j = 1, \ldots, m \end{aligned} \tag{15}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^2(\Omega) \tag{16}$$

[12] It is worth noting that from the regularity in (7) $G[u_m, u_m]$ makes sense.

$$u_m'(0) = u_{1m} \to u_1 \quad \text{in} \quad L^2(\Omega), \tag{17}$$

which by Carathéodory possesses a local solution in some interval $[0, t_m)$. Note that from $(1)_1$ and $(1)_2$ we can write

$$v_m(t) = -G[u_m(t), u_m(t)]^{\,(13)} \tag{18}$$

2ª Step: A Priori Estimate

Composing (15) with u_m' it follows from (18) that

$$(u_m''(t), u_m'(t)) + (\Delta u_m(t), \Delta u_m'(t)) - \big([u_m(t), v_m(t)], u_m'(t)\big) = (f(t), u_m'(t))$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}|\Delta u_m(t)|^2 - \big([u_m(t), v_m(t)], u_m'(t)\big) = (f(t), u_m'(t)). \tag{19}$$

Analysis of the Nonlinear Term

Consider the trilinear map

$$\begin{aligned} b\colon H_0^2(\Omega) \times H_0^2(\Omega) \times H_0^2(\Omega) &\to \mathbb{R} \\ (u, v, \omega) \mapsto b(u, v, \omega) &= \int_\Omega [u, v]\omega \, dx \end{aligned} \tag{20}$$

We claim that such map is symmetric. Indeed, initially observe that such map is well defined. From (2) and the fact that $u, v \in H_0^2(\Omega)$ we have

$$[u, v] \in L^1(\Omega). \tag{21}$$

Now since $n = 2$, we have that:

$$H_0^2(\Omega) \hookrightarrow C^0(\overline{\Omega}) \hookrightarrow L^\infty(\Omega). \tag{22}$$

Thus from (21) and (22) it is proved that the map (20) is well defined. Now, to prove symmetry observe that from the fact that $[u, v] = [v, u]$ it is sufficient to prove that

$$\int_\Omega [u, v]\omega \, dx = \int_\Omega [u, \omega] v \, dx \tag{23}$$

Indeed, let $u, v \in \mathcal{D}(\Omega)$. Then:

$$\begin{aligned} &\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial y^2} v\right) - 2\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 u}{\partial x \partial y} v\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2}{\partial x^2} v\right) \\ &= \frac{\partial^4 u}{\partial x^2 \partial y^2} v + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^4 u}{\partial x^2 \partial y^2} v - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^4 v}{\partial y^2 \partial x^2} v + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} \\ &= [u, v]. \end{aligned}$$

[13] Note that from (2), (7), (14), $[u_m, u_m] \in C^\infty(\overline{\Omega}) \subset L^2(\Omega)$, since Ω is bounded.

Thus:

$$\int_\Omega [u,v]w\,dx$$
$$= \int_\Omega \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial y^2}\,v\right)\omega - 2\int_\Omega \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2 u}{\partial x\partial y}\,v\right)\omega\,dx + \int_\Omega \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 u}{\partial x^2}\,v\right)\omega\,dx. \tag{24}$$

Using Gauss's formula twice, the equality above can be rewritten as

$$\int_\Omega \frac{\partial^2 u}{\partial y^2}\,\frac{\partial^2 \omega}{\partial x^2}\,v\,dx - 2\int_\Omega \frac{\partial^2 u}{\partial x\partial y}\,\frac{\partial^2 \omega}{\partial x\partial y}\,v\,dx + \int_\Omega \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 \omega}{\partial y^2}\,v\,dx$$

that is,

$$\int_\Omega [u,v]\omega\,dx = \int_\Omega [u,\omega]v; \quad \forall\, u,v,\omega \in \mathcal{D}(\Omega). \tag{24}$$

Let (u_ν), (v_μ) and $(\omega_\xi) \subset \mathcal{D}(\Omega)$ be such that

$$u_\nu \to u; \quad v_\mu \to v \quad \text{and} \quad \omega_\xi \to \omega \text{ in } H_0^2(\Omega). \tag{25}$$

Now from (2) and (25) it follows that

$$[u_\nu, v_\mu] \to [u,v] \quad \text{in} \quad L^1(\Omega) \tag{26}$$

and from (22) and (25) we also have that

$$\omega_\xi \to \omega \quad \text{in} \quad L^\infty(\Omega) \tag{27}$$

and from the convergences in (26) and (27) we obtain

$$\int_\Omega [u_\nu, v_\mu]\omega_\xi\,dx \to \int_\Omega [u,v]\omega\,dx.$$

Analogously

$$\int_\Omega [u_\nu, \omega_\xi]v_\mu\,dx \to \int_\Omega [u,\omega]v\,dx$$

which proves (23). It follows from this in particular that

$$\int_\Omega [u_m(t), v_m(t)]u_m'(t)\,dx = \int_\Omega [u_m(t), u_m'(t)]v_m(t)\,dx. \tag{28}$$

Substituting (28) in (19) results that

$$\frac{1}{2}\frac{d}{dt}\,|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}\,|\Delta u_m(t)|^2 - \big([u_m(t), u_m'(t)], v_m(t)\big) = (f(t), u_m'(t))$$

or even,

$$\frac{1}{2}\frac{d}{dt}\,|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}\,|\Delta u_m(t)|^2 - \frac{1}{2}\left(\frac{d}{dt}\,[u_m(t), u_m(t)], v_m(t)\right) = (f(t), u_m'(t)). \tag{29}$$

But from (13) and (18) we can write

$$\Delta^2 v_m(t) = -[u_m(t), u_m(t)]$$

which implies

$$\Delta^2 v'_m(t) = -\frac{d}{dt}\,[u_m(t), u_m(t)]. \tag{30}$$

Substituting (30) in (29) results that

$$\frac{1}{2}\,\frac{d}{dt}\,|u'_m(t)|^2 + \frac{1}{2}\,\frac{d}{dt}\,|\Delta u_m(t)|^2 + \frac{1}{2}\,(\Delta v'_m(t), \Delta v_m(t)) \leq \frac{1}{2}\,|f(t)|^2 + \frac{1}{2}\,|u'_m(t)|^2$$

that is,

$$\frac{1}{2}\,\frac{d}{dt}\,|u'_m(t)|^2 + \frac{1}{2}\,\frac{d}{dt}\,|\Delta u_m(t)|^2 + \frac{1}{4}\,\frac{d}{dt}\,|\Delta v_m(t)|^2 \leq \frac{1}{2}\,|f(t)|^2 + \frac{1}{2}\,|u'_m(t)|^2.$$

Integrating from 0 to t, $\quad t \in [0, t_m)$ we obtain

$$\begin{aligned} |u'_m(t)|^2 + |\Delta u_m(t)|^2 + \frac{1}{2}\,|\Delta v_m(t)|^2 &\leq |u_{1m}|^2 + |\Delta u_{0m}|^2 \\ &+ \frac{1}{2}\,|\Delta v_m(0)|^2 + ||f||^2_{L^2(Q)} + \int_0^t |u'_m(s)|^2\,ds. \end{aligned} \tag{31}$$

Analysis of the term $|\Delta v_m(0)|^2$

Since $n = 2$, we will prove that

$$L^1(\Omega) \hookrightarrow H^{-2}(\Omega). \tag{32}$$

Indeed, defining the operator

$$\begin{aligned} T\colon\, L^1(\Omega) &\to H^{-2}(\Omega) \\ g &\mapsto Tg \end{aligned}$$

given by

$$\langle Tg, v\rangle = \int_\Omega gv\,dx; \quad v \in H^2_0(\Omega), \tag{33}$$

we have, by virtue of (22), that

$$|\langle Tg, v\rangle| \leq \int_\Omega |g|\,|v|\,dx = ||g||_{L^1(\Omega)}\,||v||_{L^\infty(\Omega)} \leq c||g||_{L^1(\Omega)}\,||v||_{H^2_0(\Omega)}\,.$$

Therefore,

$$||Tg||_{H^{-2}(\Omega)} \leq c||g||_{L^1(\Omega)} \tag{34}$$

which proves that

$$T \in \mathcal{L}(L^1(\Omega); H^{-2}(\Omega)).$$

We note also that if $g_1, g_2 \in L^1(\Omega)$ and $T_{g_1} = T_{g_2}$ we obtain

$$\int_\Omega (g_1 - g_2)v\,dx = 0, \quad \forall\, v \in H^2_0(\Omega).$$

In particular

$$\langle g_1 - g_2, \varphi\rangle = 0, \quad \forall\, \varphi \in \mathcal{D}(\Omega)$$

which implies that $g_1 = g_2$ in $\mathcal{D}'(\Omega)$ and therefore $g_1 = g_2$ a.e. in Ω, which proves the injectivity of the map T. In this sense (32) is proved.

We will prove next that the bilinear map

$$\begin{aligned}[\,\cdot\,,\,\cdot\,]\colon H_0^2(\Omega)\times H_0^2(\Omega) &\to H^{-2}(\Omega)\\ (u,v) &\mapsto [u,v]\end{aligned} \tag{35}$$

is continuous. In fact, by virtue of (21) and (32)

$$\begin{aligned}
&||[u,v]||_{H^{-2}(\Omega)} \le c_1||[u,v]||_{L^1(\Omega)} = c_1\int_\Omega |[u,v]|\,dx\\
&\le c_1\int_\Omega\left\{\left|\frac{\partial^2 u}{\partial x^2}\right|\left|\frac{\partial^2 v}{\partial y^2}\right| + 2\left|\frac{\partial^2 u}{\partial x\partial y}\right|\left|\frac{\partial^2 v}{\partial x\partial y}\right| + \left|\frac{\partial^2 u}{\partial y^2}\right|\left|\frac{\partial^2 v}{\partial x^2}\right|\right\}dx\\
&\le c_1\left\{\left(\int\left|\frac{\partial^2 u}{\partial x^2}\right|^2\right)^{1/2}\left(\int_\Omega\left|\frac{\partial^2 v}{\partial y^2}\right|^2\right)^{1/2} + 2\left(\int_\Omega\left|\frac{\partial^2 u}{\partial x\partial y}\right|^2\right)^{1/2}\left(\int_\Omega\left|\frac{\partial^2 v}{\partial x\partial y}\right|^2\right)^{1/2}\right.\\
&\qquad\left. + \left(\int_\Omega\left|\frac{\partial^2 u}{\partial y^2}\right|^2\right)^{1/2}\left(\int_\Omega\left|\frac{\partial^2 v}{\partial x^2}\right|^2\right)^{1/2}\right\}\\
&\le c_2\left\{||u||_{H_0^2(\Omega)}\,||v||_{H_0^2(\Omega)} + 2||u||_{H_0^2(\Omega)}\,||v||_{H_o^2(\Omega)} + ||u||_{H_0^2(\Omega)}\,||v||_{H_0^2(\Omega)}\right\}^{(14)}\\
&\le c_3||u||_{H_0^2(\Omega)}\,||v||_{H_0^2(\Omega)}
\end{aligned}$$

which proves the desired result. It follows from this that

$$||[u_{0m},u_{0m}]||_{H^{-2}(\Omega)} \le c_2||u_{0m}||_{H_0^2(\Omega)}\,||u_{0m}||_{H_0^2(\Omega)}\,.$$

From (16) we guarantee the existence of a constant $c_3 > 0$ such that

$$||[u_{0m},u_{0m}]||_{H^{-2}(\Omega)} \le c_3\,;\quad \forall\, m\in\mathbb{N} \tag{36}$$

and therefore, from (13) it follows that

$$||v_m(0)||_{H_0^2(\Omega)} = ||G[u_{0m},u_{0m}]||_{H_0^2(\Omega)} \le c_4||[u_{0m},u_{0m}]||_{H^{-2}(\Omega)} \le c_5\,,\quad \forall\, m\in\mathbb{N}$$

that is,

$$|\Delta v_m(0)|_{L^2(\Omega)} \le c_6\,;\quad \forall\, m\in\mathbb{N}. \tag{37}$$

Thus, from (16), (17), (31) and (37) we obtain

$$|u_m'(t)|^2 + |\Delta u_m(t)|^2 + \frac{1}{2}\,|\Delta v_m(t)|^2 \le c_7 + \int_0^t |u_m'(s)|^2\,ds$$

and by Gronwall

$$|u_m'(t)|^2 + |\Delta u_m(t)|^2 + \frac{1}{2}\,|\Delta v_m(t)|^2 \le c;\quad \forall\, t\in[0,t_m)\ \forall\, m\in\mathbb{N}. \tag{38}$$

The estimate above allows us to extend $u_m(t)$ to the whole interval $[0,T]$, and from (18) we have the same for $v_m(t)$. Furthermore, we conclude that

$$(u_m')\quad \text{is bounded in}\quad L^\infty(0,T;L^2(\Omega)) \tag{39}$$

$$(u_m)\quad \text{is bounded in}\quad L^\infty(0,T;H_0^2(\Omega)) \tag{40}$$

$$(v_m)\quad \text{is bounded in}\quad L^\infty(0,T;H_0^2(\Omega)). \tag{41}$$

[14] Here we are using the fact that in $H_0^2(\Omega)$ the norms $|\Delta u|_{L^2}$ and $||u||_{H^2(\Omega)}$ are equivalent.

3ª Step: Passage to the Limit

From the estimates in (39), (40) and (41) we can extract subsequences (u_ν) of (u_m) and (v_ν) of (v_m) such that

$$u_\nu \overset{*}{\rightharpoonup} u \quad \text{weak-star in} \quad L^\infty(0,T;H_0^2(\Omega)) \tag{42}$$

$$u'_\nu \overset{*}{\rightharpoonup} u' \quad \text{weak-star in} \quad L^\infty(0,T;L^2(\Omega)) \tag{43}$$

$$v_\nu \overset{*}{\rightharpoonup} v \quad \text{weak-star in} \quad L^\infty(0,T;H_0^2(\Omega)) \tag{44}$$

On the other hand, from (39) and (40) we have, by virtue of the Aubin-Lions Theorem, the existence of a subsequence of (u_ν), which we will still denote by (u_ν), such that

$$u_\nu \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{45}$$

Let $j \in \mathbb{N}$ and $\theta \in \mathcal{D}(0,T)$. Then, for $\nu > j$ from (15) and (18) it follows that

$$\begin{aligned} &-\int_0^T (u'_\nu(t),\omega_j)\theta'(t)\,dt + \int_0^T (\Delta u_\nu(t),\Delta\omega_j)\theta(t)\,dt \\ &-\int_0^T ([u_\nu(t),v_\nu(t)],\omega_j)\theta(t)\,dt = \int_0^T (f(t),\omega_j)\theta(t)\,dt. \end{aligned} \tag{46}$$

From (42) it follows that

$$\langle w,u_\nu\rangle_{L^1(0,T;H^{-2}),L^\infty(0,T;H_0^2)} \longrightarrow \langle w,u\rangle_{L^1(0,T;H^{-2}),L^\infty(0,T;H_0^2)}\,; \quad \forall\, w\in L^1(0,T;H^{-2}(\Omega)).$$

In particular, if we define

$$\psi_j = \theta\omega_j \in C^\infty(\overline{\Omega}\times[0,T]) = C^\infty(\overline{Q}) \tag{47}$$

and consider $w = \Delta^2\psi_j = \theta\Delta^2\omega_j \in L^1(0,T;H^{-2}(\Omega))$ we obtain

$$\int_0^T \langle \Delta^2\psi_j(t),u_\nu(t)\rangle_{H^{-2}(\Omega);H_0^2(\Omega)}\,dt \longrightarrow \int_0^T \langle \Delta^2\psi_j(t),u(t)\rangle_{H^{-2}(\Omega);H_0^2(\Omega)}\,dt$$

that is,

$$\int_0^T (\Delta\psi_j(t),\Delta u_\nu(t))dt \longrightarrow \int_0^T (\Delta\psi_j(t),\Delta u(t))\,dt. \tag{48}$$

From (43) we obtain immediately that

$$\int_0^T (u'_\nu(t),\psi'_j(t))\,dt \longrightarrow \int_0^T (u'(t),\psi'_j(t))\,dt. \tag{49}$$

- **Passage to the Limit of the Nonlinear Term**

We have the following relation

$$\begin{aligned} \int_0^T ([u_\nu,v_\nu],\psi_j)\,dt &= \int_0^T ([u_\nu,\psi_j],v_\nu)\,dt \\ = \int_0^T ([\psi_j,u_\nu],v_\nu)\,dt &= \int_0^T ([\psi_j,v_\nu],u_\nu)\,dt. \end{aligned} \tag{50}$$

We claim that

$$[\psi_j, v_\nu] \rightharpoonup [\psi_j, v] \quad \text{weakly in} \quad L^2(Q). \tag{51}$$

Indeed, since

$$[\psi_j, v_\nu] = \frac{\partial^2 \psi_j}{\partial x^2}\,\frac{\partial^2 v_\nu}{\partial y^2} - 2\,\frac{\partial^2 \psi}{\partial x \partial y}\,\frac{\partial^2 v_\nu}{\partial x \partial y} + \frac{\partial^2 \psi_j}{\partial y^2}\,\frac{\partial^2 v_\nu}{\partial x^2}$$

it is sufficient to prove, for example, that

$$\int_Q \frac{\partial^2 \psi_j}{\partial x^2}\,\frac{\partial^2 v_\nu}{\partial y^2}\,\varphi\,dxdt \to \int_Q \frac{\partial^2 \psi_j}{\partial x^2}\,\frac{\partial^2 v}{\partial y^2}\,\varphi\,dxdt; \quad \forall\,\varphi \in L^2(Q). \tag{52}$$

since for the other terms the reasoning is analogous.

From (38) it follows that

$$(v_\nu), \left(\frac{\partial^2 v_\nu}{\partial x^2}\right), \left(\frac{\partial^2 v_\nu}{\partial y^2}\right), \left(\frac{\partial^2 v_\nu}{\partial x \partial y}\right), \left(\frac{\partial^2 v_\nu}{\partial y \partial x}\right)$$

are bounded in $L^\infty(0, T; L^2(\Omega))$.

Thus,

$$\frac{\partial^2 v_\nu}{\partial y^2} \rightharpoonup \frac{\partial^2 v}{\partial y^2} \quad \text{weakly in} \quad L^2(Q). \tag{53}$$

However from (47) we have that

$$\psi_j \in C^\infty(\overline{Q})$$

which implies that

$$\frac{\partial^2 \psi_j}{\partial x^2} = \theta_j\,\frac{\partial^2 \omega_j}{\partial x^2} \in C^\infty(\overline{Q})$$

and, consequently,

$$\frac{\partial^2 \psi_j}{\partial x^2}\,\varphi \in L^2(Q). \tag{54}$$

Thus, from (53) and (54) results the convergence (52) and consequently (51). It follows from this and from (45) that

$$\int_0^T ([\psi_j, v_\nu], u_\nu)\,dt \to \int_0^T ([\psi_j, v], u)\,dt,$$

or even

$$\int_0^T ([u_\nu, v_\nu], \psi_j)\,dt \to \int_0^T ([u, v], \psi_j)\,dt. \tag{55}$$

Finally, from (46), (48), (49) and (55) we have proved that

$$\begin{aligned} &- \int_0^T (u'(t), \omega_j)\theta'(t)\,dt + \int_0^T (\Delta u(t), \Delta \omega_j)\theta(t)\,dt \\ &- \int_0^T ([u(t), v(t)], \omega_j)\theta(t)\,dt = \int_0^T (f(t), \omega_j)\theta\,dt. \end{aligned}$$

By the totality of the sequence $\{\omega_j\}$ in $H_0^2(\Omega)$ it follows that the last expression is valid for all $v \in H_0^2(\Omega)$ and, therefore,

$$u'' + \Delta^2 u - [u,v] = f \quad \text{in} \quad \mathcal{D}'(0,T;H^{-2}(\Omega)).$$

Since $f \in L^2(0,T;L^2(\Omega))$, $\Delta^2 u \in L^\infty(0,T;H^{-2}(\Omega))$ and $[u,v] \in L^\infty(0,T;L^1(\Omega))$ we have

$$u'' \in L^2(0,T;H^{-2}(\Omega)) \tag{56}$$

and

$$u'' + \Delta^2 u - [u,v] = f \quad \text{in} \quad L^2(0,T;H^{-2}(\Omega)). \tag{57}$$

4ª Step: Initial Condition

from (42), (43) and (57) it follows that

$$u \in C^0([0,T];L^2(\Omega)) \cap C_s(0,T;H_0^2(\Omega))$$
$$u' \in C^0([0,T];H^{-2}(\Omega)) \cap C_s(0,T;L^2(\Omega))$$

making sense, therefore, to calculate $u(0)$ and $u'(0)$. From there, we prove that $u(0) = u_0$ and $u'(0) = u_1$ in the usual manner.

Remark: We know that $H^s(\Omega) \hookrightarrow C^0(\overline{\Omega})$ if $s > \dfrac{n}{2}\cdot$ In the present case $n = 2$ and, therefore,

$$H^s(\Omega) \hookrightarrow C^0(\overline{\Omega}); \quad \forall\, s > 1.$$

Thus, if $0 < \varepsilon < 1$ we have

$$H_0^{1+\varepsilon}(\Omega) \hookrightarrow C^0(\overline{\Omega}); \quad \forall\, \varepsilon \in]0,1[.$$

Repeating the previous arguments we prove that

$$L^1(\Omega) \hookrightarrow H^{-(1+\varepsilon)}(\Omega)$$

and since

$$\Delta^2 v_\nu = -[u_\nu, u_\nu]$$

it follows that

$$\Delta^2 v_\nu \in L^\infty(0,T;L^1(\Omega)) \subset L^\infty(0,T;H^{-(1+\varepsilon)}(\Omega)).$$

Using the regularity results of elliptic problems of order 2 in the spaces $H^s(\Omega)$ it follows that

$$v_\nu \in L^\infty(0,T;H^{4-(1+\varepsilon)}(\Omega)) = L^\infty(0,T;H^{3-\varepsilon}(\Omega)); \quad \forall\, \varepsilon \in]0,1[.$$

Note that from (45) it follows that

$[u_\nu, u_\nu] \to [u,u]$ in $\mathcal{D}'(Q)$ [15] ,or even, $-\Delta^2 v_\nu \to [u,u]$ in $\mathcal{D}'(Q)$.

From (44) it follows that $\Delta^2 v_\nu \to \Delta^2 v$ in $\mathcal{D}'(Q)$ and, by the uniqueness of the limit, it is concluded that:

$$\Delta^2 v = -[u,u]. \qquad \square$$

[15] Actually, $[u_\nu, \varphi] \rightharpoonup [u,\varphi]$ in $L^2(Q)$ and then $\int_\Omega [u_\nu, u_\nu]\varphi = \int [u_\nu, \varphi] u_\nu \to \int_\Omega [u,\varphi] u = \int_\Omega [u,u]\varphi$.

Chapter 8

Von Kármán System (Stationary Case)

Problem 9

Problem 9 given by

$$\left| \begin{array}{l} \Delta^2 u - [u,v] = f \quad \text{in} \quad \Omega \\ \Delta^2 v + [u,u] = 0 \quad \text{in} \quad \Omega \\ u = 0, \;\; v = 0 \quad \text{on} \quad \Gamma \\ \dfrac{\partial u}{\partial \nu} = 0, \;\; \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \end{array} \right. \tag{1}$$

where

$$f \in H^{-2}(\Omega) \tag{2}$$

admits at least one pair (u,v) as a weak solution, in the class,

$$u, v \in H_0^2(\Omega). \tag{3}$$

Proof:

1ª Step: Approximate Problem

Let $(\omega_\nu)_{\nu\in\mathbb{N}}$ be a basis of eigenfunctions of the operator $\Delta^2 \leftarrow \{H_0^2(\Omega); L^2(\Omega); ((\,\cdot\,,\,\cdot\,))_{H_0^2}\}$ where:

$$((u,v))_{H_0^2(\Omega)} = \int_\Omega \Delta u \, \Delta v \, dx.$$

As we saw in Problem 3 if (λ_ν) is the sequence of eigenvalues corresponding then

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu \leq \cdots \quad \text{and} \quad \lambda_\nu \to +\infty.$$

Furthermore,

$$(\omega_\nu) \text{ is a complete orthonormal system in } L^2(\Omega) \tag{3}$$

$$\left(\frac{\omega_\nu}{\lambda_\nu^{1/2}}\right) \text{ is a complete orthonormal system in } H_0^2(\Omega) \tag{4}$$

$$\frac{(\omega_\nu)}{\lambda_\nu} \text{ is a complete orthonormal system in } H_0^2(\Omega) \cap H^4(\Omega). \tag{5}$$

Considering that ω_ν is a solution of the problem

$$\left| \begin{aligned} \Delta^2\omega_\nu &= \lambda_\nu\omega_\nu \\ \omega_\nu|_\Gamma &= 0 \\ \frac{\partial\omega\nu}{\partial\nu}\bigg|_\Gamma &= 0, \end{aligned} \right.$$

then, by virtue of the regularity of elliptic problems of order 2, from the fact that Ω is a bounded sufficiently smooth open set and from the Sobolev Embedding Theorem, it follows that

$$(\omega_\nu) \subset \Big(\bigcap_{m\in\mathbb{N}} H^m(\Omega)\Big) \cap C^\infty(\overline{\Omega}) \cap H_0^2(\Omega). \tag{6}$$

Since the operators

$$\Delta^2 \colon H_0^2(\Omega) \cap H^4(\Omega) \to L^2(\Omega), \qquad \widetilde{\Delta^2} \colon H_0^2(\Omega) \to H^{-2}(\Omega)$$

are bijections, being the second one an isometry let us define

$$\begin{gathered} G \colon L^2(\Omega) \to H_0^2(\Omega) \cap H^4(\Omega) \quad \widetilde{G} \colon H^{-2}(\Omega) \to H_0^2(\Omega) \\ G = (\Delta^2)^{-1} \quad \text{and} \quad \widetilde{G} = (\widetilde{\Delta^2})^{-1} \end{gathered} \tag{7}$$

Consider

$$V_m = [\omega_1, \dots, \omega_m].$$

In V_m consider the approximate problem

$$u_m \in V_m \quad \Leftrightarrow \quad u_m = \sum_{i=1}^m \xi_i\omega_i \tag{8}$$

$$(\Delta u_m, \Delta\omega_j) + ([u_m, G[u_m, u_m]], \omega_j) = \langle f, \omega_j\rangle; \quad j = 1, 2, \dots, m. \tag{9}$$

We observe that from $(1)_2$ we can write that

$$v_m = -G[u_m, u_m]. \tag{10}$$

We will prove next that the algebraic system (8)–(9) possesses a solution. Note that we *cannot* use Carathéodory's Theorem since the problem is stationary.

Substituting (8) in (9) results that

$$\begin{gathered} \int_\Omega \Big(\sum_{i=1}^m \xi_i\,\Delta\omega_i\Big)\Delta\omega_j\,dx + \int_\Omega \Big[\sum_{i=1}^m \xi_i\,\omega_i, G\Big[\sum_{i=1}^m \xi_i\,\omega_i, \sum_{i=1}^m \xi_i\,\omega_i\Big]\Big]\omega_j\,dx \\ = \langle f, \omega_j\rangle, \quad j = 1, \dots, m \end{gathered} \tag{11}$$

Setting

$$a_{ij} = \int_\Omega \Delta\omega_i\,\Delta\omega_j\,dx; \quad i, j = 1, \dots, m; \quad f_j = \langle f, \omega_j\rangle; \quad j = 1, \dots, m \tag{12}$$

and

$$b_j(\xi_1, \dots, \xi_m) = \int_\Omega \Big[\sum_{i=1}^m \xi_i\,\omega_i,\ G\Big[\sum_{i=1}^m \xi_i\,\omega_i, \sum_{i=1}^m \xi_i\,\omega_i\Big]\Big]\omega_j\,dx; \quad j = 1, \dots, m, \tag{13}$$

we obtain from (11), (12) and (13)

$$\sum_{i=1}^{m} \xi_i\, a_{ij} + b_j(\xi_1, \dots, \xi_m) = f_j\,; \quad 1 \le j \le m. \tag{14}$$

To prove the existence of solution of (14) we need a result which we state below

Lemma (Visik): Let $\xi \mapsto P(\xi)$ from $\mathbb{R}^n \to \mathbb{R}^n$ be a continuous map such that for some $\rho > 0$ we have $(P(\xi), \xi)_{\mathbb{R}^n} \ge 0,\ \ \forall \xi \in \mathbb{R}^n$ with $||\xi|| = \rho$. Then, $\exists\, \xi_0 \in \overline{B_\rho(0)}$ such that $P(\xi_0) = 0$.

Proof:

Suppose, by contradiction, that

$$P(\xi) \ne 0; \quad \forall \xi \in \overline{B_\rho(0)}. \tag{15}$$

Since the map $\xi \in \mathbb{R}^n \mapsto P(\xi) \in \mathbb{R}^n$ is continuous, then the map

$$\begin{aligned} Q\colon \overline{B_\rho(0)} &\to \mathbb{R}^n \\ \xi \mapsto Q(\xi) &= -\frac{\rho}{||P(\xi)||}\, P(\xi), \end{aligned} \tag{16}$$

which is well defined by virtue of (15), is also continuous. Furthermore, for all $\xi \in \overline{B_\rho(0)}$ we have

$$||Q(\xi)|| = \left\| -\frac{\rho}{||P(\xi)||}\, P(\xi) \right\| = \rho, \tag{17}$$

which proves that Q maps $\overline{B_\rho(0)}$ into $\overline{B_\rho(0)}$. Thus, by Brouwer's fixed point theorem, $\exists\, \xi_0 \in \overline{B_\rho(0)}$ such that

$$Q(\xi_0) = \xi_0, \tag{18}$$

that is, from (16) we have equivalently that

$$-\frac{P(\xi_0)}{||P(\xi_0)||}\, \rho = \xi_0\,. \tag{19}$$

We observe that from (17) and (18) it follows that

$$||\xi_0|| = \rho > 0. \tag{20}$$

It follows from (19) and (20) that

$$\big(P(\xi_0), \xi_0\big) = -\frac{||P(\xi_0)||}{\rho}\, (\xi_0, \xi_0) = -\frac{||P(\xi_0)||}{\rho}\, ||\xi_0||^2 < 0$$

which is absurd! □

Returning to (14) let us define for each $j = 1, \dots, m$

$$\eta_j = \sum_{i=1}^{m} \xi_i\, a_{ij} + b_j(\xi_1, \dots, \xi_m) - f_j$$

and

$$\eta = (\eta_1, \dots, \eta_m).$$

We must prove that the map $P\colon \mathbb{R}^m \to \mathbb{R}^m$ given by

$$\xi \mapsto u_m = \sum_{i=1}^{m} \xi_i \omega_i \mapsto \eta \tag{21}$$

that is, $\xi \in \mathbb{R}^m \mapsto P(\xi) = \eta$ is continuous, and furthermore, that $\exists\, \rho_0 > 0$ such that:

$$(P(\xi), \xi) \geq 0; \quad \forall\, \xi \in \mathbb{R}^m \;\; |\xi| = \rho_0\,. \tag{22}$$

Indeed, we will prove initially the continuity of P. For this it is sufficient to prove that

$$\begin{aligned} P_j\colon \mathbb{R}^m &\to \mathbb{R} \\ \xi &\mapsto P_j(\xi) = \eta_j \end{aligned}$$

is continuous. In fact, let $\xi_0 \in \mathbb{R}^m$ and consider $(\xi_\nu) \subset \mathbb{R}^m$ such that

$$\xi_\nu \to \xi_0 \quad \text{in} \quad \mathbb{R}^m. \tag{23}$$

We have

$$\begin{aligned} &|P_j(\xi_\nu) - P_j(\xi_0)| \\ &= \left| \sum_{i=1}^{m} a_{ij}(\xi_{\nu i} - \xi_{0i}) - b_j(\xi_{\nu 1}, \ldots, \xi_{\nu m}) + b_j(\xi_{01}, \ldots, \xi_{0m}) \right| \\ &\leq \sum_{i=1}^{m} |a_{ij}|\, |\xi_{\nu i} - \xi_{0i}| + |b_j(\xi_{\nu 1}, \ldots, \xi_{\nu m}) - b_j(\xi_{01}, \ldots, \xi_{0m})|. \end{aligned} \tag{24}$$

However, from (23) we have

$$|\xi_{\nu i} - \xi_{0i}| \leq \sqrt{\sum_{i=1}^{m} (\xi_{\nu i} - \xi_{0i})^2} = ||\xi_\nu - \xi_0|| \to 0, \;\; \forall\, i = 1, \ldots, m,$$

which implies that

$$\xi_{\nu i} \to \xi_{0i} \quad \text{in} \quad \mathbb{R}, \quad \forall i = 1, \cdots, m.$$

Therefore

$$\sum_{i=1}^{m} \xi_{\nu i} \omega_i \to \sum_{i=1}^{m} \xi_{0i} \omega_i \quad \text{in} \quad H_0^2(\Omega). \tag{25}$$

Whence

$$\left[\sum_{i=1}^{m} \xi_{\nu i} \omega_i, \sum_{i=1}^{m} \xi_{\nu i} \omega_i \right] \to \left[\sum_{i=1}^{m} \xi_{0i} \omega_i, \sum_{i=1}^{m} \xi_{0i} \omega_i \right] \text{ in } L^1(\Omega) \hookrightarrow H^{-2}(\Omega)$$

and consequently

$$G\left[\sum_{i=1}^{m} \xi_{\nu i} \omega_i, \sum_{i=1}^{m} \xi_{\nu i} \omega_i \right] \to G\left[\sum_{i=1}^{m} \xi_{0i} \omega_i, \sum_{i=1}^{m} \xi_{0i} \omega_i \right] \text{ in } H_0^2(\Omega)$$

From the convergence above and from (25) it follows that

$$\Big[\sum_{i=1}^m \xi_{\nu i}\omega_i, G\Big[\sum_{i=1}^m \xi_{\nu i}\omega_i, \sum_{i=1}^m \xi_{\nu i}\omega_i\Big]\Big]$$
$$\to\Big[\sum_{i=1}^m \xi_{0 i}\omega_i, G\Big[\sum_{i=1}^m \xi_{0 i}\omega_i, \sum_{i=1}^m \xi_{0 i}\omega_i\Big]\Big] \text{ in } L^1(\Omega).$$

Since $\omega_j \in H_0^2(\Omega) \hookrightarrow C^0(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$ (since $n = 2$) it follows that

$$\int_\Omega \Big[\sum_{i=1}^m \xi_{\nu i}\omega_i, G\Big[\sum_{i=1}^m \xi_{\nu i}\omega_i, \sum_{i=1}^m \xi_{\nu i}\omega_i\Big]\Big]\omega_j dx$$
$$\to \int_\Omega \Big[\sum_{i=1}^m \xi_{0 i}\omega_i, G\Big[\sum_{i=1}^m \xi_{0 i}\omega_i, \sum_{i=1}^m \xi_{0 i}\omega_i\Big]\Big]\omega_j dx$$

that is,

$$|b_j(\xi_{\nu 1}, \dots, \xi_{\nu m}) - b_j(\xi_{01}, \dots, \xi_{0m})| \to 0. \tag{26}$$

Thus, the continuity of the map P is proved. We will prove next the veracity of (22). Indeed, from (13) we have

$$\begin{aligned}
&(P\xi, \xi) = (\eta, \xi) = \sum_{j=1}^m \eta_j \xi_j \\
&= \sum_{j=1}^m \Big(\sum_{i=1}^m \xi_i a_{ij} + b_j(\xi_1, \dots, \xi_m) - f_j\Big)\xi_j \\
&= \sum_{j=1}^m \Big(\sum_{i=1}^m \xi_i (\Delta\omega_i, \Delta\omega_j)_{L^2(\Omega)} + \int_\Omega [u_m, G[u_m, u_m]]\omega_j \, dx - \langle f, \omega_j\rangle\Big)\xi_j \\
&= \Big(\Delta\Big(\sum_{i=1}^m \xi_i\omega_i\Big), \Delta\Big(\sum_{j=1}^m \xi_j\omega_j\Big)\Big)_{L^2(\Omega)} \\
&+ \int_\Omega [u_m, G[u_m, u_m]]\Big(\sum_{j=1}^m \omega_j\xi_j\Big) dx - \langle f, \Big(\sum_{j=1}^m \omega_j\xi_j\Big)\rangle \\
&= (\Delta u_m, \Delta u_m) - \int_\Omega [u_m, v_m]u_m \, dx - \langle f, u_m\rangle \\
&= |\Delta u_m|^2 - \int_\Omega [u_m, v_m]u_m \, dx - \langle f, u_m\rangle.
\end{aligned} \tag{27}$$

However

$$\int_\Omega [u_m, v_m]u_m \, dx = \int_\Omega [u_m, u_m]v_m \, dx. \tag{28}$$

From (10) we have:

$$\Delta^2 v_m = -[u_m, u_m]. \tag{29}$$

Therefore, from (28) and (29) we conclude that:

$$\int_\Omega [u_m, v_m]u_m \, dx = -\int_\Omega \Delta^2 v_m v_m \, dx. \tag{30}$$

Substituting (30) in (27) results that

$$\begin{aligned}&(P\xi,\xi)\\ &= |\Delta u_m|^2 + (\Delta^2 v_m, v_m) - \langle f, u_m\rangle\\ &= |\Delta u_m|^2 + |\Delta v_m|^2 - \langle f, u_m\rangle\\ &\geq |\Delta u_m|^2 + |\Delta v_m|^2 - ||f||_{H^{-2}}\,||u_m||_{H_0^2(\Omega)}\\ &\geq |\Delta u_m|^2 + |\Delta v_m|^2 - c_1|\Delta u_m|\end{aligned}$$

that is,

$$(P\xi,\xi) \geq |\Delta u_m|^2 + |\Delta v_m|^2 - c_1|\Delta u_m| \geq |\Delta u_m|^2 - c_1|\Delta u_m|. \tag{31}$$

We have two cases to consider

(i) If $|\Delta u_m| = 0$, then from (31) it follows that $(P\xi,\xi) \geq 0$ which proves the desired in (22) for any $\rho > 0$.

(ii) If $|\Delta u_m| \neq 0$, then $(P\xi,\xi) \geq 0$ provided that $|\Delta u_m| \geq c_1$. We will prove that $\exists\,\rho_0 > 0$ such that, $\forall\,\xi \in \mathbb{R}^m$, if $||\xi|| = \rho_0$, then $|\Delta u_m| \geq c_1$ and we have the desired in (22). Indeed, we have from (3) that

$$|\Delta u_m|^2_{L^2(\Omega)} = (\Delta u_m, \Delta u_m) = \left(\sum_{i=1}^m \xi_i \Delta\omega_i, \sum_{i=1}^m \xi_i\Delta\omega_i\right) = \sum_{i=1}^m \xi_i^2 |\Delta\omega_i|^2_{L^2(\Omega)}\,. \tag{32}$$

Setting

$$\beta_m = \min\{|\Delta\omega_1|^2, \ldots, |\Delta\omega_m|^2\}$$

from (32) it follows that

$$|\Delta u_m|^2_{L^2(\Omega)} \geq \beta_m \sum_{i=1}^m \xi_i^2 = \beta_m||\xi||^2,$$

which implies that

$$|\Delta u_m|_{L^2(\Omega)} \geq \sqrt{\beta_m}||\xi||. \tag{33}$$

Choosing $\rho_0 > 0$ such that $\rho_0 > \dfrac{c_1}{\sqrt{\beta_m}}$ we obtain from (33), for all $\xi \in \mathbb{R}^m$ with $||\xi|| = \rho_0$, that:

$$|\Delta u_m|_{L^2(\Omega)} \geq \sqrt{\beta_m}\rho_0 > \sqrt{\beta_m}\cdot\frac{c_1}{\sqrt{\beta_m}} = c_1$$

which proves the desired in (22). Thus, by Visik's Lemma $\exists\,\xi_0 \in \overline{B_{\rho_0}(0)}$ such that $P(\xi_0) = 0$, that is, the system given in (11) admits a solution.

2ª Step: A Priori Estimate

Multiplying (9) by ξ_j and summing over j from t to m, we obtain from (10) that

$$(\Delta u_m, \Delta u_m) + ([u_m, v_m], u_m) = \langle f, u_m\rangle$$

or even,

$$|\Delta u_m|^2 - ([u_m, u_m], v_m) \leq ||f||_{H^{-2}}\,||u_m||_{H_0^2}\,.$$

Whence, from (29)

$$|\Delta u_m|^2 + (\Delta^2 v_m, v_m) \leq c_1||f||_{H^{-2}(\Omega)}\,|\Delta u_m|$$

and, therefore,

$$|\Delta u_m|^2 + |\Delta v_m|^2 \leq c_1 ||f||_{H^{-2}(\Omega)} \, |\Delta u_m|; \quad \forall \, m \in \mathbb{N}. \tag{34}$$

For $m \in \mathbb{N}$ such that $|\Delta u_m| = 0$ then $|\Delta u_m|$ is trivially bounded. When $|\Delta u_m| \neq 0$ we have from (34) that

$$|\Delta u_m| \leq c_2 \, ; \quad \forall \, m \in \mathbb{N} \tag{35}$$

and from (34) and (35) it follows that

$$|\Delta v_m| \leq c_3 \, ; \quad \forall \, m \in \mathbb{N}. \tag{36}$$

Thus, from (35) and (36) we have

$$(u_m) \quad \text{is bounded in} \quad H_0^2(\Omega) \tag{37}$$

$$(v_m) \quad \text{is bounded in} \quad H_0^2(\Omega). \tag{38}$$

It results from (37) and (38) the existence of subsequences (u_ν) of (u_m) and (v_ν) of (v_m) such that

$$u_\nu \rightharpoonup u \quad \text{weakly in} \quad H_0^2(\Omega) \tag{39}$$

$$v_\nu \rightharpoonup v \quad \text{weakly in} \quad H_0^2(\Omega). \tag{40}$$

On the other hand, by virtue of $H_0^2(\Omega) \overset{\text{comp.}}{\hookrightarrow} L^2(\Omega)$ and (37), we can extract from (u_ν) a subsequence, which we will still denote by the same notation, which verifies

$$u_\nu \to u \quad \text{in} \quad L^2(\Omega). \tag{41}$$

3ª Step: Passage to the Limit

Let $j \in \mathbb{N}$ and consider $\nu \geq j$. From (9) and (10) it results that:

$$(\Delta u_\nu, \Delta \omega_j) - ([u_\nu, v_\nu], \omega_j) = \langle f, \omega_j \rangle. \tag{42}$$

From (39) it follows that

$$\langle w, u_\nu \rangle_{H^{-2}, H_0^2} \longrightarrow \langle w, u \rangle_{H^{-2}, H_0^2} \, , \quad \forall \, w \in H^{-2}(\Omega).$$

In particular,

$$\langle \Delta^2 \omega_j, u_\nu \rangle \to \langle \Delta^2 \omega_j, u \rangle$$

that is,

$$(\Delta u_\nu, \Delta \omega_j) \to (\Delta u, \Delta \omega_j). \tag{43}$$

We have

$$([u_\nu, v_\nu], \omega_j) = ([u_\nu, \omega_j], v_\nu) = ([\omega_j, u_\nu], v_\nu) = ([\omega_j, v_\nu], u_\nu). \tag{44}$$

We claim that

$$[\omega_j, v_\nu] \rightharpoonup [\omega_j, v] \quad \text{in} \quad L^2(\Omega). \tag{45}$$

Indeed, we have

$$[\omega_j, v_\nu] = \frac{\partial^2 \omega_j}{\partial x^2}\,\frac{\partial^2 v_\nu}{\partial y^2} - 2\,\frac{\partial^2 \omega_j}{\partial x \partial y}\,\frac{\partial^2 v_\nu}{\partial x \partial y} + \frac{\partial^2 \omega_j}{\partial x^2}\,\frac{\partial^2 v_\nu}{\partial y^2} \tag{46}$$

and

$$[\omega_j, v] = \frac{\partial^2 \omega_j}{\partial x^2}\,\frac{\partial^2 v}{\partial y^2} - 2\,\frac{\partial^2 \omega_j}{\partial x \partial y}\,\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 \omega_j}{\partial x^2}\,\frac{\partial^2 v}{\partial y^2}. \tag{47}$$

Therefore, from (6), (10) and (40) it follows that $[\omega_j, v_\nu]$, $[\omega_j, v] \in L^2(\Omega)$. To prove (45) then, by virtue of (46) and (47), it is sufficient to prove for example that

$$\int_\Omega \frac{\partial^2 \omega_j}{\partial x^2}\,\frac{\partial^2 v_\nu}{\partial y^2}\,\varphi\,dx \to \int_\Omega \frac{\partial^2 \omega_j}{\partial x^2}\,\frac{\partial^2 v}{\partial y^2}\,\varphi\,dx, \quad \forall\,\varphi \in L^2(\Omega), \tag{48}$$

since for the other terms the procedure is analogous.

Indeed, from (36) it follows that

$$\left(\frac{\partial^2 v_\nu}{\partial y^2}\right) \quad \text{is bounded in} \quad L^2(\Omega). \tag{50}$$

Consequently, from (50) we conclude that

$$\frac{\partial^2 v_\nu}{\partial y^2} \rightharpoonup \frac{\partial^2 v}{\partial y^2} \quad \text{in} \quad L^2(\Omega). \tag{51}$$

However since

$$\left(\frac{\partial^2 \omega_j}{\partial x^2}\,\varphi\right) \in L^2(\Omega), \tag{52}$$

because from (6) we have that

$$\int_\Omega \left|\frac{\partial^2 \omega_j}{\partial x^2}\,\varphi\right|^2 dx \le c\int_\Omega |\varphi|^2\,dx < +\infty$$

from (51) and (52) the convergence in (48) follows, proving (45). It follows from this and from the convergence in (41) that

$$([\omega_j, v_\nu], u_\nu) \to ([\omega_j, v], u)$$

Thus, from (44) we can write

$$([u_\nu, v_\nu], \omega_j) = ([\omega_j, v_\nu], u_\nu) \to ([\omega_j, v), u) = ([u, v], \omega_j). \tag{53}$$

In this way, from (42), (43) and (53) it follows, in the limit situation,

$$(\Delta u, \Delta\omega_j) - \int_\Omega [u, v]\omega_j\,dx = \langle f, \omega_j\rangle, \quad \forall\,j \in \mathbb{N}.$$

By the totality of the sequence $\{\omega_j\}$ in $H_0^2(\Omega)$ it follows that

$$(\Delta u, \Delta\omega) - \int_\Omega [u, v]\omega\,dx = \langle f, \omega\rangle; \quad \forall\,\omega \in H_0^2(\Omega). \tag{54}$$

Taking $\omega = \varphi \in \mathcal{D}(\Omega)$ in (54) we obtain

$$\Delta^2 u - [u, v] = f \quad \text{in} \quad \mathcal{D}'(\Omega),$$

or even,

$$\Delta^2 u - [u, v] = f \quad \text{in} \quad H^{-2}(\Omega). \tag{55}$$

On the other hand, from (41) we have that

$$[u_\nu, u_\nu] \to [u, u] \quad \text{in} \quad L^1(\Omega) \hookrightarrow H^{-2}(\Omega),$$

which implies that

$$-G[u_\nu, u_\nu] \to -G[u, u] \quad \text{in} \quad H_0^2(\Omega),$$

or even, from (10),

$$v_\nu \to -G[u, u] \quad \text{in} \quad H_0^2(\Omega). \tag{56}$$

From the convergences in (40) and (56) and by the uniqueness of the limit we conclude that

$$v = -G[u, u]. \qquad \square \tag{57}$$

Chapter 9

Navier-Stokes System

Let Ω be a bounded and sufficiently smooth open subset of $\mathbb{R}^2$.

Let us define

$$\mathcal{V} = \{\varphi \in (\mathcal{D}(\Omega))^2;\ \operatorname{div} \varphi = 0\} \tag{1}$$

$$V = \overline{\mathcal{V}}^{(H^1(\Omega))^2} \tag{2}$$

$$H = \overline{\mathcal{V}}^{(L^2(\Omega))^2} \tag{3}$$

The set V defined in (2) can be rewritten as

$$V = \{u \in (H_0^1(\Omega))^2; \operatorname{div} u = 0\}. \tag{4}$$

We will endow H and V, respectively, with the inner products in $(L^2(\Omega))^2$ and $(H_0^1(\Omega))^2$. More precisely, we have

$$(u, v)_{(L^2(\Omega))^2} = \sum_{i=1}^{2} (u_i, v_i) \tag{5}$$

and

$$((u, v))_{(H_0^1(\Omega))^2} = \sum_{i=1}^{2} ((u_i, v_i)) = \sum_{i=1}^{2} \sum_{j=1}^{2} \left(\frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \right)_{L^2(\Omega)}. \tag{6}$$

Problem 10

Problem 10 given by

$$\left| \begin{array}{l} \dfrac{\partial u}{\partial t} - \nu \Delta u + \displaystyle\sum_{j=1}^{2} \dfrac{\partial u}{\partial x_j} u_j = f - \nabla p \quad \text{in} \quad Q \ \ (\nu > 0) \\ \operatorname{div}\, u = 0 \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(x, 0) = u_0(x); \ \ x \in \Omega, \end{array} \right. \tag{7}$$

where

$$u = (u_1, u_2); \quad \frac{\partial u}{\partial x_j} = \left(\frac{\partial u_1}{\partial x_j}, \frac{\partial u_2}{\partial x_j}\right); \quad \frac{\partial u}{\partial t} = (u_1', u_2'), \quad \Delta u = (\Delta u_1, \Delta u_2),$$

subject to the data

$$u_0 \in H \quad \text{and} \quad f \in L^2(0,T;V') \tag{8}$$

possesses a unique weak solution in the class

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H); \quad u' \in L^2(0,T;V').$$

Variational Formulation

Composing equation $(7)_1$ with an admissible function ω, we obtain

$$(u', \omega) - \nu(\Delta u, \omega) + \left(\sum_{j=1}^{2} \frac{\partial u}{\partial x_j}\, u_j, \omega\right) = (f, \omega) - (\nabla p, \omega).$$

Applying Green's Theorem formally it follows that

$$\sum_{i=1}^{2}(u_i', \omega_i)_{L^2(\Omega)} + \nu \sum_{i=1}^{2}((u_i, \omega_i))_{H_0^1(\Omega)} + \sum_{i=1}^{2} \int_\Gamma \frac{\partial u_i}{\partial \eta}\, \omega_i \, d\Gamma + \sum_{i=1}^{2}\sum_{j=1}^{2} \left(\frac{\partial u_i}{\partial x_j}\, u_j, \omega_i\right)_{L^2(\Omega)}$$
$$= \sum_{i=1}^{2}(f_i, \omega_i) - \sum_{i=1}^{2} \left(\frac{\partial p}{\partial x_i}, \omega_i\right)_{L^2(\Omega)}.$$

Now considering $\omega_i = 0$ on Γ and using formally Gauss's Theorem it results that

$$\sum_{i=1}^{2}(u_i', \omega_i)_{L^2(\Omega)} + \nu \sum_{i=1}^{2}((u_i, \omega_i))_{H_0^1(\Omega)} + \sum_{i=1}^{2}\sum_{j=1}^{2} \left(\frac{\partial u_i}{\partial x_j}\, u_j, \omega_i\right)_{L^2(\Omega)}$$
$$= \sum_{i=1}^{2}(f_i, \omega_i) + \int_\Omega p\left(\sum_{i=1}^{2} \frac{\partial \omega_i}{\partial x_i}\right) dx - \sum_{i=1}^{2} \int_\Gamma p\omega_i\, \eta_i \, d\Gamma.$$

Considering $\operatorname{div} \omega = \sum_{i=1}^{2} \frac{\partial \omega_i}{\partial x_i} = 0$ and $\omega_i = 0$ on Γ it follows that

$$\sum_{i=1}^{2}(u_i', \omega_i)_{L^2(\Omega)} + \nu \sum_{i=1}^{2}((u_i, \omega_i))_{H_0^1(\Omega)} + \sum_{i=1}^{2}\sum_{j=1}^{2} \left(\frac{\partial u_i}{\partial x_j}\, u_j, \omega_i\right)_{L^2(\Omega)} = \sum_{i=1}^{2}(f_i, \omega_i). \tag{9}$$

In truth, the variational formulation given in (9) holds for any function $\omega \in V$ given in (4).

Before proceeding to the resolution of problem (7) we will make some initial considerations that we will need in the unfolding of the problem. We have the following results

Lemma 1: The trilinear form:

$$b\colon V \times V \times V \to \mathbb{R}$$
$$(u, v, \omega) \mapsto b(u, v, \omega) = \sum_{i,j=1}^{2} \int_\Omega u_j \frac{\partial v_i}{\partial x_j}\, \omega_i \, dx$$

is continuous.

Proof. Since $n = 2$, we have by the Sobolev Embedding Theorem that

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega); \quad \forall\, q \in [2, +\infty[\,.$$

Thus and, in particular, $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Whence

$$u_j \in L^4(\Omega); \quad \frac{\partial v_i}{\partial x_j} \in L^2(\Omega) \quad \text{and} \quad \omega_i \in L^4(\Omega).$$

Now, since

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$$

it follows from the generalized Hölder inequality that $b(u, v, \omega)$ is well defined and, furthermore,

$$\left| \int_\Omega u_j \, \frac{\partial v_i}{\partial x_j} \, \omega_i \, dx \right| \leq |u_j|_{L^4(\Omega)} \, \left| \frac{\partial v_i}{\partial x_j} \right|_{L^2(\Omega)} |\omega_i|_{L^4(\Omega)} \leq c_1 ||u_j||_{H_0^1(\Omega)} \, ||v_i||_{H_0^1(\Omega)} \, ||\omega_i||_{H_0^1(\Omega)} \, .$$

Thus:

$$\begin{aligned}
|b(u, v, \omega)| &\leq \sum_{i=1}^{2} \sum_{j=1}^{2} \int_\Omega |u_j| \, \left| \frac{\partial v_i}{\partial x_j} \right| \, |\omega_i| \, dx \\
&\leq c_1 \sum_{i=1}^{2} \sum_{j=1}^{2} ||u_j||_{H_0^1(\Omega)} \, ||v_i||_{H_0^1(\Omega)} \, ||\omega_i||_{H_0^1(\Omega)} \\
&\leq c_1 \left(\sum_{j=1}^{2} ||u_j||^2_{H_0^1(\Omega)} \right)^{1/2} \left(\sum_{i=1}^{2} ||v_i||_{H_0^1(\Omega)} \, ||\omega_i||_{H_0^1(\Omega)} \right) \\
&\leq c_1 \left(\sum_{j=1}^{2} ||u_j||^2_{H_0^1(\Omega)} \right)^{1/2} \left(\sum_{i=1}^{2} ||v_i||_{H_0^1(\Omega)} \right)^{1/2} \left(\sum_{i=1}^{2} ||\omega_i||_{H_0^1(\Omega)} \right)^{1/2},
\end{aligned}$$

that is,

$$|b(u, v, \omega)| \leq c_1 ||v|| \, ||u|| \, ||\omega||, \quad \forall\, u, v, \omega \in V$$

which proves the lemma. □

Lemma 2. We have that $\quad b(u, v, \omega) = -b(u, \omega, v), \quad \forall\, u, v, \omega \in V.$

Proof. Consider, initially, $u, v, \omega \in \mathcal{V}$. Then, by Gauss's formula:

$$
\begin{aligned}
& b(u,v,\omega)+b(u,\omega,v) \\
&= \sum_{i,j=1}^{2}\left\{\int_\Omega u_j\,\frac{\partial v_i}{\partial x_j}\,\omega_i\,dx+\int_\Omega u_j\,\frac{\partial \omega_i}{\partial x_j}\,v_i\,dx\right\} \\
&= \sum_{i,j=1}^{2}\left\{\int_\Omega u_j\,\frac{\partial}{\partial x_j}\,(v_i\omega_i)\,dx\right\} \\
&= \sum_{i,j=1}^{2}\left\{-\int_\Omega \frac{\partial u_j}{\partial x_j}\,v_i\,\omega_i\,dx+\overbrace{\int_\Gamma u_j\,v_i\,\omega_i\,\eta_i\,d\Gamma}^{=0}\right\} \\
&= -\int_\Omega \underbrace{\left(\sum_{j=1}^{2}\frac{\partial u_j}{\partial x_j}\right)}_{=0}\left(\sum_{i=1}^{2} v_i\,\omega_i\right)dx=0,
\end{aligned}
$$

which proves the desired result for functions in $\mathcal{V}$. Consider, then, u, v , $\omega \in V$. Thus, from (2) it follows that there exist (u_ν), (v_ν) and $(\omega_\nu) \subset \mathcal{V}$ such that

$$u_\nu \to u; \quad v_\nu \to v \quad \text{and} \quad \omega_\nu \to \omega \quad \text{in} \quad V.$$

However

$$b(u_\nu, v_\nu, \omega_\nu) = -b(u_\nu, \omega_\nu, v_\nu). \tag{10}$$

It follows from (10) and the continuity of $b(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ (cf. Lemma 1) the desired result. □

Lemma 3. Let $\Omega \subset \mathbb{R}^2$ be a bounded and sufficiently smooth open set. Then for all $u \in H_0^1(\Omega)$ we have

$$||u||^2_{L^4(\Omega)} \leq \sqrt{2}\,||u||_{H_0^1(\Omega)}\,|u|_{L^2(\Omega)}\,.$$

Proof. Let $\varphi \in \mathcal{D}(\Omega)$ and consider ψ its extension by setting it to zero outside in $\mathbb{R}^2 \setminus \Omega$. Then,

$$\psi^2(x_1,x_2) = \int_{-\infty}^{x_1} \frac{\partial}{\partial s}\,(\psi^2(s,x_2))\,ds = 2\int_{-\infty}^{x_1} \psi(s,x_2)\,\frac{\partial}{\partial s}\,\psi(s,x_2)\,ds, \;\; \forall\,(x_1,x_2) \in \mathbb{R}^2.$$

Thus

$$
\begin{aligned}
& |\psi^2(x_1,x_2)| \\
&\leq 2\int_{-\infty}^{x_1} |\psi(s,x_2)|\,\left|\frac{\partial \psi}{\partial s}\,(s,x_2)\right|\,ds \\
&\leq 2\int_{-\infty}^{+\infty} |\psi(s,x_2)|\,\left|\frac{\partial \psi}{\partial s}\,(s,x_2)\right|\,ds.
\end{aligned}
$$

Defining

$$v(x_2) = 2\int_{-\infty}^{+\infty} |\psi(s,x_2)|\,\left|\frac{\partial \psi}{\partial s}\,(s,x_2)\right|\,ds \tag{11}$$

it follows that

$$|\psi^2(x_1,x_2)| \leq v(x_2); \quad \forall\,(x_1,x_2) \in \mathbb{R}^2. \tag{12}$$

Analogously, setting

$$v(x_1) = 2\int_{-\infty}^{+\infty} |\psi(x_1, s)| \left|\frac{\partial \psi}{\partial s}(x_1, s)\right| ds$$

we have

$$|\psi^2(x_1, x_2)| \leq v(x_1); \quad \forall\,(x_1, x_2) \in \mathbb{R}^2. \tag{13}$$

From (12) and (13) it follows that

$$|\psi^4(x_1, x_2)| \leq v(x_1) \cdot v(x_2); \quad \forall\,(x_1, x_2) \in \mathbb{R}^2. \tag{14}$$

Observing that $v_1, v_2 \in L^1(\mathbb{R})$ then by Tonelli's Theorem $(v_1 v_2) \in L^1(\mathbb{R}^2)$ and from (14) we obtain

$$\int_{\mathbb{R}^2} |\psi(x_1, x_2)|^4\, dx \leq \int_{\mathbb{R}^2} v(x_1) v(x_2)\, dx < +\infty; \quad x = (x_1, x_2).$$

It follows from the inequality above, by Fubini's Theorem, that

$$\int_{\mathbb{R}^2} |\psi(x_1, x_2)|^4\, dx \leq \Big(\int_{\mathbb{R}} v(x_1) dx_1\Big)\Big(\int_{\mathbb{R}} v(x_2) dx_2\Big). \tag{14}$$

However from (11) and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} v(x_2) dx_2 \\
&= 2\int_{\mathbb{R}}\int_{\mathbb{R}} |\psi(x_1, x_2)| \left|\frac{\partial \psi}{\partial x_1}(x_1, x_2)\right| dx_1\, dx_2 \\
&\leq 2\Big(\int_{\mathbb{R}^2} |\psi(x_1, x_2)|^2\, dx\Big)^{1/2}\Big(\int_{\mathbb{R}^2} \left|\frac{\partial \psi}{\partial x_1}(x_1, x_2)\right|^2 dx\Big)^2 \\
&\qquad = 2|\psi|_{L^2(\mathbb{R}^2)} \left|\frac{\partial \psi}{\partial x_1}\right|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

that is,

$$\int_{\mathbb{R}} v(x_2) dx_2 \leq 2|\psi|_{L^2(\mathbb{R}^2)} \left|\frac{\partial \psi}{\partial x_1}\right|_{L^2(\mathbb{R}^2)}. \tag{15}$$

Analogously,

$$\int_{\mathbb{R}} v(x_1) dx_1 \leq 2|\psi|_{L^2(\mathbb{R}^2)} \left|\frac{\partial \psi}{\partial x_2}\right|_{L^2(\mathbb{R}^2)}. \tag{16}$$

Thus, from (14), (15) and (16) we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\psi(x_1, x_2)|^4\, dx \\
&\leq 4|\psi|^2_{L^2(\mathbb{R}^2)} \left|\frac{\partial \psi}{\partial x_1}\right|_{L^2(\mathbb{R}^2)} \left|\frac{\partial \psi}{\partial x_2}\right|_{L^2(\mathbb{R}^2)} \\
&\leq 2|\psi|^2_{L^2(\mathbb{R}^2)} \left\{ \left|\frac{\partial \psi}{\partial x_1}\right|^2_{L^2(\mathbb{R}^2)} + \left|\frac{\partial \psi}{\partial x_2}\right|^2_{L^2(\mathbb{R}^2)} \right\},
\end{aligned}$$

that is,

$$||\psi||^4_{L^4(\mathbb{R}^2)} \leq 2|\psi|^2_{L^2(\mathbb{R}^2)}||\psi||^2_{H^1_0(\mathbb{R}^2)}\,.$$

Restricting ψ to Ω we have

$$||\varphi||^2_{L^4(\Omega)} \leq \sqrt{2}\,|\varphi|_{L^2(\Omega)}\,||\varphi||_{H^1_0(\Omega)}\,; \quad \forall\,\varphi \in \mathcal{D}(\Omega).$$

By density arguments and the fact that $H^1_0(\Omega) \hookrightarrow L^4(\Omega)$ the result follows. □

It follows from Lemma 1 for fixed $u, v \in V$ that the map

$$\begin{aligned} B(u,v)\colon\, V &\to \mathbb{R} \\ \omega &\mapsto \langle B(u,v),\omega\rangle = b(u,v,\omega) \end{aligned} \tag{17}$$

is a continuous bilinear form, that is, $B(u,v) \in V'$.

Lemma 4. If $u, v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ then $B(u,v) \in L^2(0,T;V')$.

Proof. For all $w \in V$ we have by Lemma 2 and by the generalized Hölder inequality

$$\begin{aligned} &|\langle B(u(t),v(t)),w\rangle| \\ &= |b(u(t),v(t),w)| = |b(u(t),w,v(t))| \\ &= c_1||u(t)||_{(L^4(\Omega))^2}\,||\omega||_V\,||v(t)||_{(L^4(\Omega))^2}\,. \end{aligned}$$

Whence

$$\begin{gathered} B(u(t),v(t)) \in V' \text{ a.e. in }]0,T[\text{ and} \\ ||B(u(t),v(t))||_{V'} \leq c_2||u(t)||_{(L^4(\Omega))^2}\,||v(t)||_{(L^4(\Omega))^2} \end{gathered} \tag{18}$$

for almost every $t \in [0,T[$.

On the other hand, by virtue of the numerical Hölder inequality

$$||u(t)||_{(L^4(\Omega))^2} = \sum_{i=1}^2 ||u_i(t)||_{L^4(\Omega)} \leq 2^{1/2}\Big(\sum_{i=1}^2 ||u_i(t)||^2_{L^4(\Omega)}\Big)^{1/2}.$$

By Lemma 3, we obtain

$$\begin{aligned} &||u(t)||_{(L^4(\Omega))^2} \leq 2^{1/2}\Big(\sum_{i=1}^2 2^{1/2}|u_i(t)|_{L^2(\Omega)}\,||u_i(t)||_{H^1_0(\Omega)}\Big)^{1/2} \\ &= 2^{1/2}2^{1/4}\Big(\sum_{i=1}^2 |u_i(t)|_{L^2(\Omega)}\,||u_i(t)||_{H^1_0(\Omega)}\Big)^{1/2} \leq 2^{3/4}\Big(||u(t)||_V \sum_{i=1}^2 |u_i(t)|_{L^2(\Omega)}\Big)^{1/2} \\ &= 2^{3/4}||u(t)||^{1/2}\Big(\sum_{i=1}^2 |u_i(t)|_{L^2(\Omega)}\Big)^{1/2} \leq 2^{3/4}||u(t)||^{1/2}\,2^{1/2}\Big\{\Big(\sum_{i=1}^2 |u_i(t)|^2_{L^2(\Omega)}\Big)^{1/2}\Big\}^{1/2} \\ &= 2^{5/4}||u(t)||^{1/2}_V\,|u(t)|^{1/2}_H\,. \end{aligned}$$

Since by hypothesis $u \in L^\infty(0,T;H)$ we have

$$||u(t)||_{(L^4(\Omega))^2} \leq 2^{5/4}\Big(\operatorname*{ess\,sup}_{t\in]0,T[}\,|u(t)|_H\Big)^{1/2}\,||u(t)||^{1/2}_V \leq c_3||u(t)||^{1/2}_V. \tag{19}$$

Analogously

$$||v(t)||_{(L^4(\Omega))^2} \leq c_4||u(t)||^{1/2}_V\,. \tag{20}$$

From (19) and (20) it follows that

$$||u(t)||_{(L^4(\Omega))^2}\,||v(t)||_{(L^4(\Omega))^2} \leq c_5||u(t)||_V^{1/2}\,||v(t)||_V^{1/2} \text{ a.e. in }]0,T[\,. \tag{21}$$

From (18) and (21) it follows that

$$||B(u(t),v(t))||_{V'} \leq c_5||u(t)||_V^{1/2}\,||v(t)||_V^{1/2} \tag{21'}$$

which implies the inequality

$$||B(u(t),v(t))||_{V'}^2 \leq c_5^2||u(t)||_V\,||v(t)||_V \leq \frac{c_5^2}{2}\left[||u(t)||_V^2 + ||v(t)||_V^2\right].$$

Thus

$$\int_0^T ||B(u(t),v(t))||_{V'}^2\,dt \leq c_6\left[\int_0^T ||u(t)||_V^2\,dt + \int_0^T ||v(t)||_V^2\,dt\right] < +\infty$$

which proves the lemma. □

Lemma 5. Let X and Y be separable Banach spaces such that $X \hookrightarrow Y$ and consider $a, b \in [-\infty, +\infty]$ with $a < b$. Setting

$$W(a,b) = \{u \mid u \in L^p(a,b,X), \quad u' \in L^q(a,b,Y)\} \quad 1 \leq p, q < +\infty$$

endowed with the topology

$$||u||_{W(a,b)} = ||u||_{L^p(a,b;X)} + ||u'||_{L^q(a,b;Y)}$$

where u' is understood in the sense of vector-valued distributions in $\mathcal{D}'(a,b;X)$, we have:

$$\mathcal{D}([a,b];X)^{(16)} \quad \text{is dense in} \quad W(a,b).$$

Proof:

1st case: $a = -\infty$; $b = +\infty$.

(a) **Truncation**

Let $u \in W(-\infty,+\infty)$ and define

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| \geq 2 \end{cases}$$

where $\psi \in C_0^\infty(\mathbb{R})$ and $0 \leq \psi(t) \leq 1, \quad \forall\, t \in \mathbb{R}$.

We claim that

$$\psi_\nu u \to u \quad \text{in} \quad W(-\infty,+\infty). \tag{22}$$

where $\psi_\nu(t) = \psi(t/\nu)$; $\forall \nu \in \mathbb{N}^*$.

Indeed, on one hand we have that

$$||\psi_\nu u - u||_{L^p(-\infty,+\infty,X)}^p = \int_{-\infty}^{+\infty} ||(\psi_\nu u)(t) - u(t)||_X^p\,dt \to 0 \text{ when } \nu \to +\infty,$$

[16] $\mathcal{D}([a,b];X) = \{u|_{[a,b]}\,;\, u \in D(-\infty,+\infty,X)\}, \quad -\infty < a < b < +\infty$

by virtue of the Lebesgue Dominated Convergence Theorem, that is,

$$\psi_\nu u \to u \quad \text{in} \quad L^p(-\infty,+\infty,X) \text{ when } \nu \to +\infty. \tag{23}$$

On the other hand, note that for $\theta \in \mathcal{D}(-\infty,+\infty)$ we have

$$\begin{aligned}\left\langle \frac{d}{dt}(\psi_\nu u),\theta\right\rangle &= -\langle \psi_\nu u,\theta'\rangle = -\int_{-\infty}^{+\infty} u(t)\psi_\nu(t)\theta'(t)\,dt\\ &= -\int_{-\infty}^{+\infty} u(t)\big[(\psi_\nu\theta)'(t)-\psi_\nu'(t)\theta(t)\big]\,dt\\ &\overset{(17)}{=} \int_{-\infty}^{+\infty} u'(t)\psi_\nu(t)\theta(t)\,dt + \int_{-\infty}^{+\infty} u(t)\psi_\nu'(t)\theta(t)\,dt\\ &= \langle u'\psi_\nu + u\psi_\nu',\theta\rangle.\end{aligned}$$

Thus

$$\begin{aligned}||(\psi_\nu u)'-u'||^q_{L^q(-\infty,+\infty,Y)} &= \int_{-\infty}^{+\infty} ||\psi_\nu u' + u\psi_\nu' - u'||^q_Y\,dt\\ \le c_1\Big\{\int_{-\infty}^{+\infty} ||\psi_\nu u'-u'||^q_Y\,dt &+ \int_{-\infty}^{+\infty} ||u\psi_\nu'||^q_Y\,dt\Big\}.\end{aligned} \tag{24}$$

In a manner analogous to (23) we prove that

$$\int_{-\infty}^{+\infty} ||\psi_\nu u'-u'||^q_Y\,dt \to 0 \text{ when } \nu \to +\infty. \tag{25}$$

Also, from the fact that $\psi_\nu'(t) = \dfrac{1}{\nu}\psi'\left(\dfrac{t}{\nu}\right)$ and ψ' is also bounded on the whole line, it follows that

$$\int_{-\infty}^{+\infty} ||u\psi_\nu'(t)||^q_Y\,dt \to 0 \text{ when } \nu \to +\infty. \tag{26}$$

From (24), (25) and (26) it follows that

$$(\psi_\nu u)' \to u' \quad \text{in} \quad L^q(-\infty,+\infty,Y). \tag{27}$$

Consequently from (23) and (27) we have proved (22). Note that

$$\operatorname{supp}(\psi_\nu u) \subset \operatorname{supp}(\psi_\nu)\cap\operatorname{supp}(u) \subset \operatorname{supp}(\psi_\nu)$$

which proves that for each $\nu \in \mathbb{N}^*$ the function $(\psi_\nu u)$ has compact support in $\mathbb{R}$. The next step is to approximate a function u of compact support by functions of $\mathcal{D}(-\infty,+\infty,X)$.

(b) **Regularization**

Let $u \in W(-\infty,+\infty)$ with compact support and consider $(\rho_\nu)_{\nu\in\mathbb{N}}$ a regularizing sequence. Define

$$u_\nu(t) = (\rho_\nu * u)(t) = \int_{-\infty}^{+\infty} \rho_\nu(s)u(t-s)\,ds. \tag{28}$$

[17] Note that $(\psi_\nu\theta) \in \mathcal{D}(-\infty,+\infty)$.

We will prove that

$$u_\nu \to u \quad \text{in} \quad W(-\infty,+\infty). \tag{29}$$

In fact, from classical integration results (Bochner) of vector-valued functions we know that

$$u_\nu \to u \quad \text{in} \quad L^p(-\infty,+\infty,X). \tag{30}$$

It remains to prove that

$$u_\nu' = (\rho_\nu * u)' \to u' \quad \text{in} \quad L^q(-\infty,+\infty,Y), \tag{31}$$

where here the derivative is in the sense of $\mathcal{D}'(-\infty,+\infty,X)$. However, to prove (31) it is sufficient to prove that

$$(\rho_\nu * u)' = \rho_\nu * u'. \tag{32}$$

Indeed, let $\theta \in \mathcal{D}(-\infty,+\infty)$. We have

$$\begin{aligned}
&\langle (\rho_\nu * u)', \theta\rangle = -\langle \rho_\nu * u, \theta'\rangle \\
&= -\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} \rho_\nu(s)u(t-s)ds\right)\theta'(t)\,dt \\
&= -\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} u(t-s)\theta'(t)dt\right)\rho_\nu(s)ds \\
&\overset{(18)}{=} \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} u'(t-s)\theta(t)dt\right)\rho_\nu(s)\,ds \\
&= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} \rho_\nu(s)u'(t-s)ds\right)\theta(t)\,dt = \langle \rho_\nu * u', \theta\rangle,
\end{aligned}$$

which proves (32) and consequently (31). From (31) and (30) (29) is proved. Furthermore, since

$$u_\nu \in C^\infty(-\infty,+\infty,X) \quad \text{and} \quad \operatorname{supp}(\rho_\nu * u) \subset \operatorname{supp}(\rho_\nu) + \operatorname{supp}(u)$$

we have that $u_\nu \in \mathcal{D}(-\infty,+\infty,X)$. From (a) and (b) the 1st Case is proved.

2nd Case: a finite and $b=+\infty$.

Without loss of generality we will consider $a=0$. Let $u \in W(0,+\infty)$ and $h>0$. We will prove that

$$\tau_h u \to u \quad \text{in} \quad W(0,+\infty) \quad \text{when} \quad h \to 0, \tag{33}$$

where $\tau_h\, u(t) = u(t+h)$.

We will prove initially that

$$\tau_h v \to v \quad \text{in} \quad L^r(0,+\infty,V) \tag{34}$$

where V is a separable Banach space, $1 \le r < +\infty$ and $v \in L^r(0,+\infty;V)$. Indeed, let $\varepsilon > 0$ be given. Since $\mathcal{D}(0,+\infty,V)$ is dense in $L^r(0,+\infty,V)$ we have that $\exists\, \varphi \in \mathcal{D}(0,+\infty,V)$ such that

$$||\varphi - v||_{L^r(0,+\infty,V)} < \frac{\varepsilon}{3}\,. \tag{35}$$

[18] setting $f(t)=u(t-s)$ then $f'(t)=u'(t-s)$ and therefore $-\int_{-\infty}^{+\infty} u(t-s)\theta'(t)dt = -\int_{-\infty}^{+\infty} f(t)\theta'(t)dt = +\int_{-\infty}^{+\infty} f'(t)\theta(t)dt = \int_{-\infty}^{+\infty} u'(t-s)\theta(t)dt$.

Thus,

$$||\tau_h v - v||_{L^r(0,+\infty,V)} \leq ||\tau_h v - \tau_h \varphi||_{L^r(0,+\infty,V)} + ||\tau_h \varphi - \varphi||_{L^r(0,+\infty,V)} + ||\varphi - v||_{L^r(0,+\infty,V)} . \tag{36}$$

However, from the fact that $\varphi \in \mathcal{D}(0,+\infty,V)$ we have that

$$||\tau_h \varphi - \varphi||_{L^r(0,+\infty,V)} \to 0 \quad \text{when} \quad h \to 0$$

and, therefore,

$$||\tau_h \varphi - \varphi||_{L^r(0,+\infty,V)} < \frac{\varepsilon}{3}; \quad 0 < h < \delta. \tag{37}$$

Also, by a change of variables it follows that

$$||\tau_h v - \tau_h \varphi||_{L^r(0,+\infty,V)} \leq ||v - \varphi||_{L^r(0,+\infty,V)} \frac{\varepsilon}{3}; \quad \forall\, h > 0. \tag{38}$$

From (35), (36), (37) and (38) we obtain

$$||\tau_h v - v||_{L^r(0,+\infty,V)} < \varepsilon; \quad 0 < h < \delta$$

which proves (34). It follows from this that

$$\tau_h u \to u \quad \text{in} \quad L^p(0,+\infty,X) \quad \text{when} \quad h \to 0$$

and, therefore, to prove (33) it is sufficient to prove that

$$(\tau_h u)' \to u' \quad \text{in} \quad L^q(0,+\infty,Y) \quad \text{when} \quad h \to 0. \tag{39}$$

However

$$(\tau_h u)' = \tau_h u' \quad \text{in} \quad \mathcal{D}'(0,+\infty,X). \tag{40}$$

Indeed, let $\theta \in \mathcal{D}(0,+\infty)$. We have

$$\langle (\tau_h u)', \theta \rangle = -\langle \tau_h u, \theta' \rangle = -\int_0^{+\infty} u(t+h)\theta'(t)\,dt = -\int_h^{+\infty} u(t)\theta'(t-h)\,dt.$$

Defining

$$\psi(t) = \theta(t-h),$$

it follows that $\psi \in \mathcal{D}(h,+\infty)$ since $\operatorname{supp}\psi = h + \operatorname{supp}(\theta)$ and from the equality above we can write

$$\begin{aligned}\langle (\tau_h u)', \theta \rangle &= -\int_h^{+\infty} u(t)\psi'(t)\,dt = \int_h^{+\infty} u'(t)\psi(t)\,dt \\ &= \int_h^{+\infty} u'(t)\theta(t-h)dt = \int_0^{+\infty} u'(t+h)\theta(t)\,dt = \langle \tau_h u', \theta \rangle,\end{aligned}$$

which proves (40). Thus, from (40) and (34), (39) is proved and consequently (33). The next step is to show that for $h > 0$ fixed we can approximate $(\tau_h u)$ by functions of $\mathcal{D}([0,+\infty);X)$. Indeed, set for $h > 0$ fixed

$$\phi(t) = \begin{cases} 1, & t \geq -\frac{h}{2} \\ 0, & t \leq -h \end{cases} ; \qquad 0 \leq \phi(t) \leq 1,\ 0 \leq \phi'(t) \leq 1 \text{ such that } \phi \in C^\infty(\mathbb{R})$$

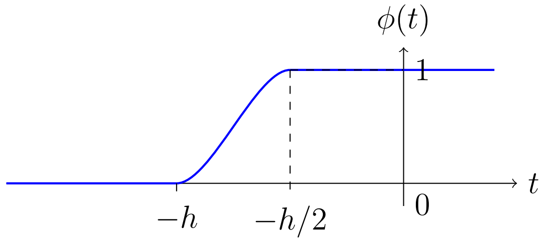

Figure 9.1: Cutoff function $\phi(t)$

cf. figure below:

Consider now $u \in W(0, +\infty)$ and define

$$v(t) = \begin{cases} \phi(t)(\tau_h u)(t); & t \geq -h \\ 0; & t < -h. \end{cases}$$

We claim that

$$v(t) = (\tau_h u)(t) \text{ a.e. in }]0, +\infty[\text{ and } v \in W(-\infty, +\infty). \tag{41}$$

Indeed, if $t > 0 \geq -\dfrac{h}{2}$ then $\phi(t) = 1$ and therefore $v(t) = (\tau_h u)(t)$. Now, from the fact that $|\phi(t)| \leq 1; \ \forall t \in \mathbb{R}$, it follows that

$$\begin{aligned} ||v||^p_{L^p(-\infty,+\infty,X)} &= \int_{-h}^{+\infty} |\phi(t)|^p \, ||(\tau_h u)(t)||^p_X \, dt \leq \int_{-h}^{+\infty} ||(\tau_h u)(t)||^p_X \, dt \\ &= \int_{-h}^{+\infty} ||u(t+h)||^p_X \, dt = \int_0^{+\infty} ||u(s)||^p_X \, ds < +\infty. \end{aligned}$$

Furthermore, from the fact that

$$v'(t) = \begin{cases} \phi'(t)(\tau_h u)(t) + \phi(t)(\tau_h u')(t); & t \geq -h \\ 0; & t < -h \end{cases}$$

and, also, since $\phi'(t) = 0; \ \forall t \geq -\dfrac{h}{2}, \quad X \hookrightarrow Y$ and $|\phi| \leq 1$, $|\phi'| \leq 1$, we have

$$\begin{aligned} &||v'||^q_{L^q(-\infty,+\infty,Y)} \leq c_1 \left\{ \int_{-h}^{-\frac{h}{2}} ||(\tau_h u)(t)||^q_X \, dt + \int_{-h}^{+\infty} ||(\tau_h u')(t)||^q_Y \, dt \right\} \\ &\leq c_1 \left(\int_{-h}^{-h/2} dt \right)^{1/(p/q)'} \left(\int_{-h}^{-h/2} ||(\tau_h u)(t)||^p_X \, dt \right)^{q/p} + c_1 \int_{-h}^{+\infty} ||(\tau_h u')(t)||^q_Y \, dt \\ &\leq c_1 \left[\frac{h}{2} \right]^{\frac{p-q}{p}} \left(\int_{-h}^{+\infty} ||u(t+h)||^p_X \, dt \right)^{q/p} + c_1 \int_{-h}^{+\infty} ||u'(t+h)||^q_Y \, dt \\ &\leq c_2(h) \left\{ \left(\int_0^{+\infty} ||u(s)||^p_X \, ds \right)^{q/p} + \int_0^{+\infty} ||u'(s)||^q_Y \, ds \right\} < +\infty \end{aligned}$$

which proves (41). It results from the inequality above and from the 1st Case, the existence of $(v_\nu) \subset \mathcal{D}(-\infty, +\infty, X)$ such that:

$$v_\nu \to v \quad \text{in} \quad W(-\infty, +\infty)$$

which implies that,

$$v_\nu|_{[0,+\infty)} \to v|_{[0,+\infty)} = \tau_h u \quad \text{in} \quad W(0,+\infty). \tag{42}$$

From (42) and from the fact that $v_\nu|_{[0,+\infty)} \subset \mathcal{D}([0,+\infty);X)$ we have the desired result.

3^rd^ Case: a, b finite

Let $\alpha, \beta \colon [a,b] \to \mathbb{R}$ be such that

$$\alpha, \beta \in \mathcal{D}([a,b]); \;\; \alpha(t) + \beta(t) = 1; \;\; \forall\, t \in [a,b] \tag{43}$$
$$\alpha \text{ (resp. } \beta\text{) vanishes in a neighborhood of } b \text{ (resp. } a\text{)} \tag{44}$$

according to the figure below:

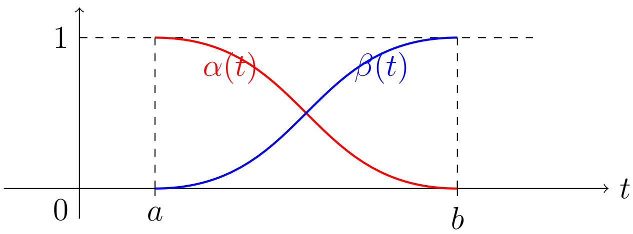

Figure 9.2: Partition of Unity on $[a,b]$

For all $u \in W(a,b)$ we can write from (43) that

$$u = \alpha u + \beta u.$$

Define

$$\widetilde{\alpha u} = \begin{cases} (\alpha u)(t); & t \in [a,b] \\ 0; & t > b \end{cases} \quad \text{and} \quad \widetilde{\beta u} = \begin{cases} (\beta u)(t); & t \in [a,b] \\ 0; & t < a. \end{cases}$$

We have

$$\widetilde{\alpha u} \in W(a,+\infty) \quad \text{and} \quad \widetilde{\beta u} \in W(-\infty,b).$$

By the 2^nd^ Case, there exist $(\alpha_\nu) \subset \mathcal{D}([a,+\infty);X)$ and $(\beta_\nu) \subset \mathcal{D}(]-\infty,b];X)$ such that

$$\alpha_\nu \to \widetilde{\alpha u} \text{ in } W(a,+\infty) \text{ and } \beta_\nu \to \widetilde{\beta u} \text{ in } W(-\infty,b).$$

Therefore

$$\alpha_\nu|_{[a,b]} \to \widetilde{\alpha u}|_{[a,b]} = \alpha u \quad \text{and} \quad \beta_\nu|_{[a,b]} \to \widetilde{\beta u}|_{[a,b]} = \beta u \text{ in } W(a,b),$$

that is,

$$(\alpha_\nu + \beta_\nu)|_{[a,b]} \to (\alpha u + \beta u) = u \quad \text{in} \quad W(a,b).$$

Since

$$(\alpha_\nu + \beta_\nu)|_{[a,b]} \in \mathcal{D}([a,b];X)$$

we have proved the desired result. □

We will proceed next to the proof of the theorem.

Proof: From (1), (2) and (3), we have

$$\mathcal{V} \subset V \subset H$$

which implies that

$$\overline{\mathcal{V}}^{(L^2(\Omega))^2} \subset \overline{V}^{(L^2(\Omega))^2} \subset H,$$

i.e.,

$$H = \overline{V}^{(L^2(\Omega))^2}. \tag{45}$$

It is not difficult to prove, by virtue of the characterization given in (4) that

$$V \overset{\text{comp.}}{\hookrightarrow} H. \tag{46}$$

Consider, then, the operator A defined by the triple $\{V, H, ((\,\cdot\,,\,\cdot\,))_V\}$. As we know

$$D(A) = \{v \in V; \exists\, f \in H \text{ s.t. } (f,v)_H = ((u,v))_V; \forall\, v \in V\}, \quad f = Au.$$

Let $u \in D(A)$. We have

$$(Au, v)_H = ((u,v))_V; \quad \forall\, v \in V. \tag{47}$$

Setting

$$Au = (\xi_1, \xi_2)$$

we have

$$\sum_{i=1}^{2} (\xi_i, v_i)_{L^2(\Omega)} = \sum_{i=1}^{2} ((u_i, v_i)).$$

In particular, taking $v = \varphi \in \mathcal{V}$ it follows that

$$\sum_{i=1}^{2} \langle \xi_i, \varphi_i \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \sum_{i=1}^{2} \langle -\Delta u_i, \varphi_i \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\,.$$

Whence

$$\langle Au, \varphi \rangle_{V',V} = \langle -\Delta u, \varphi \rangle_{V',V}$$

and, therefore,

$$Au = -\Delta u \quad \text{in} \quad V'; \quad \forall\, u \in D(A) \tag{48}$$

where $-\Delta u = (-\Delta u_1, -\Delta u_2)$.

On the other hand, according to the Spectral Theorem, we have the existence of a sequence $(\omega_\nu)_{\nu\in\mathbb{N}}$ formed by eigenfunctions of the operator A whose eigenvalues $(\lambda_\nu)_{\nu\in\mathbb{N}}$ satisfy:

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_\nu \cdot; \quad \lambda_\nu \to +\infty \text{ when } \nu \to +\infty.$$

Furthermore,

$$(\omega_\nu) \quad \text{is a complete orthonormal system in} \quad H \tag{49}$$

$$\left(\frac{\omega_\nu}{\lambda_\nu^{1/2}}\right) \quad \text{is a complete orthonormal system in} \quad V \tag{50}$$

$$\left(\frac{\omega_\nu}{\lambda_\nu}\right) \quad \text{is a complete orthonormal system in} \quad D(A). \tag{51}$$

By virtue of (47), we have

$$(A\omega_\nu, v)_H = ((\omega_\nu, v))_V; \quad \forall\, v \in V$$

i.e.,

$$\lambda_\nu(\omega_\nu, v)_H = ((\omega_\nu, v))_V\,; \quad \forall\, v \in V. \tag{52}$$

From (49) we can write that

$$u = \sum_{\nu=1}^{+\infty}(u, \omega_\nu)\omega_\nu; \quad \forall\, u \in H \tag{53}$$

and

$$|u|^2 = \sum_{\nu=1}^{+\infty} |(u, \omega_\nu)|^2; \quad \forall\, u \in H. \tag{54}$$

From (50) we have

$$v = \sum_{\nu=1}^{+\infty} \left(\left(u, \frac{\omega_\nu}{\sqrt{\lambda_\nu}}\right)\right)_V \frac{\omega_\nu}{\sqrt{\lambda_\nu}}\,; \quad \forall\, u \in V \tag{55}$$

and from (52) it follows that

$$||v||_V^2 = \sum_{\nu=1}^{+\infty} \left|\left(\left(u, \frac{\omega_\nu}{\sqrt{\lambda_\nu}}\right)\right)_V\right|^2 = \sum_{\nu=1}^{+\infty} \lambda_\nu |(v, \omega_\nu)_H|^2\,; \quad \forall\, u \in V. \tag{56}$$

It is worth recalling that the operator A admits an extension

$$\begin{aligned} \widetilde{A}\colon V &\to V' \\ u &\mapsto \widetilde{A}u \end{aligned}$$

defined by

$$\langle \widetilde{A}u, v\rangle_{V',V} = ((u, v))_V\,, \quad \forall\, v \in V;$$

extension which is an isometric bijection, self-adjoint and therefore admits an isometric inverse also self-adjoint

$$[\widetilde{A}]^{-1}\colon V' \to V.$$

We have, from the above, that if $v \in V'$ then from (56)

$$\begin{aligned} ||v||_{V'} = ||A^{-1}v||_V &= \sum_{\nu=1}^{+\infty} \lambda_\nu |(A^{-1}v, \omega_\nu)_H|^2 = \sum_{\nu=1}^{+\infty} \lambda_\nu |\langle v, A^{-1}\omega_\nu\rangle|^2 \\ &= \sum_{\nu=1}^{+\infty} \lambda_\nu \left|\left\langle v, \frac{\omega_\nu}{\lambda_\nu}\right\rangle\right|^2 = \sum_{\nu=1}^{+\infty} \frac{1}{\lambda_\nu} |\langle v, \omega_\nu\rangle|^2 \end{aligned}$$

that is,

$$||v||_{V'} = \sum_{\nu=1}^{+\infty} \frac{1}{\lambda_\nu} |\langle v, \omega_\nu\rangle_{V',V}|^2\,. \tag{57}$$

1ª Step: Approximate System

Consider the basis $(\omega_\nu)_{\nu\in\mathbb{N}}$, formed by the eigenfunctions of A, mentioned above. Set

$$V_m = [\omega_1, \omega_2, \ldots, \omega_m].$$

In V_m consider the approximate problem

$$u_m(t) \in V_m \;\Leftrightarrow\; u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i \tag{58}$$

$$(u'_m(t), \omega_j) + \nu((u_m(t), \omega_j)) + b(u_m(t), u_m(t), \omega_j) = \langle f(t), \omega_j \rangle_{V',V} \tag{59}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H, \tag{60}$$

which by Carathéodory has a local solution in some interval $[0, t_m)$.

2ª Step: A Priori Estimates

- **Estimate I**

Composing (59) with u_m results that

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu||u_m(t)||_V^2 + b(u_m(t), u_m(t), u_m(t)) = \langle f(t), u_m(t) \rangle.$$

However, from Lemma 2 we conclude that

$$b(u_m(t), u_m(t), u_m(t)) = 0.$$

Thus

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu||u_m(t)||_V^2 \le ||f(t)||_{V'}\,||u_m(t)||_V$$
$$= \frac{1}{\sqrt{\nu}}\,||f(t)||_{V'}\,\sqrt{\nu}||u_m(t)||_V \le \frac{1}{2\nu}\,||f(t)||_{V'}^2 + \frac{\nu}{2}\,||u_m(t)||_V^2\,.$$

Whence

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \frac{\nu}{2}\,||u_m(t)||_V^2 \le \frac{1}{2\nu}\,||f(t)||_{V'}^2\,.$$

Integrating from 0 to t, $t \in [0, t_m)$ results that

$$|u_m(t)|^2 + \nu\int_0^t ||u_m(s)||_V^2 \le |u_{0m}|^2 + \frac{1}{\nu}\int_0^t ||f(t)||_{V'}^2\,dt. \tag{61}$$

But from (60) it follows that

$$|u_{0m}|^2 \le c_1\,; \quad \forall\, m \in \mathbb{N}. \tag{62}$$

From (61) and (62) we obtain

$$|u_m(t)|^2 + \nu\int_0^t ||u_m(s)||_V^2\,ds \le c_1 + \frac{1}{\nu}\,||f||_{L^2(0,T;V')}^2; \quad \forall\, t \in [0, t_m),\ \forall\, m \in \mathbb{N}. \tag{63}$$

From the inequality above it follows that we can extend u_m to the whole interval $[0, T]$ with u_m absolutely continuous and u'_m existing almost everywhere. Furthermore, as proved in other problems we have that the classical and distributional derivative of u_m coincide and,

$$u'_m \in L^2(0,T;H), \quad \forall\, m \in \mathbb{N}. \tag{64}$$

Now from (63) we have that

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;H) \tag{65}$$
$$(u_m) \quad \text{is bounded in} \quad L^2(0,T;V) \tag{66}$$

- **Estimate II**

Identifying H with its dual H' we have from (59)

$$\begin{aligned}\langle u'_m(t),\omega_j\rangle_{V',V} + \nu\langle\widetilde{A}u_m(t),\omega_j\rangle_{V',V} + \langle B(u_m(t),u_m(t)),\omega_j\rangle_{V',V}\\ = \langle f(t),\omega_j\rangle_{V',V}, \quad j=1,2,\dots,m.\end{aligned}$$

Setting for each $m \in \mathbb{N}$

$$h_m(t) = f(t) - \nu\widetilde{A}u_m(t) - B(u_m(t),u_m(t)) \in V' \tag{67}$$

we have

$$\langle h_m(t),\omega_j\rangle_{V',V} = \langle u'_m(t),\omega_j\rangle; \quad j=1,2,\dots m \text{ and } \forall\, m\in\mathbb{N}.$$

Thus

$$\begin{aligned}\sum_{j=1}^{m}\frac{1}{\lambda_j}\,|\langle h_m(t),\omega_j\rangle_{V',V}|^2 &= \sum_{j=1}^{m}\frac{1}{\lambda_j}\,|\langle u'_m(t),\omega_j\rangle_{V',V}|^2\\ &= \sum_{j=1}^{m}\frac{1}{\lambda_j}\,|(u'_m(t),\omega_j)|^2;\ \ \forall\, m\in\mathbb{N}.\end{aligned}$$

Since

$$(u'_m(t),\omega_j) = 0 \quad \text{if} \quad j \ge m+1$$

we obtain from (57)

$$\begin{aligned}||u'_m(t)||^2_{V'} &= \sum_{j=1}^{+\infty}\frac{1}{\lambda_j}\,|\langle u'_m(t),\omega_j\rangle|^2 = \sum_{j=1}^{+\infty}\frac{1}{\lambda_j}\,|(u'_m(t),\omega_j)|^2\\ &= \sum_{j=1}^{m}\frac{1}{\lambda_j}\,|(u'_m(t),\omega_j)|^2 = \sum_{j=1}^{m}\frac{1}{\lambda_j}\,|\langle h_m(t),\omega_j\rangle|^2\\ &\le \sum_{j=1}^{+\infty}\frac{1}{\lambda_j}\,|\langle h_m(t),\omega_j\rangle|^2 = ||h_m(t)||^2_{V'}\end{aligned}$$

that is,

$$||u'_m(t)||^2_{V'} \le ||h_m(t)||^2_{V'}\,; \quad \forall\, m\in\mathbb{N}. \tag{68}$$

On the other hand, from (67) and (21') it results that

$$\begin{aligned}&||h_m(t)||_{V'}\\ &\le ||f(t)||_{V'} + \nu||\widetilde{A}u_m(t)||_{V'} + ||B(u_m(t),u_m(t))||_{V'}\\ &\overset{(19)}{\le} ||f(t)||_{V'} + \nu||u_m(t)||_V + c_1||u_m(t)||_V^{1/2}\,||u_m(t)||_V^{1/2}\\ &\le ||f(t)||_{V'} + (\nu+c_1)||u_m(t)||_V\,.\end{aligned}$$

It follows from the inequality above that

$$||h_m(t)||^2_{V'} \leq c_2\{||f(t)||^2_{V'} + ||u_m(t)||^2_V\}.$$

Integrating from 0 to T

$$\int_0^T ||h_m(t)||^2_{V'}\,dt \leq c_2\Big\{\int_0^T ||f(t)||^2_{V'}\,dt + \int_0^T ||u_m(t)||^2_V\,dt\Big\};\quad \forall m \in \mathbb{N} \tag{69}$$

and from (66) it follows that

$$(h_m) \quad \text{is bounded in} \quad L^2(0,T;V'). \tag{70}$$

From (68) and (69) we obtain

$$\int_0^T ||u'_m(t)||^2_{V'}\,dt \leq \int_0^T ||h_m(t)||^2_{V'}\,dt < c;\quad \forall m \in \mathbb{N}.$$

Whence

$$(u'_m) \quad \text{is bounded in} \quad L^2(0,T;V'). \tag{71}$$

3ª Step: Passage to the Limit

From (65), (66) and (71) we obtain the existence of a subsequence (u_ν) of (u_m) such that

$$u_\nu \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T;H) \tag{72}$$

$$u_\nu \rightharpoonup u \quad \text{in} \quad L^2(0,T;V) \tag{73}$$

$$u'_\nu \rightharpoonup u' \quad \text{in} \quad L^2(0,T;V'). \tag{74}$$

Let $\theta \in \mathcal{D}(0,T)$ and consider $j \in \mathbb{N}$. Multiplying (59) by θ and integrating in $[0,T]$, we obtain for $\nu \geq j$ that

$$\begin{aligned} &\int_0^T \langle u'_m(t), \omega_j\rangle_{V',V}\,\theta(t)\,dt + \nu\int_0^T ((u_m(t),\omega_j))_V\,\theta(t)\,dt \\ &+ \int_0^T b(u_m(t),u_m(t),\omega_j)\theta(t)\,dt = \int_0^T \langle f(t),\omega_j\rangle_{V',V}\,\theta(t)\,dt. \end{aligned} \tag{75}$$

The convergences in (73) and (74) are sufficient to pass to the limit in the linear part. Let's see the nonlinear part.

By Lemma 2 we have that

$$b(u_\nu(t),u_\nu(t),\omega_j) = -b(u_\nu(t),\omega_j,u_\nu(t)) = -\sum_{i,k=1}^{2}\int_\Omega u_{\nu,i}(t)\,\frac{\partial\omega_{jk}}{\partial x_i}\,u_{\nu,k}\,dx.$$

Identifying $H \equiv H'$ we have

$$V \overset{\text{comp.}}{\hookrightarrow} H \equiv H' \hookrightarrow V'.$$

[19]Note that $\widetilde{A}\colon V \to V'$ is an isometry.

Now, from (66) and (71) it follows that

$$(u_m) \text{ is bounded in } W = \{\omega \mid \omega \in L^2(0,T;V);\ ,\ \omega' \in L^2(0,T;V')\}.$$

By the Aubin-Lions Theorem there will exist a subsequence of (u_ν) which we will still denote by (u_ν) such that

$$u_\nu \to u \quad \text{strongly in} \quad L^2(0,T;H). \tag{76}$$

Therefore

$$u_{\nu,i} \to u_i \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)) = L^2(Q),\ \ i = 1,2$$

which implies that

$$u_{\nu,i}\, u_{\nu,k} \to u_i\, u_k \quad \text{a.e. in} \quad Q; \quad i,k = 1,2. \tag{77}$$

We claim that

$$(u_{\nu,i}\, u_{\nu,k}) \quad \text{is bounded in} \quad L^2(Q). \tag{78}$$

Indeed, by Schwarz, by Lemma 3 and by the fact that $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ we have for $i,k = 1,2$

$$\begin{aligned}
&\int_\Omega |u_{\nu,i}\, u_{\nu,k}|^2\, dx = \int_\Omega |u_{\nu,i}|^2\, |u_{\nu,k}|^2\, dx \\
&\le \Big(\int_\Omega |u_{\nu,i}|^4\, dx \Big)^{1/2} \Big(\int_\Omega |u_{\nu,k}|^4\, dx \Big)^{1/2} \\
&= ||u_{\nu,i}||^2_{L^4(\Omega)}\, ||u_{\nu,k}||^2_{L^4(\Omega)} \\
&\le \big(\sqrt{2}|u_{\nu,i}|_{L^2(\Omega)}\, ||u_{\nu,i}||_{H_0^1(\Omega)}\big)\big(\sqrt{2}|u_{\nu,k}|_{L^2(\Omega)}\, ||u_{\nu,k}||_{H_0^1(\Omega)}\big) \\
&= 2|u_{\nu,i}|_{L^2(\Omega)}\, |u_{\nu,k}|_{L^2(\Omega)}\, ||u_{\nu,i}||_{H_0^1(\Omega)}\, ||u_{\nu,k}||_{H_0^1(\Omega)}\, .
\end{aligned} \tag{79}$$

It follows from (65) and (79) that $\exists\, c_1 > 0$ such that

$$\int_\Omega |u_{\nu,i}\, u_{\nu,k}|^2\, dx \le 2c_1 ||u_{\nu,i}||_{H_0^1(\Omega)}\, ||u_{\nu,k}||_{H_0^1(\Omega)}\, .$$

Integrating from 0 to T, we obtain

$$\int_0^T \int_\Omega |u_{\nu,i}\, u_{\nu,k}|^2\, dxdt \le c_1 \Big\{ \int_0^T ||u_{\nu,i}(t)||^2_{H_0^1(\Omega)}\, dt + \int_0^T ||u_{\nu,k}(t)||^2_{H_0^1(\Omega)}\, dt \Big\} \le c_2$$

where such boundedness comes from the fact that (u_ν) is bounded in V (cf. (66)) and, therefore, each component is bounded in $L^2(0,T;H_0^1(\Omega))$, which proves the assertion in (78).

Thus, from (77) and (78) it follows by Lions' Lemma that

$$u_{\nu,i}\, u_{\nu,k} \rightharpoonup u_k\, u_i \quad \text{in} \quad L^2(Q),\ \ i,k = 1.2.$$

It follows from the convergence above and from the fact that $\dfrac{\partial \omega_{j,k}}{\partial x_i} \in L^2(\Omega)$ that

$$\int_\Omega u_{\nu,i}\, \frac{\partial \omega_{j,k}}{\partial x_i}\, u_{\nu,k}\, dx \to \int_\Omega u_i\, \frac{\partial \omega_j}{\partial x_i}\, u_k\, ,dx \quad i,k = 1,2. \tag{80}$$

Thus, from (73), (74), (75) and (80), in the limit situation, we obtain

$$\begin{aligned}&\int_0^T \langle u'(t),\omega_j\rangle_{V',V}\,\theta(t)\,dt+\nu\int_0^T((u(t),\omega_j))_V\,\theta(t)\,dt\\&+\int_0^T b(u(t),u(t),\omega_j)\theta(t)\,dt=\int_0^T\langle f(t),\omega_j\rangle_{V',V}\,\theta(t)\,dt,\ \ \forall\, j\in\mathbb{N}\end{aligned}\tag{81}$$

and by the totality of the $\omega_j'^s$ in V it follows that the identity in (81) is valid for all $v\in V$. Whence

$$\begin{aligned}&\Big\langle\int_0^T u'\theta\,dt,v\Big\rangle_{V',V}+\nu\Big\langle\int_0^T\widetilde{A}u\,\theta\,dt,v\Big\rangle_{V',V}\\&+\Big\langle\int_0^T B(u,u)\theta\,dt,v\Big\rangle_{V',V}\overset{(20)}{=}\Big\langle\int_0^T f\theta\,dt,v\Big\rangle_{V',V},\ \ \forall\, v\in V,\end{aligned}\tag{82}$$

which implies that

$$u'+\nu\widetilde{A}u+B(u,u)=f\quad\text{in}\quad\mathcal{D}'(0,T;V')\tag{83}$$

or even, given the regularity of the functions involved

$$u'+\nu\widetilde{A}u+B(u,u)=f\quad\text{in}\quad L^2(0,T;V').\tag{84}$$

Before proceeding to the next steps, consider the following result

Lemma 6: Let $u,v\in W(0,T)=\{u;u\in L^2(0,T;V),u'\in L^2(0,T;V')\}$. Then

$$\frac{d}{dt}(u(t),v(t))=\langle u'(t),v(t)\rangle_{V',V}+\langle u(t),v'(t)\rangle_{V,V'}\text{ in }L^1(0,T)$$

where $\dfrac{d}{dt}$ is taken in the sense of $\mathcal{D}'(0,T)$.

Proof: By Lemma 5, there exist $(u_\nu),(v_\nu)\subset\mathcal{D}([0,T];V)$ such that

$$u_\nu\to u\text{ in }L^2(0,T;V)\quad\text{and}\quad u_\nu'\to u'\text{ in }L^2(0,T;V')\tag{85}$$

$$v_\nu\to v\text{ in }L^2(0,T;V)\quad\text{and}\quad v_\nu'\to v'\text{ in }L^2(0,T;V')\tag{86}$$

Now, for each $\nu\in\mathbb{N}$, we have by virtue of the regularity of the u_ν's and v_ν's:

$$\frac{d}{dt}(u_\nu(t),v_\nu(t))=\langle u_\nu'(t),v_\nu(t)\rangle+\langle u_\nu(t),v_\nu'(t)\rangle.\tag{87}$$

Now from (85) and (86) we have

$$(u_\nu(t),v_\nu(t))\to(u(t),v(t))\quad\text{in}\quad L^1(0,T)\tag{88}$$

$$\langle u_\nu'(t),v_\nu(t)\rangle\to\langle u'(t),v(t))\rangle\quad\text{in}\quad L^1(0,T)\tag{89}$$

$$\langle u_\nu(t),v_\nu'(t)\rangle\to\langle u(t),v'(t)\rangle\quad\text{in}\quad L^1(0,T).\tag{90}$$

From (88) it follows that

$$\frac{d}{dt}(u_\nu(t),v_\nu(t))\to\frac{d}{dt}(u(t),v(t))\quad\text{in}\quad\mathcal{D}'(0,T)\tag{91}$$

[20]Follows from Lemma 4.

and from (89) and (90) we have

$$\langle u_\nu'(t), v_\nu(t)\rangle + \langle u_\nu(t), v_\nu'(t)\rangle \to \langle u'(t), v(t)\rangle + \langle u(t), v'(t)\rangle \quad \text{in} \quad L^1(0,T). \tag{92}$$

Finally from (87), (91), (92) and by the uniqueness of the limit in $\mathcal{D}'(0,T)$ we have the desired result. □

4ª Step: Initial Condition

Initially, note that by the fact that

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V) \quad \text{and} \quad u' \in L^2(0,T;V')$$

then

$$u \in C^0([0,T];V') \cap C_s(0,T;V)$$

making sense therefore to calculate $u(0)$ and $u(T)$. We will prove that

$$u(0) = u_0\,. \tag{93}$$

Indeed, let $\theta \in C^1([0,T])$ such that $\theta(0) = 1$ and $\theta(T) = 0$. From (73) in particular for $\theta'\omega_j \in L^1(0,T;H)$, we have

$$\int_0^T (u_\nu(t), \omega_j)\theta'(t)\,dt \to \int_0^T (u(t), \omega_j)\theta'(t)\,dt. \tag{94}$$

By Lemma 6

$$\frac{d}{dt}\,(u(t), \omega_j\theta) = \langle u'(t), \omega_j\,\theta(t)\rangle + \langle u(t), \omega_j\,\theta'(t)\rangle.$$

Integrating the equality above from 0 to T, we obtain

$$-(u(0), \omega_j) = \int_0^T \langle u'(t), \omega_j\rangle\theta(t)\,dt + \int_0^T (u(t), \omega_j)\theta'(t)\,dt. \tag{95}$$

Analogously

$$-(u_{0\nu}, \omega_j) = \int_0^T \langle u_\nu'(t), \omega_j\rangle\theta(t)\,dt + \int_0^T (u_\nu(t), \omega_j)\theta'(t)\,dt. \tag{96}$$

From (94), (95) and (96) we conclude that

$$-(u_{0\nu}, \omega_j) - \int_0^T \langle u_\nu'(t), \omega_j\rangle\theta(t)\,dt \to (u(0), \omega_j) - \int_0^T \langle u'(t), \omega_j\rangle\theta(t)\,dt. \tag{97}$$

But from (74) it follows that

$$\int_0^T \langle u_\nu'(t), \omega_j\rangle\theta(t)\,dt \to \int_0^T \langle u'(t), \omega_j\rangle\theta(t)\,dt$$

and from (97) it follows

$$(u_{0,\nu}, \omega_j) \to (u(0), \omega_j), \quad \forall\, j \in \mathbb{N}.$$

By the totality of the ω_j's in H we conclude that

$$(u_{0\nu}, v) \to (u(0), v), \quad \forall\, v \in H. \tag{98}$$

On the other hand, from (60) we obtain

$$(u_{0\nu}, v) \to (u_0, v), \quad \forall\, v \in H. \tag{99}$$

From (98) and (99) the desired result follows in (93). □

5ª Step: Uniqueness

Let u and v be solutions of the system in question. Then $\omega = u - v$ satisfies

$$\left|\begin{array}{l} \omega' + \nu\widetilde{A}\omega + B(u,u) - B(u,v) = 0 \quad \text{in} \quad L^2(0,T;V') \\ \omega = 0 \quad \text{on} \quad \Sigma \\ \omega(0) = 0 \end{array}\right. \tag{100}$$

Composing $(100)_1$ with $\omega(t)$ we obtain

$$\langle \omega'(t), \omega(t)\rangle_{V',V} + \nu((\omega(t),\omega(t)))_V = b(v(t),v(t),\omega(t)) - b(u(t),u(t),\omega(t)). \tag{101}$$

However,

$$\begin{aligned} & b(v(t),v(t),\omega(t)) - b(u(t),u(t),\omega(t)) \\ &= b(v(t),v(t),u(t)-v(t)) - b(u(t),u(t),u(t)-v(t)) \\ &= b(v(t),v(t),u(t)) - \overbrace{b(v(t),v(t),v(t))}^{=0} - \overbrace{b(u(t),u(t),u(t))}^{=0} + b(u(t),u(t),v(t)) \\ &= b(v(t),v(t),u(t)) - b(u(t),v(t),u(t)) \\ &= b(v(t)-u(t),v(t),u(t)) = b(-\omega(t),v(t),u(t)) \\ &= b(-\omega(t),v(t),u(t)-v(t)+v(t)) = b(-\omega(t),v(t),\omega(t)+v(t)) \\ &= b(-\omega(t),v(t),\omega(t)) + \underbrace{b(-\omega(t),v(t),v(t))}_{=0}, \end{aligned}$$

that is,

$$b(v(t),v(t),\omega(t)) - b(u(t),u(t),\omega(t)) = -b(\omega(t),v(t),\omega(t)). \tag{102}$$

From (101), (102) and Lemma 6 we obtain

$$\frac{1}{2}\frac{d}{dt}|\omega(t)|_H^2 + \nu||\omega(t)||_V^2 \leq ||\omega(t)||_{(L^4(\Omega))^2}\, ||v(t)||_V\, ||\omega(t)||_{(L^4(\Omega))^2}\,. \tag{103}$$

However, by Lemma 3

$$||\omega_i(t)||_{L^4(\Omega)}^2 \leq \sqrt{2}||\omega_i(t)||_{H_0^1(\Omega)}\, |\omega_i(t)|_{L^2(\Omega)}; \quad i = 1,2.$$

Thus

$$\begin{aligned} ||\omega(t)||_{(L^4(\Omega))^2}^2 &= \sum_{i=1}^{2} ||\omega_i(t)||_{L^4(\Omega)}^2 \leq \sqrt{2}\sum_{i=1}^{2} ||\omega_i(t)||_{H_0^1(\Omega)}\, |\omega_i(t)|_{L^2(\Omega)} \\ &\leq \sqrt{2}\Bigg\{ \underbrace{\left(\sum_{i=1}^{2} ||\omega_i(t)||_{H_0^1(\Omega)}^2\right)^{1/2}}_{||\omega(t)||_V} \underbrace{\left(\sum_{i=1}^{2} |\omega_i(t)|_{L^2(\Omega)}^2\right)^{1/2}}_{|\omega(t)|_H} \Bigg\} = \sqrt{2}||\omega(t)||_V\, |\omega(t)|_H, \end{aligned}$$

that is,

$$||\omega(t)||^2_{(L^4(\Omega))^2} \leq \sqrt{2}||\omega(t)||_V \, |\omega(t)|_H. \tag{104}$$

Thus, from (103) and (104) we arrive at

$$\frac{1}{2}\frac{d}{dt}|\omega(t)|^2_H + \nu||\omega(t)||^2_V \leq \sqrt{2}||\omega(t)||_V \, |\omega(t)|_H \, ||v(t)||_V$$
$$= \sqrt{2}\frac{\sqrt{\nu}}{\sqrt{2}}||\omega(t)||_V \, \frac{\sqrt{2}}{\sqrt{\nu}}|\omega(t)|_H \, ||v(t)||_V \leq \nu||\omega(t)||^2_V + \frac{2}{\nu}|\omega(t)|^2_H \, ||v(t)||^2_V.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}|\omega(t)|^2_H \leq \frac{2}{\nu}|\omega(t)|^2_H \, ||v(t)||^2_V.$$

Integrating the inequality above, we obtain

$$|\omega(t)|^2_H \leq \frac{4}{\nu}\int_0^t \underbrace{|\omega(s)|^2_H}_{L^\infty(0,T)} \underbrace{||v(s)||^2_V}_{L^2(0,T)} \, ds.$$

Thus, by the Gronwall-Bellman Lemma, it follows that

$$|\omega(t)|^2_H = 0; \quad \forall \, t \in [0, T]$$

which implies that

$$\omega = 0$$

and consequently that $u = v$. □

Recovery of Pressure

In what follows we will consider two results that can be found in R. Teman [13].

Lemma 7: Let $\Omega \subset \mathbb{R}^n$ be an open set and consider $T = (T_1, \ldots, T_n)$ where $T_i \in \mathcal{D}'(\Omega)$, $\forall\, i = 1, \ldots, n$. Then

$$\langle T, \phi \rangle_{(\mathcal{D}'(\Omega))^n, (\mathcal{D}(\Omega))^n} = 0, \;\; \forall\, \phi \in \mathcal{V} \Leftrightarrow \exists\, p \in \mathcal{D}'(\Omega) \text{ such that } T = \nabla p \text{ in } \mathcal{D}'(\Omega).$$

We also have

Lemma 8: Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz open set.

(i) If a distribution p possesses all first partial derivatives $\dfrac{\partial p}{\partial x_i} \in L^2(\Omega)$ then $p \in L^2(\Omega)$ and furthermore,

$$||p||_{L^2(\Omega)/\mathbb{R}^n} \leq c(\Omega)\, |\nabla p|_{L^2(\Omega)}.\ ^{(21)}.$$

(ii) If a distribution p possesses all first partial derivatives $\dfrac{\partial p}{\partial x_i} \in H^{-1}(\Omega)$ then $p \in L^2(\Omega)$ and, furthermore,

$$||p||_{L^2(\Omega)/\mathbb{R}} \leq c|\nabla p|_{H^{-1}(\Omega)}\,.$$

Remark: It follows from Lemmas 7 and 8 that if $T \in (H^{-1}(\Omega))^n$ and $\langle T, \phi \rangle = 0$ for all $\phi \in \mathcal{V}$ then $T = \nabla p$, with $p \in L^2(\Omega)$.

It follows from (84) that

$$u'(t) + \nu \widetilde{A} u(t) + B(u(t), u(t)) = f(t) \text{ in } V' \text{ a.e. in } \,]0, T[\,. \tag{105}$$

Let us set:

$$U(t) = \int_0^t u(s)\, ds, \quad F(t) = \int_0^t f(s)\, ds \quad \text{and} \quad \beta(t) = \int_0^t B(u(s), u(s))\, ds \in V'.$$

Since $u, f, B(u,u) \in L^2(0, T; V')$ then

$$U, F \text{ and } \beta \in C^0([0,T], V') \text{ (in fact they are absolutely continuous).} \tag{106}$$

Integrating (105), we obtain by virtue of (106) that

$$u(t) - u(0) + \nu \int_0^t \widetilde{A} u(s)\, ds + \int_0^t B(u(s), u(s))\, ds = \int_0^t f(s)\, ds \text{ in } V'.$$

Thus:

$$u(t) - u_0 + \nu \widetilde{A} U(t) + \beta(t) = F(t) \text{ in } V'; \quad \forall\, t \in [0, T].$$

Therefore, for all $\phi \in \mathcal{V} \subset V$ we have

$$\langle u(t) - u_0 + \nu \widetilde{A} U(t) + \beta(t) - F(t), \phi \rangle_{V', V} = 0. \tag{107}$$

[21] It is worth remembering that $L^2(\Omega)/\mathbb{R}^n$ is isomorphic (Ω bounded) to the subspace orthogonal to the constants $L^2(\Omega)/\mathbb{R} = \{p \in L^2(\Omega); \int_\Omega p(x)\, dx = 0\}$.

Let us define

$$S(t) = u(t) - u_0 + \nu \widetilde{A} U(t) + \beta(t) - F(t) \in V'. \tag{108}$$

From the fact that V is a closed subspace of $(H_0^1(\Omega))^2$ we can, thanks to the Hahn-Banach Theorem, and for each $t \in [0, T]$ extend $S(t)$ to a functional $T(t) \in (H^{-1}(\Omega))^2$ such that

$$\langle T(t), v \rangle_{(H^{-1}(\Omega))^n, (H_0^1(\Omega))^n} = \langle S(t), v \rangle_{V', V}\,; \quad \forall\, v \in V. \tag{109}$$

But from (107) and (109) we conclude that

$$\langle T(t), \phi \rangle_{(H^{-1}(\Omega))^2, (H_0^1(\Omega))^2} = 0, \quad \forall\, \phi \in \mathcal{V}.$$

By the remark after Lemma 8 it follows that $\exists\, P(t) \in L^2(\Omega)$ satisfying

$$T(t) = \nabla P(t) \quad \text{in} \quad (H^{-1}(\Omega))^n. \tag{110}$$

Thus, from (109) and (110) we obtain

$$\nabla P(t)|_V \equiv S(t) \quad \text{in} \quad V', \quad \forall\, t \in [0, T]. \tag{111}$$

Substituting (111) in (108) it follows that

$$u(t) - u_0 + \nu \tilde{A} U(t) + \beta(t) - F(t) = \nabla P(t) \text{ in } V'; \quad \forall\, t \in [0, T].$$

Since the expression on the left of the equality above belongs to $C^0([0, T], V')$ we have that $\nabla P \in C^0([0, T], V')$ and, therefore, we can differentiate the equation above distributionally obtaining

$$u' + \nu \widetilde{A} u - f + B(u, u) = \nabla \frac{\partial P}{\partial t} \quad \text{in} \quad L^2(0, T; V').$$

Consequently the equality above occurs a.e. in $]0, T[$. Setting

$$p(x, t) = -\frac{\partial P}{\partial t}(x, t)$$

it results that

$$u' + \nu \widetilde{A} u - f + B(u, u) = -\nabla p \quad \text{in} \quad L^2(0, T; V'),$$

that is,

$$u' + \nu \widetilde{A} u + B(u, u) = f - \nabla p \quad \text{in} \quad L^2(0, T; V'). \qquad \square$$

Chapter 10

Periodic Solutions of the Navier-Stokes System

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with sufficiently smooth boundary.

Problem 11

Problem 11

$$\left| \begin{aligned} &\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{j=1}^{m} u_j \frac{\partial u}{\partial x_j} + \nabla p = f \quad \text{in} \quad Q \\ &\text{div } u = 0 \quad \text{in} \quad Q \\ &u = 0 \quad \text{on} \quad \Sigma \\ &u(x,0) = u(x,T), \quad x \in \Omega, \end{aligned} \right. \tag{1}$$

where

$$f \in L^2(0,T;V'), \tag{2}$$

admits a weak solution $u \colon Q \to \mathbb{R}$ in the class

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \quad u' \in L^2(0,T;V'). \tag{3}$$

More precisely

$$\begin{aligned} &\langle u'(t), v\rangle_{V',V} + \nu((u(t),v))_V + b(u(t),u(t),v) \\ &\qquad = \langle f(t), v\rangle \text{ in } \mathcal{D}'(0,T), \quad \forall\, v \in V \end{aligned} \tag{4}$$

$$u(0) = u(T). \tag{5}$$

Proof:

1ª Step: Approximate Solution

Let $(\omega_\nu)_{\nu\in\mathbb{N}}$ be the basis formed by the eigenfunctions of the operator

$$A \leftarrow \{V, H; ((\,\cdot\,,\,\cdot\,))_V\}$$

as we saw in Problem 10. Set

$$V_m = [\omega_1, \ldots, \omega_m]$$

and in V_m consider the approximate problem

$$u_m(t) \in V_m \;\Leftrightarrow\; u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i \tag{6}$$

$$(u'_m(t), \omega_j) + \nu((u_m(t), \omega_j)) + b(u_m(t), u_m(t), \omega_j) = \langle f(t), \omega_j \rangle; \;\; j = 1, \ldots, m \tag{7}$$

$$u_m(0) = v \in V_m. \tag{8}$$

Evidently the approximate system above possesses a global solution[22] (which depends on v), whatever $v \in V_m$ is. Our goal is to show that among all the solutions of the approximate equation there exists a solution u_m (at least) that satisfies the periodicity

$$u_m(0) = u_m(T).$$

For this, it is sufficient to prove that for each $m \in \mathbb{N}$, the map

$$\begin{aligned} \tau_m \colon V_m &\to V_m \\ v &\mapsto \tau_m(v) = u_m(T) \end{aligned} \tag{9}$$

possesses a unique fixed point, because, in this case, there will exist a unique function $v \in V_m$ such that

$$u_m(T) = \tau_m(v) = v = u_m(0), \quad \forall\, m \in \mathbb{N}. \tag{10}$$

Thus from (10) we have a sequence (u_m) of approximate solutions such that all u_m satisfy the periodicity condition.

Lemma 1: There exists $\rho_0 > 0$ such that $\tau_m(\overline{B\rho_0(0)}) \subset \overline{B\rho_0(0)}$.

Proof: Using in V_m the topology induced by H it is sufficient to prove that

$$\exists\, \rho_0 > 0 \text{ such that } |\tau_m(v)|_H \leq \rho_0\,; \quad \forall\, v \in V_m \text{ with } |v|_H \leq \rho_0\,. \tag{11}$$

Indeed, composing (7) with $u_m(t)$ we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u_m(t)|_H^2 + \nu||u_m(t)||_V^2 + \overbrace{b(u_m(t), u_m(t), u_m(t))}^{=0} &= \langle f(t), u_m(t) \rangle_{V',V} \\ \leq ||f(t)||_{V'}\, ||u_m(t)||_V = \frac{1}{\sqrt{\nu}}\, ||f(t)||_{V'}\, \sqrt{\nu}||u_m(t)||_V\,. \end{aligned}$$

Whence

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu||u_m(t)||^2 \leq \frac{1}{2\nu}\,||f(t)||_{V'}^2 + \frac{\nu}{2}\,||u_m(t)||^2,$$

that is,

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \frac{\nu}{2}\,||u_m(t)||^2 \leq \frac{1}{2\nu}\,||f(t)||_{V'}^2\,. \tag{12}$$

[22]Note that: $|u_m(t)|^2 + \int_0^t ||u_m(s)||^2 ds \leq |v|^2 + \frac{1}{\nu}||f||_{L^2(0,T;V')} \leq c(m)$. Since m is fixed we can extend $u_m(t)$ to the whole interval $[0, T]$.

Now, since $V \hookrightarrow H$; $\exists\, c_0 > 0$ such that

$$c_0^2 |u_m(t)|^2 \le ||u_m(t)||^2. \tag{13}$$

Therefore from (12) and (13) we arrive at

$$\frac{d}{dt}\,|u_m(t)|^2 + c_0^2\,\nu |u_m(t)|^2 \le \frac{1}{\nu}\,||f(t)||_{V'}^2 .$$

Multiplying both sides of the inequality above by $e^{c_0^2 \nu t}$ it follows that

$$\frac{d}{dt}\left(|u_m(t)|^2\, e^{c_0^2 \nu t}\right) \le \frac{1}{\nu}\,||f(t)||_{V'}^2\, e^{c_0^2 \nu t} .$$

Integrating from 0 to T, it follows that

$$|u_m(T)|^2\, e^{c_0^2 \nu T} \le |u_m(0)|^2 + \frac{1}{\nu}\int_0^T \left(||f(t)||_{V'}^2\, e^{c_0^2 \nu t}\right) dt,$$

which implies that

$$|u_m(T)|^2 \le e^{-c_0^2 \nu T}\,|u_m(0)|^2 + \frac{1}{\nu}\, e^{-c_0^2 \nu T}\, e^{c_0^2 \nu T}\int_0^T ||f(t)||_{V'}^2\, dt,$$

that is,

$$|u_m(T)|^2 \le e^{-c_0^2 \nu T}\,|u_m(0)|^2 + \frac{1}{\nu}\,||f||_{L^2(0,T;V')}^2 . \tag{14}$$

Denoting

$$\theta = e^{-c_0^2 \nu T} \quad \text{and} \quad c = \frac{1}{\nu}\,||f||_{L^2(0,T;V')}^2$$

from (14) we can write

$$|u_m(T)|^2 \le \theta |u_m(0)|^2 + c,$$

or even,

$$|\tau_m(v)|^2 \le \theta |v|^2 + c, \quad \forall\, v \in V_m .$$

Now since $0 < \theta < 1$ then $0 < 1 - \theta < 1$. In this way $\exists\, \rho_0 > 0$, sufficiently large such that $c < (1-\theta)\rho_0^2$. Thus, if $|v| < \rho_0$ then

$$\theta |v|_{L^2(\Omega)}^2 + c \le \theta\, \rho_0^2 + (1-\theta)\rho_0^2 = \rho_0^2 .$$

Whence

$$|\tau_m(v)|^2 \le \rho_0^2 ; \quad \forall\, m \in \mathbb{N}$$

which proves the desired result. □

Lemma 2: The map $\tau_m \colon V_m \to V_m$ defined in (9) is continuous.

Proof: Let $v_1, v_2 \in V_m$ and u_m^1 and u_m^2 be the solutions of the approximate problem with initial data v_1 and v_2, respectively. Then, from (7) we have

$$((u_m^1)'(t), \omega_j) + \nu((u_m^1(t), \omega_j)) + b(u_m^1(t), u_m^2(t), \omega_j) = \langle f(t), \omega_j \rangle$$
$$((u_m^2)'(t), \omega_j) + \nu((u_m^2(t), \omega_j)) + b(u_m^2(t), u_m^2(t), \omega_j) = \langle f(t), \omega_j \rangle.$$

Considering $W_m(t) = u^1_m(t) - u^2_m(t)$ then subtracting one equation from the other results that

$$(W'_m(t), \omega_j)_H + \nu((W_m(t), \omega_j))_j + b(u^1_m(t), u^1_m(t), \omega_j) - b(u^2_m(t), u^2_m(t), \omega_j) = 0.$$

In particular

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|W_m(t)|^2_H + \nu||W_m(t)||^2_V + b(u^1_m(t), u^1_m(t), W_m(t)) \\ - b(u^2_m(t), u^2_m(t), W_m(t)) = 0. \end{aligned} \tag{15}$$

However, as we saw in the uniqueness of Problem 10

$$b(u^1_m(t), u^1_m(t), W_m(t)) - b(u^2_m(t), u^2_m(t), W_m(t)) = -b(W_m(t), u^1_m(t), W_m(t)). \tag{16}$$

Thus, substituting (16) in (11) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|W_m(t)|^2_H + \nu||W_m(t)||^2_V \le |b(W_m(t), u^1_m(t), W_m(t))| \\ &\le ||W_m(t)||_{(L^4(\Omega))^2}\, ||u^1_m(t)||_V\, ||W_m(t)||_{(L^4(\Omega))^2} \\ &= ||W_m(t)||^2_{(L^4(\Omega))^2}\, ||u^1_m(t)||_V\,. \end{aligned} \tag{17}$$

But, by Lemma 3 of the preceding section and by numerical Hölder

$$\begin{aligned} ||W_m(t)||^2_{(L^4(\Omega))^2} &= \sum_{i=1}^{2} ||W_{m,i}||^2_{L^4(\Omega)} \le \sqrt{2}\sum_{i=1}^{2} ||W_{m,i}||_{H^1_0(\Omega)}\, |W_{m,i}|_{L^2(\Omega)} \\ \le \sqrt{2}\Big(\sum_{i=1}^{2} ||W_{m,i}||^2_{H^1_0(\Omega)}\Big)^{1/2} &\Big(\sum_{i=1}^{2} |W_{m,i}|^2_{L^2(\Omega)}\Big)^{1/2} = \sqrt{2}||W_m||_V\, |W_m|_H\,. \end{aligned} \tag{18}$$

From (17) and (18) we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|W_m(t)|^2_H + \nu||W_m(t)||^2_V &\le \sqrt{2}||W_m(t)||_V\, |W_m(t)|_H\, ||u^1_m(t)||_V \\ &= \sqrt{2}\sqrt{\nu}||W_m(t)||_V\, \frac{1}{\sqrt{\nu}}\, |W_m(t)|_H\, ||u^1_m(t)||_V \\ &\le \nu||W_m(t)||^2_V + \frac{1}{2\nu}\, |W_m(t)|^2_H\, ||u^1_m(t)||^2_V\,. \end{aligned}$$

Whence

$$\frac{1}{2}\frac{d}{dt}|W_m(t)|^2_H \le \frac{1}{2\nu}\, |W_m(t)|^2_H\, ||u^2_m(t)||^2_V$$

and, therefore,

$$\frac{d}{dt}|W_m(t)|^2_H - \frac{1}{\nu}\, |W_m(t)|^2_H\, ||u^1_m(t)||^2_V \le 0.$$

Defining

$$\theta_m(t) = \frac{1}{\nu}\, ||u^1_m(t)||^2_V \in L^1(0, T)$$

we obtain

$$\frac{d}{dt}|W_m(t)|^2_H - \theta_m(t)|W_m(t)|^2_H \le 0.$$

Multiplying both sides of the inequality above by $e^{-\int_0^t \theta_m(s)ds}$ it follows that

$$\frac{d}{dt}\left(|W_m(t)|_H^2\, e^{-\int_0^t \theta_m(s)ds}\right) \leq 0.$$

Integrating the inequality above from 0 to T results that

$$|W_m(T)|_H^2\, e^{-\int_0^T \theta_m(s)ds} - |W_m(0)|_H^2 \leq 0.$$

that is,

$$|W_m(T)|_H^2 \leq e^{\int_0^T \theta_m(s)ds}\, |W_m(0)|_H^2\,. \tag{19}$$

Denoting

$$c_m = e^{\int_0^T \theta_m(s)ds}$$

from (19) we obtain

$$|u_m^1(T) - u_m^2(T)|_H^2 \leq c_m |u_m^1(0) - u_m^2(0)|_H,$$

or even,

$$|\tau_m(v_1) - \tau_m(v_2)|_H \leq c_m |v_1 - v_2|$$

which concludes the proof. □

It results from Lemmas 1 and 2 by virtue of Brouwer's Theorem that the map $\tau_m\colon \overline{B_{\rho_0}(0)} \to \overline{B_{\rho_0}(0)}$ admits a fixed point, that is, $\exists\, v \in \overline{B_{\rho_0}(0)}$ such that $\tau_m(v) = v$, that is, $u_m^v(0) = u_m^v(T)$.

Then, for each $m \in \mathbb{N}$, $\exists\, u_m\colon [0,T] \to V_m$ such that $u_m(0) \in \overline{B_{\rho_0}(0)}$, i.e., $|u_m(0)| \leq \rho_0$ and

$$\left|\begin{array}{l} (u_m'(t), \omega_j) + ((u_m(t), \omega_j)) + b(u_m(t), u_m(t), \omega_j) = \langle f(t), \omega_j\rangle, \quad j = 1, \ldots, m \\ u_m(0) = u_m(T). \end{array}\right.$$

From the fact that $u_m(0) \in \overline{B_{\rho_0}(0)}$ we can repeat the estimates obtaining a subsequence (u_ν) of (u_m) such that

$$\begin{array}{lll} u_\nu \overset{*}{\rightharpoonup} u & \text{in} & L^\infty(0,T;H) \\ u_\nu \rightharpoonup u & \text{in} & L^2(0,T;V) \\ u_\nu' \rightharpoonup u' & \text{in} & L^2(0,T;V') \end{array}$$

From the convergences above it results, by passage to the limit in the approximate equation, the desired result in (4). In a manner analogous to the proof of the initial condition in the previous case, we prove (5). This concludes the problem. □

Chapter 11

Navier-Stokes System (Stationary Case)

Problem 12

Problem 12 given by

$$\left|\begin{array}{l} -\nu\Delta u + \displaystyle\sum_{j=1}^{2} \frac{\partial u}{\partial x_j} u_j = f - \nabla p \quad \text{in} \quad \Omega \\ \text{div } u = 0 \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \Gamma, \end{array}\right. \tag{1}$$

where

$$f \in V' \tag{2}$$

possesses at least one weak solution in the class

$$u \in V. \tag{3}$$

Proof: Let $(\omega_\nu)_{\nu\in\mathbb{N}}$ be a basis of eigenfunctions of the operator

$$A \leftrightarrow \{V, H, ((\,\cdot\,,\,\cdot\,))_V\}$$

according to previous problems. Set

$$V_m = [\omega_1, \ldots, \omega_m].$$

In V_m consider the approximate problem

$$u_m \in V_m \quad \Leftrightarrow \quad u_m = \sum_{i=1}^{m} \xi_i \omega_i \tag{4}$$

$$\nu((u_m, \omega_j))_V + b(u_m, u_m, \omega_j) = \langle f, \omega_j \rangle_{V,V'}\,; \quad j = 1, 2, \ldots, m. \tag{5}$$

Substituting (4) in (5) results that

$$\nu \sum_{i=1}^{m} \xi_i((\omega_i, \omega_j)) + b\bigg(\sum_{i=1}^{m} \xi_i\omega_i, \sum_{i=1}^{m} \xi_i\omega_i, \omega_j\bigg) = \langle f, \omega_j \rangle, \quad \forall\, j = 1, \ldots, m$$

that is,

$$\nu\,\xi_j\lambda_j + b\Big(\sum_{i=1}^m \xi_i\omega_i, \sum_{i=1}^m \xi_i\omega_i, \omega_j\Big) = \langle f, \omega_j\rangle, \quad \forall\, j = 1, \dots, m.$$

Define, for each j,

$$\eta_j = \nu\,\xi_j\lambda_j + b\Big(\sum_{i=1}^m \xi_i\omega_i, \sum_{i=1}^m \xi_i\omega_i, \omega_j\Big) - \langle f, \omega_j\rangle$$

and consider the map

$$\begin{gathered} P\colon \mathbb{R}^m \to \mathbb{R}^m \\ \xi = (\xi_1, \dots, \xi_m) \mapsto P(\xi) = \eta = (\eta_1, \dots, \eta_m) \end{gathered}$$

which is clearly continuous by virtue of the continuity of the trilinear form

$$b(u, v, \omega)\colon V \times V \times V \to \mathbb{R}.$$

We will prove, next, that

$$\exists\,\rho_0 > 0 \text{ such that } (P\xi, \xi)_{\mathbb{R}^m} \geq 0, \quad \forall\,\xi \in \mathbb{R}^m \text{ with } ||\xi|| = \rho_0\,. \tag{6}$$

Indeed, we have:

$$\begin{aligned} (P\xi, \xi) = (\eta, \xi) = \sum_{j=1}^m \eta_j\xi_j \\ &= \sum_{j=1}^m (\nu\lambda_j\xi_j + b(u_m, u_m, \omega_j) - \langle f, \omega_j\rangle)\xi_j \\ &= \sum_{j=1}^m \nu\lambda_j\xi_j^2 + b\Big(u_m, u_m, \sum_{j=1}^m \xi_j\omega_j\Big) - \Big\langle f, \sum_{j=1}^m \xi_j\omega_j\Big\rangle \\ &= \sum_{j=1}^m \nu\lambda_j\xi_j^2 + \underbrace{b(u_m, u_m, u_m)}_{=0} - \langle f, u_m\rangle. \end{aligned} \tag{7}$$

Recall that

$$||u_m||_V^2 = ((u_m, u_m)) = \Big(\Big(\sum_{j=1}^m \xi_j\omega_j, \sum_{i=1}^m \xi_i\omega_i\Big)\Big) = \sum_{j=1}^m \xi_j^2\lambda_j\,. \tag{8}$$

Thus, from (7) and (8) we conclude that

$$(P\xi, \xi) \geq \nu||u_m||_V^2 - ||f||_{V'}\,||u_m||_V\,. \tag{9}$$

We have two cases to consider:

(i) If $||u_m|| = 0$ then from (9) it follows that $(P\xi, \xi) \geq 0$ which proves the desired in (6) whatever $\rho > 0$ is.

(ii) If $||u_m|| \neq 0$ then $(P\xi, \xi) \geq 0$ provided that $||u_m||_V \geq \dfrac{1}{\nu}\,||f||_{V'} = c_1$. We will prove that $\exists\,\rho_0 > 0$ such that $\forall\,\xi \in \mathbb{R}^m$ with $||\xi|| = \rho_0$ then $||u_m||_V \geq c_1$. Indeed, setting

$$\beta_m = \min\{\lambda_1, \dots, \lambda_m\}$$

from (8) it follows that

$$||u_m||_V^2 = \sum_{j=1}^{m} \xi_j^2 \lambda_j \geq \beta_m \left(\sum_{j=1}^{2} \xi_j^2 \right) = \beta_m ||\xi||^2.$$

Therefore, if $\rho_0 > 0$ is such that $\rho_0 > c_1/\sqrt{\beta_m}$ then $\forall \xi \in \mathbb{R}^m$ with $||\xi||_{\mathbb{R}^m} = \rho_0$ we have

$$||u_m||_V^2 \geq \beta_m \rho_0^2 > \beta_m \frac{c_1^2}{\beta_m} = c_1^2$$

which proves the desired and consequently (6). It results from there, by virtue of Visik's acute angle lemma, that $\exists \xi_0 \in \overline{B_{\rho_0}(0)}$ such that $P(\xi_0) = 0$, that is, the system (4) and (5) possesses a solution.

Composing (5) with u_m results that

$$\nu ||u_m||_V^2 + \underbrace{b(u_m, u_m, u_m)}_{=0} = \langle f, u_m \rangle$$
$$\leq \frac{1}{\sqrt{\nu}} ||f||_{V'} \sqrt{\nu} ||u_m|| \leq \frac{1}{2\nu} ||f||_{V'}^2 + \frac{\nu}{2} ||u_m||^2,$$

which implies that

$$\frac{\nu}{2} ||u_m||_V^2 \leq \frac{1}{2\nu} ||f||_{V'}^2 ; \quad \forall m \in \mathbb{N}.$$

Thus

$$(u_m) \quad \text{is bounded in} \quad V \tag{10}$$

and therefore, there exists a subsequence (u_ν) of (u_m) such that

$$u_\nu \rightharpoonup u \quad \text{weakly in} \quad V. \tag{11}$$

Also, from (10) and the fact that $V \overset{\text{comp.}}{\hookrightarrow} H$ we have the existence of a subsequence of (u_ν), which we will still denote by the same notation, such that

$$u_\nu \to u \quad \text{in} \quad H. \tag{12}$$

It follows from (12) that

$$u_{\nu,i}\, u_{\nu,k} \to u_i\, u_k \quad \text{a.e. in} \quad \Omega, \quad i,k = 1,2. \tag{13}$$

However, from Lemma 3 of the previous section

$$\begin{aligned} \int_\Omega |u_{\nu,i}\, u_{\nu,k}|^2\, dx &\leq \left(\int_\Omega |u_{\nu,i}|^4\, dx \right)^{1/2} \left(\int_\Omega |u_{\nu,k}|^4\, dx \right)^{1/2} \\ &\leq ||u_{\nu,i}||_{L^4(\Omega)}^2\, ||u_{\nu,k}||_{L^4(\Omega)}^2 \\ &\leq 2 ||u_{\nu,i}||_{H_0^1(\Omega)}\, |u_{\nu,i}|_{L^2(\Omega)}\, ||u_{\nu,k}||_{H_0^1(\Omega)}\, |u_{\nu,k}|_{L^2(\Omega)} \\ &\leq 2 ||u_\nu||_V^2\, |u_\nu|_H^2. \end{aligned} \tag{14}$$

Now from (10), (12) and (14) we obtain

$$\int_\Omega |u_{\nu,i}\, u_{\nu,k}|^2\, dx \leq c; \quad \forall \nu \in \mathbb{N},\ \forall i,k = 1,2. \tag{15}$$

From (13) and (15), by Lions' Lemma, it follows that

$$u_{\nu,i}\, u_{\nu,k} \rightharpoonup u_i\, u_k \quad \text{weakly in} \quad L^2(\Omega), \quad i,k = 1,2. \tag{16}$$

It results from (16) and the fact that $\dfrac{\partial \omega_j}{\partial x_i} \in L^2(\Omega)$ that

$$\sum_{i,k=1}^{2} \int_\Omega u_{\nu,i} \frac{\partial \omega_{j,k}}{\partial x_i} u_{\nu,k}\, dx \to \sum_{i,k=1}^{2} \int_\Omega u_i \frac{\partial \omega_{j,k}}{\partial x_i} u_k\, dx. \tag{17}$$

But

$$\begin{aligned} b(u_\nu, u_\nu, \omega_j) &= -b(u_\nu, \omega_j, u_\nu), \\ b(u, u, \omega_j) &= -b(u, \omega_j, u). \end{aligned} \tag{18}$$

From (17), (18) and the continuity of $b(u,v,\omega)$ it follows that

$$b(u_\nu, u_\nu, \omega_j) \to b(u, u, \omega_j). \tag{19}$$

Let $j \in \mathbb{N}$ and consider $\nu \geq j$. Then, from (5), (11) and (19) we obtain, in the limit situation

$$\nu((u, \omega_j))_V + b(u, u, \omega_j) = \langle f, \omega_j \rangle_{V',V}\, ; \quad \forall\, j \in \mathbb{N}.$$

By the totality of the ω_j's in V we conclude that

$$\nu((u, v))_V + b(u, u, v) = \langle f, v \rangle_{V',V}\, ; \quad \forall\, v \in V, \tag{20}$$

or even

$$\nu\langle \widetilde{A}u, v \rangle_{V',V} + \langle B(u,u), v \rangle_{V',V} = \langle f, v \rangle_{V',V}\, ; \quad \forall\, v \in V,$$

which implies that

$$\nu \widetilde{A}u + B(u,u) = f \quad \text{in} \quad V'. \tag{21}$$

Pressure Recovery:

Define:

$$S = \nu \widetilde{A}u + B(u,u) - f \quad \text{in} \quad V'. \tag{22}$$

Since $S \in V'$ and V is a closed subspace of $(H_0^1(\Omega))^2$ we have, by virtue of the Hahn-Banach Theorem the existence of $T \in (H^{-1}(\Omega))^2$ such that

$$\langle T, \varphi \rangle_{(H^{-1}(\Omega))^2, (H_0^1(\Omega))^2} = \langle S, \varphi \rangle_{V',V}\, , \quad \forall\, \varphi \in V. \tag{23}$$

From (21) and (23) it follows that

$$\langle T, \phi \rangle = 0, \quad \forall\, \phi \in \mathcal{V}.$$

By the remark after Lemma 8 (of the Navier-Stokes system) $\exists\, P \in L^2(\Omega)$ such that

$$T = \nabla P \quad \text{in} \quad H^{-1}(\Omega). \tag{24}$$

Thus, from (23) and (24) we obtain

$$\nabla P|_V = S \quad \text{in} \quad V'$$

and from (22) we conclude that:

$$\nu \widetilde{A}u + B(u,u) = f + \nabla P.$$

Setting $p = -\nabla P$ we can rewrite the equality above as

$$\nu \widetilde{A} u + B(u,u) = f - \nabla p. \qquad \square$$

Remark:

The linear problem

$$\left| \begin{array}{l} -\nu \Delta u = f - \nabla p \quad \text{in} \quad \Omega \\ \text{div } u = 0 \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \Gamma, \end{array} \right. \tag{1}$$

where

$$f \in V', \tag{2}$$

admits a unique weak solution $u \colon \Omega \to \mathbb{R}$ in the class

$$u \in V. \tag{3}$$

Indeed, defining

$$a(u,v) = \nu((u,v))_V\,; \quad u,v \in V; \ \ \nu > 0$$

it is easy to verify that $a(u,v)$ is a bilinear, continuous and coercive form on V. Since

$$L(v) = \langle f, v \rangle_{V',V}\,; \quad v \in V$$

belongs to V' it follows, by virtue of the Lax-Milgram Lemma, that $\exists!\, u \in V$ that verifies

$$\nu((u,v))_V = \langle f, v \rangle\,; \quad \forall\, v \in V,$$

or even,

$$\nu \widetilde{A} u = f \quad \text{in} \quad V'.$$

The pressure recovery is obtained in a manner analogous to the previous cases. $\square$

Chapter 12

Klein-Gordon System ($n \leq 3$)

Let Ω be a bounded sufficiently smooth open subset of $\mathbb{R}^n$ $(n \leq 3)$.

Problem 13

Problem 13

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + v^2 u = f_1 \quad \text{in} \quad Q \\ \dfrac{\partial^2 v}{\partial t^2} - \Delta v + u^2 v = f_2 \quad \text{in} \quad Q \\ u = 0 \quad \text{and} \quad v = 0 \quad \text{on} \quad \Sigma \\ u(0) = u_0(x), \;\; u'(0) = u_1(x), \quad x \in \Omega \\ v(0) = v_0(x), \;\; v'(0) = v_1(x), \quad x \in \Omega \end{array}\right. \tag{1}$$

subject to the initial conditions

$$u_0, v_0 \in H_0^1(\Omega), \quad u_1, v_1 \in L^2(\Omega) \quad \text{and} \quad f_1, f_2 \in L^2(0,T; L^2(\Omega)) \tag{2}$$

admits a unique pair (u, v) of weak solutions of (1) in the class

$$u, v \in L^\infty(0,T; H_0^1(\Omega)), \quad u', v' \in L^\infty(0,T; L^2(\Omega)). \tag{3}$$

Proof:

1ª Step: Approximate Problem

Let $(\omega_\nu)_{\nu \in \mathbb{N}}$ be a basis of $H_0^1(\Omega)$. Set

$$V_m = [\omega_1, \ldots, \omega_m].$$

In V_m consider the approximate problem:

$$u_m(t), v_m(t) \in V_m \;\Leftrightarrow\; u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i\,,\; v_m(t) = \sum_{i=1}^{m} h_{im}(t)\omega_i \tag{4}$$

$$(u_m''(t), \omega_j) + ((u_m(t), \omega_j)) + (v_m^2(t) u_m(t), \omega_j) = (f_1(t), \omega_j), \;\; j = 1, 2, \ldots, m \tag{5}$$

$$(v''_m(t), \omega_j) + ((v_m(t), \omega_j)) + (u_m^2(t) v_m(t), \omega_j) = (f_2(t), \omega_j), \;\; j = 1, 2, \ldots, m \tag{6}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega) \tag{7}$$

$$v_m(0) = v_{0m} \to v_0 \quad \text{in} \quad H_0^1(\Omega) \tag{8}$$

which possesses a local solution in some interval $[0, t_m)$ by virtue of Carathéodory's Theorem, with u_m , v_m , u'_m and v'_m absolutely continuous and u''_m and v''_m existing a.e. The a priori estimate will serve to extend the solution to the whole $[0, T]$.

From the Sobolev Embedding Theorems we have:

$$\begin{aligned} &H_0^1(\Omega) \hookrightarrow L^6(\Omega) \quad \text{if} \quad n = 3, \\ &H_0^1(\Omega) \hookrightarrow L^q(\Omega), \; \forall\, q \in [2, +\infty) \text{ if } n = 2, \\ &H_0^1(\Omega) \hookrightarrow C^0(\overline{\Omega}) \quad \text{if} \quad n = 1 \end{aligned}$$

In any case

$$H_0^1(\Omega) \hookrightarrow L^6(\Omega) \quad \text{and} \quad n \leq 3. \tag{9}$$

Consequently

$$u^2 v \in L^2(\Omega); \quad \forall\, u, v \in H_0^1(\Omega). \tag{10}$$

Indeed, from (9) it follows that

$$u^4 \in L^{3/2}(\Omega) \quad \text{and} \quad v^2 \in L^3(\Omega); \quad \forall\, u, v \in H_0^1(\Omega). \tag{11}$$

Now, since $\dfrac{2}{3} + \dfrac{1}{3} = 1$ then from (11) and Hölder's inequality it follows that

$$u^4 v^2 \in L^1(\Omega)$$

which proves the desired in (10). Thus the non-linear expressions in (5) and (6) are well defined.

3ª Step: A Priori Estimate

Multiplying (5) by $g'_{jm}(t)$ and (6) by $h'_{jm}(t)$ and summing over j from 1 to m it follows that

$$\frac{1}{2}\frac{d}{dt}\,|u'_m(t)|^2 + \frac{1}{2}\frac{d}{dt}\,||u_m(t)||^2 + (v_m^2(t) u_m(t), u'_m(t)) = (f_1(t), u'_m(t)) \tag{12}$$

$$\frac{1}{2}\frac{d}{dt}\,|v'_m(t)|^2 + \frac{1}{2}\frac{d}{dt}\,||v_m(t)||^2 + (u_m^2(t) v_m(t), v'_m(t)) = (f_2(t), v'_m(t)) \tag{13}$$

However

$$\int_\Omega v_m^2(t) u_m(t) u'_m(t)\, dx = \frac{1}{2}\int_\Omega v_m^2(t) (u_m^2(t))'\, dx \tag{14}$$

$$\int_\Omega u_m^2(t) v_m(t) v'_m(t)\, dx = \frac{1}{2}\int_\Omega u_m^2(t) (v_m^2(t))'\, dx \tag{15}$$

Substituting (14) and (15) in (12) and (13), respectively, and summing these two last expressions we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{|u_m'(t)|^2+|v_m'(t)|^2+||u_m(t)||^2+||v_m(t)||^2\right\}\\&\quad+\frac{1}{2}\int_\Omega\left[v_m^2(t)(u_m^2(t))'+u_m^2(t)(v_m^2(t))'\right]dx\\&\quad=(f_1(t),u_m'(t))+(f_2(t),v_m'(t)),\end{aligned}$$

or even

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{|u_m'(t)|^2+|v_m'(t)|^2+||u_m(t)||^2+||v_m(t)||^2\right\}\\&+\frac{1}{2}\int_\Omega(v_m^2(t)u_m^2(t))'\,dx\le|f_1(t)|\,|u_m'(t)|+|f_2(t)|\,|v_m'(t)|.\end{aligned}\tag{16}$$

Observing that

$$\left\langle\frac{d}{dt}\int_\Omega v_m^2u_m^2\,dx,\theta\right\rangle=\left\langle\int_\Omega(v_m^2u_m^2)'\,dx,\theta\right\rangle,\quad\forall\,\theta\in\mathcal{D}(0,t_m)\tag{17}$$

then from (16) we conclude that

$$\begin{aligned}&\frac{d}{dt}\left\{|u_m'(t)|^2+|v_m'(t)|^2+||u_m(t)||^2+||v_m(t)||^2+\int_\Omega v_m^2(t)u_m^2(t)\,dx\right\}\\&\le|f_1(t)|^2+|u_m'(t)|^2+|f_2(t)|^2+|v_m'(t)|^2.\end{aligned}$$

Integrating the inequality above from 0 to t; $t\in[0,t_m)$ results that

$$\begin{aligned}&|u_m'(t)|^2+|v_m'(t)|^2+||u_m(t)||^2+||v_m(t)||^2+\int_\Omega(v_m^2(t)u_m^2(t))\,dx\\&\le|u_{1m}|^2+|v_{1m}|^2+||u_{0m}||^2+||v_{0m}||^2+\int_\Omega v_{0m}^2\,u_{0m}^2\,dx\\&+||f_1||^2_{L^2(Q)}+||f_2||^2_{L^2(Q)}+\int_0^t\left(|u_m'(s)|^2+|v_m'(s)|^2\right)ds.\end{aligned}\tag{18}$$

But

$$\begin{aligned}\int_\Omega v_{0m}^2\,u_{0m}^2\,dx&\le\left(\int_\Omega|v_{0m}|^4\,dx\right)^{1/2}\left(\int_\Omega|u_{0m}|^4\,dx\right)^{1/2}\\&=||v_{0m}||^2_{L^4(\Omega)}\,||u_{0m}||^2_{L^4(\Omega)}\,.\end{aligned}\tag{19}$$

Now from (7), (8), (9) and (10) follows the existence of a constant $c_1>0$ such that

$$|u_{1m}|^2+|v_{1m}|^2+||u_{0m}||^2+||v_{0m}||^2+\int_\Omega u_{0m}^2\,v_{0m}^2\,dx\le c_1\,;\;\;\forall\,m\in\mathbb{N},\tag{20}$$

and from (18) and (20) we obtain

$$\begin{aligned}&|u_m'(t)|^2+|v_m'(t)|^2+||u_m(t)||^2+||v_m(t)||^2+\int_\Omega u_m^2(t)v_m^2(t)\,dx\\&\qquad\le c_2+\int_0^t\left(|u_m'(s)|^2+|v_m'(s)|^2\right)ds.\end{aligned}\tag{21}$$

Observing that $\int_\Omega u_m^2 v_m^2\, dx \geq 0$ and by virtue of Gronwall's inequality, from (21) it follows that

$$|u_m'(t)|^2 + |v_m'(t)|^2 + ||u_m(t)||^2 + ||v_m(t)||^2 \leq c; \quad \forall\, t \in [0, t_m) \text{ and } \forall\, m \in \mathbb{N} \tag{22}$$

where $c > 0$ is a constant independent of t and m. From the boundedness above it follows that we can prolong $u_m(t)$, $v_m(t)$ to the whole interval $[0,T]$ and the estimate in (22) remains valid now for all $t \in [0,T]$. Therefore,

$$(u_m) \text{ and } (v_m) \quad \text{are bounded in} \quad L^\infty(0,T;H_0^1(\Omega)) \tag{23}$$

$$(u_m') \text{ and } (v_m') \quad \text{are bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{24}$$

3ª Step: Passage to the Limit

From (23) and (24) we obtain subsequences (u_ν) and (v_ν) of (u_m) and (v_m), respectively, such that

$$u_\nu \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T;H_0^1(\Omega)) \tag{25}$$
$$v_\nu \overset{*}{\rightharpoonup} v \quad \text{in} \quad L^\infty(0,T;H_0^1(\Omega)) \tag{26}$$
$$u_\nu' \overset{*}{\rightharpoonup} u' \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \tag{27}$$
$$v_\nu' \overset{*}{\rightharpoonup} v' \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)). \tag{28}$$

Let $\theta \in \mathcal{D}(0,T)$ and consider $j \in \mathbb{N}$. Multiplying (5) and (6) by θ and integrating in $[0,T]$ we obtain for $\nu \geq j$ that

$$\begin{aligned} &-\int_0^T (u_\nu'(t), \omega_j)\theta'(t)\, dt + \int_0^T ((u_\nu(t), \omega_j))\theta(t)\, dt \\ &+ \int_0^T (v_\nu^2(t) u_\nu(t), \omega_j)\theta(t)\, dt = \int_0^T (f_1(t), \omega_j)\theta(t)\, dt \end{aligned} \tag{29}$$

and

$$\begin{aligned} &-\int_0^T (v_\nu'(t), \omega_j)\theta'(t)\, dt + \int_0^T ((v_\nu(t), \omega_j))\theta(t)\, dt \\ &+ \int_0^T (u_\nu^2(t) v_\nu(t), \omega_j)\theta(t)\, dt = \int_0^T (f_2(t), \omega_j)\theta(t)\, dt. \end{aligned} \tag{30}$$

We will perform the convergence in (29) since in (30) the procedure is analogous. The convergences in (25) and (27) are sufficient to pass the limit in the linear part. Let's see the nonlinear part.

Analysis of the Nonlinear Term

From (23) and (24) it follows that

$u_m\,, v_m$ are bounded in $W = \{u \mid u \in L^2(0,T;H_0^1(\Omega)); u' \in L^2(0,T;L^2(\Omega))\}$.

Thus, by virtue of the Aubin-Lions Theorem, there exists a subsequence of (u_ν), which we will still denote by the same notation such that

$$u_\nu \to u \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \tag{31}$$
$$v_\nu \to v \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \tag{32}$$

From (31) and (32) it follows that

$$v_\nu^2 u_\nu \to v^2 u \quad \text{a.e. in} \quad Q. \tag{33}$$

We will prove next that

$$(v_\nu^2 u_\nu) \quad \text{is bounded in} \quad L^2(Q). \tag{34}$$

Indeed, we have, by virtue of (9), (10), (11) and Hölder's inequality

$$\begin{aligned}
&\int_Q |v_\nu^2 u_\nu|^2 \, dxdt = \int_0^T \int_\Omega |v_\nu|^4 \, |u_\nu|^2 \, dxdt \\
&\leq \int_0^T ||v_\nu^4||_{L^{3/2}(\Omega)} \, ||u_\nu^2||_{L^3(\Omega)} \, dt = \int_0^T ||v_\nu||_{L^6(\Omega)}^4 \, ||u_\nu||_{L^6(\Omega)}^2 \, dt \\
&\leq c_1 \int_0^T ||v_\nu(t)||_{H_0^1(\Omega)}^4 \, ||u_\nu(t)||_{H_0^1(\Omega)}^2 \, dt \leq (c_1 T) \, ||v_\nu||_{L^\infty(0,T;H_0^1(\Omega))}^4 \, ||u_\nu||_{L^\infty(0,T;H_0^1(\Omega))}^4 .
\end{aligned}$$

Now, from (23) and the inequality above we obtain the desired in (34). It follows from (33) and (34) and Lions' Lemma that

$$v_\nu^2 u_\nu \rightharpoonup v^2 u \quad \text{weakly in} \quad L^2(Q). \tag{35}$$

Analogously we prove that

$$u_\nu^2 v_\nu \rightharpoonup u^2 v \quad \text{weakly in} \quad L^2(Q) \tag{36}$$

which is sufficient to pass the limit in the nonlinear part. Then, from (29) and (30) in the limit situation it follows that

$$\begin{aligned}
&- \int_0^T (u'(t), \omega_j)\theta'(t) \, dt + \int_0^T ((u(t), \omega_j))\theta(t) \, dt \\
&+ \int_0^T (v^2(t)u(t), \omega_j)\theta(t) \, dt = \int_0^T (f_1(t), \omega_j)\theta(t) \, dt
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&- \int_0^T (v'(t), \omega_j)\theta'(t) \, dt + \int_0^T ((v(t), \omega_j))\theta(t) \, dt \\
&+ \int_0^T (u^2(t)v(t), \omega_j)\theta(t) \, dt = \int_0^T (f_2(t), \omega_j)\theta(t) \, dt,
\end{aligned} \tag{38}$$

and by the totality of the ω_j's, the expressions above are valid for all $\omega \in H_0^1(\Omega)$. From (37) and (38) it follows then that

$$\begin{aligned}
u'' - \Delta u + v^2 u = f_1 \quad &\text{in} \quad \mathcal{D}'(0,T; H^{-1}(\Omega)) \\
v'' - \Delta v + u^2 v = f_2 \quad &\text{in} \quad \mathcal{D}'(0,T; H^{-1}(\Omega)),
\end{aligned}$$

or even,

$$u'' - \Delta u + v^2 u = f_1 \quad \text{in} \quad L^2(0,T; H^{-1}(\Omega)) \tag{39}$$

$$v'' - \Delta v + u^2 v = f_2 \quad \text{in} \quad L^2(0,T; H^{-1}(\Omega)). \tag{40}$$

4ª Step: Initial Conditions

Obtained in the usual manner.

5ª Step: Uniqueness

Let (u_1, v_1) and (u_2, v_2) be solutions of (1). Then

$$w = u_1 - u_2 \quad \text{and} \quad \hat{w} = v_1 - v_2 \tag{41}$$

verify

$$\left|\begin{array}{l} w'' - \Delta w = v_2^2 u_2 - v_1^2 u_1 \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)) \\ \hat{w}'' - \Delta \hat{w} = u_2^2 v_2 - u_1^2 v_1 \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)) \\ w = 0, \;\; \hat{w} = 0 \quad \text{on} \quad \Sigma \\ w(0) = 0, \;\; \hat{w}(0) = 0 \quad \text{in} \quad \Omega \\ w'(0) = 0, \;\; \hat{w}'(0) = 0 \quad \text{in} \quad \Omega. \end{array}\right. \tag{42}$$

Let $s \in [0,T]$. We define

$$\psi(t) = \begin{cases} -\int_t^s w(\xi)d\xi; & 0 \leq t \leq s \\ 0; & s \leq t \leq T \end{cases} \quad \rho(t) = \begin{cases} -\int_t^s \hat{w}(\xi)d\xi; & 0 \leq t \leq s \\ 0; & s \leq t \leq T. \end{cases} \tag{43}$$

Letting ψ' and ρ' be the distributional derivatives of ψ and ρ, we have

$$\psi(t) = \begin{cases} w(t); & 0 \leq t \leq s \\ 0; & s \leq t \leq T \end{cases} \qquad \rho(t) = \begin{cases} \hat{w}(t); & 0 \leq t \leq s \\ 0; & s \leq t \leq T \end{cases} \tag{44}$$

From the expressions in (43) and (44) we have that

$$\psi, \rho, \psi', \rho' \in L^\infty(0,T;H_0^1(\Omega))$$

which implies that

$$\psi, \rho \in C^0([0,T];H_0^1(\Omega)).$$

Composing $(42)_1$ with ψ and $(42)_2$ with ρ it follows that

$$\begin{aligned} &\int_0^s \langle w''(t), \psi(t)\rangle_{H^{-1},H_0^1}\,dt + \int_0^s ((w(t),\psi(t)))\,dt \\ &= \int_0^s (v_2^2(t)u_2(t) - v_1^2(t)u_1(t), \psi(t))\,dt \end{aligned} \tag{45}$$

and

$$\begin{aligned} &\int_0^s \langle \hat{w}''(t), \rho(t)\rangle_{H^{-1},H_0^1}\,dt + \int_0^s ((\hat{w}(t),\rho(t)))\,dt \\ &= \int_0^s (u_2^2(t)v_2(t) - u_1^2(t)v_1(t), \rho(t))\,dt \end{aligned} \tag{46}$$

Integrating by parts the first integrals in (45) and (46) it results that

$$\begin{aligned} \langle w'(t), \psi(t)\rangle\Big|_{t=0}^{t=s} - \int_0^s (w'(t),\psi'(t))\,dt + \int_0^s ((\psi'(t),\psi(t)))\,dt \\ = \int_0^s (v_2^2 u_2 - v_1^2 u_1, \psi)\,dt \end{aligned} \tag{47}$$

$$\langle \hat{w}'(t), \rho(t)\rangle\Big|_{t=0}^{t=s} - \int_0^s (\hat{w}'(t), \rho'(t))\,dt + \int_0^s ((\rho'(t), \rho(t)))\,dt \\ = \int_0^s (u_2^2 v_2 - u_1^2 v_1, \rho)\,dt, \tag{48}$$

where in the second integrals we used the fact that $\psi' = w$ and $\rho' = \hat{w}$ in $[0,s]$. Now since $\psi(s) = 0$, $\rho(s) = 0$, $w'(0) = 0$ and $\hat{w}'(0) = 0$ from (47) and (48) we obtain

$$-\int_0^s (w'(t), w(t))\,dt + \int_0^s ((\psi'(t), \psi(t)))\,dt = \int_0^s (v_2^2 u_2 - v_1^2 u_1, \psi)\,dt$$
$$-\int_0^s (\hat{w}'(t), \hat{w}(t))\,dt + \int_0^s ((\rho'(t), \rho(t)))\,dt = \int_0^s (u_2^2 v_2 - u_1^2 v_1, \rho)\,dt$$

where in the first integrals we used the fact that $\psi' = w$ and $\rho' = \hat{w}$ in $[0,s]$.

Whence

$$-\int_0^s \frac{1}{2}\frac{d}{dt}|w(t)|^2\,dt + \int_0^s \frac{1}{2}\frac{d}{dt}||\psi(t)||^2\,dt = \int_0^s (v_2^2 u_2 - v_1^2 u_1, \psi)\,dt$$
$$-\int_0^s \frac{1}{2}\frac{d}{dt}|\hat{w}(t)|^2\,dt + \int_0^s \frac{1}{2}\frac{d}{dt}||\rho(t)||^2\,dt = \int_0^s (u_2^2 v_2 - u_1^2 v_1, \rho)\,dt,$$

or even,

$$-\frac{1}{2}|w(s)|^2 + \frac{1}{2}|w(0)|^2 + \frac{1}{2}||\psi(s)||^2 - \frac{1}{2}||\psi(0)||^2 = \int_0^s (v_2^2 u_2 - v_1^2 u_1, \psi)dt$$
$$-\frac{1}{2}|\hat{w}(s)|^2 + \frac{1}{2}|\hat{w}(0)|^2 + \frac{1}{2}||\rho(s)||^2 - \frac{1}{2}||\rho(0)||^2 = \int_0^s (u_2^2 v_2 - u_1^2 v_1, \rho)dt$$

which implies:

$$|w(s)|^2 + ||\psi(0)||^2 = 2\int_0^s (v_1^2 u_1 - v_2^2 u_2, \psi)\,dt$$
$$|\hat{w}(s)|^2 + ||\rho(0)||^2 = 2\int_0^s (u_1^2 v_1 - u_2^2 v_2, \rho)\,dt.$$

Thus,

$$|w(s)|^2 + ||\psi(0)||^2 = 2\int_0^s (v_1^2 u_1 - v_1^2 u_2 + v_1^2 u_2 - v_2^2 u_2, \psi)\,dt \tag{49}$$

$$|\hat{w}(s)|^2 + ||\rho(0)||^2 = 2\int_0^s (u_1^2 v_1 - u_1^2 v_2 + u_1^2 v_2 - u_2^2 v_2, \rho)\,dt. \tag{50}$$

However

$$\begin{aligned} &2\int_\Omega (v_1^2 u_1 - v_1^2 u_2 + v_1^2 u_2 - v_2^2 u_2)\psi\,dx \\ &= 2\int_\Omega \{v_1^2(u_1 - u_2) + (v_1^2 - v_2^2)u_2\}\psi\,dx \\ &\le \int_\Omega |v_1|^2 |\omega|\,|\psi(t)|\,dx + \int_\Omega |u_2 v_1 + u_2 v_2|\,|\hat{\omega}(t)|\,|\psi(t)|\,dx \\ &\le k\big[||v_1||_{L^6(\Omega)}^3\,|\omega|_{L^2(\Omega)}\,||\psi(t)|| + ||u_2 v_1 + u_2 v_2||_{L^3(\Omega)}\,|\hat{\omega}|_{L^2(\Omega)}\,||\psi(t)||\big]. \end{aligned}$$

Since v_1, v_2 and $u_2 \in L^\infty(0,T;H_0^1(\Omega))$ then from (9) and the inequality above it follows that $\exists\, c_1 > 0$ such that

$$2\int_\Omega (v_1^2u_1 - v_1^2u_2 + v_1^2u_2 - v_2^2u_2)\psi\, dx \leq 2c_1\big[|w| + |\hat{w}|\big]||\psi||. \tag{51}$$

Analogously

$$\begin{aligned} &2\int_\Omega (u_1^2v_1 - u_1^2v_2 + u_1^2v_2 - u_2^2v_1)\rho\, dx \\ &\leq 2\int_\Omega \big[|u_1^2|\,|\underbrace{v_1 - v_2}_{\hat{w}}| + |v_2|\,|u_1+u_2|\,|\underbrace{u_1 - u_2}_{w}|\big]||\rho||. \end{aligned}$$

Thus, $\exists\, c_2 > 0$ such that:

$$2(u_1^2v_1 - u_1^2v_2 + u_1^2v_2 - u_2^2v_2, \rho)_{L^2(\Omega)} \leq 2c_2\big[|w| + |\hat{w}|\big]||\rho||. \tag{52}$$

From (49), (50), (51) and (52) it follows that

$$|w(s)|^2 + ||\psi(0)||^2 \leq 2c_1\int_0^s \big[|w(t)|\,||\psi(t)|| + |\hat{w}(t)|\,||\psi(t)||\big]dt \tag{53}$$

$$|\hat{w}(s)|^2 + ||\rho(0)||^2 \leq 2c_2\int_0^s \big[|w(t)|\,||\rho(t)|| + |\hat{w}(t)|\,||\psi(t)||\big]dt \tag{54}$$

Define:

$$\omega_1(t) = \int_0^s (\omega(\xi)\, d\xi.$$

We have, for all $t \in [0,s]$,

$$\psi(t) = -\int_t^s \omega(\xi)\, d\xi = -\Big[\int_0^s \omega(\xi)\, d\xi - \int_0^t \omega(\xi)\, d\xi\Big] = \omega_1(t) - \omega_1(s). \tag{55}$$

In this way

$$\psi(0) = \underbrace{\omega_1(0)}_{=0} - \omega_1(s) = \omega_1(s). \tag{56}$$

Substituting (56) and (55) in (53) it follows that

$$\begin{aligned} &|w(s)|^2 + ||w_1(s)||^2 \\ &\leq 2c_1\int_0^s \big[|w(t)|\,||w_1(t) - w_1(s)|| + |\hat{w}(t)|\,||w_1(t) - w_1(s)||\big]dt. \end{aligned} \tag{57}$$

Analogously, setting

$$\hat{\omega}_1(t) = \int_0^t \hat{\omega}(\xi)\, d\xi$$

we have

$$\begin{aligned} &|\hat{w}(s)|^2 + ||\hat{w}_1(s)||^2 \\ &\leq 2c_2\int_0^s \big[|w(t)|\,||\hat{w}_1(t) - \hat{w}_1(s)|| + |\hat{w}(t)|\,||\hat{w}_1(t) - \hat{w}_1(s)||\big]dt. \end{aligned} \tag{58}$$

From (57) we can write that

$$\begin{aligned}
&|w(s)|^2 + ||w_1(s)||^2 \\
\leq\ & 2c_1 \bigg\{ \int_0^s \big[|w(t)|\, ||w_1(t)|| + |w(t)|\, ||w_1(s)|| + |\hat{w}(t)|\, ||w_1(t)|| + |\hat{w}(t)|\, ||w_1(s)|| \big] dt \bigg\} \\
\leq\ & c_1 \bigg\{ \int_0^s |w(t)|^2 dt + \int_0^s ||w_1(t)||^2 dt + 2\int_0^s |w(t)|\, ||w_1(s)||\, dt + \int_0^t |\hat{w}(t)|^2 dt \\
&\qquad + \int_0^s ||w_1(t)||^2 dt + 2\int_0^s |\hat{w}(t)|\, ||w_1(s)||\, dt \bigg\}.
\end{aligned} \tag{59}$$

But

$$\begin{aligned}
&2c_1 \int_0^s |w(t)|\, ||w_1(s)||\, dt = 2c_1 \int_0^s \sqrt{4sc_1}\, |w(t)| \frac{1}{\sqrt{4sc_1}}\, ||\omega_1(s)||\, ds \\
&\leq 2c_1 \bigg\{ \int_0^s 2sc_1 |w(t)|^2\, dt + \int_0^s \frac{1}{8sc_1}\, ||\omega_1(s)||^2\, dt \bigg\} \\
&= 4sc_1 \int_0^s |w(t)|^2\, dt + \frac{1}{4}\, ||\omega_1(s)||^2.
\end{aligned} \tag{60}$$

Also,

$$2c_1 \int_0^s |\hat{w}(t)|\, ||w_1(s)||\, dt \leq 4sc_1 \int_0^s |\hat{w}(t)|^2\, dt + \frac{1}{4}\, ||\omega_1(s)||^2. \tag{61}$$

From (59), (60) and (61) we obtain

$$\begin{aligned}
|w(s)|^2 + ||w_1(s)||^2 \leq \frac{1}{4}\, ||w_1(s)||^2 \\
+ c_3 \int_0^s \big(|w(t)|^2 + |\hat{w}(t)|^2 + ||w_1(t)||^2 \big)\, dt.
\end{aligned} \tag{62}$$

Analogously, from (58) we arrive at

$$\begin{aligned}
|\hat{w}(s)|^2 + ||\hat{w}_1(s)||^2 \leq \frac{1}{4}\, ||\hat{w}_1(s)||^2 \\
+ c_4 \int_0^s \big(|w(t)|^2 + |\hat{w}(t)|^2 + ||\hat{w}_1(t)||^2 \big)\, dt.
\end{aligned} \tag{63}$$

Summing (62) and (63) we obtain

$$\begin{aligned}
|w(s)|^2 + \frac{1}{2}\, ||w_1(s)||^2 + \hat{|}w(s)|^2 + \frac{1}{2}\, ||\hat{w}_1(s)||^2 \\
\leq c_5 \int_0^s \big(|w(t)|^2 + |\hat{w}(t)|^2 + ||w_1(t)||^2 + ||\hat{w}_1(t)||^2 \big) dt.
\end{aligned}$$

It results from the inequality above, by Gronwall, that

$$|\omega(s)|^2 + \frac{1}{2}\, ||\omega_1(s)||^2 + |\hat{w}(s)|^2 + ||\hat{w}_1(s)||^2 = 0, \quad \forall\, s \in [0, T]$$

that is, $w(s) = 0$ and $\hat{w}(s) = 0$ in $L^2(\Omega)$. Thus: $u_1 = u_2$ and $v_1 = v_2$ which concludes the proof. $\square$

Chapter 13

The Monotonicity Method

13.1 The Browder-Minty-Visik Theorem

In this paragraph we will demonstrate an important theorem due to Browder-Minty-Visik. Before that, however, we need some definitions and preliminary results.

Definition 1. *Let V be a Banach space, V' the dual of V and $A\colon V \to V'$ a map.*
(i) We say that A is monotone when:

$$\langle A(u) - A(v), u - v\rangle_{V',V} \geq 0, \quad \forall\, u, v \in V.$$

(ii) We say that A is hemicontinuoues when for any u, v and w in V, the map $\psi\colon \mathbb{R} \to \mathbb{R}$ defined by:

$$\psi(\lambda) = \langle A(u + \lambda v), w\rangle_{V',V}$$

is continuous.
(iii) We say that A is coercive if:

$$\lim_{||v|| \to +\infty} \frac{\langle A(v), v\rangle}{||v||} = +\infty.$$

(iv) We say that A is bounded when A maps bounded sets of V into bounded sets of V', that is, for any $S \subset V$ bounded in V we have that $A(S) \subset V'$ is bounded in V'.

Lemma 1: (Visik). If the map $P\colon \mathbb{R}^m \to \mathbb{R}^m$ is continuous and $(P(\xi), \xi)_{\mathbb{R}^m} \geq 0$, $\forall\, \xi \in \mathbb{R}^m$ such that $||\xi|| = \rho$, for some $\rho > 0$; then $\exists\, \xi \in \overline{B_\rho(0)}$ such that $P(\xi) = 0$.

Proof: See page 112.

Lemma 2: Let V be a reflexive and separable Banach space and consider $A\colon V \to V'$ a map. If A is monotone, hemicontinuous and bounded (cf. definition 1) then A is continuous from $(V, \tau_{\text{strong}})$ into $(V', \tau_{\text{weak*}})$, that is, $A(u_\nu) \overset{*}{\rightharpoonup} Au$ in V' whenever $u_\nu \to u$ strongly in V.

Proof: Let $(u_\nu) \subset V$ be such that

$$u_\nu \to u \quad \text{in} \quad V \tag{2}$$

and, by contradiction, suppose that

$$Au_\nu \overset{*}{\not\rightharpoonup} Au. \tag{3}$$

It follows from (3) that $\exists\, v_0 \in V$ such that:

$$a_\nu = \langle Au_\nu, v_0\rangle \not\longrightarrow \langle Au, v_0\rangle = a.$$

Therefore, we guarantee the existence of an $\varepsilon_0 > 0$ such that $\forall\, k \in \mathbb{N}$, $\exists\, a_{\nu(k)}$ satisfying

$$|a_{\nu(k)} - a| \geq \varepsilon_0$$

that is,

$$|\langle Au_{\nu(k)}, v_0\rangle - \langle Au, v_0\rangle| \geq \varepsilon_0\,. \tag{4}$$

On the other hand, since A is bounded it follows that $(Au_{\nu(k)})_k$ is a bounded sequence in V'. Since V is a separable Banach space, it follows that there exists a subsequence of $(u_{\nu(k)})_k$ which we will still denote by the same symbol such that

$$Au_{\nu(k)} \overset{*}{\rightharpoonup} f \quad \text{in} \quad V'. \tag{5}$$

By the property of the elements of $(Au_{\nu(k)})$ given in (4), it follows that $Au \neq f$. However, setting $(Au_{\nu(k)})_k = (Au_\mu)_\mu$ we claim:

$$\langle f, u - v\rangle \geq \langle Au, u - v\rangle\,; \quad \forall\, v \in V. \tag{6}$$

Indeed, let

$$w = (1-\theta)u + \theta v; \quad \theta \in]0,1[\,.$$

We have, given the monotonicity of A, that

$$\langle Au_\mu - Aw, u_\mu - w\rangle \geq 0.$$

Whence

$$\langle Au_\mu - Aw,\ u_\mu - ((1-\theta)u + \theta v)\rangle \geq 0,$$

that is,

$$\langle Au_\mu - Aw,\ u_\mu - (u + \theta(v - u))\rangle \geq 0$$

or even,

$$\langle Au_\mu - Aw,\ u_\mu - u - \theta(v - u)\rangle \geq 0.$$

It follows from this that

$$\langle Au_\mu, u_\mu - u\rangle - \theta\langle Au_\mu, v - u\rangle - \langle Aw, u_\mu - u\rangle + \theta\langle Aw, v - u\rangle \geq 0$$

and, therefore,

$$\theta\langle Au_\mu, u - v\rangle \geq -\langle Au_\mu, u_\mu - u\rangle + \langle Aw, u_\mu - u\rangle - \theta\langle Aw, v - u\rangle.$$

Taking the limit in the inequality above as $\mu \to +\infty$ results from (2) and (5) that

$$\theta\langle f, u - v\rangle \geq -\theta\langle Aw, v - u\rangle.$$

Dividing by θ we obtain

$$\langle f, u - v\rangle \geq \langle Aw, u - v\rangle; \quad \forall\, v \in V,$$

or better,

$$\langle f, u - v\rangle \geq \langle A(u + \theta(v - u)), u - v\rangle; \quad \forall\, v \in V.$$

By the hemicontinuity of A and taking the limit as $\theta \to 0$ we obtain, from the inequality above that

$$\langle f, u - v\rangle \geq \langle Au, u - v\rangle; \quad \forall\, v \in V,$$

which proves (6).

Consider $\lambda > 0$ and $z \in V$. Then, from (6) and in particular for $v = u - \lambda z$, it follows that

$$\langle f, \lambda z\rangle \geq \langle Au, \lambda z\rangle.$$

Whence

$$\langle f, z\rangle \geq \langle Au, z\rangle; \quad \forall\, z \in V. \tag{7}$$

Analogously, taking $v = u - \lambda z$, $\lambda < 0$ and $z \in V$, we obtain

$$\langle f, z\rangle \leq \langle Au, z\rangle; \quad \forall\, z \in V. \tag{8}$$

From (7) and (8) we conclude that

$$Au = f,$$

which is a contradiction. This concludes the proof. □

Theorem 1. (Browder-Minty-Visik). Let V be a reflexive and separable Banach space and V' its dual. If $A\colon V \to V'$ is a monotone, hemicontinuous, bounded and coercive map then A is surjective.

Proof: Let $(\omega_\nu)_{\nu\in\mathbb{N}}$ be a basis of V, that is,

(i) $(w_\nu)_\nu$ constitutes a linearly independent set.

(ii) The subspace spanned by $(w_\nu)_\nu$ is dense in V.

Let $f \in V'$. Our aim is to prove that there exists $u \in V$ such that $Au = f$. Set

$$V_m = [w_1, w_2, \dots, w_m]$$

and consider, initially, the finite dimensional problem:

$$\left|\begin{aligned} &u_m \in V_m \\ &\langle Au_m, v\rangle_{V'_m, V_m} = \langle f, v\rangle_{V'_m, V_m}\,; \quad \forall\, v \in V_m. \end{aligned}\right. \tag{9}$$

We will prove next that problem (9) admits a solution u_m for all $m \in \mathbb{N}$. For this, fixed $m \in \mathbb{N}$, define the map

$$\begin{aligned} P\colon V_m &\to V'_m \\ v &\mapsto P(v) = (A(v) - f)|_{V_m} \end{aligned} \tag{10}$$

that is, we are restricting the functional $A(v) - f \in V'$ to the space V_m.

Thus,

$$P(v) \in V'_m$$

and, consequently,

$$\langle P(v), w\rangle_{V'_m,V_m} = \langle A(v) - f, w\rangle_{V',V}\,; \quad \forall\, w \in V_m\,. \tag{11}$$

It follows from (11) that:
(i) P is hemicontinuous since, A being hemicontinuous, the map

$$\begin{aligned}\psi(\lambda) &= \langle P(u+\lambda v), w\rangle_{V'_m,V_m} = \\ &= \langle A(u+\lambda v) - f, w\rangle_{V',V}\,; \quad \lambda \in \mathbb{R},\end{aligned}$$

is continuous for any u, v and $w \in V_m$.
(ii) P is monotone since, A being monotone, it follows that

$$\langle P(u) - P(v), u - v\rangle_{V'_m,V_m} = \langle A(u) - f - A(v) + f, u - v\rangle_{V',V} \geq 0; \quad \forall\, u, v, w \in V_m\,.$$

(iii) P is bounded since, $S \subset V_m$ being a bounded set, then, $\forall\, v \in S$ we have due to the inclusion " $V' \subset V'_m$" that

$$\begin{aligned}||P(v)||_{V'_m} &= ||(A(v) - f)|_{V'_m}||_{V'_m} \leq ||A(v) - f||_{V'} \\ &\leq ||A(v)||_{V'} + ||f||_{V'} \leq c + ||f||_{V'}\,.\end{aligned}$$

From (i), (ii) and (iii) it follows by Lemma 2 that the map (10) is continuous from $(V_m, \tau_{\text{strong}})$ into $(V'_m, \tau_{\text{weak*}})$. However, since V_m has finite dimension, the strong and weak-* topologies coincide. We conclude then that the map given in (10) is continuous.

Our aim now is to apply Lemma 1 and conclude that $\exists\, \rho > 0$ and $v_m \in V_m$ such that $P(v_m) = 0$, that is, $\exists\, v_m \in V_m$ such that $A(v_m) = f$ in V'_m which will prove the existence of a solution to (9). Note that at this moment we are using the fact that every vector space of finite dimension m, fixed a basis, is isomorphic to $\mathbb{R}^m$

We must prove then that $\exists\, \rho > 0$ such that

$$(P(v), v)_{V_m} \geq 0; \quad \forall\, v \in V_m \text{ with } ||v|| = \rho. \tag{12}$$

Indeed, since $f \in V'$ we have, in particular, that

$$|\langle f, v\rangle| \leq ||f||_{V'}\, ||v|| \leq c||v||, \quad \forall\, v \in V_m \quad (c > 0).$$

Whence

$$-\langle f, v\rangle \geq -c||v||; \quad \forall\, v \in V_m\,. \tag{13}$$

On the other hand, A being coercive, then

$$\lim_{||v|| \to +\infty} \frac{\langle Av, v\rangle}{||v||} = +\infty.$$

Thus, given $M > 0$, $\exists\, \delta > 0$ such that if $v \in V$ and $||v|| \geq \delta$ then

$$\frac{\langle Av, v\rangle}{||v||} \geq M.$$

In particular, for the $c > 0$ given above, $\exists\, \rho > 0$ such that if $v \in V_m$ and $||v|| \geq \rho$ then

$$\frac{\langle Av, v\rangle}{||v||} \geq c. \tag{14}$$

Thus, from (13) and (14) it follows for all $v \in V_m$ such that $||v|| \geq \rho$, that

$$\begin{aligned}(P(v), v)_{V_m} &= \langle (A(v) - f)|_{V_m}, v\rangle_{V'_m, V_m}\\ &= \langle A(v), v\rangle_{V',V} - \langle f, v\rangle_{V',V}\\ &\geq c||v|| - c||v|| = 0.\end{aligned}$$

It follows from Lemma 1 that $\exists\, v_m \in \overline{B_\rho(0)} \subset V_m$ such that $P(v_m) = 0$, that is, $Au_m = f$ which proves the existence of a solution to the finite dimensional problem in (9).

Our next step is to pass the limit in the approximate problem. For this, we need estimates as we will see next.

From (9), in particular, for $v = u_m$ it follows that

$$\langle A(u_m), u_m\rangle = \langle f, u_m\rangle \leq c||u_m||; \quad \forall\, m \in \mathbb{N}. \tag{15}$$

It follows from (15) that (u_m) is bounded in V. Indeed, otherwise, there would exist a subsequence (u_ν) of (u_m) such that $||u_\nu|| \to +\infty$ when $\nu \to +\infty$. By the coercivity of A it follows that

$$\lim_{\nu \to +\infty} \frac{\langle A(u_\nu), u_\nu\rangle}{||u_\nu||} = +\infty.$$

Thus, for the $c > 0$ given above $\exists\, \rho > 0$ such that if $||u_\nu|| > \rho$ then

$$\frac{\langle A(u_\nu) u_\nu\rangle}{||u_\nu||} > c \tag{16}$$

which contradicts (15).

Therefore

$$(u_m) \quad \text{is bounded in} \quad V. \tag{17}$$

Since A is bounded, by hypothesis, it follows from (17) that

$$(A(u_m)) \quad \text{is bounded in} \quad V'. \tag{18}$$

Since V is reflexive and separable there exists (u_μ) subsequence of (u_m) such that

$$u_\mu \rightharpoonup u \quad \text{weakly in} \quad V \tag{19}$$

and

$$A(u_\mu) \overset{*}{\rightharpoonup} \chi \quad \text{weakly * in} \quad V'. \tag{20}$$

We have from (15) that, for each $\mu \in \mathbb{N}$,

$$\langle A(u_\mu), u_\mu\rangle = \langle f, u_\mu\rangle.$$

Since the right side of the equality above converges to $\langle f, u\rangle$ it follows that the left side converges to the same limit, that is,

$$\lim_{\mu \to +\infty} \langle A(u_\mu), u_\mu\rangle = \langle f, u\rangle. \tag{21}$$

Fix $j \in \mathbb{N}$. Then, for all $\mu \geq j$ we have from (9) that

$$\langle Au_\mu, w_j \rangle = \langle f, w_j \rangle .$$

Taking the limit in the equality above it follows from (20) that

$$\langle \chi, w_j \rangle = \langle f, w_j \rangle; \quad \forall\, j \in \mathbb{N}.$$

By the density of $[w_\nu]$ in V we obtain

$$\langle \chi, v \rangle = \langle f, v \rangle; \quad \forall\, v \in V \quad \text{ and, therefore, } \quad \chi = f. \tag{22}$$

From (22), in particular, for u we have $\langle \chi, u \rangle = \langle f, u \rangle$ and from (21) it follows that

$$\lim_{\mu \to +\infty} \langle A(u_\mu), u_\mu \rangle = \langle \chi, u \rangle. \tag{23}$$

To conclude the theorem it remains to prove that

$$\chi = Au. \tag{24}$$

Indeed, by the monotonicity of A:

$$\langle A(u_\mu) - A(v), u_\mu - v \rangle \geq 0, \quad \forall\, \mu \in \mathbb{N} \text{ and } \forall\, v \in V.$$

Whence

$$\langle A(u_\mu), u_\mu \rangle - \langle A(u_\mu), v \rangle - \langle A(v), u_\mu - v \rangle \geq 0.$$

In the limit situation it follows from (19), (20) and (23) that

$$\langle \chi, u \rangle - \langle \chi, v \rangle - \langle A(v), u - v \rangle \geq 0,$$

that is,

$$\langle \chi - A(v), u - v \rangle \geq 0, \quad \forall\, v \in V.$$

Let $\lambda > 0$ and $w \in V$. We have for $v = u - \lambda w$ that

$$\langle \chi - A(u - \lambda w), w \rangle \geq 0, \quad \forall\, w \in V.$$

By the hemicontinuity of A it follows in the limit as $\lambda \to 0$ that

$$\langle \chi - A(u), w \rangle \geq 0, \quad \forall\, w \in V.$$

Analogously, considering $v = u - \lambda w$; $\lambda < 0$ and $w \in v$, it follows that

$$\langle \chi - A(u), w \rangle \leq 0, \quad \forall\, w \in V.$$

Whence

$$\langle \chi - A(u), w \rangle = 0, \quad \forall\, w \in V$$

and therefore

$$\chi = A(u)$$

which proves (24). From (22) it follows that

$$A(u) = f \quad \text{in} \quad V'. \tag{25}$$

This conclues the proof of the theorem. □

Naturally, equation (25) admits a unique solution if:

$$\langle A(u) - A(v), u - v \rangle > 0, \quad \forall\, v \in V,\; u \neq v.$$

Indeed, let $u, u^* \in V$ be solutions of (25) such that $u \neq u^*$. Then, $A(u) - A(u^*) = f - f = 0$ and therefore

$$0 = \langle A(u) - A(u^*), u - u^* \rangle > 0$$

which is a contradiction!

We will see, next, a more sophisticated uniqueness result.

Theorem 2: Under the hypotheses of Theorem 1 and, furthermore, assuming that

$$V \quad \text{is strictly convex} \tag{26}$$

and

$$A(u) = A(v) \Rightarrow ||u|| = ||v|| \tag{27}$$

Then, equation (25) admits a unique solution.

Proof: Recall that since V is strictly convex then $\forall\, u, v \in V$ with $||u|| = ||v|| = 1$ and $u \neq v$ we have

$$||\lambda u + (1-\lambda)v||_V < 1, \quad \forall\, \lambda \in]0,1[, \tag{28}$$

We will prove that

$$u \text{ is a solution of (25) if and only if } \langle A(v) - f, v - u \rangle \geq 0, \forall\, v \in V. \tag{29}$$

Indeed, if (25) occurs then $Au = f$ and, therefore,

$$\langle A(v) - f, v - u \rangle = \langle A(v) - A(u), v - u \rangle \geq 0,$$

where the last inequality is satisfied given that A is monotone.

Conversely, suppose that

$$\langle A(v) - f, v - u \rangle \geq 0, \quad \forall\, v \in V. \tag{30}$$

Consider, then $\lambda > 0, \quad w \in V$ and $v = u + \lambda w$. Then from (30) it follows that

$$\langle A(u + \lambda w) - f, w \rangle \geq 0.$$

Letting $\lambda \to 0$ we deduce, due to the hemicontinuity of A, that $\langle A(u) - f, w \rangle \geq 0$. Analogously, considering $\lambda < 0$, we deduce $\langle Au - f, w \rangle \leq 0$. Hence

$$\langle A(u) - f, w \rangle = 0, \quad \forall\, w \in V,$$

that is, $Au = f$ in V'. This proves (29).

Let us define, for each $v \in V$, the following set

$$S_v = \{u \in V;\; \langle A(v) - f, v - u \rangle \geq 0\}. \tag{31}$$

We claim that S_v is convex. Indeed, let $u_1, u_2 \in S_v$ and $\lambda \in [0,1]$. We have:

$$\begin{aligned}&\langle A(v) - f, v - (\lambda u_1 + (1-\lambda)u_2)\rangle \\ &= \langle A(v) - f, \lambda v + (1-\lambda)v - \lambda u_1 - (1-\lambda)u_2\rangle \\ &= \lambda\langle A(v) - f, v - u_1\rangle + (1-\lambda)\langle A(v) - f, v - u_2\rangle \geq 0\end{aligned}$$

which proves the convexity of S_v.

Letting

$$E = \bigcap_{v\in V} S_v \tag{32}$$

it follows from (29) and (31) that:

$$E = \bigcap_{v\in V} S_v = \{u \in V;\ Au = f\}.$$

Since S_v is convex it follows that E is also convex.

Consider, finally, $u, u^* \in E$ solutions of (25) and suppose that $u \neq u^*$. We have

$$A(u) = f \quad \text{and} \quad A(u^*) = f.$$

From (27) it follows that

$$||u|| = ||u^*||. \tag{33}$$

If $\lambda \in]0,1[$ then by the convexity of the set E given in (32) it follows that

$$\lambda u + (1-\lambda)u^* \in E.$$

Consequently

$$A(\lambda u + (1-\lambda)u^*) = f$$

and from (27) we obtain:

$$||\lambda u + (1-\lambda)u^*|| = ||u|| = ||u^*|| = \rho. \tag{34}$$

We have two cases to consider:

(1$^{\text{st}}$) $\rho \neq 0$.

In this case, $\dfrac{u}{||u||} \neq \dfrac{u^*}{||u^*||}$ and from (28) it follows that

$$\left\|\lambda \frac{u}{||u||} + (1-\lambda)\frac{u^*}{||u^*||}\right\| < 1$$

and from (33) and (34) it follows that

$$\rho = ||\lambda u + (1-\lambda)u^*|| < \rho$$

which is absurd!

(2$^{\text{nd}}$) $\rho = 0$

In this case, from (34) it follows that

$$||u|| = ||u^*|| = 0$$

and, therefore, $u = u^* = 0$. But this is absurd since $u \neq u^*$. The proof is concluded. □

13.2 Duality Mappings

Let E be a Banach space over $\mathbb{R}$, endowed with the norm $||\cdot||$, let $||\cdot||_*$ be the dual norm on the (dual) Banach space E' and consider $\langle\,,\,\rangle$ the duality between E' and E.

Let $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+,\quad r \mapsto \phi(r)$ be a continuous, monotone and strictly increasing map, such that

$$\phi(0) = 0, \quad \text{and} \quad \phi(r) \to +\infty \text{ if } r \to +\infty. \tag{1}$$

Definition 1: A map $J\colon E \to E'$ is called a *duality mapping relative to* ϕ if the following conditions are verified:

$$\langle J(u), u\rangle = ||J(u)||_*\,||u||, \quad \forall\, u \in E. \tag{2}$$

$$||J(u)||_* = \phi(||u||), \quad \forall\, u \in E. \tag{3}$$

Naturally this notion depends on the choice of the norm on E.

Proposition 1: Every duality mapping is monotone.

Proof: Let $u, v \in E$ and $J\colon E \to E'$ a duality mapping. We have, from (2) and (3):

$$\begin{aligned}
&\langle J(u) - J(v), u - v\rangle \\
&= \langle J(u), u\rangle - \langle J(u), v\rangle - \langle J(v), u\rangle + \langle J(v), v\rangle \\
&= ||J(u)||_*\,||u|| - \langle J(u), v\rangle - \langle J(v), u\rangle + ||J(v)||_*\,||v|| \\
&\geq ||J(u)||_*\,||u|| - ||J(u)||_*\,||v|| - ||J(v)||_*\,||u|| + ||J(v)||_*\,||v|| \\
&= (||J(u)||_* - ||J(v)||_*)(||u|| - ||v||) \\
&= (\phi(||u||) - \phi(||v||))(||u|| - ||v||) \geq 0
\end{aligned}$$

where the last inequality holds since ϕ is strictly increasing. This proves the proposition. □

Proposition 2: Let E be a strictly convex Banach space and $J\colon E \to E'$ a duality mapping. Then, J is strictly monotone.

Proof: According to Proposition 1,

$$\langle J(u) - J(v), u - v\rangle \geq 0, \quad \forall\, u, v \in E.$$

Thus, it suffices to prove that

$$\langle J(u) - J(v), u - v\rangle > 0, \quad \forall\, u, v \in E;\ u \neq v. \tag{4}$$

Suppose, by contradiction, that there exist, $u, v \in E,\ u \neq v$, such that

$$\langle J(u) - J(v), u - v\rangle = 0. \tag{5}$$

However, as in the proof of Proposition 1, from (5) it follows that

$$0 = \langle J(u) - J(v), u - v\rangle \geq (\phi(||u||) - \phi(||v||))(||u|| - ||v||) \geq 0.$$

Thus

$$\big(\phi(||u||) - \phi(||v||)\big)\big(||u|| - ||v||\big) = 0. \tag{6}$$

We have two cases to consider:
(i) $\phi(||u||) - \phi(||v||) = 0$.
In this case, $\phi(||u||) = \phi(||v||)$ and therefore $||u|| = ||v||$.
(ii) $||u|| - ||v|| = 0$.
Also we have $||u|| = ||v||$.
In any case, (6) implies that

$$||u|| = ||v||. \tag{7}$$

Note that $||u|| \neq 0$ (and $||v|| \neq 0$) because, otherwise, $u = v = 0$, which is absurd, since $u \neq v$. Thus, from (7) it follows that

$$\frac{u}{||u||} \neq \frac{v}{||v||} \cdot \tag{8}$$

On the other hand, note that from the fact that E is strictly convex and $J(u) \in E'$ ($J(u) \not\equiv 0$) it follows that $||J(u)||_*$ is attained at a unique point of the unit ball. Indeed, from (2) we have that

$$||J(u)||_* = \frac{\langle J(u), u\rangle}{||u||} = \left\langle J(u), \frac{u}{||u||}\right\rangle \tag{9}$$

which shows that $||J(u)||_* = \sup\limits_{||v||\leq 1} \langle J(u), v\rangle$ is attained at the point $w = \dfrac{u}{||u||} \cdot$ We claim that this point is unique. Indeed, suppose there exists $w^* \in E;\ \ ||w^*|| = 1$ such that

$$||J(u)||_* = \langle J(u), w\rangle = \langle J(u), w^*\rangle. \tag{10}$$

Now, since the ball $\overline{B_1(0)}$ is convex it follows that the convex combination $(1 - \lambda)w + \lambda w^* \in \overline{B_1(0)};\ \ \lambda \in]0, 1[$. Since E is strictly convex it follows that

$$||(1 - \lambda)w + \lambda w^*|| < 1. \tag{11}$$

But, from (10) it follows that

$$\begin{aligned} \langle J(u), (1-\lambda)w + \lambda w^*\rangle &= (1-\lambda)\langle J(u), w\rangle + \lambda\langle J(u), w^*\rangle \\ &= \langle J(u), w\rangle = ||J(u)||_*\,, \end{aligned} \tag{12}$$

Thus, from (11) and (12) we have

$$||J(u)||_* = \langle J(u), (1-\lambda)w + \lambda w^*\rangle \leq ||J(u)||_*\, ||(1-\lambda)w + \lambda w^*|| < ||J(u)||_*$$

which is absurd! This proves that the attained point is unique.
It follows from (9) and the above that

$$||J(u)||_* = \left\langle J(u), \frac{u}{||u||}\right\rangle > \left\langle J(u), \frac{v}{||v||}\right\rangle.$$

and from (7) we obtain

$$\langle J(u), v\rangle < \langle J(u), u\rangle. \tag{13}$$

Analogously, we also show that

$$\langle J(v), u\rangle < \langle J(v), v\rangle. \tag{14}$$

Therefore, from (13) and (14) we arrive at

$$\begin{aligned}0 &= \langle J(u) - J(v), u - v\rangle = \langle J(u), u\rangle - \langle J(u), v\rangle - \langle J(v), u\rangle + \langle J(v), v\rangle \\ &> \langle J(u), u\rangle - \langle J(u), u\rangle - \langle J(v), v\rangle + \langle J(v), v\rangle = 0,\end{aligned}$$

which is a contradiction! □

Proposition 3: Let E be a Banach space. There always exists a duality mapping relative to ϕ. This map is uniquely defined if E' is strictly convex.

Proof: Let B_1 be the unit ball of E. For all $u \in \partial B_1$ there exists, according to the Hahn-Banach Theorem an element $u^* \in E'$ such that:

$$||u^*||_* = ||u|| = 1 \quad \text{and} \quad \langle u^*, u\rangle = ||u||^2 = 1. \tag{15}$$

On the other hand, given $v \in E$; $\exists\, \lambda \geq 0$ and $u \in \partial B_1$ such that

$$v = \lambda u. \tag{16}$$

Indeed, if $v = 0$, just take $\lambda = 0$. Now, if $v \neq 0$ then $\dfrac{v}{||v||} \in \partial B_1$ and furthermore,

$$v = ||v|| \cdot \frac{v}{||v||}\cdot \tag{17}$$

Thus, from (17) it follows that $\lambda = ||v|| > 0$ and $u = \dfrac{v}{||v||}\cdot$

Consider, then

$$J\colon E \to E'$$

defined according to (15) and (16) by

$$J(v) = J(\lambda u) = \phi(\lambda) \cdot u^* \tag{18}$$

where we are making a unique choice of u^* so that we have a defined map.

We will prove next, that the operator J defined in (18) satisfies (2) and (3). In fact, from the above it follows that

$$\begin{aligned}\langle J(v), v\rangle &= \langle \phi(\lambda)u^*, \lambda u\rangle = \phi(\lambda)\lambda\langle u^*, u\rangle \\ &= \phi(\lambda)\lambda = \phi(\lambda)\cdot\lambda\cdot||u|| \\ &= \phi(\lambda)\,||\lambda u|| = \phi(\lambda)||u^*||_*\,||\lambda u|| \\ &= ||\phi(\lambda)u^*||_*\,||\lambda u|| \\ &= ||J(v)||_*\,||v||, \qquad \forall\, v \in E.\end{aligned} \tag{i}$$

$$\begin{aligned}||J(v)||_* &= ||\phi(\lambda)u^*||_* = \phi(\lambda)||u^*||_* = \phi(\lambda) \\ &= \phi(\lambda||u||) = \phi(||\lambda u||) = \phi(||v||), \quad \forall\, v \in E.\end{aligned} \tag{ii}$$

Suppose, now, that E' is strictly convex and suppose, by contradiction, that there exist J_1, J_2 duality mappings relative to ϕ with $J_1 \neq J_2$. Thus, there exists $u \in E,\ u \neq 0$ such that $J_1(u) \neq J_2(u)$ ($u \neq 0$ since if $u = 0$ then $||J_1(u)||_* = ||J_2(u)||_* = \phi(||u||) = \phi(0) = 0$ and therefore $J_1(u) = J_2(u)$). Furthermore, since $u \neq 0$ then $||J_1(u)||_* = ||J_2(u)||_* = \phi(||u||) \neq 0$. Whence: $\dfrac{J_1(u)}{||J_1(u)||_*} \neq \dfrac{J_2(u)}{||J_2(u)||_*}$.

Then, if $\lambda \in]0,1[$ we have from (2) and (3)

$$\begin{aligned}
&\left\langle \lambda \frac{J_1(u)}{||J_1(u)||_*} + (1-\lambda)\frac{J_2(u)}{||J_2(u)||_*}, u \right\rangle = \frac{\lambda}{||J_1(u)||_*}\langle J_1(u), u\rangle \\
&+ \frac{1}{||J_2(u)||_*}\langle J_2(u), u\rangle - \frac{\lambda}{||J_2(u)||_*}\langle J_2(u), u\rangle = \frac{1}{||J_2(u)||_*}\langle J_2(u), u\rangle \\
&= \frac{||J_2(u)||_*\,||u||}{||J_2(u)||_*} = ||u||.
\end{aligned} \tag{19}$$

From (19) and the fact that E' is strictly convex it follows that:

$$\begin{aligned}
||u|| &= \left\langle \lambda \frac{J_1(u)}{||J_1(u)||_*} + (1-\lambda)\frac{J_2(u)}{||J_2(u)||_*}, u \right\rangle \\
&\leq \left\| \lambda \frac{J_1(u)}{||J_1(u)||_*} + (1-\lambda)\frac{J_2(u)}{||J_2(u)||_*} \right\|_* ||u|| < ||u||
\end{aligned}$$

which is absurd! Thus, the duality mapping is unique. Thus, for each $u \in B_1$ there exists a unique $u^* \in E'$ satisfying (15). In this way, the map (18) is uniquely defined. This concludes the proof. □

Proposition 4: Let E be a reflexive separable Banach space whose dual E' is strictly convex. The duality mapping J relative to ϕ is hemicontinuous.

Proof: We will prove a more general result:

$$\text{If } v_\nu \to v \text{ in } E \text{ then } J(v_\nu) \overset{*}{\rightharpoonup} J(v) \text{ in } E'. \tag{20}$$

Recall that since E' is strictly convex then the duality mapping J is uniquely defined. In fact such map is given as in (18). Furthermore, according to the construction given in (18) it is sufficient to verify that:

$$\text{If } (u_\nu) \subset \partial B_1 \text{ and } u_\nu \to u \ (u \in \partial B_1) \text{ then } J(u_\nu) \overset{*}{\rightharpoonup} J(u) \text{ in } E'. \tag{21}$$

Indeed, suppose for a moment that (21) holds and consider $v_\nu \to v$ in E. For each $\nu \in \mathbb{N}$ we can write:

$$v_\nu = ||v_\nu||\,\frac{v_\nu}{||v_\nu||} \quad \text{and} \quad v = ||v||\,\frac{v}{||v||},$$

assuming $v_\nu \neq 0$ and $v \neq 0$. Setting

$$u_\nu = \frac{v_\nu}{||v_\nu||} \quad \text{and} \quad u = \frac{v}{||v||}$$

then $(u_\nu) \subset \partial B_1$, $\ u \in \partial B_1$ and furthermore $u_\nu \to u$ in E. It follows from (21) that $J(u_\nu) \overset{*}{\rightharpoonup} J(u)$ in E', that is,

$$\phi(1)\langle u_\nu^*, w\rangle \to \phi(1)\langle u^*, w\rangle; \quad \forall\, w \in E.$$

Since $\phi(1) > 0$. it follows that

$$\langle u_\nu^*, w\rangle \to \langle u^*, w\rangle; \quad \forall\, w \in E. \tag{22}$$

Since ϕ is continuous and since $v_\nu \to v$ in E it follows that:

$$\phi(||v_\nu||) \to \phi(||v||). \tag{23}$$

From (22) and (23) we obtain, then:

$$\phi(||v_\nu||)\langle u_\nu^*, w\rangle \to \phi(||v||)\langle u^*, w\rangle, \forall w \in E.$$

Therefore

$$\langle \phi(||v_\nu||)u_\nu^*, w\rangle \to \langle \phi(||v||)u^*, w\rangle, \forall w \in E$$

that is,

$$\langle J(v_\nu), w\rangle \to \langle J(v), w\rangle, \forall w \in E. \tag{24}$$

Thus, from (24) it follows that

$$J(v_\nu) \overset{*}{\rightharpoonup} J(v) \quad \text{in} \quad E'. \tag{25}$$

Consider, now, the other possibilities:

$$\begin{array}{llll} v \neq 0 & \text{and} & v_\nu = 0 & \text{for a finite number of indices} \\ v = 0 & \text{and} & v_\nu = 0; & \forall\, \nu \in \mathbb{N} \\ v = 0 & \text{and} & v_\nu = 0 & \text{for a finite number of indices} \\ v = 0 & \text{and} & v_\nu = 0 & \text{for an infinite number of indices.} \end{array}$$

In the first case, we disregard the finite number of indices and proceed as above. Let us analyze the other cases: When $v = 0$, we claim that $J(v_\nu) \to 0$ strongly in E'. Indeed, since

$$v_\nu \to 0 \quad \text{in} \quad E$$

then, by the continuity of ϕ it follows that

$$\phi(||v_\nu||) \to \phi(0) = 0.$$

Thus,

$$||J(v_\nu)||_* = \phi(||v_\nu||) \to 0$$

that is, $J(v_\nu) \to 0$ in E', which proves the desired result.

In this way, it is sufficient to prove the claim made in (21).

Consider, then, $(u_\nu) \subset \partial B_1$, $u_\nu \to u_0$, $(u_0 \in \partial B_1)$ and suppose, by contradiction, that $J(u_\nu) \not\rightharpoonup J(u_0)$ weak-star in E'. It follows from this that there exists $v_0 \in E$ and $\varepsilon_0 > 0$ such that $\forall\, k \in \mathbb{N}$, there exists a unique index $\nu(k)$ satisfying

$$|\langle J(u_{\nu(k)}), v_0\rangle - \langle J(u_0), v_0\rangle| \geq \varepsilon_0\,.$$

However, from (3) and from the fact that $(u_{\nu(k)}) \subset (u_\nu) \subset \partial B_1$ we have

$$||J(u_{\nu(k)})||_* = \phi(||u_{\nu(k)}||) = \phi(1) < +\infty.$$

Therefore, we can extract a subsequence (u_μ) of $(u_{\nu(k)})_k$ such that

$$J(u_\mu) \rightharpoonup \chi \quad \text{weakly-* in} \quad E' \tag{26}$$

and by the contradiction hypothesis it follows that $\chi \neq J(u_0)$.

Since $u_\mu \to u_0$, by hypothesis, then from (26) it follows that

$$\langle J(u_\mu), u_\mu \rangle \to \langle \chi, u_0 \rangle \quad \text{when} \quad \mu \to +\infty. \tag{27}$$

However, from (2) we can write that

$$\langle J(u_\mu), u_\mu \rangle = ||J(u_\mu)||_* \, ||u_\mu|| = ||J(u_\mu)||_* \, . \tag{28}$$

On the other hand, from (26) we have that

$$||\chi||_* \leq \underline{\lim} \, ||J(u_\mu)||_* \, . \tag{29}$$

Thus, from (27), (28) and (29) we obtain:

$$||\chi||_* \leq \lim ||J(u_\mu)||_* = \lim \langle J(u_\mu), u_\mu \rangle = \langle \chi, u_0 \rangle \leq ||\chi||_* \, ||u_0|| = ||\chi||_* \, .$$

Whence

$$\langle \chi, u_0 \rangle = ||\chi||_* \, ||u_0|| = ||\chi||_* \tag{30}$$

and

$$||\chi||_* = \lim ||J(u_\mu)||_* = \phi(1) = \phi(||u_0||). \tag{31}$$

From (30) and (31) it follows that

$$\chi = J(u_0). \tag{32}$$

Indeed, suppose the contrary, that is, suppose that $\chi \neq J(u_0)$. From (31) and from (3) it follows that:

$$||\chi||_* = \phi(||u_0||) = ||J(u_0)||_* \, . \tag{33}$$

Since $\phi(||u_0||) \neq 0$ it follows from (33) that

$$\frac{\chi}{||\chi||_*} \neq \frac{J(u_0)}{||J(u_0)||_*} \, . \tag{34}$$

On the other hand, from (2), (30) and (33) we can write that

$$\langle J(u_0), u_0 \rangle = ||J(u_0)||_* = ||\chi||_* = \langle \chi, u_0 \rangle. \tag{35}$$

Now, if $\lambda \in]0, 1[$ then from (35) we obtain

$$\begin{aligned} &\left\langle \lambda \frac{J(u_0)}{||J(u_0)||_*} + (1-\lambda) \frac{\chi}{||\chi||_*} \, , u_0 \right\rangle \\ &= \frac{\lambda}{||J(u_0)||_*} \langle J(u_0), u_0 \rangle + \frac{1}{||\chi||_*} \langle \chi, u_0 \rangle - \frac{\lambda}{||\chi||_*} \langle \chi, u_0 \rangle \\ &= \frac{1}{||\chi||_*} \langle \chi, u_0 \rangle = 1. \end{aligned}$$

In this way, since E' is strictly convex,

$$1 = \left\langle \lambda \frac{J(u_0)}{||J(u_0)||_*} + (1-\lambda)\frac{\chi}{||\chi||_*}, u_0 \right\rangle \le \left\| \lambda \frac{J(u_0)}{||J(u_0)||_*} + (1-\lambda)\frac{\chi}{||\chi||_*} \right\|_* ||u_0|| < 1$$

which is a contradiction! Thus (32) holds. This concludes the proof. □

Theorem 1: Let E be a separable, reflexive Banach space, strictly convex with strictly convex dual. Let J be the duality mapping from E to E' relative to ϕ. Then given $f \in E'$ there exists a unique $u \in E$ such that

$$J(u) = f$$

that is, the map $J\colon E \to E'$ is a bijection.

Proof: Initially, observe that the existence of the duality mapping J is given by Proposition 3. Furthermore, such map is the unique duality mapping. From (3) we have that

$$||J(u)||_* = \phi(||u||), \quad \forall\, u \in E. \tag{36}$$

Since $S \subset E$ is a bounded subset, it follows from (36) that $J(S)$ is bounded in E'. Indeed, from the boundedness of S it follows that there exists $c > 0$ such that $||u|| \le c$; $\forall\, u \in S$. Whence $\phi(||u||) \le \phi(c)$; $\forall\, u \in S$, which proves that

$$J\colon E \to E' \quad \text{is bounded.} \tag{37}$$

On the other hand, from (1), (2) and (3) we also have that:

$$\lim_{||u|| \to +\infty} \frac{\langle J(u), u\rangle}{||u||} = \lim_{||u|| \to +\infty} ||J(u)||_* = \lim_{||u|| \to +\infty} \phi(||u||) = +\infty \tag{38}$$

that is, J is coercive. By Proposition 1 we have that

$$J\colon E \to E' \quad \text{is monotone} \tag{39}$$

and by Proposition 4 it follows that

$$J\colon E \to E' \quad \text{is hemicontinuous.} \tag{40}$$

Thus, by Browder's Theorem (Theorem 1 §1) and by Proposition 2 of this paragraph, we have that given $f \in E'$; $\exists!\, u \in E$ such that

$$Ju = f$$

which concludes the proof. □

Example 1: Let $E = L^p(\Omega)$, $2 \le p < +\infty$ and $\phi(r) = r^{p-1}$.

Let $L^{p'}(\Omega)$ be the topological dual of $L^p(\Omega)$. Thus

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Define

$$\begin{aligned} J\colon L^p(\Omega) &\to L^{p'}(\Omega) \\ u &\mapsto Ju = |u|^{p-2}\, u. \end{aligned}$$

Note that J is well defined, since if $u \in L^p(\Omega)$, we have that

$$||J(u)||^{p'}_{L^{p'}(\Omega)} = \int_\Omega \left||u|^{p-2}\, u\right|^{p'}\, dx = \int_\Omega \left(|u|^{p-1}\right)^{\frac{p}{p-1}}\, dx = ||u||^p_{L^p(\Omega)}\,.$$

It follows from this that:

$$||J(u)||_{L^{p'}(\Omega)} = ||u||^{p-1}_{L^p(\Omega)}, \tag{41}$$

that is,

$$||J(u)||_{L^{p'}(\Omega)} = \phi(||u||_{L^p(\Omega)}). \tag{42}$$

Furthermore, from (41)

$$\begin{aligned} \langle J(u), u\rangle_{L^{p'},L^p} &= \int_\Omega |u|^{p-2}u^2\, dx = \int_\Omega |u|^p\, dx = ||u||^p_{L^p(\Omega)} \\ &= ||u||^{p-1}_{L^p(\Omega)}\, ||u||_{L^p(\Omega)} \\ &= ||J(u)||_{L^{p'}(\Omega)}\, ||u||_{L^p(\Omega)} \end{aligned} \tag{43}$$

Therefore, from (42) and (43) we conclude that J is a duality mapping relative to ϕ.

Example 2: Let $E = W^{1,p}_0(\Omega)$; $\quad 2 \le p < +\infty$; $\quad \phi(r) = r^{p-1}$, where we are endowing E with the topology

$$||u|| = \left(\sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|^p_{L^p(\Omega)}\, dx\right)^{1/p}.$$

Define

$$\begin{aligned} J\colon W^{1,p}_0(\Omega) &\to W^{-1,p'}(\Omega) \\ u &\mapsto Ju = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\, \frac{\partial u}{\partial x_i}\right) \end{aligned}$$

Observe that J is well defined since

$$\left\|\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\, \frac{\partial u}{\partial x_i}\right\|^{p'}_{L^{p'}(\Omega)} = \int_\Omega \left|\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\, \frac{\partial u}{\partial x_i}\right|^{\frac{p}{p-1}}\, dx = \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p\, dx < +\infty$$

and, therefore,

$$Ju \in W^{-1,p'}(\Omega).$$

On the other hand, for all $\varphi \in \mathcal{D}(\Omega)$ we obtain

$$\begin{aligned} \langle J(u), \varphi\rangle &= \left\langle -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\, \frac{\partial u}{\partial x_i}\right), \varphi\right\rangle \\ &= \sum_{i=1}^n \left\langle \left|\frac{\partial u}{\partial x_i}\right|^{p-2}\, \frac{\partial u}{\partial x_i}, \frac{\partial \varphi}{\partial x_i}\right\rangle \end{aligned}$$

that is,

$$\langle J(u), \varphi\rangle = \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2}\, \frac{\partial u}{\partial x_i}\, \frac{\partial \varphi}{\partial x_i}\, dx. \tag{44}$$

Consider, then, $v \in W_0^{1,p}(\Omega)$. Thus, $\exists\,(\varphi_\nu) \subset \mathcal{D}(\Omega)$ such that

$$\varphi_\nu \to v \quad \text{in} \quad W_0^{1,p}(\Omega). \tag{45}$$

Therefore

$$\frac{\partial \varphi_\nu}{\partial x_i} \to \frac{\partial v}{\partial x_i} \quad \text{in} \quad L^p(\Omega). \tag{46}$$

But from (44), for each $\nu \in \mathbb{N}$, we can write that

$$\langle J(u), \varphi_\nu \rangle = \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial u \varphi_\nu}{\partial x_i}\, dx.$$

Taking the limit in the expression above it follows from (45) and (46) that

$$\langle J(u), v \rangle = \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx,$$

that is,

$$\begin{aligned}
|\langle J(u), v \rangle| &\le \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-1} \left| \frac{\partial v}{\partial x_i} \right| dx \\
&\le \sum_{i=1}^n \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p dx \right)^{1/p'} \left(\int_\Omega \left| \frac{\partial v}{\partial x_i} \right|^p dx \right)^{1/p} \\
&= \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}^{p/p'} \cdot \left\| \frac{\partial v}{\partial x_i} \right\|_{L^p(\Omega)} \\
&\le \left(\sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}^p \right)^{1/p'} \left(\sum_{i=1}^n \left\| \frac{\partial v}{\partial x_i} \right\|_{L^p(\Omega)}^p \right)^{1/p} \\
&\le ||u||_{W_0^{1,p}(\Omega)}^{p-1} ||v||_{W_0^{1,p}(\Omega)}\,; \quad \forall\, v \in W_0^{1,p}(\Omega),
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Thus, we obtained that, $||J(u)||_{W^{-1,p'}(\Omega)} \le ||u||_{W_0^{1,p}(\Omega)}^{p-1}$. On the other hand, since we also have the inverse inequality, given that,

$$\begin{aligned}
||u||_{W_{1,p}(\Omega)}^p &= \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p dx = \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\, dx \\
&= -\sum_{i=1}^n \left\langle \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right), u \right\rangle = \langle Ju, u \rangle \\
&\le ||Ju||_{W^{-1,p'}(\Omega)}\, ||u||_{W_0^{1,p}(\Omega)},
\end{aligned}$$

that is,

$$||u||_{W_0^{1,p}(\Omega)}^{p-1} \le ||Ju||_{W^{-1,p'}(\Omega)},$$

then

$$||J(u)||_{W^{-1,p'}(\Omega)} = ||u||_{W_0^{1,p}(\Omega)}^{p-1} = \phi(||u||_{W_0^{1,p}(\Omega)}), \quad \forall\, u \in W_0^{1,p}(\Omega).$$

The relations above show that the map J is a duality mapping relative to ϕ.

Remark: Theorem 1 leads us naturally to "reflexive" spaces and "strictly convex spaces as well as their dual". In truth, this last hypothesis is not a special restriction as the following result shows, whose proof we will omit in this text.

Theorem 2: Let E be a reflexive Banach space with norm $||\cdot||$. There exists a norm $|||\cdot|||$ equivalent to $||\cdot||$ such that, for this new norm, E is strictly convex as well as its dual endowed with the dual norm $|||\cdot|||_*$.

In order to complement the result above consider the next result.

Theorem 3: (Brézis-Crandall-Pazy). Let E be a reflexive Banach space with norm $||\cdot||$. For all $a > 1$ there exists a norm $||\cdot||_a$ on E that verifies the following conditions:

(i) Endowed with the norm $||\cdot||_a$, E is strictly convex as well as its dual (endowed with the dual norm $||\cdot||_{a,*}$);

(ii) $\frac{1}{a}||\cdot||_a \le ||\cdot|| \le a||\cdot||_a\,; \quad \frac{1}{a}||\cdot||_{a,*} \le ||\cdot||_* \le a||\cdot||_{a,*}\,.$ □

Example 3: Let

$$A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$
$$u \mapsto A(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)$$

be the operator of Example 2. From the above, given $f \in W^{-1,p'}(\Omega)$ there exists by Theorem 1 a unique $u \in W_0^{1,p}(\Omega)$ such that

$$A(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f.$$

Thus, the stationary problem is solved

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \Gamma \end{cases} \tag{48}$$

when $f \in W^{-1,p'}(\Omega)$.

13.3 Gateaux Derivative - Stationary Problems

In this paragraph, we will present a technique to solve stationary problems that do not involve duality operators relative to maps. In truth, we will use Browder's Theorem conjugated to a new type of operator, namely the Gateaux derivative (or differential). This is what we will see next.

Definition 1: Let E be a Banach space and $J\colon E \to \mathbb{R}$ a map. If for each $u, v \in E$ there exists the limit

$$\lim_{\lambda\to 0} \frac{J(u+\lambda v) - J(u)}{\lambda}$$

which we will denote by $J'(u,v)$, we say that $J'(u,v)$ is the *first variation of J at u in the direction v.*

Definition 2: Let E be a Banach space and $J\colon E \to \mathbb{R}$ a map. If for each fixed $u \in E$ there exists $u^* \in E'$ such that

$$\langle u^*, v\rangle_{E',E} = J'(u,v); \quad \forall\, v \in E$$

we say that J *is Gateaux differentiable at* u and u^* is called the *Gateaux derivative (or differential) of J at u* and we denote

$$u^* = J'(u).$$

If $J\colon E \to \mathbb{R}$ is Gateaux differentiable for all $u \in E$, then the operator is defined

$$\begin{aligned} J'\colon E &\to E' \\ u &\mapsto J'(u) \end{aligned}$$

where

$$\langle J'(u), v\rangle = J'(u,v) = \lim_{\lambda\to 0} \frac{J(u+\lambda v) - J(u)}{\lambda}; \quad \forall\, u,v \in E. \tag{1}$$

Proposition 1: Let E be a Banach space and K a convex subset of E. Consider $J\colon K \to \mathbb{R}$ a Gateaux differentiable map. Then, each of the statements are equivalent:

(i) J is convex

(ii) $J(v) - J(u) \geq \langle J'(u), v-u\rangle; \quad \forall\, u,v \in K$

(iii) $\langle J'(v) - J'(u), v-u\rangle \geq 0; \quad \forall\, u,v \in K$, that is , J' is a monotone operator.

Proof:
(i) $\Rightarrow$ (ii)

Suppose that $J\colon K \to \mathbb{R}$ is convex and let $u,v \in K$ and $\lambda \in]0,1]$. By the convexity of K it follows that $(1-\lambda)u + \lambda v \in K$ and by the convexity of J it follows that

$$J((1-\lambda)u + \lambda v) \leq (1-\lambda)J(u) + \lambda J(v)$$

or even

$$J(u + \lambda(v-u)) \leq J(u) + \lambda(J(v) - J(u)).$$

Thus:

$$\frac{J(u+\lambda(v-u)) - J(u)}{\lambda} \leq J(v) - J(u).$$

Since J is Gateaux differentiable, by hypothesis, taking the limit in the inequality above as $\lambda \to 0$ we obtain

$$\langle J'(u), v-u\rangle \leq J(v) - J(u)$$

which proves (ii).
(ii) $\Rightarrow$ (iii)

Suppose that (ii) holds and let $u, v \in K$. Then

$$\begin{aligned} \langle J'(u), v-u \rangle &\leq J(v) - J(u) \\ \langle J'(v), u-v \rangle &\leq J(u) - J(v). \end{aligned}$$

Summing the inequalities above member by member it follows that

$$\langle J'(u), v-u \rangle + \langle J'(v), u-v \rangle \leq 0.$$

Whence

$$\langle J'(u), v-u \rangle - \langle J'(v), v-u \rangle \leq 0,$$

that is,

$$\langle J'(v) - J'(u), v-u \rangle \geq 0$$

which proves (iii).

(iii) $\Rightarrow$ (i)

Suppose that (iii) happens and let $u, v \in K$. Set

$$[u,v] = \{(1-\lambda)u + \lambda v;\ \ \lambda \in [0,1]\} \subset K.$$

Define, then

$$\begin{aligned} \phi \colon [0,1] &\to \mathbb{R} \\ \lambda &\mapsto \phi(\lambda) = J(u + \lambda(v-u)) \end{aligned}$$

that is $J|_{[u,v]}$. Now, for each $\lambda \in]0,1[$ let $h > 0$ be sufficiently small such that $(\lambda + h) \in]0,1[$. In this way:

$$\begin{aligned} \phi'(\lambda) &= \lim_{h\to 0} \frac{\phi(\lambda+h) - \phi(\lambda)}{h} \\ &= \lim_{h\to 0} \frac{J(u + (\lambda+h)(v-u)) - J(u+\lambda(v-u))}{h} \\ &= \lim_{h\to 0} \frac{J[(u+\lambda(v-u)) + h(v-u)] - J(u+\lambda(v-u))}{h}\,. \end{aligned}$$

Since J is Gateaux differentiable in K then the limit above exists and it follows that

$$\phi'(\lambda) = \langle J'(u+\lambda(v-u)), v-u \rangle. \quad \forall\, \lambda \in]0,1[\,. \tag{2}$$

Now, if $\lambda = 0$ or $\lambda = 1$ then consider, respectively, the limit from the left and from the right so as to obtain:

$$\phi'(0) = \lim_{\substack{h\to 0 \\ h>0}} \frac{J(u + h(v-u)) - J(u)}{h} = \langle J'(u), v-u \rangle \tag{3}$$

and

$$\phi'(1) = \lim_{\substack{h\to 0 \\ h<0}} \frac{J(v + h(v-u)) - J(v)}{h} = \langle J'(v), v-u \rangle \tag{4}$$

From (2), (3) and (4) we can write

$$\phi'(\lambda) = \langle J'(u+\lambda(v-u)), v-u \rangle; \quad \forall\, \lambda \in [0,1]. \tag{5}$$

We will prove next that ϕ' is increasing. Indeed, let $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 < \lambda_2$. Then, from (5) it follows that

$$\begin{aligned} &\phi'(\lambda_2) - \phi'(\lambda_1) \\ &= \langle J'(u + \lambda_2(v-u)), v-u\rangle - \langle J'(u+\lambda_1(v-u)), v-u\rangle \\ &= \langle J'(u+\lambda_2(v-u)) - J'(u+\lambda_1(v-u)), v-u\rangle \end{aligned} \tag{6}$$

Setting

$$w_1 = u + \lambda_1(v-u) \in K \quad \text{and} \quad w_2 = u + \lambda_2(v-u) \in K$$

it follows that

$$w_2 - w_1 = (\lambda_2 - \lambda_1)(v-u).$$

However, by hypothesis

$$\langle J'(w_2) - J'(w_1), w_2 - w_1\rangle \geq 0$$

that is,

$$\langle J'(u+\lambda_2(v-u)) - J'(u+\lambda_1(v-u)), (\lambda_2-\lambda_1)(v-u)\rangle \geq 0.$$

Since $(\lambda_2 - \lambda_1) > 0$ it follows from the inequality above that

$$\langle J'(u+\lambda_2(v-u)) - J'(u+\lambda_1(v-u)), v-u\rangle \geq 0$$

and from (6) it follows that

$$\phi'(\lambda_2) \geq \phi'(\lambda_1).$$

Thus, ϕ' is increasing and, consequently, ϕ is convex.

Therefore

$$\phi((1-\lambda)\cdot 0 + \lambda \cdot 1) \leq (1-\lambda)\phi(0) + \lambda\phi(1); \quad \forall\, \lambda \in [0,1],$$

that is,

$$\phi(\lambda) \leq (1-\lambda)\phi(0) + \lambda\phi(1); \quad \forall\, \lambda \in [0,1],$$

or even,

$$J((1-\lambda)u + \lambda v) \leq (1-\lambda)J(u) + \lambda J(v); \quad \forall\, \lambda \in [0,1],$$

which proves (i) and concludes the proof of the proposition. □

In what follows, we will prove the hemicontinuity of the operator $J'\colon E \to E'$ when $J\colon E \to \mathbb{R}$ is convex and Gateaux differentiable. Before that, however, we need some preliminary results.

Lemma 1: Let E be a Banach space and $A\colon E \to E'$ a map satisfying the following property: For each $v \in E$,

$$\langle A(u), u - v\rangle_{E',E} \text{ is bounded below on bounded sets} \tag{7}$$

(as a function of u).

Then, for each $u_0 \in E$, there exist $\varepsilon, c > 0$ such that if $u \in E$ and $||u - u_0|| \leq \varepsilon \Rightarrow \langle A(u), u - u_0\rangle \leq c$.

Proof: Suppose, by contradiction, that there exists u_0 such that for each $\varepsilon, c > 0$, $\exists\, u_{\varepsilon,c} \in E$ such that $||u_{\varepsilon,c} - u_0|| \leq \varepsilon$ and yet $\langle A(u_{\varepsilon,c}), u_{\varepsilon,c} - u_0\rangle > c$.

In particular, for each $n \in \mathbb{N}$, $\exists\, u_n \in E$ such that

$$||u_n - u_0|| \leq \frac{1}{n} \quad \text{and} \quad \langle A(u_n), u_n - u_0\rangle > n.$$

Then,

$$||u_n - u_0|| \to 0 \quad \text{and} \quad \langle A(u_n), u_n - u_0\rangle \to +\infty, \text{ when } n \to +\infty. \tag{8}$$

Since $0 < n < \langle A(u_n), u_n - u_0\rangle \leq ||A(u_n)||_{E'}\, ||u_n - u_0||$ then

$$0 < \frac{1}{||u_n - u_0||} \leq \frac{||A(u_n)||_{E'}}{\langle A(u_n), u_n - u_0\rangle}\,. \tag{9}$$

But from (8) it follows that

$$\frac{1}{||u_n - u_0||} \to +\infty$$

and, therefore, from (9) it follows that

$$v_n = \frac{A(u_n)}{\langle A(u_n), u_n - u_0\rangle} \quad \text{is unbounded in } E'.$$

By the Banach-Steinhaus Theorem there exists $w \in E$ such that

$$\sup_{n\in\mathbb{N}} |\langle v_n, w\rangle| = +\infty.$$

Thus, there exists (v_ν) subsequence of (v_n) such that

$$\langle v_\nu, w\rangle \to +\infty \quad \text{or} \quad \langle v_\nu, w\rangle \to -\infty.$$

Without loss of generality we can assume that only the 1^{st} case happens because, otherwise, if we replace w by $-w$ the analysis is the same as the first case. Thus, suppose that

$$\langle v_\nu, w\rangle \to +\infty \tag{10}$$

Therefore,

$$\begin{aligned}\langle A(u_\nu), (u_\nu - u_0) - w\rangle &= \langle A(u_\nu), u_\nu - u_0\rangle - \langle A(u_\nu), w\rangle \\ &= \langle A(u_\nu), u_\nu - u_0\rangle - \frac{1}{\langle A(u_\nu), u_\nu - u_0\rangle}\,\langle A(u_\nu), u_\nu - u_0\rangle\langle A(u_\nu), w\rangle \\ = \langle A(u_\nu), u_\nu - u_0\rangle &\left[1 - \frac{\langle A(u_\nu), w\rangle}{A(u_\nu), u_\nu - u_0\rangle}\right] = \langle A(u_\nu), u_\nu - u_0\rangle \cdot (1 - \langle v_\nu, w\rangle).\end{aligned}$$

However, from (8) and (10) it follows that the last expression above tends to $-\infty$, when $\nu \to +\infty$, that is,

$$\langle A(u_\nu), (u_\nu - u_0) - w\rangle \to -\infty, \text{ when } \nu \to +\infty.$$

But this contradicts (7), which concludes the proof. $\square$

Proposition 2: Let E be a Banach space and $A\colon E \to E'$ a map verifying the property mentioned in (7). Then, A is locally bounded, that is,

$$\text{Given } u_0 \in E,\ \exists\, \varepsilon_0 > 0 \quad \text{such that} \quad ||A(u)||_{E'} \text{ is bounded for all } u \in B_{\varepsilon_0}(u_0).$$

Proof: Let $u_0 \in E$. Since A satisfies (7) then by Lemma 1 there exist $\varepsilon > 0$ and $c > 0$ such that

$$\text{If } u \in E \quad \text{and} \quad ||u - u_0|| \leq \varepsilon_0 \Rightarrow \langle A(u), u - u_0 \rangle \leq c. \tag{11}$$

Consider $w \in E$. We have

$$\langle A(u), (u - u_0) - w \rangle = \langle A(u), u - u_0 \rangle - \langle A(u), w \rangle.$$

Whence

$$\langle A(u), w \rangle = \langle A(u), u - u_0 \rangle - \langle A(u), u - u_0 - w \rangle.$$

From (7) and (11) it follows that the right side of the equality above is bounded above, that is, there exists $c_1 > 0$ such that

$$\langle A(u), w \rangle \leq c_1\,.$$

Replacing w by $-w$ in the inequality above, it follows that

$$\langle A(u), w \rangle \geq -c_1,$$

which leads us to conclude that for each $w \in E$, there exists $c_1(w) > 0$ such that

$$|\langle A(u), w \rangle| \leq c_1(w)\,; \quad \forall\, u \in B_{\varepsilon_0}(u_0),$$

where $B_{\varepsilon_0}(u_0)$ designates the closed ball centered at u_0 with radius ε_0.

This means that the image of the ball $B_{\varepsilon_0}(u_0)$ is weak-$*$ bounded. By the Banach-Steinhaus Theorem it follows that

$$\sup_{u \in B_{\varepsilon_0}(u_0)} ||A(u)||_{E'} < +\infty,$$

which proves the desired result. □

Proposition 3: Let E be a Banach space and $A\colon E \to E'$ a monotone map. Then A is locally bounded.

Proof: By the monotonicity of A it follows that:

$$\langle A(u) - A(v), u - v \rangle_{E',E} \geq 0; \quad \forall\, u, v \in E.$$

Thus, for each $v \in E$

$$\langle A(u), u - v \rangle \geq \langle A(v), u - v \rangle; \quad \forall\, u \in E.$$

Thus, fixing $v \in E$, if we let "u" traverse a bounded set of E it follows that the right side of the last inequality is bounded below with respect to u. Indeed, we have

$$-||A(v)||_{E'}||u - v|| \leq \langle A(v), u - v \rangle.$$

Whence

$$-||A(v)||_{E'}\big(||u||+||v||\big)\leq\langle A(v),u-v\rangle. \tag{12}$$

But, considering

$$||u||\leq L;\quad\forall\,u\in S,\ \text{where}\ \ S\ \text{is a bounded set of}\ E,$$

then from (12) it follows that

$$-||A(v)||_{E'}(L+||v||)\leq\langle A(v),u-v\rangle$$

which proves the desired result. Then, the property in (7) is verified and, consequently, from Proposition 2 we conclude that A is locally bounded. □

Proposition 4: Let E be a Banach space and K a convex subset of E. Consider $J\colon K\to\mathbb{R}$ a convex and Gateaux differentiable map on K. Then, given $u,v\in K$, the map

$$\psi(\lambda)=\langle J'(u+\lambda(v-u)),v-u\rangle,\quad \lambda\in[0,1],$$

is continuous on $[0,1]$.

Proof: Set, according to the proof of Proposition 1 the following map

$$\begin{aligned}\phi\colon[0,1]&\to\mathbb{R}\\ \lambda&\mapsto\phi(\lambda)=J(u+\lambda(v-u)).\end{aligned}$$

We have

$$\phi'(\lambda)=\langle J'(u+\lambda(v-u)),v-u\rangle;\quad\forall\,\lambda\in[0,1].$$

Since J is convex, it follows by Proposition 1 that J' is monotone. Thus, if $\lambda_1,\lambda_2\in[0,1]$ and $\lambda_1<\lambda_2$ then

$$\phi'(\lambda_2)\geq\phi'(\lambda_1).$$

Thus ϕ' is increasing, besides being defined on the whole $[0,1]$. It follows from this that ϕ' does not admit discontinuities of any kind, that is, $\phi'=\psi$ is continuous. This proves the proposition. □

Theorem 1: Let E be a separable Banach space and $J\colon E\to\mathbb{R}$ a convex and Gateaux differentiable map. Then, the map $u\mapsto J'(u)$ from E to E' is hemicontinuoues.

Proof: We will prove, in truth, something more general, that is, that J' is continuous from (E,τ_{strong}) into $(E',\tau_{\text{weak*}})$. Indeed, let $(u_n)\subset E$ be such that

$$u_n\to u\quad\text{in}\quad E \tag{13}$$

and, by contradiction, suppose that

$$J'(u_n)\overset{*}{\not\rightharpoonup}J'(u). \tag{14}$$

According to Proposition 1, J' is a monotone map. From Proposition 3 it follows then that J' is locally bounded, that is, for all $u\in E$; there exists $\varepsilon_u>0$ such that $||J'(v)||_{E'}$ is bounded; for all $v\in B_{\varepsilon_u}(u)$.

In particular, for $u \in E$ given in (13), there exists $\varepsilon > 0$ such that

$$||J'(v)||_{E'} \leq c; \quad \forall\, v \in B_\varepsilon(u), \tag{15}$$

where $c > 0$. From (13) it follows that $\exists\, n_0 \in \mathbb{N}$ such that $\forall\, n \geq n_0$ we have

$$u_n \in B_\varepsilon(u),$$

where $B_\varepsilon(u)$ designates the open ball centered at u with radius ε. It follows from (15) that

$$||J'(u_n)||_{E'} \leq c; \quad \forall\, n \geq n_0\,. \tag{16}$$

Let us define

$$(u_\nu) = (u_n)_{n \geq n_0}\,.$$

Evidently, this subsequence, except for a finite number of terms, continues verifying the properties given in (13) and (14).

It results from (14) the existence of $v_0 \in E$ such that

$$a_\nu = \langle J'(u_\nu), v_0 \rangle \not\to \langle J'(u), v_0 \rangle = a.$$

Therefore, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$ $\exists\, a_{\nu(k)}$ such that

$$|a_{\nu(k)} - a| \geq \delta,$$

that is,

$$|\langle J'(u_{\nu(k)}), v_0 \rangle - \langle J'(u), v_0 \rangle| \geq \delta. \tag{17}$$

On the other hand, from (16) it follows that $(J'(u_{\nu(k)}))_{k \in \mathbb{N}}$ is a bounded sequence of E'. Since E is separable Banach, there exists $(u_\mu)_{\mu \in \mathbb{N}}$ subsequence of $(u_{\nu(k)})_{k \in \mathbb{N}}$ such that

$$J'(u_\mu) \overset{*}{\rightharpoonup} f \quad \text{in} \quad E'. \tag{18}$$

By the property of the elements of $(J'(u_\mu))$ given in (17) it follows that

$$J'(u) \neq f. \tag{19}$$

We will prove, next that

$$\langle f, u - v \rangle \geq \langle J'(u), u - v \rangle; \quad \forall\, v \in E. \tag{20}$$

Indeed, let

$$w = (1 - \theta)u + \theta v; \quad \theta \in]0, 1[\,.$$

We have, by the monotonicity of J' that

$$\langle J'(u_\mu) - J'(w), u_\mu - w \rangle \geq 0.$$

Whence

$$\langle J'(u_\mu) - J'(w), u_\mu - u - \theta(v - u) \rangle \geq 0.$$

It follows from this that

$$\langle J'(u_\mu), u_\mu - u \rangle - \theta \langle J'(u_\mu), v - u \rangle - \langle J'(w), u_\mu - u \rangle + \theta \langle J'(w), v - u \rangle \geq 0$$

and, therefore,

$$\theta\langle J'(u_\mu), u-v\rangle \geq -\langle J'(u_\mu), u_\mu - u\rangle + \langle J'(w), u_\mu - u\rangle - \theta\langle J'(w), v-u\rangle.$$

Taking the limit in the inequality above as $\mu \to +\infty$ results from (13) and (18) that

$$\theta\langle f, u-v\rangle \geq -\theta\langle J'(w), v-u\rangle.$$

Dividing by θ, we obtain

$$\langle f, u-v\rangle \geq \langle J'(w), u-v\rangle; \quad \forall\, v \in E$$

or better,

$$\langle f, u-v\rangle \geq \langle J'(u+\theta(v-u)), u-v\rangle; \quad \forall\, v \in E.$$

Taking the limit in the inequality above as $\theta \to 0$ results from Proposition 4 that

$$\langle f, u-v\rangle \geq \langle J'(u), u-v\rangle, \quad \forall\, v \in E,$$

which proves (20).

Consider, now, $\lambda > 0$ and $z \in E$. Then, taking in (20) $v = u - \lambda z$, it results that

$$\langle f, \lambda z\rangle \geq \langle J'(u), \lambda z\rangle.$$

Whence

$$\langle f, z\rangle \geq \langle J'(u), z\rangle; \quad \forall\, z \in E. \tag{21}$$

Analogously, taking $v = u - \lambda z, \quad \lambda < 0$, in (20) we obtain

$$\langle f, z\rangle \leq \langle J'(u), z\rangle; \quad \forall\, z \in E. \tag{22}$$

From (21) and (22) we conclude that

$$J'(u) = f,$$

which contradicts (19). This proves the theorem. □

It follows from Propositions 1 and 3 and from Theorem 1 the central result of this paragraph which we state in the form of the following Theorem:

Theorem 2: Let E be a separable Banach space and $J\colon E \to \mathbb{R}$ a convex and Gateaux differentiable map. Then, the map $u \mapsto J'(u)$ from E to E' is monotone, hemicontinuous and locally bounded.

Example 1: Consider $E = \mathbb{R}$ and $J\colon E \to \mathbb{R}$ differentiable. Then, for each $x \in E$ and $h > 0$ we have

$$\begin{aligned}\langle J'(x), h\rangle &= \lim_{\substack{\lambda\to 0\\ \lambda\neq 0}} \frac{J(x+\lambda h) - J(x)}{\lambda} \\ &= h \cdot \lim_{\substack{\lambda\to 0\\ \lambda\neq 0}} \frac{J(x+\lambda h) - J(x)}{(\lambda h)} = J'(x)\cdot h,\end{aligned}$$

that is,

$$\begin{aligned} J'(x)\colon E &\to \mathbb{R} \\ h &\mapsto \langle J'(x), h\rangle = J'(x)\cdot h, \end{aligned}$$

where $J'(x)$ is the derivative of J at x.

Example 2: Let $E = \mathbb{R}^n$ and $J\colon E \to \mathbb{R}$ be differentiable in E. Then, for all $u, v \in \mathbb{R}^n$ we have

$$\langle J'(u), v\rangle = \lim_{\lambda\to 0} \frac{J(u+\lambda v) - J(u)}{\lambda} = \langle \nabla J(u), v\rangle = \sum_{i=1}^{n} \frac{\partial J}{\partial x_i}(u)\cdot v_i$$

that is,

$$\begin{aligned} J'(u)\colon E &\to \mathbb{R} \\ v &\mapsto \langle J'(u), v\rangle = \langle \nabla J(u), v\rangle. \end{aligned}$$

Example 3: Let $E = L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$ open and $2 \le p < +\infty$. Consider $g\colon \mathbb{R} \to \mathbb{R}$ of class $C^1(\mathbb{R})$ such that

(i) $|g(s)| \le \alpha |s|^p\,; \quad \alpha > 0$,

(ii) $|g'(s)| \le \beta |s|^{p-1}\,; \quad \beta > 0$.

Define

$$\begin{aligned} J\colon E &\to \mathbb{R} \\ u &\mapsto J(u) = \int_\Omega g(u(x))dx. \end{aligned}$$

J is well defined since, from item (i) it follows that

$$|g(u(x))| \le \alpha |u(x)|^p; \quad \forall\, x \in \Omega,$$

and since $(g \circ u)$ is measurable and $u \in L^p(\Omega)$ it follows that $(g\circ u) \in L^1(\Omega)$. We will calculate, next, the Gateaux derivative of J. Given $u, v \in L^p(\Omega)$, let us evaluate the first variation of J at u in the direction v. We have

$$\begin{aligned} J'(u,v) &= \lim_{\lambda\to 0} \frac{J(u+\lambda v) - J(u)}{\lambda} \\ &= \lim_{\lambda\to 0} \frac{1}{\lambda}\left\{ \int_\Omega \big[g(u(x)+\lambda v(x)) - g(u(x))\big]\, dx \right\}. \end{aligned} \tag{23}$$

However, given $\xi, \eta \in \mathbb{R}$ such that $\eta < \xi$, by the Mean Value Theorem there exists $\xi_0 \in]\eta, \xi[$ such that

$$g(\xi) - g(\eta) = g'(\xi_0)(\xi - \eta).$$

Since $\xi_0 \in]\eta, \xi[$, then $\xi_0 = (1-\theta)\eta + \theta\xi = (\xi - \eta)\theta + \eta$ for some $\theta \in]0,1[$. In particular, supposing without loss of generality that $v(x) > 0$ for each $x \in \Omega$ and $\lambda > 0$, there exists $\theta_\lambda(x)$ with $0 < \theta_\lambda(x) < 1$ such that

$$g(\underbrace{u(x)+\lambda v(x)}_{\xi}) - g(\underbrace{u(x)}_{\eta}) = g'(\lambda v(x)\theta_\lambda(x) + u(x))(\lambda v(x)).$$

Thus, the last expression in (23) becomes:

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \left\{ \int_\Omega \left[g'(u(x) + \lambda \theta_\lambda(x) v(x)) \right] \lambda v(x)\, dx \right\}. \tag{24}$$

Since $\theta_\lambda(x)$ is bounded, the product $\lambda \cdot \theta_\lambda(x) \to 0$ when $\lambda \to 0$; whatever $x \in \Omega$ is. By the continuity of g' it follows that, for all $x \in \Omega$,

$$g'(u(x) + \lambda \theta_\lambda(x) v(x)) v(x) \xrightarrow{\lambda \to 0} g'(u(x)) v(x). \tag{25}$$

However, for each $\lambda \in [0,1]$, we have from (ii) that

$$\begin{aligned} |g'(u(x) + \lambda \theta_\lambda(x) v(x))|\, |v(x)| &\le \beta |u(x) + \lambda \theta_\lambda(x) v(x)|^{p-1} |v(x)| \\ &\le c(p) \cdot \beta \cdot \left\{ |u(x)|^{p-1} + |v(x)|^{p-1} \right\} |v(x)| \\ &= c(p) \cdot \beta \cdot \left\{ |u(x)|^{p-1} \cdot |v(x)| + |v(x)|^p \right\}. \end{aligned} \tag{26}$$

Note that $|u|^{p-1} \in L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, since

$$\int_\Omega \left[|u(x)|^{p-1} \right]^{p'} dx = \int_\Omega |u(x)|^p\, dx < +\infty.$$

Thus, by Hölder's inequality the product $|u|^{p-1} |v| \in L^1(\Omega)$. Therefore, the last expression in (26) is integrable. Thus, from (25), (26) and from the Lebesgue Dominated Convergence Theorem it follows that the integral in (24) converges to

$$\int_\Omega g'(u(x)) v(x)\, dx.$$

Therefore

$$J'(u,v) = \int_\Omega g'(u(x)) \cdot v(x)\, dx, \quad \forall\, u, v \in L^p(\Omega). \tag{27}$$

Define,

$$\begin{aligned} u^* \colon L^p(\Omega) &\to \mathbb{R} \\ v &\mapsto \langle u^*, v \rangle = J'(u,v). \end{aligned}$$

We will prove that $u^* \in L^{p'}(\Omega) = [L^p(\Omega)]'$. Indeed, u^* is clearly linear by virtue of the linearity of the integral in (27). Now, let $(v_\nu) \subset L^p(\Omega)$ be such that $v_\nu \to 0$ in $L^p(\Omega)$. We have, by Hölder's inequality that

$$\begin{aligned} |\langle u^*, v_\nu \rangle| &\le \int_\Omega |g'(u(x))| \cdot |v_\nu(x)|\, dx \\ &\le \beta \int_\Omega \underbrace{|u(x)|^{p-1}}_{L^{p'}} \cdot \underbrace{|v_\nu(x)|}_{L^p}\, dx \\ &\le \beta \left(\int_\Omega |u(x)|^p\, dx \right)^{1/p'} \left(\int_\Omega |v_\nu(x)|^p\, dx \right)^{1/p} \\ &= \beta ||u||_{L^p(\Omega)}^{p/p'} ||v_\nu||_{L^p(\Omega)} \to 0, \text{ when } \nu \to +\infty, \end{aligned}$$

that is,

$$v_\nu \to 0 \text{ in } L^p(\Omega) \Rightarrow \langle u^*, v_\nu \rangle \to 0 \text{ in } \mathbb{R}.$$

Thus, $u^* \in L^{p'}(\Omega)$. Setting $u^* = J'(u)$ results that

$$\begin{aligned} \langle J'(u), v \rangle &= \int_\Omega g'(u(x))v(x)\,dx; \quad \forall\, u, v \in L^p(\Omega) \\ &= \langle g' \circ u, v \rangle \end{aligned} \tag{28}$$

Example 4: Let $A\colon L^p(\Omega) \to L^p(\Omega)$ be a linear operator of $L^p(\Omega)$ whose domain is given by

$$W(\Omega, A) = \{u \in L^p(\Omega); Au \in L^p(\Omega)\}.$$

We will endow $W(\Omega, A)$ with the graph norm

$$||v||_W^p = ||v||_{L^p(\Omega)}^p + ||Av||_{L^p(\Omega)}^p.$$

Let $g\colon \mathbb{R} \to \mathbb{R}$, be continuously differentiable satisfying the following properties:

(i) $|g(s)| \le \alpha |s|^p; \quad \alpha > 0.$

(ii) $|g'(s)| \le \beta |s|^{p-1}; \quad \beta > 0.$

Define, then the functional

$$\begin{aligned} J\colon W(\Omega, A) &\to \mathbb{R} \\ u &\mapsto J(u) = \int_\Omega g(Au(x))\,dx. \end{aligned}$$

Note that J is well defined. Indeed, from (i) we have

$$|g(Au(x))| \le \alpha |Au(x)|^p; \quad \forall\, x \in \Omega. \tag{29}$$

Since $u \in W(\Omega, A)$ then $Au \in L^p(\Omega)$ and, consequently, $|Au|^p \in L^1(\Omega)$. Furthermore, since $g \circ Au$ is measurable and, by (29), bounded by an integrable function then $g \circ Au \in L^1(\Omega)$.

We will calculate, next, the first variation of J at u in the direction v, where $u, v \in W(\Omega, A)$. We have

$$\begin{aligned} J'(u, v) &= \lim_{\lambda \to 0} \frac{J(u + \lambda v) - J(u)}{\lambda} \\ &= \lim_{\lambda \to 0} \frac{1}{\lambda} \left\{ \int_\Omega \left[g(Au(x) + \lambda Av(x)) - g(Au(x)) \right] dx \right\}. \end{aligned}$$

By the Mean Value Theorem this last expression becomes

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \left\{ \int_\Omega \left[g'(Au(x) + \lambda \theta_\lambda(x) Av(x)) \right] \lambda Av(x) \right] dx \right\},$$

where $0 < \theta_\lambda(x) < 1$. In a manner analogous to what was done in the previous example we prove, given the Lebesgue Dominated Convergence Theorem, that the integral above converges to

$$\int_\Omega g'(Au(x))Av(x)\,dx.$$

Therefore

$$J'(u,v)=\int_\Omega g'(Au(x))Av(x)\,dx;\quad \forall\, u,v\in W(\Omega,A).$$

Define

$$\begin{aligned} u^*\colon W(\Omega,A) &\to \mathbb{R}\\ v &\mapsto \langle u^*,v\rangle = J'(u,v).\end{aligned}$$

We will prove that u^* is a linear and continuous form on $W(\Omega,A)$, that is, $u^*\in (W(\Omega,A))'$. Indeed, the linearity is obvious. Let us prove continuity. Consider, then $(v_\nu)\subset W(\Omega,A)$ such that $v_\nu\to 0$ in $W(\Omega,A)$. We have

$$\begin{aligned} |\langle u^*,v_\nu\rangle| &\le \int_\Omega |g'(A(u))|\,|Av_\nu(x)|\,dx\\ &\le \beta\int_\Omega \underbrace{|Au(x)|^{p-1}}_{L^{p'}}\,\underbrace{|Av_\nu(x)|}_{L^p}\,dx\\ &\le \beta\Big(\int_\Omega |Au(x)|^p\,dx\Big)^{1/p'}\Big(\int_\Omega |Av_\nu(x)|^p\,dx\Big)^{1/p}\\ &= \beta||Au||^{p/p'}_{L^p(\Omega)}\,||Av_\nu||_{L^p(\Omega)}\\ &\le \beta||Au||^{p/p'}_{L^p(\Omega)}\,||v_\nu||_{W(\Omega,A)}\to 0,\ \text{when } \nu\to+\infty.\end{aligned}$$

Thus, $\langle u^*,v_\nu\rangle\to 0$ which proves that

$$u^*\in (W(\Omega,A))'.$$

Therefore, for each $u\in W(\Omega,A)$ we have that

$$\langle J'(u),v\rangle=\int_\Omega g'(Au(x))Av(x)\,dx,\quad \forall\, v\in W(\Omega,A), \tag{30}$$

where $J'(u)$ is defined by the operator

$$\begin{aligned} J'\colon W(\Omega,A) &\to (W(\Omega,A))'\\ u &\mapsto J'(u)=u^*.\end{aligned}$$

We will see, next, some particular cases of the previous example.

Example 5: *The Pseudo-Laplacian operator.*
Consider for each $i=1,\ldots,n$

$$W_i=W(\Omega,A_i)=\Big\{v\in L^p(\Omega);\frac{\partial v}{\partial x_i}\in L^p(\Omega)\Big\};\ \ 2\le p<+\infty,$$

endowed with the topology: $||u||^p_{W_i}=||v||^p_{L^p(\Omega)}+\left\|\frac{\partial v}{\partial x_i}\right\|^p_{L^p(\Omega)}$. Define

$$\begin{aligned} A_i\colon W(\Omega,A_i) &\to L^p(\Omega)\\ v &\mapsto A_iv=\frac{\partial v}{\partial x_i}.\end{aligned}$$

Let $\rho > 0$ and consider

$$f(x) = |s|^\rho s = \begin{cases} s^\rho s; & s \geq 0 \\ (-s)^\rho s; & s < 0 \end{cases} = \begin{cases} s^\rho s; & s \geq 0 \\ -(-s)^\rho(-s); & s < 0 \end{cases} = \begin{cases} s^{\rho+1}; & s \geq 0 \\ -(-s)^{\rho+1}; & s < 0. \end{cases}$$

Whence

$$f'(s) = \begin{cases} (\rho+1)s^\rho; & s \geq 0 \\ -(\rho+1)\cdot(-s)^\rho\cdot(-1); & s < 0 \end{cases} = \begin{cases} (\rho+1)s^\rho; & s \geq 0 \\ (\rho+1)(-s)^\rho; & s < 0 \end{cases} = (\rho+1)|s|^\rho.$$

Thus, setting

$$g(s) = |s|^p = \left(|s|^{p-2}\, s^2\right) = \left(|s|^{p-2}\, s\right)s,$$

it follows that

$$g'(s) = [(p-2)+1]|s|^{p-2}\, s + \left(|s|^{p-2}\, s\right) = p|s|^{p-2}\, s.$$

In this way, $g \in C^1(\mathbb{R})$ since $g' \in C^0(\mathbb{R})$. Furthermore,

$$\begin{aligned} &|g(s)| = |s|^\rho \\ &|g'(s)| = |p|s|^{p-2}\, s| = p|s|^{p-1}, \end{aligned}$$

which proves that g satisfies properties (i) and (ii) alluded to in the previous example.

We are, then, within the hypotheses of Example 4. Thus, defining for each $i = 1, \ldots, n$

$$\begin{aligned} J_i \colon W(\Omega, A_i) &\to \mathbb{R} \\ u \mapsto J_i(u) &= \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx, \end{aligned}$$

then J_i is Gateaux differentiable and for each $u \in W(\Omega, A_i)$ we have that

$$\langle J_i'(u), v\rangle = p \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx; \quad \forall\, v \in W(\Omega, A_i). \tag{31}$$

Consider, now,

$$E = \bigcap_{i=1}^n W(\Omega, A_i) = \left\{u \in L^p(\Omega); \frac{\partial u}{\partial x_i} \in L^p(\Omega); i = 1, \ldots, n\right\},$$

that is,

$$E = W^{1,p}(\Omega).$$

We will endow E with the natural topology

$$||u||_E^p = ||u||_{L^p(\Omega)}^p + \sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}^p.$$

Define

$$\begin{aligned} J \colon E &\to \mathbb{R} \\ u \mapsto J(u) &= \frac{1}{p}\sum_{i=1}^n J_i(u). \end{aligned} \tag{32}$$

J is clearly well defined. We will calculate, next, the first variation of J at u in the direction v, where $u, v \in E$. We have

$$\begin{aligned}
\langle J'(u), v\rangle &= \lim_{\lambda\to 0} \frac{J(u+\lambda v) - J(u)}{\lambda} \\
&= \lim_{\lambda\to 0} \frac{1}{p} \sum_{i=1}^{n} \frac{J_i(u+\lambda v) - J_i(u)}{\lambda} \\
&= \frac{1}{p} \sum_{i=1}^{n} \lim_{\lambda\to 0} \frac{J_i(u+\lambda v) - J_i(u)}{\lambda} \\
&= \frac{1}{p} \sum_{i=1}^{n} \langle J_i'(u), v\rangle \\
&= \frac{1}{p}\, p \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx.
\end{aligned}$$

Set

$$\begin{aligned}
u^* \colon\ & E \to \mathbb{R} \\
& v \mapsto \langle u^*, v\rangle = J'(u,v) = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx.
\end{aligned}$$

We have that u^* is clearly linear. By the Hölder and Minkowski inequalities it follows that

$$\begin{aligned}
|\langle u^*, v\rangle| &\le \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-1} \left|\frac{\partial v}{\partial x_i}\right| dx \\
&\le \sum_{i=1}^{n} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p} dx\right)^{1/p'} \left(\int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^{p} dx\right)^{1/p} \\
&= \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}^{p/p'} \cdot \left\|\frac{\partial v}{\partial x_i}\right\|_{L^p(\Omega)} \\
&\le \left(\sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}^{p}\right)^{1/p'} \left(\sum_{i=1}^{n} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^p(\Omega)}^{p}\right)^{1/p} \\
&\le ||u||_E^{p/p'} ||v||_E,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, which proves the continuity of u^*. Therefore, J is Gateaux differentiable and, furthermore,

$$\langle J'(u), v\rangle = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx; \quad \forall\, u, v \in E, \tag{33}$$

where

$$\begin{aligned}
J' \colon\ & E \to E' \\
& u \mapsto J'(u) = u^*.
\end{aligned}$$

Remark: It is worth noting that the dual of $E = W^{1,p}(\Omega)$ is not a "space of distributions". We can then consider

$$E = W_0^{1,p}(\Omega)$$

whose dual $E' = W^{-1,p'}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. thus, if $u \in E = W_0^{1,p}(\Omega)$ then $J'(u) \in E' = W^{-1,p'}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. Furthermore, for all $\varphi \in \mathcal{D}(\Omega)$ we obtain

$$\begin{aligned}
\langle J'(u), \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} &= \sum_{i=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx \\
&= - \sum_{i=1}^{n} \left\langle \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right), \varphi \right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\
&= \left\langle - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right), \varphi \right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} .
\end{aligned}$$

By density we conclude:

$$\langle J'(u), v \rangle_{W^{-1,p'}, W_0^{1,p}} = \left\langle - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right), v \right\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} ,$$

in this way,

$$J'(u) = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) \text{ in } W^{-1,p'}(\Omega) \tag{34}$$

Finally, note that the second derivative of the function $g(s) = |s|^p$ mentioned at the beginning of this example is given by

$$\begin{aligned}
g''(s) &= p\big\{ \big(|s|^{p-2}\big)' s + |s|^{p-2} \big\} \\
&= p\big\{ (p-2)|s|^{p-4} s^2 + |s|^{p-2} \big\} \\
&= p\big\{ (p-2)|s|^{p-2} + |s|^{p-2} \big\} \\
&= p(p-1)|s|^{p-2} .
\end{aligned}$$

Thus,

$$g''(s) \geq 0; \quad \forall \, s \in \mathbb{R},$$

which implies that

$$g(s) = |s|^p,$$

is a convex function. Thus, the functional given in (32) is convex. Indeed, whatever $\forall \, u \in E$

$$J(u) = \frac{1}{p} \sum_{i=1}^{n} J_i(u)$$

where

$$J_i(u) = \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p dx.$$

Let $u, v \in E$ and $\lambda \in [0,1]$. We have

$$\begin{aligned}
J_i(\lambda u + (1-\lambda)v) &= \int_\Omega \left| \frac{\partial}{\partial x_i} \big(\lambda u + (1-\lambda) v \big) \right|^p dx \\
&= \int_\Omega \left| \lambda \frac{\partial u}{\partial x_i} + (1-\lambda) \frac{\partial v}{\partial x_i} \right|^p dx,
\end{aligned}$$

which by the convexity of g is less than or equal to

$$\int_\Omega \left\{\lambda \left|\frac{\partial u}{\partial x_i}\right|^p + (1-\lambda)\left|\frac{\partial v}{\partial x_i}\right|^p\right\} dx;$$

that is, for each $i = 1, \ldots, n$

$$J_i(\lambda u + (1-\lambda)v) \le \lambda\, J_i(u) + (1-\lambda) J_i(v), \quad \forall\, u, v \in E.$$

Therefore,

$$\begin{aligned} J(\lambda u + (1-\lambda)v) &= \frac{1}{p}\sum_{i=1}^{n} J_i(\lambda u + (1-\lambda)v) \\ &\le \frac{1}{p}\left\{\sum_{i=1}^{n}\big(\lambda\, J_i(u) + (1-\lambda)J_i(v)\big)\right\} \\ &= \lambda\, J(u) + (1-\lambda)J(v). \end{aligned}$$

Conclusion:

The map

$$\begin{aligned} J\colon W_0^{1,p}(\Omega) &\to \mathbb{R} \\ u &\mapsto J(u) = \frac{1}{p}\sum_{i=1}^{n} J_i(u), \end{aligned}$$

where

$$J_i(u) = \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx; \quad 2 \le p < +\infty,$$

is convex and Gateaux differentiable. Then, by Theorem 2, the map

$$\begin{aligned} J'\colon W_0^{1,p}(\Omega) &\to W^{-1,p}(\Omega) \\ u &\mapsto J'(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) \end{aligned}$$

is monotone and hemicontinuous.

Example 6: *The Pseudo-biharmonic operator.*

Consider $2 \le p < +\infty$ and define

$$W(\Omega, A) = \{u \in L^p(\Omega); -\Delta u \in L^p(\Omega)\},$$

endowed with the topology

$$||u||_W^p = ||u||_{L^p(\Omega)}^p + ||\Delta u||_{L^p(\Omega)}^p\,,$$

and

$$\begin{aligned} A\colon W(\Omega, A) &\to L^p(\Omega) \\ u &\mapsto Au = -\Delta u. \end{aligned}$$

Let

$$\begin{aligned} g\colon\ \mathbb{R} &\to \mathbb{R} \\ s &\mapsto g(s) = \frac{1}{p}\,|s|^p \end{aligned}$$

whose derivative is given by

$$g'(s) = |s|^{p-2}\, s.$$

In this way, $g \in C^1(\mathbb{R})$. Furthermore,

$$\begin{aligned} |g(s)| &= \frac{1}{p}\,|s|^p \\ |g'(s)| &= |s|^{p-1}, \end{aligned}$$

which proves that g satisfies properties (i) and (ii) alluded to in Example 4. We are, therefore, within the hypotheses of that example. Defining

$$\begin{aligned} J\colon W(\Omega, A) &\to \mathbb{R} \\ u &\mapsto J(u) = \frac{1}{p}\int_\Omega |\Delta u|^p\, dx, \end{aligned} \tag{35}$$

then J is Gateaux differentiable and given $u, v \in W(\Omega, A)$ it follows that

$$\langle J'(u), v\rangle = \int_\Omega |\Delta u|^{p-2}\, \Delta u\, \Delta v\, dx. \tag{36}$$

In particular, if we consider $W_0^{2,p}(\Omega)$ instead of $W(\Omega, A)$ whose dual is $W^{-2,p'}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ then, given $u \in W_0^{2,p}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, it follows that

$$\begin{aligned} \langle J'(u), \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} &= \int_\Omega |\Delta u|^{p-2}\, \Delta u\, \Delta\varphi\, dx \\ &= \big\langle \Delta\big(|\Delta u|^{p-2}\, \Delta u\big), \varphi\big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}, \end{aligned}$$

that is, by density we have that

$$\langle J'(u), v\rangle_{W^{-2,p'},W_0^{2,p}} = \big\langle \Delta\big(|\Delta u|^{p-2}\, \Delta u\big), v\big\rangle_{W^{-2,p'},W_0^{2,p}} \text{ for all } v \in W_0^{2,p}(\Omega).$$

Thus,

$$J'(u) = \Delta\big(|\Delta u|^{p-2}\, \Delta u\big) \quad \text{in} \quad W^{-2,p'}(\Omega). \tag{37}$$

Analogously, as in the previous example we also have that the map $g(s) = \dfrac{1}{p}|s|^p$ is convex. Therefore, the functional given in (35) is also convex.

Conclusion:

The map

$$\begin{aligned} J\colon W_0^{2,p}(\Omega) &\to \mathbb{R} \\ u \mapsto J(u) &= \frac{1}{p}\int_\Omega |\Delta u|^p\, dx \end{aligned}$$

is convex and Gateaux differentiable. Then, by Theorem 2, the map

$$\begin{aligned} J'\colon W_0^{2,p}(\Omega) &\to W^{-2,p'}(\Omega) \\ u \mapsto J'(u) &= \Delta\big(|\Delta u|^{p-2}\, \Delta u\big) \end{aligned}$$

is monotone and hemicontinuous.

We will see, next, some applications of the exposed theory for the resolution of stationary problems.

Application 1:

Consider the stationary problem

$$\left| \begin{aligned} & -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) = f \quad \text{in} \quad \Omega \\ & u|_\Gamma = 0, \end{aligned} \right. \tag{38}$$

where $f \in W^{-1,p'}(\Omega)$; $2 \leq p < +\infty$. It follows from Example 5 that the operator

$$\begin{aligned} J'\colon W_0^{1,p}(\Omega) &\to W^{-1,p'}(\Omega) \\ u \mapsto J'(u) &= -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) \end{aligned}$$

is monotone and hemicontinuous, since it is the Gateaux differential of the convex functional

$$\begin{aligned} J\colon W_0^{1,p}(\Omega) &\to \mathbb{R} \\ u \mapsto J(u) &= \frac{1}{p}\sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p \, dx. \end{aligned}$$

Our aim is to use Browder's Theorem and conclude that the stationary problem (38) possesses a weak solution. We already have the monotonicity and hemicontinuity of J'. It remains to prove that J' is bounded and coercive. Indeed:

(i) J' is bounded. Indeed, we have that

$$||J'(u)||_{W^{-1,p'}(\Omega)} = \sup_{||v||_{W_0^{1,p}(\Omega)}\le 1} |\langle J'(u), v\rangle|.$$

However, from (33) it follows that

$$\begin{aligned} |\langle J'(u), v\rangle| &\le \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-1} \left|\frac{\partial v}{\partial x_i}\right| dx \\ &\le \sum_{i=1}^n \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx\right)^{1/p'} \left(\int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^p dx\right)^{1/p} \\ &= \sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}^{p/p'} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^p(\Omega)} \\ &\le \left(\sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}^p\right)^{1/p'} \left(\sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|_{L^p(\Omega)}^p\right)^{1/p} \\ &= ||u||_{W_0^{1,p}(\Omega)}^{p/p'} ||v||_{W_0^{1,p}(\Omega)} \\ &= ||u||_{W_0^{1,p}(\Omega)}^{p-1} ||v||_{W_0^{1,p}(\Omega)}. \end{aligned}$$

Whence

$$||J'(u)||_{W^{-1,p'}(\Omega)} \le ||u||_{W_0^{1,p}(\Omega)}^{p-1}; \quad \forall\, u \in W_0^{1,p}(\Omega). \tag{39}$$

The inequality above proves the desired result; that is, that J' maps bounded sets of $W_0^{1,p}(\Omega)$ into bounded sets of $W^{-1,p'}(\Omega)$.

(ii) J' is coercive. Indeed, we have from (33) that

$$\langle J'(u), u\rangle = \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \left(\frac{\partial u}{\partial x_i}\right)^2 dx = \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx,$$

that is,

$$\langle J'(u), u\rangle = ||u||_{W_0^{1,p}(\Omega)}^p.$$

Whence:

$$\lim_{||u||_{W_0^{1,p}(\Omega)}\to+\infty} \frac{\langle J'(u), u\rangle}{||u||_{W_0^{1,p}(\Omega)}} = \lim_{||u||_{W_0^{1,p}(\Omega)}\to+\infty} ||u||_{W_0^{1,p}(\Omega)}^{p-1} = +\infty. \qquad \square$$

Application 2:

Consider the stationary problem

$$\left|\begin{array}{l}\Delta\big(|\Delta u|^{p-2}\,\Delta u\big)=f \quad \text{in} \quad \Omega\\ u|_\Gamma=0,\end{array}\right. \tag{40}$$

where $f\in W^{-2,p'}(\Omega)$; $2\le p<+\infty$. It follows from Example 6 that the operator

$$\begin{aligned}J'\colon W_0^{2,p}(\Omega)&\to W^{-2,p'}(\Omega)\\ u&\mapsto J'(u)=\Delta\big(|\Delta u|^{p-2}\,\Delta u\big)\end{aligned}$$

is monotone and hemicontinuous, since it is the Gateaux differential of the convex functional

$$\begin{aligned}J\colon W_0^{2,p}(\Omega)&\to\mathbb{R}\\ u&\mapsto J(u)=\frac{1}{p}\int_\Omega|\Delta u|^p\,dx.\end{aligned}$$

We have

(i) J' is bounded. Indeed, from (36) it follows that

$$\langle J'(u),v\rangle=\int_\Omega|\Delta u|^{p-2}\,\Delta u\,\Delta v\,dx;\quad \forall\,u,v\in W_0^{2,p}(\Omega).$$

Whence

$$\begin{aligned}|\langle J'(u),v\rangle|&\le\int_\Omega|\Delta u|^{p-1}\,|\Delta v|\,dx\\ &\le\Big(\int_\Omega|\Delta u|^p\,dx\Big)^{\frac{1}{p'}}\Big(\int_\Omega|\Delta v|^p\,dx\Big)^{\frac{1}{p}}\\ &=||\Delta u||^{\frac{p}{p'}}_{L^p(\Omega)}||\Delta v||_{L^p(\Omega)}\\ &=||\Delta u||^{p-1}_{L^p(\Omega)}||\Delta v||_{L^p(\Omega)}=||u||^{p-1}_{W_0^{2,p}(\Omega)}||v||_{W_0^{2,p}(\Omega)}\,.\end{aligned}$$

Thus,

$$||J'(u)||_{W^{-2,p'}(\Omega)}=\sup_{||v||_{W_0^{2,p}}\le 1}|\langle J'(u),v\rangle|\le||u||^{p-1}_{W_0^{2,p}(\Omega)}. \tag{41}$$

The inequality in (41) shows us that J' is bounded.

(ii) J' is coercive. Indeed, we have:

$$\begin{aligned}\langle J'(u),u\rangle&=\int_\Omega|\Delta u|^{p-2}\,\Delta u\,\Delta u\,dx\\ &=\int_\Omega|\Delta u|^p\,dx=||u||^p_{W_0^{2,p}(\Omega)}\,.\end{aligned}$$

Thus

$$\lim_{||u||_{W_0^{2,p}(\Omega)}\to+\infty}\frac{\langle J'(u),u\rangle}{||u||_{W_0^{2,p}(\Omega)}}=\lim_{||u||_{W_0^{2,p}(\Omega)}\to+\infty}||u||^{p-1}_{W_0^{2,p}(\Omega)}=+\infty.$$

By Browder's Theorem we conclude then that problem (40) possesses a weak solution. □

Chapter 14

Evolution Problems

14.1 Monotone Parabolic Problem

Consider the parabolic problem

$$\left| \begin{array}{l} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) = f \quad \text{in} \quad Q = \Omega \times (0,T) \ \ (p \geq 2) \\ u = 0 \quad \text{on} \quad \Sigma = \partial\Omega \times (0,T) \\ u(x,0) = u_0(x); \quad x \in \Omega, \end{array} \right. \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$.

We have the following result:

Theorem 1: Given $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $u_0 \in L^2(\Omega)$ there exists a unique function $u\colon Q \to \mathbb{R}$, weak solution of (1) in the class

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)); u' \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \tag{2}$$

verifying

$$\frac{d}{dt}(u(t),w) - \left\langle \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right), w \right\rangle = \langle f(t), w \rangle \quad \text{in} \quad \mathcal{D}'(0,T), \tag{3}$$

for all $w \in W_0^{1,p}(\Omega)$ and

$$u(0) = u_0\,. \tag{4}$$

Proof:

1ª Step: Approximate Solution.

Since $W_0^{1,p}(\Omega)$ is a separable Banach space [1] $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ by definition. there exists $(v_\nu)_{\nu\in\mathbb{N}}$ a countable dense subset in $W_0^{1,p}(\Omega)$. We can, from this, construct a new sequence $(w_\nu)_{\nu\in\mathbb{N}}$ orthonormal in $L^2(\Omega)$, by the Gram-Schmidt process, such that

(i) $(w_\nu)_{\nu\in\mathbb{N}}$ constitutes a linearly independent set.

(ii) $[(w_\nu)_{\nu\in\mathbb{N}}]$ is dense in $W_0^{1,p}(\Omega)$.

[1] (*)

Indeed, note that the sequence $(w_\nu)_{\nu\in\mathbb{N}}$ is obtained from $(v_\nu)_{\nu\in\mathbb{N}}$ through the Gram-Schmidt orthogonalization process. Thus, the w_ν's can be written in terms of the v_ν's and consequently we also have the reciprocal. From there, it follows that

$$(v_\nu)_{\nu\in\mathbb{N}} \subset [(w_\nu)_{\nu\in\mathbb{N}}] \subset W_0^{1,p}(\Omega).$$

Whence, taking the closure in $W_0^{1,p}(\Omega)$ it follows that

$$\overline{(v_\nu)_\nu} \subset \overline{[(w_\nu)_{\nu\in\mathbb{N}}]} \subset W_0^{1,p}(\Omega)$$

that is,

$$\overline{[(w_\nu)_{\nu\in\mathbb{N}}]} = W_0^{1,p}(\Omega),$$

which proves (ii).

According to the preceding chapter we know that the pseudo-Laplacian operator is defined by

$$\begin{aligned} A\colon W_0^1(\Omega) &\to W^{-1,p'}(\Omega) \\ u \mapsto Au &= -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right). \end{aligned}$$

Set, then

$$V_m = [w_1, \ldots, w_m]$$

and consider the approximate problem: Fixed $m \in \mathbb{N}$, determine $u_m(t) \in V_m$ such that

$$(u_m'(t), w_j) + \langle Au_m(t), w_j\rangle = \langle f(t), w_j\rangle; \quad j = 1, \ldots, m \tag{5}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad L^2(\Omega). \tag{6}$$

Since $u_m(t) \in V_m$ then

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i\,; \quad g_{im}(t) \text{ to be determined.} \tag{7}$$

Substituting (7) in (5) we obtain

$$\left(\sum_{i=1}^{m} g_{im}'(t)w_i, w_j\right) + \left\langle A\left(\sum_{i=1}^{m} g_{im}(t)w_i\right), w_j\right\rangle = \langle f(t), w_j\rangle; \quad 1 \le j \le m.$$

By the orthonormality of the w_j's in $L^2(\Omega)$ it follows that

$$g_{jm}'(t) + \left\langle A\left(\sum_{i=1}^{m} g_{im}(t)w_i\right), w_j\right\rangle = \langle f(t), w_j\rangle; \quad 1 \le j \le m. \tag{8}$$

Now, from (6) and (7) it follows that

$$u_m(0) = \sum_{i=1}^{m} g_{im}(0)w_i = u_{0m} \to u_0 \quad \text{in} \quad L^2(\Omega). \tag{9}$$

However, since $[(w_\nu)_{\nu\in\mathbb{N}}]$ is dense in $W_0^{1,p}(\Omega)$ and this in turn is dense in $L^2(\Omega)$ then $[(w_\nu)_{\nu\in\mathbb{N}}]$ is dense in $L^2(\Omega)$. Furthermore, since the $(w_\nu)_\nu$ are orthonormal in $L^2(\Omega)$ we have

$$u_0 = \sum_{\nu=1}^{+\infty}(u_0, w_\nu)w_\nu\,. \tag{10}$$

From (9) and (10) it follows that

$$\sum_{i=1}^{m} g_{im}(0)w_i = \sum_{i=1}^{m}(u_0, w_i)w_i$$

and from the orthonormality of the w_i's we obtain

$$g_{jm}(0) = (u_0, w_j);\quad \forall\, j = 1,\dots,m. \tag{11}$$

Thus, from (8) and (11) the system of O.D.E.:

$$\left|\begin{aligned}&g'_{jm}(t) + \left\langle A\left(\sum_{i=1}^{m} g_{im}(t)w_i\right), w_j\right\rangle = \langle f(t), w_j\rangle\\ &g_{jm}(0) = (u_0, w_j);\quad 1 \le j \le m.\end{aligned}\right. \tag{12}$$

We will use Carathéodory's Criterion to determine a local solution of (12). Set

$$Y(t) = \begin{bmatrix} g_{1m}(t)\\ \vdots \\ g_{mm}(t)\end{bmatrix};\qquad Y(0) = \begin{bmatrix}(u_0, w_1)\\ \vdots \\ (u_0, w_m)\end{bmatrix} = Y_0\,. \tag{13}$$

Define

$$h\colon\ ]0,T[\ \times\mathbb{R}^m \to \mathbb{R}^m \tag{14}$$

$$(t,y) \mapsto h(t,y) = \left[\langle f(t), w_j\rangle - \left\langle A\left(\sum_{i=1}^{m} y_i w_i\right), w_j\right\rangle\right]_{1\le j\le m}, \tag{14.1}$$

that is,

$$h(t,y) = \begin{pmatrix}\langle f(t), w_1\rangle - \left\langle A\left(\sum\limits_{i=1}^{m} y_i w_i\right), w_1\right\rangle\\ \vdots\\ \langle f(t), w_m\rangle - \left\langle A\left(\sum\limits_{i=1}^{m} y_i w_i\right), w_m\right\rangle.\end{pmatrix}$$

Then, from (12), (13) and (14) we can write

$$\left|\begin{aligned}&Y'(t) = h(t, Y(t))\\ &Y(0) = Y_0\end{aligned}\right.$$

We will prove, next, that the map (14) is in the Carathéodory conditions. Indeed, since $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ then the map

$$t \mapsto \langle f(t), v\rangle_{W^{-1,p'};W_0^{1,p}}\quad \text{from}\quad [0,T[\quad \text{to}\quad \mathbb{R} \tag{15}$$

is measurable $\forall\, v \in W_0^{1,p}(\Omega)$ and, furthermore, the map

$$t \mapsto ||f(t)||_{W^{-1,p'}(\Omega)} \quad \text{from} \quad]0,T[\quad \text{to} \quad \mathbb{R}$$

belongs to $L^{p'}(0,T)$.

Thus, for almost every $y \in \mathbb{R}^m$ fixed, the map h given in (14) is measurable in t because the map (15) is.

On the other hand, from what was seen previously, we have that the operator A is monotone, hemicontinuous, bounded and coercive. Thus, A is *continuous* from

$$(W_0^{1,p}(\Omega), \tau_{\text{strong}}) \quad \text{into} \quad (W^{-1,p'}(\Omega), \tau_{\text{weak}^*}).$$

This being so, if $(y_\nu) \subset \mathbb{R}^m$ and $y_\nu \to y$ in $\mathbb{R}^m$ then

$$\sum_{i=1}^m y_{\nu,i}\, w_i \xrightarrow{\nu\to+\infty} \sum_{i=1}^m y_i\, w_i \quad \text{in} \quad W_0^{1,p}(\Omega)$$

and, therefore,

$$\left\langle A\Big(\sum_{i=1}^m y_{\nu,i}\, w_i\Big), w_j\right\rangle \xrightarrow{\nu\to+\infty} \left\langle A\Big(\sum_{i=1}^m y_i\, w_i\Big), w_j\right\rangle; \quad \forall\, j = 1, \ldots, m. \tag{16}$$

Thus, for almost every $t \in]0,T[$ fixed, the map h given in (14) is continuous in y by virtue of (16).

On the other hand, setting

$$g(t) = \begin{bmatrix} \langle f(t), w_1\rangle \\ \vdots \\ \langle f(t), w_m\rangle \end{bmatrix} \qquad \text{and} \qquad D(y) = \begin{bmatrix} \left\langle A\Big(\sum_{i=1}^m y_i w_i\Big), w_1\right\rangle \\ \vdots \\ \left\langle A\Big(\sum_{i=1}^m y_i w_i\Big), w_m\right\rangle \end{bmatrix}$$

then

$$h(t,y) = g(t) - D(y).$$

Thus,

$$||h(t,y)||_{\mathbb{R}^m} = ||g(t) - D(y)||_{\mathbb{R}^m} \le ||g(t)||_{\mathbb{R}^m} + ||D(y)||_{\mathbb{R}^m}\,. \tag{17}$$

However, K being a compact of $]0,T[\times\mathbb{R}^m$, we have for all $(t,y) \in K$ that

$$||D(y)||^2 = \sum_{j=1}^m \left|\left\langle A\Big(\sum_i y_i w_i\Big), w_j\right\rangle\right|^2 \le \sum_{j=1}^m \left\|A\Big(\sum_i y_i w_i\Big)\right\|^2_{W^{-1,p'}(\Omega)} ||w_j||^2_{W_0^{1,p}(\Omega)}\,. \tag{18}$$

Recall that A maps bounded sets into bounded sets. Thus, since $y \in \text{proj}_y K$ then $||y|| \le c_1$ and, therefore,

$$\left\|\sum_i y_i w_i\right\|_{W_0^{1,p}(\Omega)} \le \sum_i |y_i|\, ||w_i||_{W_0^{1,p}(\Omega)} \le \max_{1\le i\le m} ||w_i||_{W_0^{1,p}(\Omega)} \sum_{i=1}^m |y_i| \le c_2\,.$$

Thus,

$$\left\| A\Big(\sum_i y_i w_i\Big)\right\|^2_{W^{-1,p'}} \leq c_3\,,$$

and from (13) it follows that

$$||D(y)||^2_{\mathbb{R}^m} \leq m\big(\max_{1\leq j\leq m} ||w_j||^2_{W_0^{1,p}(\Omega)}\big) c_3 = c_4^2\,.$$

It follows from (17) that

$$\begin{aligned} ||h(t,y)||_{\mathbb{R}^m} &\leq \sum_{j=1}^m |\langle f(t), w_j\rangle|^2 + c_4 \leq \sum_{j=1}^m ||f(t)||^2_{W^{-1,p'}}\, ||w_j||^2_{W_0^{1,p}} + c_4 \qquad (19)\\ &\leq \big(\max_{1\leq j\leq m} ||w_j||^2_{W_0^{1,p}}\big)\, ||f(t)||^2_{W^{-1,p'}} + c_4\,. \qquad (14.2)\end{aligned}$$

Since $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ it follows from the inequality above that the map on the right of the inequality in (19) is integrable on $\text{proj}_t\, K$.

Thus, by Carathéodory's Theorem there exists a solution $Y(t)$ of the problem

$$\left|\begin{aligned} &Y'(t) = h(t, Y(t))\\ &Y(0) = Y_0, \end{aligned}\right. \qquad (20)$$

in some interval $[0,t_m)$; $0 < t_m \leq T$. Thus, $Y(t)$ is absolutely continuous and differentiable almost everywhere in $[0,t_m)$. This entails that the maps $g_{jm}(t)$ are absolutely continuous and differentiable a.e. in $[0,t_m)$. We will make, next, a priori estimates that will help us extend the solution $Y(t)$ to the whole interval $[0,T]$.

2ª Step: A Priori Estimate.

Multiplying (5) by $g_{jm}(t)$, $t\in[0,t_m)$ and summing over j, we obtain

$$(u'_m(t), u_m(t)) + \langle Au_m(t), u_m(t)\rangle = \langle f(t), u_m(t)\rangle. \qquad (21)$$

However,

$$\begin{aligned} \langle Au_m(t), u_m(t)\rangle &= \sum_{i=1}^n \int_\Omega \left|\frac{\partial u_m}{\partial x_i}\right|^{p-2} \frac{\partial u_m}{\partial x_i}\,\frac{\partial u_m}{\partial x_i}\,dx\\ &= \sum_{i=1}^n \int_\Omega \left|\frac{\partial u_m}{\partial x_i}\right|^{p} dx = ||u_m(t)||^p_{W_0^{1,p}(\Omega)} \end{aligned}$$

and from (21) it follows that

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + ||u_m(t)||^p_{W_0^{1,p}} \leq ||f(t)||_{W^{-1,p'}(\Omega)}\, ||u_m(t)||_{W_0^{1,p}(\Omega)}\,.$$

From Young's inequality

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + ||u_m(t)||^p_{W_0^{1,p}(\Omega)} \leq \frac{1}{p'}\, ||f(t)||^{p'}_{W^{1,p'}(\Omega)} + \frac{1}{p}\, ||u_m(t)||^p_{W_0^{1,p}(\Omega)}\,.$$

Whence

$$\frac{1}{2}\,\frac{d}{dt}\,|u_m(t)|^2 + \left(1 - \frac{1}{p}\right)||u_m(t)||^p_{W_0^{1,p}(\Omega)} \leq \frac{1}{p'}\,||f(t)||^{p'}_{W^{-1,p'}(\Omega)}\,.$$

Integrating this last inequality from 0 to $t < t_m$; it follows that

$$\frac{1}{2}|u_m(t)|^2 + \frac{1}{p'}\int_0^t ||u_m(s)||^p_{W_0^{1,p}(\Omega)}\,ds \leq \frac{1}{2}|u_m(0)|^2 + \frac{1}{p'}\int_0^t ||f(s)||^{p'}_{W^{-1,p'}(\Omega)}\,ds. \tag{22}$$

From (6) we guarantee the existence of a constant $c_1 > 0$ such that

$$|u_{0m}|^2 \leq c_1\,; \quad \forall\, m.$$

Thus, from (22) we arrive at

$$|u_m(t)|^2 + \frac{2}{p'}\int_0^t ||u_m(s)||^p_{W_0^{1,p}}\,ds \leq c_1 + \frac{2}{p'}\,||f||^{p'}_{L^{p'}(0,T;W^{1,p'}(\Omega))} \tag{23}$$

$\forall\, t \in [0, t_m)$.

Thus, there exists $c_2 > 0$ such that

$$|u_m(t)|^2 + \frac{1}{p'}\int_0^t ||u_m(s)||^p_{W_0^{1,p}(\Omega)}\,ds \leq c_2\,; \quad \forall\, t \in [0, t_m). \tag{24}$$

It follows from (24) that:

$$\sum_{j=1}^m g^2_{jm}(t) = (u_m(t), u_m(t)) = |u_m(t)|^2 \leq c_2\,; \quad \forall\, t \in [0, t_m);\ \forall\, m.$$

From there it follows that

$$||Y(t)||^2_{\mathbb{R}^m} = \sum_{j=1}^m g^2_{jm}(t) \leq c_2\,; \quad \forall\, t \in [0, t_m) \text{ and } \forall\, m$$

It follows from this that the solution $Y(t)$ of (20) can be prolonged to the whole interval $[0, T]$. Thus, for each $m \in \mathbb{N}$ there exists a solution $u_m(t)$ of (5) and (6) in $[0, T]$. Using what we did to obtain (24) we obtain, analogously

$$|u_m(t)|^2 + \frac{1}{p'}\int_0^t ||u_m(s)||^p_{W_0^{1,p}}\,ds \leq c; \quad \forall\, t \in [0, T],\ \forall\, m. \tag{25}$$

Thus

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)) \tag{26}$$

$$(u_m) \quad \text{is bounded in} \quad L^p(0, T; W_0^{1,p}(\Omega)) \tag{27}$$

$$(u_m(T)) \quad \text{is bounded in} \quad L^2(\Omega) \tag{28}$$

Recall that

$$\langle Au, v\rangle \leq ||u||^{p-1}_{W_0^{1,p}(\Omega)}\,||v||_{W_0^{1,p}(\Omega)}\,; \quad \forall\, u, v \in W_0^{1,p}(\Omega).$$

Whence

$$||Au||_{W^{-1,p'}(\Omega)} = \sup_{\substack{||v|| \leq 1 \\ v \neq 0}} \frac{\langle |Au, v\rangle|}{||v||} \leq ||u||^{p-1}_{W_0^{q,p}(\Omega)}; \quad \forall\, u \in W_0^{1,p}(\Omega).$$

Also,

$$||Au||_{W^{-1,p'}(\Omega)} \geq \frac{|\langle Au, u\rangle|}{||u||} = ||u||^{p-1}_{W_0^{1,p}(\Omega)}; \quad \forall\, u \in W_0^{1,p}(\Omega);\ u \neq 0$$

Therefore,

$$||Au||_{W^{-1,p'}(\Omega)} = ||u||^{p-1}_{W_0^{1,p}(\Omega)}; \quad \forall\, u \in W_0^{1,p}(\Omega). \tag{29}$$

In particular,

$$||Au_m(t)||_{W^{-1,p'}(\Omega)} = ||u_m(t)||^{p-1}_{W_0^{1,p}(\Omega)}$$

and from (27) it follows that

$$\int_0^T ||Au_m(t)||^{p'}_{W^{-1,p'}(\Omega)}\,dt = \int_0^T ||u_m(t)||^{(p-1)p'}_{W_0^{1,p}(\Omega)}\,dt = \int_0^T ||u_m(t)||^{p}_{W_0^{1,p}(\Omega)}\,dt < c.$$

Thus,

$$(Au_m) \quad \text{is bounded in} \quad L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{30}$$

From (26), (27), (28) and (30) we obtain a subsequence $(u_\mu)_{\mu\in\mathbb{N}}$ of $(u_\nu)_{\nu\in\mathbb{N}}$ such that

$$u_\mu \overset{*}{\rightharpoonup} u \quad \text{weak* in} \quad L^\infty(0,T;L^2(\Omega)) \tag{31}$$

$$Au_\mu \overset{*}{\rightharpoonup} \chi \quad \text{weak* in} \quad L^{p'}(0,T;W^{-1,p'}(\Omega)) \tag{32}$$

$$u_\mu \rightharpoonup u \quad \text{weakly in} \quad L^p(0,T;W_0^{1,p}(\Omega)) \tag{33}$$

$$u_\mu(T) \rightharpoonup \xi \quad \text{weakly in} \quad L^2(\Omega) \tag{34}$$

3ª Step: Passage to the Limit

Let $\theta \in \mathcal{D}(0,T)$ and $j \in \mathbb{N}$. Multiplying (5) by θ and integrating in $[0,T]$, we obtain, in particular, for the sequence (u_μ) $\mu > j$:

$$\int_0^T (u'_\mu(t), w_j)\theta(t)\,dt + \int_0^T \langle Au_\mu(t), w_j\rangle\theta(t)\,dt = \int_0^T \langle f(t), w_j\rangle\theta(t)\,dt.$$

Whence

$$-\int_0^T (u_\mu(t), w_j)\theta'(t)\,dt + \int_0^T \langle Au_\mu(t), w_j\rangle\theta(t)\,dt = \int_0^T \langle f(t), w_j\rangle\theta(t)\,dt. \tag{35}$$

From (31) we obtain

$$\int_0^T (u_\mu(t), w_j)\theta'(t)\,dt \xrightarrow{\mu\to+\infty} \int_0^T (u(t), w_j)\theta'(t)\,dt \tag{36}$$

and from (32) we have that

$$\int_0^T \langle Au_\mu(t), w_j\rangle\theta(t)\,dt \xrightarrow{\mu\to+\infty} \int_0^T \langle\chi(t), w_j\rangle\theta(t)\,dt. \tag{37}$$

In this way, from (35), (36) and (37) in the limit situation, it follows that

$$-\int_0^T (u(t), w_j)\theta'(t)\,dt + \int_0^T \langle\chi(t), w_j\rangle\theta(t)\,dt = \int_0^T \langle f(t), w_j\rangle\theta(t)\,dt. \tag{38}$$

Since the finite linear combinations of the elements of the basis $(w_\nu)_{\nu\in\mathbb{N}}$ are dense in $W_0^{1,p}(\Omega)$, it follows that the equality in (38) is valid for all $w \in W_0^{1,p}(\Omega)$. Thus,

$$-\int_0^T (u(t), w)\theta'(t)\,dt + \int_0^T \langle\chi(t), w\rangle\theta(t)\,dt = \int_0^T \langle f(t), w\rangle\theta(t)\,dt, \tag{39}$$

for all $w \in W_0^{1,p}(\Omega)$, for all $\theta \in \mathcal{D}(0,T)$.

Identifying $L^2(\Omega)$ with its dual we have the following chain:

$$W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \approx (L^2(\Omega))' \hookrightarrow W^{1,p'}(\Omega),$$

so that

$$\langle h, g\rangle_{W^{-1,p'}(\Omega);W_0^{1,p}(\Omega)} = (h,g); \quad \forall\, h, g \in L^2(\Omega).$$

It follows from this that (39) can be rewritten as

$$-\int_0^T \langle u(t), w\rangle\theta'(t)\,dt + \int_0^T \langle\chi(t), w\rangle\theta(t)\,dt = \int_0^T \langle f(t), w\rangle\theta(t)\,dt.$$

Therefore,

$$\left\langle -\int_0^T u(t)\theta'(t)\,dt, w\right\rangle + \left\langle \int_0^T \chi(t)\theta(t)\,dt, w\right\rangle = \left\langle \int_0^T f(t)\theta(t)\,dt, w\right\rangle,$$

for all $w \in W_0^{1,p}(\Omega)$ and for all $\theta \in \mathcal{D}(0,T)$. It follows from the identity above that

$$-\int_0^T u(t)\theta'(t)\,dt + \int_0^T \chi(t)\theta(t)\,dt = \int_0^T f(t)\theta(t)\,dt$$

in $W^{-1,p'}(\Omega)$ for all $\theta \in \mathcal{D}(0,T)$. In this way,

$$\left\langle \frac{du}{dt}, \theta\right\rangle + \langle\chi, \theta\rangle = \langle f, \theta\rangle \quad \text{in} \quad W^{-1,p'}(\Omega);\ \forall\,\theta \in \mathcal{D}(0,T)$$

that is,

$$\frac{du}{dt} + \chi = f \quad \text{in} \quad \mathcal{D}'(0,T; W^{-1,p'}(\Omega)).$$

Considering $f, \chi \in L^{p'}(0,T; W^{-1,p'}(\Omega))$ it follows that

$$\frac{du}{dt} \in L^{p'}(0,T; W^{-1,p'}(\Omega)) \tag{40}$$

and

$$\frac{du}{dt} + \chi = f \quad \text{in} \quad L^{p'}(0,T; W^{-1,p'}(\Omega)). \tag{41}$$

The next step is to prove the initial condition

4ª Step: Initial Condition.

Since $u \in L^p(0,T;W_0^{1,p}(\Omega))$ and $u' \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ then

$$u \in C_s(0,T;W_0^{1,p}(\Omega)) \cap C^0([0,T];W^{-1,p'}(\Omega)).$$

Let $\theta \in C^1([0,T])$ and $w \in W_0^{1,p}(\Omega)$. Then $\theta w \in L^p(0,T;W_0^{1,p}(\Omega))$ and from (41) we obtain

$$\int_0^T \langle u'(t), w\rangle\theta(t)\,dt + \int_0^T \langle \chi(t), w\rangle\theta(t)\,dt = \int_0^T \langle f(t), w\rangle\theta(t)\,dt. \tag{42}$$

We claim that

$$\frac{d}{dt}\big[\langle u(t), w\rangle\theta(t)\big] = \langle u'(t), w\rangle\theta(t) + \langle u(t), w\rangle\,\frac{d\theta(t)}{dt}\,; \quad \forall\, w \in W_0^{1,p}(\Omega). \tag{43}$$

Indeed, set

$$W(0,T) = \left\{ u \in L^p(0,T;W_0^{1,p}(\Omega)); \quad \frac{du}{dt} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \right\}$$

endowed with the topology

$$||u||_W = ||u||_{L^p(0,T;W_0^{1,p}(\Omega))} + \left\|\frac{du}{dt}\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))},$$

which makes it a Banach space.

Since $\mathcal{D}([0,T];W_0^{1,p}(\Omega))$ is dense in $W(0,T)$, given $u \in W(0,T)$ there exists $(u_\nu) \subset \mathcal{D}([0,T];W_0^{1,p}(\Omega))$ such that

$$u_\nu \to u \quad \text{in} \quad W(0,T).$$

Whence

$$u_\nu \to u \text{ in } L^p(0,T;W_0^{1,p}(\Omega)) \quad \text{and} \quad u'_\nu \to u' \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{44}$$

We have, given the regularity of the u_ν's,

$$\frac{d}{dt}\,[(u_\nu(t), w)\theta(t)] = (u'_\nu(t), w)\theta(t) + (u_\nu(t), w)\,\frac{d\theta(t)}{dt}\,.$$

Now, identifying $L^2(\Omega)$ with its dual we can write

$$\frac{d}{dt}\,[\langle u_\nu(t), w\rangle\theta(t)] = \langle u'_\nu(t), w\rangle\theta(t) + \langle u_\nu(t), w\rangle\theta'(t). \tag{45}$$

The next step is to pass the limit in (45) to obtain (43). Given the convergences in (44), we have that the right side of (45) converges.

Indeed, we have that

$$\langle u'_\nu(t), w\rangle\theta(t) \to \langle u'(t), w\rangle\theta(t) \quad \text{in} \quad L^1(0,T) \tag{46}$$

since

$$\int_0^T |\langle u_\nu'(t)-u'(t),w\theta\rangle|\,dt \le \int_0^T ||u_\nu'(t)-u'(t)||_{W^{-1,p'}(\Omega)}\,||w\theta||_{W_0^{1,p}(\Omega)}\,dt$$
$$\le\Big(\int_0^T ||u_\nu'(t)-u'(t)||^{p'}_{W^{-1,p'}(\Omega)}\Big)^{1/p'}\Big(\int_0^T ||w\theta||^p_{W_0^{1,p}(\Omega)}\Big)^{1/p}\to 0,$$

since $u_\nu'\to u'$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Analogously,

$$\langle u_\nu(t),w\rangle\theta'(t)\to\langle u(t),w\rangle\theta'(t)\quad\text{in}\quad L^1(0,T) \tag{47}$$

because

$$\int_0^T |\langle u_\nu(t)-u(t),w\theta'\rangle|\,dt \le \int_0^T ||u_\nu(t)-u(t)||_{W^{-1,p'}(\Omega)}\,||w\theta'(t)||_{W_0^{1,p}(\Omega)}\,dt$$
$$\le\Big(\int_0^T ||u_\nu(t)-u(t)||^{p'}_{W^{-1,p'}(\Omega)}\,dt\Big)^{1/p'}\Big(\int_0^T ||w\theta'(t)||^p_{W_0^{1,p}(\Omega)}\,dt\Big)^{1/p}\to 0,$$

since if $u_\nu\to u$ in $L^p(0,T;W_0^{1,p}(\Omega))$ then $u_\nu\to u$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ (note that $L^p(0,T;W_0^{1,p}(\Omega))\hookrightarrow L^{p'}(0,T;W^{-1,p'}(\Omega))$.

From (46) and (47) we conclude that

$$\langle u_\nu'(t),w\rangle\theta(t)+\langle u_\nu(t),w\rangle\theta'(t)\to\langle u'(t),w\rangle\theta(t)+\langle u(t),w\rangle\theta'(t) \tag{48}$$

in $L^1(0,T)$.

On the other hand, we also have in a manner analogous to (46) that

$$\langle u_\nu(t),w\rangle\theta(t)\to\langle u(t),w\rangle\theta(t)\quad\text{in}\quad L^1(0,T)$$

and, consequently,

$$\frac{d}{dt}\big[\langle u_\nu(t),w\rangle\theta(t)\big]\to\frac{d}{dt}\big[\langle u(t),w\rangle\theta(t)\big]\quad\text{in}\quad\mathcal{D}'(0,T). \tag{49}$$

Thus, from (45), (48) and (49) we conclude, given the uniqueness of the limit in $\mathcal{D}'(0,T)$, that

$$\frac{d}{dt}\big[\langle u(t),w\rangle\theta(t)\big]=\langle u'(t),w\rangle\theta(t)+\langle u(t),w\rangle\theta'(t)$$

which proves (43). Assuming that $\theta(0)=1$ and $\theta(T)=0$, considering the fact that $u\in C^0(0,T;W^{-1,p'}(\Omega))$ and integrating in $[0,T]$, it follows that

$$\langle u(T),w\rangle\theta(T)-\langle u(0),w\rangle\theta(0)=\int_0^T\langle u'(t),w\rangle\theta(t)\,dt+\int_0^T\langle u(t),w\rangle\theta'(t)\,dt,$$

that is,

$$\int_0^T\langle u'(t),w\rangle\theta(t)\,dt=-\langle u(0),w\rangle-\int_0^T\langle u(t),w\rangle\theta'(t)\,dt. \tag{50}$$

Substituting (50) in (42) we obtain

$$-\langle u(0),w\rangle-\int_0^T\langle u(t),w\rangle\theta'(t)\,dt+\int_0^T\langle\chi(t),w\rangle\theta(t)\,dt=\int_0^T\langle f(t),w\rangle\theta(t)\,dt. \tag{51}$$

Let $j \in \mathbb{N}$. Multiplying the approximate equation given in (5) by θ and integrating in $[0,T]$ we obtain in particular for (u_μ)[23], $\mu > j$,

$$\int_0^T (u'_\mu(t), w_j)\theta(t)\,dt + \int_0^T \langle Au_\mu(t), w_j\rangle\theta(t)\,dt = \int_0^T \langle f(t), w_j\rangle\theta(t)\,dt. \tag{52}$$

However,

$$\frac{d}{dt}(u_\mu(t), w_j)\theta(t) = (u'_\mu(t), w_j)\theta(t) + (u_\mu(t), w_j)\theta'(t).$$

Integrating in $[0,T]$ we obtain

$$-(u_\mu(0), w_j) = \int_0^T (u'_\mu(t), w_j)\theta(t)\,dt + \int_0^T (u_\mu(t), w_j)\theta'(t)\,dt. \tag{53}$$

Substituting (53) in (52) results that

$$\begin{aligned}-\langle u_{0\mu}, w_j\rangle - \int_0^T \langle u_\mu(t), w_j\rangle\theta'(t)\,dt + \int_0^T \langle Au_\mu(t), w_j\rangle\theta(t)\,dt \\ = \int_0^T \langle f(t), w_j\rangle\theta(t)\,dt.\end{aligned}$$

In the limit situation, given the convergences in (6), (31) and (32), we obtain

$$\begin{aligned}-\langle u_0, w_j\rangle - \int_0^T \langle u(t), w_j\rangle\theta'(t)\,dt + \int_0^T \langle \chi(t), w_j\rangle\theta(t)\,dt \\ = \int_0^T \langle f(t), w_j\rangle\theta(t)\,dt.\end{aligned}$$

By density the expression above remains valid for all $w \in W_0^{1,p}(\Omega)$. Whence

$$-\langle u_0, w\rangle - \int_0^T \langle u(t), w\rangle\theta(t)\,dt + \int_0^T \langle \chi(t), w\rangle\theta(t)\,dt = \int_0^T \langle f(t), w\rangle\theta(t)\,dt. \tag{54}$$

Finally, from (51) and (54) we obtain

$$\langle u(0), w\rangle = \langle u_0, w\rangle; \qquad \forall\, w \in W_0^{1,p}(\Omega),$$

that is,

$$u(0) = u_0 \quad \text{in} \quad W^{-1,p'}(\Omega), \quad \text{in truth, in } L^2(\Omega). \tag{55}$$

The Theorem will be proved, except for uniqueness, if we show that

$$\chi = Au. \tag{56}$$

This is what we will do next. Consider (u_μ) the subsequence of (u_ν) given in (31)-(34). Then, by the monotonicity of the operator A we can write

$$0 \le \int_0^T \langle Au_\mu - Av, u_\mu - v\rangle\,dt; \quad \forall\, v \in L^p(0,T; W_0^{1,p}(\Omega)).$$

[23] Here (u_μ) is the sequence obtained in (31)-(34).

Whence

$$0 \leq \int_0^T \langle Au_\mu, u_\mu \rangle \, dt - \int_0^T \langle Au_\mu, v \rangle \, dt - \int_0^T \langle Av, u_\mu - v \rangle \, dt. \tag{57}$$

On the other hand, returning to the approximate equation given in (5) we obtain

$$\frac{1}{2} \frac{d}{dt} |u_\mu(t)|^2 + \langle Au_\mu, u_\mu \rangle = \langle f, u_\mu \rangle.$$

Integrating the expression above from 0 to T it follows that

$$\int_0^T \langle Au_\mu, u_\mu \rangle \, dt = \frac{1}{2} |u_\mu(0)|^2 - \frac{1}{2} |u_\mu(T)|^2 + \int_0^T \langle f, u_\mu \rangle \, dt. \tag{58}$$

Substituting (58) in (57) results that

$$\begin{aligned} 0 \leq \frac{1}{2} |u_{0\mu}|^2 - \frac{1}{2} |u_\mu(T)|^2 + \int_0^T \langle f, u_\mu \rangle \, dt - \int_0^T \langle Au_\mu, v \rangle \, dt \\ - \int_0^T \langle Av, u_\mu - v \rangle \, dt. \end{aligned} \tag{59}$$

We observe that if we use the same technique applied to prove that $u(0) = u_0$, we can also prove that

$$u(T) = \xi, \tag{60}$$

where ξ is given in (34). Indeed, it suffices to consider $\theta \in C^1[0, T]$ such that $\theta(0) = 0$ and $\theta(T) = 1$ and proceed as in the $4^{\underline{a}}$ step.

Thus from (34) and (60) we obtain

$$u_\nu(T) \rightharpoonup u(T) \quad \text{in} \quad L^2(\Omega). \tag{61}$$

Taking the $\overline{\lim}$ in (59) results that

$$\begin{aligned} 0 \leq \overline{\lim} \frac{1}{2} |u_{0\mu}|^2 - \underline{\lim} \frac{1}{2} |u_\mu(T)|^2 + \overline{\lim} \int_0^T \langle f, u_\mu \rangle \, dt \\ - \underline{\lim} \int_0^T \langle Au_\mu, v \rangle \, dt - \underline{\lim} \int_0^T \langle Av, u_\mu - v \rangle \, dt. \end{aligned}$$

However from the convergences in (6), (32), (33) and (61)[24] we obtain

$$0 \leq \frac{1}{2} |u_0|^2 - \frac{1}{2} |u(T)|^2 + \int_0^T \langle f, u \rangle \, dt - \int_0^T \langle \chi, v \rangle \, dt - \int_0^T \langle Av, u - v \rangle \, dt. \tag{62}$$

On the other hand, from (41) we can write

$$\int_0^T \langle u'(t), u(t) \rangle \, dt + \int_0^T \langle \chi(t), u(t) \rangle \, dt = \int_0^T \langle f(t), u(t) \rangle \, dt. \tag{63}$$

[24] Use Banach-Steinhaus: $|u(T)| \leq \underline{\lim} |u_\nu(T)| \Rightarrow |u(T)|^2 \leq \underline{\lim} |u_\nu(T)|^2$
$(\underline{\lim} |u_\nu|)^2 = (\underline{\lim} |u_\nu|)(\underline{\lim} |u_\nu|) \leq \underline{\lim} |u_\nu|^2$

However, proceeding in a manner analogous to the proof of the identity in (43) we prove, identifying $L^2(\Omega)$ with its dual, that

$$\frac{d}{dt}\,(u(t),u(t)) = 2\langle u'(t),u(t)\rangle$$

that is,

$$\frac{1}{2}\,\frac{d}{dt}\,|u(t)|^2 = \langle u'(t),u(t)\rangle. \tag{64}$$

Substituting (64) in (63) we conclude that

$$\frac{1}{2}\int_0^T \frac{d}{dt}\,|u(t)|^2\,dt + \int_0^T \langle \chi(t),u(t)\rangle\,dt = \int_0^T \langle f(t),u(t)\rangle\,dt,$$

that is,

$$\frac{1}{2}\,|u(T)|^2 - \frac{1}{2}\,|u(0)|^2 + \int_0^T \langle \chi(t),u(t)\rangle\,dt = \int_0^T \langle f(t),u(t)\rangle\,dt. \tag{65}$$

Thus, from (55), (62) and (65) it results that

$$0 \le \int_0^T \langle \chi,u\rangle\,dt - \int_0^T \langle \chi,v\rangle\,dt - \int_0^T \langle Av,u-v\rangle\,dt,$$

that is,

$$0 \le \int_0^T \langle \chi - Av,u-v\rangle\,dt; \qquad \forall\, v \in L^p(0,T;W_0^{1,p}(\Omega)). \tag{66}$$

Let $w \in L^p(0,T;W_0^{1,p}(\Omega))$ and $\lambda > 0$. Then, taking $v = u - \lambda w$ in (66) it follows that

$$0 \le \int_0^T \langle \chi - A(u-\lambda w),w\rangle\,dt$$

and, by the hemicontinuity of A, we conclude when $\lambda \to 0$ that

$$0 \le \int_0^T \langle \chi - Au,w\rangle\,dt, \qquad \forall\, w \in L^p(0,T;W_0^{1,p}(\Omega)). \tag{67}$$

In the same way, taking $v = u - \lambda w$; where $\lambda < 0$, we obtain the opposite inequality

$$0 \ge \int_0^T \langle \chi - Au,w\rangle\,dt; \qquad \forall\, w \in L^p(0,T;W_0^{1,p}(\Omega)). \tag{68}$$

Whence, from (67) and (68)

$$\int_0^T \langle \chi - Au,w\rangle\,dt = 0; \qquad \forall\, w \in L^p(0,T;W_0^{1,p}(\Omega)),$$

that is,

$$\langle \chi - Au,w\rangle_{L^{p'}(0,T;W^{-1,p'}(\Omega)),L^p(0,T;W_0^{1,p}(\Omega))} = 0; \quad \forall\, w \in L^p(0,T;W_0^{1,p}(\Omega)).$$

The equality above leads us to conclude that

$$\chi = Au \quad \text{in} \quad L^{p'}(0,T;W^{-1,p'}(\Omega)),$$

which proves (56).

5ª Step: Uniqueness

Let u_1 and u_2 be solutions of Theorem 1. Then, from (41)

$$W = u_1 - u_2$$

satisfies

$$\frac{dW}{dt} = \frac{d}{dt}(u_1 - u_2) = (f - Au_1) - (f - Au_2) \quad \text{in} \quad L^{p'}(0, T, W^{-1,p'}(\Omega)).$$

Whence

$$\frac{dW}{dt} + Au_1 - Au_2 = 0 \quad \text{in} \quad L^{p'}(0, T; W^{-1,p'}(\Omega)). \tag{69}$$

Also

$$W(0) = 0. \tag{70}$$

It follows from (69) that for almost all $q \in [0, T]$

$$W'(t) + Au_1(t) - Au_2(t) = 0 \quad \text{in} \quad W^{-1,p'}(\Omega).$$

Composing the identity above with $W(t)$ results that

$$\langle W'(t), W(t)\rangle + \langle Au_1(t) - Au_2(t), W(t)\rangle = 0. \tag{71}$$

However, in a manner analogous to (64), we can write that

$$\frac{1}{2}\frac{d}{dt}|W(t)|^2 = \langle W'(t), W(t)\rangle$$

and, from (71), it results that

$$\frac{1}{2}\frac{d}{dt}|W(t)|^2 + \langle Au_1(t) - Au_2(t), W(t)\rangle = 0. \tag{72}$$

By the monotonicity of the operator A we have that

$$\langle Au_1(t) - Au_2(t), W(t)\rangle \geq 0; \qquad \text{a.e. in } [0, T[$$

and, from (72), it follows that

$$\frac{1}{2}\frac{d}{dt}|W(t)|^2 \leq 0 \quad \text{a.e. in} \quad]0, T[\,.$$

Integrating the inequality above we obtain

$$|W(t)|^2 - |W(0)|^2 \leq 0; \qquad \forall\, t \in [0, T]$$

and, from (70), it results that

$$0 \leq |W(t)|^2 \leq 0; \qquad \forall\, t \in [0, T].$$

Whence,

$$W(t) = 0; \qquad \forall\, t \in [0, T],$$

that is, $u_1(t) = u_2(t)$; $\forall\, t \in [0, T]$. □

14.2 Hyperbolic Problem with Viscosity

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded subset with sufficiently regular boundary Γ. Consider the problem

$$\left| \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) - \Delta\left(\frac{\partial u}{\partial t}\right) = f \text{ in } Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(x,0) = u_0(x); \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x); \;\; x \in \Omega, \end{array} \right. \tag{1}$$

where $2 \leq p < +\infty$.

Before we determine a solution for (1) we will make some initial considerations essential to its understanding. Initially, we will prove that

$$W_0^{1,p}(\Omega) \subset H_0^1(\Omega); \quad \text{for all} \quad 2 \leq p < +\infty. \tag{2}$$

Indeed, let $u \in W_0^{1,p}(\Omega)$. Then, $\exists\,(\varphi_\nu) \subset \mathcal{D}(\Omega)$ such that $\varphi_\nu \to u$ in $W^{1,p}(\Omega)$. Since $2 \leq p < +\infty$ and Ω is bounded it results that $W^{1,p}(\Omega) \hookrightarrow H^1(\Omega)$ since $L^p(\Omega) \hookrightarrow L^2(\Omega)$. Thus, $\varphi_\nu \to u$ in $H^1(\Omega)$ which proves that $u \in H_0^1(\Omega)$ and, consequently, (2) is proved.

Moreover, we will prove that the embedding is continuous. In fact, let $u \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} ||u||_{H_0^1(\Omega)} &= \left(\sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2_{L^2(\Omega)} \right)^{1/2} \leq \left\{ |u|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2_{L^2(\Omega)} \right\}^{1/2} \\ &\leq c_1 \left\{ |u|_{L^2(\Omega)} + \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|_{L^2(\Omega)} \right\} \\ &\leq c_2 \left\{ |u|_{L^p(\Omega)} + \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|_{L^p(\Omega)} \right\} \\ &\leq c_3 \left\{ |u|^p_{L^p(\Omega)} + \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^p_{L^p(\Omega)} \right\}^{1/p} \\ &\leq c_4 \left(\sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^p_{L^p(\Omega)} \right)^{1/p} = c_4 ||u||_{W_0^{1,p}(\Omega)}, \end{aligned}$$

that is,

$$||u||_{H_0^1(\Omega)} \leq c_4 ||u||_{W_0^{1,p}(\Omega)}, \quad \forall\, u \in W_0^{1,p}(\Omega); \;\; 2 \leq p < +\infty. \tag{3}$$

Since $W_0^{1,p}(\Omega)$ is a reflexive Banach space then, identifying $L^2(\Omega)$ with its dual, we have the following chain of embeddings:

$$W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \approx (L^2(\Omega))' \hookrightarrow H^{-1}(\Omega) \hookrightarrow W^{-1,p'}(\Omega), \tag{4}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Our goal is to obtain a "*special basis*" to solve problem (1). However, since $W_0^{1,p}(\Omega)$ is Banach we cannot make use of the Spectral Theorem. We must, then, obtain a Hilbert space contained in $W_0^{1,p}(\Omega)$ in order to apply the Spectral Theorem and thus obtain a special basis. From Sobolev embeddings we have

$$W_0^{s,q}(\Omega) \hookrightarrow W_0^{s-r,q_r}(\Omega),$$

where $\dfrac{1}{q_r} = \dfrac{1}{q} - \dfrac{r}{n} \cdot$ In this way, choosing

$$q = 2; \quad q_r = p$$

we obtain

$$\frac{r}{n} = \frac{1}{2} - \frac{1}{p} \Rightarrow r = \frac{(p-2)\cdot n}{2p} > 0.$$

Consequently, setting

$$s = r + 1$$

we have $s - r = 1 > 0$ and, therefore,

$$H_0^s(\Omega) = W_0^{s,2}(\Omega) \hookrightarrow W_0^{1,p}(\Omega). \tag{5}$$

Since $H_0^s(\Omega)$ is a Hilbert space we have from (4) and (5) the following chain of continuous and dense embeddings:

$$H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow W^{-1,p'}(\Omega) \hookrightarrow H^{-s}(\Omega). \tag{6}$$

However, since the embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact (thanks to Rellich's Theorem), it follows that the embedding of $H_0^s(\Omega)$ in $L^2(\Omega)$ is also compact. In this way, letting B be the operator defined by the triple

$$\{H_0^s(\Omega),\ L^2(\Omega);\ ((\,\cdot\,,\,\cdot\,))_s\},$$

where $((\,\cdot\,,\,\cdot\,))_s$ denotes the inner product in $H_0^s(\Omega)$, we have by the "Spectral Theorem" the existence of a collection $(w_\nu)_{\nu\in\mathbb{N}}$ of eigenvectors of the operator B whose associated eigenvalues $(\lambda_\nu)_{\nu\in\mathbb{N}}$ are positive and such that $\lambda_\nu \le \lambda_{\nu+1}$ and $\lambda_\nu \to +\infty$ when $\nu \to \infty$. Furthermore,

$$(w_\nu) \quad \text{is a complete orthonormal system in} \quad L^2(\Omega) \tag{7}$$

$$\left(\frac{w_\nu}{\sqrt{\lambda_\nu}}\right) \quad \text{is a complete orthonormal system in} \quad H_0^s(\Omega). \tag{8}$$

Thus, for each $\nu \in \mathbb{N}$, we have

$$(Bw_\nu, v) = ((w_\nu, v))_s\,; \quad \forall\, v \in H_0^s(\Omega),$$

that is,

$$\lambda_\nu (w_\nu, v) = ((w_\nu, v))_s\,; \quad \forall\, v \in H_0^s(\Omega). \tag{9}$$

In what follows we will prove the result below.

Theorem 2: Given

$$u_0 \in W_0^{1,p}(\Omega); \;\; u_1 \in L^2(\Omega) \quad \text{and} \quad f \in L^2(0,T;L^2(\Omega))$$

there exists a unique function $u\colon Q = \Omega\times]0,T[\to \mathbb{R}$ in the class

$$u \in L^\infty(0,T;W_0^{1,p}(\Omega)) \tag{10}$$

$$u' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \tag{11}$$

$$u'' \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \tag{12}$$

verifying

$$\frac{d}{dt}(u'(t),w) - \left\langle \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-1} \frac{\partial u}{\partial x_i}\right), w \right\rangle + (\nabla u'(t), \nabla w) = (f(t),w), \tag{13}$$

$\forall\, w \in W_0^{1,p}(\Omega)$,

$$u(0) = u_0\,; \quad u'(0) = u_1\,. \tag{14}$$

Proof:
1ª Step: Approximate System.

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be the "special basis" of $H_0^s(\Omega)$ mentioned previously and consider

$$V_m = [w_1, \ldots, w_m].$$

In V_m we have the approximate problem

$$u_m(t) \in V_m \tag{15}$$

$$(u''_m(t), w_j) + \langle A u_m(t), w_j\rangle + a(u'_m(t), w_j) = (f(t), w_j);\ j = 1, \ldots, m \tag{16}$$

$$u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad W_0^{1,p}(\Omega);\ u_{0m} \in V_m \tag{17}$$

$$u'_m(0) = u_{1m} \to u_1 \quad \text{in} \quad L^2(\Omega);\ \ u_{1m} \in V_m. \tag{18}$$

From (15) it follows that

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i\,; \quad g_{im}(t) \text{ to be determined.} \tag{19}$$

Substituting (19) in (15), it follows from (7) that

$$\begin{aligned} g''_{jm}(t) &+ \left\langle A\left(\sum_{i=1}^{m} g_{im}(t) w_i\right), w_j\right\rangle \\ &+ a\left(\sum_{i=1}^{m} g'_{im}(t) w_i, w_j\right) = (f(t), w_j),\ \ j = 1, \ldots, m. \end{aligned} \tag{20}$$

However from (8) we have that (w_ν) is complete in $H_0^s(\Omega)$. Since this in turn is dense in $W_0^{1,p}(\Omega)$ then we also have that (w_ν) is total (or complete) in $W_0^{1,p}(\Omega)$.

Thus, since u_0 is in $W_0^{1,p}(\Omega)$, there will exist $(\eta_\nu)_\nu\,,\ \ \eta_\nu = \sum_{j=1}^{m(\nu)} \alpha_{jm} w_j \in [(w_\nu)_\nu]$ such that

$$\sum_{j=1}^{m(\nu)} \alpha_{jm} w_j \xrightarrow{\nu\to+\infty} u_0 \quad \text{in} \quad W_0^{1,p}(\Omega). \tag{21}$$

Analogously, since $u_1 \in L^2(\Omega)$ and (w_ν) is total in $L^2(\Omega)$ it follows that there will exist $(\xi_\nu)_\nu\,;\ \ \xi_\nu = \sum_{j=1}^{m(\nu)} \beta_{jm} w_j$ such that

$$\sum_{j=1}^{m(\nu)} \beta_{jm} w_j \xrightarrow{\nu\to+\infty} u_1 \quad \text{in} \quad L^2(\Omega). \tag{22}$$

But, from (17) and (18) we have that

$$u_{0m} = \sum_{j=1}^{m} g_{jm}(0) w_j \to u_0 \quad \text{in} \quad W_0^{1,p}(\Omega) \tag{23}$$

$$u_{1m} = \sum_{j=1}^{m} g'_{jm}(0) w_j \to u_1 \quad \text{in} \quad L^2(\Omega). \tag{24}$$

It results from (21)-(24) that

$$\sum_{j=1}^{m(\nu)} \alpha_{jm} w_j = \sum_{j=1}^{m} g_{jm}(0) w_j \tag{25}$$

and

$$\sum_{j=1}^{m(\nu)} \beta_{jm} w_j = \sum_{j=1}^{m} g'_{jm}(0) w_j \,. \tag{26}$$

Note, initially, that since the w_j's are linearly independent we can consider, without loss of generality, $m = m(\nu)$ and, by the orthonormality of the system in $L^2(\Omega)$, we have from (25) and (26) that

$$g_{jm}(0) = \alpha_{jm}\,; \quad \forall\, j = 1, \ldots, m \tag{27}$$

$$g'_{jm}(0) = \beta_{jm}\,; \quad \forall\, j = 1, \ldots, m. \tag{28}$$

Thus, from (20), (27) and (28) we have the system of O.D.E.

$$\left| \begin{array}{l} g''_{jm}(t) + \left\langle A\Big(\sum_{i=1}^{m} g_{im}(t), w_i\Big), w_j \right\rangle + \sum_{i=1}^{m} g'_{im}(t) a(w_i, w_j) = (f(t), w_j) \\ g_{jm}(0) = \alpha_{jm}\,;\ g'_{jm}\,(0) = \beta_{jm}; \quad j = 1, \ldots, m. \end{array} \right. \tag{29}$$

The problem above is equivalent to

$$\begin{bmatrix} g''_{im}(t) \\ \vdots \\ g''_{mm}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \langle A(\sum_{i=1}^{m} g_{im} w_i), w_1 \rangle \\ \vdots \\ \langle A(\sum_{i=1}^{m} g_{im} w_i), w_m \rangle \end{bmatrix}}_{G(Z(t))}$$

$$+ \underbrace{\begin{bmatrix} a(w_1, w_1) \ldots a(w_m, w_1) \\ \vdots \\ a(w_1, w_m) \ldots a(w_m, w_m) \end{bmatrix}}_{C} \begin{bmatrix} g'_{1m}(t) \\ \vdots \\ g'_{mm}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} (f(t), w_1) \\ \vdots \\ (f(t), w_m) \end{bmatrix}}_{F(t)},$$

$$\begin{bmatrix} g_{1m}(0) \\ \vdots \\ g_{mm}(0) \end{bmatrix} = \begin{bmatrix} \alpha_{1m} \\ \vdots \\ \alpha_{mm} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} g'_{1m}(0) \\ \vdots \\ g'_{mm}(0) \end{bmatrix} = \begin{bmatrix} \beta_{1m} \\ \vdots \\ \beta_{mm} \end{bmatrix}.$$

Defining

$$Z(t) = \begin{bmatrix} g_{1m}(t) \\ \vdots \\ g_{mm}(t) \end{bmatrix}$$

it follows that

$$\left| \begin{aligned} & Z''(t) + G(Z(t)) + CZ'(t) = F(t) \\ & Z(0) = Z_0\,; \quad Z'(0) = Z_1; \end{aligned} \right. \tag{30}$$

where

$$Z_0 = \begin{bmatrix} \alpha_{1m} \\ \vdots \\ \alpha_{mm} \end{bmatrix} \quad \text{and} \quad Z_1 = \begin{bmatrix} \beta_{1m} \\ \vdots \\ \beta_{mm} \end{bmatrix}. \tag{31}$$

Setting

$$Y_1(t) = Z(t), \quad Y_2(t) = Z'(t) \quad \text{and} \quad Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}$$

it follows that

$$\begin{aligned} Y'(t) = \begin{bmatrix} Y_1'(t) \\ Y_2'(t) \end{bmatrix} &= \begin{bmatrix} Z'(t) \\ Z''(t) \end{bmatrix} = \begin{bmatrix} Z'(t) \\ F(t) - G(Z(t)) - CZ'(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -G(Z(t)) \end{bmatrix} + \begin{bmatrix} Z'(t) \\ -CZ'(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}. \end{aligned}$$

Thus,

$$Y'(t) = \overbrace{\begin{bmatrix} 0 \\ -G(Z(t)) \end{bmatrix}}^{\mathcal{G}(Y(t))} + \overbrace{\begin{bmatrix} 0 & \mathbf{1} \\ 0 & -C \end{bmatrix}}^{D} \overbrace{\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}}^{Y} + \overbrace{\begin{bmatrix} 0 \\ F(t) \end{bmatrix}}^{\mathcal{F}(t)}$$

and

$$Y(0) = \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix} = \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} = Y_0,$$

that is,

$$\left| \begin{aligned} & Y'(t) = \mathcal{G}(Y(t)) + DY(t) + \mathcal{F}(t) \\ & Y(0) = Y_0. \end{aligned} \right.$$

Let us define, now, the auxiliary function

$$\begin{aligned} h \colon [0,T] \times \mathbb{R}^{2m} &\to \mathbb{R}^{2m} \\ (t,y) &\mapsto h(t,y) = \mathcal{G}(y) + Dy + \mathcal{F}(t). \end{aligned} \tag{32}$$

We will prove, next, that h satisfies the Carathéodory conditions. Indeed, since $f \in L^2(0,T;L^2(\Omega)$, the map

$$t \mapsto (f(t), w_j) \tag{33}$$

is measurable; for all $j = 1, \ldots, m$. Moreover, the map

$$t \mapsto |f(t)|^2_{L^2(\Omega)}$$

belongs to $L^1(0,T)$. Thus, for almost every $y \in \mathbb{R}^{2m}$ fixed, the map h given in (32) is measurable in t because map (33) is.

On the other hand, we saw in Chapter 1, that the pseudo-Laplacian operator A is monotone, hemicontinuous, bounded and coercive. Thus, A is *continuous* from

$$(W_0^{1,p}, \tau_{\text{strong}}) \quad \text{to} \quad (W^{-1,p'}; \tau_{\text{weak}^*}).$$

This being so, if $(y_\nu) \subset \mathbb{R}^{2m}$ and $y_\nu \to y$ in $\mathbb{R}^{2m}$ then

$$\sum_{i=1}^{m} y_{\nu,i} w_i \to \sum_{i=1}^{m} y_i w_i \quad \text{in} \quad W_0^{1,p}(\Omega)$$

and, therefore,

$$\left\langle A\Big(\sum_{j=1}^{m} y_{\nu,i} w_i\Big), w_j \right\rangle \to \left\langle A\Big(\sum_{i=1}^{m} y_i w_i\Big), w_j \right\rangle, \quad \forall\, j = 1, \ldots, m. \tag{34}$$

It follows from this that

$$\mathcal{G}(y_\nu) \to \mathcal{G}(y)$$

and, from the fact that

$$D \cdot y_\nu \to D \cdot y,$$

we have that the map h given in (32) is continuous in y.

Finally, let K be a compact of $]0, T[\, \times \mathbb{R}^{2m}$. For all $(t, y) \in K$, we have

$$||h(t,y)||_{2m} = ||\mathcal{G}(y) + D(y) + \mathcal{F}(t)|| \leq ||\mathcal{G}(y)|| + ||D||\, ||y|| + ||\mathcal{F}(t)||. \tag{35}$$

But

$$\begin{aligned} ||\mathcal{G}(y)||_{2m}^2 = ||G(y)||_m^2 &= \sum_{j=1}^{m} \left| \left\langle A\Big(\sum_i y_i w_i\Big), w_j \right\rangle \right|^2 \\ &\leq \sum_{j=1}^{m} \left\| A\Big(\sum_i y_i w_i\Big) \right\|_{W^{-1,p'}}^2 ||w_j||_{W_0^{1,p}}^2 . \end{aligned} \tag{36}$$

However, since $\text{proj}_y K$ is a compact set then $||y||_m \leq c_1$ and then

$$\left\| \sum_i y_i w_i \right\|_{W_0^{1,p}} \leq \sum_i |y_i|\, ||w_i||_{W_0^{1,p}} \leq c_1 \Big(\sum_i |y_i|^2 \Big) \leq c_2 \,.$$

Since A maps bounded sets into bounded sets, from (36) we have

$$||\mathcal{G}(y)||_{2m} \leq c_3 \,.$$

Therefore, from (35) it follows that

$$||h(t,y)||_{2m} \leq k + ||\mathcal{F}(t)||. \tag{37}$$

However

$$\begin{aligned} ||\mathcal{F}(t)||_{2m}^2 = ||F(t)||_m^2 = \sum_{j=1}^{m} |(f(t), w_j)|^2 &\leq \sum_{j=1}^{m} |f(t)|_{L^2}^2\, |w_j|_{L^2}^2 \\ &\leq \bar{c}_1 |f(t)|_{L^2}^2 \,, \end{aligned}$$

whence

$$||\mathcal{F}(t)||_{2m} \leq c_2 |f(t)|_{L^2(\Omega)}\,.$$

Since $f \in L^2(Q)$ it follows from the inequality above that $||\mathcal{F}(t)||_{2m}$ is integrable on $\operatorname{proj}_t K$.

Thus, by Carathéodory's Theorem there exists a solution $Y(t)$ of the problem

$$\left| \begin{array}{l} Y'(t) = h(t, Y(t)) \\ Y(0) = Y_0 \end{array} \right.$$

in some interval $[0, t_m)$, $0 < t_m \leq T$. Thus, $Y(t)$ is absolutely continuous and differentiable a.e. in $[0, t_m)$. This entails that the maps $g_{jm}(t)$ and $g'_{jm}(t)$ are absolutely continuous and $g''_{jm}(t)$ exists a.e. in $[0, t_m)$. We will make, next, an a priori estimate that will allow us to extend the solution $Y(t)$ to the whole interval $[0, T]$.

2ª Step: A Priori Estimate

Multiplying (19) by $g'_{jm}(t)$, $t \in [0, t_m)$ and summing over j, we obtain

$$\frac{1}{2}\frac{d}{dt}\,|u'_m(t)|^2 + \langle Au_m(t), u'_m(t)\rangle + |\nabla u'_m(t)|^2 = (f(t), u'_m(t)). \tag{38}$$

But,

$$\langle A(u), v\rangle = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx; \quad \forall\, u, v \in W_0^{1,p}(\Omega). \tag{39}$$

In particular,

$$\langle Au_m(t), u'_m(t)\rangle = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u_m}{\partial x_i}\right|^{p-2} \frac{\partial u_m}{\partial x_i}\,\frac{\partial u'_m}{\partial x_i}\,dx$$

and

$$\langle Au_m(t), u_m(t)\rangle = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u_m}{\partial x_i}\right|^{p} dx = ||u_m(t)||^p_{W_0^{1,p}(\Omega)}\,. \tag{40}$$

From this last expression it follows that

$$\begin{aligned} \frac{d}{dt}\,||u_m(t)||^p_{W_0^{1,p}(\Omega)} &= \sum_{i=1}^{n} \int_\Omega \frac{d}{dt}\left|\frac{\partial u_m(t)}{\partial x_i}\right|^p dx \\ &= p\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p-2} \frac{\partial u_m(t)}{\partial x_i}\,\frac{\partial u'_m(t)}{\partial x_i}\,dx \\ &= p\langle Au_m(t), u'_m(t)\rangle, \end{aligned}$$

whence

$$\langle Au_m(t), u'_m(t)\rangle = \frac{1}{p}\frac{d}{dt}\,||u_m(t)||^p_{W_0^{1,p}(\Omega)}\,. \tag{41}$$

Substituting (41) in (38), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\,|u'_m(t)|^2 + \frac{1}{p}\frac{d}{dt}\,||u_m(t)||^p_{W_0^{1,p}(\Omega)} + ||u'_m(t)||^2 &= (f(t), u'_m(t)) \\ \leq |f(t)|\,|u'_m(t)| \leq \frac{1}{2}\big(|f(t)|^2 + |u'_m(t)|^2\big)&. \end{aligned} \tag{42}$$

Integrating (42) from 0 to t, $t \in [0, t_m)$, it follows from (17) and (18) that

$$\begin{aligned}\frac{1}{2}\,|u_m'(t)|^2 + \frac{1}{p}\,||u_m(t)||^p_{W_0^{1,p}} + \int_0^t ||u_m'(s)||^2\,ds \leq \frac{1}{2}\,|u_{1m}|^2 \\ + \frac{1}{p}\,||u_{0m}||^p_{W_0^{1,p}} + \frac{1}{2}\int_0^T |f(s)|^2\,ds + \frac{1}{2}\int_0^t |u_m'(s)|^2\,ds.\end{aligned} \tag{43}$$

But, from (17) and (18) the existence of a positive constant $c_1 > 0$ follows such that

$$\frac{1}{2}\,|u_{1m}|^2 + \frac{1}{p}\,||u_{0m}||^p_{W_0^{1,p}} \leq c_1\,. \tag{44}$$

Thus, from (43) and (44) we conclude that

$$\begin{aligned}&\frac{1}{2}\,|u_m'(t)|^2 + \frac{1}{p}\,||u_m(t)||^p_{W_0^{1,p}} + \int_0^t ||u_m'(s)||^2\,ds \\ &\leq c_2 + \int_0^t \left\{\frac{1}{2}\,|u_m'(s)|^2 + \frac{1}{p}\,||u_m(s)||^p_{W_0^{1,p}} + \int_0^s ||u_m'(\tau)||^2\,d\tau\right\}ds\end{aligned}$$

and from Gronwall's inequality it follows that

$$|u_m'(t)|^2 + ||u_m(t)||^p_{W_0^{1,p}} + \int_0^t ||u_m'(s)||^2\,ds \leq c; \quad \forall\, t \in [0, t_m) \text{ and } \forall\, m. \tag{45}$$

Thus, from (19) and (45) it follows that

$$\sum_{j=1}^m g_{jm}^2(t) = (u_m(t), u_m(t)) = |u_m(t)|^2_{L^2(\Omega)} \leq k||u_m(t)||^p_{W_0^{1,p}} \leq \bar{c}, \tag{46}$$

$\forall\, t \in [0, t_m)$ and $\forall\, m \in \mathbb{N}$.

Also, from (45) it follows that

$$\sum_{j=1}^m \left(g_{jm}'(t)\right)^2 = (u_m'(t), u_m'(t)) = |u_m'(t)|^2_{L^2(\Omega)} \leq c; \quad \forall\, t \in [0, t_m) \text{ and } \forall\, m \in \mathbb{N}. \tag{47}$$

Thus, from (46) and (47) we obtain

$$\begin{aligned}||Y(t)||^2_{2m} &= ||Y_1(t)||^2 + ||Y_2(t)||^2 = ||z(t)||^2 + ||z'(t)||^2 \\ &= \sum_{j=1}^m |g_{jm}(t)|^2 + \sum_{j=1}^m |g_{jm}'(t)|^2 \leq c',\end{aligned}$$

$\forall\, t \in [0, t_m)$ and $\forall\, m$.

It results from this that the solution $Y(t)$ of problem (29) can be prolonged to the whole interval $[0, T]$. The same happens then for $u_m(t)$. Thus, for each $m \in \mathbb{N}$ there exists a solution $u_m(t)$ of (15)-(18), absolutely continuous, with u_m' absolutely continuous and u_m'' existing a.e. in $]0, T[$. Carrying out the same calculation we did to obtain (45) we conclude, analogously, that

$$|u_m'(t)|^2 + ||u_m(t)||^p_{W_0^{1,p}(\Omega)} + \int_0^t ||u_m'(s)||^2\,ds \leq c; \quad \forall\, t \in [0, T] \text{ and } m \in \mathbb{N}. \tag{48}$$

On the other hand, as seen previously

$$||Au_m(t)||_{W^{-1,p'}(\Omega)} = ||u_m(t)||^{p-1}_{W_0^{1,p}(\Omega)}. \tag{49}$$

Thus, from (48) and (49) we conclude that

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;W_0^{1,p}(\Omega)) \tag{50}$$

$$(u'_m) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \tag{51}$$

$$(Au_m) \quad \text{is bounded in} \quad L^\infty(0,T;W_0^{-1,p'}(\Omega)) \tag{52}$$

$$(u'_m(T)) \quad \text{is bounded in} \quad L^2(\Omega) \tag{53}$$

$$(u_m(T)) \quad \text{is bounded in} \quad W_0^{1,p}(\Omega) \tag{54}$$

Consequently there will exist (u_μ) a subsequence of (u_m) such that

$$u_\mu \overset{*}{\rightharpoonup} u \quad \text{weak* in} \quad L^\infty(0,T;L^2(\Omega)) \tag{55}$$

$$Au_\mu \overset{*}{\rightharpoonup} \chi \quad \text{weak* in} \quad L^{p'}(0,T;W^{-1,p'}(\Omega)) \tag{56}$$

$$u_\mu \rightharpoonup u \quad \text{weakly in} \quad L^p(0,T;W_0^{1,p}(\Omega)) \tag{57}$$

$$u_\mu(T) \rightharpoonup \xi \quad \text{weakly in} \quad L^2(\Omega) \tag{58}$$

3ª Step: Passage to the Limit

Let $\theta \in \mathcal{D}(0,T)$ and $\mu \in \mathbb{N}$ such that $j \leq \mu$. Multiplying (16) by θ and integrating in $[0,T]$ it results that

$$\int_0^T (u''_\mu(t), w_j)\theta(t)\,dt + \int_0^T \langle Au_\mu(t), w_j\rangle\theta(t)\,dt + \int_0^T a(u'_\mu(t), w_j)\theta(t)\,dt = \int_0^T (f(t), w_j)\theta(t)\,dt.$$

Whence

$$-\int_0^T (u'_\mu(t), w_j)\theta'(t)\,dt + \int_0^T \langle Au_\mu(t), w_j\rangle\theta(t)\,dt + \int_0^T a(u'_\mu(t), w_j)\theta(t)\,dt = \int_0^T (f(t), w_j)\theta(t)\,dt$$

From (56)-(58) it follows, in the limit situation, that

$$-\int_0^T (u'(t), w_j)\theta'(t)\,dt + \int_0^T \langle \chi(t), w_j\rangle\theta(t)\,dt + \int_0^T a(u'(t), w_j)\theta(t)\,dt = \int_0^T (f(t), w_j)\theta(t)\,dt \tag{61}$$

$\forall\, \theta \in \mathcal{D}(0,T);\ \forall\, j \in \mathbb{N}$.

Since the finite linear combinations of the elements of the basis (w_ν) are dense in $W_0^{1,p}(\Omega)$, it follows from (61) that

$$-\int_0^T (u'(t), w)\theta'(t)\,dt + \int_0^T \langle \chi(t), w\rangle\theta(t)\,dt + \int_0^T a(u'(t), w)\theta(t)\,dt = \int_0^T (f(t), w)\theta(t)\,dt \tag{62}$$

$\forall\, w \in W_0^{1,p}(\Omega)$ and $\forall\, \theta \in \mathcal{D}(0,T)$.

Identifying $L^2(\Omega)$ with its dual we have that

$$(u,v) = \langle u,v\rangle, \quad \forall\, u \in L^2(\Omega),\ \ \forall\, v \in W_0^{1,p}(\Omega) \tag{63}$$

where $\langle \cdot\,,\,\cdot \rangle$ designates the duality $W^{-1,p'}, \ \ W_0^{1,p}$.

On the other hand, recall also that

$$a(u,v) = \langle -\Delta u, v\rangle; \quad \forall\, u,v \in H_0^1(\Omega). \tag{64}$$

In particular, from (63) and (64) we obtain

$$(u'(t),w) = \langle u'(t),w\rangle_{W^{-1,p'},W_0^{1,p}} \quad \text{and} \quad (f(t),w) = \langle f(t),w\rangle_{W^{-1,p'},W_0^{1,p}}\,; \quad \forall\, w \in W_0^{1,p}(\Omega) \tag{65}$$

and

$$a(u(t),w) = \langle -\Delta u'(t),w\rangle_{H^{-1};H_0^1} = \langle -\Delta u'(t),w\rangle_{W^{-1,p'};W_0^{1,p}}\,; \quad \forall\, w \in W_0^{1,p}(\Omega). \tag{66}$$

This last equality follows from the fact that

$$W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow W^{-1,p'}(\Omega).$$

Thus, from (62), (65) and (66) it follows that

$$\begin{aligned}\Big\langle -\int_0^T u'(t)\theta'(t)dt, w\Big\rangle + \Big\langle \int_0^T \chi(t)\theta(t), w\Big\rangle + \Big\langle \int_0^T -\Delta u'(t)\theta(t)dt, w\Big\rangle \\ = \Big\langle \int_0^T f(t),\theta(t)\,dt, w\Big\rangle\end{aligned}$$

that is,

$$\begin{aligned}-\int_0^T u'(t)\theta'(t)\,dt + \int_0^T \chi(t)\theta(t)\,dt + \int_0^T -\Delta u'(t)\theta(t)\,dt \\ = \int_0^T f(t)\theta(t)\,dt \quad \text{in} \quad W^{-1,p'}(\Omega),\end{aligned}$$

for all $\theta \in \mathcal{D}(0,T)$.

Therefore,

$$\langle u'',\theta\rangle + \langle \chi,\theta\rangle + \langle \Delta u',\theta\rangle = \langle f,\theta\rangle \quad \text{in} \quad W^{-1,p'}(\Omega)$$

for all $\theta \in \mathcal{D}(0,T)$, whence

$$u'' + \chi - \Delta u' = f \quad \text{in} \quad \mathcal{D}'(0,T;W^{-1,p'}(\Omega)). \tag{67}$$

In truth, since

$$\Delta u' \in L^2(0,T;H^{-1}(\Omega)); \quad \chi \in L^\infty(0,T;W^{-1,p'}(\Omega)) \quad \text{and} \quad f \in L^2(0,T;L^2(\Omega))$$

from (67) we have

$$u'' \in L^2(0,T;W^{-1,p'}(\Omega)) \tag{68}$$

and, moreover,

$$u'' + \chi - \Delta u' = f \quad \text{in} \quad L^2(0,T; W^{-1,p'}(\Omega)). \tag{69}$$

4ª Step: Initial Conditions

These are proved in the usual manner as in the previous problem. Analogously we also prove from (59) and (60) that

$$\xi = u(T) \quad \text{and} \quad \eta = u'(T). \tag{70}$$

We will obtain, next, an estimate for the second derivative using a technique different from the usual one without it being necessary to impose stronger initial conditions.

Since V_m is a closed subspace of $L^2(\Omega)$, given that it has finite dimension, we can write

$$L^2(\Omega) = V_m \oplus V_m^{\perp}.$$

Consider then, for each $m \in \mathbb{N}$, the projection P_m on the subspace V_m, that is, consider the linear map

$$\begin{aligned} P_m \colon L^2(\Omega) &\to V_m \\ u &\to P_m u = \sum_{j=1}^{m} (u, w_j) w_j \,. \end{aligned} \tag{71}$$

Note that

$$\begin{aligned} |P_m u|^2_{L^2(\Omega)} &= \left| \sum_{j=1}^{m} (u, w_j) w_j \right|^2_{L^2(\Omega)} = \sum_{j=1}^{m} |(u, w_j)|^2 \\ &\leq \sum_{j=1}^{+\infty} |(u, w_j)|^2 \leq |u|^2_{L^2(\Omega)}, \end{aligned}$$

where the last inequality is due to Bessel. Thus,

$$|P_m u|^2_{L^2(\Omega)} \leq |u|^2_{L^2(\Omega)} \,; \quad \forall\, u \in L^2(\Omega),$$

which proves the continuity of the map P_m.

On the other hand, if $u \in H_0^s(\Omega)$ then from (9) it follows that

$$\begin{aligned} P_m u &= \sum_{j=1}^{m} (u, w_j) w_j = \sum_{j=1}^{m} \lambda_j \left(u, \frac{w_j}{\sqrt{\lambda_j}} \right) \frac{w_j}{\sqrt{\lambda_j}} \\ &= \sum_{j=1}^{m} \left(\left(u, \frac{w_j}{\sqrt{\lambda_j}} \right) \right)_s \frac{w_j}{\sqrt{\lambda_j}}. \end{aligned}$$

From (8) it results that

$$\begin{aligned} ||P_m u||^2_{H_0^s(\Omega)} &= \left\| \sum_{j=1}^{m} \left(\left(u, \frac{w_j}{\sqrt{\lambda_j}} \right) \right)_s \frac{w_j}{\sqrt{\lambda_j}} \right\|^2_{H_0^s(\Omega)} = \sum_{j=1}^{m} \left| \left(\left(u, \frac{w_j}{\sqrt{\lambda_j}} \right) \right)_s \right|^2 \\ &\leq \sum_{j=1}^{+\infty} \left| \left(\left(u, \frac{w_j}{\sqrt{\lambda_j}} \right) \right)_s \right|^2 \leq ||u||^2_{H_0^s(\Omega)} \,. \end{aligned}$$

Thus,

$$||P_m u||_{H_0^s(\Omega)} \leq ||u||_{H_0^s(\Omega)}\,; \quad \forall\, u \in H_0^s(\Omega) \tag{72}$$

and consequently,

$$||P_m||_{\mathcal{L}(H_0^s,H_0^s)} = \sup_{\substack{u\in H_0^s\\ ||u||_s\neq 0}} \frac{||P_m u||_{H_0^s}}{||u||_{H_0^s}} \leq 1; \quad \forall\, m \in \mathbb{N}. \tag{73}$$

Consider, now, P_m^* the adjoint map of P_m. Recall that

$$\begin{aligned} P_m^*\colon H^{-s}(\Omega) &\to H^{-s}(\Omega)\\ f &\to P_m f \end{aligned}$$

where

$$\langle P_m^* f, u\rangle = \langle f, P_m u\rangle; \quad \forall\, u \in H_0^s(\Omega) \text{ and } \forall\, f \in H^{-s}(\Omega). \tag{74}$$

We have, from (72) that

$$\begin{aligned} ||P_m^* f||_{H^{-s}(\Omega)} &= \sup_{||u||_s=1} |\langle P_m^* f, u\rangle| = \sup_{||u||_s=1} |\langle f, P_m u\rangle| \\ &\leq \sup_{||u||_s=1} ||f||_{H^{-s}}\, ||P_m u||_{H_0^s} \leq \sup_{||u||_s=1} ||f||_{H^{-s}}\, ||u||_{H_0^s} = ||f||_{H^{-s}} \quad \forall\, m \in \mathbb{N}, \end{aligned}$$

that is,

$$||P_m^* f||_{H^{-s}(\Omega)} \leq ||f||_{H^{-s}(\Omega)}\,; \quad \forall\, m \in \mathbb{N}. \tag{75}$$

In this way,

$$||P_m^*||_{\mathcal{L}(H^{-s},H^{-s})} = \sup_{\substack{f\in H^{-s}\\ ||f||\neq 0}} \frac{||P_m^* f||_{H^{-s}}}{||f||_{H^{-s}}} \leq 1; \quad \forall\, m \in \mathbb{N}. \tag{76}$$

Returning to (16) we obtain

$$(u_m''(t), w_j) = \langle -Au_m(t), w_j\rangle + \langle \Delta u_m'(t), w_j\rangle + (f(t), w_j); \quad j = 1, \ldots, m$$

or even, identifying $L^2(\Omega)$ with its dual

$$\langle u_m''(t), w\rangle_{H^{-s};H_0^s} = \langle -Au_m(t) + \Delta u_m'(t) + f(t), w\rangle_{H^{-s};H_0^s}\,; \quad \forall\, w \in V_m\,. \tag{77}$$

However, if $v \in H_0^s(\Omega)$ we can write

$$\begin{aligned} \langle u_m''(t), v\rangle_{H^{-s};H_0^s} &= \left(\sum_{j=1}^{m} g_{jm}''(t) w_j, \sum_{i=1}^{+\infty} (v_i, w_i) w_i\right) = \left(\sum_{j=1}^{m} g_{jm}''(t) w_j, \sum_{i=1}^{m} (v, w_i) w_i\right) \\ &= (u_m''(t), P_m v) = \langle u_m''(t), P_m v\rangle_{H^{-s};H_0^s}, \end{aligned}$$

that is,

$$\langle u_m''(t), v\rangle_{H^{-s};H_0^s} = \langle u_m''(t), P_m v\rangle_{H^{-s};H_0^s}\,; \quad \forall\, v \in H_0^s(\Omega). \tag{78}$$

On the other hand, given $v \in H_0^s(\Omega)$ we have that $P_m v \in V_m$ and from (77) it follows that

$$\langle u_m''(t), P_m v\rangle = \langle -Au_m(t) + \Delta u_m'(t) + f(t), P_m v\rangle$$

and from (78) and (74) it results that

$$\langle u''_m(t), v\rangle = \langle P^*_m(-Au_m(t) + \Delta u'_m(t) + f(t)), v\rangle; \quad \forall\, v \in H^s_0 .$$

Consequently

$$u''_m(t) = -P^*_m(Au_m(t)) + P^*_m(\Delta u'_m(t)) + P^*_m(f(t)) \text{ in } H^{-s}(\Omega). \tag{79}$$

In this way, $u''_m(t) \in H^{-s}(\Omega)$ for a.e. $t \in]0, T[$. Furthermore, we have,

$$\begin{aligned} ||u''_m(t)||_{H^{-s}(\Omega)} &\le ||P^*_m(Au_m(t))||_{H^{-s}(\Omega)} + ||P^*_m(\Delta u'_m(t))||_{H^{-s}(\Omega)} + ||P^*_m(f(t))||_{H^{-s}(\Omega)} \\ &\le ||P^*_m||_{\mathcal{L}(H^{-s})}\big\{||Au_m(t)||_{H^{-s}(\Omega)} + ||\Delta u'_m(t)||_{H^{-s}(\Omega)} + ||f(t)||_{H^{-s}(\Omega)}\big\}. \end{aligned}$$

Then, from (76) it results that

$$||u''_m(t)||_{H^{-s}} \le ||Au_m(t)||_{H^{-s}} + ||\Delta u'_m(t)||_{H^{-s}} + ||f(t)||_{H^{-s}} ; \qquad \text{a.e. in } [0, T[\,.$$

Thus,

$$\begin{aligned} ||u''_m(t)||^2_{H^{-s}(\Omega)} &\le c\big\{||Au_m(t)||^2_{H^{-s}(\Omega)} + ||\Delta u'_m(t)||^2_{H^{-s}(\Omega)} + ||f(t)||^2_{H^{-s}(\Omega)}\Big\} \\ &\le c'\big\{||Au_m(t)||^2_{W^{-1,p}(\Omega)} + ||\Delta u'_m(t)||^2_{H^{-1}(\Omega)} + ||f(t)||^2_{L^2(\Omega)}\big\} \text{ a.e. in }]0, T[. \end{aligned}$$

From the inequality above and from (51), (52) and the fact that $f \in L^2(Q)$ it follows that

$$\int_0^T ||u''_m(t)||^2_{H^{-s}(\Omega)}\, dt < +\infty.$$

Thus,

$$(u''_m) \quad \text{is bounded in} \quad L^2(0, T; H^{-s}(\Omega)). \tag{80}$$

In particular, for the subsequence (u_μ) of (u_m) given previously we have the same boundedness.

It remains to prove that

$$\chi = Au.$$

In fact, by the monotonicity of A we have that

$$0 \le \int_0^T \langle A(u_\mu) - A(v), u_\mu - v\rangle\, dt; \quad \forall\, v \in L^2(0, T; W^{1,p}_0(\Omega)),$$

whence,

$$0 \le \int_0^T \langle A(u_\mu), u_\mu\rangle\, dt - \int_0^T \langle A(u_\mu), v\rangle\, dt - \int_0^T \langle A(v), u_\mu - v\rangle\, dt \tag{81}$$

$\forall\, v \in L^2(0, T; W^{1,p}_0(\Omega))$.

Returning to the approximate equation given in (16), we obtain

$$(u''_\mu(t), u_\mu(t)) + \langle Au_\mu(t), u_\mu(t)\rangle + a(u'_\mu(t), u_\mu(t)) = (f(t), u_\mu(t)).$$

Integrating the last identity in $[0, T]$ follows that

$$\int_0^T (u''_\mu(t), u_\mu(t))\, dt + \int_0^T \langle Au_\mu(t), u_\mu(t)\rangle\, dt + \frac{1}{2}\int_0^T \frac{d}{dt}\, ||u_\mu(t)||^2\, dt = \int_0^T (f(t), u_\mu(t))\, dt$$

that is,

$$\int_0^T (u''_\mu(t), u_\mu(t))dt + \int_0^T \langle Au_\mu(t), u_\mu(t)\rangle dt + \frac{1}{2}\,||u_\mu(T)||^2 - \frac{1}{2}\,||u_\mu(0)||^2 = \int_0^T (f(t), u_\mu(t))dt. \tag{82}$$

But,

$$\int_0^T \frac{d}{dt}\,(u'_\mu(t), u_\mu(t))\,dt = \int_0^T (u''_\mu(t), u_\mu(t))\,dt + \int_0^T (u'_\mu(t), u'_\mu(t))\,dt.$$

Thus,

$$\int_0^T (u''_\mu(t), u_\mu(t))\,dt = (u'_\mu(T), u_\mu(T)) - (u'_\mu(0), u_\mu(0)) - \int_0^T |u'_\mu(t)|^2\,dt.$$

Substituting the equality above in (82) results that

$$\begin{aligned}(u'_\mu(T), u_\mu(T)) - (u'_\mu(0), u_\mu(0)) - \int_0^T |u'_\mu(t)|^2\,dt + \int_0^T \langle Au_\mu(t), u_\mu(t)\rangle\,dt + \\ + \frac{1}{2}\,||u_\mu(T)||^2 - \frac{1}{2}\,||u_\mu(0)||^2 = \int_0^T (f(t), u_\mu(t))\,dt.\end{aligned} \tag{83}$$

However, we have from (51) and (80) that

$$\begin{array}{lll}(u'_\mu) & \text{is bounded in} & L^2(0,T;H_0^1(\Omega)), \\ (u''_\mu) & \text{is bounded in} & L^2(0,T;H^{-s}(\Omega)).\end{array}$$

Since $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$ then, by Aubin-Lions, it follows that there exists a subsequence of (u_μ) which we will still denote by the same symbol such that

$$u'_\mu \to u' \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{84}$$

On the other hand, substituting (83) in (81) results that

$$\begin{aligned}0 \le \int_0^T (f(t), u_\mu(t))dt + \frac{1}{2}\,||u_\mu(0)||^2 - \frac{1}{2}\,||u_\mu(T)||^2 + (u'_\mu(0), u_\mu(0)) - (u'_\mu(T), u_\mu(T)) \\ + \int_0^T |u'_\mu(t)|^2\,dt - \int_0^T \langle Au_\mu(t), v\rangle\,dt - \int_0^T \langle Av(t), u_\mu(t) - v(t)\rangle\,dt\end{aligned}$$

for all $v \in L^{p'}(0,T;W_0^{1,p}(\Omega))$.

Taking the $\overline{\lim}$ in the inequality above we obtain

$$\begin{aligned}0 \le\, & \overline{\lim} \int_0^T (f(t), u_\mu(t))dt + \frac{1}{2}\,\overline{\lim}\,||u_{0\mu}||^2 \\ & - \frac{1}{2}\,\underline{\lim}\,||u_\mu(T)||^2 + \overline{\lim}(u_{1\mu}, u_{0\mu}) \\ & - \underline{\lim}(u'_\mu(T), u_\mu(T)) + \overline{\lim} \int_0^T |u'_\mu(t)|^2\,dt \\ & - \overline{\lim} \int_0^T \langle Au_\mu(t), v(t)\rangle\,dt - \underline{\lim} \int_0^T \langle Av(t), u_\mu(t) - v(t)\rangle\,dt.\end{aligned} \tag{85}$$

However,

$$u_{0\mu} \to u_0 \quad \text{in} \quad W_0^{1,p} \hookrightarrow H_0^1 \Rightarrow ||u_{0\mu}|| \to ||u_0||. \tag{86}$$

But, from (60) and by Banach-Steinhaus

$$||u(T)|| \leq \underline{\lim}\, ||u_\mu(T)||$$

and, therefore,

$$||u(T)||^2 \leq \underline{\lim}\, ||u_\mu(T)||^2,$$

or even,

$$-\underline{\lim}\, ||u_\mu(T)||^2 \leq -||u(T)||^2. \tag{87}$$

Now, since $u_{0\mu} \to u_0$ in $W_0^{1,p} \hookrightarrow L^2$ and $u_{1\mu} \to u_1$ in L^2 it results that

$$(u_{1\mu}, u_{0\mu}) \xrightarrow{\mu\to+\infty} (u_1, u_0). \tag{88}$$

Finally, from (54) we have that

$$(u_\mu(T)) \quad \text{is bounded in} \quad W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega).$$

Thus, the embedding of $W_0^{1,p}(\Omega)$ in $L^2(\Omega))$ is compact there will exist a subsequence of (u_μ), which we will still denote by the same symbol, such that

$$u_\mu(T) \xrightarrow{\mu\to+\infty} u(T) \quad \text{in} \quad L^2(\Omega). \tag{89}$$

In this way, from (89) and (59) we have that

$$(u'_\mu(T), u_\mu(T)) \xrightarrow{\mu\to+\infty} (u'(T), u(T)). \tag{90}$$

Then, from (55), (58), (84), (85), (87), (88), (89) and (90) we conclude that

$$\begin{aligned} 0 \leq & \int_0^T (f(t), u(t))dt + \frac{1}{2}\,||u_0||^2 - \frac{1}{2}\,||u(T)||^2 + (u_1, u_0) - (u'(T), u(T)) \\ & + \int_0^T |u'(t)|^2\, dt + \int_0^T \langle \chi(t), v(t)\rangle\, dt - \int_0^T \langle Av(t), u(t) - v(t)\rangle\, dt. \end{aligned} \tag{91}$$

On the other hand, from (69) we obtain

$$\langle u'', u\rangle + \langle \chi, u\rangle - \langle \Delta u', u\rangle = \langle f, u\rangle,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ designates the duality $L^2(0,T;W^{-1,p'}(\Omega)) \times L^2(0,T;W_0^{1,p}(\Omega))$.

Equivalently, we have

$$\int_0^T \langle u''(t), u(t)\rangle\, dt + \int_0^T \langle \chi(t), u(t)\rangle\, dt + \int_0^T \langle -\Delta u'(t), u(t)\rangle\, dt = \int_0^T \langle f(t), u(t)\rangle\, dt. \tag{92}$$

However, analogously to the parabolic case

$$\frac{d}{dt}\,(u'(t), u(t)) = \langle u''(t), u(t)\rangle + (u'(t), u'(t)).$$

Integrating the last identity in $[0, T]$, we obtain

$$(u'(T), u(T)) - (u'(0), u(0)) = \int_0^T \langle u''(t), u(t)\rangle\, dt + \int_0^T |u'(t)|^2\, dt. \tag{93}$$

Substituting (93) in (92) results that

$$\begin{aligned}(u'(T), u(T)) - (u'(0), u(0)) - \int_0^T |u'(t)|^2\, dt + \int_0^T \langle \chi(t), u(t)\rangle\, dt \\ + \int_0^T \langle -\Delta u'(t), u(t)\rangle\, dt = \int_0^T \langle f(t), u(t)\rangle\, dt.\end{aligned}$$

However,

$$\int_0^T \langle -\Delta u'(t), u(t)\rangle\, dt = \int_0^T a(u'(t), u(t))\, dt = \int_0^T \frac{1}{2}\frac{d}{dt}\,||u(t)||^2\, dt = \frac{1}{2}\,||u(T)||^2 - \frac{1}{2}\,||u(0)||^2. \tag{94}$$

From (93) and (94) it follows that

$$\begin{aligned}(u'(T), u(T)) - (u_1, u_0) + \frac{1}{2}\,||u(T)||^2 - \frac{1}{2}\,||u_0||^2 - \int_0^T |u'(t)|^2\, dt \\ + \int_0^T \langle \chi(t), u(t)\rangle\, dt = \int_0^T \langle f(t), u(t)\rangle\, dt.\end{aligned} \tag{95}$$

Finally, substituting (95) in (91) it follows that

$$\int_0^T \langle \chi(t) - Av(t), u(t) - v(t)\rangle\, dt \geq 0. \tag{96}$$

Let $w \in W_0^{1,p}(\Omega)$. Considering, initially, v equal to $u - \lambda w$, with $\lambda > 0$, and, next, with $\lambda < 0$, it results that

$$\int_0^T \langle \chi - A(u - \lambda w), w\rangle\, dt = 0; \quad \forall\, w \in W_0^{1,p}(\Omega).$$

By the hemicontinuity of A it follows that

$$\chi = Au. \qquad \square$$

14.3 Elasticity System

In this section, we will address an elliptic problem fundamental regarding its applications in Solid Mechanics: The elasticity system.

Let Ω be a bounded connected open set of $\mathbb{R}^n$ with a smooth boundary Γ, representing the volume occupied by an elastic body. Let Γ_0 be a part of Γ, with strictly positive surface measure, and let Γ_1 be the complement of Γ_0 in Γ. Let us assume that the body is fixed along Γ_0 and that a force $\vec{f} = (f_i)_{1\leq i\leq n}$ acts on the body and that a surface force $\vec{g} = (g_i)_{1\leq i\leq n}$ acts on Γ_1 as illustrated in the figure below:

Let $\vec{u} = (u_j);\quad 1 \leq j \leq n$, be the displacement vector. The *Strain Tensor* (ε_{ij}) is defined by

$$\varepsilon_{ij}(\vec{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); \quad 1 \leq i, j \leq n. \tag{1}$$

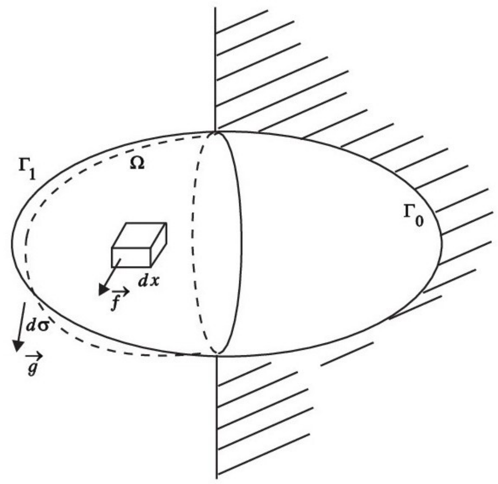

Figure 14.1: Elastic body configuration

If σ_{ij} is the *stress tensor*, then we need a law relating both tensors; a law that will describe the properties constituting the material. Assuming that the solid is elastic, homogeneous, and isotropic, the law relating the tensors is linear; more precisely, it is Hooke's Law

$$\sigma_{ij}(\vec{u}) = \lambda\Big(\sum_{k=1}^{n}\varepsilon_{kk}(\vec{u})\Big)\delta_{ij} + 2\mu\,\varepsilon_{ij}(\vec{u}); \quad 1 \le i, j \le n, \tag{2}$$

where $\lambda \ge 0$ and $\mu > 0$ are called *Lamé Coefficients*. Here, δ_{ij} is the Kronecker delta. The elasticity system consists of the following boundary value problem:

$$\begin{cases} -\displaystyle\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\sigma_{ij}(\vec{u}) = f_i \quad \text{in} \quad \Omega, \quad 1 \le i \le n \\ \vec{u} = 0 \quad \text{on} \quad \Gamma_0 \\ \displaystyle\sum_{j=1}^{n}\sigma_{ij}(\vec{u})\nu_j = g_i \quad \text{on} \quad \Gamma_1\,; \quad 1 \le i \le n. \end{cases} \tag{3}$$

Let us set

$$H = (L^2(\Omega))^n \tag{4}$$

endowed with the inner product

$$(u, v)_H = \sum_{i=1}^{n}(u_i, v_i)_{L^2(\Omega)} \tag{5}$$

and

$$V = (H^1(\Omega))^n \tag{6}$$

endowed with the inner product

$$((u,v))_V = \sum_{i=1}^{n} ((u_i, v_i))_{H^1(\Omega)} \,. \tag{7}$$

Define

$$V_0 = \{v \in V;\ \vec{\gamma}_0 v = 0 \text{ on } \Gamma_0\}, \tag{8}$$

where $\vec{\gamma}_0$ is the trace map given by

$$\begin{aligned} \vec{\gamma}_0 \colon V &\to (H^{1/2}(\Gamma))^n \\ v &\mapsto \vec{\gamma}_0 v = (\gamma_0 v_1, \gamma_0 v_2, \ldots, \gamma_0 v_n), \end{aligned} \tag{9}$$

with $\gamma_0 \colon H^1(\Omega) \to H^{1/2}(\Gamma)$ being the trace map for functions in $H^1(\Omega)$.

Our goal is to show that the map

$$v \in V_0 \mapsto [v] = \left(\sum_{i,j=1}^{n} \int_\Omega |\varepsilon_{ij}(v)|^2 \, dx \right)^{1/2} = \left(\sum_{i,j} |\varepsilon_{ij}(v)|^2_{L^2(\Omega)} \right)^{1/2} \tag{10}$$

defines a norm on V_0, and that in V_0 this norm is equivalent to the norm induced by V. However, we need some preliminary results as we will see next.

Lemma 1. *Let $v \in (H^1(\Omega))^n$. Then, for all $1 \le i,j \le n$ we have*

$$\varepsilon_{ij}(\vec{v}) = 0 \quad \Leftrightarrow \quad \vec{v}(x) = a + b \cdot x,$$

where $a \in \mathbb{R}^n$ and $b \in \mathcal{L}(\mathbb{R}^n)$ with $b = -b^*$, where b^* is the transpose of b.

Proof. Let $\vec{v}(x) = (v_1, v_2, \ldots, v_n)$ and suppose that $\vec{v}(x) = a + b \cdot x$, that is,

$$v_i(x) = a_i + \sum_{j=1}^{n} b_{ij}\, x_j \,; \quad 1 \le i \le n.$$

From the hypothesis on b, i.e., from the fact that $b = -b^*$, it follows that

$$b_{ij} = -b_{ji}$$

and, therefore,

$$\frac{\partial v_i}{\partial x_j} = b_{ij} = -b_{ji} = -\frac{\partial v_j}{\partial x_i}\,; \quad 1 \le i,j \le n.$$

Thus

$$\varepsilon_{ij}(\vec{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = 0; \quad 1 \le i,j \le n.$$

Conversely, suppose that

$$\varepsilon_{ij}(u) = 0, \qquad 1 \le i,j \le n; \tag{11}$$

then

$$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=0; \qquad 1\le i,j\le n.$$

In this way,

$$\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=0; \qquad 1\le i,j,k\le n,$$

that is,

$$\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_j}\right)+\frac{\partial}{\partial x_k}\left(\frac{\partial u_j}{\partial x_i}\right)=0; \qquad 1\le i,j,k\le n.$$

But, in the sense of distributions, we can also write that

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_k}\right)=0. \tag{12}$$

On the other hand, from (11) we have that

$$\frac{\partial u_i}{\partial x_k}=-\frac{\partial u_k}{\partial x_i} \quad \text{and} \quad \frac{\partial u_j}{\partial x_k}=-\frac{\partial u_k}{\partial x_j}\,. \tag{13}$$

Thus, substituting (13) into (12) implies that

$$\frac{\partial}{\partial x_j}\left(-\frac{\partial u_k}{\partial x_i}\right)+\frac{\partial}{\partial x_i}\left(-\frac{\partial u_k}{\partial x_j}\right)=0,$$

that is,

$$2\,\frac{\partial^2 u_k}{\partial x_i\partial x_j}=0\Rightarrow\frac{\partial^2 u_k}{\partial x_i\partial x_j}=0; \quad 1\le i,j,k\le n,$$

which implies that

$$\frac{\partial u_k}{\partial x_j}=b_{kj}\,, \quad 1\le j,k\le n; \tag{14}$$

where b_{kj} is a constant that depends on k and j.

But from (11) it follows that

$$b_{kj}=-b_{jk}\,. \tag{15}$$

Integrating (14) with respect to x_j we obtain

$$u_k=b_{kj}\,x_j+a_{kj}+f(x_1,x_2,\ldots,x_{j-1},x_{j+1},\ldots,x_n);$$

where a_{kj} is a constant that depends on k and j.

It results for $i\ne j$ that

$$\frac{\partial u_k}{\partial x_i}=\frac{\partial f}{\partial x_i}$$

and from (14) it follows that

$$\frac{\partial f}{\partial x_i}=b_{ki}\,; \quad \forall\, i\ne j.$$

Thus,

$$f=\sum_{i=1,i\ne j}^{n}(b_{ki}\,x_i+a_{ki})$$

and, therefore,

$$u_k = b_{kj}\, x_j + a_{kj} + \sum_{i=1, i\neq j} (b_{ki}\, x_i + a_{ki})$$

that is,

$$u_k = \sum_{j=1}^{n} (b_{kj}\, x_j + a_{kj}) = \sum_{j=1}^{n} b_{kj} + a_k\,; \quad \text{where} \quad a_k = \sum_{j=1}^{n} a_{kj}\,.$$

Then,

$$\begin{aligned} u = (u_1, \dots, u_n) = (a_1, \dots, a_n) + \Big(\sum_{j=1}^{n} b_{1j}\, x_j, \dots, \sum_{j=1}^{n} b_{nj}\, x_j\Big) = \\ = \vec{a} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = a + b \cdot x. \end{aligned} \tag{14.3}$$

According to (15) it follows that $b = -b^*$, which proves the lemma. □

Lemma 2. *Let Ω be a bounded open set with a smooth boundary Γ. Let $\Gamma_0 \subset \Gamma$ such that the surface measure of Γ_0 is strictly positive. If $v(x) = a + b \cdot x; \quad \forall\, x \in \overline{\Omega}$, where $a \in \mathbb{R}^n$, $b \in \mathcal{L}(\mathbb{R}^n)$ such that $b = -b^*$ and $v(x) = 0, \quad \forall\, x \in \Gamma_0$ then $v(x) = \vec{0} \quad \forall\, x \in \overline{\Omega}$.*

Proof. We will perform the proof in the cases where $n = 2$ or $n = 3$. However, generally, due to the fact that the surface measure is positive, we guarantee the existence of n linearly independent vectors $\vec{x}_1, \dots, \vec{x}_n$ in $\mathbb{R}^n$ such that $x_1, x_2, \dots, x_n \in \Gamma_0$. We claim that

$$a = \vec{0} \quad \text{and} \quad b = (0)_{n\times n}\,.$$

Indeed,

• **Case $n = 2$**

Let $\vec{x}_1$ and $\vec{x}_2$ be l.i. in $\mathbb{R}^2$ such that $x_1, x_2 \in \Gamma_0$. Then

$$\begin{aligned} v(x_1) &= a + b \cdot x_1 = 0 \\ v(x_2) &= a + b \cdot x_2 = 0 \end{aligned}$$

and, therefore,

$$b \cdot (x_1 - x_2) = 0.$$

Since b is a skew-symmetric matrix we have that $b = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$, whence being $x_1 = (a_1, b_1)$ and $x_2 = (a_2, b_2)$ it follows that

$$\begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix} \begin{bmatrix} a_1 - a_2 \\ b_1 - b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

that is,

$$k(b_1 - b_2) = 0$$
$$-k(a_1 - a_2) = 0.$$

Since x_1 and x_2 are l.i. we have that $x_1 \neq x_2$ and, therefore, $a_1 - a_2 \neq 0$ or $b_1 - b_2 \neq 0$. Consequently, $k = 0$ and thus

$$b = 0.$$

Thus, $v(x) = a; \quad \forall\, x \in \overline{\Omega}$ and, in particular, $v(x) = a; \quad \forall\, x \in \Gamma_0$. Since $v(x) = 0$ on Γ_0 it results that $a = 0$ and in this way,

$$v(x) = 0; \quad \forall\, x \in \overline{\Omega}.$$

- **Case $n = 3$**

Let $x_1 = (a_1, b_1, c_1); \quad x_2 = (a_2, b_2, c_2)$ and $x_3 = (a_3, b_3, c_3)$ be l.i. vectors in $\mathbb{R}^3$ such that $x_1, x_2, x_3 \in \Gamma_0$. Let, further,

$$b = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix}.$$

Since $v(x) = 0$ on Γ_0, it follows that $v(x_i) = a + b\,x_i = 0; \quad i = 1, 2, 3$. Therefore,

$$b(x_i - x_j) = 0; \quad i, j = 1, 2, 3 \quad \text{and} \quad i \neq j$$

that is,

$$b(x_1 - x_2) = b(x_2 - x_3) = b(x_1 - x_3) = 0.$$

Thus

$$\begin{cases} k_1(b_1 - b_2) + k_2(c_1 - c_2) = 0 \\ -k_1(a_1 - a_2) + k_3(c_1 - c_2) = 0 \\ -k_2(a_1 - a_2) - k_3(b_1 - b_2) = 0 \end{cases} \tag{16}$$

$$\begin{cases} k_1(b_2 - b_3) + k_2(c_2 - c_3) = 0 \\ -k_1(a_2 - a_3) + k_3(c_2 - c_3) = 0 \\ -k_2(a_2 - a_3) - k_3(b_2 - b_3) = 0 \end{cases} \tag{17}$$

$$\begin{cases} k_1(b_1 - b_3) + k_2(c_2 - c_3) = 0 \\ -k_1(a_1 - a_3) + k_3(c_1 - c_3) = 0 \\ -k_2(a_1 - a_3) - k_3(b_1 - b_3) = 0 \end{cases} \tag{18}$$

Since $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ are linearly independent vectors we have that

$$x_1 - x_2 = (a_1 - a_2, b_1 - b_2, c_1 - c_2); \quad x_2 - x_3 = (a_2 - a_3, b_2 - b_3, c_2 - c_3);$$

$$x_1 - x_3 = (a_1 - a_3, b_1 - b_3, c_1 - c_3)$$

possess at least one non-zero coordinate because, otherwise, $x_1 = x_2$ and/or $x_2 = x_3$ and/or $x_1 = x_3$. Suppose, then, without loss of generality that $a_1 - a_2 \neq 0$. From (16) it follows

$$k_1 = k_3 \frac{c_1 - c_2}{a_1 - a_2} \tag{19}$$

$$k_2 = -k_3 \frac{b_1 - b_2}{a_1 - a_2} \cdot \tag{20}$$

We have, now, two cases to consider:

1st Case: $a_2 - a_3 \neq 0$.
From (17) it results that

$$k_1 = k_3 \frac{c_2 - c_3}{a_2 - a_3} \tag{21}$$

$$k_2 = -k_3 \frac{b_2 - b_3}{a_2 - a_3} \cdot \tag{22}$$

We have two subcases to consider

$$a_1 - a_3 \neq 0 \quad \text{or} \quad a_1 - a_3 = 0.$$

If $a_1 - a_3 \neq 0$, from (18) it follows that

$$k_1 = k_3 \frac{c_1 - c_3}{a_1 - a_3} \tag{23}$$

and

$$k_2 = -k_3 \frac{b_1 - b_3}{a_1 - a_3} \cdot \tag{24}$$

From (19), (21) and (23) we obtain

$$k_3 \frac{c_1 - c_2}{a_1 - a_2} = k_3 \frac{c_2 - c_3}{a_2 - a_3} = k_3 \frac{c_1 - c_3}{a_1 - a_3} \cdot \tag{25}$$

From (20), (22) and (24) it results that

$$k_3 \frac{b_1 - b_2}{a_1 - a_2} = k_3 \frac{b_2 - b_3}{a_2 - a_3} = k_3 \frac{b_1 - b_3}{a_1 - a_3} \cdot \tag{26}$$

(27) If $k_3 = 0$ then $k_1 = k_2 = 0$ and, therefore, $b = [0]_{3\times 3}$.
If $k_3 \neq 0$ we obtain from (25) and (26) that

$$\frac{c_1 - c_2}{a_1 - a_2} = \frac{c_2 - c_3}{a_2 - a_3} = \frac{c_1 - c_3}{a_1 - a_3} = m_1$$

$$\frac{b_1 - b_2}{a_1 - a_2} = \frac{b_2 - b_3}{a_2 - a_3} = \frac{b_1 - b_3}{a_1 - a_3} = m_2,$$

or even,

$$\begin{aligned} c_1 - c_2 &= m_1(a_1 - a_2) & b_1 - b_2 &= m_2(a_1 - a_2) \\ c_2 - c_3 &= m_1(a_2 - a_3) \quad \text{and} & b_2 - b_3 &= m_2(a_2 - a_3) \\ c_1 - c_3 &= m_1(a_1 - a_3) & b_1 - b_3 &= m_2(a_1 - a_3). \end{aligned} \tag{28}$$

If $m_1 = m_2 = 0$ we have that

$$\begin{aligned} c_1 - c_2 = c_2 - c_3 = c_1 - c_3 = 0 \\ b_1 - b_2 = b_2 - b_3 = b_1 - b_3 = 0 \end{aligned}$$

and, therefore,

$$x_1 - x_2 = (a_1 - a_2, 0, 0) \quad \text{and} \quad x_2 - x_3 = (a_2 - a_3, 0, 0).$$

Thus, there exists a constant k satisfying

$$x_1 - x_2 = k(x_2 - x_3),$$

which is a contradiction, since the vectors are linearly independent.

Consider, now, the case where $m_1 = 0$ or $m_2 = 0$. Suppose, without loss of generality, that $m_1 = 0$ and $m_2 \neq 0$, from (28) it comes that

$$c_1 - c_2 = c_2 - c_3 = c_1 - c_3 = 0,$$

whence,

$$\begin{aligned} x_1 - x_2 &= (a_1 - a_2, m_2(a_1 - a_2), 0) = (a_1 - a_2)(1, m_2, 0) \\ x_2 - x_3 &= (a_2 - a_3, m_2(a_2 - a_3), 0) = (a_2 - a_3)(1, m_2, 0) \\ x_1 - x_3 &= (a_1 - a_3, m_2(a_1 - a_3), 0) = (a_1 - a_3)(1, m_2, 0). \end{aligned}$$

Thus,

$$x_1 - x_2 = \frac{a_1 - a_2}{a_2 - a_3}\,(x_2 - x_3),$$

which is a contradiction!

Suppose, now, that $m_1, m_2 \neq 0$. From (18) it comes that:

$$\begin{aligned} x_1 - x_2 &= (a_1 - a_2, m_2(a_1 - a_2), m_1(a_1 - a_2)) = (a_1 - a_2)(1, m_2, m_1) \\ x_2 - x_3 &= (a_2 - a_3, m_2(a_2 - a_3), m_1(a_2 - a_3)) = (a_2 - a_3)(1, m_2, m_1) \\ x_1 - x_3 &= (a_1 - a_3, m_2(a_1 - a_3), m_1(a_1 - a_3)) = (a_1 - a_3)(1, m_2, m_1). \end{aligned}$$

Then,

$$x_2 - x_3 = \frac{a_1 - a_2}{a_1 - a_3}\,(x_1 - x_3),$$

which is a contradiction!

Thus, $k_3 \neq 0$ cannot occur. We must have, then, $k_3 = 0$ and from (27) it comes that

$$b = [0]_{3\times 3}.$$

Let's pass to the case where $a_1 - a_3 = 0$. We have two subcases to consider:

$$b_1 - b_3 \neq 0 \quad \text{or} \quad b_1 - b_3 = 0.$$

If $b_1 - b_3 \neq 0$ then from $(18)_3$ we obtain

$$k_3(b_1 - b_3) = 0.$$

Thus, $k_3 = 0$ and from (21) and (22) it follows that $k_1 = k_2 = 0$. Whence

$$b = [0]_{3\times 3}.$$

On the other hand, if $b_1 - b_3 = 0$ then $c_1 - c_3 \neq 0$ since $a_1 - a_3 = 0$, by hypothesis and the vectors x_1 and x_3 are linearly independent. From $(17)_2$ it results that $k_3 = 0$ and therefore from (21) and (22) it comes that $k_1 = k_2 = 0$, that is,

$$b = [0]_{3\times 3}.$$

2nd Case: $a_2 - a_3 = 0$

We have two subcases to consider

$$b_2 - b_3 \neq 0 \quad \text{and} \quad b_2 - b_3 = 0.$$

If $b_2 - b_3 \neq 0$ from $(17)_3$ we obtain $k_3 = 0$ and, therefore, from (21) and (22) we have that $k_1 = k_2 = 0$ and then,

$$b = [0]_{3\times 3}.$$

If $b_2 - b_3 = 0$, since $a_2 - a_3 = 0$ then $c_2 - c_3 \neq 0$, given that the vectors x_2 and x_3 are linearly independent. From $(17)_2$ we obtain $k_3 = 0$ and, then, from (21) and (22), we conclude that $k_1 = k_2 = 0$. Thus,

$$b = [0]_{3\times 3}. \qquad \square$$

$\square$

Lemma 3. *Let $v \in V_0 = \{v \in (H^1(\Omega))^n;\ \vec{\gamma}_0 v = 0$ on $\Gamma_0\}$ where the surface measure of Γ_0 is positive. Then, $\varepsilon_{ij}(\vec{v}) = 0;\quad i,j = 1,\dots,n \Leftrightarrow \vec{v} = \vec{0}$.*

Proof. If $\vec{v} = \vec{0}$ then, trivially, $\varepsilon_{ij}(\vec{v}) = 0;\quad \forall\, i,j = 1,\dots,n$.

Conversely, suppose that $\varepsilon_{ij}(\vec{v}) = 0;\quad \forall\, i,j = 1,\dots,n$. Then, by Lemma 1, there exist $a \in \mathbb{R}^n$ and $b \in \mathcal{L}(\mathbb{R}^n);\quad b = (b_{ij})$ with $b_{ij} = -b_{ji}$ such that

$$v(x) = a + b\cdot x;\quad \forall\, x \in \Omega.$$

Note that the function $\tilde{v}(x) = a + b\cdot x,\quad x \in \overline{\Omega}$ is such that

$$\tilde{v} \in (C^\infty(\overline{\Omega}))^n \cap (H^1(\Omega))^n.$$

Thus,

$$\vec{\gamma}_0 \tilde{v} = \tilde{v}|_\Gamma\,.$$

However, since $\tilde{v}(x) = v(x)$ in Ω, we have that $\vec{\gamma}_0\tilde{v} = \vec{\gamma}_0 v$ and, therefore,

$$\tilde{v}(x) = (\vec{\gamma}_0 v)(x);\quad \text{for almost every}\ \ x \in \Gamma.$$

In particular,

$$\tilde{v}(x) = (\vec{\gamma}_0 v)(x);\quad \text{for almost every}\ \ x \in \Gamma_0\,.$$

Since $v \in V_0$, then $(\gamma_0 v)(x) = 0$ for almost every $x \in \Gamma_0$ and then

$$\tilde{v}(x) = 0;\quad \forall\, x \in \Gamma_0.^{(*)}$$

By Lemma 2 it follows that $\tilde{v}(x) = 0,\ \forall\, x \in \overline{\Omega}$ which implies that $v = 0$ in Ω. $\square$

[13] Note that $\tilde{v} \in (C^\infty(\overline{\Omega}))^n$ and $\tilde{v} = 0$ a.e. on Γ_0 then $\tilde{v} = 0,\ \forall\, x \in \Gamma$.

From the previous lemmas we obtain the following result.

Proposition 1. *The map* $v \in V_0 \mapsto [v] = \left(\sum_{i,j=1}^{n} |\varepsilon_{ij}(v)|^2_{L^2(\Omega)} \right)^{1/2}$ *where*

$$\varepsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

is a norm on V_0.

Next, we will prove that in V_0 the norms $||\cdot||_V$ and $[\cdot]$ are equivalent. Before that, however, we need two results that we will state in the form of lemmas.

Lemma 4. *Let* $\Omega = \mathbb{R}^n$; $\Omega = \mathbb{R}^n_+$ *or a bounded open set with smooth boundary of* $\mathbb{R}^n$. *If* $v \in \mathcal{D}'(\Omega)$ *such that* $v \in H^{-1}(\Omega)$ *and* $\frac{\partial v}{\partial x_i} \in H^{-1}(\Omega)$; $i = 1, \ldots, n$ *then* $v \in L^2(\Omega)$.

Proof. We will perform the proof for the case $\Omega = \mathbb{R}^n$. The proof for the other cases will be omitted as it is beyond the scope of these notes, but can be found in Duvaut-Lions.

Initially, recall that for each $s > 0$ the set

$$H^{-s}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); (1 + ||\xi||^2)^{-s/2}\, \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}$$

endowed with the topology,

$$||u||^2_{H^{-s}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + ||\xi||^2)^{-s}\, |\hat{u}(\xi)|^2\, d\xi$$

is a Hilbert space.

Let, then, $v \in \mathcal{D}'(\mathbb{R}^n)$ such that $v, \frac{\partial v}{\partial x_i} \in H^{-1}(\Omega)$; $i = 1, \ldots, n$. Thus, for $s = 1$,

$$\int_{\mathbb{R}^n} (1 + ||\xi||^2)^{-1}\, |\hat{v}(\xi)|^2\, d\xi < +\infty \tag{29}$$

and

$$\int_{\mathbb{R}^n} (1 + ||\xi||^2)^{-1} \left| \widehat{\frac{\partial v}{\partial x_j}}(\xi) \right|^2 < +\infty; \; j = 1, \ldots, n.$$

However

$$\widehat{\frac{\partial v}{\partial x_j}}(\xi) = (2\pi i)\xi_j\; \hat{v}(\xi).$$

Thus, for all $j = 1, \ldots, n$ we have

$$2\pi \int_{\mathbb{R}^n} (1 + ||\xi||^2)^{-1}\, |\xi_j|^2\, |\hat{v}(\xi)|^2\, d\xi < +\infty$$

whence, summing over j and dividing by 2π results

$$\int_{\mathbb{R}^n} (1 + ||\xi||^2)^{-1}\, ||\xi||^2\, |\hat{v}(\xi)|^2\, d\xi < +\infty. \tag{30}$$

Summing (29) and (30) we obtain

$$\int_{\mathbb{R}^n} (1+||\xi||^2)^{-1}\,(1+||\xi||^2)\,|\hat{v}(\xi)|^2\,d\xi < +\infty,$$

that is,

$$\hat{v} \in L^2(\mathbb{R}^n).$$

Due to Plancherel's Theorem it follows that

$$v \in L^2(\mathbb{R}^n).$$

Admitting the veracity of the lemma for the case $\Omega = \mathbb{R}^n_+$, we prove, via local charts, that the same continues to be valid for Ω a bounded open set with smooth boundary. □

Lemma 5 (Korn's Inequality). *There exists $c > 0$ such that for all $v \in V = (H^1(\Omega))^n$ we have*

$$\sum_{i,j=1}^{n} \int_\Omega (\varepsilon_{ij}(v(x)))^2\,dx + \sum_{i=1}^{n} \int_\Omega (v_i(x))^2\,dx \geq c||v||^2.$$

Proof. Consider

$$E = \{v \in (L^2(\Omega))^n;\ \varepsilon_{ij}(v) \in L^2(\Omega),\ \forall\, i,j = 1,\dots,n\}$$

and

$$V = (H^1(\Omega))^n = \left\{v \in (L^2(\Omega))^n; v_i, \frac{\partial v_i}{\partial x_j} \in L^2(\Omega),\, \forall\, i,j = 1,\dots,n\right\}.$$

I claim: $V = E$. Indeed,

Evidently, $V \subset E$. Endowing E with the topology

$$||v||_E^2 = \sum_{i,j=1}^{n} \int_\Omega (\varepsilon_{ij}(v))^2\,dx + \sum_{i=1}^{n} \int_\Omega v_i^2\,dx,$$

it results that the canonical map

$$\begin{aligned} \tau\colon V &\to E \\ u &\mapsto \tau u = u, \end{aligned}$$

which is clearly linear and injective, is also continuous, since

$$||\tau u||_E^2 = ||u||_E^2 \leq c_1||u||^2;\ \forall\, u \in V.$$

We will prove, next, that $E \subset V$. Indeed, let $v \in E$. Then

$$v \in (L^2(\Omega))^n \quad \text{and} \quad \varepsilon_{ij}(v) \in L^2(\Omega);\ \forall\, i,j = 1,\dots,n.$$

However, the following identity is valid

$$\frac{\partial}{\partial x_k}(\varepsilon_{ij}(v)) + \frac{\partial}{\partial x_j}(\varepsilon_{jk}(v)) - \frac{\partial}{\partial x_i}(\varepsilon_{jk}(v)) = \frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_k}\right)$$

$\forall\, v \in E$ and $\forall\, i,j,k = 1,\dots,n$.

Since $\varepsilon_{ij}(v) \in L^2(\Omega)$ and, therefore, any derivative of it of 1st order belongs to $H^{-1}(\Omega)$, then from the identity above it comes that

$$\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_k}\right) \in H^{-1}(\Omega); \quad \forall\, i, j, k = 1, \ldots, n.$$

Furthermore, since

$$\frac{\partial v_i}{\partial x_k} \in H^{-1}(\Omega); \quad \forall\, i, k = 1, \ldots, n,$$

by Lemma 4 it results that $\dfrac{\partial v_i}{\partial x_k} \in L^2(\Omega)$; $\forall\, i, k = 1 \ldots, n$. Furthermore, since $v_i \in L^2(\Omega)$; $\forall\, i = 1, \ldots, n$ it follows that $v \in V$. Which proves that $V = E$.

Therefore, $\tau \colon V \to E$ is a linear, continuous and bijective map. Thus, $\exists\, \tau^{-1} \colon E \to V$ which is linear and continuous.

Thus, $\exists\, c_2 > 0$ such that

$$||v|| = ||\tau^{-1} v|| \leq c_2 ||v||_E\,; \quad \forall\, v \in E,$$

that is,

$$||v||_E \geq \frac{1}{c_2}\, ||v||; \quad \forall\, v \in V.$$

□

Lemma 6. *The space* $V_0 = \{v \in V; \vec{\gamma}_0 v = 0 \text{ on } \Gamma_0\}$ *endowed with the inner product* $((\,\cdot\,,\,\cdot\,))$ *given in (7) is a closed subspace of* V.

Proof. Let $(v_\nu) \subset V_0$ such that $v_\nu \to v$ in V. Then,

$$\vec{\gamma}_0 v_\nu \to \vec{\gamma}_0 v \quad \text{in} \quad (H^{1/2}(\Gamma))^n \hookrightarrow (L^2(\Gamma))^n.$$

Thus, there exists a subsequence $(v_\mu) \subset (v_\nu)$ verifying

$$(\vec{\gamma}_0 v_\mu)(x) \to (\vec{\gamma}_0 v)(x) \quad \text{for almost every} \quad x \in \Gamma.$$

In particular,

$$(\vec{\gamma}_0 v_\mu)(x) \to (\vec{\gamma}_0 v)(x) \quad \text{for almost every} \quad x \in \Gamma_0.$$

Since $\vec{\gamma}_0 v_\mu = 0$ a.e. on Γ_0, $\forall\, \mu \in \mathbb{N}$, we have that $\vec{\gamma}_0 v = 0$ on Γ_0, which proves the desired result. □

Proposition 2. *The norms* $||\cdot||$ *and* $[\,\cdot\,]$ *are equivalent in* V_0*, where*

$$||v|| = \left(\sum_{j=1}^{n} ||v_i||^2_{H^1(\Omega)}\right)^{1/2}$$

and

$$[v] = \left(\sum_{i,j=1}^{n} |\varepsilon_{ij}(v)|^2_{L^2(\Omega)}\right)^{1/2}.$$

Proof. As already seen previously (Proposition 1), the map $v \in V_0 \to [v]$ defines a norm on V_0. We will prove, next, the equivalence between $||\cdot||$ and $[\cdot]$. Indeed, let $v \in V_0$. On one hand, we have that

$$\begin{aligned}
[v]^2 &= \sum_{i,j=1}^{n} \frac{1}{4} \left| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right|^2_{L^2(\Omega)} \\
&\leq c_1 \sum_{i,j=1}^{n} \left\{ \left| \frac{\partial v_i}{\partial x_j} \right|^2_{L^2(\Omega)} + \left| \frac{\partial v_j}{\partial x_i} \right|^2_{L^2(\Omega)} \right\} \\
&\leq c_1 \sum_{i,j=1}^{n} \left\{ ||v_i||^2_{H^1(\Omega)} + ||v_j||^2_{H^1(\Omega)} \right\} \\
&\leq nc_1 \left\{ \sum_{i=1}^{n} ||v_i||^2_{H^1(\Omega)} + \sum_{j=1}^{n} ||v_j||^2_{H^1(\Omega)} \right\} = 2nc_1 ||v||^2.
\end{aligned}$$

Thus, $\exists\, c > 0$ such that

$$[v] \leq c||v||. \tag{31}$$

Conversely, suppose that there exists $k_0 > 0$ such that

$$[v] \geq k_0 |v|; \quad \forall\, v \in V_0\,. \tag{32}$$

By Korn's inequality, $\exists\, k_1 > 0$ such that

$$[v]^2 + |v|^2 \geq k_1 ||v||^2. \tag{33}$$

Thus, from (32) and (33) we obtain

$$k_1 ||v||^2 \leq [v]^2 + |v|^2 \leq [v]^2 + \left(\frac{1}{k_0}\right)^2 [v]^2 = \left(1 + \frac{1}{k_0^2}\right)[v]^2$$

that is, $\exists\, k_2 > 0$ verifying

$$k_2 ||v|| \leq [v]. \tag{34}$$

Therefore, from (31) and (34) the desired result follows. It remains, then, to show inequality (32). Indeed, note that if $v = 0$ the inequality in (32) follows trivially. Consider $v \neq 0$. In this case, (32) is equivalent to

$$\frac{[v]}{|v|} \geq k_0 \Leftrightarrow \left[\frac{v}{|v|}\right] \geq k_0 \quad \text{where} \quad \left|\frac{v}{|v|}\right| = 1.$$

From the above it is sufficient to prove that

$$\exists\, k_0 > 0;\ \forall\, v \in V_0 \text{ s.t. } |v| = 1 \text{ we have } [v] \geq k_0\,. \tag{35}$$

Suppose, by contradiction, that (35) does not happen. Thus, for each $n \in \mathbb{N}$, $\exists\, v_n \in V_0\,;\quad |v_n| = 1$ and $[v_n] < \dfrac{1}{n}\cdot$

It follows from this that

$$\lim_{n \to +\infty} [v_n] = 0. \tag{36}$$

But, for each $n \in \mathbb{N}$, from (33) we have that

$$k_1||v_n||^2 \leq [v_n]^2 + |v_n|^2 < \frac{1}{n^2} + 1 < 2; \quad \forall\, n \in \mathbb{N},$$

which implies that

$$(v_n) \subset V_0 \quad \text{is a bounded sequence in the norm of} \quad V. \tag{37}$$

Since the topological space $(V_0, ||\cdot||)$ is a Hilbert space, given that it is closed, according to Lemma 6, we have that $\exists\,(v_\nu) \subset (v_n)$ and $v \in V_0$ such that

$$v_\nu \rightharpoonup v \quad \text{weakly in} \quad V_0\,. \tag{38}$$

Furthermore, the map $v \in V_0 \mapsto [v]$ is a seminorm on V_0, which implies that such map is convex and l.s.c. on V_0 endowed with the weak topology. It results that

$$[v] \leq \liminf_{\nu \to +\infty} [v_\nu].$$

But, from (36) we conclude that

$$\liminf_{\nu \to +\infty} [v_\nu] = 0,$$

consequently,

$$[v] = 0. \tag{39}$$

From (39), from the fact that $v \in V_0$ and since $[\,\cdot\,]$ is a norm on V_0 it follows that

$$v = 0. \tag{40}$$

On the other hand, since $H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$ we have that $V \overset{c}{\hookrightarrow} H$ and, therefore, from (37) we conclude that there exists a subsequence of (v_ν), which we will continue denoting by (v_ν), such that

$$v_\nu \to v \quad \text{in} \quad H.$$

Thus

$$|v_\nu| \to |v| \quad \text{in} \quad \mathbb{R}.$$

Since we have that $|v_\nu| = 1; \quad \forall\, \nu \in \mathbb{N}$, it follows that $|v| = 1$. But this contradicts what was obtained in (40), proving inequality (32), which concludes the proof. □

Next, we will solve the mathematical problem given in (3).

Let $\Omega \subset \mathbb{R}^n$, be a bounded open set, with sufficiently smooth boundary Γ. Let $\Gamma_0, \Gamma_1 \subset \Gamma$, such that Γ_0 has positive surface measure and $\Gamma_1 = \Gamma \backslash \Gamma_0$. Given

$$f = (f_1, \dots, f_n) \in (L^2(\Omega))^n \quad \text{and} \quad g = (g_1, \dots, g_n) \in (L^2(\Gamma))^n,$$

determine $u\colon \Omega \to \mathbb{R}^n$ verifying

$$\begin{cases} -\displaystyle\sum_{j=1}^n \frac{\partial \sigma_{ij}(u)}{\partial x_j} = f_i \quad \text{in} \quad \Omega, \quad 1 \leq i \leq n \\ u_i = 0 \quad \text{on} \quad \Gamma_0 \\ \displaystyle\sum_{j=1}^n \sigma_{ij}(u)\nu_j = g_i \quad \text{on} \quad \Gamma_1\,; \quad 1 \leq i \leq n. \end{cases} \tag{41}$$

where $\sigma_{ij}(u) = \lambda \operatorname{div} u \, \delta_{ij} + 2\mu \, \varepsilon_{ij}(u)$; $\quad \lambda, \mu > 0$ and $\varepsilon_{ij}(u) = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$.

Let us set as before

$$H = (L^2(\Omega))^n; \quad V = (H^1(\Omega))^n \text{ and } V_0 = \{v \in V; \vec{\gamma}_0 v = 0\}.$$

In what follows we will proceed formally. Multiplying equation $(41)_1$ by v_i, where $v = (v_1, \dots, v_n) \in V_0$, summing over i and integrating over Ω, we obtain

$$-\sum_{i,j=1} \int_\Omega \frac{\partial \sigma_{ij}(u)}{\partial x_j} \, v_i \, dx = \sum_{i=1}^n \int_\Omega f_i v_i \, dx. \tag{42}$$

However, by Gauss

$$\int_\Omega \frac{\partial}{\partial x_j} \, (\sigma_{ij}(u) v_i) dx = \int_\Gamma \sigma_{ij}(u) v_i \nu_j \, d\Gamma = \int_{\Gamma_0} \sigma_{ij}(u) v_i \nu_j \, d\Gamma + \int_{\Gamma_1} \sigma_{ij}(u) v_i \nu_j \, d\Gamma,$$

that is,

$$\int_\Omega \frac{\partial}{\partial x_j} \, (\sigma_{ij}(u)) v_i \, dx + \int_\Omega \sigma_{ij}(u) \, \frac{\partial v_i}{\partial x_j} \, dx = \int_{\Gamma_1} \sigma_{ij}(u) v_i \nu_j \, d\Gamma. \tag{43}$$

Substituting (43) in (42) results that

$$\sum_{i,j=1}^n \int_\Omega \sigma_{ij}(u) \, \frac{\partial v_i}{\partial x_j} \, dx = \sum_{i=1}^n \int_\Omega f_i v_i \, dx + \sum_{i,j=1}^n \int_{\Gamma_1} \sigma_{ij}(u) v_i \nu_j \, d\Gamma. \tag{44}$$

Analogously,

$$\begin{aligned} \sum_{i,j=1}^n \int_\Omega \sigma_{ji}(u) \, \frac{\partial v_j}{\partial x_i} \, dx &= \sum_{j=1}^n \int_\Omega f_j v_j \, dx + \sum_{i,j=1}^n \int_{\Gamma_1} \sigma_{ji}(u) v_j \nu_i \, d\Gamma \\ &= \sum_{i=1}^n \int_\Omega f_i v_i \, dx + \sum_{i,j=1}^n \int_{\Gamma_1} \sigma_{ij}(u) v_i \nu_j \, d\Gamma. \end{aligned} \tag{45}$$

Summing (44) and (45) and observing that $\sigma_{ij}(u) = \sigma_{ji}(u)$ we obtain

$$\sum_{i,j=1}^n \int_\Omega \sigma_{ij}(u) \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] dx = 2\left[\sum_{i=1}^n \int_\Omega f_i v_i \, dx + \sum_{i,j=1}^n \int_{\Gamma_1} \sigma_{ij}(u) v_i \nu_j \, d\Gamma\right].$$

Whence

$$\sum_{i,j=1}^n \int_\Omega \sigma_{ij}(u) \, \frac{1}{2} \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] dx = \sum_{i=1}^n \int_\Omega f_i v_i \, dx + \sum_{i=1}^n \int_{\Gamma_1} \left(\sum_{j=1}^n \sigma_{ij}(u) \nu_j\right) v_i \, d\Gamma,$$

or even,

$$\sum_{i,j=1}^n \int_\Omega \sigma_{ij}(u) \varepsilon_{ij}(v) dx = \sum_{i=1}^n \int_\Omega f_i v_i \, dx + \sum_{i=1}^n \int_{\Gamma_1} g_i v_i \, d\Gamma. \tag{46}$$

Now, setting

$$a(u, v) = \sum_{i,j=1}^n \int_\Omega \sigma_{ij}(u) \varepsilon_{ij}(v) \, dx \tag{47}$$

$$L(v) = \sum_{i=1}^n \left(\int_\Omega f_i v_i \, dx + \int_{\Gamma_1} g_i v_i \, d\Gamma \right), \tag{48}$$

we arrive at the weak problem

$$\begin{cases} \text{Determine } u \in V_0 \text{ such that} \\ \qquad a(u,v) = L(v); \;\; \forall\, v \in V_0\,. \end{cases} \tag{49}$$

Our goal is to use the Lax-Milgram Lemma. For this, we must show that $a(u,v)$ is a bilinear, continuous and coercive form on V_0 and $L \in V_0'$. Indeed, from (47) and the definition of σ_{ij} it follows that

$$a(u,v) = \sum_{i,j=1}^{n} \int_\Omega \lambda \,\text{div}\; u\,\delta_{ij}\,\varepsilon_{ij}(v)dx + \sum_{i,j=1}^{n} \int_\Omega 2\mu\varepsilon_{ij}(u)\varepsilon_{ij}(v)dx, \tag{50}$$

whence,

$$a(u,v) = \lambda \int_\Omega \text{div}\, u \bigg(\sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \bigg) dx + \sum_{i,j=1}^{n} \int_\Omega 2\mu\varepsilon_{ij}(u)\varepsilon_{ij}(v)dx,$$

that is,

$$a(u,v) = \lambda \int_\Omega (\text{div}\, u)(\text{div}\, v)dx + \sum_{i,j=1}^{n} \int_\Omega 2\mu\varepsilon_{ij}(u)\varepsilon_{ij}(v)dx. \tag{51}$$

The bilinearity of $a(u,v)$ is clear. We will prove its continuity and coercivity. Let $u, v \in V_0$. Note, initially, that

$$|\,\text{div}\, u| \leq \sum_{i=1}^{n} \left| \frac{\partial u_i}{\partial x_i} \right| = \sum_{i=j=1}^{n} |\varepsilon_{ij}(u)| \leq \sum_{i=j=1}^{n} |\varepsilon_{ij}(u)| + \sum_{\substack{i \neq j \\ i,j=1}}^{n} |\varepsilon_{ij}(u)|$$

that is,

$$|\,\text{div}\, u| \leq \sum_{i,j=1}^{n} |\varepsilon_{ij}(u)|. \tag{52}$$

Thus, from (50) and (52) given the Hölder and Minkowski inequalities, we obtain

$$\begin{aligned}
|a(u,v)| &\leq \lambda \sum_{i,j=1}^{n} \int_\Omega \bigg(\sum_{i,j=1}^{n} |\varepsilon_{ij}(u)| \bigg) |\varepsilon_{ij}(v)| dx + 2\mu \sum_{i,j=1}^{n} \int_\Omega |\varepsilon_{ij}(u)|\,|\varepsilon_{ij}(v)| dx \\
&= \lambda \int_\Omega \bigg(\sum_{i,j=1}^{n} |\varepsilon_{ij}(u)| \bigg) \bigg(\sum_{i,j=1}^{n} |\varepsilon_{ij}(v)| \bigg) dx + 2\mu \sum_{i,j=1}^{n} \int_\Omega |\varepsilon_{ij}(u)|\,|\varepsilon_{ij}(v)| dx \\
&\leq \lambda \bigg(\int_\Omega \bigg(\sum_{i,j=1}^{n} |\varepsilon_{ij}(u)| \bigg)^2 dx \bigg)^{1/2} \bigg(\int_\Omega \bigg(\sum_{i,j=1}^{n} |\varepsilon_{ij}(v)| \bigg)^2 dx \bigg)^{1/2} \\
&+ 2\mu \sum_{i,j=1}^{n} \bigg(\int_\Omega |\varepsilon_{ij}(u)|^2 dx \bigg)^{1/2} \bigg(\int_\Omega |\varepsilon_{ij}(v)|^2 dx \bigg)^{1/2} \\
&\leq c_1 \bigg\{ \bigg(\int_\Omega \sum_{i,j=1}^{n} |\varepsilon_{ij}(u)|^2 dx \bigg)^{1/2} \bigg(\int_\Omega \sum_{i,j=1}^{n} |\varepsilon_{ij}(v)|^2 dx \bigg)^{1/2} \\
&+ \bigg(\sum_{i,j=1}^{n} \int_\Omega |\varepsilon_{ij}(u)|^2 dx \bigg)^{1/2} \bigg(\sum_{i,j=1}^{n} \int_\Omega |\varepsilon_{ij}(v)|^2 dx \bigg)^{1/2} \bigg\}.
\end{aligned}$$

Thus,

$$|a(u,v)| \leq c_2 \Big(\sum_{i,j=1}^{n} \int_\Omega |\varepsilon_{ij}(u)|^2 dx \Big)^{1/2} \Big(\sum_{i,j=1}^{n} \int_\Omega |\varepsilon_{ij}(v)|^2 dx \Big)^{1/2} = c_2 [u][v]$$

which proves continuity.

We will show, now, that $a(u,v)$ is coercive. Indeed, let $v \in V_0$. Then from (51) it follows that

$$\begin{aligned} a(v,v) &= \lambda \int_\Omega (\operatorname{div} v)^2\, dx + 2\mu \sum_{i,j=1}^{n} \int_\Omega |\varepsilon_{ij}(v)|^2\, dx \\ &\geq 2\mu \sum_{i,j=1}^{n} \int_\Omega |\varepsilon_{ij}(v)|^2\, dx = 2\mu [v]^2, \end{aligned}$$

which proves coercivity. It remains to show that $L \in V_0'$. Indeed, linearity is obvious. We will prove, then that L is bounded in V_0. We have, $\forall\, v \in V_0$:

$$\begin{aligned} |L(v)| &\leq \sum_{i=1}^{n} \int_\Omega |f_i|\, |v_i| dx + \sum_{i=1}^{n} \int_{\Gamma_1} |g_i|\, |\gamma_0 v_i| d\Gamma \\ &\leq \sum_{i=1}^{n} \Big(\int_\Omega |f_i|^2 dx \Big)^{1/2} \Big(\int_\Omega |v_i|^2 dx \Big)^{1/2} \\ &+ \sum_{i=1}^{n} \Big(\int_{\Gamma_1} |g_i|^2 d\Gamma \Big)^{1/2} \Big(\int_{\Gamma_1} |\gamma_0 v_i|^2 d\Gamma \Big)^{1/2} \\ &\leq \Big(\sum_{i=1}^{n} \int_\Omega |f_i|^2 dx \Big)^{1/2} \Big(\sum_{i=1}^{n} \int_\Omega |v_i|^2 dx \Big)^{1/2} \\ &+ \Big(\sum_{i=1}^{n} \int_{\Gamma_1} |g_i|^2 d\Gamma \Big)^{1/2} \Big(\sum_{i=1}^{n} \int_{\Gamma_1} |\gamma_0 v_i|^2 d\Gamma \Big)^{1/2} \\ &= |f|\, |v| + |g|_{(L^2(\Gamma))^n}\, |\vec{\gamma}_0 v|_{(L^2(\Gamma))^n} \\ &\leq c_1 \big\{ |f|\, |v| + |g|_{(L^2(\Gamma))^n}\, |\vec{\gamma}_0 v|_{(H^{1/2}(\Gamma))^n} \big\} \\ &\leq c_2 \big\{ |f|\, ||v||_V + |g|_{(L^2(\Gamma))^n} ||v||_V \big\} \\ &= c_2 \big\{ |f| + |g|_{(L^2(\Gamma))^n} \big\} ||v||. \end{aligned}$$

Whence,

$$|L(v)| \leq c_3 \big\{ |f|_H + |g|_{(L^2(\Gamma))^n} \big\} [v], \tag{53}$$

where the last inequality follows from Proposition 2.

In this way, we have by the Lax-Milgram Lemma

$$\left| \begin{array}{l} \exists! u \in V_0 \quad \text{such that} \\ \quad a(u,v) = L(v);\ \forall\, v \in V_0 \end{array} \right.$$

or, equivalently,

$$\left| \begin{array}{l} \exists! u \in V_0 \quad \text{verifying} \\ \displaystyle\sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u) \varepsilon_{ij}(v) dx = \sum_{i=1}^{n} \Big(\int_\Omega f_i v_i dx + \int_{\Gamma_1} g_i v_i d\Gamma \Big);\ \forall\, v \in V_0\,. \end{array} \right. \tag{54}$$

Characterization of the Obtained Result

Let $\varphi \in (\mathcal{D}(\Omega))^n$. From (54) it follows that

$$\sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u)\, \frac{1}{2} \left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i} \right) dx = \sum_{i=1}^{n} \int_\Omega f_i \varphi_i \, dx. \tag{55}$$

However, from the fact that $\sigma_{ij}(u) = \sigma_{ji}(u)$,

$$\begin{aligned} &\sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u) \frac{1}{2} \left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i} \right) dx \\ &\quad = \frac{1}{2} \left\{ \sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u)\, \frac{\partial \varphi_i}{\partial x_j} + \sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u)\, \frac{\partial \varphi_j}{\partial x_i}\, dx \right\} \\ &\quad = \frac{1}{2} \left\{ \sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u)\, \frac{\partial \varphi_i}{\partial x_j} + \sum_{i,j=1}^{n} \int_\Omega \sigma_{ji}(u)\, \frac{\partial \varphi_j}{\partial x_i}\, dx \right\} \\ &\quad = \frac{1}{2} \left\{ \sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u)\, \frac{\partial \varphi_i}{\partial x_j} + \sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u)\, \frac{\partial \varphi_i}{\partial x_j}\, dx \right\} \\ &\quad = \sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u)\, \frac{\partial \varphi_i}{\partial x_j} \end{aligned} \tag{56}$$

Thus, from (55) and (56) we obtain

$$\sum_{i,j=1}^{n} \int_\Omega \sigma_{ij}(u)\, \frac{\partial \varphi_i}{\partial x_j}\, dx = \sum_{i=1}^{n} \int_\Omega f_i \varphi_i \, dx,$$

that is,

$$\sum_{i=1}^{n} \left\langle -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\, \sigma_{ij}(u), \varphi_i \right\rangle = \sum_{i=1}^{n} \langle f_i, \varphi_i \rangle.$$

Then, setting

$$\sigma_i = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\, \sigma_{ij}(u) \quad \text{and} \quad \sigma = (\sigma_1, \dots, \sigma_n)$$

we have

$$\sum_{i=1}^{n} \langle \sigma_i, \varphi_i \rangle = \sum_{i=1}^{n} \langle f_i, \varphi_i \rangle.$$

Whence

$$\langle \sigma, \varphi \rangle = \langle f, \varphi \rangle, \quad \forall\, \varphi \in (\mathcal{D}(\Omega))^n$$

and, therefore,

$$\sigma = f \quad \text{in} \quad (\mathcal{D}'(\Omega))^n;$$

or even,

$$\sigma_i = f_i \quad \text{in} \quad \mathcal{D}'(\Omega), \quad i = 1, 2, \dots, n.$$

Since $f_i \in L^2(\Omega)$ we conclude that

$$\sigma_i = f_i \quad \text{in} \quad L^2(\Omega).$$

Thus,

$$-\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\,\sigma_{ij}(u) = f_i \quad \text{in} \quad L^2(\Omega) \tag{57}$$

and then a.e. in Ω.

- **Boundary Condition**

Returning to (54) we have

$$\frac{1}{2}\sum_{i,j=1}^{n}\int_\Omega \sigma_{ij}(u)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)dx = \sum_{i=1}^{n}\int_\Omega f_i v_i\,dx + \sum_{i=1}^{n}\int_{\Gamma_1} g_i v_i\,d\Gamma.$$

Using the same arguments used previously, we obtain

$$\sum_{i,j=1}^{n}\int_\Omega \sigma_{ij}(u)\,\frac{\partial v_i}{\partial x_j}\,dx = \sum_{i=1}^{n}\int_\Omega f_i v_i\,dx + \sum_{i=1}^{n}\int_{\Gamma_1} g_i v_i\,d\Gamma. \tag{58}$$

Substituting (57) in (58) it follows that

$$\sum_{i,j=1}^{n}\int_\Omega \sigma_{ij}(u)\,\frac{\partial v_i}{\partial x_j}\,dx = \sum_{i=1}^{n}\int_\Omega \left(-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\,\sigma_{ij}(u)\right)v_i\,dx + \sum_{i=1}^{n}\int_{\Gamma_1} g_i v_i\,d\Gamma. \tag{59}$$

Suppose that $\dfrac{\partial}{\partial x_j}\,\sigma_{ij}(u) \in L^2(\Omega);\quad i,j = 1,\dots,n$ we have that

$$\sigma_{ij}u \in H^1(\Omega);\quad i,j = 1,\dots,n. \tag{60}$$

Given the hypothesis above, in what follows we will proceed formally. From equality (59), by virtue of the Gauss Lemma it follows that

$$\sum_{i,j=1}^{n}\left\{-\int_\Omega \frac{\partial}{\partial x_j}\,\sigma_{ij}(u)v_i\,dx + \int_\Gamma \sigma_{ij}(u)v_i\nu_j\,d\Gamma\right\} = \sum_{i,j=1}^{n} -\int_\Omega \frac{\partial}{\partial x_j}\,\sigma_{ij}(u)v_i\,dx + \sum_{i=1}^{n}\int_{\Gamma_1} g_i v_i\,d\Gamma,$$

that is,

$$\sum_{i=1}^{n}\int_{\Gamma_1}\left(\sum_{j=1}^{n}\gamma_0(\sigma_{ij}(u))\nu_j\right)(\gamma_0 v_i)\,d\Gamma = \sum_{i=1}^{n}\int_{\Gamma_1} g_i(\gamma_0 v_i)\,d\Gamma.$$

Define

$$\overline{\sigma} = (\overline{\sigma}_1,\dots,\overline{\sigma}_n) \text{ where } \overline{\sigma}_i = \sum_{j=1}^{n}\gamma_0(\sigma_{ij}(u))\nu_j \in H^{1/2}(\Gamma).$$

Then, $\overline{\sigma} \in (H^{1/2}(\Gamma))^n$ and, furthermore,

$$\sum_{i=1}^{n}\int_{\Gamma_1}\overline{\sigma}_i(\gamma_0 v_i)\,d\Gamma = \sum_{i=1}^{n}\int_{\Gamma_1} g_i(\gamma_0 v_i)\,d\Gamma,$$

that is,

$$(\overline{\sigma}, \vec{\gamma}_0 v)_{(L^2(\Gamma_1))^n} = (g, \vec{\gamma}_0 v)_{(L^2(\Gamma_1))^n}.$$

In this way,

$$(\overline{\sigma}, \vec{\gamma}_0 v)_{(L^2(\Gamma))^n} = (g, \vec{\gamma}_0 v)_{(L^2(\Gamma))^n}, \tag{61}$$

since $\vec{\gamma}_0 v = 0$ on Γ_0.

On the other hand, since V_0 is a closed subspace of V it follows that $\vec{\gamma}_0(V_0)$ is closed in $(H^{1/2}(\Gamma))^n$. Indeed, let $(v_\nu) \subset V_0$ such that

$$\vec{\gamma}_0(v_\nu) \to w \quad \text{in} \quad (H^{1/2}(\Gamma))^n.$$

Then, by the surjectivity of $\tilde{\gamma}_0$ it follows that $\exists\, v \in V$ such that $\vec{\gamma}_0 v = w$ and, therefore,

$$\vec{\gamma}_0(v_\nu) \to \vec{\gamma}_0(v) \quad \text{in} \quad (H^{1/2}(\Gamma))^n.$$

It remains to prove that $v \in V_0$. In fact, from the convergence above it follows that

$$\vec{\gamma}_0(v_\nu) \to \vec{\gamma}_0(v) \quad \text{a.e. on} \quad \Gamma$$

and since $(v_\nu) \subset V_0$ it follows that $\vec{\gamma}_0(v_\nu)(x) = 0$ a.e. on Γ_0, for all $\nu \in \mathbb{N}$, and then

$$(\vec{\gamma}_o v)(x) = 0 \quad \text{a.e. on} \quad \Gamma_0.$$

Thus, $w = \vec{\gamma}_0 v$ where $v \in V_0$ and therefore

$$\vec{\gamma}_0(V_0) \quad \text{is closed in} \quad (H^{1/2}(\Gamma))^n.$$

Since $(H^{1/2}(\Gamma))^n$ is a Hilbert space then $\vec{\gamma}_0(V_0)$ is also one. Identifying $L^2(\Gamma)$ with its dual, we have the following embeddings

$$\vec{\gamma}_0(V_0) \hookrightarrow (L^2(\Gamma))^n \hookrightarrow (\vec{\gamma}_0(V_0))'$$

and from (61) it follows that

$$\langle \overline{\sigma}, \vec{\gamma}_0 v \rangle_{(\vec{\gamma}_0(V_0))', \vec{\gamma}_0(V_0)} = \langle g, \vec{\gamma}_0 v \rangle_{(\gamma_0(V_0))', \vec{\gamma}(V_0)}\,; \quad \forall\, v \in V_0.$$

Consequently,

$$\overline{\sigma} = g \quad \text{in} \quad (\vec{\gamma}_0(V_0))'$$

and since $\overline{\sigma} \in (H^{1/2}(\Gamma))^n$ we have that

$$\overline{\sigma}_i = g_i \quad \text{in} \quad H^{1/2}(\Gamma),$$

that is,

$$\sum_j \gamma_0(\sigma_{ij}(u))\nu_j = g_i \quad \text{in} \quad H^{1/2}(\Gamma); \quad i = 1, 2, \ldots, n.$$

Observe that g was considered in $(L^2(\Gamma))^n$. However, according to the equality above and the hypothesis of regularity given in (60), we should choose g in $(H^{1/2}(\Gamma))^n$. □

14.4 Hyperbolic-Parabolic Problem

Let $k_1(x), k_2(x) \in L^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary Γ. Suppose that

$$k_2(x) \geq 0; \quad k_1(x) \geq \beta > 0 \quad \text{a.e. in} \quad \Omega. \tag{1}$$

Let $T > 0$, $Q = \Omega \times]0,T[$ and $\Sigma = \Gamma \times]0,T[$. In the cylinder Q consider the problem

$$\begin{cases} k_2(x)\,\dfrac{\partial^2 u}{\partial t^2} + k_1(x)\,\dfrac{\partial u}{\partial t} - \Delta u = f \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(0) = u_0(x); \quad (k_2 u')(0) = (k_2 u_1)(x). \end{cases} \tag{2}$$

Theorem: Given $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ and k_1 and k_2 verifying the previous hypotheses, there exists a unique $u\colon Q \to \mathbb{R}$, weak solution of (2) in the class

$$u \in L^\infty(0,T;H_0^1(\Omega)); \quad u' \in L^\infty(0,T;L^2(\Omega)); \quad k_2 u'' \in L^2(0,T;H^{-1}(\Omega)).$$

Proof: Let $0 < \varepsilon < 1$. Consider the perturbed problem

$$\begin{cases} (k_2(x)+\varepsilon)u_\varepsilon'' + k_1(x)u_\varepsilon' - \Delta u_\varepsilon = f \quad \text{in} \quad Q \\ u_\varepsilon = 0 \quad \text{on} \quad \Sigma \\ u_\varepsilon(0) = u_0(x); \quad ((k_2(x)+\varepsilon)u_\varepsilon')(0) = (k_2(x)+\varepsilon)u_1. \end{cases} \tag{P$_2$}$$

• **Solution of (P$_2$)**:

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H_0^1(\Omega)$ which we can consider, without loss of generality, orthonormal in $L^2(\Omega)$. Furthermore, consider

$$V_m = [w_1, \ldots, w_m].$$

In V_m consider the approximate problem, where $a(.,.)$ is the Dirichlet form.

$$\text{(AP)}\begin{cases} u_{\varepsilon m}(t) \in V_m \Leftrightarrow u_{\varepsilon m}(t) = \displaystyle\sum_{i=1}^m g_{i\varepsilon m}(t) w_i \\ ([k_2+\varepsilon]u_{\varepsilon m}''(t), w_j) + (k_1 u_{\varepsilon m}'(t), w_j) + a(u_{\varepsilon m}(t), w_j) = (f(t), w_j) \\ u_{\varepsilon m}(0) = u_{0m} \to u_0 \text{ in } H_0^1(\Omega); (k_2+\varepsilon)u_{\varepsilon m}'(0) \to (k_2+\varepsilon)u_1 \text{ in } L^2(\Omega)^{(*)} \\ j = 1, \ldots, m \end{cases}$$

Justification for $(\text{AP})_3^{(*)}$:

Since (w_ν) is total in $H_0^1(\Omega)$, there exist $(u_{0m}), (u_{1m}) \subset [w_\nu]_{\nu\in\mathbb{N}}$ such that $u_{0m} \to u_0$ in $H_0^1(\Omega)$ and $u_{1m} \to u_1$ in $L^2(\Omega)$. Thus, let us set

$$u_{\varepsilon m}(0) = u_{0m} \quad \text{and} \quad u_{\varepsilon m}'(0) = u_{1m}\,. \tag{3}$$

Now, since $u_{1m} \to u_1$ in $L^2(\Omega)$ and $k_2 \in L^\infty(\Omega)$ and $k_2 \geq 0$, then

$$\sqrt{k_2+\varepsilon}\, u_{1m} \to \sqrt{k_2+\varepsilon}\, u_1 \quad \text{in} \quad L^2(\Omega)$$

and from (3) it follows that, for each $0 < \varepsilon < 1$, we have

$$\sqrt{k_2 + \varepsilon}\, u'_{\varepsilon m}(0) \to \sqrt{k_2 + \varepsilon}\, u_1 \quad \text{in} \quad L^2(\Omega), \tag{4}$$

as well as,

$$(k_2 + \varepsilon) u'_{\varepsilon m}(0) \to (k_2 + \varepsilon) u_1 \quad \text{in} \quad L^2(\Omega). \tag{5}$$

From $(\mathrm{AP})_1$ and $(\mathrm{AP})_2$ we obtain

$$\begin{aligned}&\sum_{i=1}^{m} \big([k_2 + \varepsilon] g''_{i\varepsilon m}(t) w_i, w_j\big) + \sum_{i=1}^{m} (k_1 g'_{i\varepsilon m}(t) w_i, w_j) + \sum_{i=1}^{m} ((g_{i\varepsilon m}(t) w_i, w_j))\\ &= (f(t), w_j) \;\; j = 1, \ldots, m\end{aligned}$$

Since (u_{0m}), $(u_{1m}) \subset [w_\nu]_{\nu \in \mathbb{N}}$, we can write

$$u_{0m} = \sum_{j=1}^{m} \alpha_{jm} w_j \quad \text{and} \quad u_{1m} = \sum_{j=1}^{m} \beta_{jm} w_j\,.$$

From the orthogonality of the w_ν's in $L^2(\Omega)$ it follows that

$$g_{j\varepsilon m}(0) = \alpha_{jm} \quad \text{and} \quad g'_{j\varepsilon m}(0) = \beta_{jm}\,.$$

In this way we arrive at the system

$$\begin{cases} \displaystyle\sum_{i=1}^{m} ([k_2(x) + \varepsilon] g''_{i\varepsilon m}(t) w_i, w_j) + \sum_{i=1}^{m} (k_1(x) g'_{i\varepsilon m}(t) w_i, w_j) \\ \qquad\displaystyle + \sum_{i=1}^{m} ((g_{i\varepsilon m}(t) w_i, w_j)) = (f(t), w_j) \\ g_{j\varepsilon m}(0) = \alpha_{jm}\,; \quad g'_{j\varepsilon m}(0) = \beta_{jm}; \qquad j = 1, \ldots, m \end{cases} \tag{6}$$

or, equivalently,

$$\begin{cases} \begin{bmatrix} ((k_2(x)+\varepsilon) w_1, w_1) & \ldots & ((k_2(x)+\varepsilon) w_m, w_1) \\ \vdots & & \vdots \\ ((k_2(x)+\varepsilon) w_1, w_m) & \ldots & ((k_2(x)+\varepsilon) w_m, w_m) \end{bmatrix} \begin{bmatrix} g''_{1\varepsilon m}(t) \\ \vdots \\ g''_{m\varepsilon m}(t) \end{bmatrix} + \\ \quad + \begin{bmatrix} (k_1(x) w_1, w_1) & \ldots & (k_1(x) w_m, w_1) \\ \vdots & & \vdots \\ (k_1(x) w_1, w_m) & \ldots & (k_1(x) w_m, w_m) \end{bmatrix} \begin{bmatrix} g'_{1\varepsilon m}(t) \\ \vdots \\ g'_{m\varepsilon m}(t) \end{bmatrix} \\ + \begin{bmatrix} ((w_1, w_1)) & \ldots & ((w_m, w_1)) \\ \vdots & & \vdots \\ ((w_1, w_m)) & \ldots & ((w_m, w_m)) \end{bmatrix} \begin{bmatrix} g_{1\varepsilon m}(t) \\ \vdots \\ g_{m\varepsilon m}(t) \end{bmatrix} = \begin{bmatrix} (f(t), w_1) \\ \vdots \\ (f(t), w_m) \end{bmatrix} \\ \begin{bmatrix} g_{1\varepsilon m}(0) \\ \vdots \\ g_{m\varepsilon m}(0) \end{bmatrix} = \begin{bmatrix} \alpha_{1m} \\ \vdots \\ \alpha_{mm} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} g'_{1\varepsilon m}(0) \\ \vdots \\ g'_{m\varepsilon m}(0) \end{bmatrix} = \begin{bmatrix} \beta_{1m} \\ \vdots \\ \beta_{mm} \end{bmatrix} \end{cases} \tag{7}$$

Let us set

$$A = \begin{bmatrix} ((k_2(x)+\varepsilon)w_1, w_1) & \dots & ((k_2(x)+\varepsilon)w_m, w_1) \\ \vdots & & \vdots \\ ((k_2(x)+\varepsilon)w_1, w_m) & \dots & ((k_2(x)+\varepsilon)w_m, w_m) \end{bmatrix};$$

$$B = \begin{bmatrix} (k_1(x)w_1, w_1) & \dots & (k_1(x)w_m, w_1) \\ \vdots & & \vdots \\ (k_1(x)w_1, w_m) & \dots & (k_1(x)w_m, w_m) \end{bmatrix};$$

$$C = \begin{bmatrix} ((w_1, w_1) & \dots & ((w_m, w_1)) \\ \vdots & & \vdots \\ ((w_1, w_m) & \dots & ((w_m, w_m)) \end{bmatrix}; \qquad F(t) = \begin{bmatrix} (f(t), w_1) \\ \vdots \\ (f(t), w_m) \end{bmatrix}$$

$$Z(t) = \begin{bmatrix} g_{1\varepsilon m}(t) \\ \vdots \\ g_{m\varepsilon m}(t) \end{bmatrix}; \quad Z(0) = \begin{bmatrix} \alpha_{1m} \\ \vdots \\ \alpha_{mm} \end{bmatrix} = Z_0 \quad \text{and} \quad Z'(0) = \begin{bmatrix} \beta_{1m} \\ \vdots \\ \beta_{mm} \end{bmatrix} = Z_1$$

Thus, from (7) we arrive at the system of equations

$$\begin{cases} AZ''(t) + BZ'(t) + CZ(t) = F(t) \\ Z(0) = Z_0\,; \quad Z'(0) = Z_1. \end{cases} \tag{8}$$

Note that A is invertible. Indeed, let $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ and suppose that

$$A\xi = 0,$$

that is,

$$\sum_{i=1}^{m} ((k_2(x)+\varepsilon)w_i, w_j)\xi_i = 0; \quad \forall\, j = 1, \dots, m.$$

Whence

$$\left(\sum_{i=1}^{m} (k_2(x)+\varepsilon)w_i\xi_i, w_j \right) = 0; \quad \forall\, j = 1, \dots, m.$$

Multiplying by ξ_j and summing over j results that

$$\left(\sum_{i=1}^{m} (k_2(x)+\varepsilon)^{1/2} w_i\xi_i, \sum_{j=1}^{m} (k_2(x)+\varepsilon)^{1/2} w_j\xi_j \right) = 0,$$

which implies

$$(k_2(x)+\varepsilon) \sum_{i=1}^{m} w_i\xi_i = 0 \quad \text{in} \quad L^2(\Omega)$$

and, therefore, a.e. in Ω. Considering that $(k_2(x)+\varepsilon) > 0, \quad \forall\, x \in \Omega$, we obtain

$$\sum_{i=1}^{m} w_i\xi_i = 0 \quad \text{in} \quad L^2(\Omega).$$

Since the w_i's are linearly independent in $L^2(\Omega)$ it follows that

$$\xi_i = 0; \quad \forall\, i = 1, \dots, m$$

which proves that $\xi = 0$. Therefore, the matrix A is invertible. It follows from this that we can rewrite (8) in the form

$$\begin{cases} Z''(t) + A^{-1}BZ'(t) + A^{-1}CZ(t) = A^{-1}F(t) \\ Z(0) = Z_0\,; \quad Z'(0) = Z_1. \end{cases} \tag{9}$$

Next, we reduce the order of the system above. For this, consider

$$Y_1(t) = Z'(t); \quad Y_2(t) = Z(t) \quad \text{and} \quad Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}. \tag{10}$$

Thus

$$\begin{aligned} Y'(t) = \begin{bmatrix} Y_1'(t) \\ Y_2'(t) \end{bmatrix} = \begin{bmatrix} Z''(t) \\ Z'(t) \end{bmatrix} &= \begin{bmatrix} A^{-1}F(t) - A^{-1}BZ'(t) - A^{-1}CZ(t) \\ Z'(t) \end{bmatrix} \\ &= \begin{bmatrix} A^{-1}F(t) - A^{-1}BY_1(t) - A^{-1}CY_2(t) \\ Y_1(t) \end{bmatrix}, \end{aligned}$$

or even,

$$Y'(t) = \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} + \begin{bmatrix} A^{-1}F(t) \\ 0 \end{bmatrix}. \tag{11}$$

Denoting

$$D = \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}; \qquad G(t) = \begin{bmatrix} A^{-1}F(t) \\ 0 \end{bmatrix}$$

from (9), (10) and (11) it follows that

$$\begin{cases} Y'(t) = DY(t) + G(t) \\ Y(0) = Y_0, \end{cases} \tag{12}$$

where $Y_0 = \begin{bmatrix} Z_1 \\ Z_0 \end{bmatrix}$. Consider the map

$$\begin{aligned} h\colon [0,T] \times \mathbb{R}^{2m} &\to \mathbb{R}^{2m} \\ (t, Y) &\mapsto DY + G(t). \end{aligned}$$

The function h enjoys the following properties:

(i) For all $Y \in \mathbb{R}^{2m}$ fixed, $h(t, Y)$ is measurable in t;

(ii) For a.e. $t \in [0,T]$ fixed, $h(t, Y)$ is continuous in Y;

(iii) If $K \subset [0,T] \times \mathbb{R}^{2m}$ is compact, then $\forall\,(t, , y) \in K$ we have

$$|h(t, Y)|_{2m} = |DY + G(t)|_{2m} \leq ||D||\,|Y|_{2m} \leq k_1 + |G(t)|_{2m}$$

where $k_1 > 0$. However, the function on the right of the inequality above is integrable on $\operatorname{proj}_t K$, since $f \in L^2(0,T; L^2(\Omega))$.

Thus, from (i), (ii) and (iii) we have, by Carathéodory's Theorem that $\exists\, Y\colon [0, t_{\varepsilon m}) \to \mathbb{R}^{2m}$ local solution of (12), $0 < t_{\varepsilon m} \le T$, where $Y(t)$ is absolutely continuous in $[0, t_{\varepsilon m})$ and differentiable a.e. in $[0, t_{\varepsilon m})$. It follows that the system of ordinary differential equations given in (8) possesses a local solution in the same interval, with $Z(t)$, $Z'(t)$ being absolutely continuous and $Z''(t)$ existing a.e.. The regularity of the function $Z(t)$ is inherited by the $g_{j\varepsilon m}(t)$. The a priori estimate below will serve to extend the solution to the whole interval $[0, T]$.

A Priori Estimate

Multiplying $(\text{AP})_2$ by $g'_{j\varepsilon m}(t)$ and summing over j, we obtain

$$\begin{aligned}\big([k_2+\varepsilon]u''_{\varepsilon m}(t), u'_{\varepsilon m}(t)\big) + (k_1 u'_{\varepsilon m}(t), u'_{\varepsilon m}(t)) + ((u_{\varepsilon m}(t), u'_{\varepsilon m}(t)))\\ = (f(t), u'_{\varepsilon m}(t)).\end{aligned} \tag{13}$$

Note that if $k \in L^\infty(\Omega)$ and $k \ge 0$ the following equality is valid:

$$(k u''_{\varepsilon m}(t), u'_{\varepsilon m}(t)) = (k^{1/2}\, u''_{\varepsilon m}(t), k^{1/2}\, u'_{\varepsilon m}(t)).$$

On the other hand,

$$\frac{d}{dt}\,(k^{1/2}\, u'_{\varepsilon m}(t), k^{1/2}\, u'_{\varepsilon m}(t)) = 2(k^{1/2}\, u''_{\varepsilon m}(t), k^{1/2}\, u'_{\varepsilon m}(t)).$$

Whence

$$(k\, u''_{\varepsilon m}(t), u'_{\varepsilon m}(t)) = \frac{1}{2}\,\frac{d}{dt}\,|k^{1/2}\, u'_{\varepsilon m}(t)|^2. \tag{14}$$

Thus, from (13) and (14) it follows that

$$\frac{1}{2}\,\frac{d}{dt}\,|(k_2+\varepsilon)^{1/2}\, u'_{\varepsilon m}(t)|^2 + |k_1^{1/2}\, u'_{\varepsilon m}(t)|^2 + \frac{1}{2}\,\frac{d}{dt}\,||u_{\varepsilon m}(t)||^2 = (f(t), u'_{\varepsilon m}(t)).$$

Integrating from 0 to t with $t < t_{\varepsilon m}$ we obtain

$$\begin{aligned}&|(k_2+\varepsilon)^{1/2}\, u'_{\varepsilon m}(t)|^2 + ||u_{\varepsilon m}(t)||^2 + 2\int_0^t |k_1^{1/2}\, u'_{\varepsilon m}(s)|^2\, ds\\ &= |(k_2+\varepsilon)^{1/2}\, u'_{\varepsilon m}(0)|^2 + ||u_{\varepsilon m}(0)||^2 + 2\int_0^t (f(s), u'_{\varepsilon m}(s))ds.\end{aligned}$$

Since $u'_{\varepsilon m}(0) = u_{1m} \to u_1$ in $L^2(\Omega)$; $\exists\, c_1 > 0$ such that

$$|u'_{\varepsilon m}(0)|^2 \le c_1\,.$$

Now, since, $0 < \varepsilon < 1$ and $k \in L^\infty(\Omega)$ $\exists\, c_2 > 0$ independent of ε and m verifying

$$|(k_2+\varepsilon)^{1/2}\, u'_{\varepsilon m}(0)|^2 \le |(k_2+1)^{1/2}\, u'_{\varepsilon m}(0)|^2 \le c_2\,; \quad \forall\, m \in \mathbb{N};\ \forall\, \varepsilon \in]0,1[\,.$$

Also, since $u_{\varepsilon m}(0) = u_{0m} \to u_0$ in $H_0^1(\Omega)$, $\exists\, c_3 > 0$ such that

$$||u_{\varepsilon m}(0)||^2 \le c_3\,; \quad \forall\, m \in \mathbb{N};\ \forall\, \varepsilon \in]0,1[\,.$$

Therefore, there exists $c_4 > 0$, independent of ε and m satisfying

$$\begin{aligned} |(k_2+\varepsilon)^{1/2}\, u'_{\varepsilon m}(t)|^2 + ||u_{\varepsilon m}(t)||^2 + 2\int_0^t |k_1^{1/2}\, u'_{\varepsilon m}(s)|^2\, ds \\ \le c_4 + \int_0^t (f(s), u'_{\varepsilon m}(s))ds. \end{aligned}$$

But, by hypothesis, $k_1(x) \ge \beta > 0$ and, therefore,

$$\int_0^t |k_1^{1/2}\, u'_{\varepsilon m}(s)|^2\, ds \ge \int_0^t k_1 |u'_{\varepsilon m}(s)|^2\, ds \ge \beta \int_0^t |u'_{\varepsilon m}(s)|^2\, ds.$$

Thus,

$$\begin{aligned} |(k_2+\varepsilon)^{1/2}\, u'_{\varepsilon m}(t)|^2 + ||u_{\varepsilon m}(t)||^2 + 2\beta \int_0^t |u'_{\varepsilon m}(s)|^2\, ds \le c_4 \\ + 2\int_0^t (f(s), u'_{\varepsilon m}(s))ds. \end{aligned} \tag{15}$$

On the other hand, let $\lambda > 0$. Then

$$\begin{aligned} 2\int_0^t (f(s), u'_{\varepsilon m}(s))ds = 2\int_0^t \left(\frac{1}{\lambda^{1/2}}\, f(s), \lambda^{1/2}\, u'_{\varepsilon m}(s)\right) ds \\ \le \frac{1}{\lambda}\int_0^T |f(s)|^2 ds + \lambda \int_0^t |u'_{\varepsilon m}(s)|^2 ds. \end{aligned} \tag{16}$$

Choosing $\lambda = \beta$ it results from (15) and (16) that $\exists\, c > 0$ such that

$$|(k_2+\varepsilon)^{1/2}\, u'_{\varepsilon m}(t)|^2 + ||u_{\varepsilon m}(t)||^2 + \beta \int_0^t |u'_{\varepsilon m}(s)|^2\, ds \le c; \tag{17}$$

$\forall\, t \in [0, t_m)$; $\forall\, m \in \mathbb{N}$ and $\forall\, \varepsilon \in]0,1[$.

From (17), we obtain the existence of a constant $c(\varepsilon) > 0$ which verifies

$$|u'_{\varepsilon m}(t)|^2 + ||u_{\varepsilon m}(t)||^2 + \int_0^t |u'_{\varepsilon m}(s)|^2\, ds \le c(\varepsilon), \quad \forall\, t \in [0, t_m); \forall\, m \in \mathbb{N}. \tag{18}$$

We have

$$|u_{\varepsilon m}(t)|^2_{L^2(\Omega)} = \left| \sum_{j=1}^m g_{j\varepsilon m}(t) w_j \right|^2_{L^2(\Omega)} = \sum_{j=1}^m |g_{j\varepsilon m}(t)|^2_m .$$

Thus, from (17) we have then that there exists $c > 0$ such that

$$|Z(t)|^2_m = \sum_{j=1}^m |g_{j\varepsilon m}(t)|^2 = |u_{\varepsilon m}(t)|^2_{L^2(\Omega)} \le c' ||u_{\varepsilon m}(t)||^2 \le c \tag{19}$$

$\forall\, t \in [0, t_{\varepsilon m})$, $\forall\, m \in \mathbb{N}$, $\forall\, \varepsilon \in]0,1[$.

Also

$$|u'_{\varepsilon m}(t)|^2_{L^2(\Omega)} = \left| \sum_{j=1}^m g'_{j\varepsilon m}(t) w_j \right|^2_{L^2(\Omega)} = \sum_{j=1}^m |g'_{j\varepsilon m}(t)|^2_m .$$

From (18) it results that there exists $c(\varepsilon) > 0$ verifying

$$|Z'(t)|_m^2 = \sum_{j=1}^{m} |g'_{j\varepsilon m}(t)|_m^2 = |u'_{\varepsilon m}(t)|^2_{L^2(\Omega)} \leq c(\varepsilon) \tag{20}$$

$\forall\, t \in [0, t_{\varepsilon m})$, $\forall\, m \in \mathbb{N}$ and $\forall\, \varepsilon \in]0, 1]$.

From (19) and (20) it follows that $\exists\, \bar{c}(\varepsilon) > 0$ such that

$$|Y(t)|^2_{2m} = |Y_1(t)|^2_m + |Y_2(t)|^2_m = |Z(t)|^2_m + |Z'(t)|^2_m \leq \bar{c}(\varepsilon),$$

$\forall\, t \in [0, t_{\varepsilon m})$, $\forall\, m \in \mathbb{N}$ and $\forall\, \varepsilon \in]0, 1[$.

It results that we can extend the solution $Y(t)$ of the system (12) to the whole interval $[0, T]$. Consequently, for each $0 < \varepsilon < 1$, we can extend $g_{j\varepsilon m}(t)$ and therefore $u_{\varepsilon m}(t)$ to the whole interval $[0, T]$.

Repeating the calculations made to obtain (17) we conclude that

$$|(k_2 + \varepsilon)^{1/2}\, u'_{\varepsilon m}(t)|^2 + ||u_{\varepsilon m}(t)||^2 + \int_0^t |u'_{\varepsilon m}(s)|^2\, ds \leq c; \tag{21}$$

$\forall\, t \in [0, T]$; $\forall\, m \in \mathbb{N}$; and $\forall\, \varepsilon \in]0, 1[$.

Consequently

$$(u_{\varepsilon m}) \quad \text{is bounded in} \quad L^\infty(0, T; H^1_0(\Omega)). \tag{22}$$

$$(u'_{\varepsilon m}) \quad \text{is bounded in} \quad L^2(0, T; L^2(\Omega)). \tag{23}$$

Passage to the Limit

From the estimates (22) and (23) there exists $(u_{\varepsilon\nu}) \subset (u_{\varepsilon m})$ such that, for each $\varepsilon \in]0, 1[$,

$$u_{\varepsilon\nu} \overset{*}{\rightharpoonup} u_\varepsilon \quad \text{weak * in} \quad L^\infty(0, T; H^1_0(\Omega)) \tag{24}$$

$$u'_{\varepsilon\nu} \rightharpoonup u'_\varepsilon \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)). \tag{25}$$

Let $j \in \mathbb{N}$ and $\nu \geq j$. Consider $\theta \in \mathcal{D}(0, T)$. Multiplying $(\text{AP})_2$ by θ and integrating in $[0, T]$, results that

$$\int_0^T ([k_2 + \varepsilon]u''_{\varepsilon\nu}(t), w_j)\theta(t)dt + \int_0^T (k_1 u'_{\varepsilon\nu}(t), w_j)\theta(t)dt + \int_0^T ((u_{\varepsilon\nu}(t), w_j))\theta(t)dt = \int_0^T (f(t), w_j)\theta(t)dt$$

Whence

$$-\int_0^T ([k_2 + \varepsilon]u'_{\varepsilon\nu}(t), w_j)\theta'(t)dt + \int_0^T (k_1 u'_{\varepsilon\nu}(t), w_j)\theta(t)dt + \int_0^T ((u_{\varepsilon\nu}(t), w_j))\theta(t)dt = \int_0^T (f(t), w_j)\theta(t)dt. \tag{26}$$

Note that

$$\begin{aligned} &(k_2 + \varepsilon)w_j\theta' \in L^2(0, T; L^2(\Omega)) \\ &k_1 w_j \theta \in L^2(0, T; L^2(\Omega)) \\ &-\Delta w_j \theta \in L^1(0, T; H^{-1}(\Omega)), \end{aligned}$$

where the last inclusion comes from the fact that $\Delta \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

Therefore, from (24) and (25) we conclude that

$$\int_0^T ([k_2+\varepsilon]u'_{\varepsilon\nu}(t), w_j)\theta'(t)dt \xrightarrow{\nu\to+\infty} \int_0^T ([k_2+\varepsilon]u'_{\varepsilon}(t), w_j)\theta'(t)dt \tag{27}$$

$$\int_0^T (k_1 u'_{\varepsilon\nu}(t), w_j)\theta(t)dt \xrightarrow{\nu\to+\infty} \int_0^T (k_1 u'_{\varepsilon}(t), w_j)\theta(t)dt \tag{28}$$

$$\int_0^T \langle -\Delta w_j, u_{\varepsilon\nu}(t)\rangle\theta(t)dt \xrightarrow{\nu\to+\infty} \int_0^T \langle -\Delta w_j, u_{\varepsilon}(t)\rangle\theta(t)dt. \tag{29}$$

Since $\langle -\Delta w_j, v\rangle = a(w_j, v), \quad \forall\, v \in H_0^1(\Omega)$, it follows from (26), (27), (28) and (29), in the limit situation, when $\nu \to +\infty$, that

$$-\int_0^T ([k_2+\varepsilon]u'_{\varepsilon}(t), w_j)\theta'(t)dt + \int_0^T (k_1 u'_{\varepsilon}(t), w_j)\theta(t)dt + \int_0^T ((u_\varepsilon(t), w_j))\theta(t)dt$$
$$= \int_0^T (f(t), w_j)\theta(t)dt.$$

By the arbitrariness of j and the totality of the $(w_\nu)'_{\nu\in\mathbb{N}}s$ in $H_0^1(\Omega)$ it follows that

$$\begin{aligned} &-\int_0^T ([k_2+\varepsilon]u'_{\varepsilon}(t), v)\theta'(t)dt + \int_0^T (k_1 u'_{\varepsilon}(t), v)\theta(t)dt \\ &+\int_0^T ((u_\varepsilon(t), v))\theta(t)dt = \int_0^T (f(t), v)\theta(t)dt \end{aligned} \tag{30}$$

$\forall\, v \in H_0^1(\Omega)$, or even,

$$\frac{d}{dt}([k_2+\varepsilon]u'_{\varepsilon}(t), v) + (k_1 u'_{\varepsilon}(t), v) + ((u_\varepsilon(t), v)) = (f(t), v) \tag{31}$$

in $\mathcal{D}'(0,T), \ \forall\, v \in H_0^1(\Omega)$.

Taking $v = \varphi \in \mathcal{D}(\Omega)$ in (30) we obtain

$$\left\langle \frac{d}{dt}[(k_2+\varepsilon)u'_\varepsilon], \varphi\theta \right\rangle + \langle k_1 u'_\varepsilon, \varphi\theta\rangle + \langle -\Delta u_\varepsilon, \varphi\theta\rangle = \langle f, \varphi\theta\rangle;$$

$\forall\, \varphi \in \mathcal{D}(\Omega)$ and $\forall\, \theta \in \mathcal{D}(0,T)$ and, by the totality of $\{\varphi\theta; \varphi \in \mathcal{D}(\Omega), \theta \in \mathcal{D}(0,T)\}$ in $\mathcal{D}(\Omega\times]0,T[)$, it follows that

$$\frac{d}{dt}([k_2+\varepsilon]u'_\varepsilon) + k_1 u'_\varepsilon - \Delta u_\varepsilon = f \quad \text{in} \quad \mathcal{D}'(Q). \tag{32}$$

Recalling that $\Delta \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$, $u_\varepsilon \in L^\infty(0,T,H_0^1(\Omega))$, $\quad k_1 \in L^\infty(\Omega)$, $u'_\varepsilon \in L^2(0,T;L^2(\Omega))$ and $f \in L^2(0,T;L^2(\Omega))$ from (32) we conclude that

$$\frac{d}{dt}([k_2+\varepsilon]u'_\varepsilon) \in L^2(0,T;H^{-1}(\Omega)) \tag{33}$$

and, therefore,

$$\frac{d}{dt}([k_2+\varepsilon]u'_\varepsilon) + k_1 u'_\varepsilon - \Delta u_\varepsilon = f \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)). \tag{34}$$

- **Initial Conditions**

Note, initially, that from (24) and (25) we have that

$$u_\varepsilon \in C^0([0,T];L^2(\Omega)) \cap C_s([0,T];H_0^1(\Omega))$$

and, from (25) and (33) we also have that

$$(k_2+\varepsilon)u'_\varepsilon \in C^0([0,T];H^{-1}(\Omega)) \cap C_s([0,T];L^2(\Omega));$$

making sense to speak of $u_\varepsilon(0)$, $((k_2+\varepsilon)u'_\varepsilon)(0)$, $u_\varepsilon(T)$ and $((k_2+\varepsilon)u'_\varepsilon)(T)$.

Lemma 1: Let $k \in L^\infty(\Omega)$ and $T \in \mathcal{D}'(0,T;L^2(\Omega))$. Then the map

$$\begin{aligned} kT\colon \mathcal{D}(0,T) &\to L^2(\Omega) \\ \theta &\mapsto \langle kT,\theta\rangle = k\langle T,\theta\rangle \end{aligned}$$

is linear and continuous, that is, $kT \in \mathcal{D}'(0,T;L^2(\Omega))$.

Proof: Observe that the map above is well defined since as $k \in L^\infty(\Omega)$ and $\langle T,\theta\rangle \in L^2(\Omega)$ then $k\cdot\langle T,\theta\rangle \in L^2(\Omega)$. Furthermore, kT is clearly linear and if $\theta_\nu \to 0$ in $\mathcal{D}(0,T)$ then $\langle kT,\theta_\nu\rangle \to 0$ in $L^2(\Omega)$. Indeed, from the fact that $\theta_\nu \to 0$ in $\mathcal{D}(0,T)$ it results that $\langle T,\theta_\nu\rangle \to 0$ in $L^2(\Omega)$ and therefore $k\langle T,\theta\nu\rangle \to 0$ in $L^2(\Omega)$. □

Lemma 2: Under the previous hypotheses we have that

$$(kT)' = kT'.$$

Proof: Let $\theta \in \mathcal{D}(0,T)$. We have:

$$\langle (kT)',\theta\rangle = -\langle kT,\theta'\rangle = -k\langle T,\theta'\rangle = k(-\langle T,\theta'\rangle) = k\langle T',\theta\rangle = \langle kT',\theta\rangle. \qquad \square$$

By virtue of Lemma 2 we can write

$$(k_2+\varepsilon)u'_\varepsilon = ((k_2+\varepsilon)u_\varepsilon)'.$$

Furthermore, in particular,

$$[(k_2+\varepsilon)u'_\varepsilon](0) = ((k_2+\varepsilon)u_\varepsilon)'(0) \quad \text{and} \quad [(k_2+\varepsilon)u'_\varepsilon](T) = ((k_2+\varepsilon)u_\varepsilon)'(T).$$

Remark: It is worth observing that if $T \in \mathcal{D}'(0,T;L^2(\Omega))$ then $T \in \mathcal{D}'(0,T;H^{-1}(\Omega))$ and from Lemma 2 it follows that

$$(kT)' = kT',$$

where now the derivative is in the sense of vector-valued distributions in $\mathcal{D}'(0,T;H^{-1}(\Omega))$.

(i) $u_\varepsilon(0) = u_0$

Let $\theta \in C^1([0,T])$ such that $\theta(0)=1$ and $\theta(T)=0$ and consider $v \in L^2(\Omega)$. Then $v\theta \in L^2(0,T;L^2(\Omega))$ and from (25) it follows that

$$\int_0^T (u'_{\varepsilon\nu}(t),v)\theta(t)dt \xrightarrow{\nu\to+\infty} \int_0^T (u'_\varepsilon(t),v)\theta(t)dt.$$

Integrating by parts we have

$$-(u_{\varepsilon\nu}(0),v)-\int_0^T (u_{\varepsilon\nu}(t),v)\theta'(t)dt \xrightarrow{\nu\to+\infty} -(u_\varepsilon(0),v)-\int_0^T (u_\varepsilon(t),v)\theta'(t)dt.$$

Since

$$\int_0^T (u_{\varepsilon\nu}(t),v)\theta'(t)dt \xrightarrow{\nu\to+\infty} \int_0^T (u_\varepsilon(t),v)\theta'(t)dt$$

then:

$$(u_{\varepsilon\nu}(0),v) \xrightarrow{\nu\to+\infty} (u_\varepsilon(0),v).$$

Since $u_{\varepsilon\nu}(0)=u_{0\nu}\to u_0$ in $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$, then

$$(u_{\varepsilon\nu}(0),v) \xrightarrow{\nu\to+\infty} (\tilde{u}_0,v);\quad \forall\, v\in L^2(\Omega).$$

Thus

$$(u_\varepsilon(0),v)=(\tilde{u}_0,v);\quad \forall\, v\in L^2(\Omega)$$

and, then,

$$u_\varepsilon(0)=u_0\,.$$

(ii) $((k_2+\varepsilon)u'_\varepsilon)(0)=(k_2+\varepsilon)u_1$

Let $\theta\in C^1([0,T])$ such that $\theta(0)=1$ and $\theta(T)=0$. Consider $j\in\mathbb{N}$ and $\nu\geq j$. Multiplying $(\mathrm{AP})_2$ by θ and integrating in $[0,T]$ we obtain

$$\int_0^T ((k_2+\varepsilon)u''_{\varepsilon\nu}(t),w_j)\theta(t)dt+\int_0^T (k_1u'_{\varepsilon\nu}(t),w_j)\theta(t)dt+\int_0^T ((u_{\varepsilon\nu}(t),w_j))\theta(t)dt \quad (14.4)$$

$$=\int_0^T (f(t),w_j)\theta(t)dt. \quad (14.5)$$

Integrating by parts the first integral on the left of the equality above results that

$$-((k_2+\varepsilon)u'_{\varepsilon\nu}(0),w_j)-\int_0^T ((k_2+\varepsilon)u'_{\varepsilon\nu}(t),w_j)\theta'(t)dt+\int_0^T (k_1u'_{\varepsilon\nu}(t),w_j)\theta(t)dt \quad (14.6)$$

$$+\int_0^T ((u_{\varepsilon\nu}(t),w_j))\theta(t)dt=\int_0^T (f(t),w_j)\theta(t)dt. \quad (14.7)$$

Taking the limit when $\nu\to+\infty$ then from (5), (24) and (25) it follows that

$$\begin{aligned}-((k_2+\varepsilon)u_1,w_j)-\int_0^T ((k_2+\varepsilon)u'_{\varepsilon\nu}(t),w_j)\theta'(t)dt+\int_0^T (k_1u'_{\varepsilon\nu}(t),w_j)\theta(t)dt\\ +\int_0^T ((u_\varepsilon(t),w_j))\theta(t)dt=\int_0^T (f(t),w_j)\theta(t)dt.\end{aligned} \quad (35)$$

On the other hand, from (34) it follows that

$$\int_0^T \left\langle \frac{d}{dt}([k_2+\varepsilon]u'_\varepsilon(t),w_j\right\rangle\theta(t)dt+\int_0^T (k_1u'_\varepsilon(t),w_j)\theta(t)dt \quad (14.8)$$

$$+\int_0^T ((u_\varepsilon(t),w_j))\theta(t)dt=\int_0^T (f(t),w_j)\theta(t)dt. \quad (14.9)$$

Integrating by parts the first integral on the left of the equality follows that

$$
\begin{aligned}
-(([k_2+\varepsilon]u'_\varepsilon)(0), w_j) - \int_0^T ([k_2+\varepsilon]u'_\varepsilon(t), w_j)\theta'(t)dt + \int_0^T (k_1 u'_\varepsilon(t), w_j)\theta(t)dt \\
+ \int_0^T ((u_\varepsilon(t), w_j))\theta(t)dt = \int_0^T (f(t), w_j)\theta(t)dt.
\end{aligned} \tag{36}
$$

From (35) and (36) and from the fact that $\{w_\nu\}$ is total in $L^2(\Omega)$ it follows that

$$((k_2+\varepsilon)u_\varepsilon)'(0) = (k_2+\varepsilon)u_1\,.$$

In this way, for each $\varepsilon \in]0,1[$, $\quad \exists\, u_\varepsilon\colon Q \to \mathbb{R}$ in the class

$$u_\varepsilon \in L^\infty(0,T;H_0^1(\Omega)); \quad u'_\varepsilon \in L^2(0,T;L^2(\Omega)) \quad \text{and} \quad [k_2+\varepsilon]u''_\varepsilon \in L^2(0,T;H^{-1}(\Omega)) \tag{37}$$

weak solution of (P_2), that is,

$$
\left|
\begin{aligned}
&\frac{d}{dt}\,([k_2+\varepsilon]u'_\varepsilon) + k_1 u'_\varepsilon - \Delta u_\varepsilon = f \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega))\\
&u_\varepsilon = 0 \quad \text{on} \quad \Sigma\\
&u_\varepsilon(0) = u_0\,; \quad ((k_2+\varepsilon)u'_\varepsilon)(0) = (k_2+\varepsilon)u_1
\end{aligned}
\right. \tag{38}
$$

From (24) and (25) we have

$$||u_\varepsilon||_{L^\infty(0,T;H_0^1(\Omega))} \le \liminf_{\nu\to+\infty} ||u_{\varepsilon\nu}||_{L^\infty(0,T;H_0^1(\Omega))}.$$

and

$$||u'_\varepsilon||_{L^\infty(0,T;L^2(\Omega))} \le \liminf_{\nu\to+\infty} ||u'_{\varepsilon\nu}||_{L^\infty(0,T;L^2(\Omega))}$$

It follows from the inequalities above, from (22) and (23) that

$$
\begin{aligned}
&(u_\varepsilon) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega))\\
&(u'_\varepsilon) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)).
\end{aligned} \tag{39}
$$

Thus, there will exist a subnet of (u_ε) which we will still designate by (u_ε) such that

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{weak * in} \quad L^\infty(0,T;H_0^1(\Omega)) \tag{40}$$

$$u'_\varepsilon \overset{*}{\rightharpoonup} u' \quad \text{weak * in} \quad L^\infty(0,T;L^2(\Omega)). \tag{41}$$

Note also that due to the fact that $k_2 \in L^\infty(\Omega)$, given $v \in L^2(\Omega)$ and $\theta \in L^2(0,T)$, we have that

$$(k_2+\varepsilon)v\theta \to k_2 v\theta \quad \text{in} \quad L^2(0,T;L^2(\Omega)), \quad \text{when } \varepsilon \to 0. \tag{42}$$

Indeed,

$$\int_Q |(k_2+\varepsilon)v\theta - k_2 v\theta|^2\,dxdt = \int_Q (\varepsilon v\theta)^2\,dxdt = \varepsilon^2 \int_Q |v\theta|^2\,dxdt = c\varepsilon^2 \to 0, \text{ when } \varepsilon \to 0.$$

Thus, from (41) and (42) we conclude that

$$(u'_\varepsilon, (k_2+\varepsilon)v\theta)_{L^2(0,T;L^2(\Omega))} \overset{\varepsilon\to 0}{\longrightarrow} (u', k_2 v\theta)_{L^2(0,T;L^2(\Omega))}\,; \quad \forall\, v \in L^2(\Omega) \text{ and } \forall\, \theta \in L^2(0,T),$$

that is,

$$\int_0^T ((k_2+\varepsilon)u'_\varepsilon, v)\theta(t)dt \xrightarrow{\varepsilon\to 0} \int_0^T (k_2u', v)\theta(t)dt; \quad \forall\, v\in L^2(\Omega) \text{ and } \forall\,\theta\in L^2(0,T). \quad (43)$$

Consider, then, $v\in H_0^1(\Omega)$ and $\theta\in\mathcal{D}(0,T)$. Then $v\theta\in L^2(0,T;H_0^1(\Omega))$ and from (34) we can write

$$\int_0^T \left\langle \frac{d}{dt}\,[(k_2+\varepsilon)u'_\varepsilon](t), v\right\rangle\theta(t)dt + \int_0^T \langle k_1u'_\varepsilon, v\rangle\theta(t)dt + \int_0^T \langle -\Delta u_\varepsilon(t), v\rangle\theta(t)dt \quad (14.10)$$

$$= \int_0^T \langle f(t), v\rangle\theta(t)dt, \quad (14.11)$$

or even,

$$-\int_0^T ((k_2+\varepsilon)u'_\varepsilon(t), v)\theta'(t)dt + \int_0^T (k_1u'_\varepsilon(t), v)\theta(t)dt + \int_0^T ((u_\varepsilon(t), v))\theta(t)dt \quad (14.12)$$

$$= \int_0^T (f(t), v)\theta(t)dt. \quad (14.13)$$

From (40), (41) and (43), in the limit situation, we obtain

$$\begin{aligned} -\int_0^T (k_2u'(t), v)\theta'(t)dt + \int_0^T (k_1u'(t), v)\theta(t)dt + \int_0^T ((u(t), v))\theta(t)dt \\ = \int_0^T (f(t), v)\theta(t)dt, \end{aligned} \quad (44)$$

that is,

$$\frac{d}{dt}\,(k_2u'(t), v) + (k_1u'(t), v) + ((u(t), v)) = (f(t), v) \text{ in } \mathcal{D}'(0,T); \;\; \forall\, v\in H_0^1(\Omega). \quad (45)$$

Taking $v=\varphi\in\mathcal{D}(\Omega)$ in (44) results that

$$\frac{d}{dt}\,(k_2u') + k_1u' - \Delta u = f \quad \text{in} \quad \mathcal{D}'(Q).$$

But since $f\in L^2(0,T;L^2(\Omega))$, $\Delta u\in L^\infty(0,T;H^{-1}(\Omega))$ and $k_1u'\in L^\infty(0,T;L^2(\Omega))$ it follows that

$$\frac{d}{dt}\,(k_2u') \in L^2(0,T;H^{-1}(\Omega)) \quad (46)$$

and, therefore,

$$\frac{d}{dt}\,(k_2u') + k_1u' - \Delta u = f \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)). \quad (47)$$

But, by Remark 1, we have

$$\frac{d}{dt}\,(k_2u') = k_2u'',$$

whence from (46) and (47) we conclude that

$$k_2 u'' \in L^2(0,T;H^{-1}(\Omega)) \tag{48}$$

and

$$k_2 u'' + k_1 u' - \Delta u = f \quad \text{and} \quad L^2(0,T;H^{-1}(\Omega)). \tag{49}$$

Initial Conditions

Note that from (40), (41) and (46) we have

$$u \in C^0([0,T];L^2(\Omega)) \cap C_s([0,T];H_0^1(\Omega)) \tag{14.14}$$

$$k_2 u' \in C^0([0,T];H^{-1}(\Omega)) \cap C_s([0,T];L^2(\Omega)) \tag{14.15}$$

making sense therefore to speak of $u(0)$ and $(k_2u')(0) = (k_2u)'(0)$; $u(T)$ and $(k_2u')(T) = (k_2u)'(T)$.

(i) $u(0) = u_0$.

Let $\theta \in C^1([0,T])$ such that $\theta(0) = 1$ and $\theta(T) = 0$ and consider $v \in L^2(\Omega)$. Then, from (21) it follows that

$$\int_0^T (u_\varepsilon'(t), v)\theta(t)dt \xrightarrow{\varepsilon\to 0} \int_0^T (u'(t), v)\theta(t)dt.$$

Integrating by parts we have

$$-(u_\varepsilon(0), v) - \int_0^T (u_\varepsilon(t), v)\theta'(t)dt \xrightarrow{\varepsilon\to 0} -(u(0), v) - \int_0^T (u(t), v)\theta'(t)dt.$$

Since

$$\int_0^T (u_\varepsilon(t), v)\theta'(t)dt \xrightarrow{\varepsilon\to 0} \int_0^T (u(t), v)\theta'(t)dt$$

then:

$$(u_\varepsilon(0), v) \xrightarrow{\varepsilon\to 0} (u(0), v).$$

Since $u_\varepsilon(0) = u_0$; $\forall\, \varepsilon > 0$ we have that

$$(u_0, v) = (u(0), v); \quad \forall\, v \in L^2(\Omega),$$

that is,

$$u(0) = u_0.$$

(ii) $(k_2u')(0) = k_2u_1$.

Let $\theta \in C^1([0,T])$ such that $\theta(0) = 1$ and $\theta(T) = 0$ and consider $v \in H_0^1(\Omega)$. Then, $v\theta \in L^2(0,T;H_0^1(\Omega))$ and from (34) it results that

$$\int_0^T \left\langle \frac{d}{dt}([k_2+\varepsilon]u_\varepsilon'), v\right\rangle \theta(t)dt + \int_0^T (k_1u_\varepsilon', v)\theta(t)dt + \int_0^T ((u_\varepsilon(t), v))\theta(t)dt \tag{14.16}$$

$$= \int_0^T (f(t), v)\theta(t)dt. \tag{14.17}$$

Integrating by parts

$$-(([k_2+\varepsilon]u'_\varepsilon)(0),v) - \int_0^T ([k_2+\varepsilon]u'_\varepsilon, v)\theta'(t)dt + \int_0^T (k_1 u'_\varepsilon, v)\theta(t)dt \tag{14.18}$$

$$+\int_0^T ((u_\varepsilon(t),v))\theta(t)dt = \int_0^T (f(t),v)\theta(t)dt. \tag{14.19}$$

Since $([k_2+\varepsilon]u'_\varepsilon)(0) = [k_2+\varepsilon]u_1$, taking the limit in the equality above we obtain from (40), (41) and (43) that

$$\begin{aligned} -(k_2u_1,v) - \int_0^T (k_2u'(t),v)\theta'(t)dt + \int_0^T (k_1u'(t),v)\theta(t)dt \\ + \int_0^T ((u(t),v))\theta(t)dt = \int_0^T (f(t),v)\theta(t)dt. \end{aligned} \tag{50}$$

On the other hand, from (49) we can write

$$\int_0^T \left\langle \frac{d}{dt}(k_2u'), v\right\rangle \theta(t)dt \ + \int_0^T (k_1u'(t),v)\theta(t)dt + \int_0^T ((u(t),v))\theta(t)dt \tag{14.20}$$

$$= \int_0^T (f(t),v)\theta(t)dt. \tag{14.21}$$

Integrating by parts

$$\begin{aligned} -((k_2u')(0),v) - \int_0^T (k_2u'(t),v)\theta'(t)dt + \int_0^T (k_1u'(t),v)\theta(t)dt \\ + \int_0^T ((u(t),v))\theta(t)dt = \int_0^T (f(t),v)\theta(t)dt. \end{aligned} \tag{51}$$

From (50) and (51) it follows that

$$(k_2u')(0) = k_2u_1\,.$$

Uniqueness

Let u and v be weak solutions of (2) in the class

$$u,v \in L^\infty(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega)) \tag{14.22}$$

$$u',v' \in L^2(0,T;L^2(\Omega)) \tag{14.23}$$

$$k_2u'',k_2v'' \in L^2(0,T;H^{-1}(\Omega)). \tag{14.24}$$

Then, $w=u-v$ satisfies

$$\left| \begin{array}{l} k_2w''+k_1w'-\Delta w = 0 \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)) \\ w=0 \quad \text{on} \quad \Sigma \\ w(0)=0, \ \ (k_2w')(0)=0. \end{array} \right. \tag{52}$$

Let $s \in [0, T]$ be arbitrary, but fixed, and define

$$z(t) = \begin{cases} -\int_t^s w(\xi)d\xi; & 0 \leq t \leq s \\ 0; & s < t \leq T. \end{cases} \tag{53}$$

We have, for all $t \in [0, T]$,

$$||z(t)|| \leq \int_t^s ||w(\xi)||d\xi \leq \int_0^T ||w(\xi)||d\xi \leq T \cdot \underset{t\in[0,T]}{\text{supess}}||w(t)|| < +\infty.$$

Thus,

$$z \in L^\infty(0, T; H_0^1(\Omega)). \tag{54}$$

We claim that

$$z'(t) = \begin{cases} w(t); & 0 \leq t \leq s \\ 0; & s < t \leq T. \end{cases}$$

Indeed, let $\varphi \in \mathcal{D}(0, T)$. Then

$$\langle z', \varphi \rangle = -\langle z, \varphi' \rangle = -\int_0^T z(t)\varphi'(t)dt = -\int_0^s z(t)\varphi'(t)dt. \tag{55}$$

However, for $t \in [0, s]$,

$$z(t) = -\int_t^s w(\xi)d\xi. \tag{56}$$

Whence

$$z'(t) = w(t), \quad \text{a.e. in} \quad t \in [0, s] \tag{57}$$

where z' is the classical (Dini) derivative.

Let $t_1, t_2 \in [0, s]$ such that $t_1 < t_2$, without loss of generality. From (56) it follows that

$$||z(t_1) - z(t_2)|| \tag{14.25}$$

$$= \left\| \int_{t_1}^s w(\xi)d\xi - \int_{t_2}^s w(\xi)d\xi \right\| = \left\| \int_{t_1}^{t_2} w(\xi)d\xi + \int_{t_2}^s w(\xi)d\xi - \int_{t_2}^s w(\xi)d\xi \right\| \tag{14.26}$$

$$= \left\| \int_{t_1}^{t_2} w(\xi)d\xi \right\| \leq \int_{t_1}^{t_2} ||w(\xi)||d\xi \leq ||w||_{L^\infty(0,T;H_0^1(\Omega))}|t_2 - t_1| \tag{14.27}$$

that is, z is Lipschitz and, therefore, absolutely continuous in $[0, s]$.

But,

$$(z\varphi)' = z'\varphi + z\varphi' \quad \text{a.e. in} \quad [0, s].$$

Integrating in $[0, s]$ results that

$$\int_0^s (z\varphi)'dt = \int_0^s z'\varphi dt + \int_0^s z\varphi' dt.$$

Since $z\varphi$ is absolutely continuous and $H_0^1(\Omega)$ is reflexive, we obtain

$$(z\varphi)(s) - (z\varphi)(0) = \int_0^s z'\varphi\, dt + \int_0^s z\varphi'\, dt.$$

Since $z(s) = 0$ and $\varphi(0) = 0$ then

$$-\int_0^s z(t)\varphi'(t)\,dt = \int_0^s z'(t)\varphi(t)\,dt. \tag{58}$$

From (55), (57) and (58) we conclude that

$$\langle z', \varphi\rangle = \int_0^s z'(t)\varphi(t)dt = \int_0^s w(t)\varphi(t)dt.$$

Let us define

$$v(t) = \begin{cases} w(t); & t \in [0, s] \\ 0; & t \in [s, T]; \end{cases} \tag{59}$$

then:

$$\int_0^s w(t)\varphi(t)dt = \int_0^T v(t)\varphi(t)dt$$

and, therefore,

$$\langle z', \varphi\rangle = \int_0^T v(t)\varphi(t)dt = \langle v, \varphi\rangle; \quad \forall\,\varphi \in \mathcal{D}(0, T),$$

that is, $z' = v$, or even,

$$z'(t) = \begin{cases} w(t); & t \in [0, s] \\ 0; & t \in [s, T], \end{cases} \tag{60}$$

as we wanted to demonstrate. It results from (40) that for all $t \in [0, T]$

$$||z'(t)|| \leq ||w(t)|| \leq_{\underset{0\leq t\leq T}{\rightarrow}} \text{supess}||w(t)|| = ||w(t)|| < +\infty,$$

that is,

$$z' \in L^\infty(0, T; H_0^1(\Omega)). \tag{61}$$

Thus, from (54) and (61) we have that

$$z \in C^0([0, T]; H_0^1(\Omega)). \tag{62}$$

Composing $(52)_1$ with z in the duality $L^2(0, T; H^{-1}(\Omega)) \times L^2(0, T; H_0^1(\Omega))$ we obtain

$$\int_0^T \langle (k_2 w'')(t), z(t)\rangle dt + \int_0^T (k_1 w'(t), z(t))dt + \int_0^T ((w(t), z(t)))dt = 0.$$

Due to the fact that $z = 0$ in $[s, T]$, we rewrite the expression above as

$$\int_0^s \langle (k_2 w'')(t), z(t)\rangle dt + \int_0^s (k_1 w'(t), z(t))dt + \int_0^s ((w(t), z(t)))dt = 0. \tag{63}$$

Next, we will make some evaluations of the integrals in (63). We have by Remark 1,

$$\frac{d}{dt}\langle k_2 w'(t), z(t)\rangle = \langle (k_2 w'')(t), z(t)\rangle + \langle k_2 w'(t), z'(t)\rangle,$$

or even,

$$\frac{d}{dt}(k_2 w'(t), z(t)) = \langle (k_2 w'')(t), z(t)\rangle + (k_2 w'(t), z'(t)).$$

Integrating in $[0,s]$ it comes that

$$\int_0^s \frac{d}{dt}(k_2 w'(t), z(t))dt = \int_0^s \langle (k_2 w'')(t), z(t)\rangle dt + \int_0^s (k_2 w'(t), z'(t))dt. \tag{64}$$

Since $(k_2 w'(t), z(t)) \in H^1(0,T)$ we have that $(k_2 w'(t), z(t)) \in C^0([0,T])$ and $\frac{d}{dt}(k_2 w'(t), z(t)) \in L^2(0,T)$. Thus, $(k_2 w'(t), z(t))$ is absolutely continuous and, therefore,

$$\int_0^s \frac{d}{dt}(k_2 w'(t), z(t))dt = (k_2 w'(s), z(s)) - ((k_2 w')(0), z(0)) = 0 \tag{65}$$

since $z(s) = 0$ and $(k_2 w')(0) = 0$.

Substituting (65) in (64) and observing that $z'(t) = w(t)$ in $[0,s]$, we obtain

$$\int_0^s \langle (k_2 w'')(t), z(t)\rangle dt = -\int_0^s (k_2 w'(t), z'(t))dt = -\int_0^s (k_2 w'(t), w(t))dt \tag{14.28}$$

$$= -\int_0^s (\sqrt{k_2}\, w'(t), \sqrt{k_2}\, w(t))dt = -\int_0^s \frac{1}{2}\frac{d}{dt}|\sqrt{k_2} w(t)|^2 dt \tag{14.29}$$

$$= -\frac{1}{2}\{|\sqrt{k_2} w(s)|^2 - |\sqrt{k_2} w(0)|^2\} = -\frac{1}{2}|\sqrt{k_2} w(s)|^2, \tag{14.30}$$

since $w(0) = 0$. Thus,

$$\int_0^s \langle (k_2 w'')(t), z(t)\rangle dt = -\frac{1}{2}|\sqrt{k_2} w(s)|^2. \tag{66}$$

We have

$$\frac{d}{dt}(k_1 w(t), z(t)) = (k_1 w'(t), z(t)) + (k_1 w(t), z'(t)).$$

Integrating in $[0,s]$ it comes that

$$(k_1 w(s), z(s)) - (k_1 w(0), z(0)) = \int_0^s (k_1 w'(t), z(t))dt + \int_0^s (k_1 w(t), z'(t))dt.$$

But, since $z(s) = 0, \quad w(0) = 0$ and $z'(t) = w(t)$ in $[0,s]$; we obtain

$$\int_0^s (k_1 w'(t), z(t))dt = -\int_0^s (k_1 w(t), w(t))dt = -\int_0^s (\sqrt{k_1}\, w(t), \sqrt{k_1}\, w(t))dt \tag{14.31}$$

$$= -\int_0^s |\sqrt{k_1}\, w(t)|^2 dt \tag{14.32}$$

that is,

$$\int_0^s (k_1 w'(t), z(t))dt = -\int_0^s |\sqrt{k_1}\, w(t)|^2\, dt. \tag{67}$$

Finally, since $z'(t) = w(t)$ a.e. in $[0,s]$ we have

$$\int_0^s ((w(t), z(t)))dt = \int_0^s ((z'(t), z(t)))dt = \frac{1}{2}\int_0^s \frac{d}{dt}||z(t)||^2\, dt \tag{14.33}$$

$$= \frac{1}{2}\left[||z(s)||^2 - ||z(0)||^2\right] = -\frac{1}{2}||z(0)||^2 \tag{14.34}$$

since $z(s) = 0$, that is,

$$\int_0^s ((w(t), z(t)))\, dt = -\frac{1}{2}\, ||z(0)||^2. \tag{68}$$

In this way, from (63), (66), (67) and (68) it follows that

$$\frac{1}{2}\, |\sqrt{k_2}\, w(s)|^2 + \int_0^s |\sqrt{k_1}\, w(t)|^2\, dt + \frac{1}{2}\, ||z(0)||^2 = 0.$$

Therefore

$$\int_0^s |\sqrt{k_1}\, w(t)|^2\, dt = 0$$

and, due to the fact that $\sqrt{k_1} \geq \sqrt{\beta} > 0$ a.e. in Ω, it follows that

$$\int_0^s |w(t)|^2\, dt = 0.$$

Consequently

$$w(t) = 0 \quad \text{a.e. in} \quad [0, s].$$

By the arbitrariness of $s \in [0, T]$ we conclude that

$$w(t) = 0 \quad \text{a.e. in} \quad [0, T],$$

that is, $w = 0$, which proves the desired result. □

14.5 Problems in Non-Cylindrical Domains

Consider the linear problem

$$\left|\begin{array}{l}\dfrac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(0) = u_0(x); \ \dfrac{\partial u}{\partial t}(0) = u_1(x) \quad \text{in} \quad \Omega_0,\end{array}\right. \tag{1}$$

where Q is a non-cylindrical domain, which we will need below. Σ will denote the lateral boundary of Q and Ω_0 is the "base" of Q, as illustrated in the figure below.

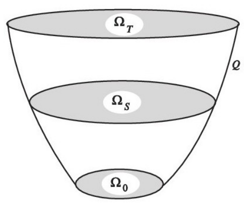

Figure 14.2: Figure 2

- **About the Domain Q**

  Let Q be a bounded connected open set of $\mathbb{R}^n_x \times \mathbb{R}^+_t$. Consider, for each $s \in]0, T[$:

$$\Omega_s = \{t = s\} \cap Q$$

and let Ω_0 and Ω_T, respectively, be the "open ends", corresponding to $t = 0$ and $t = T$. Let, also,

$$\Gamma_s = \partial\Omega_s; \quad 0 \leq s \leq T$$

and

$$\Sigma = \bigcup_{s \in]0,T[} \Gamma_s$$

be the lateral boundary of Q, so that

$$\partial Q = \overline{\Omega}_0 \cup \Sigma \cup \overline{\Omega}_T.$$

Let $\mathcal{O}$ be a bounded open set of $\mathbb{R}^n_x$, with regular boundary, such that

$$Q \subset \mathcal{O} \times]0, T[\,.$$

We will denote by Ω_t^* the projection of Ω_t onto the hyperplane $t = 0$. Note that $L^2(\Omega_t^*)$ (respectively $H_0^1(\Omega_t^*)$) is a *closed subspace* of $L^2(\mathcal{O})$ (respectively $H_0^1(\mathcal{O})$). In this way, we can identify $L^2(\Omega_t)$ (respectively $H_0^1(\Omega_t)$) with a closed subspace of $L^2(\mathcal{O})$ (respectively $H_0^1(\mathcal{O})$).

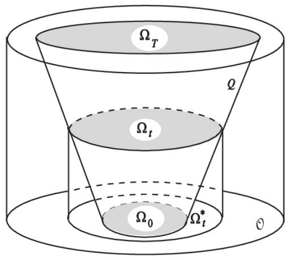

Figure 14.3: Figure 3

We define, for $1 \leq p \leq +\infty$

$$L^p(0,T;L^2(\Omega_t)) = \{u \in L^p(0,T;L^2(\mathcal{O})); u(t) \in L^2(\Omega_t) \text{ a.e. in }]0,T[\}$$
$$L^p(0,T;H_0^1(\Omega_t)) = \{u \in L^p(0,T;H_0^1(\mathcal{O})); u(t) \in H_0^1(\Omega_t) \text{ a.e. in }]0,T[\}$$

We make the following hypotheses on Q

| Ω_t increases with t, that is, if Ω_t^* is the projection of Ω_t onto the hyperplane $t = 0$, then, $\Omega_t^* \subset \Omega_{t'}^*$, if $t \leq t'$ (see fig. above) (2)

| $\forall\, t \in]0,T[$ if $v \in H_0^1(\mathcal{O})$ and $v = 0$ a.e. in $\mathcal{O}\backslash\Omega_t^*$ then $v \in H_0^1(\Omega_t)$ (3)

It follows from (3) that if $v \in H_0^1(\mathcal{O})$ and $v = 0$ a.e. in $\mathcal{O}\backslash\Omega_{t_0}^*$ then

$$v \in H_0^1(\Omega_t); \quad \forall\, t \geq t_0\,.$$

Indeed, let $t \geq t_0$. Then, by property (2), $\Omega_{t_0}^* \subset \Omega_t^*$ and, therefore, $\mathcal{O}\backslash\Omega_t^* \subset \mathcal{O}\backslash\Omega_{t_0}^*$. Thus, if $v = 0$ a.e. in $\mathcal{O}\backslash\Omega_{t_0}^*$ it follows that $v = 0$ a.e. in $\mathcal{O}\backslash\Omega_t^*$. Thus, $v \in H_0^1(\mathcal{O})$ and $v = 0$ a.e. in $\mathcal{O}\backslash\Omega_t^*$. Whence, by property (3) it follows that $v \in H_0^1(\Omega_t)$.

We have the following result

Theorem: Given

$$u_0 \in H_0^1(\Omega_0),\ u_1 \in L^2(\Omega_0) \quad \text{and} \quad f \in L^2(Q)$$

there exists $u\colon \mathcal{O}\times]0,T[\to \mathbb{R}$ such that

$$u \in L^\infty(0,T;H_0^1(\Omega_t)); \quad u' \in L^\infty(0,T;L^2(\Omega_t))$$

weak solution of problem 1.

Proof: Let

$$M(x,t) = \begin{cases} 0 & \text{in} \quad Q \\ 1 & \text{in} \quad Q^c = \mathcal{O}\times]0,T[\setminus Q \end{cases}$$

Then, $M \in L^\infty(\mathcal{O}\times]0,T[)$. Consider $\tilde{u}_0$, $\tilde{u}_1$ extensions of u_0 and u_1 zero outside Ω_0 and $\tilde{f}$ extension of f zero outside Q. For each $\varepsilon > 0$, consider the cylindrical problem

$$\left| \begin{aligned} & u_\varepsilon'' - \Delta u_\varepsilon + \frac{1}{\varepsilon} M\, u_\varepsilon' = \tilde{f} \quad \text{in} \quad \mathcal{O}\times]0,T[\\ & u_\varepsilon = 0 \quad \text{on} \quad \Sigma' \text{ (lateral boundary of } \mathcal{O}\times]0,T[) \\ & u_\varepsilon(0) = \tilde{u}_0\,; \quad u_\varepsilon'(0) = \tilde{u}_1 \end{aligned} \right. \tag{P$_\varepsilon$}$$

- **Resolution of $\mathbf{P}_\varepsilon$**

Fix $\varepsilon > 0$ and let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H_0^1(\mathcal{O})$. Set

$$V_m = [w_1, \ldots, w_m]$$

and consider in V_m the approximate problem

$$\text{(AP)} \begin{cases} u_{\varepsilon m}(t) \in V_m \Leftrightarrow u_{\varepsilon m}(t) = \displaystyle\sum_{i=1}^m g_{\varepsilon m i}(t) w_i \\ (u_{\varepsilon m}''(t), w_j) + a(u_{\varepsilon m}(t), w_j) + \dfrac{1}{\varepsilon}\displaystyle\int_{\mathcal{O}} M(x,t) u_{\varepsilon m}'(t) w_j\, dx = (\tilde{f}(t), w_j) \\ \qquad\qquad j = 1, \ldots, m \\ u_{\varepsilon m}(0) = u_{0m} \to \tilde{u}_0 \quad \text{in} \quad H_0^1(\mathcal{O}) \\ u_{\varepsilon m}'(0) = u_{1m} \to \tilde{u}_1 \quad \text{in} \quad L^2(\mathcal{O}), \end{cases}$$

which possesses a local solution $u_{\varepsilon m}(t)$ in some interval $[0, t_{\varepsilon m}[$, where $u_{\varepsilon m}$ and $u_{\varepsilon m}'$ are absolutely continuous and $u_{\varepsilon m}''$ exists a.e., by Carathéodory's Theorem. The a priori estimates will serve to extend $u_{\varepsilon m}(t)$ to the whole interval $[0,T]$.

- **A Priori Estimate**

Multiplying the approximate equation by $g_{\varepsilon m j}'(t)$ and summing over j results that

$$\begin{aligned} \frac{d}{dt}\left\{ \frac{1}{2}\, |u_{\varepsilon m}'(t)|^2 + \frac{1}{2}\, ||u_{\varepsilon m}(t)||^2 \right\} + \frac{1}{\varepsilon}\int_{\mathcal{O}} M(x,t)\big(u_{\varepsilon m}'(t)\big)^2\, dx &= (\tilde{f}(t), u_{\varepsilon m}'(t)) \\ &\le \frac{1}{2}\, |\tilde{f}(t)|^2 + \frac{1}{2}\, |u_{\varepsilon m}'(t)|^2. \end{aligned}$$

Integrating in $]0,T[$, with $0 < t < t_{\varepsilon m}$, it follows that

$$\begin{aligned} & \frac{1}{2}\, |u_{\varepsilon m}'(t)|^2 + \frac{1}{2}\, ||u_{\varepsilon m}(t)||^2 + \frac{1}{\varepsilon}\int_0^t \int_{\mathcal{O}} M(u_{\varepsilon m}'(s))^2\, dx ds \\ \le\; & \frac{1}{2}\, |u_{1m}|^2 + \frac{1}{2}\, ||u_{0m}||^2 + \frac{1}{2}\, ||\tilde{f}||_{L^2(0,T;L^2(\mathcal{O}))} + \frac{1}{2}\int_0^t |u_{\varepsilon m}'(s)|^2\, ds. \end{aligned}$$

However from (AP) we obtain $c_1 > 0$ such that

$$\frac{1}{2}\, |u_{1m}|^2 + \frac{1}{2}\, ||u_{0m}||^2 + \frac{1}{2}\, ||\tilde{f}||_{L^2((0,T)\times\mathcal{O})} \le \frac{c_1}{2}\,.$$

Thus

$$\frac{1}{2}\,|u'_{\varepsilon m}(t)|^2 + \frac{1}{2}\,||u_{\varepsilon m}(t)||^2 + \frac{1}{\varepsilon}\int_0^t \int_{\mathcal{O}} M(u'_{\varepsilon m})^2\,dxds$$
$$\leq \frac{c_1}{2} + \frac{1}{2}\int_0^t \left\{ |u'_{\varepsilon m}(s)|^2 + ||u_{\varepsilon m}(s)||^2 + \frac{1}{\varepsilon}\int_0^s \int_{\mathcal{O}} M(u'_{\varepsilon m})^2\,dxd\sigma \right\} ds.$$

By Gronwall's Inequality it follows that

$$\frac{1}{2}\,|u'_{\varepsilon m}(t)|^2 + \frac{1}{2}\,||u_{\varepsilon m}(t)||^2 + \frac{1}{\varepsilon}\int_0^t \int_{\mathcal{O}} M(x,s)(u'_{\varepsilon m}(s))^2\,dxds \leq k, \tag{4}$$

$\forall\, t \in [0, t_{\varepsilon m})$; $\forall\, \varepsilon > 0$; $\forall\, m \in \mathbb{N}$ where k is independent of t, ε and m. It results that we can extend $u_{\varepsilon m}(t)$ to the whole interval $[0, T]$ and proceeding as before we obtain the same inequality obtained in (4), being now valid for all $t \in [0, T], \varepsilon > 0$ and $m \in \mathbb{N}$.

- **Passage to the Limit**

We obtain, therefore, from (4) the existence of a subsequence $(u_{\varepsilon\nu}) \subset (u_{\varepsilon m})$ such that

$$u_{\varepsilon\nu} \overset{*}{\rightharpoonup} u_\varepsilon \quad \text{weak-* in} \quad L^\infty(0, T; H_0^1(\mathcal{O})) \tag{5}$$
$$u'_{\varepsilon\nu} \overset{*}{\rightharpoonup} u'_\varepsilon \quad \text{weak-* in} \quad L^\infty(0, T; L^2(\mathcal{O})) \tag{6}$$

Let $j \in \mathbb{N}$ and consider $\nu \geq j$. Then from (AP) it follows that

$$(u''_{\varepsilon\nu}(t), w_j) + a(u_{\varepsilon\nu}(t), w_j) + \frac{1}{\varepsilon}\int_{\mathcal{O}} M(x,t)u'_{\varepsilon\nu}(t)w_j\,dx = (\tilde{f}(t), w_j).$$

Multiplying the equation above by $\theta \in \mathcal{D}(0, T)$ and integrating from 0 to T we have

$$\int_0^T (u''_{\varepsilon\nu}(t), w_j)\theta(t)dt + \int_0^T a(u_{\varepsilon\nu}(t), w_j)\theta(t)dt + \frac{1}{\varepsilon}\int_0^T \int_{\mathcal{O}} M(x,t)u'_{\varepsilon\nu}(t)w_j\,dx\,\theta(t)dt$$
$$= \int_0^T (\tilde{f}(t), w_j)\theta(t)dt.$$

Whence

$$-\int_0^T (u'_{\varepsilon\nu}(t), w_j)\theta'(t)dt + \int_0^T a(u_{\varepsilon\nu}(t), w_j)\theta(t)dt + \frac{1}{\varepsilon}\int_0^T \int_{\mathcal{O}} M(x,t)u'_{\varepsilon\nu}(t)w_j\,dx\,\theta(t)dt$$
$$= \int_0^T (\tilde{f}(t), w_j)\theta(t)dt.$$

Taking the limit in the equality above as $\nu \to +\infty$ we obtain from (5) and (6) that

$$-\int_0^T (u'_\varepsilon(t), w_j)\theta'(t)dt + \int_0^T a(u_\varepsilon(t), w_j)\theta(t)dt + \frac{1}{\varepsilon}\int_0^T \int_{\mathcal{O}} M(x,t)u'_\varepsilon(t)w_j\,dx\,\theta(t)dt$$
$$= \int_0^T (\tilde{f}(t), w_j)\theta(t)dt. \tag{7}$$

Since j was taken arbitrarily in $\mathbb{N}$, we conclude that (7) is valid for any $j \in \mathbb{N}$. From the totality of $(w_j)_{j\in\mathbb{N}}$ in $H_0^1(\mathcal{O})$ it follows that the equality above remains valid for all $v \in H_0^1(\mathcal{O})$, that is,

$$\begin{aligned} &-\int_0^T (u_\varepsilon'(t), v)\theta'(t)dt + \int_0^T a(u_\varepsilon(t), v)\theta(t)dt + \\ &+\frac{1}{\varepsilon}\int_0^T \int_{\mathcal{O}} M(x,t)u_\varepsilon'(t)v(x)dx\,\theta(t)dt = \int_0^T (\tilde{f}(t), v)\theta(t)dt; \end{aligned} \tag{8}$$

$\forall\, v \in H_0^1(\mathcal{O})$ and $\forall\, \theta \in \mathcal{D}(0,T)$.

Whence

$$\frac{d}{dt}(u_\varepsilon'(t), v) + a(u_\varepsilon(t), v) + \frac{1}{\varepsilon}(Mu_\varepsilon'(t), v) = (\tilde{f}(t), v) \tag{9}$$

in $\mathcal{D}'(0,T)$; $\quad\forall\, v \in H_0^1(\mathcal{O})$.

Resuming (8) with $v = \varphi \in \mathcal{D}(\mathcal{O})$ we also obtain

$$\langle u_\varepsilon'', \theta\varphi\rangle + \langle -\Delta u_\varepsilon, \theta\varphi\rangle + \left\langle \frac{1}{\varepsilon} Mu_\varepsilon', \theta\varphi\right\rangle = \langle \tilde{f}, \theta\varphi\rangle; \tag{10}$$

$\forall\, \theta \in \mathcal{D}(0,T)$ and $\forall\, \varphi \in \mathcal{D}(\mathcal{O})$. Since the set $\{\varphi\theta; \varphi \in \mathcal{D}(\mathcal{O})$ and $\theta \in \mathcal{D}(0,T)\}$ is total in $\mathcal{D}(\mathcal{O}\times]0,T[)$ it follows that the equality in (10) is valid for all $\psi \in \mathcal{D}(\mathcal{O}\times]0,T[)$ and, therefore,

$$u_\varepsilon'' - \Delta u_\varepsilon + \frac{1}{\varepsilon} Mu_\varepsilon' = \tilde{f} \quad \text{in} \quad \mathcal{D}'(\mathcal{O}\times]0,T[). \tag{11}$$

Since $\Delta u_\varepsilon \in L^\infty(0,T;H^{-1}(\mathcal{O}))$, $\frac{1}{\varepsilon} Mu_\varepsilon' \in L^\infty(0,T;L^2(\mathcal{O}))$ and $\tilde{f} \in L^2(0,T;L^2(\mathcal{O}))$ it follows that

$$u_\varepsilon'' \in L^2(0,T;H^{-1}(\mathcal{O})), \tag{12}$$

that is,

$$u_\varepsilon'' - \Delta u_\varepsilon + \frac{1}{\varepsilon} Mu_\varepsilon' = \tilde{f} \quad \text{in} \quad L^2(0,T;H^{-1}(\mathcal{O})). \tag{13}$$

- **Initial Conditions**

Note, initially, that due to the fact that

$$u_\varepsilon \in L^\infty(0,T;H_0^1(\mathcal{O})), \quad u_\varepsilon' \in L^\infty(0,T;L^2(\mathcal{O})) \quad \text{and} \quad u_\varepsilon'' \in L^2(0,T;H^{-1}(\mathcal{O}))$$

then

$$\begin{aligned} u_\varepsilon &\in C^0([0,T];L^2(\mathcal{O})) \cap C_s(0,T;H_0^1(\mathcal{O})), \\ u_\varepsilon' &\in C^0([0,T];H^{-1}(\mathcal{O})) \cap C_s(0,T;L^2(\mathcal{O})), \end{aligned}$$

making sense therefore to speak of $u_\varepsilon(0)$ and $u_\varepsilon'(0)$. We will prove, next, that

$$u_\varepsilon(0) = \tilde{u}_0 \tag{i}$$
$$u_\varepsilon'(0) = \tilde{u}_1 \tag{ii}$$

Proof of (i)

Let $\theta \in C^1([0,T])$ such that $\theta(T) = 0$ and $\theta(0) = 1$. From (6), in particular, for $w = v\theta$, with $v \in L^2(\mathcal{O})$ it follows that

$$\left\langle u'_{\varepsilon\nu}, w \right\rangle_{L^\infty(0,T;L^2(\mathcal{O})),L^1(0,T;L^2(\mathcal{O}))} \xrightarrow{\nu\to+\infty} \left\langle u'_{\varepsilon}, w \right\rangle_{L^\infty(0,T;L^2(\mathcal{O})),L^1(0,T;L^2(\mathcal{O}))}$$

that is,

$$\int_0^T (u'_{\varepsilon\nu}(t), v)\theta(t)dt \xrightarrow{\nu\to+\infty} \int_0^T (u'_{\varepsilon}(t), v)\theta(t)dt; \quad \forall\, v \in L^2(\mathcal{O}).$$

Integrating by parts

$$-(u_{\varepsilon\nu}(0), v) - \int_0^T (u_{\varepsilon\nu}(t), v)\theta'(t)dt \xrightarrow{\nu\to+\infty} -(u_{\varepsilon}(0), v) - \int_0^T (u_{\varepsilon}(t), v)\theta'(t)dt. \tag{14}$$

From (5) we have that

$$\int_0^T (u_{\varepsilon\nu}(t), v)\theta'(t)dt \xrightarrow{\nu\to+\infty} \int_0^T (u_{\varepsilon}(t), v)\theta'(t)dt$$

and, therefore, from (14) it follows that

$$(u_{\varepsilon\nu}(0), v) \to (u_{\varepsilon}(0), v); \quad \forall\, v \in L^2(\mathcal{O}).$$

But, we also have that

$$u_{\varepsilon\nu}(0) = u_{0\nu} \to \tilde{u}_0 \quad \text{in} \quad H_0^1(\mathcal{O}).$$

Whence

$$(u_{\varepsilon\nu}(0), v) \xrightarrow{\nu\to+\infty} (\tilde{u}_0, v); \quad \forall\, v \in L^2(\mathcal{O}).$$

Thus

$$(u_{\varepsilon}(0), v) = (\tilde{u}_0, v); \quad \forall\, v \in L^2(\mathcal{O})$$

and, then,

$$u_{\varepsilon}(0) = \tilde{u}_0.$$

Proof of (ii)

Consider, analogously to item (i), $\theta \in C^1([0,T])$, $\theta(0) = 1$ and $\theta(T) = 0$. Let $j \in \mathbb{N}$ and $\nu \geq j$. Multiplying the approximate equation by θ and integrating in $[0,T]$, we obtain

$$\begin{aligned}\int_0^T (u''_{\varepsilon\nu}(t), w_j)\theta(t)\,dt &+ \int_0^T a(u_{\varepsilon\nu}(t), w_j)\theta(t)dt \\ + \frac{1}{\varepsilon}\int_0^T \int_{\mathcal{O}} M(x,t)u'_{\varepsilon\nu}(t)w_j\,dx\,\theta(t)dt &= \int_0^T (\tilde{f}(t), w_j)\theta(t)dt;\end{aligned}$$

Integrating the 1$^{\text{st}}$ term of the expression above by parts, it follows that

$$\begin{aligned}-(u'_{\varepsilon\nu}(0), w_j) - \int_0^T (u'_{\varepsilon\nu}(t), w_j)\theta'(t)dt &+ \int_0^T a(u_{\varepsilon\nu}(t), w_j)\theta(t)dt \\ + \frac{1}{\varepsilon}\int_0^T \int_{\mathcal{O}} M(x,t)u'_{\varepsilon\nu}(t)w_j\,dx\,\theta(t)dt &= \int_0^T (\tilde{f}(t), w_j)\theta(t)dt.\end{aligned}$$

Taking the limit in the equality above when $\nu \to +\infty$, it follows from (5) and (6) that

$$\begin{aligned} &-(\tilde{u}_1, w_j) - \int_0^T (u'_\varepsilon(t), w_j)\theta'(t)dt + \int_0^T a(u_\varepsilon(t), w_j)\theta(t)dt \\ &+ \frac{1}{\varepsilon}\int_0^T \int_{\mathcal{O}} M(x,t)u'_\varepsilon(t)w_j\, dx\, \theta(t)dt = \int_0^T (\tilde{f}(t), w_j)\theta(t)dt. \end{aligned} \tag{15}$$

On the other hand, from (13) and the fact that $(w_j\theta) \in L^2(0,T;H_0^1(\mathcal{O}))$ it follows that

$$\left\langle u''_\varepsilon - \Delta u_\varepsilon + \frac{1}{\varepsilon} M u'_\varepsilon, w_j\theta \right\rangle = \langle \tilde{f}, w_j\theta\rangle,$$

or even,

$$\begin{aligned} \int_0^T \langle u''_\varepsilon(t), w_j\rangle\theta(t)dt + \int_0^T \langle -\Delta u_\varepsilon(t), w_j\rangle\theta(t)dt + \frac{1}{\varepsilon}\int_0^T \langle M u'_\varepsilon(t), w_j\rangle\theta(t)dt \\ = \int_0^T \langle \tilde{f}(t), w_j\rangle\theta(t)dt. \end{aligned}$$

Integrating by parts the 1st term of the equality above and recalling that $\langle -\Delta u_\varepsilon(t), w_j\rangle = a(u_\varepsilon(t), w_j)$, we obtain

$$\begin{aligned} &-(u'_\varepsilon(0), w_j) - \int_0^T (u'_\varepsilon(t), w_j)\theta'(t)dt + \int_0^T a(u_\varepsilon(t), w_j)\theta(t)dt \\ &+ \frac{1}{\varepsilon}\int_0^T \int_{\mathcal{O}} M(x,t)u'_\varepsilon(t)w_j\, dx\, \theta(t)dt = \int_0^T (\tilde{f}(t), w_j)\theta(t)dt. \end{aligned} \tag{16}$$

From (15) and (16) we conclude that

$$(\tilde{u}_1, w_j) = (u'_\varepsilon(0), w_j); \quad \forall\, j \in \mathbb{N}.$$

By the "totality" of $\{w_j\}_{j\in\mathbb{N}}$ in $L^2(\mathcal{O})$ it follows that

$$u'_\varepsilon(0) = \tilde{u}_1\,.$$

Thus, for each $\varepsilon > 0$, $\exists\, u_\varepsilon \colon \mathcal{O}\times]0,T[\to \mathbb{R}$ in the class

$$u_\varepsilon \in L^\infty(0,T;H_0^1(\mathcal{O})); \quad u'_\varepsilon \in L^\infty(0,T;L^2(\mathcal{O})) \quad \text{and} \quad u''_\varepsilon \in L^2(0,T;H^{-1}(\mathcal{O})) \tag{17}$$

solution of (P$_\varepsilon$), that is,

$$\left| \begin{aligned} &u''_\varepsilon - \Delta u_\varepsilon + \frac{1}{\varepsilon} M u'_\varepsilon = \tilde{f} \quad \text{in} \quad L^2(0,T;H^{-1}(\mathcal{O})) \\ &u_\varepsilon = 0 \quad \text{on} \quad \Sigma' \\ &u_\varepsilon(0) = \tilde{u}_0\,; \quad u'_\varepsilon(0) = \tilde{u}_1. \end{aligned} \right. \tag{18}$$

• **Uniqueness**

Fix $\varepsilon > 0$ and let $u = u_\varepsilon$ and $v = v_\varepsilon$ in the class (17), weak solutions of (P$_\varepsilon$), that is, satisfying (18). Then $w = u - v$ satisfies

$$\left| \begin{aligned} &w'' - \Delta w + \frac{1}{\varepsilon} M w' = 0 \quad \text{in} \quad L^2(0,T;H^{-1}(\mathcal{O})) \\ &w = 0 \quad \text{on} \quad \Sigma' \\ &w(0) = w'(0) = 0 \end{aligned} \right. \tag{19}$$

Let $s \in [0,T]$ and define

$$z(t) = \begin{cases} at2 - \displaystyle\int_t^s w(\tau)d\tau; & 0 \le t \le s \\ 0; & s < t \le T. \end{cases} at \tag{20}$$

We have that $z \in L^\infty(0,T;H_0^1(\mathcal{O}))$ [2] since w also belongs to this set. Indeed,

$$\begin{aligned} ||z(t)|| &= \left\| \int_t^s w(\tau)d\tau \right\| \le \int_t^s ||w(\tau)||\, d\tau \le \int_0^T ||w(\tau)||\, d\tau \\ &\le T \operatorname*{ess\ sup}_{\tau\in[0,T]} ||w(\tau)|| = T \cdot ||w||_{L^\infty(0,T;H_0^1(\mathcal{O}))}; \quad \forall\, t \in [0,T]. \end{aligned}$$

Thus

$$\operatorname*{ess\ sup}_{t\in[0,T]} ||z(t)|| \le T \cdot ||w||_{L^\infty(0,T;H_0^1(\mathcal{O}))} < +\infty.$$

Therefore, from $(19)_1$ we can write that

$$\left\langle w'' - \Delta w + \frac{1}{\varepsilon} M w', z \right\rangle_{L^2(0,T;H^{-1}(\mathcal{O})),L^2(0,T;H_0^1(\mathcal{O}))} = 0,$$

that is,

$$\int_0^T \langle w''(t), z(t)\rangle dt + \int_0^T \langle -\Delta w(t), z(t)\rangle dt + \frac{1}{\varepsilon}\int_0^T \langle M w'(t), z(t)\rangle dt = 0.$$

Since $z(t) = 0, \quad \forall\, t \in [s,T]$ it follows that

$$\int_0^s \langle w''(t), z(t)\rangle dt + \int_0^s ((w(t), z(t)))dt + \frac{1}{\varepsilon}\int_0^s (M w'(t), z(t))dt = 0. \tag{21}$$

Set

$$w_1(s) = \int_0^s w(\tau)d\tau. \tag{22}$$

Note that if $t \in [0,s]$ we have

$$\int_0^s w(\tau)d\tau = \int_0^t w(\tau)d\tau + \int_t^s w(\tau)d\tau.$$

Whence

$$-\int_t^s w(\tau)d\tau = \int_0^t w(\tau)d\tau - \int_0^s w(\tau)d\tau,$$

that is,

$$z(t) = w_1(t) - w_1(s). \tag{23}$$

- **Calculation of the Integrals in (21)**

We have, integrating by parts

$$\int_0^s \langle w''(t), z(t)\rangle dt = (w'(s), z(s)) - (w'(0), z(0)) - \int_0^s (w'(t), z'(t))dt.$$

[2]In fact since $z'(t) = w(t)$ and $w \in L^\infty(0,T;H_0^1(\mathcal{O}))$ then $z' \in L^\infty(0,T;H_0^1(\mathcal{O}))$ and, therefore, $z \in C^0([0,T];H_0^1(\mathcal{O}))$.

However, from $(19)_3$ and (23) it follows that $w'(0) = 0$ and $z(s) = 0$. Thus

$$\int_0^s \langle w''(t), z(t)\rangle dt = -\int_0^s (w'(t), z'(t))dt.$$

But from (20) it follows that $z'(t) = w(t)$. Returning to the equality above results that

$$\int_0^s \langle w''(t), z(t)\rangle dt = -\int_0^s (w'(t), w(t))dt$$
$$= -\frac{1}{2}\int_0^s \frac{d}{dt}\,|w(t)|^2\,dt = -\frac{1}{2}\left[|w(s)|^2 - |w(0)|^2\right] = -\frac{1}{2}\,|w(s)|^2$$

since $w(0) = 0$ according to $(19)_3$.

Then

$$\int_0^s \langle w''(t), z(t)\rangle dt = -\frac{1}{2}\,|w(s)|^2. \tag{24}$$

Using again the fact that $z'(t) = w(t)$ in $[0, s]$ it follows that

$$\int_0^s ((w(t), z(t))dt = \int_0^s ((z'(t), z(t)))dt = \frac{1}{2}\int_0^s \frac{d}{dt}\,||z(t)||^2 dt = \frac{1}{2}\left[||z(s)||^2 - ||z(0)||^2\right].$$

But from (23) we obtain

$$\int_0^s ((w(t), z(t)))dt = -\frac{1}{2}\,||z(0)||^2. \tag{25}$$

Finally

$$\int_0^s (Mw'(t), z(t))dt = \int_0^s \int_{\mathcal{O}} M(x,t)w'(x,t)z(x,t)dxdt$$
$$= \int_0^s \int_{\mathcal{O}} w'(x,t)z(x,t)dxdt.$$

Integrating the last integral by parts it follows that

$$\int_0^s (w'(t), z(t))\,dt = \overbrace{(w(s), z(s))}^{=0} - \overbrace{(w(0), z(0))}^{=0} - \int_0^s (w(t), z'(t))dt$$
$$= -\int_0^s (w(t), w(t))dt = -\int_0^s |w(t)|^2\,dt.$$

Thus,

$$\int_0^s (w'(t), z(t))dt = -\int_0^s |w(t)|^2\,dt. \tag{26}$$

Therefore, from (21), (24), (25) and (26) we conclude that

$$-\frac{1}{2}\,|w(s)|^2 - \frac{1}{2}\,||z(0)||^2 - \frac{1}{\varepsilon}\int_0^s |w(t)|^2\,dt = 0.$$

Thus

$$|w(s)|^2 = 0 \quad \Rightarrow \quad w(s) = 0.$$

By the arbitrariness of $s \in [0,T]$ we have that $w(s) = 0, \quad \forall\, s \in [0,T]$, which proves uniqueness.

We have proved then that for each $\varepsilon > 0$ there exists a unique function $u_\varepsilon \colon \mathcal{O}\times]0,T[\to \mathbb{R}$, weak solution of (P_ε), in the class given in (17). □

- **Passage to the Limit in $(\mathbf{P}_\varepsilon)$**

It results from (5) and (6), given the Banach-Steinhaus Theorem

$$||u_\varepsilon||_{L^\infty(0,T;H_0^1(\mathcal{O}))} \le \underline{\lim}\, ||u_{\varepsilon\nu}||_{L^\infty(0,T;H_0^1(\mathcal{O}))}, \tag{27}$$

$$||u'_\varepsilon||_{L^\infty(0,T;L^2(\mathcal{O}))} \le \underline{\lim}\, ||u'_{\varepsilon\nu}||_{L^\infty(0,T;L^2(\mathcal{O}))}, \tag{28}$$

and from (4) it follows that

$$\begin{aligned} &(u_\varepsilon) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\mathcal{O})),\\ &(u'_\varepsilon) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\mathcal{O})). \end{aligned}$$

Thus, there exists a subsequence of the "net" (u_ε) which we will still denote by (u_ε) such that

$$u_\varepsilon \overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty(0,T;H_0^1(\mathcal{O})), \tag{29}$$

$$u'_\varepsilon \overset{*}{\rightharpoonup} w' \quad \text{in} \quad L^\infty(0,T;L^2(\mathcal{O})). \tag{30}$$

However, from (4) we have that

$$\frac{1}{\varepsilon}\int_0^T\int_{\mathcal{O}} M(x,t)|u'_{\varepsilon\nu}(x,t)|^2\,dxdt \le k; \quad \forall\, \nu \in \mathbb{N} \text{ and } \varepsilon > 0,$$

and from (6) it follows that

$$Mu'_{\varepsilon\nu} \overset{*}{\rightharpoonup} Mu'_\varepsilon \quad \text{in} \quad L^\infty(0,T;L^2(\mathcal{O}))$$

and, in this way, we conclude that

$$Mu'_{\varepsilon\nu} \overset{*}{\rightharpoonup} Mu'_\varepsilon \quad \text{in} \quad L^2(0,T;L^2(\mathcal{O})).$$

Thus

$$\frac{1}{\varepsilon}\int_0^T\int_{\mathcal{O}} |Mu'_\varepsilon|^2\,dxdt \le \inf_\nu \sup \frac{1}{\varepsilon}\int_0^T\int_{\mathcal{O}} |Mu'_{\varepsilon\nu}|^2\,dxdt \le k; \quad \forall\, \varepsilon > 0. \tag{31}$$

Thus,

$$0 \le \int_0^T\int_{\mathcal{O}} |Mu'_\varepsilon|^2\,dxdt \le k\varepsilon; \quad \forall\, \varepsilon > 0.$$

It follows then that

$$Mu'_\varepsilon \xrightarrow{\varepsilon\to 0} 0 \quad \text{strongly in} \quad L^2(0,T;L^2(\mathcal{O})). \tag{32}$$

However from (30)

$$Mu'_\varepsilon \xrightarrow{\varepsilon\to 0} Mw' \quad \text{weakly * in} \quad L^\infty(0,T;L^2(\mathcal{O})). \tag{33}$$

From (32) and (33) it follows that

$$Mw' = 0 \quad \text{in} \quad L^2(\mathcal{O}\times\,]0,T[).$$

Therefore

$$M(x,t)w'(x,t) = 0 \quad \text{a.e. in} \quad \mathcal{O}\times\,]0,T[\,. \tag{34}$$

Now, since $M = 0$ in Q and $M = 1$ in Q^c it results from (34) that

$$w'(x,t) = 0 \quad \text{a.e. in} \quad Q^c. \tag{35}$$

On the other hand from $(18)_1$, we have

$$u_\varepsilon'' - \Delta u_\varepsilon + \frac{1}{\varepsilon}\, M u_\varepsilon' = \tilde{f} \quad \text{in} \quad L^2(0,T;H^{-1}(\mathcal{O})).$$

It follows from (27), (31) and from the fact that $-\Delta \in \mathcal{L}(H_0^1(\mathcal{O}), H^{-1}(\mathcal{O}))$ that

$$||u_\varepsilon''||_{L^2(0,T;H^{-1}(\mathcal{O}))} \leq c_1||u_\varepsilon||_{L^2(0,T;H_0^1(\mathcal{O}))} + c_2 + c_3||\tilde{f}||_{L^2(0,T;L^2(\mathcal{O}))} \leq k_0\,; \quad \forall\,\varepsilon > 0.$$

Thus

$$u_\varepsilon'' \rightharpoonup w'' \quad \text{in} \quad L^2(0,T;H^{-1}(\mathcal{O})). \tag{36}$$

From (29), (30) and (36) it follows that

$$\begin{aligned} &w \in C^0([0,T];L^2(\mathcal{O})) \cap C_s(0,T;H_0^1(\mathcal{O})),\\ &w' \in C^0([0,T];H^{-1}(\mathcal{O})) \cap C_s(0,T;L^2(\mathcal{O})). \end{aligned}$$

Therefore, it makes sense to speak of $w(0)$, $w(T)$, $w'(0)$ and $w'(T)$. We will prove next that

$$w(0) = \tilde{u}_0, \tag{37}$$

$$w'(0) = \tilde{u}_1. \tag{38}$$

Indeed, let $\theta \in C^1([0,T])$; $\theta(0) = 1$ and $\theta(T) = 0$. Then, given $v \in L^2(\mathcal{O})$, $(v\theta) \in L^1(0,T;L^2(\mathcal{O}))$ and from (30) we obtain

$$\int_0^T (u_\varepsilon'(t), v)_{L^2(\mathcal{O})}\, \theta(t)dt \xrightarrow{\varepsilon\to 0} \int_0^T (w'(t), v)_{L^2(\mathcal{O})}\, \theta(t)dt.$$

Integrating by parts, it follows that

$$-(u_\varepsilon(0), v) - \int_0^T (u_\varepsilon(t), v)\theta'(t)dt \xrightarrow{\varepsilon\to 0} (w(0), v) - \int_0^T (w(t), v)\theta'(t)dt.$$

Now, from (29) we have that

$$\int_0^T (u_\varepsilon(t), v)\theta'(t)dt \xrightarrow{\varepsilon\to 0} \int_0^T (w(t), v)\theta'(t)dt.$$

Then

$$(u_\varepsilon(0), v) \xrightarrow{\varepsilon\to 0} (w(0), v); \quad \forall\, v \in L^2(\mathcal{O}).$$

Since $u_\varepsilon(0) = \tilde{u}_0\,; \quad \forall\, \varepsilon > 0$, it results that

$$w(0) = \tilde{u}_0\,.$$

Considering θ in the same previous conditions and $v \in H_0^1(\mathcal{O})$, from (36) it follows that

$$\int_0^T \langle u_\varepsilon''(t), v\rangle \theta(t)dt \xrightarrow{\varepsilon\to 0} \int_0^T \langle w''(t), v\rangle \theta(t)dt.$$

Integrating by parts, we obtain

$$-(u_\varepsilon'(0), v) - \int_0^T (u_\varepsilon'(t), v)\theta'(t)dt \xrightarrow{\varepsilon\to 0} -(w'(0), v) - \int_0^T (w'(t), v)\theta'(t)dt.$$

However, due to the fact that

$$\int_0^T (u_\varepsilon'(t), v)\theta'(t)dt \xrightarrow{\varepsilon\to 0} \int_0^T (w'(t), v)\theta'(t)dt$$

we have that

$$(u_\varepsilon'(0), v) \to (u'(0), v); \quad \forall\, v \in H_0^1(\mathcal{O}).$$

By the density of $H_0^1(\mathcal{O})$ in $L^2(\mathcal{O})$ and by the fact that $u_\varepsilon'(0) = \tilde{u}_1$ we obtain

$$w'(0) = \tilde{u}_1\,.$$

On the other hand, from (35), we obtain

$$w(x,t) = w(x) \quad \text{a.e. in} \quad Q^c = \big(\mathcal{O}\times\,]0,T[\big)\backslash Q.$$

Since $w \in C^0([0,T]; L^2(\mathcal{O}))$ it follows that

$$w(x,t) = w(x), \text{ for almost all } (x,t) \in \mathcal{O} \times \{t\}\backslash\Omega_t \text{ and } \forall\, t \in [0,T]. \tag{39}$$

But from (37) we have then that

$$w(x,0) = w(x) = \tilde{u}_0\,, \quad \text{for a.e.} \quad x \in \mathcal{O}\backslash\Omega_0\,.$$

However,

$$\tilde{u}_0(x) = \begin{cases} u_0(x); & x \in \Omega_0 \\ 0; & x \in \mathcal{O}\backslash\Omega_0 \end{cases}$$

which implies that

$$w(x) = 0 \quad \text{a.e. in} \quad \mathcal{O}\backslash\Omega_0\,.$$

It follows from (39) that

$$w(t)(x) = w(x,t) = 0, \text{ for a.e. } (x,t) \in \mathcal{O} \times \{t\}\backslash\Omega_t \text{ and } \forall\, t \in [0,T]. \tag{40}$$

Recalling that Ω_t is identified with Ω_t^* it follows that

$$w(t)(x) = 0 \quad \text{a.e.} \quad x \in \mathcal{O}\backslash\Omega_t^* \text{ and } \forall\, t \in [0,T].$$

This together with hypothesis (3) leads us to conclude that if we define u as the restriction of w to Q then

$$u \in L^\infty(0,T; H_0^1(\Omega_t)). \tag{41}$$

Also from (39) it results that

$$u' \in L^\infty(0,T;L^2(\Omega_t)). \tag{42}$$

Consider, now, $\psi \in \mathcal{D}(Q)$ and define

$$\widetilde{\psi} = \begin{cases} \psi & \text{in} \quad Q \\ 0 & \text{in} \quad Q^c \end{cases}$$

Then, $\widetilde{\psi} \in \mathcal{D}\ (\mathcal{O}\times]0,T[)$ and therefore $\widetilde{\psi} \in L^2(0,T;H^1(\mathcal{O}))$, $\widetilde{\psi}' \in L^2(0,T;L^2(\mathcal{O}))$. It results from (13) that

$$\left\langle u_\varepsilon'' - \Delta u_\varepsilon + \frac{1}{\varepsilon} M u_\varepsilon', \widetilde{\psi} \right\rangle = \langle \tilde{f}, \widetilde{\psi} \rangle$$

where $\langle \cdot \rangle$ designates the duality $L^2(0,T;H^{-1}(\mathcal{O})) \times L^2(0,T;H_0^1(\mathcal{O}))$, or even,

$$\begin{aligned} &\overbrace{(u_\varepsilon'(T), \widetilde{\psi}(T))}^{=0} - \overbrace{(u_\varepsilon'(0), \widetilde{\psi}(0))}^{=0} - \int_0^T (u_\varepsilon'(t), \widetilde{\psi}'(t))_{L^2(\mathcal{O})}\, dt \\ &+ \int_0^T a(u_\varepsilon(t), \widetilde{\psi}(t)) dt + \frac{1}{\varepsilon} \int_0^T (M u_\varepsilon'(t), \widetilde{\psi}(t))_{L^2(\mathcal{O})}\, dt \\ &= \int_0^T (\tilde{f}(t), \widetilde{\psi}(t))_{L^2(\mathcal{O})}\, dt. \end{aligned} \tag{43}$$

But, since $M(x,t) = 0$ if $(x,t) \in Q$ and $\psi = 0$ in Q^c, it follows that

$$\begin{aligned} &\frac{1}{\varepsilon} \int_0^T (M u_\varepsilon'(t), \widetilde{\psi}(t))_{L^2(\mathcal{O})}\, dt \\ &= \frac{1}{\varepsilon} \left[\int_Q M(x,t) u_\varepsilon'(x,t) \widetilde{\psi}(x,t) dx dt + \int_{Q^c} M(x,t) u_\varepsilon'(x,t) \widetilde{\psi}(x,t) dx dt \right] = 0. \end{aligned}$$

Whence, from (43)

$$- \int_0^T (u_\varepsilon'(t), \widetilde{\psi}'(t))_{L^2(\mathcal{O})}\, dt + \int_0^T a(u_\varepsilon(t), \widetilde{\psi}(t)) dt = \int_0^T (\tilde{f}(t), \widetilde{\psi}(t))_{L^2(\mathcal{O})}\, dt.$$

Taking the limit as $\varepsilon \to 0$ results from (29) and (30) that

$$- \int_0^T (w'(t), \widetilde{\psi}'(t))_{L^2(\mathcal{O})}\, dt + \int_0^T a(w(t), \widetilde{\psi}(t)\, dt = \int_0^T (\tilde{f}(t), \widetilde{\psi}(t))_{L^2(\mathcal{O})}\, dt,$$

that is,

$$- \int_Q u'(x,t) \psi'(x,t) dx dt + \sum_{i=1}^n \int_Q \frac{\partial u}{\partial x_i}(x,t) \frac{\partial \psi}{\partial x_i}(x,t) dx dt = \int_Q f(x,t) \psi(x,t) dx dt.$$

It follows from there that

$$\langle u'' - \Delta u, \psi \rangle = \langle f, \psi \rangle, \quad \forall\, \psi \in \mathcal{D}(Q).$$

Therefore

$$u'' - \Delta u = f \quad \text{in} \quad \mathcal{D}'(Q).$$

14.6 Problem with Nonlinear Vibrations

Let Ω be a bounded open set with sufficiently smooth boundary Γ. Let $Q = \Omega \times]0,T[$ and $\Sigma = \Gamma \times]0,T[$, with $T > 0$ and $M\colon [0,+\infty[\to \mathbb{R}$, $\quad M \in C^1([0,+\infty[)$ and $M \geq m_0 > 0$.

We wish to find $u\colon Q \to \mathbb{R}$ weak solution of

$$\left| \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - M\Big(\displaystyle\int_\Omega |\nabla u|^2 \, dx \Big) \Delta u = f \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(0) = u_1. \end{array} \right. \tag{1}$$

Instead of solving problem 1 specifically, we will obtain a solution for the abstract problem. For this, let V and H be separable Hilbert spaces such that $\dim H = +\infty$, $V \subsetneqq H$,

$$V \overset{c}{\to} \hookrightarrow H \quad \text{and} \quad V \quad \text{is dense in} \quad H.$$

Let $((\,\cdot\,,\,\cdot\,))$ and $(\,\cdot\,,\,\cdot\,)$, respectively, be the inner products of V and H, consider

$$A \leftarrow \{V, H, ((\,\cdot\,,\,\cdot\,))\}.$$

As is well known A is a self-adjoint, positive and unbounded operator of H. We also have the existence of a sequence of eigenvectors $(w_\nu)_{\nu\in\mathbb{N}}$ of A and corresponding eigenvalues $(\lambda_\nu)_{\nu\in\mathbb{N}}$ such that

$$\begin{array}{ll} (w_\nu) & \text{is a complete orthonormal system of} \quad H \\ \left(\dfrac{w_\nu}{\sqrt{\lambda_\nu}}\right) & \text{is a complete orthonormal system of} \quad V. \end{array}$$

Also

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{and} \quad \lambda_\nu \to +\infty \quad \text{when} \quad \nu \to +\infty. \tag{2}$$

Now, if $\alpha \in \mathbb{R}$, we define the powers of A by

$$D(A^\alpha) = \left\{ u \in H; \sum_{\nu=1}^{+\infty} \lambda_\nu^{2\alpha} |(u, w_\nu)|^2 < +\infty \right\} \tag{3}$$

and

$$A^\alpha u = \sum_{\nu=1}^{+\infty} \lambda_\nu^\alpha (u, w_\nu) w_\nu \, . \tag{4}$$

We have that A^α is equally self-adjoint and positive, $\forall\, \alpha \in \mathbb{R}$, making sense therefore to speak of the root of A^α. It follows from this that if we define

$$D(T) = \left\{ u \in H; \sum_{\nu=1}^{+\infty} \lambda_\nu^{\alpha} |(u, w_\nu)|^2 < +\infty \right\}$$

and

$$Tu = \sum_{\nu=1}^{+\infty} \lambda_\nu^{\alpha/2} (u, w_\nu) w_\nu$$

then,

$$T = A^{\alpha/2}.$$

We can also prove that T is the unique positive self-adjoint operator that verifies

$$T^2 = A^\alpha. \tag{5}$$

Thus

$$(A^\alpha u, u) = (T^2 u, u) = (Tu, Tu) = (A^{\alpha/2}u, A^{\alpha/2}u); \quad \forall\, u \in D(A^\alpha). \tag{6}$$

We note further that the powers of A satisfy the following property

$$\text{If} \quad \alpha_1 \leq \alpha_2 \quad \text{then} \quad D(A^{\alpha_2}) \subset D(A^{\alpha_1}). \tag{7}$$

We can prove from (3) and (4) that

$$D(A^{\alpha_1+\alpha_2}) = \big\{u \in H; u \in D(A^{\alpha_1}) \cap D(A^{\alpha_2}), A^{\alpha_1}u \in D(A^{\alpha_2}) \text{ and } A^{\alpha_2}u \in D(A^{\alpha_1})\big\}. \tag{8}$$

It follows from (8) and (4) that:

$$A^{\alpha_1} \circ A^{\alpha_2} = A^{\alpha_1+\alpha_2} = A^{\alpha_2+\alpha_1} = A^{\alpha_2} \circ A^{\alpha_1} \quad \text{in} \quad D(A^{\alpha_1+\alpha_2}). \tag{9}$$

Now, if $\alpha \geq 0$ it is verified that there exists $c > 0$ such that

$$(A^\alpha u, u) \geq c|u|^2; \quad \forall\, u \in D(A^\alpha). \tag{10}$$

We endow $D(A^\alpha)$ with the inner product

$$(u, v)_{D(A^\alpha)} = (u, v) + (A^\alpha u, A^\alpha v); \quad u, v \in D(A^\alpha),$$

which makes it a Hilbert space since A^α is closed given that it is self-adjoint. Since $\alpha \geq 0$, it follows from (10) that the norms

$$||u||_{D(A^\alpha)} = \big(|u|^2 + |A^\alpha u|^2\big)^{1/2} \tag{11}$$

and

$$|||u|||_{D(A^\alpha)} = |A^\alpha u| \tag{12}$$

are equivalent in $D(A^\alpha)$. Therefore, the topological space $(D(A^\alpha); |||\cdot|||_{D(A^\alpha)})$ is also a Hilbert space. In what follows, we will work with the topology given in (12). Furthermore, endowed with this topology, if $\alpha_1 \leq \alpha_2$ then $D(A^{\alpha_2}) \hookrightarrow D(A^{\alpha_1})$.

If $\alpha > 0$ we note that

$$A^{-\alpha} \quad \text{is a compact operator of} \quad H. \tag{13}$$

Finally, we also observe that the following properties are satisfied

$$\text{The embedding of } D(A^\alpha),\ \alpha > 0, \text{ in } H \text{ is compact.} \tag{14}$$

$$\text{If } \rho \geq 0,\ \alpha > 0 \text{ then the embedding of } D(A^{\alpha+\rho}) \text{ in } D(A^\rho) \text{ is compact.} \tag{15}$$

Consider, now, the abstract problem

$$\left| \begin{array}{l} u'' + M(a(u))Au = f \\ u(0) = u_0\,, \;\; u'(0) = u_1, \end{array} \right. \tag{16}$$

where $a(u,v) = ((u,v)); \quad u, v \in V$.

A natural question that arises is whether we can solve the problem above subject to the following initial data

$$u_0 \in D(A^{1/2}) = V; \quad u_1 \in H \quad \text{and} \quad f \in L^2(0,T;H). \tag{17}$$

In what follows we will proceed formally. Composing $(16)_1$ with $2u'$ results that

$$2(u'', u') + M(a(u))(Au, 2u') = 2(f, u'). \tag{18}$$

We have

$$2(Au, u') = 2a(u, u') = 2((u, u')) = \frac{d}{dt}\,||u||^2.$$

Setting

$$\widehat{M}(\lambda) = \int_0^\lambda M(\xi)d\xi; \quad \lambda \in [0, +\infty[\tag{19}$$

then

$$\frac{d}{dt}\,(\widehat{M}(a(u)) = M(a(u)) \cdot \frac{d}{dt}\,a(u) = M(a(u))\,\frac{d}{dt}\,||u||^2 = M(a(u))2(Au, u'). \tag{20}$$

Thus, from (18) and (20) we conclude that:

$$\frac{d}{dt}\left\{|u'|^2 + \widehat{M}(a(u))\right\} = 2(f, u').$$

Integrating from 0 to t, we obtain

$$|u'|^2 + \widehat{M}(a(u)) \le |u_1|^2 + \widehat{M}(a(u_0)) + \int_0^T |f|^2\,ds + \int_0^t |u'|^2\,ds. \tag{21}$$

However, due to the fact that

$$\widehat{M}(a(u)) = \int_0^{a(u)} M(\xi)d\xi \ge m_0\,a(u)$$

from (21) it follows that

$$|u'|^2 + m_0\,a(u) \le |u_1|^2 + \widehat{M}(a(u_0)) + \int_0^T |f|^2\,ds + \int_0^t |u'|^2\,ds.$$

By Gronwall's Lemma it follows that

$$|u'|^2 + a(u) \le c; \quad \forall\, t. \tag{22}$$

We observe that, until now, the choice of initial data as in (17) has been satisfactory and sufficient to pass the limit in the linear part of the problem. However, to pass the limit in the nonlinear part of the problem it is necessary that we have strong convergence in $L^2(0,T;D(A^{1/2}))$. However, the estimate in (22) guarantees us only, given the Aubin-Lions theorem, strong convergence in $L^2(0,T;H)$ which is insufficient to pass the limit. We need, therefore, a new a priori estimate.

Let $0<\varepsilon<1$ to be determined later. Consider the scalar product of $(16)_1$ with $2\,A^\varepsilon u'$. We have

$$2(u'',A^\varepsilon u')+2M(a(u))(Au,A^\varepsilon u')=2(f,A^\varepsilon u'). \tag{23}$$

But,

$$\begin{aligned}2(u'',A^\varepsilon u')&=2(A^{\varepsilon/2}u'',A^{\varepsilon/2}u')=2(u'',u')_{D(A^{\varepsilon/2})}\\&=\frac{d}{dt}\,|u'(t)|^2_{D(A^{\varepsilon/2})}=\frac{d}{dt}\,|A^{\varepsilon/2}u'|^2\end{aligned} \tag{24}$$

Also,

$$\begin{aligned}2(Au,A^\varepsilon u')&=2(A^{1/2}A^{1/2}u,A^{\varepsilon/2}A^{\varepsilon/2}u')=2\big(A^{\frac{\varepsilon+1}{2}}u,A^{\frac{\varepsilon+1}{2}}u'\big)\\&=2(u,u')_{D(A^{\frac{1+\varepsilon}{2}})}=\frac{d}{dt}\,|u|_{D(A^{\frac{\varepsilon+1}{2}})}=\frac{d}{dt}\,|A^{\frac{\varepsilon+1}{2}}\,u|^2.\end{aligned} \tag{25}$$

Thus, from (23), (24) and (25) we obtain

$$\frac{d}{dt}\,|A^{\varepsilon/2}u'|^2+M(a(u))\,\frac{d}{dt}\,|A^{\frac{\varepsilon+1}{2}}u|^2=2(f,A^\varepsilon u'). \tag{26}$$

Since

$$\frac{d}{dt}\,\big(M(a(u))|A^{\frac{\varepsilon+1}{2}}u|^2\big)=\frac{d}{dt}\,M(a(u))|A^{\frac{\varepsilon+1}{2}}u|^2+M(a(u))\cdot\frac{d}{dt}\,|A^{\frac{\varepsilon+1}{2}}u|^2$$

then from (26) it follows that

$$\frac{d}{dt}\left\{|A^{\varepsilon/2}u'|^2+M(a(u))|A^{\frac{\varepsilon+1}{2}}u|^2\right\}=2(f,A^\varepsilon u')+\frac{d}{dt}\,M(a(u))|A^{\frac{\varepsilon+1}{2}}u|^2. \tag{27}$$

However,

$$\begin{aligned}\frac{d}{dt}\,(M(a(u)))&=M'(a(u))\,\frac{d}{dt}\,a(u)=M'(a(u))\,\frac{d}{dt}\,||u||^2=M'(a(u))2((u,u'))\\&=2M'(a(u))(Au,u').\end{aligned}$$

Therefore, from (22)

$$\left|\frac{d}{dt}\,(M(a(u))\right|=2|M'(a(u))|\,|(Au,u')|\le2\bigg(\max_{0\le a(u)\le c}|M'(a(u))|\bigg)|(Au,u')|,$$

that is, there exists $c_1>0$ such that:

$$\left|\frac{d}{dt}\,(M(a(u))\right|\le c_1|(Au,u')|. \tag{28}$$

Consider, now, $0<\gamma<1$. We have

$$(Au,u')=(A^\gamma A^{1-\gamma}u,u')=(A^{1-\gamma}u,A^\gamma u'). \tag{29}$$

From (28) and (29) we arrive at

$$\left|\frac{d}{dt}\left(M(a(u))\right)\right| \leq c_1|A^{1-\gamma}u|\,|A^{\gamma}u'| \tag{30}$$

and from (27) and (30) we can write

$$\begin{aligned}
&\frac{d}{dt}\left\{|A^{\varepsilon/2}u'|^2 + M(a(u))|A^{\frac{\varepsilon+1}{2}}u|^2\right\}\\
&\leq 2(f, A^{\varepsilon}u') + c_1|A^{1-\gamma}u|\,|A^{\gamma}u'|\,|A^{\frac{\varepsilon+1}{2}}u|^2\\
&= 2(A^{\varepsilon/2}f, A^{\varepsilon/2}u') + c_1|A^{1-\gamma}u|\,|A^{\gamma}u'|\,|A^{\frac{\varepsilon+1}{2}}u|^2\\
&\leq |A^{\varepsilon/2}f|^2 + |A^{\varepsilon/2}u'|^2 + c_1|A^{1-\gamma}u|\,|A^{\gamma}u'|\,|A^{\frac{\varepsilon+1}{2}}u|^2.
\end{aligned}$$

Integrating the previous inequality we obtain

$$\begin{aligned}
&|A^{\varepsilon/2}u'|^2 + M(a(u))|A^{\frac{\varepsilon+1}{2}}u|^2\\
&\leq |A^{\varepsilon/2}u_1|^2 + M(a(u_0))|A^{\frac{\varepsilon+1}{2}}u_0|^2 + \int_0^T |A^{\varepsilon/2}f|^2\,dt\\
&+ \int_0^t |A^{\varepsilon/2}u'|^2\,ds + c_1\int_0^t |A^{1-\gamma}u|\,|A^{\gamma}u'|\,|A^{\frac{\varepsilon+1}{2}}u|^2\,ds.
\end{aligned}$$

We would like there to exist $k_0 > 0$ and $k_1 > 0$ satisfying

$$\begin{gathered}
|A^{1-\gamma}u| \leq k_0|A^{\frac{\varepsilon+1}{2}}u|,\\
\text{and}\\
|A^{\gamma}u'| \leq k_1|A^{\varepsilon/2}u'|.
\end{gathered} \tag{31}$$

Assuming that (31) is true we obtain

$$\begin{aligned}
&|A^{\varepsilon/2}u'|^2 + M(a(u))|A^{\frac{\varepsilon+1}{2}}u|^2\\
&\leq |A^{\varepsilon/2}u_1|^2 + M(a(u_0))|A^{\frac{\varepsilon+1}{2}}u_0|^2 + \int_0^T |A^{\varepsilon/2}f|^2\,dt\\
&+ \int_0^t |A^{\varepsilon/2}u'|^2\,ds + c_2\int_0^t |A^{\frac{\varepsilon+1}{2}}u|\,|A^{\varepsilon/2}u'|\,|A^{\frac{\varepsilon+1}{2}}u|^2\,ds.
\end{aligned} \tag{32}$$

Therefore, for us to have (31) it is necessary that the following embeddings hold:

$$D\left(A^{\frac{\varepsilon+1}{2}}\right) \hookrightarrow D(A^{1-\gamma}),$$

and

$$D(A^{\varepsilon/2}) \hookrightarrow D(A^{\gamma}).$$

For the embeddings above to occur we must have that

$$1-\gamma \leq \frac{1+\varepsilon}{2} \qquad \text{and} \qquad \gamma \leq \frac{\varepsilon}{2}.$$

Summing the two inequalities we must have

$$1 \leq \frac{1}{2} + \varepsilon \quad \Rightarrow \quad \varepsilon \geq \frac{1}{2}\cdot$$

Choosing $\varepsilon = \dfrac{1}{2}$ (which is the best choice since the smaller the ε the larger the set $D(A^{\varepsilon/2})$ and, therefore, we are being less restrictive) it follows that $\gamma \leq \dfrac{1}{4}\cdot$

Returning to (32) with the choice above, we obtain

$$\begin{aligned}
&|A^{1/4}u'|^2 + M(a(u))|A^{3/4}u|^2 \\
&\leq |A^{1/4}u_1|^2 + M(a(u_0))|A^{3/4}u_0|^2 + \int_0^t |A^{1/4}f|^2\,dt \\
&+ \int_0^t |A^{1/4}u'|^2\,ds + c_2 \int_0^t |A^{3/4}u|\,|A^{1/4}u'|\,|A^{3/4}u|^2\,ds.
\end{aligned}$$

The inequality above indicates the ideal place to consider the initial data, that is, we must consider

$$u_0 \in D(A^{3/4}), \quad u_1 \in D(A^{1/4}) \quad \text{and} \quad f \in L^2(0,T;D(A^{1/4})). \tag{34}$$

Note that

$$D(A^{3/4}) \hookrightarrow D(A^{1/2}) = V \hookrightarrow D(A^{1/4})$$

Suppose, then (34), there exists $c_3 > 0$ such that

$$|A^{1/4}u'|^2 + M(a(u))|A^{3/4}u|^2 \leq c_3 + \int_0^t |A^{1/4}u'|^2\,ds + c_2 \int_0^t |A^{3/4}u|\,|A^{1/4}u'|\,|A^{3/4}u|^2\,ds.$$

Since $M(\lambda) \geq m_0 > 0; \quad \forall\,\lambda \in [0,+\infty[$ we obtain

$$|A^{1/4}u'|^2 + |A^{3/4}u|^2 \leq k_3 + k_1 \int_0^t |A^{1/4}u'|^2 + k_2 \int_0^t |A^{3/4}u|\,|A^{1/4}u'|\,|A^{3/4}u|^2\,ds, \tag{35}$$

where

$$k_3 = \frac{c_3}{\min\{1,m_0\}}\,; \quad k_2 = \frac{c_2}{\min\{1,m_0\}} \quad \text{and} \quad k_1 = \frac{1}{\min\{1,m_0\}}\cdot$$

However, despite the new choice of initial data, we cannot bound the expression on the left of the inequality in (35) for all t belonging to the field of definition of u, as we will see next.

Set

$$Y = |A^{1/4}u'|^2 + |A^{3/4}u|^2.$$

We have

$$|A^{3/4}u|^2 \leq Y \quad \text{and} \quad |A^{1/4}u'| \leq Y.$$

Thus, from (35) it follows that

$$\begin{aligned}
Y(t) &\leq k_3 + k_2 \int_0^t Y^{1/2}(s)\,Y^{1/2}(s)\,Y\,ds + k_1 \int_0^t Y(s)ds \\
&\leq k_3 + k_4 \int_0^t (Y^2(s) + Y(s))ds,
\end{aligned}$$

that is,

$$Y(t) \leq k_3 + k_4 \int_0^t (Y^2 + Y)ds. \tag{36}$$

Setting

$$h(t) = \int_0^t (Y^2 + Y)ds \tag{37}$$

it follows from (36) that

$$Y(t) \leq k_3 + k_4\, h(t). \tag{38}$$

From (37) and (38) we obtain

$$h'(t) = Y^2 + Y \leq (k_3 + k_4\, h(t))^2 + (k_3 + k_4\, h(t)),$$

or even,

$$k_4\, h'(t) \leq k_4\big\{(k_3 + k_4\, h(t))^2 + (k_3 + k_4\, h(t))\big\}. \tag{39}$$

Since

$$(k_3 + k_4\, h(t))' = k_4\, h'(t)$$

we have from (39) that

$$(k_3 + k_4\, h(t))' \leq k_4\big\{(k_3 + k_4\, h(t))^2 + (k_3 + k_4\, h(t))\big\}.$$

Setting

$$\psi(t) = k_3 + k_4\, h(t) \tag{40}$$

then we have

$$\psi'(t) \leq k_4\{\psi(t)^2 + \psi(t)\}.$$

Whence

$$\psi'(t) - k_4\, \psi(t) \leq k_4\, \psi^2(t).$$

Multiplying the inequality above by $e^{-k_4 t}$ it follows that

$$\Big(\psi'(t)e^{-k_4 t} - k_4\, e^{-k_4 t}\, \psi(t)\Big) \leq k_4\, \psi^2(t)e^{-k_4 t},$$

that is,

$$\big(\psi(t)e^{-k_4 t}\big)' \leq k_4\, \psi^2(t)e^{-k_4 t}.$$

Integrating from 0 to t, we obtain from (37) and (40) that

$$\psi(t)e^{-k_4 t} - k_3 \leq \int_0^t k_4\, \psi^2(s)e^{-k_4 s}\, ds.$$

Whence

$$\psi(t) \leq e^{k_4 t}\bigg\{k_3 + k_4 \int_0^t \psi^2(s)e^{-k_4 s}\, ds\bigg\}. \tag{41}$$

Now, defining

$$z(t) = \int_0^t e^{-k_4 s}\, \psi^2(s)ds, \tag{42}$$

from (41) we can write

$$\psi(t) \leq e^{k_4 t}\{k_3 + k_4\, z(t)\}. \tag{43}$$

From (42) and (43) we obtain

$$z'(t) = e^{-k_4 t}\, \psi^2(t) \leq e^{-k_4 t}\, e^{2k_4 t}\{k_3 + k_4\, z(t)\}^2 = e^{k_4 t}\{k_3 + k_4\, z(t)\}^2$$

and, therefore, from the fact that $k_3, k_4 > 0$, we have

$$\frac{z'(t)}{(k_3 + k_4\, z(t))^2} \leq e^{k_4 t}.$$

Integrating from 0 to t we arrive at:

$$\int_0^t \frac{z'(s)}{(k_3 + k_4\, z(s))^2}\, ds \leq \int_0^t e^{k_4 s}\, ds.$$

Consider the following change of variables:

$$u = k_3 + k_4\, z(s) \quad \Rightarrow \quad du = k_4\, z'(s) ds$$

and when

$$\begin{aligned} s = 0 \quad &\Rightarrow \quad u = k_3 \\ s = t \quad &\Rightarrow \quad u = k_3 + k_4\, z(t). \end{aligned}$$

Thus

$$\begin{aligned} \int_0^t \frac{z'(s)}{(k_3 + k_4\, z(s))^2}\, ds &= \frac{1}{k_4} \int_{k_3}^{k_3 + k_4\, z(t)} u^{-2}\, du = -\frac{u^{-1}}{k_4}\bigg|_{k_3}^{k_3 + k_4\, z(t)} \\ &= \frac{1}{k_4}\left\{\frac{1}{k_3} - \frac{1}{k_3 + k_4\, z(t)}\right\}. \end{aligned}$$

Also,

$$\int_0^t e^{k_4 s}\, ds = \frac{1}{k_4} e^{k_4 s}\bigg|_0^t = \frac{1}{k_4}\,(e^{k_4 t} - 1)$$

whence

$$\frac{1}{k_3} - \frac{1}{k_3 + k_4\, z(t)} \leq e^{k_4 t} - 1$$

that is,

$$-\frac{1}{k_3 + k_4\, z(t)} \leq -\frac{1}{k_3} + e^{k_4 t} - 1.$$

Therefore,

$$\frac{1}{k_3 + k_4\, z(t)} \geq 1 - e^{k_4 t} + \frac{1}{k_3}\,\cdot \tag{44}$$

On the other hand,

$$\begin{aligned} 1 - e^{k_4 t} + \frac{1}{k_3} > 0 \Leftrightarrow e^{k_4 t} < 1 + \frac{1}{k_3} \Leftrightarrow \ell n(e^{k_4 t}) < \ell n\left(1 + \frac{1}{k_3}\right) \Leftrightarrow \\ \Leftrightarrow k_4 t < \ell n\left(1 + \frac{1}{k_3}\right) \Leftrightarrow t < \frac{1}{k_4}\,\ell n\left(1 + \frac{1}{k_3}\right). \end{aligned}$$

Setting

$$T^* = \frac{1}{k_4}\,\ell n\left(\frac{1}{k_3} + 1\right)$$

then if $t \leq T_0$ and $T_0 < T^*$ we have

$$\frac{1}{k_3} - e^{k_4 t} + 1 > 0; \quad \forall\, t \in [0, T_0].$$

From (44) it follows that

$$k_3 + k_4\, z(t) \le \left(1 - e^{k_4 t} + \frac{1}{k_3}\right)^{-1}; \quad \forall\, t \in [0, T_0]$$

and, from (43), it results that

$$\psi(t) \le e^{k_4 T_0}\left(1 - e^{k_4 T_0} + \frac{1}{k_3}\right)^{-1} = L; \quad \forall\, t \in [0, T_0].$$

From this last inequality and from (40) we obtain

$$h(t) \le \frac{L - k_3}{k_4} = M; \quad \forall\, t \in [0, T_0].$$

Finally from (38) we conclude that

$$Y(t) \le k_3 + k_4\, M = c; \quad \forall\, t \in [0, T_0],$$

that is,

$$|A^{1/4}u'|^2 + |A^{3/4}u|^2 \le c; \quad \forall\, t \in [0, T_0]. \tag{45}$$

From the above, we have the following result

Theorem: Given:

$$u_0 \in D(A^{3/4}); \quad u_1 \in D(A^{1/4}) \quad \text{and} \quad f \in L^2(0, T; D(A^{1/4}))$$

there exists $0 < T_0 \le T$ and a unique solution $u\colon [0, T_0] \to H$ of (16) in the class

$$u \in L^\infty(0, T_0; D(A^{3/4})); \quad u' \in L^\infty(0, T_0; D(A^{1/4})) \quad \text{and} \quad u'' \in L^2(0, T_0; V')$$

verifying

$$\begin{aligned} &\frac{d}{dt}\,(u'(t), v) + M(a(u(t)))a(u(t), v) = (f(t), v) \text{ in } L^2(0, T_0); \quad \forall\, v \in V \\ &u(0) = u_0\,;\ u'(0) = u_1\,. \end{aligned}$$

Proof:

1ª Step: Approximate Solution

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be the sequence of eigenvectors associated with the operator

$$A \leftarrow \{V, H; ((\cdot\,,\cdot))\}$$

whose corresponding eigenvalues $(\lambda_\nu)_{\nu\in\mathbb{N}}$ verify

$$0 < \lambda_1 \le \lambda_2 \le \dots \quad \text{and} \quad \lambda_\nu \to +\infty, \text{ when } \nu \to +\infty. \tag{46}$$

We know that, by virtue of the Spectral Theorem

$$(w_\nu) \quad \text{is a complete orthonormal system of} \quad H \tag{47}$$

$$\left(\frac{w_\nu}{\sqrt{\lambda_\nu}}\right) \quad \text{is a complete orthonormal system of} \quad V = D(A^{1/2}) \tag{48}$$

$$\left(\frac{w_\nu}{\lambda_\nu}\right) \quad \text{is a complete orthonormal system of} \quad D(A) \tag{49}$$

Note that

$$D(A) \hookrightarrow D(A^{3/4}) \hookrightarrow D(A^{1/2}) = V \hookrightarrow D(A^{1/4}) \hookrightarrow H. \tag{50}$$

We claim that

$$(w_\nu)_{\nu\in\mathbb{N}} \quad \text{is orthogonal complete in} \quad D(A^\alpha);\ \ 0 \leq \alpha \leq 1. \tag{51}$$

Indeed, let $\nu, \mu \in \mathbb{N}$ such that $\nu \neq \mu$. Then:

$$(w_\nu, w_\mu)_{D(A^\alpha)} = (A^\alpha w_\nu, A^\alpha w_\mu) = \lambda_\nu^\alpha \lambda_\mu^\alpha (w_\nu, w_\mu) - 0.$$

Furthermore, let $u \in D(A^\alpha)$ such that $(u, w_\nu)_{D(A^\alpha)} = 0;\quad \forall\, \nu \in \mathbb{N}$. We have:

$$0 = (A^\alpha u, A^\alpha w_\nu) = \lambda_\nu^\alpha (A^\alpha u, w_\nu) = \lambda_\nu^\alpha (u, A^\alpha w_\nu) = \lambda_\nu^{2\alpha}(u, w_\nu);\quad \forall\, \nu \in \mathbb{N}.$$

Since $\lambda_\nu > 0,\quad \forall\, \nu \in \mathbb{N}$ it follows that $(u, w_\nu) = 0;\quad \forall\, \nu \in \mathbb{N}$. Therefore, from (47) it follows that $u = 0$ which proves (51).

But,

$$||w_\nu||^2_{D(A^\alpha)} = (A^\alpha w_\nu, A^\alpha w_\nu) = \lambda_\nu^{2\alpha}|w_\nu|^2 = \lambda_\nu^{2\alpha}.\ \forall\, \nu \in \mathbb{N}.$$

Whence

$$\left(\frac{w_\nu}{\lambda_\nu^\alpha}\right) \quad \text{is orthonormal complete in} \quad D(A^\alpha). \tag{52}$$

It results from this that for all $u \in D(A^\alpha)$ we have

$$u = \sum_{\nu=1}^{+\infty} \left(\left(u, \frac{w_\nu}{\lambda_\nu^\alpha}\right)\right)_{D(A^\alpha)} \frac{w_\nu}{\lambda_\nu^\alpha}$$

that is,

$$\sum_{\nu=1}^{n} \left(\left(u, \frac{w_\nu}{\lambda_\nu^\alpha}\right)\right)_{D(A^\alpha)} \frac{w_\nu}{\lambda_\nu^\alpha} \quad \xrightarrow{n\to+\infty} \quad u \text{ in } D(A^\alpha). \tag{53}$$

However, since

$$\left(\left(u, \frac{w_\nu}{\lambda_\nu^{2\alpha}}\right)\right)_{D(A^\alpha)} = \left(A^\alpha u, \frac{A^\alpha w_\nu}{\lambda_\nu^{2\alpha}}\right) = \frac{1}{\lambda_\nu^\alpha}(A^\alpha u, w_\nu) = \frac{1}{\lambda_\nu^\alpha}(u, A^\alpha w_\nu) = (u, w_\nu)$$

we have from (53) that

$$\sum_{\nu=1}^{n} (u, w_\nu) w_\nu \quad \xrightarrow{n\to+\infty} \quad u \quad \text{in} \quad D(A^\alpha)$$

that is,

$$\lim_{n\to+\infty} \sum_{\nu=1}^{n} (u, w_\nu) w_\nu = u;\quad \forall\, u \in D(A^\alpha),\ 0 \leq \alpha \leq 1. \tag{54}$$

Set

$$V_m = [w_1, \ldots, w_m].$$

In V_m and by virtue of (54) consider the approximate problem

$$
\text{(AP)}\begin{cases} u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i \in V_m, \\ (u''_m(t), w_j) + M(a(u_m(t)))a(u_m(t), w_j) = (f(t), w_j); \quad j = 1, 2, \dots, m, \\ u_m(0) = u_{0m} = \sum_{i=1}^{m} (u_0, w_\nu) w_\nu \to u_0 \quad \text{in} \quad D(A^{3/4}), \\ u'_m(0) = u_{1m} = \sum_{i=1}^{m} (u_1, w_\nu) w_\nu \to u_1 \quad \text{in} \quad D(A^{1/4}). \end{cases}
$$

Whence

$$
\begin{cases} g''_{jm}(t) + M\left(\sum_{i=1}^{m} g^2_{im}(t) \lambda_i^2 \right) g_{jm} \lambda_j = (f(t), w_j) \\ g_{jm}(0) = (u_0, w_j) \\ g'_{jm}(0) = (u_1, w_j), \qquad j = 1, \dots, m, \end{cases} \tag{55}
$$

that is,

$$
\begin{cases} \begin{bmatrix} g''_{1m}(t) \\ \vdots \\ g''_{mm}(t) \end{bmatrix} + \begin{pmatrix} M\left(\sum_{j=1}^{m} g^2_{jm}(t) \lambda_j^2 \right) g_{1m} \lambda_1 \\ \vdots \\ M\left(\sum_{j=1}^{m} g^2_{jm}(t) \lambda_j^2 \right) g_{mm} \lambda_m \end{pmatrix} = \begin{pmatrix} (f(t), w_1) \\ \vdots \\ (f(t), w_m) \end{pmatrix} \\ g_{jm}(0) = \begin{pmatrix} (u_0, w_1) \\ \vdots \\ (u_0, w_m) \end{pmatrix}; \qquad g'_{jm}(0) = \begin{pmatrix} (u_1, w_1) \\ \vdots \\ (u_1, w_m) \end{pmatrix}. \end{cases} \tag{56}
$$

Set

$$
z(t) = \begin{pmatrix} g_{1m}(t) \\ \vdots \\ g_{mm}(t) \end{pmatrix};
$$

$$
A(z(t)) = \begin{pmatrix} M\left(\sum_{j=1}^{m} g^2_{jm}(t) \lambda_j^2 \right) g_{1m} \lambda_1 & 0 & \dots & 0 \\ 0 & M\left(\sum_{j=1}^{m} g^2_{jm}(t) \lambda_j^2 \right) g_{2m} \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \dots M\left(\sum_{j=1}^{m} g^2_{jm}(t) \lambda_j^2 \right) g_{mm} \lambda_m & \end{pmatrix}
$$

and

$$
F(t) = \begin{pmatrix} (f(t), w_1) \\ \vdots \\ (f(t), w_m) \end{pmatrix}; \quad z_0 = \begin{pmatrix} (u_0, w_1) \\ \vdots \\ (u_0, w_m) \end{pmatrix} \text{ and } z_1 = \begin{pmatrix} (u_1, w_1) \\ \vdots \\ (u_1, w_m) \end{pmatrix}. \tag{57}
$$

From (56) and (57) it follows that

$$\begin{cases} z''(t) + A(z(t)) \cdot z(t) = F(t) \\ z(0) = z_0\,; \;\; z'(0) = z_1. \end{cases} \tag{58}$$

Set

$$Y_1(t) = z'(t); \quad Y_2(t) = z(t) \tag{59}$$

and

$$Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} \tag{60}$$

In this way, it follows from (58), (59) and (60) that

$$Y'(t) = \begin{bmatrix} Y_1'(t) \\ Y_2'(t) \end{bmatrix} = \begin{bmatrix} z''(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} F(t) - A(z(t)).z(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} F(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -A(z(t)) \\ I & 0 \end{bmatrix} \begin{bmatrix} z'(t) \\ z(t) \end{bmatrix}.$$

Thus

$$\begin{cases} Y'(t) = \mathcal{F}(t) + \mathcal{A}(Y(t)).Y(t) \\ Y(0) = Y_0 \end{cases} \tag{61}$$

where

$$\mathcal{F}(t) = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}, \quad \mathcal{A}(Y(t)) = \begin{bmatrix} 0 & -A(z(t)) \\ I & 0 \end{bmatrix} \quad \text{and} \quad Y_0 = \begin{bmatrix} z_1 \\ z_0 \end{bmatrix}.$$

Define the following function

$$\begin{aligned} h\colon [0,T] \times \mathbb{R}^{2m} &\to \mathbb{R}^{2m} \\ (t,y) &\to h(t,y) = \mathcal{F}(t) + \mathcal{A}(y).y \end{aligned} \tag{62}$$

where

$$\mathcal{A}(y) = \begin{bmatrix} 0 & -A(\bar{y}) \\ I & 0 \end{bmatrix} \text{ where } \bar{y} = (y_{m+1}, \ldots, y_{2m}) \text{ if } y = (y_1, \ldots, y_m, y_{m+1}, \ldots, y_{2m}).$$

Then

$$\begin{cases} y'(t) = h(t, y(t)) \\ y(0) = Y_0. \end{cases} \tag{63}$$

Note that

(i) For each fixed $y \in \mathbb{R}^{2m}$, $h(t,y)$ is measurable in t, since $\mathcal{F}(t)$ is measurable since $f \in L^2(0,T; D(A^{1/4})$ and therefore the coordinate functions $(f(t), w_j)$ are measurable.

(ii) For a.e. $t \in [0,T]$, the map $h(t,y)$ is continuous in y because the product $\mathcal{A}(y) \cdot y$ is. We observe that the continuity of $\mathcal{A}(y).y$ comes from the fact that $\mathcal{A}(y)$ is continuous and the continuity of this results from the fact that $M \in C^1(\mathbb{R}_+)$.

(iii) Let K be a compact set of $[0,T] \times \mathbb{R}^{2m}$. Thus, $\forall\,(t,y) \in K$ we have that $\exists\, c > 0$ such that

$$||h(t,y)||_{2m} \leq ||\mathcal{F}(t)||_{2m} + ||\mathcal{A}(y)||\;||y||_{2m} \leq ||\mathcal{F}(t)||_{2m} + c.$$

Since $||\mathcal{F}(t)||_{2m} \in L^1(\text{proj}_t\, K)$, it results from Carathéodory's Theorem that the system of O.D.E. given in (63) admits a local solution $y(t)$ in some interval $[0, t_m)$, such that $y(t)$ is absolutely continuous and $y'(t)$ exists a.e. in $[0, t_m)$. It follows, that the system (58) has a local solution $z(t)$ in the same interval considered and from (59) and (60) it follows that $z(t)$ and $z'(t)$ are absolutely continuous and z'' exists a.e. in $[0, t_m)$. Finally, from (57) we conclude that the functions $g_{jm}(t)$, $g'_{jm}(t)$ are absolutely continuous and $g''_{jm}(t)$ exists a.e. in $[0, t_m)$, verifying the system (55). The a priori estimates will serve to extend $g_{jm}(t)$ and therefore $u_m(t)$, to the whole interval $[0, T]$.

2ª Step: A Priori Estimates

It is worth observing that from (50) and from the fact that $(u_m) \subset D(A)$ everything that was done formally remains valid in the interval $[0, t_m)$, that is, from (22) we obtain

$$|u'_m(t)|^2 + ||u_m(t)||^2 \leq c; \quad \forall\, t \in [0, t_m) \text{ and } \forall\, m \in \mathbb{N}. \tag{64}$$

From this it follows that

$$c \geq ((u_m(t), u_m(t))) = \sum_{j=1}^{m} g_{jm}^2(t) \cdot \lambda_j \geq \left(\sum_{j=1}^{m} g_{jm}^2(t) \right) \lambda_1$$

that is,

$$|z(t)|_m^2 = \sum_{j=1}^{m} g_{jm}^2(t) \leq c_1 = \frac{c}{\lambda_1}; \quad \forall\, t \in [0, t_m). \tag{65}$$

Also,

$$c \geq (u'_m(t), u'_m(t)) = \sum_{j=1}^{m} g'^2_{jm}(t)$$

whence

$$|z'(t)|^2 = \sum_{j=1}^{m} |g'_{jm}(t)|^2 \leq c. \tag{66}$$

From (65) and (66) we conclude that

$$|Y(t)|_{2m}^2 = |Y_1(t)|_m^2 + |Y_2(t)|_m^2 = |z'(t)|_m^2 + |z(t)|_m^2 \leq K; \quad \forall\, t \in [0, t_m) \text{ and } \forall\, m \in \mathbb{N}.$$

Given this last inequality, we can prolong $Y(t)$ to the whole interval $[0, T]$. It follows from this that $g_{jm}(t)$ and therefore $u_m(t)$ can be prolonged to the whole interval $[0, T]$. Thus, we can retrace the same calculations of the 1ª a priori estimate and obtain as in (64)

$$|u'_m(t)|^2 + ||u_m(t)||^2 \leq c; \quad \forall\, t \in [0, T] \text{ and } \forall\, m \in \mathbb{N}. \tag{67}$$

However, despite extending u_m to the whole interval $[0, T]$, the second a priori estimate is only valid in an interval $[0, T_0]$, as we obtain in (45), that is,

$$|A^{1/4}\, u'_m(t)|^2 + |A^{3/4}\, u_m(t)|^2 \leq c_0\,; \quad \forall\, t \in [0, T_0] \text{ and } \forall\, m \in \mathbb{N} \tag{68}$$

where $T_0 < T^* = \dfrac{1}{k_4} \ell n \left(1 + \dfrac{1}{k_3} \right)$.

It follows from (67) and (68) that

$$\begin{aligned}
&(u_m) \quad \text{is bounded in} \quad L^\infty(0, T_0, D(A^{3/4})),\\
&(u_m) \quad \text{is bounded in} \quad L^\infty(0, T, V),\\
&(u'_m) \quad \text{is bounded in} \quad L^\infty(0, T_0, D(A^{1/4})),\\
&(u'_m) \quad \text{is bounded in} \quad L^\infty(0, T; H).
\end{aligned}$$

Consequently, $\exists\,(u_\nu) \subset (u_m)$ such that

$$u_\nu \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T_0,D(A^{3/4})) \tag{69}$$

$$u_\nu \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T;V) \tag{70}$$

$$u'_\nu \overset{*}{\rightharpoonup} u' \quad \text{in} \quad L^\infty(0,T_0,D(A^{1/4})) \tag{71}$$

$$u'_\nu \rightharpoonup u' \quad \text{in} \quad L^\infty(0,T;H) \tag{72}$$

3ª Step: Passage to the Limit

Let $j \in \mathbb{N}$ and $\nu \geq j$. Consider $\theta \in \mathcal{D}(0,T_0)$. Multiplying $(\text{AP})_1$ by θ and integrating over $[0,T_0]$ results that

$$\begin{aligned} -\int_0^{T_0} (u'_\nu(t),w_j)\theta'(t)dt + \int_0^{T_0} M(a(u_\nu(t)))a(u_\nu(t),w_j)\theta(t)dt \\ = \int_0^{T_0} (f(t),w_j)\theta(t)dt. \end{aligned} \tag{73}$$

From (72) we obtain

$$\int_0^{T} (u'_\nu(t),w_j)\theta'(t)dt \xrightarrow{\nu\to+\infty} \int_0^{T_0} (u'(t),w_j)\theta'(t)dt. \tag{74}$$

Analysis of the Nonlinear Term

From (15), in particular, for $\alpha = \dfrac{1}{4}$ and $\rho = \dfrac{1}{2}$ we have that

$$D(A^{3/4}) \overset{c}{\hookrightarrow} D(A^{1/2}). \tag{75}$$

Set

$$B_0 = D(A^{3/4}); \quad B = D(A^{1/2}); \quad B_1 = H$$

and consider the space

$$W = \{v \in L^2(0,T_0,B_0);\ v' \in L^2(0,T_0,B_1)\}$$

endowed with the topology

$$||v||_W = ||v||_{L^2(0,T_0,B_0)} + ||v'||_{L^2(0,T_0,B_1)}\,.$$

From (75) due to the Aubin-Lions Theorem we have that:

$$W \overset{c}{\hookrightarrow} L^2(0,T_0,D(A^{1/2})).$$

It results from this and the above that there will exist a subsequence of (u_ν), which we will continue denoting by (u_ν), such that

$$u_\nu \to u \quad \text{strongly in} \quad L^2(0,T_0,V) \tag{76}$$

and then

$$||u_\nu(t)||^2 \to ||u(t)||^2 \quad \text{in} \quad L^1(0,T_0).$$

On the other hand, since $M \in C^1(\mathbb{R}_+)$ we have that:

$$M(a(u_\nu(t))) \to M(a(u(t))) \quad \text{in} \quad L^2(0,T_0). \tag{77}$$

Indeed, since $M \in C^1(\mathbb{R}_+)$ we have for a.e. $t \in [0,T_0]$

$$M(a(u_\nu(t))) - M(a(u(t))) = \int_{a(u_\nu(t))}^{a(u(t))} M'(\xi)d\xi. \tag{78}$$

However, from (67) we have that $\exists\, c_1 > 0$ such that

$$a(u_\nu(t)) \le c_1\,; \quad \forall\, t \in [0,T_0] \text{ and } \forall\, \nu \in \mathbb{N},$$

and from the fact that $u \in L^\infty(0,T_0;V)$ it follows that $\exists\, c_2 > 0$ such that

$$a(u(t)) \le c_2\,; \quad \text{for a.e.} \quad t \in [0,T_0].$$

Taking $c = \max\{c_1, c_2\}$ we have

$$0 \le a(u_\nu(t)), a((u(t)) \le c; \quad \text{a.e.} \quad t \in [0,T_0] \text{ and } \forall\, \nu \in \mathbb{N}, \tag{79}$$

that is,

$$a(u_\nu(t)), a(u(t)) \in [0,c]; \quad \text{a.e.} \quad t \in [0,T_0] \text{ and } \nu \in \mathbb{N}.$$

On the other hand, since $M' \in C^0(\mathbb{R}_+)$, $\exists\, L > 0$ such that $|M'(\xi)| \le L$; $\forall\, \xi \in [0,c]$.

Thus, from (78) it follows that:

$$|M(a(u_\nu(t))) - M(a(u(t)))| \le L|a(u_\nu(t)) - a(u(t))|;\ \forall\, \nu \in \mathbb{N} \text{ and a.e. } t \in [0,T_0]. \tag{80}$$

So, from (80) it comes that:

$$\begin{aligned}
&|M(a(u_\nu(t))) - M(a(u(t)))|^2 \le L^2|\,||u_\nu(t)||^2 - ||u(t)||^2\,|^2 \\
&= L^2|(||u_\nu(t)|| - ||u(t)||)(||u_\nu(t)|| + ||u(t)||)|^2 \\
&= L^2|\,||u_\nu(t)|| - ||u(t)||\,|^2\,[\,||u_\nu(t)|| + ||u(t)||\,]^2 \\
&\le 4cL^2|\,||u_\nu(t)|| - ||u(t)||\,|^2 \le 4cL||u_\nu(t) - u(t)||^2,
\end{aligned}$$

$\forall\, \nu \in \mathbb{N}$ and a.e. $t \in [0,T_0]$.

Integrating the last inequality from 0 to T_0 it comes that

$$\begin{aligned}
&\int_0^{T_0} |M(a(u_\nu(t))) - M(a(u(t)))|^2\,dt \\
&\le 4cL^2 \int_0^{T_0} ||u_\nu(t) - u(t)||^2\,dt = 4cL^2||u_\nu - u||^2_{L^2(0,T_0;V}.
\end{aligned}$$

But from (76), we have that the right side of the inequality above converges to zero when $\nu \to +\infty$. Thus

$$\int_0^{T_0} |M(a(u_\nu(t))) - M(a(u(t)))|^2\,dt \to 0 \text{ when } \nu \to +\infty$$

which proves the desired result.

On the other hand, from (76) we also have

$$u_\nu \rightharpoonup u \quad \text{in} \quad L^2(0,T_0;V).$$

Whence

$$\langle \eta, u_\nu \rangle \to \langle \eta, u \rangle; \quad \forall\, \eta \in L^2(0,T_0,V').$$

In particular, if $\eta = Aw_j\, v$ where $v \in L^2(0,T_0)$ results that

$$\int_0^{T_0} (Aw_j, u_\nu(t))v(t)dt \xrightarrow{\nu\to+\infty} \int_0^{T_0} (Aw_j, u(t))v(t)dt,$$

that is,

$$\int_0^{T_0} a(u_\nu(t), w_j)v(t)dt \xrightarrow{\nu\to+\infty} \int_0^{T_0} a(u(t), w_j)v(t)dt.$$

Thus,

$$a(u_\nu(t), w_j) \rightharpoonup a(u(t), w_j) \quad \text{in} \quad L^2(0,T_0). \tag{81}$$

It follows from (77) that:

$$\omega\, M(a(u_\nu(t))) \to \omega\, M(a(u(t))) \text{ in } L^2(0,T_0);\ \forall\, w \in L^\infty(0,T_0). \tag{82}$$

From (81) and (82) we obtain

$$(\omega M(a(u_\nu(t))), a(u_\nu(t), w_j))_{L^2(0,T_0)} \xrightarrow{\nu\to+\infty} (\omega M(a(u(t))), a(u(t), w_j))_{L^2(0,T_0)}\,,$$

for all $w \in L^\infty(0,T_0)$.

In particular, for $w = \theta \in \mathcal{D}(0,T_0)$ it follows that

$$\int_0^{T_0} M(a(u_\nu(t)))a(u_\nu(t), w_j)\theta(t)dt \to \int_0^{T_0} M(a(u(t)))a(u(t), w_j)\theta(t)dt. \tag{83}$$

From (73), (74) and (83) in the limit situation, we obtain

$$\begin{aligned} -\int_0^{T_0} (u'(t), w_j)\theta'(t)dt + \int_0^{T_0} M(a(u(t)))a(u(t), w_j)\theta(t)dt \\ = \int_0^{T_0} (f(t), w_j)\theta(t)dt,\ \ \forall\, j \in \mathbb{N}. \end{aligned} \tag{84}$$

Since the system $(w_\nu)_\nu$ is complete in V it follows that

$$\begin{aligned} -\int_0^{T_0} (u'(t), v)\theta'(t)dt + \int_0^{T_0} M(a(u(t)))a(u(t), v)\theta(t)dt = \\ = \int_0^{T_0} (f(t), v)\theta(t)dt; \quad \forall\, v \in V, \end{aligned} \tag{85}$$

or even,

$$\frac{d}{dt}\,(u'(t), v) + M(a(u(t))a(u(t), v) = (f(t), v) \quad \text{in} \quad \mathcal{D}'(0,T_0),\ \ \forall\, v \in V. \tag{86}$$

Furthermore, identifying H with its dual comes that

$$\langle u, v \rangle_{V',V} = a(u,v); \quad \forall\, u \in H \text{ and } \forall\, v \in V.$$

In what follows $\widetilde{A}\colon V \to V'$ will represent the isometric extension of the operator $A\colon D(A) \to H$ defined by

$$\langle \widetilde{A}u, v\rangle = a(u,v) = ((u,v)); \quad \forall\, u, v \in V.$$

From the above and from (85) we can write

$$\Big\langle -\int_0^{T_0} u'(t)\theta'(t)dt, v\Big\rangle + \Big\langle \int_0^{T_0} M(a(u(t)))\widetilde{A}u(t)\theta(t)dt, v\Big\rangle = \Big\langle \int_0^{T_0} f(t)\theta(t)dt, v\Big\rangle; \quad \forall\, v \in V \text{ and } \forall\, \theta \in \mathcal{D}(0,T_0),$$

that is,

$$u'' + M(a(u))\widetilde{A}u = f \quad \text{in} \quad \mathcal{D}'(0,T_0;V'). \tag{87}$$

But, since $f \in L^2(0,T_0;D(A^{1/4}))$ and $M(a(u))\widetilde{A}u \in L^\infty(0,T_0,V')$, (since $a(u) \in L^\infty(0,T_0)$ and, therefore, $M(a(u)) \in L^\infty(0,T_0)$) we have from (87) that

$$u'' \in L^2(0,T_0,V') \tag{88}$$

and

$$u'' + M(a(u))\widetilde{A}u = f \quad \text{in} \quad L^2(0,T_0,V'). \tag{89}$$

4ª Step: Initial Conditions

Note initially that

$$u \in C^0([0,T_0],D(A^{1/4})) \cap C_s([0,T_0];D(A^{3/4}))$$
$$u' \in C^0([0,T_0];V') \cap C_s([0,T_0],D(A^{1/4})),$$

making sense therefore to speak of $u(0)$, $u'(0)$, $u(T)$ and $u'(T)$.

(i) $u(0) = u_0$

Let $\theta \in C^1([0,T_0])$ such that $\theta(0) = 1$ and $\theta(T_0) = 0$. Consider $v \in H$; then $v\theta \in L^2(0,T_0,H)$ and, consequently, from (72) comes that

$$\int_0^{T_0} (u_\nu'(t),v)\theta(t)dt \xrightarrow{\nu\to+\infty} \int_0^{T_0} (u'(t),v)\theta(t)dt.$$

Integrating by parts

$$-(u_\nu(0),v) - \int_0^{T_0} (u_\nu(t),v)\theta(t)dt \xrightarrow{\nu\to+\infty} -(u(0),v) - \int_0^{T_0} (u(t),v)\theta'(t)dt.$$

Since

$$\int_0^{T_0} (u_\nu(t),v)\theta(t)dt \xrightarrow{\nu\to+\infty} \int_0^{T_0} (u(t),v)\theta(t)dt$$

it results that

$$(u_{0\nu},v) \to (u(0),v); \quad \forall\, v \in H. \tag{90}$$

But, $u_{0\nu} \to u_0$ in $D(A^{3/4}) \hookrightarrow H$. Thus,

$$(u_{0\nu},v) \to (u_0,v); \quad \forall\, v \in H. \tag{91}$$

From (90) and (91) we conclude that

$$u(0) = u_0 \,. \tag{92}$$

(ii) $u'(0) = u_1$
Let $\delta > 0$. Consider the auxiliary function

$$\theta_\delta(t) = \begin{cases} -\frac{t}{\delta} + 1; & \text{if} \quad 0 \le t \le \delta \\ \ \ 0; & \text{if} \quad \delta \le t \le T_0. \end{cases}$$

Let $j \in \mathbb{N}$ and consider $\nu \ge j$. Multiplying both sides of $(\text{AP})_2$ by θ_δ and integrating in $[0, T_0]$ results that

$$\begin{aligned} \int_0^\delta (u_\nu''(t), w_j)\theta_\delta(t)dt + \int_0^\delta M(a(u_\nu(t)))a(u_\nu(t), w_j)\theta_\delta(t)dt \\ = \int_0^\delta (f(t), w_j)\theta_\delta(t)dt. \end{aligned} \tag{93}$$

Recalling that if $g \in H^1(0, T_0)$ we have

$$\big\langle \big(g|_{[0,\delta]}\big)', \theta \big\rangle = \big\langle g'|_{[0,\delta]}, \theta \big\rangle; \quad \forall\, \theta \in \mathcal{D}(0, \delta),$$

then since $(u_\nu'(t), w_j) \in H^1(0, T_0)$ comes that

$$\frac{d}{dt}\,(u_\nu'(t), w_j) = \frac{d}{dt}\,\big(u_\nu'|_{[0,\delta]}, w_j\big) = \left(\frac{d}{dt}\,\big(u_\nu'|_{[0,\delta]}\big), w_j\right) = \big(u_\nu''|_{[0,\delta]}, w_j\big).$$

Furthermore, since $\theta_\delta \in C^1([0, \delta])$ then the derivative of θ_δ in the sense of distributions in $[0, \delta]$ coincides with the classical derivative. Thus,

$$\frac{d}{dt}\,\big[(u_\nu'(t), w_j)\theta_\delta(t)\big] = (u_\nu''(t), w_j)\theta_\delta(t) + (u_\nu'(t), w_j)\theta_\delta'(t).$$

Integrating by parts the first integral of (93) comes

$$\begin{aligned} &\overbrace{(u_\nu'(t), w_j)\theta_\delta(\delta)}^{=0} - (u_\nu'(0), w_j)\underbrace{\theta_\delta(0)}_{=1} - \int_0^\delta (u_\nu'(t), w_j)\theta_\delta'(t)dt \\ &+ \int_0^\delta M(a(u_\nu(t)))a(u_\nu(t), w_j)\theta_\delta(t)dt = \int_0^\delta (f(t), w_j)\theta_\delta(t)dt, \end{aligned}$$

that is,

$$\begin{aligned} -(u_\nu'(0), w_j) - \int_0^\delta (u_\nu'(t), w_j)\theta_\delta'(t)dt + \int_0^\delta M(a(u_\nu(t))a(u_\nu(t), w_j)\theta_\delta(t)dt \\ = \int_0^\delta (f(t), w_j)\theta_\delta(t)dt. \end{aligned}$$

Taking the limit in ν in the expression above comes that

$$\begin{aligned} -(u_1, w_j) - \int_0^\delta (u'(t), w_j)\theta_\delta'(t)dt + \int_0^\delta M(a(u(t))a(u(t), w_j)\theta_\delta(t)dt \\ = \int_0^\delta (f(t), w_j)\theta_\delta(t)dt. \end{aligned}$$

Observing that in $[0,\delta], \theta_\delta'(t) = -\dfrac{1}{\delta}$ and $\theta_\delta(t) = -\dfrac{t}{\delta} + 1$ we obtain

$$\begin{aligned} &-(u_1, w_j) + \frac{1}{\delta}\int_0^\delta (u'(t), w_j)dt + \int_0^\delta M(a(u(t))a(u(t), w_j)dt \\ &-\frac{1}{\delta}\int_0^\delta M(u(t)))a(u(t), w_j)t\,dt = \int_0^\delta (f(t), w_j)dt - \frac{1}{\delta}\int_0^\delta (f(t), w_j)t\,dt. \end{aligned} \tag{94}$$

Observing that

$$\left|\frac{1}{\delta}\int_0^\delta g(t)t\,dt\right| \le \frac{1}{\delta}\int_0^\delta |g(t)|t\,dt \le \frac{1}{\delta}\int_0^\delta |g(t)|\delta\,dt = \int_0^\delta |g(t)|dt;\ \forall\, g \in L^1(0, T_0)$$

we have that

$$\frac{1}{\delta}\int_0^\delta g(t)t\,dt \to 0 \quad \text{when} \quad \delta \to 0^+$$

since

$$\int_0^\delta |g(t)|dt \to 0 \quad \text{when} \quad \delta \to 0^+.$$

Furthermore, since $u' \in C_s([0,T_0]; D(A^{1/4})) \subset C_s([0,T]; H)$ we have that every $t \in [0,T]$ is a Lebesgue point of the function $(u'(t), w_j)$ and, therefore, in particular for $t = 0$, we have that

$$\frac{1}{\delta}\int_0^\delta (u'(t), w_j)dt \xrightarrow{\delta \to 0^+} (u'(0), w_j).$$

In this way, taking the limit in (94) when $\delta \to 0^+$ we obtain

$$-(u_1, w_j) + (u'(0), w_j) = 0; \quad \forall\, j \in \mathbb{N}$$

that is,

$$(u'(0), w_j) = (u_1, w_j); \quad \forall\, j \in \mathbb{N}.$$

By the totality of the $(w_j)_{\nu\in\mathbb{N}}$ in H it follows that:

$$u'(0) = u_1\,. \qquad \square \tag{95}$$

APPENDIX

1. Let I be a bounded interval of the real line and $f \in W^{1,p}(I), \quad 1 \leq p \leq +\infty$. Then f is absolutely continuous.

Indeed, observe that $W^{1,p}(I) \hookrightarrow W^{1,1}(I)$ since I is bounded. Thus, if $f \in W^{1,p}(I)$ then $f \in W^{1,1}(I)$ and, therefore, $f, f' \in L^1(I)$.

Let us define

$$v(x) = \int_a^x f'(\xi)d\xi; \quad a \in I.$$

Then, v is absolutely continuous, v' (Dini derivative) exists a.e. and $v'(x) = f'(x)$ a.e.. Furthermore, the Dini derivative of v and the derivative in the sense of distributions coincide. Thus

$$(v - f)' = 0 \quad \text{(derivative in the sense of distributions)}$$

and, therefore, $v - f = \text{ constant } = c$, a.e. in I. Due to the fact that $W^{1,1}(I) \hookrightarrow C^0(\overline{I})$ we have that $f \in C^0(\overline{I})$ and then $v - f \in C^0(\overline{I})$. Whence,

$$v - f = c \quad \text{in} \quad I,$$

that is,

$$f = v + c \quad \text{in} \quad I.$$

Since v is absolutely continuous, $v + c$ is absolutely continuous and in this way f is also.

2. Let f be absolutely continuous on $[a, b]$ such that $f' \in L^1(a, b)$ where f' represents the Dini derivative. Let $\varphi \in \mathcal{D}(a, b)$, then $(f\varphi)$ is absolutely continuous and

$$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_a^b f(\xi)\varphi'(\xi)d\xi = -(f\varphi)\Big|_a^b + \int_a^b f'(\xi)\varphi(\xi)d\xi = \langle f', \varphi\rangle.$$

Thus, the distributional derivative of f coincides with the classical one.

In particular, considering the approximate problem

$$(u''_m(t), w_j) + ((u_m(t), w_j)) = (f(t), w_j); \quad f \in L^2(0, T; L^2(\Omega)),$$

where $(w_\nu)_\nu$ is a basis of $H^1_0(\Omega)$ orthonormal in $L^2(\Omega)$ and $u_m(t) = \sum_{i=1}^{m} g_{im}(t)w_i$. By Carathéodory's Theorem we have that

$$g_{jm}(t), g'_{jm}(t) \quad \text{are absolutely continuous and} \quad g''_{jm}(t) \text{ exists a.e.}$$

We claim that $g''_{jm}(t) \in L^2(0, T)$. Indeed, let $j = 1, \cdots, m$, we have

$$g''_{jm}(t) = \sum_{i=1}^{m} g''_{im}(t)(w_i, w_j) = (u''_m(t), w_j) = (f(t), w_j) - ((u_m(t), w_j)) \in L^2(0, T).$$

Thus, g'_{jm} is absolutely continuous and $g''_{jm} \in L^2(0, T)$. From what was seen previously, g''_{jm} in the classical sense coincides with g''_{jm} in the sense of distributions. Furthermore, since g_{jm}, g'_{jm} and $g''_{jm} \in L^2(0, T)$ then $g_{jm} \in H^2(0, T)$. □

APPENDIX

1. Let I be a bounded interval of the real line and $f \in W^{1,p}(I), \quad 1 \leq p \leq +\infty$. Then f is absolutely continuous.

Indeed, observe that $W^{1,p}(I) \hookrightarrow W^{1,1}(I)$ since I is bounded. Thus, if $f \in W^{1,p}(I)$ then $f \in W^{1,1}(I)$ and, therefore, $f, f' \in L^1(I)$.

Let us define

$$v(x) = \int_a^x f'(\xi)d\xi; \quad a \in I.$$

Then, v is absolutely continuous, v' (Dini derivative) exists a.e. and $v'(x) = f'(x)$ a.e.. Furthermore, the Dini derivative of v and the derivative in the sense of distributions coincide. Thus

$$(v - f)' = 0 \quad \text{(derivative in the sense of distributions)}$$

and, therefore, $v - f = \text{ constant } = c$, a.e. in I. Due to the fact that $W^{1,1}(I) \hookrightarrow C^0(\overline{I})$ we have that $f \in C^0(\overline{I})$ and then $v - f \in C^0(\overline{I})$. Whence,

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Since v is absolutely continuous, $v + c$ is absolutely continuous and in this way f is also.

2. Let f be absolutely continuous on $[a, b]$ such that $f' \in L^1(a, b)$ where f' represents the Dini derivative. Let $\varphi \in \mathcal{D}(a, b)$, then $(f\varphi)$ is absolutely continuous and

$$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = -\int_a^b f(\xi)\varphi'(\xi)d\xi = -(f\varphi)\Big|_a^b + \int_a^b f'(\xi)\varphi(\xi)d\xi = \langle f', \varphi \rangle.$$

Thus, the distributional derivative of f coincides with the classical one.

In particular, considering the approximate problem

$$(u_m''(t), w_j) + ((u_m(t), w_j)) = (f(t), w_j); \quad f \in L^2(0, T; L^2(\Omega)),$$

where $(w_\nu)_\nu$ is a basis of $H_0^1(\Omega)$ orthonormal in $L^2(\Omega)$ and $u_m(t) = \sum_{i=1}^{m} g_{im}(t)w_i$. By Carathéodory's Theorem we have that

$$g_{jm}(t), g_{jm}'(t) \quad \text{are absolutely continuous and} \quad g_{jm}''(t) \text{ exists a.e.}$$

We claim that $g_{jm}''(t) \in L^2(0, T)$. Indeed, let $j = 1, \cdots, m$, we have

$$g_{jm}''(t) = \sum_{i=1}^{m} g_{im}''(t)(w_i, w_j) = (u_m''(t), w_j) = (f(t), w_j) - ((u_m(t), w_j)) \in L^2(0, T).$$

Thus, g_{jm}' is absolutely continuous and $g_{jm}'' \in L^2(0, T)$. From what was seen previously, g_{jm}'' in the classical sense coincides with g_{jm}'' in the sense of distributions. Furthermore, since g_{jm}, g_{jm}' and $g_{jm}'' \in L^2(0, T)$ then $g_{jm} \in H^2(0, T)$. □

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