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Preface

The theory of semigroups of operators is undoubtedly one of the most powerful and elegant tools in modern functional analysis for the study of evolution equations. From the classical heat diffusion to complex wave propagation phenomena and quantum mechanics, the abstract language of semigroups allows us to unify diverse problems under a common framework, providing robust methods for establishing existence, uniqueness, and asymptotic behavior of solutions.

This book, *Linear and Nonlinear Semigroups and Applications*, is the result of years of teaching and research at the State University of Maringá. It has been conceived to serve both as a textbook for graduate students in Mathematics and as a reference for researchers interested in the analysis of partial differential equations.

The text is structured to guide the reader from the foundations to the frontiers of the theory. We begin with a review of differential and integral calculus in Banach spaces, setting the stage for the theory of C_0-semigroups of linear operators. Here, the classical theorems of Hille-Yosida and Lumer-Phillips are presented not just as abstract results, but as operational tools essential for solving linear evolution problems.

However, nature is inherently nonlinear. A distinctive feature of this volume is the substantial treatment dedicated to nonlinear analysis. We introduce the theory of monotone and accretive operators, multivalued mappings, and the crucial Crandall-Liggett Theorem, which generalizes the generation of semigroups to the nonlinear setting. This transition is handled with care, highlighting the geometric and analytic subtleties that arise when linearity is abandoned.

Throughout the book, the abstract theory is constantly motivated by and applied to concrete problems. We explore in detail the heat equation, the wave equation with various types of damping (frictional, viscoelastic, and boundary damping), and the Schrödinger equation. Special attention is given to the regularity of solutions and to the concept of weak and generalized solutions, bridging the gap between abstract functional analysis and applied mathematics.

We assume the reader has a background in basic functional analysis and Lebesgue integration theory. Our goal is that, by the end of this journey, the reader will not only understand the "how" and "why" of semigroup theory but will also be equipped to apply these powerful techniques to their own research problems.

We are grateful to our colleagues and students whose questions and feedback over the years have helped shape this material. We hope this book serves as a solid foundation for those venturing into the vast and dynamic field of evolution equations.

Maringá, 2025

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Contents

Chapter 1

Linear Semigroups

1.1 A Review of Differential and Integral Calculus in Banach Spaces

In this section we recall some preliminary results concerning Differential and Integral Calculus in Banach spaces , which will be of fundamental importance throughout this text. We start with the notion of series in Banach spaces. Throughout this section, E will denote a Banach space with norm $\|\cdot\|$.

Definition 1.1 *Let (x_n) be a sequence in E. From it we form a new sequence (s_n) whose elements are the sums*

$$s_1 = x_1, \;\; s_2 = x_1 + x_2, \;\; s_n = x_1 + \cdots + x_n,$$

which we shall call the partial sums of the series $\sum_{n=1}^{\infty} x_n$. If the limit

$$s = \lim s_n = \lim_{n\to\infty} (x_1 + \cdots + x_n),$$

exists, we say that the series $\sum_{n=1}^{\infty} x_n$ is convergent and the limit s is called the sum of the series. If the sequence of partial sums does not converge, we say that the series $\sum_{n=1}^{\infty} x_n$ is divergent. We say that a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent in E if $\sum_{n=1}^{\infty} \|x_n\|$ converges.

Proposition 1.2 *Every absolutely convergent series $\sum_{n=1}^{\infty} x_n$ is convergent.*

Proof: Let $s_n = \sum_{k=1}^{n} x_k$, $n \in \mathbb{N}$, be the sequence of partial sums of the given series. Since E is a Banach space, it suffices to prove that

$$(s_n) \text{ is a Cauchy sequence in } E. \tag{1.1.1}$$

Indeed, let $\varepsilon > 0$ and consider $m, n \in \mathbb{N}$ with $m > n$. We have

$$\begin{aligned} \|s_m - s_n\| &= \left\| \sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k \right\| \\ &= \left\| \sum_{k=n+1}^{m} x_k \right\| \\ &\leq \sum_{k=n+1}^{m} \|x_k\| \\ &\leq \left| \sum_{k=1}^{m} \|x_k\| - \sum_{k=1}^{n} \|x_k\| \right| \\ &= |\tilde{s}_m - \tilde{s}_n|, \end{aligned}$$

where $\tilde{s}_n = \sum_{k=1}^{n} \|x_k\|$ is the n-th partial sum of the convergent series $\sum_{k=1}^{\infty} \|x_k\|$, which is therefore a Cauchy sequence in $\mathbb{R}$. Hence, for the given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, if $m, n \in \mathbb{N}$ with $m > n > n_0$, then

$$\|s_m - s_n\| \leq |\tilde{s}_m - \tilde{s}_n| < \varepsilon,$$

which proves the claim in (1.1.1). □

Another very important result for determining convergence of series is the Weierstrass test, which we state next.

Proposition 1.3 (Comparison Test) *Let $M_n \geq 0$ be such that $\|x_n\| \leq M_n$ for every $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ is convergent, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.*

Proof: Since $\|x_n\| \leq M_n$ for all $n \in \mathbb{N}$, it follows that

$$\sum_{n=1}^{\infty} \|x_n\| \leq \sum_{n=1}^{\infty} M_n.$$

By the comparison test for series of real numbers, it follows that $\sum_{n=1}^{\infty} \|x_n\|$ is convergent, since $\sum_{n=1}^{\infty} M_n$ is convergent. Hence, by definition, the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. □

Proposition 1.4 (Weierstrass M-test) *Suppose that $(E, \|\cdot\|)$ is a Banach space, (Y, d) is a metric space, and for each $n \in \mathbb{N}$, $f_n : Y \to E$ is a function. Assume that there exists a sequence (M_n) such that $\|f_n(y)\|_E \leq M_n$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} M_n < \infty$. Then $\overline{f}_N(y) = \sum_{n=1}^{N} f_n(y)$ converges absolutely and uniformly to $\overline{f}(y) = \sum_{n=1}^{\infty} f_n(y)$.*

Proof: Define $\overline{f}_N(y) = f_1(y) + \cdots + f_N(y)$. For $M > N$ we have

$$\begin{aligned} \|\overline{f}_M(y) - \overline{f}_N(y)\| &= \|f_{N+1}(y) + \cdots + f_M(y)\| \\ &\leq \sum_{k=N+1}^{M} M_k \qquad \text{for each } y \in Y. \end{aligned}$$

Since $\sum_{k=1}^{\infty} M_k$ is convergent, $(\overline{f}_N(y))$ is a Cauchy sequence in E. Thus there exists an element $\xi \in E$ with

$\xi = \lim_{N\to\infty} \overline{f}_N(y)$. Define $\overline{f}(y) = \xi$; this gives us a function $\overline{f} : Y \to E$. Now,

$$\begin{aligned} \|\overline{f}(y) - \overline{f}_N(y)\| &= \left\| \sum_{k=N+1}^{\infty} f_k(y) \right\| \\ &\leq \sum_{k=N+1}^{\infty} \|f_k(y)\| \\ &\leq \sum_{k=N+1}^{\infty} M_k. \end{aligned}$$

Since $\sum\limits_{k=1}^{\infty} M_k$ is convergent, given $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that $\sum\limits_{k=N+1}^{\infty} M_k < \varepsilon$ whenever $N \geq n_0$. Hence $\|\overline{f}(y) - \overline{f}_N(y)\| < \varepsilon$ for all $y \in Y$ whenever $N \geq n_0$. □

From now on, we denote by $\mathcal{L}(E, F)$ the family of bounded linear operators with domain E and range F, where E and F are Banach spaces, that is, the family of linear operators $A : E \to F$ such that

$$\|A\| = \sup_{x\in E, \|x\|\leq 1} \|Ax\| = \sup_{x\in E, \|x\|=1} \|Ax\|.$$

With this norm, $\mathcal{L}(E, F)$ is a Banach space. In the case where $E = F$ we simply write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$. If $A, B \in \mathcal{L}(E)$, the product of A and B is defined by $AB = A \circ B$.

An *algebra* A over a field $\mathbb{K}$ is a vector space over $\mathbb{K}$ such that for each ordered pair $(x, y) \in A \times A$ we can define a unique product $xy \in A$ with the following properties:

i) $(xy)z = x(yz)$

ii) $x(y + z) = xy + xz$

iii) $(x + y)z = xz + yz$

iv) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

for all $x, y, z \in A$ and every scalar $\alpha \in \mathbb{K}$.

It follows that $\mathcal{L}(E)$ is an algebra and that, for $A, B \in \mathcal{L}(E)$, we have $AB \in \mathcal{L}(E)$ and $\|AB\| \leq \|A\|\|B\|$, that is, $\mathcal{L}(E)$ is a Banach algebra.

Proposition 1.5 *Let $A \in \mathcal{L}(E)$. Then $\sum\limits_{n=0}^{\infty} \frac{A^n}{n!}$ is absolutely convergent in $\mathcal{L}(E)$. By analogy with Calculus, we define:*

$$\exp(A) = e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}, \quad \textit{with } A^0 = I.$$

Moreover, the following inequality holds:

$$\|e^A\| \leq e^{\|A\|}.$$

Proof: We apply the comparison test. First recall the following property, valid in $\mathcal{L}(E)$:

$$\|AB\| \leq \|A\|\,\|B\|, \quad \text{for all } A, B \in \mathcal{L}(E).$$

Thus, it suffices to note that, if $n \in \mathbb{N}$ and $A \in \mathcal{L}(E)$ is given, then

$$\left\| \frac{A^n}{n!} \right\| = \frac{1}{n!}\|A^n\| \leq \frac{1}{n!}\|A\|^n \leq M_n,$$

where $M_n = \frac{\|A\|^n}{n!}$. Since the series $\sum\limits_{n=0}^{\infty} \frac{\|A\|^n}{n!} = e^{\|A\|}$ converges, it follows from the comparison test that the series $\sum\limits_{n=0}^{\infty} \frac{A^n}{n!}$ is absolutely convergent and, moreover,

$$\|e^A\| = \left\| \sum_{n=0}^{\infty} \frac{A^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{\|A\|^n}{n!} = e^{\|A\|},$$

which completes the proof. □

Proposition 1.6 (Neumann's Theorem) *Let $A \in \mathcal{L}(E)$ with $\|A\| < 1$. Then the series $\sum\limits_{n=0}^{\infty} A^n$ converges to $(I-A)^{-1}$ in $\mathcal{L}(E)$ and, moreover,*

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}.$$

Proof: By Proposition 1.2, it suffices to show that the series is absolutely convergent. Note first that, since $\|A\| < 1$,

$$\sum_{n=0}^{\infty} \|A\|^n = \frac{1}{1-\|A\|}, \tag{1.1.2}$$

as this is a geometric series. By the comparison test, taking $M_n = \|A\|^n$, $n \in \mathbb{N}$, and observing that

$$\|A^n\| \leq \|A\|^n, \quad \text{for all } n \in \mathbb{N},$$

we obtain that the series $\sum\limits_{n=0}^{\infty} A^n$ is absolutely convergent and therefore convergent. Furthermore,

$$\left\| \sum_{n=0}^{\infty} A^n \right\| \leq \sum_{n=0}^{\infty} \|A^n\| \leq \sum_{n=0}^{\infty} \|A\|^n = \frac{1}{1-\|A\|}. \tag{1.1.3}$$

To conclude the proof, we show that $\sum\limits_{n=0}^{\infty} A^n = (I-A)^{-1}$. Indeed,

$$\begin{aligned} (I-A)\sum_{n=0}^{\infty} A^n &= \sum_{n=0}^{\infty} A^n - \sum_{n=1}^{\infty} A^n \\ &= A^0 + \sum_{n=1}^{\infty} A^n - \sum_{n=1}^{\infty} A^n = I. \end{aligned}$$

Also,

$$\begin{aligned} \sum_{n=0}^{\infty} A^n(I-A) &= \lim_{k\to\infty} \sum_{n=0}^{k} A^n(I-A) \\ &= \lim_{k\to\infty} (I - A^{k+1}) = I, \end{aligned}$$

since $\|A^{k+1}\| \leq \|A\|^{k+1} \to 0$ as $k \to \infty$. We conclude that $\sum\limits_{n=0}^{\infty} A^n = (I-A)^{-1}$, and from (1.1.2) and

(1.1.3) we deduce

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|},$$

which finishes the proof. □

Proposition 1.7 *Let $A \in \mathcal{L}(E)$ be such that $\|A\| < 1$. Then $\sum_{n=1}^{\infty} \frac{1}{n} A^n$ converges in $\mathcal{L}(E)$. We denote the limit of this series by* $\log(I-A)$.

Proof: We once again use the comparison test to verify that the given series is absolutely convergent and hence convergent. Indeed,

$$\sum_{n=1}^{\infty} \left\| \frac{1}{n} A^n \right\| \leq \sum_{n=1}^{\infty} \frac{\|A\|^n}{n} \leq \sum_{n=1}^{\infty} \|A\|^n,$$

and since the series on the right-hand side converges, as $\|A\| < 1$, the result follows. □

Let I be an interval in $\mathbb{R}$ and consider the function

$$\begin{aligned} x : I &\to E \\ t &\mapsto x(t). \end{aligned}$$

We say that x is *continuous* at $t_0 \in I$ if

$$\lim_{t \to t_0} \|x(t) - x(t_0)\| = 0.$$

We say that x is *continuous* if it is continuous at every point of I. We define

$$\mathcal{C}([a,b],E) = \{x : [a,b] \to E; x(t) \text{ is continuous}\},$$

the space of "curves" in E defined on $[a,b]$.

Proposition 1.8 $\mathcal{C}([a,b],E)$ *is a Banach space equipped with the norm*

$$\|x\|_{\mathcal{C}([a,b],E)} = \sup_{t \in [a,b]} \|x(t)\|.$$

Proof: Let $(x_n) \subset \mathcal{C}([a,b],E)$ be a Cauchy sequence. Then, given $\varepsilon > 0$, there exists $N_1(\varepsilon)$ such that, for $m,n > N_1$,

$$\|x_m - x_n\|_{\mathcal{C}([a,b],E)} = \sup_{t \in [a,b]} \|x_m(t) - x_n(t)\| < \frac{\varepsilon}{2}. \tag{1.1.4}$$

Therefore, for a fixed $t_0 \in [a,b]$ we have

$$\|x_m(t_0) - x_n(t_0)\| < \frac{\varepsilon}{2}, \tag{1.1.5}$$

whenever $m,n \geq N_1$. This shows that $(x_n(t_0))$ is a Cauchy sequence in E. Since E is complete, there exists $x(t_0) \in E$ such that $x_n(t_0) \to x(t_0)$ as $n \to \infty$. Thus we can associate to each $t \in [a,b]$ a unique element $x(t) \in E$, which defines a function $x : [a,b] \to E$. Fixing $n \geq N_1$ and letting $m \to \infty$ in (1.1.5), we obtain

$$\|x(t_0) - x_n(t_0)\| < \frac{\varepsilon}{2},$$

for each $t_0 \in [a,b]$, whenever $n \geq N_1$. This shows that (x_n) converges uniformly to x in $[a,b]$ and hence x is continuous. We also have the inclusion $\{\|x_n(t_0) - x(t_0)\|; t_0 \in [a,b]\} \subset [0, \frac{\varepsilon}{2}]$. Hence $\sup_{t \in [a,b]} \|x(t) - x_n(t)\| \leq \frac{\varepsilon}{2} < \varepsilon$, which shows that (x_n) converges to x in $\mathcal{C}([a,b],E)$. □

We say that $x : (a,b) \to E$ is *right differentiable* at $t_0 \in (a,b)$ if there exists $y \in E$ such that

$$\lim_{h\to 0^+} \left\| \frac{x(t_0+h)-x(t_0)}{h} - y \right\| = 0.$$

Similarly, we define *left differentiability* . When both one-sided derivatives exist and are equal, we say that $x : (a,b) \to E$ is *differentiable* at $t_0 \in (a,b)$ and we denote y by $x'(t_0)$.

Proposition 1.9 *Let $x : (a,b) \to E$ be differentiable at $t_0 \in (a,b)$. Then x is continuous at t_0.*

Proof: Let $\varepsilon > 0$ and $t_0 \in (a,b)$. Since x is differentiable at t_0, there exists $x^* := x'(t_0) \in E$ such that

$$\lim_{h\to 0} \left\| \frac{x(t_0+h)-x(t_0)}{h} - x^* \right\| = 0.$$

Thus, for the given $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |h| < \delta$, then

$$\|x(t_0+h)-x(t_0)\| < \varepsilon|h| + \|x'(t_0)\|\,|h|.$$

Fix $\varepsilon = 1$. Then there exists $\delta_1 > 0$ such that

$$\|x(t_0+h)-x(t_0)\| < (1+\|x'(t_0)\|)|h|, \quad \text{whenever } 0 < |h| < \delta_1.$$

Set $t = t_0 + h$. Then

$$\|x(t)-x(t_0)\| \le C|t-t_0|, \quad \text{whenever } |t-t_0| < \delta_1,$$

where $C = 1 + \|x'(t_0)\|$. For a given $\varepsilon > 0$ define $\delta = \min\{\delta_1, \varepsilon/C\}$. Thus, if $|t-t_0| < \delta$, it follows that

$$\|x(t)-x(t_0)\| \le C|t-t_0| < C\delta \le C\varepsilon/C = \varepsilon,$$

whenever $|t-t_0| < \delta$, which proves the claim. □

As before, given $A \in \mathcal{L}(E)$, we define $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$. One can also prove that $e^{tA} \in \mathcal{L}(E)$ and $\|e^{tA}\| \le e^{|t|\|A\|}$.

Proposition 1.10 *For $t \in \mathbb{R}$, let $T(t) = \exp(tA)$, where $A \in \mathcal{L}(E)$. Then:*

(*i*) $\lim_{t\to 0} \|T(t) - I\|_{\mathcal{L}(E)} = 0$ ($T(t)$ *is continuous at* $t = 0$ *and* $T(0) = I$).

(*ii*) $\lim_{t\to 0} \left\| \frac{T(t)-I}{t} - A \right\|_{\mathcal{L}(E)} = 0$ ($T(t)$ *is differentiable at* $t = 0$ *and* $T'(0) = A$).

Proof:

(*i*) Note that $T(t) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$. Thus

$$T(t) - I = A^0 + \sum_{n=1}^{\infty} \frac{t^n A^n}{n!} - I = \sum_{n=1}^{\infty} \frac{t^n A^n}{n!}.$$

Consider the series $\sum_{n=1}^{\infty} \frac{t^n A^n}{n!}$. We have

$$\left\| \sum_{n=1}^{\infty} \frac{t^n A^n}{n!} \right\| \leq \sum_{n=1}^{\infty} \left\| \frac{t^n A^n}{n!} \right\| \leq \sum_{n=1}^{\infty} \frac{|t|^n \|A\|^n}{n!}.$$

If $|t| < 1$, then the series $\sum_{n=1}^{\infty} \frac{|t|^n \|A\|^n}{n!}$ converges uniformly. Moreover, for each $n \in \mathbb{N}$,

$$\lim_{t \to 0} \frac{|t|^n \|A\|^n}{n!} = 0.$$

Therefore,

$$\lim_{t \to 0} \sum_{n=1}^{\infty} \frac{|t|^n \|A\|^n}{n!} = \sum_{n=1}^{\infty} \lim_{t \to 0} \frac{|t|^n \|A\|^n}{n!} = 0. \tag{1.1.6}$$

On the other hand,

$$\|T(t) - I\| = \left\| \sum_{n=1}^{\infty} \frac{t^n A^n}{n!} \right\| \leq \sum_{n=1}^{\infty} \frac{|t|^n \|A\|^n}{n!}. \tag{1.1.7}$$

Combining (1.1.6) and (1.1.7) we obtain

$$0 \leq \lim_{t \to 0} \|T(t) - I\| \leq \sum_{n=1}^{\infty} \lim_{t \to 0} \frac{|t|^n \|A\|^n}{n!} = 0,$$

which proves (i).

(ii) Note that

$$\begin{aligned} \frac{T(t) - I}{t} - A &= \frac{1}{t} \left[\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} - A^0 \right] - A \\ &= \frac{1}{t} \left[\sum_{n=1}^{\infty} \frac{t^n A^n}{n!} + A^0 - A^0 \right] - A \\ &= \sum_{n=1}^{\infty} \frac{t^{n-1} A^n}{n!} - A \\ &= \sum_{n=2}^{\infty} \frac{t^{n-1} A^n}{n!} + A - A = \sum_{n=2}^{\infty} \frac{t^{n-1} A^n}{n!}. \end{aligned}$$

As in part (i), assume $|t| < 1$. Then the series $\sum_{n=2}^{\infty} \frac{|t|^{n-1} \|A\|^n}{n!}$ converges uniformly and, in addition,

$$\lim_{t \to 0} \sum_{n=2}^{\infty} \frac{|t|^{n-1} \|A\|^n}{n!} = 0. \tag{1.1.8}$$

However,

$$\left\| \frac{T(t) - I}{t} - A \right\| \leq \sum_{n=2}^{\infty} \frac{|t|^{n-1} \|A\|^n}{n!}, \tag{1.1.9}$$

and (1.1.8)–(1.1.9) yield the desired result. □

Let $f : (a,b) \to X$, where X is a Banach space, be a continuous function. Given a partition π of $[a,b]$, that is, $n+1$ real numbers $t_0, \dots, t_n$ satisfying $a = t_0 < t_1 < \cdots < t_n = b$, and n real numbers ξ_i with $\xi_i \in (t_{i-1}, t_i)$, $i = 1, \dots, n$, we define a Riemann sum of f by

$$\sigma_\pi(f) = \sum_{i=1}^{n} (t_i - t_{i-1}) f(\xi_i).$$

Clearly, $\sigma_\pi(f) \in X$. Set

$$|\pi| = \max_{1 \le i \le n} \{t_i - t_{i-1}\}.$$

Arguing as in the scalar case, one proves that $\sigma_\pi(f)$ has a limit $x \in X$ as $|\pi| \to 0$. More precisely, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\sigma_\pi(f) - x\| < \varepsilon,$$

for every partition π with $|\pi| < \delta$. As in the numerical case, we say that x is the integral of f on $[a,b]$ and we write

$$x = \lim_{|\pi| \to 0} \sigma_\pi(f) = \int_a^b f(t)\, dt.$$

Proposition 1.11 *The following properties hold for the integral of a vector-valued function:*

i) *If K is a constant, then* $\int_a^b K f(t)\, dt = K \int_a^b f(t)\, dt.$

ii) $\int_a^b (f+g)(t)\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt.$

iii) *If $a \le c \le b$, then* $\int_a^b f(t)\, dt = \int_a^c f(t)\, dt + \int_c^b f(t)\, dt.$

iv) $\left\| \int_a^b f(t)\, dt \right\| \le \int_a^b \|f(t)\|\, dt.$

v) $\left\| \int_a^b f(t)\, dt \right\| \le \max_{a \le t \le b} \|f(t)\| (b-a).$

Proof: This follows immediately from the definition. □

Proposition 1.12 *Let E and F be Banach spaces, $A \in \mathcal{L}(E,F)$ and let $x \in C([a,b];E)$. Then*

$$A \int_a^b x(t)\, dt = \int_a^b Ax(t)\, dt.$$

Proof: Consider the partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$, where $t_i = a + \frac{i(b-a)}{n}$, and let $\xi_i \in (t_{i-1}, t_i)$. Then,

$$x_n = \sum_{i=1}^{n} (t_i - t_{i-1}) x(\xi_i) \in E, \quad \text{for each } n \in \mathbb{N},$$

since $x(\xi_i) \in E$ for $i = 1, \dots, n$. As x and Ax are continuous by hypothesis, we have

$$x_n \to \int_a^b x(t)\, dt \quad \text{and}$$

$$Ax_n = \sum_{i=1}^{n}(t_i - t_{i-1})Ax(\xi_i) \to \int_a^b Ax(t)\,dt.$$

Therefore

$$A\int_a^b x(t)\,dt = \int_a^b Ax(t)\,dt.$$

□

Lemma 1.13 *Let $x, y \in C([a,b];E)$ be curves which are differentiable on $[a,b]$ and such that $y'(t) = x'(t)$ for every $t \in [a,b]$. Then there exists $\xi \in E$ such that $y(t) = x(t) + \xi$ for all $t \in [a,b]$.*

Proof: We first claim that if $w \in C([a,b];E)$ is differentiable on $[a,b]$ and $w'(t) = 0$ for all $t \in [a,b]$, then w is constant on $[a,b]$. Indeed, let $c \in (a,b)$ and $\varepsilon > 0$. Since $w'_+ = 0$, we have

$$\|w(t) - w(c)\| \leq \varepsilon(t - c) \tag{1.1.10}$$

for $t > c$ sufficiently close to c.

Let $[c, t_0)$ be the maximal subinterval of $[c, b)$ on which (1.1.10) is valid. We must have $t_0 = b$. Suppose on the contrary that $t_0 < b$. Since $w'_+ = 0$, we have

$$\|w(t) - w(t_0)\| \leq \varepsilon(t - t_0), \tag{1.1.11}$$

for all $t > t_0$ sufficiently close to t_0. Let $t > t_0$ be such that (1.1.11) holds. From (1.1.10) and (1.1.11) we obtain

$$\begin{aligned} \|w(t) - w(c)\| &\leq \|w(t) - w(t_0)\| + \|w(t_0) - w(c)\| \\ &\leq \varepsilon(t - t_0) + \varepsilon(t_0 - c) = \varepsilon(t - c), \end{aligned}$$

that is, (1.1.10) is valid for all $t > t_0$ sufficiently close to t_0, which contradicts the definition of t_0. Hence $t_0 = b$ and we have $\|w(t) - w(c)\| \leq \varepsilon(t - c)$ for all $t \in [c, b)$. By the arbitrariness of ε, $w(t) = w(c)$ for all $t \in [c, b)$. Since c is an arbitrary point in (a, b), it follows that w is constant on (a, b) and, by continuity of w on $[a, b]$, the claim follows.

Now consider x, y continuous curves satisfying the assumptions of the lemma. Defining $w = y - x$, we have $w \in C([a,b];E)$ and $w'(t) = y'(t) - x'(t) = 0$ for all $t \in [a,b]$. By what we have just proved, there exists $\xi \in E$ such that $w(t) = \xi$ for all $t \in [a,b]$, which completes the proof. □

Proposition 1.14 *Let $x \in C([a,b];E)$ and set*

$$y(t) = \int_a^t x(s)\,ds.$$

Then $y \in C^1([a,b];E)$ and $y'(t) = x(t)$ for all $t \in [a,b]$. Moreover, if $x \in C^1([a,b];E)$, then

$$x(b) - x(a) = \int_a^b x'(s)\,ds.$$

Proof: In order to prove that $y \in C^1([a,b];E)$, given that $x \in C([a,b];E)$, it suffices to show that $y'(t) = x(t)$ for all $t \in [a,b]$. Indeed, let $t_0 \in [a,b]$ and $\varepsilon > 0$. Since $x \in C([a,b];E)$, there exists

$\delta = \delta(\varepsilon) > 0$ such that if $0 < |h| < \delta$, then $\|x(t_0+h) - x(t_0)\| < \varepsilon$. Hence, for all $0 < h < \delta$ we have

$$\begin{aligned}
\left\|\frac{y(t_0+h)-y(t_0)}{h} - x(t_0)\right\| &= \left\|\frac{\int_a^{t_0+h} x(s)\,ds - \int_a^{t_0} x(s)\,ds}{h} - x(t_0)\right\| \\
&= \left\|\frac{\int_{t_0}^{t_0+h} x(s)\,ds}{h} - x(t_0)\right\| \\
&= \left\|\int_0^h \frac{x(t_0+\xi)-x(t_0)}{h}\,d\xi\right\| \\
&\leq \frac{1}{h}\int_0^h \|x(t_0+\xi)-x(t_0)\|\,d\xi < \frac{1}{h}\int_0^h \varepsilon\,d\xi = \varepsilon,
\end{aligned}$$

since $0 < \xi < h < \delta$. This shows that the right derivative $\frac{d^+y}{dt}(t_0)$ exists and $\frac{d^+y}{dt}(t_0) = x(t_0)$. Similarly one proves that the left derivative $\frac{d^-y}{dt}(t_0)$ exists and $\frac{d^-y}{dt}(t_0) = x(t_0)$. Therefore y is differentiable at t_0 and $y'(t_0) = x(t_0)$. By the arbitrariness of $t_0 \in [a,b]$ we conclude that y is differentiable on $[a,b]$ and $y'(t) = x(t)$ for all $t \in [a,b]$. When $t_0 = a$ or $t_0 = b$, we consider only the corresponding one-sided derivative. Thus we conclude that $y \in C^1([a,b];E)$.

Now suppose that $x \in C^1([a,b];E)$. Define

$$y(t) = x(a) + \int_a^t x'(s)\,ds, \quad t \in [a,b].$$

Then $y'(t)$ exists for all $t \in [a,b]$ and $y'(t) = x'(t)$ for all $t \in [a,b]$. By Lemma 1.13 there exists $\xi \in E$ such that $y(t) = \xi + x(t)$ for all $t \in [a,b]$. In particular, for $t = a$ we obtain

$$\xi + x(a) = y(a) = x(a),$$

which implies $\xi = 0$, that is, $y(t) = x(t)$ for all $t \in [a,b]$. In particular, for $t = b$ we obtain

$$x(b) = y(b) = x(a) + \int_a^b x'(s)\,ds,$$

from which

$$x(b) - x(a) = \int_a^b x'(s)\,ds$$

follows, completing the proof. □

From now on, we are interested in functions defined on the unbounded interval $[a,+\infty)$ with values in a Banach space. Let $x \in C([a,+\infty);E)$. We say that x is integrable on $[a,+\infty)$ if the limit in E

$$\lim_{t\to+\infty}\int_a^t x(s)\,ds$$

exists.

Proposition 1.15 (Cauchy criterion) *Let $f : [a,+\infty) \to E$. A necessary and sufficient condition for the limit $\lim_{t\to+\infty} f(t)$ to exist is that for every $\varepsilon > 0$ there exists $t_0 > 0$ such that, if $t,s > t_0$, then $\|f(t) - f(s)\| < \varepsilon$.*

Proof: Suppose that

$$\lim_{t\to+\infty} f(t) = x_0.$$

Then, given $\varepsilon > 0$, there exists $t_0 > 0$ such that if $t > t_0$ then $\|f(t) - x_0\| < \varepsilon/2$. Hence, for $t, s > t_0$ we have

$$\|f(t) - f(s)\| \leq \|f(t) - x_0\| + \|x_0 - f(s)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Conversely, assume that for every $\varepsilon > 0$ there exists $t_0 > 0$ such that if $t, s > t_0$, then $\|f(t) - f(s)\| < \varepsilon$. Taking $\varepsilon = 1/n$, $n \in \mathbb{N}$, there exists $t_n \in (0, +\infty)$ such that if $t, s > t_n$ then $\|f(t) - f(s)\| < 1/n$. Note that we may assume, without loss of generality, that the sequence (t_n) is increasing, that is, $t_m > t_n$ whenever $m > n$. Define $(x_n) \subset E$ by $x_n = f(t_n)$. We claim that (x_n) is a Cauchy sequence. Indeed, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < \varepsilon$. Then

$$m, n > n_0 \Longrightarrow t_m > t_n > t_{n_0} \Longrightarrow \|x_m - x_n\| = \|f(t_m) - f(t_n)\| < \frac{1}{n_0} < \varepsilon,$$

which proves the claim. Since E is complete, there exists $x_0 \in E$ such that $x_n \to x_0$. Thus, given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$n > n_1 \Longrightarrow \|x_n - x_0\| < \varepsilon/2,$$

that is, there exists $t_{n_1} > 0$ such that

$$t_n > t_{n_1} \Longrightarrow n > n_1 \Longrightarrow \|f(t_n) - x_0\| = \|x_n - x_0\| < \varepsilon/2.$$

On the other hand, there exists $n_2 \in \mathbb{N}$ and hence $t_{n_2} > 0$ such that

$$t, s > t_{n_2} \Longrightarrow \|f(t) - f(s)\| < \varepsilon/2.$$

Taking $n_0 = \max\{n_1, n_2\}$, it follows that there exists $t_0 = t_{n_0} > 0$ such that

$$t > t_0 \Longrightarrow \|f(t) - x_0\| \leq \|f(t) - f(t_n)\| + \|f(t_n) - x_0\|,$$

where $n \in \mathbb{N}$ is chosen so that $n > n_0$, i.e. $t_n > t_{n_0} = t_0$. Hence

$$t > t_0 \Longrightarrow \|f(t) - x_0\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which completes the proof. □

Proposition 1.16 *Let $x \in C([0, \infty); E)$. Suppose there exist positive constants C, ω such that*

$$\|x(t)\| \leq Ce^{\omega t}, \quad \text{for all } t \geq 0. \tag{1.1.12}$$

Then we can define the <u>Laplace transform</u> of x by

$$\mathcal{L}(x)(\lambda) = \int_0^\infty e^{-\lambda s} x(s)\, ds, \quad \text{for all } \lambda > \omega.$$

Moreover, if $x \in C^1([0, \infty); E)$ and (1.1.12) holds, then

$$\mathcal{L}(x')(\lambda) = -x(0) + \lambda \mathcal{L}(x)(\lambda). \tag{1.1.13}$$

Proof: We shall use Proposition 1.15. Consider the auxiliary function $f : [0, \infty) \to E$ defined by

$$f(t) = \int_0^t e^{-\lambda s} x(s)\, ds, \quad t \in [0, \infty) \text{ and } \lambda > \omega > 0 \text{ fixed}.$$

We shall prove that f satisfies Proposition 1.15. First note that for all $t > s > 0$, it follows from

(1.1.12) that

$$\begin{aligned}
\|f(t)-f(s)\| &= \left\|\int_0^t e^{-\lambda s}x(s)\,ds - \int_0^s e^{-\lambda s}x(s)\,ds\right\| \\
&= \left\|\int_s^t e^{-\lambda s}x(s)\,ds\right\| \\
&\leq \int_s^t e^{-\lambda s}\|x(s)\|\,ds \\
&\leq C\int_s^t e^{-(\lambda-\omega)s}\,ds \\
&= \frac{Ce^{-(\lambda-\omega)s}}{\lambda-\omega}\left(1-\underbrace{\frac{e^{-(\lambda-\omega)t}}{e^{-(\lambda-\omega)s}}}_{<1}\right) < \frac{Ce^{-(\lambda-\omega)s}}{\lambda-\omega}.
\end{aligned}$$

For every $s > t_0$ we have $e^{-(\lambda-\omega)s} < e^{-(\lambda-\omega)t_0}$ and from the inequality above it follows that

$$\|f(t)-f(s)\| < \frac{Ce^{-(\lambda-\omega)t_0}}{\lambda-\omega} \quad \text{for all } t > s > t_0. \tag{1.1.14}$$

From the above and given $\varepsilon > 0$ such that $0 < \varepsilon(\lambda-\omega) < C$, or equivalently $\frac{1}{\varepsilon(\lambda-\omega)} > C$, there exists $t_0 > \left[\ln\left(\frac{C}{\varepsilon(\lambda-\omega)}\right)\right]/(\lambda-\omega)$. It then follows from (1.1.14) that

$$\|f(t)-f(s)\| < \varepsilon \quad \text{for all } t > s > t_0.$$

By Proposition 1.15 we obtain

$$\lim_{t\to\infty}\int_0^t e^{-\lambda s}x(s)\,ds = \int_0^\infty e^{-\lambda s}x(s)\,ds,$$

so that the Laplace transform of x is well defined. It remains to verify that if $x \in C^1([0,\infty);E)$ and (1.1.12) holds, then (1.1.13) also holds. Indeed, integrating by parts the integral $\int_0^t e^{-\lambda s}x(s)\,ds$, we obtain

$$\int_0^t e^{-\lambda s}x(s)\,ds = \frac{x(t)}{-\lambda}e^{-\lambda t} + \frac{x(0)}{\lambda} + \frac{1}{\lambda}\int_0^t e^{-\lambda s}x'(s)\,ds.$$

Taking the limit in the identity above as t tends to infinity, we obtain

$$\lambda\mathcal{L}(x)(\lambda) = x(0) + \lim_{t\to\infty}\int_0^t e^{-\lambda s}x'(s)\,ds.$$

Thus the limit $\lim_{t\to\infty}\int_0^t e^{-\lambda s}x'(s)\,ds$ exists and is precisely $\mathcal{L}(x')(\lambda)$, which shows (1.1.13) and completes the proof. □

Proposition 1.17 *Let $k \in \mathbb{R}$. Define the space*

$$X_k = \left\{u \in C([0,\infty);E);\ \|u(t)\| \leq Ce^{kt},\ \textit{for some } C > 0 \textit{ and for all } t \geq 0\right\}.$$

Then

$$\|u\|_{X_k} := \sup_{t\geq 0} e^{-kt}\|u(t)\|$$

is a norm with respect to which X_k is a Banach space.

Proof: First note that $\|u\|_{X_k}$ is well defined. In fact, if $u \in X_k$ then $u \in C([0,\infty);E)$ and $\|u(t)\| \le Ce^{kt}$ for some $C > 0$ and for all $t \ge 0$. Hence there exists $C > 0$ such that $\|u(t)\|e^{-kt} \le C$ for all $t \ge 0$. Therefore $\sup_{t\ge 0}\|u(t)\|e^{-kt}$ makes sense. Moreover, $\|u\|_{X_k} \ge 0$ for every $u \in X_k$. We now prove that

$$\|u\|_{X_k} = 0 \text{ if and only if } u \equiv 0. \tag{1.1.15}$$

Indeed, if $u \equiv 0$ then clearly $\|u\|_{X_k} = 0$. Conversely, if $\|u\|_{X_k} = 0$, then, since $0 \le e^{-kt}\|u(t)\| \le \|u\|_{X_k} = 0$ for all $t \ge 0$, it follows that $u \equiv 0$, which proves (1.1.15).

Let $u \in X_k$ and $\alpha \in \mathbb{R}$. From $\|(\alpha u)(t)\| = |\alpha|\,\|u(t)\|$ for all $t \ge 0$ it follows that

$$\|\alpha u\|_{X_k} = |\alpha|\,\|u\|_{X_k}. \tag{1.1.16}$$

Let $u, v \in X_k$. Then

$$\begin{aligned}\|u+v\|_{X_k} &= \sup_{t\ge 0}\left(e^{-kt}\|u(t)+v(t)\|\right) \\ &\le \sup_{t\ge 0}\left(e^{-kt}(\|u(t)\|+\|v(t)\|)\right) \\ &\le \sup_{t\ge 0} e^{-kt}\|u(t)\| + \sup_{t\ge 0} e^{-kt}\|v(t)\| = \|u\|_{X_k} + \|v\|_{X_k}.\end{aligned} \tag{1.1.17}$$

From (1.1.15), (1.1.16) and (1.1.17) we conclude that $\|u\|_{X_k}$ is indeed a norm.

We now prove that

$$(X_k, \|\cdot\|_{X_k}) \text{ is a Banach space.} \tag{1.1.18}$$

Let (u_n) be a Cauchy sequence in X_k, so that

$$\sup_{t\in[0,\infty)} e^{-kt}\|u_n(t) - u_m(t)\| \to 0$$

as $m, n \to \infty$. Define $v_{n,l} : [0,l] \to E$ by $v_{n,l}(t) = e^{-kt}u_n(t)$, where $t \in [0,l]$, $l \in \mathbb{N}$. Since

$$\begin{aligned}\|v_{n,l} - v_{m,l}\|_{C([0,l];E)} &= \sup_{t\in[0,l]} e^{-kt}\|u_n(t) - u_m(t)\| \\ &\le \sup_{t\in[0,\infty)} e^{-kt}\|u_n(t) - u_m(t)\|\end{aligned}$$

it follows that $(v_{n,l})$ is a Cauchy sequence in the Banach space $C([0,l];E)$; hence there exists $v_l \in C([0,l];E)$ such that

$$v_{n,l} \to v_l$$

in $C([0,l];E)$. Define

$$\begin{aligned}&v : [0,\infty) \to E, \\ &v(t) = v_l(t) \quad \text{for some } l \in \mathbb{N},\ l > t.\end{aligned}$$

Note that v is well defined, since

$$v_l(t) = v_{l'}(t) \quad \text{if } l < l',$$

and v is continuous: given $t \in [0,\infty)$, the function v coincides with v_l, for some $l \in \mathbb{N}$, on a neighbourhood of t, and since v_l is continuous, so is v.

Define

$$u : [0,\infty) \to E,$$
$$u(t) = e^{kt}v(t).$$

Then u is continuous. We claim that $u \in X_k$ and that

$$u_n \to u \text{ in } X_k.$$

Indeed, given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|e^{-kt}u_n(t) - e^{-kt}u_m(t)\| < \epsilon$$

for all $t \in [0,\infty)$ and all $m, n > n_0$. Letting $n \to \infty$,

$$\|e^{-kt}u(t) - e^{-kt}u_m(t)\| < \epsilon,$$

(where we have used that $v(t) = e^{-kt}u(t)$). Since $u = u - u_m + u_m$ for some $m \in \mathbb{N}$ sufficiently large, we have, for this m,

$$\begin{aligned} \sup_{t\in[0,\infty)} \|e^{-kt}u(t)\| &\leq \sup_{t\in[0,\infty)} e^{-kt}\|u(t) - u_m(t)\| + \sup_{t\in[0,\infty)} e^{-kt}\|u_m(t)\| \\ &< \infty, \end{aligned}$$

hence $u \in X_k$ and, since

$$\sup_{t\in[0,\infty)} \|e^{-kt}u(t) - e^{-kt}u_m(t)\| \leq \epsilon$$

for any $\epsilon > 0$, we conclude that

$$u_n \to u \text{ in } X_k.$$

□

Proposition 1.18 *Let $F : E \to E$ be a Lipschitz function, that is,*

$$\|F(u) - F(v)\| \leq \alpha\|u - v\|, \quad (\alpha > 0).$$

Let $\phi : X_\omega \to X_\omega$ (where X_ω is defined in Proposition 1.17) be given by

$$\phi(u)(t) = u_0 + \int_0^t F(u(s))\,ds, \quad u_0 \in E.$$

If $\omega > \alpha$ then ϕ is a contraction on X_ω.

Proof: Let $u, v \in X_\omega$ and $t \in [0,\infty)$. We have

$$\begin{aligned} \|\phi(u)(t) - \phi(v)(t)\| &\leq \int_0^t \|F(u(s)) - F(v(s))\|\,ds \\ &\leq \alpha\int_0^t \|u(s) - v(s)\|\,ds. \end{aligned} \tag{1.1.19}$$

On the other hand, since $e^{-\omega t}\|u(t) - v(t)\| \leq \|u - v\|_{X_\omega}$ for all $t \geq 0$, it follows that

$$\|u(t) - v(t)\| \leq \|u - v\|_{X_\omega} e^{\omega t}, \quad \text{for all } t \geq 0. \tag{1.1.20}$$

Combining (1.1.19) and (1.1.20) we obtain

$$\begin{aligned} \|\phi(u)(t)-\phi(v)(t)\| &\leq \alpha\|u-v\|_{X_\omega}\int_0^t e^{\omega s}\,ds \\ &= \frac{\alpha}{\omega}(e^{\omega t}-1)\|u-v\|_{X_\omega} \\ &\leq \frac{\alpha}{\omega}e^{\omega t}\|u-v\|_{X_\omega}, \end{aligned} \tag{1.1.21}$$

and hence

$$\|\phi(u)-\phi(v)\|_{X_\omega} = \sup_{t\geq 0}\left(e^{-\omega t}\|\phi(u)(t)-\phi(v)(t)\|\right) \leq \frac{\alpha}{\omega}\|u-v\|_{X_\omega},$$

which completes the proof. □

A function $u \in C([0,\infty), E)$ is said to be a solution of the initial value problem in the Banach space E

$$\begin{cases} u'(t) = F(u(t)), & t>0, \\ u(0) = u_0, \end{cases} \tag{1.1.22}$$

if and only if u is a solution of the integral equation

$$u(t) = u_0 + \int_0^t F(u(s))\,ds.$$

It is not difficult to verify that the unique fixed point of the mapping ϕ defined in Proposition 1.18 is a solution of the initial value problem given in (1.1.22). We leave this fact to the reader.

1.1.1 Exercises

1.1.1)Prove that the series $\sum\limits_{n=0}^{\infty}\frac{2^n}{n!}\cos nt$ converges absolutely in $E = \mathcal{C}([-\pi,\pi],\mathbb{R}) = \{f : [-\pi,\pi] \to \mathbb{R} \mid f \text{ is continuous}\}$, endowed with the norm $\|f\|_E = \sup\limits_{t\in[-\pi,\pi]}|f(t)|$.

1.1.2) Does the identity $\exp(\log(I-A)) = I-A$ hold?

1.1.3) Let (A_n) and (B_n) be sequences in $\mathcal{L}(X)$ such that

$$\begin{aligned} &(i)\ \sum_{n=0}^{\infty}A_n \text{ converges absolutely,} \\ &(ii)\ \sum_{n=0}^{\infty}B_n \text{ converges,} \\ &(iii)\ C_n = \sum_{k=0}^{n}A_kB_{n-k}, \quad n=0,1,2,\ldots \end{aligned}$$

Prove that

$$\sum_{n=0}^{\infty}C_n = \left(\sum_{n=0}^{\infty}A_n\right)\left(\sum_{n=0}^{\infty}B_n\right).$$

1.1.4) Let $x : (a,b) \to E$ be a function which is continuously differentiable on (a,b). Prove that

$$\|x(d)-x(c)\|_E \leq (d-c)\|x'\|_{\mathcal{C}([c,d],E)}, \quad a<c<d<b.$$

1.1.5) Let $f, g \in C([a,b];E)$, where E is a Banach space. Prove that:

(i) $\displaystyle\int_a^b Cf(t)\,dt = C\int_a^b f(t)\,dt,$ where C is a constant.

(ii) $\displaystyle\int_a^b (f+g)(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.$

(iii) If $a \le c \le b$, then $\displaystyle\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$

(iv) One has $\displaystyle\int_a^b f(t)\,dt = (b-a)\tilde{x},$ for some $\tilde{x} \in \overline{\operatorname{conv} f(a,b)}$,

where $\overline{\operatorname{conv} f(a,b)}$ denotes the closure of the convex combinations of the elements of the set of values of f on $[a,b]$.

(v) [Mean value theorem].For every $t \in [a,b]$ one has

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(\tau)\,d\tau = f(t).$$

Hint: use item (iv).

1.1.6) Let $A \in \mathcal{L}(E)$. Prove that the initial value problem

$$\begin{cases} u'(t) = A(u(t)), & t > 0, \\ u(0) = u_0 \end{cases}$$

admits a unique solution $u \in C([0,\infty);E)$.

1.1.6) Let $T(t) : E \to E$ be the linear operator defined by $T(t)u_0 = u(t)$, where u is the unique solution of

$$\begin{cases} u'(t) = A(u(t)), & t > 0, \\ u(0) = u_0. \end{cases}$$

(i) Show that $T(0) = I$ and that $T(t+s) = T(t)\circ T(s)$ for all $t, s \in [0,\infty)$. Use Gronwall's inequality to show that $T(t) \in \mathcal{L}(E)$ for all $t \ge 0$ and that $\|T(t)\| \le e^{\|A\|t}$.

(ii) Show that $T(t)$ is continuous in t with respect to the norm of $\mathcal{L}(E)$, that is,

$$\lim_{t\to t_0}\|T(t) - T(t_0)\|_{\mathcal{L}(E)} = 0.$$

(iii) Show that $T(t)$ is differentiable (in the space $\mathcal{L}(E)$) and that

$$T'(t) = AT(t), \text{ that is,}$$

$$\lim_{h\to 0}\left\|\frac{T(t+h)-T(t)}{h} - AT(t)\right\|_{\mathcal{L}(E)} = 0.$$

(iv) Consider the Laplace transform of $T(t)$, namely

$$\mathcal{L}(T)(\lambda) = \int_0^\infty e^{-\lambda s}T(s)\,ds, \quad \lambda > \|A\|.$$

Show that $\mathcal{L}(T)(\lambda) = (\lambda I - A)^{-1}$.

(v) Let $B = I - A/\lambda$, $\lambda > \|A\|$. Since $\|I - B\| < 1$, use C. Neumann's theorem to show that $B^{-1} \in \mathcal{L}(E)$ and that

$$B^{-1} = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^k}.$$

Recalling that

$$\mathcal{L}(t^n) = \int_0^{\infty} e^{-\lambda t} t^n \, dt = \frac{n!}{\lambda^{n+1}},$$

and that if the Laplace transform of a continuous function is zero then the function itself is identically zero, show that $T(t) = e^{tA}$.

1.2 The Exponential Function

The exponential function e^{tA}, where A is a real number and t a real variable, may be defined by the formula

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}. \tag{1.2.23}$$

The series on the right-hand side of (1.2.23) converges for all values of t and therefore defines a real-valued function. Without much difficulty this definition extends to the case where A is a bounded (that is, continuous) linear operator on a Banach space (as seen in the preceding section) and, in this case, the series (1.2.23) converges in norm and, consequently, for each $t \in \mathbb{R}$ its "sum" is a bounded linear operator on that space. A rather delicate problem, however, is to define the "exponential function" when A is unbounded. One of the reasons for the interest in such a function lies in the fact that it is, formally, a solution of the Cauchy problem: given an unbounded linear operator A on a Banach space X, determine a function $u(t)$ defined on $[0, \infty)$, taking values in $D(A)$ ($D(A)$ = domain of A), which satisfies the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} = A(u(t)), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{1.2.24}$$

where u_0 is a given element of X.

When $A \in \mathbb{R}$ and $t \geq 0$, the exponential function $E : \mathbb{R}_+ \to \mathbb{R}$ has the following properties:

$$E(0) = 1, \tag{1.2.25}$$

$$E(t+s) = E(t)E(s), \tag{1.2.26}$$

$$\lim_{t \to 0_+} E(t) = 1, \tag{1.2.27}$$

and, as will be shown below, it is the unique function defined on $\mathbb{R}_+$ with values in $\mathbb{R}$ having such properties. The same occurs when E takes values in the algebra of linear operators on any finite-dimensional space (recalling that every linear map defined on a finite-dimensional space is continuous). In this case, the number 1 appearing in (1.2.25) and (1.2.27) should be interpreted as the identity operator $I : X \to X$, and the product in (1.2.26) as the composition of linear operators. To understand what happens when X is infinite-dimensional, one must take into account that, in this case, three topologies are usually introduced on the algebra $\mathcal{L}(X)$ of bounded linear operators on X, each one giving a different meaning to the limit in (1.2.27). Thus, we may interpret this limit as a *uniform*, *strong*, or *weak* limit. Recall that I is the uniform limit of $E(t)$ as $t \to 0_+$ if $\|E(t) - I\|_{\mathcal{L}(X)} \to 0$; it is the strong limit if $\|E(t) - I\|_X \to 0$ for all $x \in X$, and it is the weak limit if $\langle [E(t) - I]x, x' \rangle_{X,X'} \to 0$ for all $x \in X$ and all $x' \in X'$, where X' is the topological dual of X. When the limit (1.2.27) is taken in the sense of the uniform topology, the situation is rather simple, as shown by the following theorem.

Theorem 1.19 *A function $E : \mathbb{R}_+ \to \mathcal{L}(X)$ satisfies*

(a) $E(0) = I$,

(b) $E(t+s) = E(t)E(s)$,

(c) $\|E(t) - I\|_{\mathcal{L}(X)} \to 0$ *as* $t \to 0_+$,

if and only if $E(t) = e^{tA}$, *where* $A \in \mathcal{L}(X)$ *and* e^{tA} *is defined by (1.2.23).*

Proof: Assume first that $A \in \mathcal{L}(X)$ and that

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

Since, for each $t \geq 0$, the series $\sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$ converges absolutely and $\mathcal{L}(X)$ is a Banach space, we have that e^{tA} defines, for each $t \geq 0$, a linear and continuous operator on X. Thus $E(t) = e^{tA}$ is such that $E : \mathbb{R}_+ \to \mathcal{L}(X)$. It remains to prove that $E(t)$ satisfies conditions (a), (b) and (c). Indeed,

(a)

$$E(0) = \sum_{n=0}^{\infty} \frac{(0\,A)^n}{n!} = I.$$

(b) Observe that, by the binomial theorem,

$$\begin{aligned}(t+s)^p &= \sum_{k=0}^{p} \binom{p}{k} t^k s^{p-k} \\ &= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} t^k s^{p-k},\end{aligned}$$

which implies

$$\frac{(t+s)^p}{p!} = \sum_{k=0}^{p} \frac{t^k}{k!} \frac{s^{p-k}}{(p-k)!}.$$

Hence

$$\begin{aligned}e^{(t+s)A} &= \sum_{n=0}^{\infty} \frac{(t+s)^n}{n!} A^n \\ &= \lim_{n\to\infty} \sum_{p=0}^{n} \frac{(t+s)^p}{p!} A^p \\ &= \lim_{n\to\infty} \sum_{p=0}^{n} \sum_{k=0}^{p} \frac{(tA)^k}{k!} \frac{(sA)^{p-k}}{(p-k)!}.\end{aligned}$$

Since $\sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$ converges absolutely, it follows from Exercise 1.1.3 of the previous section that $e^{(t+s)A} = e^{tA} e^{sA}$, that is, $E(t+s) = E(t)E(s)$.

(c) We have

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = I + \sum_{n=1}^{\infty} \frac{(tA)^n}{n!}.$$

Thus

$$\begin{aligned} e^{tA} - I &= \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} \\ &= \frac{tA}{1!} + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \cdots \\ &= tA\left(I + \frac{tA}{2!} + \frac{(tA)^2}{3!} + \cdots\right) \\ &= tA\sum_{n=0}^{\infty} \frac{(tA)^n}{(n+1)!}. \end{aligned}$$

Note that the series $\sum_{n=0}^{\infty} \frac{(tA)^n}{(n+1)!}$ converges absolutely, since

$$\left\|\sum_{n=0}^{\infty} \frac{(tA)^n}{(n+1)!}\right\| \leq \sum_{n=0}^{\infty} \frac{\|(tA)^n\|}{(n+1)!} \leq \sum_{n=0}^{\infty} \frac{\|(tA)^n\|}{n!} = e^{\|tA\|} = e^{t\|A\|}.$$

Hence

$$\|e^{tA} - I\|_{\mathcal{L}(X)} \leq t\|A\|_{\mathcal{L}(X)} e^{t\|A\|_{\mathcal{L}(X)}}.$$

As $t \to 0_+$ we have $e^{t\|A\|} \to 1$ and, consequently, $\|e^{tA} - I\|_{\mathcal{L}(X)} \to 0$ as $t \to 0_+$.

Conversely, suppose that $E : \mathbb{R}_+ \to \mathcal{L}(X)$ satisfies (a), (b) and (c). We first show that $\|E(t)\|$ is bounded on every bounded interval. Indeed, given $\varepsilon = 1$, there exists, by property (c), a $\delta > 0$ such that if $0 \leq t \leq \delta$ then $\|E(t) - I\| < 1$. Since $\|E(t)\| - \|I\| \leq \|E(t) - I\|$, it follows that $\|E(t)\| < 2$ for $0 \leq t \leq \delta$. Now let $t \geq 0$ be arbitrary. Then there exists $n \in \mathbb{N}$ such that $t = n\delta + r$, where $0 \leq r < \delta$. Hence, by property (b),

$$E(t) = E(n\delta + r) = E(n\delta)E(r) = E(\delta)^n E(r),$$

and therefore

$$\|E(t)\| \leq \|E(\delta)\|^n \|E(r)\| < 2^{n+1}.$$

Since $t = n\delta + r$, we have $t \geq n\delta$, i.e. $n \leq t/\delta$. Consequently,

$$\|E(t)\| \leq 2^n 2 \leq 2^{t/\delta} 2 = 2^{t/\delta + 1}.$$

Setting $\omega = (1/\delta)\ln 2$, the inequality above may be written as

$$\|E(t)\| \leq 2e^{\omega t}, \quad \text{for all } t \geq 0, \text{ where } \omega = (1/\delta)\ln 2. \tag{1.2.28}$$

Now let $t \in [T_0, T]$, where $0 \leq T_0 < T < +\infty$. Then, from (1.2.28),

$$\|E(t)\| \leq 2e^{\omega T}, \quad \text{for all } t \in [T_0, T],$$

that is, $\|E(t)\|$ is bounded on bounded intervals, which proves the claim.

We now prove that E is continuous with respect to the uniform topology on $\mathcal{L}(X)$. Let $h > 0$ and $t \geq 0$. From property (c) and the boundedness of $\|E(t)\|$ on bounded intervals we have

$$\begin{aligned}\|E(t+h) - E(t)\|_{\mathcal{L}(X)} &= \|E(t)E(h) - E(t)\|_{\mathcal{L}(X)} \\ &\leq \|E(t)\|_{\mathcal{L}(X)} \|E(h) - I\|_{\mathcal{L}(X)} \to 0 \quad \text{as } h \to 0_+.\end{aligned}$$

Similarly, if $0 < h \leq t$, i.e. $0 \leq t - h < t$, we obtain

$$\begin{aligned}\|E(t-h) - E(t)\|_{\mathcal{L}(X)} &= \|E(t-h) - E(t-h)E(h)\|_{\mathcal{L}(X)} \\ &\leq \|E(t-h)\|_{\mathcal{L}(X)} \|E(h) - I\|_{\mathcal{L}(X)} \to 0 \quad \text{as } h \to 0_+.\end{aligned}$$

From these convergences we conclude that E is continuous in the uniform topology of $\mathcal{L}(X)$. It follows that E is Riemann integrable with respect to the uniform topology of $\mathcal{L}(X)$ and, moreover, from the mean value theorem (see Exercise 1.1.5 (v) of the previous section) we have

$$\lim_{h \to 0_+} \frac{1}{h} \int_0^h E(t)\, dt = E(0) = I \quad \text{in } \mathcal{L}(X).$$

Thus, given $\varepsilon = 1$, there exists $\delta > 0$ such that

$$\left\| \frac{1}{\delta} \int_0^\delta E(t)\, dt - I \right\|_{\mathcal{L}(X)} < 1,$$

and therefore $\frac{1}{\delta} \int_0^\delta E(t)\, dt$ is invertible in $\mathcal{L}(X)$, by Proposition 1.6, and, consequently, so is $\int_0^\delta E(t)\, dt$. With this in mind, let $0 < h < \delta$. Then

$$\begin{aligned}\left[\frac{E(h) - I}{h}\right] \int_0^\delta E(t)\, dt &\overset{Prop.(1.12)}{=} \frac{1}{h} \int_0^\delta E(t+h)\, dt - \frac{1}{h} \int_0^\delta E(t)\, dt \\ &= \frac{1}{h} \left[\int_h^{\delta+h} E(t)\, dt - \int_0^\delta E(t)\, dt \right] \\ &= \frac{1}{h} \left[\int_h^\delta E(t)\, dt + \int_\delta^{\delta+h} E(t)\, dt - \int_0^h E(t)\, dt - \int_h^\delta E(t)\, dt \right] \\ &= \frac{1}{h} \left[\int_\delta^{\delta+h} E(t)\, dt - \int_0^h E(t)\, dt \right],\end{aligned}$$

which implies

$$\frac{E(h) - I}{h} = \left[\frac{1}{h} \int_\delta^{\delta+h} E(t)\, dt - \frac{1}{h} \int_0^h E(t)\, dt \right] \left[\int_0^\delta E(t)\, dt \right]^{-1}.$$

Since the right-hand side of the last identity converges in norm to $(E(\delta) - I)\big(\int_0^\delta E(t)\, dt\big)^{-1}$ as $h \to 0_+$, the same occurs for the left-hand side. We denote by A the uniform limit of $\frac{E(h)-I}{h}$ in $\mathcal{L}(X)$ as $h \to 0_+$. Thus,

$$\frac{d^+ E(0)}{dt} = A.$$

(We use the notation $\dfrac{d^+ E(0)}{dt}$ for $\displaystyle\lim_{h \to 0_+} \frac{E(h) - I}{h}$.)

Moreover, for $t > 0$ and $h > 0$ we have

$$\frac{E(t+h) - E(t)}{h} = E(t)\left(\frac{E(h) - I}{h}\right),$$

and since $\frac{E(h)-I}{h}$ converges in norm to A, it follows that $\frac{E(t+h)-E(t)}{h}$ converges in norm to $E(t)A$ as $h \to 0_+$. Hence E is right differentiable for all $t \geq 0$ with respect to the uniform topology of $\mathcal{L}(X)$ and

$$\frac{d^+E(t)}{dt} = E(t)A. \tag{1.2.29}$$

Similarly, if $t, h > 0$ and $0 < h < t$, then

$$\frac{E(t-h) - E(t)}{-h} = \frac{E(t) - E(t-h)}{h} = E(t-h)\left[\frac{E(h) - I}{h}\right].$$

Since E is continuous in the uniform topology of $\mathcal{L}(X)$, we have that $E(t-h)$ converges to $E(t)$ in $\mathcal{L}(X)$ as $h \to 0_+$. Also, $\frac{E(h)-I}{h}$ converges to A as $h \to 0_+$, and therefore

$$\frac{d^-E(t)}{dt} = \lim_{h \to 0_+} \frac{E(t-h) - E(t)}{-h} = E(t)A. \tag{1.2.30}$$

Thus, from (1.2.29) and (1.2.30) we conclude that

$$\frac{dE(t)}{dt} = E(t)A, \quad \text{for all } t \geq 0. \tag{1.2.31}$$

Finally, consider the function

$$\varphi(t) = E(t)e^{-tA}, \quad t \geq 0. \tag{1.2.32}$$

Recalling that differentiation in $\mathcal{L}(X)$ has the same properties as classical differentiation, we obtain from (1.2.31) that

$$\begin{aligned} \varphi'(t) &= E'(t)e^{-tA} - E(t)Ae^{-tA} \\ &= E(t)Ae^{-tA} - E(t)Ae^{-tA} = 0. \end{aligned}$$

Consequently, φ is constant. But $\varphi(0) = I$ and, therefore, $E(t)e^{-tA} = I$ for all $t \geq 0$, so $E(t) = e^{tA}$, which completes the proof. □

Remark 1.20 *In finite-dimensional spaces, the uniform, strong and weak topologies all coincide with the usual topology and, since the proof of Theorem 1.19 did not involve the dimension of the space, this theorem remains valid in the finite-dimensional setting.*

As uniform convergence implies strong convergence, Theorem 1.19 shows that the definition given below (in the next section) generalises the usual definition of the exponential function.

1.3 Semigroups of class C_0

Definition 1.21 *Let $(X, \|\cdot\|)$ be a Banach space and let $\mathcal{L}(X)$ denote the algebra of bounded linear operators on X. We say that a mapping $S : \mathbb{R}_+ \to \mathcal{L}(X)$ is a semigroup of bounded operators on X if:*

(i) $S(0) = I$, *where I is the identity operator on X.*

(ii) $S(t+s) = S(t)S(s)$, *for all $t, s \in \mathbb{R}_+$.*

The semigroup is said to be of class C_0 if

$$(iii) \lim_{t \to 0_+} \|(S(t) - I)x\| = 0, \quad \textit{for all } x \in X.$$

Proposition 1.22 *If S is a semigroup of class C_0, then $\|S(t)\|_{\mathcal{L}(X)}$ is a bounded function on every bounded interval $[0, T]$.*

Proof: We first claim that there exists an interval of the form $[0, \delta]$ on which the function $\|S(t)\|$ is bounded, that is,

$$\text{There exist } \delta > 0 \text{ and } M > 0 \text{ such that } \|S(t)\| \leq M, \text{ for all } t \in [0, \delta]. \tag{1.3.33}$$

Indeed, suppose by contradiction that this is not the case, that is, for every interval of the form $[0, \delta]$ the function $\|S(t)\|$ is unbounded. In other words, for every $\delta > 0$ and $M > 0$ there exists $t_{\delta,M} \in [0, \delta]$ such that $\|S(t_{\delta,M})\| > M$. Hence, for $\delta = 1/n$ and $M = n$, $n \in \mathbb{N}$, there exists $t_n \in [0, 1/n]$ such that $\|S(t_n)\| > n$. Thus there exists a sequence $t_n \to 0_+$ with $\|S(t_n)\| > n$ for all $n \in \mathbb{N}$. By the Uniform Boundedness Principle (Banach–Steinhaus theorem), there exists $x \in X$ for which $\|S(t_n)x\|$ is unbounded in $n \in \mathbb{N}$, which contradicts property (iii) of S, since S is assumed to be of class C_0. This proves the claim in (1.3.33). Moreover, note that $M \geq 1$, because $\|S(t)\| \leq M$ for all $t \in [0, \delta]$ and, in particular, $\|S(0)\| = \|I\| = 1 \leq M$.

Now let $t \in [0, T]$, where $T > 0$ is arbitrary. Then $t = n\delta + r$ for some $n \in \mathbb{N}$ and $0 \leq r < \delta$. Hence

$$\begin{aligned} \|S(t)\| &= \|S(n\delta + r)\| = \|S(\delta)^n S(r)\| \\ &\leq \|S(\delta)\|^n \|S(r)\| \\ &\leq M^n M \\ &\leq M^{t/\delta} M = Me^{\omega t} \leq Me^{\omega T}, \end{aligned}$$

where $\omega := \frac{1}{\delta} \ln M$, which completes the proof. □

Corollary 1.23 *Every semigroup of class C_0 is strongly continuous on $\mathbb{R}_+$, that is, if $t \in \mathbb{R}_+$, then*

$$\lim_{s \to t} S(s)x = S(t)x, \quad \textit{for all } x \in X.$$

Proof: Let $t \in \mathbb{R}_+$ and $x \in X$. If $h \geq 0$, then

$$\begin{aligned} \|S(t+h)x - S(t)x\| &= \|S(t)[S(h) - I]x\| \\ &\leq \|S(t)\|_{\mathcal{L}(X)} \|[S(h) - I]x\|. \end{aligned} \tag{1.3.34}$$

Since S is of class C_0, we have

$$\|[S(h) - I]x\| \to 0 \quad \text{as } h \to 0_+, \tag{1.3.35}$$

and therefore, from (1.3.34), (1.3.35) and Proposition 1.22 we obtain

$$\lim_{s \to t_+} S(s)x = S(t)x. \tag{1.3.36}$$

Now, if $0 < h < t$, we have

$$\begin{aligned} \|S(t-h)x - S(t)x\| &= \|S(t-h)[I - S(h)]x\| \\ &\leq \|S(t-h)\|_{\mathcal{L}(X)}\|[S(h) - I]x\|. \end{aligned} \tag{1.3.37}$$

Again, since S is of class C_0, we have

$$\|[S(h) - I]x\| \to 0 \quad \text{as } h \to 0_+, \tag{1.3.38}$$

and, using (1.3.37), (1.3.38) and Proposition 1.22 in the same way, we infer that

$$\lim_{s \to t_-} S(s)x = S(t)x. \tag{1.3.39}$$

Combining (1.3.36) and (1.3.39) yields the desired conclusion. □

Remark 1.24 *Semigroups of class C_0 are also called <u>continuous</u> semigroups, which is justified by Corollary 1.23. We have seen in the proof of Proposition 1.22 that if S is a semigroup of class C_0, then there exist real numbers M and ω such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad \textit{for all } t \geq 0.$$

A more refined result will be proved later. As a preliminary step we shall prove the following result about <u>subadditive</u> functions, that is, functions $p : \mathbb{R} \to \mathbb{R}$ such that $p(t+s) \leq p(t) + p(s)$ for all $t, s \in \mathbb{R}$.

Lemma 1.25 *Let p be a subadditive function defined on $\mathbb{R}_+$ and bounded from above on every bounded interval. Then $\frac{p(t)}{t}$ has a limit as $t \to +\infty$, and*

$$\lim_{t \to \infty} \frac{p(t)}{t} = \inf_{t > 0} \frac{p(t)}{t}.$$

Proof: Set

$$\omega_0 := \inf_{t>0} \frac{p(t)}{t}.$$

Then $\omega_0 \geq -\infty$. We first consider the case $\omega_0 > -\infty$. Given $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that

$$\frac{p(T)}{T} < \omega_0 + \varepsilon. \tag{1.3.40}$$

Let $t \in \mathbb{R}_+$. Then there exists $n \in \mathbb{N}$ such that $t = nT + r$ with $0 \leq r < T$. By the subadditivity of p and (1.3.40) we obtain

$$\begin{aligned} \omega_0 \leq \frac{p(t)}{t} = \frac{p(nT+r)}{t} &\leq \frac{p(nT) + p(r)}{t} \\ &\leq \frac{np(T) + p(r)}{t} \\ &= \frac{nTp(T)}{tT} + \frac{p(r)}{t} \\ &\leq \frac{nT}{t}(\omega_0 + \varepsilon) + \frac{p(r)}{t}. \end{aligned} \tag{1.3.41}$$

Since p is bounded from above on $[0, T)$, there exists $c \in \mathbb{R}$ such that

$$p(r) \leq c, \quad \text{for all } r \in [0, T). \tag{1.3.42}$$

Moreover, because $t = nT + r$, we have

$$\frac{nT}{t} = \frac{t-r}{t} \to 1 \quad \text{as } t \to +\infty. \tag{1.3.43}$$

Therefore, from (1.3.41), (1.3.42) and (1.3.43) we get

$$\omega_0 \le \liminf_{t\to+\infty} \frac{p(t)}{t} \le \omega_0 + \varepsilon,$$
$$\omega_0 \le \limsup_{t\to+\infty} \frac{p(t)}{t} \le \omega_0 + \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$ it follows that

$$\omega_0 = \liminf_{t\to+\infty} \frac{p(t)}{t} = \limsup_{t\to+\infty} \frac{p(t)}{t},$$

and hence the desired limit exists.

Now consider the case $\omega_0 = -\infty$. In this case, for each real number ω there exists $T = T(\omega) > 0$ such that

$$\frac{p(T)}{T} \le \omega.$$

Let $t \in \mathbb{R}_+$ and $\omega \in \mathbb{R}$. Then there exist $T = T(\omega) > 0$ and $n \in \mathbb{N}$ such that

$$\frac{p(T)}{T} \le \omega \quad \text{and} \quad t = nT + r, \ \text{with } 0 \le r < T.$$

Proceeding as in the previous case, we obtain

$$\frac{p(t)}{t} \le \omega + \frac{c}{t}, \quad \text{for some } c > 0.$$

Hence

$$\liminf_{t\to+\infty} \frac{p(t)}{t} \le \omega \quad \text{and} \quad \limsup_{t\to+\infty} \frac{p(t)}{t} \le \omega.$$

By the arbitrariness of ω we conclude that

$$\lim_{t\to+\infty} \frac{p(t)}{t} = -\infty,$$

which completes the proof. □

Proposition 1.26 *Let S be a semigroup of class C_0. Then*

$$\lim_{t\to+\infty} \frac{\ln \|S(t)\|}{t} = \inf_{t>0} \frac{\ln \|S(t)\|}{t} = \omega_0, \tag{1.3.44}$$

and for each $\omega > \omega_0$ there exists a constant $M \ge 1$ such that

$$\|S(t)\| \le M e^{\omega t}, \quad \text{for all } t \ge 0. \tag{1.3.45}$$

Proof: We have

$$\begin{aligned}\ln\|S(t+s)\| = \ln\|S(t)S(s)\| &\leq \ln\big(\|S(t)\|\,\|S(s)\|\big)\\ &= \ln\|S(t)\| + \ln\|S(s)\|,\end{aligned}$$

since the logarithm function is increasing. Thus $\ln\|S(t)\|$ is subadditive. By Proposition 1.22 we know that $\|S(t)\|$ is bounded on every bounded interval, hence $\ln\|S(t)\|$ is bounded from above there as well. Setting $p(t) = \ln\|S(t)\|$, we infer from Lemma 1.25 that

$$\lim_{t\to+\infty}\frac{\ln\|S(t)\|}{t} = \inf_{t>0}\frac{\ln\|S(t)\|}{t} = \omega_0.$$

Let $\omega > \omega_0$. We claim that there exists $t_0 \in \mathbb{R}_+$ such that

$$\frac{\ln\|S(t)\|}{t} < \omega, \quad \text{for all } t \geq t_0. \tag{1.3.46}$$

Indeed, if $\omega_0 < +\infty$, take $\varepsilon = \omega - \omega_0 > 0$. By the definition of the limit, there exists $t_0 \in \mathbb{R}_+$ such that

$$\left|\frac{\ln\|S(t)\|}{t} - \omega_0\right| < \varepsilon, \quad \text{for all } t \geq t_0,$$

which yields (1.3.46). If $\omega_0 = -\infty$, the desired inequality in (1.3.46) follows directly from the definition of infinite limit.

On the other hand, since $\|S(t)\|$ is bounded on $[0, t_0]$ and $\|S(0)\| = 1$, there exists $M_0 \geq 1$ such that

$$\|S(t)\| \leq M_0, \quad \text{for all } t \in [0, t_0].$$

Thus

$$\ln\|S(t)\| \leq \ln M_0, \quad \text{for all } t \in [0, t_0]. \tag{1.3.47}$$

Let $\omega \geq 0$. From (1.3.46) and (1.3.47) we obtain

$$\ln\|S(t)\| \leq \omega t + \ln M_0, \quad \text{for all } t \geq 0,$$

and hence

$$\|S(t)\| \leq M_0 e^{\omega t}, \quad \text{for all } t \geq 0.$$

Setting $M = M_0$, we obtain the desired estimate. If $\omega < 0$, then $-\omega t_0 > 0$ and, therefore, by (1.3.46) we have

$$\ln\|S(t)\| < \omega t - \omega t_0, \quad \text{for all } t \geq t_0. \tag{1.3.48}$$

From (1.3.47), (1.3.48) and the fact that $\omega(t - t_0) > 0$ on $[0, t_0]$ we conclude that

$$\ln\|S(t)\| \leq \omega(t - t_0) + \ln M_0, \quad \text{for all } t \geq 0.$$

Hence

$$\|S(t)\| \leq M_0 e^{\omega(t-t_0)}, \quad \text{for all } t \geq 0.$$

Setting $M = M_0 e^{-\omega t_0}$, the proposition follows. □

Remark 1.27 *When $\omega_0 < 0$, we may choose $\omega_0 < \omega < 0$ and, from (1.3.45), obtain $M \geq 1$ such that*

$$\|S(t)\| \leq M, \quad \textit{for all } t \geq 0.$$

In this case S is called a uniformly bounded semigroup of class C_0. If, in addition, $M = 1$, then S is called a semigroup of contractions of class C_0 .

Definition 1.28 *Let S be a semigroup of class C_0. The operator $A : D(A) \to X$ defined by*

$$D(A) = \left\{ x \in X;\ \lim_{h \to 0_+} \left(\frac{S(h) - I}{h} \right) x \textit{ exists} \right\},$$

and

$$Ax = \lim_{h \to 0_+} \left(\frac{S(h) - I}{h} \right) x, \quad \textit{for all } x \in D(A),$$

is called the infinitesimal generator of the semigroup S.

Proposition 1.29 *$D(A)$ is a vector subspace of X and A is a linear operator.*

Proof: This is an immediate consequence of Definition 1.28 and the properties of limits. □

Proposition 1.30 *Let S be a semigroup of class C_0 and let A be the infinitesimal generator of S. Then*

(i) If $x \in D(A)$, then $S(t)x \in D(A)$ for all $t \geq 0$ and the following identities hold:

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax, \quad \forall t \geq 0, \tag{1.3.49}$$

where $\displaystyle\frac{d}{dt} S(t)x = \lim_{h \to 0} \frac{S(t+h)x - S(t)x}{h}$ and, when $t = 0$, this limit is understood as a right-hand limit only.

(ii) If $x \in D(A)$, then

$$S(t)x - S(s)x = \int_s^t AS(\xi)x \, d\xi = \int_s^t S(\xi)Ax \, d\xi, \quad 0 \leq s \leq t. \tag{1.3.50}$$

(iii) If $x \in X$, then $\displaystyle\int_0^t S(\xi)x \, d\xi \in D(A)$ and

$$A \int_0^t S(\xi)x \, d\xi = S(t)x - x. \tag{1.3.51}$$

Proof:

(i) If $t = 0$ then $S(0) = I$, so

$$S(0)x = x \in D(A).$$

Consequently, by Definition 1.28 we have

$$\frac{d}{dt}^+ S(0)x = \lim_{h\to 0_+} \left(\frac{S(h)x - S(0)x}{h}\right) = Ax.$$

Now let $t > 0$. We shall prove that $S(t)x \in D(A)$, i.e. that the limit

$$\lim_{h\to 0_+} \left(\frac{S(h) - I}{h}\right) S(t)x \tag{1.3.52}$$

exists. Indeed, for $h > 0$ we have

$$\left(\frac{S(h) - I}{h}\right) S(t)x = \frac{(S(t+h) - S(t))x}{h} = S(t)\left(\frac{S(h) - I}{h}\right) x. \tag{1.3.53}$$

Since $x \in D(A)$ we know that

$$\lim_{h\to 0_+} \left(\frac{S(h) - I}{h}\right) x = Ax,$$

and, as $S(t) \in \mathcal{L}(X)$, we obtain from (1.3.53) that

$$\lim_{h\to 0_+} S(t)\left(\frac{S(h) - I}{h}\right) x = S(t)\lim_{h\to 0_+}\left(\frac{S(h) - I}{h}\right) x = S(t)Ax,$$

which implies that $S(t)x \in D(A)$ and, therefore, by the very definition of A,

$$AS(t)x = S(t)Ax.$$

We now prove identity (1.3.49). For $h > 0$ and $t > 0$, from the above we have

$$\frac{d}{dt}^+ S(t)x = \lim_{h\to 0_+} \frac{(S(t+h) - S(t))x}{h} = S(t)Ax = AS(t)x. \tag{1.3.54}$$

Now suppose $0 < h < t$. Then

$$\begin{aligned}
\frac{S(t-h)x - S(t)x}{-h} &= S(t-h)\left(\frac{x - S(h)x}{-h}\right) \\
&= S(t-h)\left(\frac{S(h)x - x}{h}\right) \\
&= S(t-h)\left[\left(\frac{S(h) - I}{h}\right) x - Ax + Ax\right] \\
&= S(t-h)\left[\left(\frac{S(h) - I}{h}\right) x - Ax\right] + S(t-h)Ax.
\end{aligned} \tag{1.3.55}$$

Since $\|S(t-h)\|$ is bounded on bounded intervals and

$$\lim_{h\to 0_+} \left(\frac{S(h) - I}{h}\right) x = Ax \quad (\text{because } x \in D(A)),$$

we obtain

$$\lim_{h\to 0} \left[\left(\frac{S(h) - I}{h}\right) x - Ax\right] = 0. \tag{1.3.56}$$

Moreover, since S is strongly continuous, we have

$$\lim_{h\to 0_+} S(t-h)Ax = S(t)Ax. \tag{1.3.57}$$

Combining (1.3.55), (1.3.56) and (1.3.57) we conclude that

$$\frac{d}{dt}^{-} S(t)x = S(t)Ax, \tag{1.3.58}$$

and consequently $S(t)x$ is left differentiable for all $t \geq 0$. From (1.3.54) and (1.3.58) it follows that

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax,$$

which proves item (i).

(ii) Let $x \in D(A)$. From item (i) we know that $\frac{d}{dt} S(t)x$ is a continuous function of t for all $x \in D(A)$, since S is strongly continuous. Hence we may integrate over compact intervals in $\mathbb{R}_+$ and obtain

$$\int_s^t \frac{d}{d\xi} S(\xi)x \, d\xi = \int_s^t AS(\xi)x \, d\xi = \int_s^t S(\xi)Ax \, d\xi,$$

that is,

$$S(t)x - S(s)x = \int_s^t AS(\xi)x \, d\xi = \int_s^t S(\xi)Ax \, d\xi,$$

which proves (ii).

(iii) Let $x \in X$. We shall prove that

$$\lim_{h\to 0_+} \left(\frac{S(h) - I}{h} \right) \int_0^t S(\xi)x \, d\xi = S(t)x - x. \tag{1.3.59}$$

Indeed, let $0 < h < t$. By linearity and continuity of the operator $\frac{S(h)-I}{h}$ we have

$$\begin{aligned}
&\frac{S(h) - I}{h} \left(\int_0^t S(\xi)x \, d\xi \right) \qquad (1.3.60)\\
&= \frac{1}{h} \left(\int_0^t S(\xi + h)x \, d\xi - \int_0^t S(\xi)x \, d\xi \right)\\
&= \frac{1}{h} \int_h^{t+h} S(\xi)x \, d\xi - \frac{1}{h} \int_0^t S(\xi)x \, d\xi\\
&= \frac{1}{h} \int_h^t S(\xi)x \, d\xi + \frac{1}{h} \int_t^{t+h} S(\xi)x \, d\xi - \frac{1}{h} \int_0^h S(\xi)x \, d\xi - \frac{1}{h} \int_h^t S(\xi)x \, d\xi\\
&= \frac{1}{h} \int_t^{t+h} S(\xi)x \, d\xi - \frac{1}{h} \int_0^h S(\xi)x \, d\xi.
\end{aligned}$$

On the other hand, by the mean value theorem (see Exercise 1.1.5 of Section 1.1),

$$\lim_{h\to 0_+} \frac{1}{h} \int_t^{t+h} S(\xi)x \, d\xi = S(t)x, \quad \text{and} \quad \lim_{h\to 0_+} \frac{1}{h} \int_0^h S(\xi)x \, d\xi = S(0)x. \tag{1.3.61}$$

From (1.3.60) and (1.3.61) we obtain (1.3.51), and, by the definition of A,

$$A\int_0^t S(\xi)x\,d\xi = S(t)x - x.$$

□

Proposition 1.31 *The infinitesimal generator of a semigroup of class C_0 is a closed linear operator and $D(A)$ is dense in X.*

Proof: We first prove that $D(A)$ is dense in X by constructing a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ converging to $x \in X$. Let $x \in X$ and, for each $n \in \mathbb{N}^*$, define

$$x_n = \frac{1}{1/n}\int_0^{\frac{1}{n}} S(t)x\,dt.$$

Note that for each $n \in \mathbb{N}^*$ we have $x_n \in D(A)$ in view of Proposition 1.29 and Proposition 1.2.24(iii). Moreover, by the mean value theorem we have

$$x_n = \frac{1}{1/n}\int_0^{\frac{1}{n}} S(t)x\,dt \to S(0)x = x \quad \text{as } n \to \infty,$$

which proves the density. We now show that A is closed. Let $(x_n)_{n\in\mathbb{N}} \subset D(A)$ be such that

$$x_n \to x \quad \text{and} \quad Ax_n \to y \text{ in } X. \tag{1.3.62}$$

From (1.3.50) we may write

$$S(h)x_n - x_n = \int_0^h S(t)Ax_n\,dt, \quad h > 0. \tag{1.3.63}$$

By (1.3.45) there exists $C > 0$ such that

$$\begin{aligned} \|S(t)Ax_n - S(t)y\| &\le \|S(t)\|_{\mathcal{L}(X)}\|Ax_n - y\| \\ &\le C\|Ax_n - y\|, \quad \text{for all } t \in [0,h]. \end{aligned} \tag{1.3.64}$$

From (1.3.62) and (1.3.64) we conclude that

$$S(t)Ax_n \to S(t)y \quad \text{as } n \to \infty. \tag{1.3.65}$$

Thus, from (1.3.62), (1.3.63) and (1.3.65), and using that $S(h) \in \mathcal{L}(X)$, we obtain in the limit that

$$S(h)x - x = \int_0^h S(t)y\,dt,$$

whence

$$\frac{S(h)x - x}{h} = \frac{1}{h}\int_0^h S(t)y\,dt.$$

Letting $h \to 0_+$ in the identity above and using the mean value theorem, we conclude that $x \in D(A)$ and $Ax = y$, which shows that A is closed and completes the proof. □

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Example 1.32 Let S be a C_0–semigroup, A the infinitesimal generator of S, and $\lambda \in \mathbb{C}$. Then

$$\tilde{S}(t) := e^{-\lambda t} S(t), \ t \geq 0,$$

is a C_0–semigroup whose infinitesimal generator is $(A - \lambda I)$.

Indeed, it is clear that for every $t \geq 0$, $\tilde{S}(t) \in \mathcal{L}(X)$ since $S(t) \in \mathcal{L}(X)$. Moreover:

(i) $\tilde{S}(0) = S(0) = I$.

(ii) $\tilde{S}(t+s) = e^{-\lambda(t+s)} S(t+s) = e^{-\lambda t} e^{-\lambda s} S(t) S(s) = \tilde{S}(t)\tilde{S}(s)$.

(iii)

$$\begin{aligned} \|\tilde{S}(t)x - x\| &= \|e^{-\lambda t} S(t)x - x\| \\ &\leq \|e^{-\lambda t} S(t)x - e^{-\lambda t} x\| + \|e^{-\lambda t} x - x\| \\ &= |e^{-\lambda t}| \, \|S(t)x - x\| + |e^{-\lambda t} - 1| \, \|x\| \to 0 \text{ as } t \to 0_+, \end{aligned}$$

since S is a C_0–semigroup and hence

$$\lim_{t \to 0_+} \|S(t)x - x\| = 0, \qquad \lim_{t \to 0_+} |e^{-\lambda t} - 1| = 0.$$

Therefore, $\tilde{S}$ is a C_0–semigroup. On the other hand, if $\tilde{A}$ denotes the infinitesimal generator of $\tilde{S}$, then

$$D(\tilde{A}) = \left\{ x \in X; \ \lim_{h \to 0_+} \frac{\tilde{S}(h)x - x}{h} \text{ exists} \right\}.$$

For all $x \in X$ we have

$$\begin{aligned} \frac{\tilde{S}(h)x - x}{h} &= \frac{e^{-\lambda h} S(h)x - x}{h} \\ &= \frac{e^{-\lambda h} S(h)x - e^{-\lambda h} x}{h} + \frac{e^{-\lambda h} x - x}{h} \\ &= \frac{e^{-\lambda h}(S(h)x - x)}{h} + \frac{(e^{-\lambda h} - 1)x}{h}. \end{aligned} \tag{1.3.66}$$

Thus, for every $x \in D(\tilde{A})$ it follows from (1.3.66) that

$$\frac{S(h)x - x}{h} = e^{\lambda h} \left(\frac{\tilde{S}(h)x - x}{h} \right) - e^{\lambda h} \left(\frac{e^{-\lambda h} - 1}{h} \right) x \to \tilde{A}x + \lambda x \quad \text{as } h \to 0_+,$$

since

$$\lim_{h \to 0_+} e^{\lambda h} = 1, \qquad \lim_{h \to 0_+} \left(\frac{e^{-\lambda h} - 1}{h} \right) = -\lambda, \qquad \lim_{h \to 0_+} \frac{\tilde{S}(h)x - x}{h} = \tilde{A}x.$$

Hence if $x \in D(\tilde{A})$, then $x \in D(A)$ and

$$Ax = \tilde{A}x + \lambda x,$$

that is, $\tilde{A}x = Ax - \lambda x$.

Conversely, for $x \in D(A)$, using (1.3.66) we analogously obtain

$$\frac{\tilde{S}(h)x - x}{h} = \frac{e^{-\lambda h}(S(h)x - x)}{h} + \frac{(e^{-\lambda h} - 1)x}{h} \to Ax - \lambda x, \quad h \to 0_+,$$

showing that $x \in D(\tilde{A})$ and $\tilde{A}x = Ax - \lambda x$. Thus,

$$D(\tilde{A}) = D(A) \quad \text{and} \quad \tilde{A}x = Ax - \lambda x.$$

1.3.1 Exercises

Exercise 1.3.1 Let S_1 and S_2 be C_0–semigroups with infinitesimal generator A for both S_1 and S_2. Prove that $S_1 = S_2$.

Exercise 1.3.2 Exponential functions are examples of C_0–semigroups, which follows from Theorem 1.1 and from the fact that uniform convergence implies strong convergence. Prove that the infinitesimal generator of e^{tA}, $A \in \mathcal{L}(X)$, is A.

Exercise 1.3.3 Let X be a Banach space and $f : (a,b) \to X$ a continuous function such that $f'_+(t) = 0$ for every $t \in (a,b)$. Prove that f is constant on (a,b).

Exercise 1.3.4 (Dini's lemma) Let X be a Banach space and let $f : (a,b) \to X$ be a continuous function on (a,b) which admits a right derivative f'_+ continuous on (a,b). Prove that f is of class $C^1(a,b)$. [Kosaku Yosida – Functional Analysis]

Exercise 1.3.5 Let $C_b(\mathbb{R})$ be the Banach space of bounded and uniformly continuous functions on $\mathbb{R}$, with the norm $\|u\| = \sup\limits_{x\in\mathbb{R}}|u(x)|$. Consider the mapping

$$S : \mathbb{R}_+ \to \mathcal{L}(C_b(\mathbb{R})),$$

defined by

$$(S(t)u)(x) = u_t(x) = u(x+t), \quad \text{for all } x \in \mathbb{R}.$$

Prove that:

- S is well defined.
- $S(t)$ is an isometry.
- S is a C_0–semigroup.
- Determine the infinitesimal generator A of S (use Dini's lemma – Exercise 1.3.4).

Exercise 1.3.6 Let N_t, $t > 0$, be the function on $\mathbb{R}^n$ defined by

$$N_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4t}}.$$

Define

$$S : [0,\infty) \to \mathcal{L}(L^2(\mathbb{R}^n)),$$
$$[S(0)u](x) = u(x), \quad \forall x \in \mathbb{R}^n,$$
$$[S(t)u](x) = (N_t * u)(x), \quad \forall x \in \mathbb{R}^n, \ \forall t > 0.$$

Prove that:

- S is well defined and

$$\|S(t)u\|_{L^2(\mathbb{R}^\ltimes)} \le \|u\|_{L^2(\mathbb{R}^n)}, \quad \forall u \in L^2(\mathbb{R}^n).$$

- S is a C_0–semigroup.
- Determine the infinitesimal generator A of S.

Exercise 1.3.7 Let S be a C_0–semigroup with infinitesimal generator A. Define $A^0 = I$, $A^1 = A$, and, assuming A^{n-1} is defined, define A^n by

$$D(A^n) = \{x \in X;\ x \in D(A^{n-1}) \text{ and } A^{n-1}x \in D(A)\},$$

$$A^n x = A(A^{n-1}x), \quad \forall x \in D(A^n).$$

Prove that:

- $D(A^n)$ is a subspace of X and A^n is a linear operator on X.
- If $x \in D(A^n)$ then $S(t)x \in D(A^n)$ for all $t \geq 0$ and
$$\frac{d^n}{dt^n} S(t)x = A^n S(t)x = S(t)A^n x, \quad \forall n \in \mathbb{N}.$$
- If $x \in D(A^n)$, prove that Taylor's formula holds:
$$S(t)x = \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} A^k S(a)x + \frac{1}{(n-1)!} \int_a^t (t-u)^{n-1} A^n S(u)x\, du.$$
- Prove that
$$(S(t)-I)^n x = \int_0^t \cdots \int_0^t S(u_1 + \cdots + u_n) A^n x\, du_1 \cdots du_n, \quad \forall x \in D(A^n).$$
- Prove that $\bigcap_{n=1}^{\infty} D(A^n)$ is dense in X.

1.4 The Hille–Yosida Theorem

In this section we present a necessary and sufficient condition for a linear operator A to be the infinitesimal generator of a C_0–semigroup. Before that, however, we make some preliminary considerations.

Let A be a linear operator on a Banach space X. The set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - A$ is invertible, its inverse is bounded and densely defined, is called the *resolvent set of* A and is denoted by $\rho(A)$. The set $\sigma(A) = \mathbb{C} \backslash \rho(A)$ is called the *spectrum of* A.

If $\lambda \in \rho(A)$, the operator $(\lambda I - A)^{-1}$, denoted by $R(\lambda, A)$, is called the *resolvent of* A. Hence $R(\lambda, A)$ is, by definition, a linear and bounded operator and densely defined. Observe that $R(\lambda, A)$ is an operator defined on $Im(\lambda I - A)$ with values in $D(A)$, where the closure of $Im(\lambda I - A)$ equals X.

Proposition 1.33 *Let A be a closed linear operator on a Banach space X and let $\lambda \in \rho(A)$. Then $D(R(\lambda, A)) = X$ and hence $R(\lambda, A)$ is closed.*

Proof: Let $y \in X$. Since $D(R(\lambda, A))$ is dense in X, there exists a sequence $(y_n)_{n \in \mathbb{N}} \subset D(R(\lambda, A))$ such that

$$y_n \to y \quad \text{in } X. \tag{1.4.67}$$

However, for each $n \in \mathbb{N}$, there exists $x_n \in D(\lambda I - A)$ such that

$$y_n = (\lambda I - A)x_n. \tag{1.4.68}$$

On the other hand, for every $x \in D(A)$, by the continuity of $R(\lambda, A)$ we have

$$\|x\| = \|R(\lambda, A)(\lambda I - A)x\| \leq C_1\|(\lambda I - A)x\|,$$

where C_1 is a positive constant. Hence

$$\|(\lambda I - A)x\| \geq C_2\|x\|, \quad \text{for all } x \in D(A), \tag{1.4.69}$$

where $C_2 > 0$ is a constant. In particular, for the sequence $(x_n)_{n\in\mathbb{N}}$, it follows from (1.4.69) that

$$\begin{aligned} \|(\lambda I - A)x_n - (\lambda I - A)x_m\| &= \|(\lambda I - A)(x_n - x_m)\| \\ &\geq C_2\|x_n - x_m\|, \quad \text{for all } m, n \in \mathbb{N}. \end{aligned} \tag{1.4.70}$$

Thus, from (1.4.67) and (1.4.70) it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X. Hence there exists $x \in X$ such that

$$x_n \to x \quad \text{in } X. \tag{1.4.71}$$

Moreover, from (1.4.67) and (1.4.68) we have

$$(\lambda I - A)x_n \to y \quad \text{in } X. \tag{1.4.72}$$

Since A is closed, $(\lambda I - A)$ is also closed and from (1.4.71) and (1.4.72) we obtain

$$x \in D(A) \quad \text{and} \quad (\lambda I - A)x = y,$$

that is, $y \in Im(\lambda I - A) = D(R(\lambda, A))$, which proves $D(R(\lambda, A)) = X$.

Therefore, $R(\lambda, A)$ is a continuous operator defined on the whole space X and hence closed, which completes the proof. □

Proposition 1.34 *Let S be a C_0–semigroup with infinitesimal generator A. If $\lambda \in \mathbb{C}$ is such that* $\operatorname{Re}\lambda > \omega_0$*, where*

$$\omega_0 = \lim_{t\to\infty} \frac{\ln\|S(t)\|}{t},$$

then the integral $\int_0^\infty e^{-\lambda t}S(t)x\,dt$ exists for every $x \in X$ and $\lambda \in \rho(A)$. Moreover,

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}S(t)x\,dt, \quad \textit{for all } x \in X.$$

Proof: Let $x \in X$ and $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda > \omega_0$. Choose ω with $\operatorname{Re}\lambda > \omega > \omega_0$. Then, from (1.3.45) there exists $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad \text{for all } t \geq 0. \tag{1.4.73}$$

It follows from (1.4.73) that

$$\begin{aligned} \|e^{-\lambda t}S(t)x\| &\leq M\|x\|e^{-(\mathrm{Re}\,\lambda)t}e^{\omega t} \\ &= M\|x\|e^{-(\mathrm{Re}\,\lambda-\omega)t}. \end{aligned} \tag{1.4.74}$$

The function $t \mapsto M\|x\|e^{-(\mathrm{Re}\,\lambda)t}e^{\omega t}$ is continuous on $[0, \infty)$ and integrable, since

$$\begin{aligned} \int_0^\infty M\|x\|e^{-(\mathrm{Re}\,\lambda)t}e^{\omega t}\,dt &= M\|x\|\left[\frac{e^{-(\mathrm{Re}\,\lambda-\omega)t}}{-(\mathrm{Re}\,\lambda-\omega)}\right]_{t=0}^{t=\infty} \\ &= \frac{M\|x\|}{\mathrm{Re}\,\lambda-\omega} < +\infty, \end{aligned} \tag{1.4.75}$$

since $\mathrm{Re}\,\lambda > \omega$. Now the mapping $t \mapsto e^{-\lambda t}S(t)x$ is continuous on $[0, \infty)$ with values in X and is thus Bochner–integrable on each interval $[0, b]$, $b > 0$. From (1.4.74), (1.4.75) and the Weierstrass test it follows that

$$\int_0^\infty \|e^{-\lambda t}S(t)x\|\,dt < +\infty,$$

and consequently the integral $\int_0^\infty e^{-\lambda t}S(t)x\,dt$ exists. For each $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \omega > \omega_0$, define the linear operator on X:

$$R_\lambda x = \int_0^\infty e^{-\lambda t}S(t)x\,dt.$$

From (1.4.74) and (1.4.75) we get

$$\|R_\lambda x\| \leq \left(\frac{M}{\mathrm{Re}\,\lambda-\omega}\right)\|x\|,$$

that is,

$$R_\lambda \in \mathcal{L}(X) \quad \text{and} \quad \|R_\lambda\|_{\mathcal{L}(X)} \leq \frac{M}{\mathrm{Re}\,\lambda-\omega}. \tag{1.4.76}$$

We claim that

$$\lim_{h\to 0_+}\left(\frac{S(h)-I}{h}\right)R_\lambda x = \lambda R_\lambda x - x, \quad \text{for all } x \in X. \tag{1.4.77}$$

Indeed, let $h > 0$. Then

$$\begin{aligned}
\left(\frac{S(h)-I}{h}\right) R_\lambda x &= \left(\frac{S(h)-I}{h}\right) \int_0^\infty e^{-\lambda t} S(t)x\,dt \\
&= \frac{1}{h}\int_0^\infty e^{-\lambda t} S(t+h)x\,dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} S(t)x\,dt \\
&= \frac{1}{h}\int_h^\infty e^{-\lambda(\xi-h)} S(\xi)x\,d\xi - \frac{1}{h}\int_0^\infty e^{-\lambda t} S(t)x\,dt \\
&= \frac{1}{h}\int_h^\infty e^{-\lambda(t-h)} S(t)x\,dt + \frac{1}{h}\int_0^h e^{-\lambda(t-h)} S(t)x\,dt \\
&\quad -\frac{1}{h}\int_0^h e^{-\lambda(t-h)} S(t)x\,dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} S(t)x\,dt \\
&= \frac{e^{\lambda h}}{h}\int_h^\infty e^{-\lambda t} S(t)x\,dt + \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)x\,dt \\
&\quad -\frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)x\,dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} S(t)x\,dt \\
&= \left(\frac{e^{\lambda h}-1}{h}\right)\int_0^\infty e^{-\lambda t} S(t)x\,dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)x\,dt,
\end{aligned}$$

that is,

$$\left(\frac{S(h)-I}{h}\right) R_\lambda x = \left(\frac{e^{\lambda h}-1}{h}\right)\int_0^\infty e^{-\lambda t} S(t)x\,dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)x\,dt. \tag{1.4.78}$$

By l'Hôpital's rule,

$$\left(\frac{e^{\lambda h}-1}{h}\right)\int_0^\infty e^{-\lambda t} S(t)x\,dt \to \lambda\int_0^\infty e^{-\lambda t} S(t)x\,dt = \lambda R_\lambda x \quad \text{in } X \text{ as } h \to 0_+,$$

and by the Mean Value Theorem

$$\frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)x\,dt \to x \quad \text{in } X \text{ as } h \to 0_+.$$

From these convergences and (1.4.78) we obtain (1.4.77). It follows that

$$R_\lambda x \in D(A) \quad \text{and} \quad AR_\lambda x = \lambda R_\lambda x - x, \quad \text{for all } x \in X. \tag{1.4.79}$$

Thus, from (1.4.79) we deduce

$$x = \lambda R_\lambda x - AR_\lambda x = (\lambda I - A)R_\lambda x, \quad \text{for all } x \in X, \tag{1.4.80}$$

that is, R_λ is a right inverse of $\lambda I - A$. It remains to prove that R_λ is also a left inverse of $\lambda I - A$. Let $x \in D(A)$. Then

$$S(t)x \in D(A) \quad \text{and} \quad AS(t)x = S(t)Ax,$$

which implies that, for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega > \omega_0$ we may write

$$R_\lambda Ax = \int_0^\infty e^{-\lambda t} S(t)Ax\,dt = \int_0^\infty e^{-\lambda t} AS(t)x\,dt. \tag{1.4.81}$$

Since A is a closed linear operator, we use the theorem which says:
"Let A be a closed operator on X (that is, an operator with domain and image contained in X) and

let f be a continuous function on $[a,b]$ with values in $D(A)$ such that Af is continuous on $[a,b]$. Then $\int_a^b f(t)\,dt \in D(A)$ and $A\int_a^b f(t)\,dt = \int_a^b Af(t)\,dt$." This guarantees that

$$\int_0^\infty e^{-\lambda t} AS(t)x\,dt = A\int_0^\infty e^{-\lambda t} S(t)x\,dt. \tag{1.4.82}$$

From (1.4.81) and (1.4.82) we conclude that

$$R_\lambda Ax = AR_\lambda x, \quad \text{for all } x \in D(A). \tag{1.4.83}$$

Finally, combining (1.4.79) and (1.4.83) we obtain

$$R_\lambda Ax = \lambda R_\lambda x - x, \quad \text{for all } x \in D(A),$$

that is,

$$x = \lambda R_\lambda x - R_\lambda Ax = R_\lambda(\lambda I - A)x,$$

which shows that R_λ is a left inverse of $\lambda I - A$ and therefore

$$R_\lambda = (\lambda I - A)^{-1}, \quad \text{for all } \lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda > \omega_0.$$

Consequently, $(\lambda I - A)^{-1}$ exists, is bounded (by the Open Mapping Theorem and the fact that R_λ is bounded) and, in addition,

$$D((\lambda I - A)^{-1}) = D(R_\lambda) = X,$$

so that $(\lambda I - A)^{-1}$ is densely defined. Hence $\lambda \in \rho(A)$ and

$$R(\lambda, A)x = R_\lambda x = \int_0^\infty e^{-\lambda t} S(t)x\,dt, \quad \text{for all } x \in X,$$

which completes the proof. □

Corollary 1.35 *Under the same assumptions as in Proposition 1.34, we have*

$$(i) \quad \tfrac{d^n}{d\lambda^n} R(\lambda, A)x = (-1)^n n!\, R(\lambda, A)^{n+1}x, \quad \textit{for every } x \in X.$$

$$(ii) \quad \tfrac{d^n}{d\lambda^n} R(\lambda, A)x = \textstyle\int_0^\infty e^{-\lambda t}(-t)^n S(t)x\,dt, \quad \textit{for every } x \in X.$$

Proof: (i) We first show that

$$\lim_{\mu\to\lambda} R(\mu, A)x = R(\lambda, A)x, \quad \text{for every } x \in X. \tag{1.4.84}$$

Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}(\lambda) > \omega_1 > \omega > \omega_0$, and consider a sequence $(\mu_\nu) \subset \mathbb{C}$ such that $\mu_\nu \to \lambda$ as $\nu \to +\infty$ and $\operatorname{Re}(\mu_\nu) > \omega_1$. We claim that

$$\begin{aligned} &\text{For each } x \in X \text{ and } t \in \mathbb{R}_+, \text{ we have} \\ &\lim_{\nu\to+\infty} e^{-\mu_\nu t} S(t)x = e^{-\lambda t} S(t)x \quad \text{in } X. \end{aligned} \tag{1.4.85}$$

Indeed,

$$\|e^{-\mu_\nu t} S(t)x - e^{-\lambda t} S(t)x\| = |e^{-\mu_\nu t} - e^{-\lambda t}|\,\|S(t)x\| \to 0, \quad \text{as } \nu \to +\infty,$$

since the exponential function is continuous, which proves (1.4.85).

On the other hand, from (1.4.73) we have

$$\begin{aligned}\|e^{-\mu_\nu t}S(t)x\| &= |e^{-\mu_\nu t}|\,\|S(t)x\| \\ &\leq e^{-\operatorname{Re}(\mu_\nu)t}\|S(t)\|_{\mathcal{L}(X)}\|x\| \\ &\leq Me^{-(\operatorname{Re}\mu_\nu-\omega)t}\|x\| \\ &\leq Me^{-(\omega_1-\omega)t}\|x\|.\end{aligned}$$

Since $-(\omega_1-\omega)<0$, it follows that

$$\int_0^\infty M\|x\|e^{-(\omega_1-\omega)t}\,dt<+\infty,$$

and furthermore,

$$\int_0^C e^{-\mu_\nu t}S(t)x\,dt<+\infty \quad \text{and} \quad \int_0^C e^{-\lambda t}S(t)x\,dt<+\infty, \quad \text{for all } \nu\in\mathbb{N} \text{ and } C>0.$$

Hence, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{\nu\to\infty}\underbrace{\int_0^\infty e^{-\mu_\nu t}S(t)x\,dt}_{=R(\mu_\nu,A)}=\underbrace{\int_0^\infty e^{-\lambda t}S(t)x\,dt}_{=R(\lambda,A)},$$

which proves (1.4.84).

Now, if $\operatorname{Re}\lambda>\omega_0$ and $\operatorname{Re}\mu>\omega_0$, we have

$$[(\lambda I-A)(\mu I-A)]^{-1}=(\mu I-A)^{-1}(\lambda I-A)^{-1},$$

and since $(\lambda I-A)(\mu I-A)=(\mu I-A)(\lambda I-A)$, it follows that

$$[(\lambda I-A)(\mu I-A)]^{-1}=[(\mu I-A)(\lambda I-A)]^{-1}=(\lambda I-A)^{-1}(\mu I-A)^{-1},$$

hence

$$(\mu I-A)^{-1}(\lambda I-A)^{-1}=(\lambda I-A)^{-1}(\mu I-A)^{-1}.$$

Thus,

$$\begin{aligned}R(\lambda,A)-R(\mu,A) &= (\lambda I-A)^{-1}-(\mu I-A)^{-1} \\ &= (\mu I-A)(\mu I-A)^{-1}(\lambda I-A)^{-1} \\ &\quad -(\lambda I-A)(\lambda I-A)^{-1}(\mu I-A)^{-1} \\ &= [(\mu I-A)-(\lambda I-A)](\lambda I-A)^{-1}(\mu I-A)^{-1} \\ &= (\mu-\lambda)I(\lambda I-A)^{-1}(\mu I-A)^{-1} \\ &= (\mu-\lambda)\,R(\lambda,A)R(\mu,A).\end{aligned}$$

Therefore,

$$R(\lambda,A)-R(\mu,A)=(\mu-\lambda)R(\lambda,A)R(\mu,A),$$

which implies

$$\frac{R(\lambda, A) - R(\mu, A)}{\mu - \lambda} = R(\lambda, A)R(\mu, A), \quad (\mu \neq \lambda),$$

or equivalently,

$$\frac{R(\mu, A) - R(\lambda, A)}{\mu - \lambda} = -R(\lambda, A)R(\mu, A), \quad (\mu \neq \lambda). \tag{1.4.86}$$

Let $\mu \in \mathbb{C}$ and $\mu \to \lambda$. Then, from (1.4.84) and (1.4.86), and using the continuity of $R(\lambda, A)$, we obtain

$$\begin{aligned}
\lim_{\mu\to\lambda} \left[\frac{R(\mu, A) - R(\lambda, A)}{\mu - \lambda}\right] x &= \lim_{\mu\to\lambda} [-R(\lambda, A)R(\mu, A)]x \\
&= -R(\lambda, A) \lim_{\mu\to\lambda} R(\mu, A)x \\
&= -R(\lambda, A)^2 x.
\end{aligned}$$

Therefore,

$$\frac{d}{d\lambda} R(\lambda, A)x = -R(\lambda, A)^2 x, \quad \text{for every } x \in X, \tag{1.4.87}$$

which proves item (i) for $n = 1$. We now use induction on n. Assume that (i) holds for n and let us prove it for $n + 1$. From the induction hypothesis we have

$$\begin{aligned}
\frac{d^{n+1}}{d\lambda^{n+1}} R(\lambda, A)x &= \frac{d}{d\lambda}\left(\frac{d^n}{d\lambda^n} R(\lambda, A)\right) x \qquad (1.4.88)\\
&= \frac{d}{d\lambda}\big((-1)^n n! R(\lambda, A)^{n+1} x\big) \\
&= (-1)^n n! \frac{d}{d\lambda} R(\lambda, A)^{n+1} x.
\end{aligned}$$

We claim that

$$\frac{d}{d\lambda} R(\lambda, A)^n x = nR(\lambda, A)^{n-1} \frac{d}{d\lambda} R(\lambda, A)x \underbrace{=}_{\text{by (1.4.87)}} -nR(\lambda, A)^{n+1} x. \tag{1.4.89}$$

For $n = 1$, identity (1.4.89) follows directly from (1.4.87). Assume (1.4.89) holds for n and let us prove it for $n + 1$. Then

$$\begin{aligned}
\frac{d}{d\lambda} R(\lambda, A)^{n+1} x &= \frac{d}{d\lambda}\big(R(\lambda, A)R(\lambda, A)^n x\big) \\
&= \frac{d}{d\lambda} R(\lambda, A)\, R(\lambda, A)^n x + R(\lambda, A)\, \frac{d}{d\lambda} R(\lambda, A)^n x \\
&= -R(\lambda, A)^2 R(\lambda, A)^n x + R(\lambda, A)\big(-nR(\lambda, A)^{n+1} x\big) \\
&= -R(\lambda, A)^{n+2} x - nR(\lambda, A)^{n+2} x \\
&= -(n + 1)R(\lambda, A)^{n+2} x,
\end{aligned}$$

which proves (1.4.89). Combining (1.4.88) and (1.4.89) we obtain

$$\begin{aligned}
\frac{d^{n+1}}{d\lambda^{n+1}} R(\lambda, A)x &= (-1)^n n! \frac{d}{d\lambda} R(\lambda, A)^{n+1} x \\
&= (-1)^n n!\big(-(n + 1)R(\lambda, A)^{n+2} x\big) \\
&= (-1)^{n+1}(n + 1)! R(\lambda, A)^{n+2} x,
\end{aligned}$$

which completes the proof of item (i).

(ii) First note that the function $t^n e^{-\lambda t} S(t)x$ is continuous in λ, as is its derivative with respect to λ, namely $-t^{n+1}e^{-\lambda t}S(t)x$. Moreover, for $\mathrm{Re}(\lambda) > \omega_1 > \omega > \omega_0$ we have

$$\begin{aligned} \|t^n e^{-\lambda t} S(t)x\| &\leq t^n |e^{-\lambda t}| \, \|S(t)\|_{\mathcal{L}(X)} \|x\| \\ &\leq t^n e^{-\mathrm{Re}(\lambda) t} M e^{\omega t} \|x\| \\ &= M t^n e^{-(\mathrm{Re}(\lambda) - \omega) t} \|x\| \\ &\leq M t^n e^{-(\omega_1 - \omega) t} \|x\|. \end{aligned} \tag{1.4.90}$$

We claim that

$$\int_0^\infty t^n e^{-(\omega_1 - \omega)t} \, dt < +\infty \quad \text{for } \omega_1 > \omega > \omega_0. \tag{1.4.91}$$

Indeed, we proceed by induction on n. For $n = 0$ we have

$$\int_0^\infty e^{-(\omega_1 - \omega)t} \, dt = \frac{1}{\omega_1 - \omega} < +\infty.$$

Assume that (1.4.91) holds for n and let us prove it for $n+1$. Let $b > 0$ and consider

$$\int_0^b t^{n+1} e^{-(\omega_1 - \omega)t} \, dt.$$

Integrating by parts, we obtain

$$\begin{aligned} \int_0^b t^{n+1} e^{-(\omega_1 - \omega)t} \, dt &= \left. \frac{-t^{n+1} e^{-(\omega_1 - \omega)t}}{\omega_1 - \omega} \right|_0^b \\ &\quad + \frac{n+1}{\omega_1 - \omega} \int_0^b t^n e^{-(\omega_1 - \omega)t} \, dt. \end{aligned} \tag{1.4.92}$$

Note that

$$\left. \frac{-t^{n+1} e^{-(\omega_1 - \omega)t}}{\omega_1 - \omega} \right|_0^b = \frac{-b^{n+1} e^{-(\omega_1 - \omega)b}}{\omega_1 - \omega},$$

and therefore, by L'Hôpital's rule,

$$\lim_{b \to \infty} \frac{-b^{n+1} e^{-(\omega_1 - \omega)b}}{\omega_1 - \omega} = \frac{-1}{\omega_1 - \omega} \lim_{b \to \infty} \frac{b^{n+1}}{e^{(\omega_1 - \omega)b}} = 0.$$

Moreover, by the induction hypothesis,

$$\lim_{b \to \infty} \int_0^b t^n e^{-(\omega_1 - \omega)t} \, dt < +\infty.$$

Hence, from (1.4.92) and the above, we conclude that

$$\int_0^\infty t^n e^{-(\omega_1 - \omega)t} \, dt < +\infty,$$

which proves (1.4.91). From (1.4.90) we obtain

$$\|t^n e^{-\lambda t} S(t)x\| \leq M t^n e^{-(\omega_1 - \omega)t} \|x\|, \quad \text{for all } \lambda \in \mathbb{C} \text{ with } \mathrm{Re}(\lambda) > \omega_1 > \omega > \omega_0,$$

and

$$\int_0^\infty M t^n e^{-(\omega_1-\omega)t}\|x\|\,dt < +\infty.$$

Thus, by the Weierstrass M–test, the integral

$$\int_0^\infty t^n e^{-\lambda t} S(t)x\,dt$$

converges absolutely and uniformly for $\mathrm{Re}(\lambda) > \omega_1 > \omega > \omega_0$ and $n = 0, 1, \ldots$. Consequently,

$$\int_0^\infty t^{n+1} e^{-\lambda t} S(t)x\,dt$$

also converges absolutely and uniformly in the same region, and therefore it is legitimate to differentiate

$$\int_0^\infty t^n e^{-\lambda t} S(t)x\,dt$$

with respect to λ, obtaining

$$\begin{aligned} \frac{d}{d\lambda}\int_0^\infty t^n e^{-\lambda t} S(t)x\,dt &= \int_0^\infty \frac{d}{d\lambda}\big(t^n e^{-\lambda t} S(t)x\big)\,dt \\ &= -\int_0^\infty t^{n+1} e^{-\lambda t} S(t)x\,dt. \end{aligned} \tag{1.4.93}$$

We now prove (ii) by induction on n. For $n = 0$, Proposition 1.34 gives

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x\,dt,$$

so the formula holds. Assume (ii) holds for n and let us prove it for $n+1$. From the induction hypothesis and (1.4.93) we obtain

$$\begin{aligned} \frac{d^{n+1}}{d\lambda^{n+1}} R(\lambda, A)x &= \frac{d}{d\lambda}\left(\frac{d^n}{d\lambda^n} R(\lambda, A)x\right) \\ &= \frac{d}{d\lambda}\left(\int_0^\infty e^{-\lambda t}(-t)^n S(t)x\,dt\right) \\ &= (-1)^n \frac{d}{d\lambda}\left(\int_0^\infty e^{-\lambda t} t^n S(t)x\,dt\right) \\ &= (-1)^n(-1)\int_0^\infty t^{n+1} e^{-\lambda t} S(t)x\,dt \\ &= \int_0^\infty e^{-\lambda t}(-t)^{n+1} S(t)x\,dt, \end{aligned}$$

which completes the proof. □

We now prove the main result of this section, which provides a characterisation of the infinitesimal generator of a C_0–semigroup.

Theorem 1.36 [*Hille–Yosida*] *For a linear operator A, defined on $D(A) \subset X$ with values in X, to be the infinitesimal generator of a C_0–semigroup it is necessary and sufficient that:*
(i) A be closed and its domain be dense in X.
(ii) There exist real numbers M and ω such that, for each real $\lambda > \omega$, we have $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^n}, \quad \text{for all } n \in \mathbb{N}.$$

In this case, $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$, $t \geq 0$.

Proof:

(1) Necessity.

Assume that a linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0–semigroup. Then item (i) of the theorem follows immediately from Proposition 1.31. We now prove item (ii). Let $\omega > \omega_0 = \lim_{t\to\infty} \frac{\ln \|S(t)\|}{t}$. Since S is a C_0–semigroup, it follows from (1.3.45) that there exists $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{1.4.94}$$

Thus, if $\lambda > \omega$, then by Proposition 1.34, $\lambda \in \rho(A)$ and, by item (i) of Corollary 1.35, we have

$$R(\lambda, A)^n x = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, A)x, \quad \text{for all } x \in X,$$

which, by item (ii) of Corollary 1.35, is equal to

$$\begin{aligned} R(\lambda, A)^n x &= \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty e^{-\lambda t}(-t)^{n-1} S(t)x\, dt \\ &= \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} S(t)x\, dt. \end{aligned} \tag{1.4.95}$$

Hence, for each $x \in X$, from (1.4.94) and (1.4.95) we obtain

$$\|R(\lambda, A)^n x\| \leq \frac{M\|x\|}{(n-1)!} \int_0^\infty t^{n-1} e^{-(\lambda-\omega)t}\, dt. \tag{1.4.96}$$

We now prove that

$$\int_0^\infty t^{n-1} e^{-(\lambda-\omega)t}\, dt = \frac{(n-1)!}{(\lambda-\omega)^n}. \tag{1.4.97}$$

Indeed, for $n = 1$ we have

$$\int_0^\infty e^{-(\lambda-\omega)t}\, dt = \frac{1}{\lambda - \omega}.$$

Assume that (1.4.97) holds for n and let us prove it for $n + 1$. For any $b > 0$, integrating by parts yields

$$\begin{aligned}\int_0^b t^n e^{-(\lambda-\omega)t}\,dt &= \left[t^n \frac{e^{-(\lambda-\omega)t}}{-(\lambda-\omega)}\right]_{t=0}^{t=b} - \int_0^b nt^{n-1}\frac{e^{-(\lambda-\omega)t}}{-(\lambda-\omega)}\,dt \\ &= \left[t^n \frac{e^{-(\lambda-\omega)t}}{-(\lambda-\omega)}\right]_{t=0}^{t=b} + \frac{n}{\lambda-\omega}\int_0^b t^{n-1}e^{-(\lambda-\omega)t}\,dt.\end{aligned}$$

Taking the limit as $b \to +\infty$ in the last identity and using the induction hypothesis, we obtain

$$\int_0^\infty t^n e^{-(\lambda-\omega)t}\,dt = \frac{n}{\lambda-\omega}\,\frac{(n-1)!}{\lambda-\omega} = \frac{n!}{(\lambda-\omega)^{n+1}},$$

which proves (1.4.97). From (1.4.96) and (1.4.97) it follows that

$$\|R(\lambda,A)^n x\| \le \frac{M}{(\lambda-\omega)^n}\|x\|, \quad \text{for all } x \in X,$$

that is,

$$\|R(\lambda,A)^n\|_{\mathcal{L}(X)} \le \frac{M}{(\lambda-\omega)^n},$$

which proves necessity.

(2) Sufficiency.

Assume now that there exist real numbers M and ω such that, for each real $\lambda > \omega$, we have

$$\lambda \in \rho(A) \quad \text{and} \quad \|R(\lambda,A)^n\|_{\mathcal{L}(X)} \le \frac{M}{(\lambda-\omega)^n}, \quad \text{for all } n \in \mathbb{N}, \tag{1.4.98}$$

and, in addition, that A is closed and densely defined. For each $\lambda > \omega$, we define

$$B_\lambda := \lambda^2 R(\lambda,A) - \lambda I. \tag{1.4.99}$$

The operator defined in (1.4.99) is known as the *Yosida approximation of A*. Since $\lambda \in \rho(A)$, $R(\lambda,A)$ is bounded and hence B_λ is also bounded. We shall prove that the exponential e^{tB_λ} converges, as $\lambda \to \infty$, to a C_0–semigroup whose infinitesimal generator is A. The proof is organised in several steps.

1st step.

We first show that

$$\lim_{\lambda\to\infty} B_\lambda x = Ax, \quad \text{for all } x \in D(A). \tag{1.4.100}$$

Indeed, let $x \in D(A)$. Then

$$R(\lambda,A)(\lambda I - A)x = x,$$

and consequently

$$\lambda R(\lambda,A)x - x = R(\lambda,A)Ax. \tag{1.4.101}$$

From (1.4.98) and (1.4.101) we obtain

$$\|\lambda R(\lambda,A)x - x\| = \|R(\lambda,A)Ax\| \le \frac{M}{\lambda-\omega}\|Ax\|.$$

Since the right-hand side tends to zero as $\lambda \to \infty$, we deduce that

$$\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x, \quad \text{for all } x \in D(A). \tag{1.4.102}$$

We now show that the convergence in (1.4.102) actually holds for every $x \in X$. From (1.4.98) we have

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda - \omega},$$

and hence

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{|\lambda|}{\lambda - \omega} M. \tag{1.4.103}$$

Since $\frac{|\lambda|}{\lambda-\omega} \to 1$ as $\lambda \to \infty$, we obtain $\frac{|\lambda| M}{\lambda-\omega} \to M$ as $\lambda \to \infty$. From this and the fact that $M > 0$, there exists $\eta > 0$ such that, if $\lambda > \eta$, then

$$\left| \frac{|\lambda| M}{\lambda - \omega} - M \right| < M,$$

and from (1.4.103) we conclude that

$$\begin{aligned} \|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} - M &\leq \frac{|\lambda| M}{\lambda - \omega} - M \\ &\leq \left| \frac{|\lambda| M}{\lambda - \omega} - M \right| < M, \quad \text{if } \lambda > \eta, \end{aligned}$$

that is,

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} < 2M, \quad \text{if } \lambda > \eta. \tag{1.4.104}$$

Now let $x \in X$. Since $D(A)$ is dense in X, there exists a sequence $(x_n) \subset D(A)$ such that

$$x_n \to x \quad \text{in } X \text{ as } n \to \infty. \tag{1.4.105}$$

Let $\varepsilon > 0$ be given. From (1.4.105) there exists $n_0 \in \mathbb{N}$ such that

$$\|x_n - x\| < \frac{\varepsilon}{2M+2}, \quad \text{for all } n \geq n_0, \tag{1.4.106}$$

and from (1.4.102) there exists $\delta > 0$ such that

$$\|\lambda R(\lambda, A)x_{n_0} - x_{n_0}\| < \frac{\varepsilon}{2M+2}, \quad \text{if } \lambda > \delta. \tag{1.4.107}$$

Hence, from (1.4.104)–(1.4.107), setting $\xi = \max\{\eta, \delta\}$, we obtain

$$\begin{aligned} \|\lambda R(\lambda, A)x - x\| &\leq \|\lambda R(\lambda, A)x - \lambda R(\lambda, A)x_{n_0}\| + \|\lambda R(\lambda, A)x_{n_0} - x_{n_0}\| \\ &\quad + \|x_{n_0} - x\| \\ &< 2M\frac{\varepsilon}{2M+2} + \frac{\varepsilon}{2M+2} + \frac{\varepsilon}{2M+2} = \varepsilon, \end{aligned}$$

which proves that

$$\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x, \quad \text{for all } x \in X. \tag{1.4.108}$$

From (1.4.99) and (1.4.101) we can write

$$B_\lambda x = \lambda^2 R(\lambda, A)x - \lambda x = \lambda\big[\lambda R(\lambda, A)x - x\big] = \lambda R(\lambda, A)Ax, \quad \text{for all } x \in D(A).$$

From the last identity and the convergence in (1.4.108) we obtain (1.4.100).

2nd step.

The next step is to establish an estimate for e^{tB_λ}. More precisely, we shall prove that

$$\begin{gathered}\text{Given } \gamma > \omega \text{ there exists } \lambda_0 > \omega \text{ such that if } \lambda > \lambda_0 \\ \|e^{tB_\lambda}\|_{\mathcal{L}(X)} \leq Me^{t\gamma}.\end{gathered} \tag{1.4.109}$$

Indeed, let $x \in X$. We have

$$\begin{aligned}\|e^{tB_\lambda}x\| &= \|e^{t\lambda^2 R(\lambda,A) - t\lambda I}x\| \\ &= \|e^{t\lambda^2 R(\lambda,A)}e^{-t\lambda I}x\| \\ &\leq \|e^{t\lambda^2 R(\lambda,A)}\|_{\mathcal{L}(X)}\|e^{-t\lambda I}x\|.\end{aligned} \tag{1.4.110}$$

Now,

$$\begin{aligned}\|e^{-t\lambda I}x\| &= \left\|\sum_{n=0}^{\infty}\frac{(-t\lambda)^n}{n!}x\right\| \\ &= \left|\sum_{n=0}^{\infty}\frac{(-t\lambda)^n}{n!}\right| \|x\| = e^{-t\lambda}\|x\|,\end{aligned} \tag{1.4.111}$$

and

$$\begin{aligned}\|e^{t\lambda^2 R(\lambda,A)}\|_{\mathcal{L}(X)} &= \left\|\sum_{n=0}^{\infty}\frac{(t\lambda^2)^n}{n!}R(\lambda, A)^n\right\| \\ &\leq \sum_{n=0}^{\infty}\frac{(t\lambda^2)^n}{n!}\|R(\lambda, A)^n\|.\end{aligned} \tag{1.4.112}$$

From (1.4.98) and (1.4.112) we have

$$\begin{aligned}\|e^{t\lambda^2 R(\lambda,A)}\|_{\mathcal{L}(X)} &\leq M\sum_{n=0}^{\infty}\frac{(t\lambda^2)^n}{n!}(\lambda - \omega)^{-n} \\ &= M\sum_{n=0}^{\infty}\frac{\big(t\lambda^2(\lambda-\omega)^{-1}\big)^n}{n!} = Me^{t\lambda^2(\lambda-\omega)^{-1}}.\end{aligned} \tag{1.4.113}$$

From (1.4.110), (1.4.111) and (1.4.113), for all $x \in X$, we obtain

$$\begin{aligned}\|e^{tB_\lambda}x\| &\leq Me^{t\lambda^2(\lambda-\omega)^{-1}}e^{-t\lambda}\|x\| \\ &= Me^{t\lambda^2(\lambda-\omega)^{-1} - t\lambda}\|x\| \\ &= Me^{t(-\lambda + \lambda^2(\lambda-\omega)^{-1})}\|x\|.\end{aligned} \tag{1.4.114}$$

Note that

$$-\lambda+\lambda^2(\lambda-\omega)^{-1} = \frac{-\lambda(\lambda-\omega)+\lambda^2}{\lambda-\omega} = \frac{-\lambda^2+\lambda\omega+\lambda^2}{\lambda-\omega} = \frac{\lambda\omega}{\lambda-\omega}. \tag{1.4.115}$$

Hence, from (1.4.114) and (1.4.115) we arrive at

$$\|e^{tB_\lambda}x\| \le Me^{t\lambda\omega(\lambda-\omega)^{-1}}\|x\|, \quad \forall x\in X. \tag{1.4.116}$$

However, $\frac{\lambda\omega}{\lambda-\omega}\to\omega$ as $\lambda\to\infty$. Let $\gamma>\omega$ and set $\varepsilon=\gamma-\omega>0$. From this convergence there exists $\lambda_0>\omega$ such that, if $\lambda>\lambda_0$, then

$$\frac{\lambda\omega}{\lambda-\omega}-\omega<\varepsilon=\gamma-\omega,$$

that is,

$$\lambda\omega(\lambda-\omega)^{-1}<\gamma. \tag{1.4.117}$$

From (1.4.116) and (1.4.117) we obtain the desired estimate (1.4.109).

3rd step.

We now show that e^{tB_λ} converges to a bounded linear operator as $\lambda\to\infty$. For this purpose we define

$$S_\lambda(t)=e^{tB_\lambda}, \quad \text{for all } t\ge 0 \text{ and } \lambda>\omega. \tag{1.4.118}$$

We shall prove that

$$\{S_\lambda(t)x\}_{\lambda>\omega} \text{ is a Cauchy family in } X \text{ uniformly on bounded intervals of } [0,\infty). \tag{1.4.119}$$

Observe that

$$(e^{tB_\lambda}-e^{tB_\mu})x=\int_0^t \frac{d}{d\tau}(e^{(t-\tau)B_\mu}e^{\tau B_\lambda})x\,d\tau, \quad \text{for all } x\in X, \tag{1.4.120}$$

that is, from (1.4.118) and (1.4.120) we may write

$$(S_\lambda(t)-S_\mu(t))x=\int_0^t \frac{d}{d\tau}\big(S_\mu(t-\tau)S_\lambda(\tau)\big)x\,d\tau, \quad \text{for all } x\in X. \tag{1.4.121}$$

But

$$\begin{aligned} \frac{d}{d\tau}\big(S_\mu(t-\tau)S_\lambda(\tau)\big)x &= \frac{d}{d\tau}\big(e^{(t-\tau)B_\mu}e^{\tau B_\lambda}\big)x \\ &= \frac{d}{d\tau}\big(e^{tB_\mu+\tau(B_\lambda-B_\mu)}\big)x \\ &= (B_\lambda-B_\mu)e^{tB_\mu+\tau(B_\lambda-B_\mu)}x \\ &= (B_\lambda-B_\mu)e^{(t-\tau)B_\mu}e^{\tau B_\lambda}x \\ &= (B_\lambda-B_\mu)S_\mu(t-\tau)S_\lambda(\tau)x. \end{aligned} \tag{1.4.122}$$

Substituting (1.4.122) into (1.4.121) and observing that B_λ and B_μ commute with $S_\mu(t)$ (we leave the verification of this fact to the reader), we obtain

$$\|S_\lambda(t)x - S_\mu(t)x\| \le \int_0^t \|S_\mu(t-\tau)S_\lambda(\tau)\| \, \|B_\lambda x - B_\mu x\| \, d\tau. \tag{1.4.123}$$

Let $\gamma > \omega$. Then, from (1.4.109) and (1.4.123), for $\lambda, \mu > \lambda_0$, we get

$$\begin{aligned} \|S_\lambda(t)x - S_\mu(t)x\| &\le \int_0^t (Me^{(t-\tau)\gamma})(Me^{\tau\gamma})\|B_\lambda x - B_\mu x\| \, d\tau \\ &= M^2 t e^{\gamma t}\|B_\lambda x - B_\mu x\|. \end{aligned}$$

In the particular case in which $x \in D(A)$, it follows from (1.4.100) that the right-hand side of the last inequality tends to zero as $\lambda, \mu \to \infty$, uniformly in t on any bounded interval, that is,

$$\{S_\lambda(t)x\}_{\lambda>\omega} \text{ is a Cauchy family on bounded intervals of } t \text{ for every } x \in D(A). \tag{1.4.124}$$

Now let $x \in X$. By the density of $D(A)$ in X there exists a sequence $(x_n) \subset D(A)$ such that

$$x_n \to x \quad \text{in } X \text{ as } n \to \infty. \tag{1.4.125}$$

Let J be a bounded interval of $[0, \infty)$ and $\gamma > \omega$. From (1.4.109) and (1.4.118) we infer that

$$\|S_\lambda(t)\|_{\mathcal{L}(X)} = \|e^{tB_\lambda}\|_{\mathcal{L}(X)} \le Me^{t\gamma} \le C, \quad \text{for all } t \in J \text{ and } \lambda > \lambda_0. \tag{1.4.126}$$

Let $\varepsilon > 0$ be given. From (1.4.125) there exists $n_0 \in \mathbb{N}$ such that

$$\|x_n - x\| < \frac{\varepsilon}{2C+1}, \quad \text{for all } n \ge n_0, \tag{1.4.127}$$

where C is the constant in (1.4.126).

On the other hand, from (1.4.124), applied to the interval J, there exists $\alpha > 0$ (coming from the Cauchy property of $\{S_\lambda(t)x_{n_0}\}$ with parameter $\eta = \frac{\varepsilon}{2C+1}$) such that if $\lambda, \mu > \max\{\omega, \lambda_0, \alpha\} := \beta$, then

$$\|S_\lambda(t)x_{n_0} - S_\mu(t)x_{n_0}\| < \frac{\varepsilon}{2C+1}, \quad \text{for all } t \in J. \tag{1.4.128}$$

Hence, from (1.4.125), (1.4.126), (1.4.127) and (1.4.128) we conclude that

$$\begin{aligned} &\|S_\lambda(t)x - S_\mu(t)x\| \\ &\le \|S_\lambda(t)x - S_\lambda(t)x_{n_0}\| + \|S_\lambda(t)x_{n_0} - S_\mu(t)x_{n_0}\| + \|S_\mu(t)x_{n_0} - S_\mu(t)x\| \\ &\le \|S_\lambda(t)\| \, \|x - x_{n_0}\| + \|S_\lambda(t)x_{n_0} - S_\mu(t)x_{n_0}\| + \|S_\mu(t)\| \, \|x_{n_0} - x\| \\ &\le 2C\|x_{n_0} - x\| + \|S_\lambda(t)x_{n_0} - S_\mu(t)x_{n_0}\| < 2C\frac{\varepsilon}{2C+1} + \frac{\varepsilon}{2C+1} = \varepsilon, \end{aligned}$$

for all $\lambda, \mu > \beta$ and all $t \in J$, which proves (1.4.119). Since X is a Banach space, we deduce the existence of a linear mapping $S(t) : X \to X$ such that, for every $x \in X$,

$$S(t)x = \lim_{\lambda\to\infty, \lambda>\omega} S_\lambda(t)x \quad \text{in } X, \text{ uniformly on bounded intervals of the real line.} \tag{1.4.129}$$

We now prove that $S(t) \in \mathcal{L}(X)$. Indeed, from (1.4.129), for each $x \in X$ we have

$$\sup_{\lambda>\omega} \|S_\lambda(t)x\| < +\infty.$$

By the Banach–Steinhaus Uniform Boundedness Principle it follows that

$$\sup_{\lambda>\omega} \|S_\lambda(t)\|_{\mathcal{L}(X)} < +\infty,$$

or equivalently, there exists $C > 0$ such that

$$\|S_\lambda(t)x\| \leq C\|x\|, \quad \text{for all } x \in X \text{ and } \lambda > \omega.$$

Taking limits as $\lambda \to \infty$ and using (1.4.129) we obtain

$$\|S(t)x\| \leq C\|x\|, \quad \text{for all } x \in X,$$

and therefore $S(t) \in \mathcal{L}(X)$, as claimed.

4th step.

We now show that S is a C_0–semigroup. From (1.4.129) we have

$$S(0)x = \lim_{\lambda\to\infty} S_\lambda(0)x = \lim_{\lambda\to\infty} x = x, \quad \text{for all } x \in X. \tag{1.4.130}$$

Moreover, given $t, s \geq 0$ and $x \in X$, we have

$$S(t+s)x = \lim_{\lambda\to\infty} S_\lambda(t+s)x = \lim_{\lambda\to\infty} S_\lambda(t)S_\lambda(s)x. \tag{1.4.131}$$

We claim that

$$\lim_{\lambda\to\infty} S_\lambda(t)S_\lambda(s)x = S(t)S(s)x. \tag{1.4.132}$$

Indeed, let $\varepsilon > 0$ and $\gamma > \omega$. From (1.4.109) we have

$$\|S_\lambda(t)\|_{\mathcal{L}(X)} \leq Me^{t\gamma}, \quad \text{for all } \lambda > \lambda_0. \tag{1.4.133}$$

Thus, if J is a bounded interval of $[0, \infty)$ containing t and s, from (1.4.133) we infer that

$$\|S_\lambda(\xi)\|_{\mathcal{L}(X)} \leq C, \quad \text{for all } \lambda > \lambda_0 \text{ and all } \xi \in J. \tag{1.4.134}$$

On the other hand, from (1.4.129) and the given $\varepsilon > 0$, there exist $\lambda_1, \lambda_2 > \omega$ such that

$$\|S_\lambda(s)x - S(s)x\| < \frac{\varepsilon}{C+1}, \quad \text{for all } \lambda \geq \lambda_1, \tag{1.4.135}$$

and

$$\|S_\lambda(t)S(s)x - S(t)S(s)x\| < \frac{\varepsilon}{C+1}, \quad \text{for all } \lambda \geq \lambda_2. \tag{1.4.136}$$

From (1.4.134), (1.4.135) and (1.4.136) we obtain

$$\begin{aligned}
&\|S_\lambda(t)S_\lambda(s)x - S(t)S(s)x\| \\
&\leq \|S_\lambda(t)S_\lambda(s)x - S_\lambda(t)S(s)x\| + \|S_\lambda(t)S(s)x - S(t)S(s)x\| \\
&\leq \|S_\lambda(t)\|_{\mathcal{L}(X)}\|S_\lambda(s)x - S(s)x\| + \|S_\lambda(t)S(s)x - S(t)S(s)x\| \\
&< C\frac{\varepsilon}{C+1} + \frac{\varepsilon}{C+1} = \varepsilon,
\end{aligned}$$

for all $\lambda \geq \lambda_0^* := \max\{\lambda_0, \lambda_1, \lambda_2\}$, which proves (1.4.132). Combining (1.4.131) and (1.4.132) we conclude that

$$S(t+s) = S(t)S(s), \quad \text{for all } t, s \geq 0. \tag{1.4.137}$$

Let $\varepsilon > 0$, $x \in X$ and $0 < h < 1$. Then, by (1.4.129), there exists $\lambda_0 > \omega$ such that

$$\|S_\lambda(h)x - S(h)x\| < \frac{\varepsilon}{2}, \quad \text{for all } \lambda \geq \lambda_0 \text{ and } h \in (0,1). \tag{1.4.138}$$

Since S_{λ_0} is a C_0–semigroup, there exists $\delta > 0$ such that, if $0 < h < \delta$, then

$$\|S_{\lambda_0}(h)x - x\| < \frac{\varepsilon}{2}. \tag{1.4.139}$$

Therefore, for $0 < h < \min\{1, \delta\}$, from (1.4.138) and (1.4.139) we obtain

$$\begin{aligned}
\|S(h)x - x\| &\leq \|S(h)x - S_{\lambda_0}(h)x\| + \|S_{\lambda_0}(h)x - x\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which proves that

$$\lim_{h \to 0_+} S(h)x = x \quad \text{in } X. \tag{1.4.140}$$

Thus, from (1.4.130), (1.4.137) and (1.4.140), we have shown that S is a C_0–semigroup.

5th step.

To complete the proof it remains to show that A is the infinitesimal generator of S. Let B denote the infinitesimal generator of S. We first show that $D(A) \subset D(B)$. Indeed, let $x \in D(A)$, $\lambda > \omega$ and $h > 0$.

We have

$$S_\lambda(h)x - x = \int_0^h \frac{d}{dt}(S_\lambda(t)x)\,dt.$$

But

$$\frac{d}{dt}(S_\lambda(t)x) = \frac{d}{dt}(e^{tB_\lambda}x) = B_\lambda e^{tB_\lambda}x = B_\lambda S_\lambda(t)x,$$

so that

$$S_\lambda(h)x - x = \int_0^h S_\lambda(t)B_\lambda x\,dt, \quad h > 0, \tag{1.4.141}$$

since $S_\lambda(t)$ and B_λ commute for $\lambda \geq 0$.

We claim that

$$\lim_{\lambda\to\infty} S_\lambda(t)B_\lambda x = S(t)Ax, \tag{1.4.142}$$

uniformly on bounded intervals of the real line.

Indeed, let J be a bounded interval of the real line, $\gamma > \omega$ and $\varepsilon > 0$. From (1.4.129) there exists $\lambda_1 > \omega$ such that

$$\|S_\lambda(t)Ax - S(t)Ax\| < \frac{\varepsilon}{C+1}, \quad \text{for all } \lambda \geq \lambda_1 \text{ and } t \in J, \tag{1.4.143}$$

where C is the constant appearing in (1.4.134). Now, from (1.4.100), (1.4.134) and (1.4.143), for $\lambda > \max\{\lambda_0, \lambda_1, \lambda_2\}$ we have

$$\begin{aligned}
&\|S_\lambda(t)B_\lambda x - S(t)Ax\| \\
&\leq \|S_\lambda(t)B_\lambda x - S_\lambda(t)Ax\| + \|S_\lambda(t)Ax - S(t)Ax\| \\
&\leq \|S_\lambda(t)\|_{\mathcal{L}(X)}\|B_\lambda x - Ax\| + \|S_\lambda(t)Ax - S(t)Ax\| \\
&< C\frac{\varepsilon}{C+1} + \frac{\varepsilon}{C+1} = \varepsilon,
\end{aligned}$$

which proves (1.4.142). From (1.4.129) and (1.4.141), passing to the limit as $\lambda \to \infty$, we obtain

$$S(h)x - x = \int_0^h S(t)Ax\,dt, \quad h > 0.$$

From this identity and the Mean Value Theorem we deduce that

$$Bx = \lim_{h\to 0_+} \frac{S(h)x - x}{h} = \lim_{h\to 0_+} \frac{1}{h}\int_0^h S(t)Ax\,dt = Ax, \quad \text{for all } x \in D(A). \tag{1.4.144}$$

The relation (1.4.144) shows that

$$D(A) \subset D(B) \quad \text{and} \quad A \equiv B \text{ on } D(A). \tag{1.4.145}$$

We now prove, in fact, that

$$D(A) = D(B). \tag{1.4.146}$$

By hypothesis, if $\lambda > \omega$ then $\lambda \in \rho(A)$. Now, since B is the infinitesimal generator of S, it follows from Proposition 1.34 that if $\lambda > \omega_0 = \lim_{t\to\infty} \frac{\ln\|S(t)\|}{t}$, then $\lambda \in \rho(B)$. Therefore, if $\lambda > \max\{\omega, \omega_0\}$, then $\lambda \in \rho(A) \cap \rho(B)$. For such values of λ we have

$$(\lambda I - A)D(A) = X \qquad \text{and} \qquad (\lambda I - B)D(B) = X, \tag{1.4.147}$$

since $D((\lambda I - A)^{-1}) = Im(\lambda I - A) = X$, by Proposition 1.33 (because A is closed).

On the other hand, from (1.4.145) and (1.4.147) we may write

$$(\lambda I - B)D(B) = (\lambda I - A)D(A),$$

which implies

$$D(B) = (\lambda I - B)^{-1}(\lambda I - A)D(A) = (\lambda I - B)^{-1}(\lambda I - B)D(A) = D(A),$$

which proves (1.4.146). □

Corollary 1.37 *Let A be the infinitesimal generator of a C_0–semigroup T. If B_λ is the Yosida approximation of A, then*

$$T(t)x = \lim_{\lambda\to\infty} e^{tB_\lambda}x, \quad \textit{for all } x \in X.$$

Proof: From the proof of the Hille–Yosida Theorem (Theorem 1.36) it follows that the right-hand side of the above identity defines a C_0–semigroup S whose infinitesimal generator is A. By Exercise 1.3.1 we conclude that $T = S$. □

Theorem 1.38 [*Hille–Yosida for Contractions*] *A linear operator A is the infinitesimal generator of a contraction semigroup S if and only if*
(i) A is closed and densely defined.
(ii) For every $\lambda > 0$, we have $\lambda \in \rho(A)$ and, moreover,

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{1}{\lambda}.$$

Proof: The proof is analogous to that of Theorem 1.36, with the obvious adaptations. □

To simplify the terminology we shall write

$$A \in G(M, \omega)$$

to express that A is the infinitesimal generator of a C_0–semigroup satisfying

$$\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}, \quad t \ge 0.$$

We have the following result:

Proposition 1.39 $(A - \omega I) \in G(M, 0)$ *if and only if* $A \in G(M, \omega)$.

Proof: Let $A \in G(M, \omega)$. Then A is the infinitesimal generator of a C_0–semigroup S such that

$$\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}, \quad t \ge 0.$$

Setting

$$\tilde{S}(t) = e^{-\omega t}S(t),$$

it follows, in view of Example 1.3.1, that $\tilde{S}$ is a C_0–semigroup whose infinitesimal generator is $A - \omega I$. Moreover,

$$\|\tilde{S}(t)\| = e^{-\omega t}\|S(t)\| \le e^{-\omega t}Me^{\omega t} = M,$$

which shows that $A - \omega I \in G(M, 0)$.

Conversely, suppose that $A - \omega I \in G(M, 0)$. Then $A - \omega I$ is the infinitesimal generator of a C_0–semigroup S satisfying

$$\|S(t)\| \le M, \quad t \ge 0.$$

Defining

$$\tilde{S}(t) = e^{\omega t}S(t),$$

it follows, by the same reasoning, that $\tilde{S}$ is a C_0–semigroup whose infinitesimal generator is $A-\omega I+\omega I = A$. Also,

$$\|\tilde{S}(t)\| \leq e^{\omega t}\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0,$$

which completes the proof. □

1.5 The Lumer–Phillips Theorem

In this section we present a result due to Lumer and Phillips which gives a necessary and sufficient condition for a linear operator A to be the infinitesimal generator of a contraction semigroup. The proof of this result follows from the Hille–Yosida Theorem, as we shall see below, and its advantage in comparison with the Hille–Yosida Theorem is that its hypotheses are easier to verify. Before that, however, we need some definitions and preliminary results, which we state next.

Let X be a Banach space, X' its topological dual, and $\langle \cdot, \cdot \rangle$ the duality pairing between X' and X. For each $x \in X$ we define

$$F(x) = \{x^* \in X';\ \langle x^*, x\rangle = \|x^*\|^2 = \|x\|^2\}.$$

As a consequence of the Hahn–Banach Theorem , $F(x) \neq \emptyset$ for each $x \in X$. This leads to the notion of a *duality mapping*, that is, a mapping $j : X \to X'$ such that, for each $x \in X$, we have $j(x) \in F(x)$. We immediately obtain

$$\|j(x)\| = \|x\| = \left(\langle j(x), x\rangle\right)^{1/2}. \tag{1.5.148}$$

Observe that if X is a Hilbert space, the duality can be expressed in terms of the inner product (via the Riesz Representation Theorem). In this case, $F(x) = \{x\}$.

Definition 1.40 *A linear operator A is said to be dissipative with respect to a duality mapping j if*

$$\mathrm{Re}\ \langle j(x), Ax\rangle \leq 0, \quad \textit{for all } x \in D(A).$$

Definition 1.41 *A dissipative operator A which satisfies $Im(I-A) = X$ is called m-dissipative.*

Remark: If A is dissipative, then λA is dissipative for every $\lambda > 0$.

Proposition 1.42 *If A is a linear dissipative operator with respect to some duality mapping, then*

$$\|(\lambda I - A)x\| \geq \mathrm{Re}\,\lambda\,\|x\|, \quad \textit{for all } \lambda \in \mathbb{C} \textit{ and } x \in D(A).$$

Proof: Let $\lambda \in \mathbb{C}$ and $x \in D(A)$. Let $j : X \to X'$ be the duality mapping with respect to which A is dissipative. From (1.5.148) we have

$$\begin{aligned}\langle j(x), (\lambda I - A)x\rangle &= \langle j(x), \lambda x\rangle - \langle j(x), Ax\rangle \\ &= \overline{\lambda}\|x\|^2 - \langle j(x), Ax\rangle\,,\end{aligned}$$

whence

$$(\mathrm{Re}\,\lambda)\|x\|^2 = \mathrm{Re}\,\langle j(x), (\lambda I - A)x\rangle + \mathrm{Re}\,\langle j(x), Ax\rangle\,.$$

By Definition 1.40 and again by (1.5.148), it follows that

$$\begin{array}{rcl}(\operatorname{Re}\lambda)\|x\|^2 & \leq & \operatorname{Re}\langle j(x),(\lambda I-A)x\rangle \\ & \leq & |\langle j(x),(\lambda I-A)x\rangle| \\ & \leq & \|j(x)\|_{X'}\|(\lambda I-A)x\| \\ & = & \|x\|\,\|(\lambda I-A)x\|,\end{array}$$

which implies

$$(\operatorname{Re}\lambda)\|x\| \leq \|(\lambda I-A)x\|, \quad \text{if } x\neq 0.$$

If $x=0$, the inequality is trivial, which completes the proof. □

Proposition 1.43 *Let $A:D(A)\subset X\to X$ be a linear, closed and dissipative operator with respect to some duality mapping. Then $\rho(A)\cap(0,\infty)$ is an open subset of $\mathbb{R}$.*

Proof: If $\rho(A)\cap(0,\infty)=\emptyset$, there is nothing to prove. Assume, therefore, that $\rho(A)\cap(0,\infty)\neq\emptyset$ and let $\lambda_0\in\rho(A)\cap(0,\infty)$. Now, given $\lambda\in\mathbb{C}$ and $f\in X$, consider the identity

$$\lambda u - Au = f, \tag{1.5.149}$$

which can be rewritten as

$$\lambda_0 u - Au = f + (\lambda_0-\lambda)u,$$

or, equivalently,

$$(\lambda_0 I - A)u = f + (\lambda_0-\lambda)u. \tag{1.5.150}$$

Since $\lambda_0 I - A$ is invertible, (1.5.150) yields

$$u = (\lambda_0 I - A)^{-1}(f + (\lambda_0-\lambda)u). \tag{1.5.151}$$

Define

$$\begin{array}{c}G: X\to X \\ u\mapsto Gu := (\lambda_0 I - A)^{-1}(f+(\lambda_0-\lambda)u).\end{array} \tag{1.5.152}$$

Note that G is well-defined, since A is closed, and G is continuous, because $(\lambda_0 I - A)^{-1}$ is continuous. Moreover, for all $u,v\in X$, we have

$$\begin{array}{rcl}\|Gu-Gv\| & = & \|(\lambda_0 I-A)^{-1}(f+(\lambda_0-\lambda)u)-(\lambda_0 I-A)^{-1}(f+(\lambda_0-\lambda)v)\| \\ & = & \|(\lambda_0 I-A)^{-1}[(\lambda_0-\lambda)(u-v)]\| \\ & \leq & \|(\lambda_0 I-A)^{-1}\|\,|\lambda_0-\lambda|\,\|u-v\|.\end{array}$$

If we assume that

$$|\lambda-\lambda_0| < \frac{1}{\|(\lambda_0 I-A)^{-1}\|} := r_0, \tag{1.5.153}$$

then, in view of (1.5.153), the mapping defined in (1.5.152) is a contraction, and by the Banach Fixed Point Theorem, there exists a unique $u\in X$ which solves (1.5.151), and hence a unique solution of (1.5.149). In other words, the operator $(\lambda I - A)$ is bijective for every λ satisfying condition (1.5.153), and therefore admits an inverse $(\lambda I - A)^{-1}$ for every $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| < r_0$. Note that, since $\lambda_0 > 0$, we may choose r_0 sufficiently small so that $|\lambda - \lambda_0| < r_0$ implies $\operatorname{Re}\lambda > 0$. It follows that, if $\lambda \in B_{r_0}(\lambda_0)$ (where $B_{r_0}(\lambda_0)$ denotes the open ball in the complex plane centred at $(\lambda_0, 0)$ with radius $r_0 > 0$) and $x \in X$, then $(\lambda I - A)^{-1}x \in D(A)$ and, by Proposition 1.42,

$$\|x\| = \|(\lambda I - A)(\lambda I - A)^{-1}x\| \geq \operatorname{Re}\lambda\, \|(\lambda I - A)^{-1}x\|,$$

that is,

$$\|(\lambda I - A)^{-1}x\| \leq \frac{1}{\operatorname{Re}\lambda}\|x\|, \quad \text{for all } x \in X \text{ and } \lambda \in B_{r_0}(\lambda_0),$$

which shows the continuity of the family of operators $(\lambda I - A)^{-1}$ for every $\lambda \in B_{r_0}(\lambda_0)$. Hence $(\lambda_0 - r_0, \lambda_0 + r_0) \subset \rho(A) \cap (0, \infty)$, which completes the proof. □

Theorem 1.44 [*Lumer–Phillips Theorem*] *If $A \in G(1,0)$ then*
(i) A is dissipative with respect to any duality mapping.
(ii) $Im(\lambda I - A) = X$, for every $\lambda > 0$.
Conversely, if
(iii) $D(A)$ is dense in X.
(iv) A is dissipative with respect to some duality mapping.
(v) $Im(\lambda_0 I - A) = X$, for some $\lambda_0 > 0$,
then $A \in G(1,0)$.

Proof: Assume that $A \in G(1,0)$. Thus, A is the infinitesimal generator of a contraction semigroup S, that is,

$$\|S(t)\| \leq 1, \quad \text{for all } t \geq 0. \tag{1.5.154}$$

Let $j : X \to X'$ be a duality mapping and consider $x \in D(A)$ and $t \geq 0$. From (1.5.148) and (1.5.154) we get

$$\begin{aligned}
\operatorname{Re}\langle j(x), S(t)x - x\rangle &= \operatorname{Re}\langle j(x), S(t)x\rangle - \operatorname{Re}\langle j(x), x\rangle \\
&\leq |\langle j(x), S(t)x\rangle| - \|x\|^2 \\
&\leq \|j(x)\|\,\|S(t)\|\,\|x\| - \|x\|^2 \\
&\leq \|x\|^2 - \|x\|^2 = 0.
\end{aligned}$$

From the last inequality we deduce

$$\operatorname{Re}\left\langle j(x), \frac{S(t)x - x}{t}\right\rangle \leq 0, \quad \text{for all } t > 0.$$

Taking the limit as $t \to 0_+$ and using the fact that $\frac{S(t)x - x}{t} \to Ax$ as $t \to 0_+$, we obtain

$$\operatorname{Re}\langle j(x), Ax\rangle \leq 0,$$

which proves item (i).

On the other hand, according to the Hille–Yosida Theorem for contractions (Theorem 1.38), we infer that $(0, \infty) \subset \rho(A)$. It follows that $R(\lambda, A) = (\lambda I - A)^{-1}$ exists, is continuous, and has domain equal to the whole space X, since A is closed, for every $\lambda > 0$, which proves item (ii).

Conversely, let $A : D(A) \subset X \to X$ be a linear operator satisfying items (iii), (iv) and (v) of the theorem. We shall prove that $A \in G(1,0)$. To this end we shall again use the Hille–Yosida Theorem for contractions. We first prove that

$$A \text{ is closed,} \tag{1.5.155}$$

since A is densely defined by hypothesis. Indeed, by (iv) A is dissipative with respect to some duality mapping j. By Proposition 1.42 we have

$$\|(\lambda I - A)x\| \geq \lambda \|x\|, \quad \text{for all } \lambda > 0 \text{ and } x \in D(A),$$

which shows that $\{(\lambda I - A)\}_{\lambda > 0}$ is a family of injective operators. Now, from (v) we have

$$Im(\lambda_0 I - A) = X,$$

for some $\lambda_0 > 0$. In this particular case, it follows that $(\lambda_0 I - A)$ is a bijection from $D(A)$ onto the whole space X. Therefore,

$$(\lambda_0 I - A)^{-1} x \in D(A), \quad \text{for all } x \in X,$$

and, again by Proposition 1.42, we can write

$$\|(\lambda_0 I - A)^{-1} x\| \leq \frac{1}{\lambda_0} \|x\|, \quad \text{for all } x \in X,$$

that is,

$$(\lambda_0 I - A)^{-1} \in \mathcal{L}(X, D(A)) \quad (D(A) \text{ endowed with the topology of } X). \tag{1.5.156}$$

Now consider $(x_\nu)_\nu \subset D(A)$ such that

$$x_\nu \to x \text{ in } X \text{ and } Ax_\nu \to y \text{ in } X \text{ as } \nu \to \infty. \tag{1.5.157}$$

From (1.5.157) we have

$$-Ax_\nu \to -y \text{ in } X \quad \text{and} \quad \lambda_0 x_\nu \to \lambda_0 x \text{ in } X \text{ as } \nu \to \infty,$$

and hence

$$(\lambda_0 I - A)x_\nu \to \lambda_0 x - y \text{ in } X \text{ as } \nu \to \infty. \tag{1.5.158}$$

From (1.5.156) and (1.5.158) we conclude that

$$(\lambda_0 I - A)^{-1}(\lambda_0 I - A)x_\nu \to (\lambda_0 I - A)^{-1}(\lambda_0 x - y) \text{ in } X \text{ as } \nu \to \infty,$$

that is,

$$x_\nu \to (\lambda_0 I - A)^{-1}(\lambda_0 x - y) \text{ in } X \text{ as } \nu \to \infty. \tag{1.5.159}$$

From (1.5.157) and (1.5.159), by uniqueness of limits, we obtain

$$x = (\lambda_0 I - A)^{-1}(\lambda_0 x - y),$$

which shows that $x \in D(A)$. Moreover, from this relation we also have

$$(\lambda_0 I - A)x = \lambda_0 x - y,$$

that is, $y = Ax$, which proves (1.5.155).

To conclude the theorem it remains to prove that

$$\text{Given } \lambda > 0, \text{ we have } \lambda \in \rho(A) \text{ and } \|R(\lambda, A)\| \leq \frac{1}{\lambda}. \tag{1.5.160}$$

Indeed, let

$$\Lambda = (0, \infty) \cap \rho(A),$$

which is non-empty because, by item (v), there exists $\lambda_0 \in \rho(A)$ such that $\lambda_0 > 0$. By Proposition 1.43 it follows that

$$\Lambda \text{ is open in } (0, \infty), \tag{1.5.161}$$

since Λ is an open subset of $\mathbb{R}$ contained in $(0, \infty)$. We now prove that

$$\Lambda \text{ is closed in } (0, \infty). \tag{1.5.162}$$

Let $(\lambda_\nu)_\nu \subset \Lambda$ be such that

$$\lambda_\nu \to \lambda \text{ in } \mathbb{R}, \quad \text{with } \lambda \in (0, \infty). \tag{1.5.163}$$

Since $(\lambda_\nu)_\nu \subset \rho(A)$, then, by (1.5.155) and Proposition 1.33, for each $\nu \in \mathbb{N}$,

$$Im(\lambda_\nu I - A) = X. \tag{1.5.164}$$

Let $y \in X$ be arbitrary. From (1.5.164), for each $\nu \in \mathbb{N}$ there exists $x_\nu \in D(A)$ such that

$$\lambda_\nu x_\nu - Ax_\nu = y.$$

By Proposition 1.42 we infer

$$\|x_\nu\| \leq \frac{1}{\lambda_\nu}\|(\lambda_\nu I - A)x_\nu\| = \frac{1}{\lambda_\nu}\|y\|, \tag{1.5.165}$$

since $\lambda_\nu > 0$. From (1.5.163) we see that $(1/\lambda_\nu)$ is bounded and from (1.5.165) there exists $C > 0$ such that

$$\|x_\nu\| \leq C, \quad \text{for all } \nu \in \mathbb{N}, \tag{1.5.166}$$

where C is a constant depending on y.

Let $\nu, \mu \in \mathbb{N}$ with $\mu > \nu$. By Proposition 1.42 we have

$$\begin{aligned} \lambda_\mu \|x_\mu - x_\nu\| &\leq \|(\lambda_\mu I - A)(x_\mu - x_\nu)\| \\ &= \|\lambda_\mu (x_\mu - x_\nu) - A(x_\mu - x_\nu)\|. \end{aligned} \tag{1.5.167}$$

However, since

$$\lambda_\nu x_\nu - Ax_\nu = y \quad \text{and} \quad \lambda_\mu x_\mu - Ax_\mu = y,$$

we obtain

$$\lambda_\mu x_\mu - \lambda_\nu x_\nu = Ax_\mu - Ax_\nu,$$

which implies

$$\lambda_\mu(x_\mu - x_\nu) + (\lambda_\mu - \lambda_\nu)x_\nu = A(x_\mu - x_\nu). \tag{1.5.168}$$

From (1.5.167) and (1.5.168) we can write

$$\lambda_\mu \|x_\mu - x_\nu\| \leq |\lambda_\mu - \lambda_\nu| \, \|x_\nu\|,$$

and from (1.5.166) we deduce

$$\lambda_\mu \|x_\mu - x_\nu\| \leq C|\lambda_\mu - \lambda_\nu|. \tag{1.5.169}$$

From (1.5.163), (1.5.169) and the boundedness of $(1/\lambda_\nu)$, it follows that $(x_\nu)_\nu$ is a Cauchy sequence in X. Thus, there exists $x \in X$ such that

$$x_\nu \to x \text{ in } X \text{ as } \nu \to \infty. \tag{1.5.170}$$

From (1.5.163) and (1.5.170) we obtain

$$\lambda_\nu x_\nu \to \lambda x \text{ in } X \text{ as } \nu \to \infty,$$

and consequently

$$Ax_\nu = \lambda_\nu x_\nu - y \to \lambda x - y \text{ in } X \text{ as } \nu \to \infty, \tag{1.5.171}$$

Hence, from (1.5.155), (1.5.170) and (1.5.171) we conclude that $x \in D(A)$ and $Ax = \lambda x - y$, i.e.

$$(\lambda I - A)x = y. \tag{1.5.172}$$

From (1.5.172), reasoning as in the proof of (1.5.155), we obtain that $(\lambda I - A)^{-1}$ exists and is continuous (using Proposition 1.42), that is, $\lambda \in \rho(A)$, which proves that $\lambda \in \Lambda$ and hence (1.5.162). From (1.5.161) and (1.5.162) we deduce that $\Lambda = (0, \infty)$ and, since

$$\Lambda = (0, \infty) \cap \rho(A) \subset \rho(A),$$

it follows that $(0, \infty) \subset \rho(A)$. It remains to prove that

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}, \quad \text{for all } \lambda > 0.$$

Indeed, since $(0, \infty) \subset \rho(A)$, for every $\lambda > 0$ we have $Im(\lambda I - A) = X$ and, therefore, by Proposition 1.42,

$$\|R(\lambda, A)x\| = \|(\lambda I - A)^{-1}x\| \leq \frac{1}{\lambda}\|x\|, \quad \text{for all } x \in X,$$

which completes the proof. □

Remark 1.45 *In terms of m-dissipative operators, the Lumer–Phillips Theorem can be reformulated as follows: A densely defined operator A is the infinitesimal generator of a C_0–contraction semigroup if and only if A is m-dissipative.*

Remark 1.46 *It follows from the proof of the Lumer–Phillips Theorem that if A is m-dissipative, then $Im(\lambda I - A) = X$ for every $\lambda > 0$.*

1.5.1 Exercises

1.5.1) Let $A \in G(1,0)$ and B be dissipative with respect to some duality mapping. If $D(A) \subset D(B)$ and there exist constants a and b with $0 \leq a < 1$ and $b \geq 0$ such that $\|Bx\| \leq a\|Ax\| + b\|x\|$ for all $x \in D(A)$, prove that $A + B \in G(1,0)$.

1.5.2) Use Exercise 1.5.1 to prove the following result: If $A \in G(1,0)$ and $B \in \mathcal{L}(X)$, prove that $A + B \in G(1, \|B\|)$.

1.6 Stone's Theorem

In this section we present a necessary and sufficient condition for a linear operator A to be the infinitesimal generator of a C_0–group. To this end we first define what we mean by a group of bounded operators. Throughout, X denotes a Banach space.

Definition 1.47 *A function $S : \mathbb{R} \to \mathcal{L}(X)$ is called a group of bounded operators if*
(1) $S(0) = I$.
(2) $S(t+s) = S(t)S(s)$, *for all* $t, s \in \mathbb{R}$.
We say that S is of class C_0 if
(3) $\lim_{h \to 0} \|S(h)x - x\| = 0$ *for all* $x \in X$.
The operator A defined by

$$D(A) = \left\{ x \in X;\ \lim_{h \to 0} \frac{S(h)x - x}{h} \text{ exists} \right\},$$

and

$$Ax = \lim_{h \to 0} \frac{S(h)x - x}{h}, \quad \text{for all } x \in D(A),$$

is called the infinitesimal generator of S.

Before stating Stone's Theorem, we make some preliminary remarks that will be needed later. Let $A : D(A) \subset X \to X$ be a linear operator. Defining

$$D(A^*) = \{u^* \in X';\ \text{there exists } v^* \in X' \text{ such that } \langle u^*, Au \rangle = \langle v^*, u \rangle \text{ for all } u \in D(A)\},$$

it is well known that if $D(A)$ is dense in X, then the v^* corresponding to a given u^* is unique, which allows us to define the adjoint operator A^* by

$$\begin{aligned} A^* : D(A^*) \subset X' &\to X' \\ u^* &\mapsto A^* u^* = v^*. \end{aligned}$$

Some relevant conclusions are:

(A1) A^* is clearly linear and is also closed. A proof can be found in [23, Proposition 2.45].

(A2) If X is a reflexive Banach space and $A : D(A) \subset X \to X$ is a closed linear operator with $D(A)$ dense in X, then $D(A^*)$ is also dense in X'. A proof can be found in [83, Lemma 10.5].

(A3) If $A : D(A) \subset X \to X$ is a closed, densely defined linear operator, then the following properties are equivalent:

$$\begin{aligned} &(i) \qquad D(A) = X. \\ &(ii) \qquad A \text{ is continuous.} \\ &(iii) \qquad D(A^*) = X'. \\ &(iv) \qquad A^* \text{ is continuous.} \end{aligned} \tag{1.6.173}$$

Under these conditions we have

$$\|A\|_{\mathcal{L}(X)} = \|A^*\|_{\mathcal{L}(X')}. \tag{1.6.174}$$

A proof of this fact can be found in [14, Théorème II.21].

Lemma 1.48 *Let $T : D(T) \subset X \to X$ be a bijective linear operator with $D(T)$ dense in X. If T^{-1} is closed, then $(T^*)^{-1}$ exists and $(T^*)^{-1} = (T^{-1})^*$.*

Proof: Since $\overline{D(T)} = X$, the adjoint T^* is well defined. On the other hand, as T is bijective, T^{-1} exists and $D(T^{-1}) = X$. Therefore, $(T^{-1})^*$ is also well defined. Moreover, since T^{-1} is closed, it follows from (1.6.173)(iv) that $D((T^{-1})^*) = X'$.

Let $u^* \in D(T^*)$ and $u \in X$. Then, by the definition of adjoint operator, in particular for $v = T^*u^*$, we have

$$\begin{aligned} \left\langle (T^{-1})^* T^* u^*, u \right\rangle &= \left\langle T^* u^*, T^{-1} u \right\rangle \\ &= \left\langle u^*, T(T^{-1}u) \right\rangle = \langle u^*, u \rangle . \end{aligned}$$

From the density of $u^* \in D(T^*)$ and $u \in X$ it follows that

$$(T^{-1})^* T^* u^* = u^*, \quad \text{for all } u^* \in D(T^*). \tag{1.6.175}$$

On the other hand, let $u^* \in X'$ and $u \in D(T)$. Then

$$\left\langle (T^{-1})^* u^*, Tu \right\rangle = \left\langle u^*, T^{-1}(Tu) \right\rangle = \langle u^*, u \rangle ,$$

from which we deduce that

$$(T^{-1})^* u^* \in D(T^*) \quad \text{and} \quad T^*(T^{-1})^* u^* = u^*, \quad \text{for all } u^* \in X'. \tag{1.6.176}$$

From (1.6.175) and (1.6.176) the desired identity follows. □

Proposition 1.49 *Let X be a reflexive Banach space and S a C_0–semigroup with infinitesimal generator A. Define $S^* : \mathbb{R}_+ \to \mathcal{L}(X')$ by $S^*(t) = [S(t)]^*$ for all $t \in \mathbb{R}_+$. Then S^* is a C_0–semigroup whose infinitesimal generator is A^*.*

Proof: First observe that S^* is well defined because, for each $t \in \mathbb{R}_+$, we have $S(t) \in \mathcal{L}(X)$ and $D(S(t)) = X$, and thus, by (1.6.173)(iv), $[S(t)]^* \in \mathcal{L}(X')$. Moreover, since X is a reflexive Banach space and A is closed and densely defined, it follows from (A1) and (A2) that A^* is closed and densely defined. Our aim is to apply the Hille–Yosida Theorem to A^*, and hence we must show that there exist $M, \omega \in \mathbb{R}$ such that, if $\lambda > \omega$, then $\lambda \in \rho(A^*)$ and $\|R(\lambda, A^*)^n\| \leq \frac{M}{(\lambda-\omega)^n}$ for all $n \in \mathbb{N}$.

Indeed, if $\lambda \in \rho(A)$ then $\overline{\lambda} \in \rho(A^*)$, because if $\lambda \in \rho(A)$, then $(\lambda I - A)^{-1}$ exists and $(\lambda I - A)^{-1} \in \mathcal{L}(X)$. By Lemma 1.48 we have that $[(\lambda I - A)^*]^{-1}$ exists and, moreover,

$$[(\lambda I - A)^{-1}]^* = [(\lambda I - A)^*]^{-1}.$$

From (1.6.173)(iv) it follows that $[(\lambda I - A)^{-1}]^* \in \mathcal{L}(X')$ and $D\{[(\lambda I - A)^{-1}]^*\} = X'$, since $D(\lambda I - A)^{-1} = X$. Thus $[(\lambda I - A)^*]^{-1} \in \mathcal{L}(X')$ and $D\{[(\lambda I - A)^*]^{-1}\} = X'$. Furthermore,

$$(\lambda I - A)^* = \overline{\lambda} I - A^*,$$

and therefore $\overline{\lambda} \in \rho(A^*)$.

Since A is the infinitesimal generator of a C_0–semigroup, by the Hille–Yosida Theorem there exist real constants M, ω such that, if $\lambda > \omega$, then $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \quad \text{for all } n \in \mathbb{N}.$$

Thus, for the above M and ω, let $\lambda > \omega$. Since $\lambda \in \rho(A)$, we have $\lambda \in \rho(A^*)$ (because $\lambda \in \mathbb{R}$) and, moreover, from

$$(\lambda I - A^*)^{-1} = [(\lambda I - A)^*]^{-1} = [(\lambda I - A)^{-1}]^* \quad \text{(note that } \lambda \in \mathbb{R}\text{)},$$

and by (1.6.174) we obtain

$$\begin{aligned} \|R(\lambda, A^*)^n\| &= \|([R(\lambda, A)]^*)^n\| \\ &= \|[R(\lambda, A)^n]^*\| \\ &= \|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}. \end{aligned}$$

From the above we conclude that:

(i) A^* is closed and densely defined.

(ii) There exist real constants M and ω such that, if $\lambda > \omega$, then $\lambda \in \rho(A^*)$ and $\|R(\lambda, A^*)^n\| \leq \frac{M}{(\lambda - \omega)^n}$.

Hence, by the Hille–Yosida Theorem, A^* is the infinitesimal generator of a C_0–semigroup T. By Corollary 1.37 we may write

$$T(t)x^* = \lim_{\lambda \to \infty} e^{t(\lambda^2 R(\lambda, A^*) - \lambda I)} x^*, \quad \text{for all } x^* \in X'.$$

Setting $B_\lambda := \lambda^2 R(\lambda, A) - \lambda I$, we have $B_\lambda^* = (\lambda^2 R(\lambda, A) - \lambda I)^* = \lambda^2 R(\lambda, A^*) - \lambda I$. Thus

$$T(t)x^* = \lim_{\lambda \to \infty} e^{tB_\lambda^*} x^*, \quad \text{for all } x^* \in X'. \tag{1.6.177}$$

Recall that $B_\lambda \in \mathcal{L}(X)$ and, therefore, from (1.6.173)(iv) it follows that $B_\lambda^* \in \mathcal{L}(X')$.

We claim that

$$\text{if } L_n \to L \text{ in } \mathcal{L}(X), \text{ then } L_n^* \to L^* \text{ in } \mathcal{L}(X'). \tag{1.6.178}$$

Indeed, by (1.6.174),

$$\|L_n^* - L^*\|_{\mathcal{L}(X')} = \|(L_n - L)^*\|_{\mathcal{L}(X')} = \|L_n - L\|_{\mathcal{L}(X)},$$

which proves (1.6.178).

Hence, for $t \geq 0$,

$$T_{\lambda,n}(t) = \sum_{i=0}^{n} \frac{(tB_\lambda)^i}{i!} \longrightarrow S_\lambda(t) = e^{tB_\lambda} \quad \text{as } n \to \infty,$$

and therefore, by (1.6.178),

$$T^*_{\lambda,n}(t) = \left(\sum_{i=0}^{n} \frac{(tB_\lambda)^i}{i!} \right)^* \longrightarrow [S_\lambda(t)]^* = \left(e^{tB_\lambda}\right)^* \quad \text{as } n \to \infty. \tag{1.6.179}$$

On the other hand,

$$T^*_{\lambda,n}(t) = \sum_{i=0}^{n} \frac{(tB^*_\lambda)^i}{i!} \longrightarrow T_\lambda(t) = e^{tB^*_\lambda} \quad \text{as } n \to \infty. \tag{1.6.180}$$

Since

$$T_\lambda(t) = e^{tB^*_\lambda} \longrightarrow T(t) \quad \text{as } \lambda \to \infty$$

and

$$[S_\lambda(t)]^* \longrightarrow [S(t)]^* \quad \text{as } \lambda \to \infty,$$

it follows, by uniqueness of limits, that $[S(t)]^* = T(t)$ for all $t \geq 0$. □

Proposition 1.50 *For a linear operator A on a Banach space X to be the infinitesimal generator of a C_0–group S it is necessary and sufficient that both A and $-A$ be infinitesimal generators of C_0–semigroups.*

Proof: Assume first that A is the infinitesimal generator of a C_0–group S. The restriction of S to $\mathbb{R}_+$, which we denote by S_+, is clearly a C_0–semigroup whose infinitesimal generator is A. The same holds for the mapping $S_- : \mathbb{R}_+ \to \mathcal{L}(X)$ defined by $S_-(t) = S(-t)$, which has $-A$ as its infinitesimal generator. This proves the necessity.

Conversely, suppose that A and $-A$ are, respectively, the infinitesimal generators of C_0–semigroups S_+ and S_-. By Corollary 1.37, for all $x \in X$,

$$S_+(t)x = \lim_{\lambda\to\infty} e^{tB_\lambda}x \quad \text{and} \quad S_-(t)x = \lim_{\lambda\to\infty} e^{t\tilde{B}_\lambda}x, \tag{1.6.181}$$

where

$$\begin{aligned} B_\lambda &= \lambda^2 R(\lambda, A) - \lambda I, \quad \lambda > \omega > \omega_0 = \lim_{t\to\infty} \frac{\ln \|S_+(t)\|}{t}, \\ \tilde{B}_\lambda &= \lambda^2 R(\lambda, -A) - \lambda I, \quad \lambda > \omega > \tilde{\omega_0} = \lim_{t\to\infty} \frac{\ln \|S_-(t)\|}{t}, \end{aligned}$$

are the Yosida approximations of A and $-A$, respectively. Now, since $R(\lambda, A)$ commutes with $R(\mu, -A)$ for $\lambda, \mu > \omega > \max\{\omega_0, \tilde{\omega_0}\}$, it follows that

$$e^{tB_\lambda}e^{t\tilde{B}_\mu}x = e^{t\tilde{B}_\mu}e^{tB_\lambda}x, \quad \text{for all } x \in X \text{ and } \lambda, \mu \text{ sufficiently large.}$$

Fixing such a μ, it follows from (1.6.181) and from the fact that $e^{t\tilde{B}_\mu} \in \mathcal{L}(X)$ that, in the limit as $\lambda \to \infty$,

$$S_+(t)e^{t\tilde{B}_\mu}x = e^{t\tilde{B}_\mu}S_+(t)x.$$

Now, taking the limit as $\mu \to \infty$ in the last identity and using that $S_+(t) \in \mathcal{L}(X)$, we obtain

$$S_+(t)S_-(t)x = S_-(t)S_+(t)x, \quad \text{for all } x \in X \text{ and } t \geq 0. \tag{1.6.182}$$

Define

$$T(t) = S_+(t)S_-(t), \quad t \geq 0. \tag{1.6.183}$$

It follows immediately that T is a C_0–semigroup, since S_+ and S_- are C_0–semigroups and satisfy (1.6.182). Let B denote the infinitesimal generator of T. We claim that

$$D(A) \subset D(B) \quad \text{and} \quad Bx = 0, \quad \text{for all } x \in D(A). \tag{1.6.184}$$

Indeed, let $x \in D(A)$ and $h > 0$. From (1.6.183) we have

$$\begin{aligned} \frac{T(h)x - x}{h} &= \frac{S_+(h)S_-(h)x - x}{h} \\ &= \frac{S_+(h)S_-(h)x - S_+(h)x + S_+(h)x - x}{h} \\ &= S_+(h)\left[\frac{S_-(h)x - x}{h}\right] + \frac{S_+(h)x - x}{h}. \end{aligned} \tag{1.6.185}$$

Taking the limit in (1.6.185) as $h \to 0_+$, we obtain

$$\lim_{h \to 0_+} \frac{T(h)x - x}{h} = -Ax + Ax = 0,$$

which proves (1.6.184).

Now let $x \in D(B)$. Since $D(A)$ is dense in X, there exists a sequence $(x_\nu)_\nu \subset D(A)$ such that $x_\nu \to x$ in X. But from (1.6.184) we have $Bx_\nu = 0$ for all $\nu \in \mathbb{N}$, and hence $Bx_\nu \to 0$ as $\nu \to \infty$. Since B is closed, we conclude that $Bx = 0$, that is,

$$Bx = 0, \quad \text{for all } x \in D(B). \tag{1.6.186}$$

However, by Proposition 1.30(iii), for all $x \in X$ we have

$$\int_0^t T(s)x\,ds \in D(B), \quad t \geq 0,$$

and

$$T(t)x - x = B\int_0^t T(s)x\,ds. \tag{1.6.187}$$

From (1.6.186) and (1.6.187) it follows that $T(t)x = x$ for all $x \in X$ and $t \geq 0$, that is,

$$T(t) = I, \quad \text{for all } t \geq 0. \tag{1.6.188}$$

From (1.6.182), (1.6.183) and (1.6.188) we obtain

$$S_+(t)S_-(t) = S_-(t)S_+(t) = I. \tag{1.6.189}$$

Identity (1.6.189) shows that $S_+(t)$ is invertible and

$$(S_+(t))^{-1} = S_-(t), \quad \text{for all } t \geq 0. \tag{1.6.190}$$

Define

$$S(t) = \begin{cases} S_+(t), & t \geq 0, \\ S_-(-t), & t < 0. \end{cases} \tag{1.6.191}$$

We now prove that S is a C_0–group with infinitesimal generator A. Clearly,

$$S(0) = I, \tag{1.6.192}$$

and

$$\lim_{h \to 0} \|S(h)x - x\| = 0. \tag{1.6.193}$$

It remains to show that

$$S(t+s) = S(t)S(s), \quad \text{for all } t, s \in \mathbb{R}. \tag{1.6.194}$$

We consider several cases:

(i) $t, s \geq 0$ (trivial).

(ii) $t, s < 0$ (trivial).

(iii) $t \geq 0$, $s \leq 0$ and $t + s \geq 0$. From (1.6.190) we can write

$$\begin{aligned} S(t+s) &= S_+(t+s) \\ &= S_+(t+s)S_+(-s)[S_+(-s)]^{-1} \\ &= S_+(t)S_-(-s) = S(t)S(s). \end{aligned}$$

The remaining cases are analogous to (iii).

Next we show that A is the infinitesimal generator of S. Let $\tilde{A}$ denote the infinitesimal generator of S. Then, for all $x \in D(A)$,

$$\begin{aligned} \lim_{h \to 0_+} \frac{S(h)x - x}{h} &= \lim_{h \to 0_+} \frac{S_+(h)x - x}{h} = Ax, \\ \lim_{h \to 0_-} \frac{S(h)x - x}{h} &= \lim_{h \to 0_-} \frac{S_-(-h)x - x}{h} \\ &= -\lim_{h \to 0_-} \frac{S_-(-h)x - x}{-h} = -(-Ax) = Ax, \end{aligned}$$

which shows that $x \in D(\tilde{A})$ and $\tilde{A}x = Ax$, that is,

$$D(A) \subset D(\tilde{A}) \quad \text{and} \quad Ax = \tilde{A}x, \quad \text{for all } x \in D(A).$$

Conversely, $D(\tilde{A}) \subset D(A)$, because if $x \in D(\tilde{A})$, then the limit

$$\lim_{h \to 0_+} \frac{S_+(h)x - x}{h} = \lim_{h \to 0_+} \frac{S(h)x - x}{h} = \tilde{A}x$$

exists, which completes the proof. □

Proposition 1.51 *Let X be a Banach space and S a C_0–semigroup. If, for some $t_0 > 0$, the inverse $S(t_0)^{-1}$ exists and $S(t_0)^{-1} \in \mathcal{L}(X)$, then $S(t)^{-1}$ exists for all $t \geq 0$ and $S(t)^{-1} \in \mathcal{L}(X)$.*

Proof: Suppose there exists $t_0 > 0$ such that $S(t_0)^{-1}$ exists and $S(t_0)^{-1} \in \mathcal{L}(X)$. Then $S(t_0)$ is bijective and continuous. Hence, for each $n \in \mathbb{N}$, $[S(t_0)]^n$ is bijective and continuous, and since

$$S(nt_0) = S(\underbrace{t_0 + \cdots + t_0}_{n \text{ times}}) = \underbrace{S(t_0) \cdots S(t_0)}_{n \text{ factors}} = [S(t_0)]^n, \tag{1.6.195}$$

it follows that $S(nt_0)$ is bijective and continuous.

Now let $t > 0$. Then there exists $n \in \mathbb{N}$ such that $nt_0 > t$. Let $x \in X$ be such that $S(t)x = 0$. Then

$$S(nt_0)x = (S(nt_0 - t)S(t))x = S(nt_0 - t)(S(t)x) = 0,$$

and by the injectivity of $S(nt_0)$ we obtain $x = 0$, that is, $N(S(t)) \subset \{0\}$, which proves that $S(t)$ is injective for all $t \geq 0$ (the case $t = 0$ is trivial). Moreover, from the surjectivity of $S(nt_0)$ we have

$$X = S(nt_0)X = (S(t)S(nt_0 - t))X = S(t)(\underbrace{S(nt_0 - t)X}_{=Y}).$$

Thus $S(t)Y = X$, where $Y = S(nt_0 - t)X$, i.e. $X \subset S(t)X \subset X$, which implies $S(t)X = X$, and therefore $S(t)$ is surjective for all $t \geq 0$. Hence $S(t)$ is bijective for all $t \geq 0$ and therefore invertible for all $t \geq 0$. In addition, since $S(t) \in \mathcal{L}(X)$ for all $t \geq 0$, we deduce that

$$S(t)^{-1} \text{ is closed, for all } t \geq 0. \tag{1.6.196}$$

Indeed, let $(x_n)_n \subset D(S(t)^{-1}) = X$ be such that

$$x_n \to x \quad \text{and} \quad S(t)^{-1}x_n \to y, \quad \text{as } n \to \infty. \tag{1.6.197}$$

It remains to prove that $y = S(t)^{-1}x$. Since $S(t)$ is surjective, we have, for each $n \in \mathbb{N}$, $x_n = S(t)y_n$, and thus, from (1.6.197), we infer

$$S(t)y_n \to x \quad \text{and} \quad y_n = S(t)^{-1}(S(t)y_n) \to y \quad \text{as } n \to \infty. \tag{1.6.198}$$

By the continuity of $S(t)$, from (1.6.198) we obtain

$$S(t)y_n \to S(t)y \quad \text{as } n \to \infty, \tag{1.6.199}$$

and by uniqueness of limits, from (1.6.198) and (1.6.199) we conclude that $S(t)y = x$, i.e. $y = S(t)^{-1}x$, which proves (1.6.196). Furthermore, we conclude that $S(t)^{-1} \in \mathcal{L}(X)$ for all $t \geq 0$, which completes the proof. □

Proposition 1.52 *Let X be a Banach space and S a C_0–semigroup with infinitesimal generator A. If for some $t_0 > 0$ the inverse $S(t_0)^{-1}$ exists and $S(t_0)^{-1} \in \mathcal{L}(X)$, then A is the infinitesimal generator of a C_0–group.*

Proof: Since A is the infinitesimal generator of a C_0–semigroup, by Proposition 1.50 it suffices to show that $-A$ is also the infinitesimal generator of a C_0–semigroup. Indeed, by Proposition 1.51 we have that for every $t \geq 0$, $S(t)$ is invertible and $S(t)^{-1} \in \mathcal{L}(X)$. Define

$$\begin{aligned} T : \mathbb{R}_+ &\to \mathcal{L}(X) \\ t &\mapsto T(t) = [S(t)]^{-1}. \end{aligned} \tag{1.6.200}$$

We now prove that the mapping (1.6.200) is a C_0–semigroup whose infinitesimal generator is $-A$.

Indeed,

$$\begin{aligned}
&(i) \quad T(0) = [S(0)]^{-1} = I. \\
&(ii) \quad T(s+t) = [S(t+s)]^{-1} = [S(s)S(t)]^{-1} \\
&\qquad\qquad = [S(t)]^{-1}[S(s)]^{-1} = T(t)T(s), \quad \text{for all } t, s \in \mathbb{R}_+.
\end{aligned}$$

It remains to show that:

$$(iii) \quad \lim_{h\to 0_+} \|T(h)x - x\| = 0, \quad \text{for all } x \in X.$$

Let $x \in X$ and $r > 1$. Since $S(t)$ is invertible and $S(t)^{-1} \in \mathcal{L}(X)$ for all $t \geq 0$, we have that $S(t)$ is bijective for all $t \geq 0$. Therefore $S(t)X = X$ for all $t \geq 0$ and, consequently, there exists $y \in X$ such that $S(r)y = x$. Let $0 < h < 1$. Then

$$x = S(r)y = S(h)S(r-h)y,$$

and hence

$$T(h)x = T(h)S(h)S(r-h)y = [S(h)]^{-1}S(h)S(r-h)y = S(r-h)y,$$

that is,

$$T(h)x = S(r-h)y.$$

Since S is strongly continuous, it follows that

$$\lim_{h\to 0_+} S(r-h)y = S(r)y,$$

and therefore

$$\lim_{h\to 0_+} T(h)x = S(r)y = x,$$

which proves item (iii) and consequently that the mapping defined in (1.6.200) is a C_0–semigroup.

It remains to prove that $-A$ is the infinitesimal generator of T. Let B be the infinitesimal generator of T and consider $x \in D(-A) = D(A)$. Note that

$$\begin{aligned}
\frac{T(h)x - x}{h} &= \frac{[S(h)]^{-1}x - x}{h} \\
&= \frac{[S(h)]^{-1}x - [S(h)]^{-1}S(h)x}{h} \\
&= [S(h)]^{-1}\left(\frac{x - S(h)x}{h}\right) \\
&= -[S(h)]^{-1}\left(\frac{S(h)x - x}{h}\right) \\
&= -T(h)\left(\frac{S(h)x - x}{h}\right).
\end{aligned}$$

From the last identity we obtain

$$\begin{aligned}\left\|\frac{T(h)x-x}{h}+Ax\right\| &= \left\|-T(h)\left(\frac{S(h)x-x}{h}\right)+Ax\right\| \\ &\leq \left\|-T(h)\left(\frac{S(h)x-x}{h}\right)+T(h)Ax\right\|+\|-T(h)Ax+Ax\| \\ &\leq \|T(h)\|\left\|\frac{S(h)x-x}{h}-Ax\right\|+\|T(h)Ax-Ax\|.\end{aligned} \tag{1.6.201}$$

Since $\|T(h)\|$ is bounded on bounded intervals and

$$\frac{S(h)x-x}{h}\to Ax \quad \text{as } h\to 0_+,$$

it follows that

$$\|T(h)\|\left\|\frac{S(h)x-x}{h}-Ax\right\|\longrightarrow 0 \quad \text{as } h\to 0_+. \tag{1.6.202}$$

Moreover, since T is strongly continuous we obtain

$$\|T(h)Ax-Ax\|\longrightarrow 0 \quad \text{as } h\to 0_+. \tag{1.6.203}$$

Thus, from (1.6.201), (1.6.202) and (1.6.203) we conclude that

$$\left\|\frac{T(h)x-x}{h}+Ax\right\|\longrightarrow 0 \quad \text{as } h\to 0_+.$$

This last convergence shows that if $x\in D(A)$ then the limit $\lim_{h\to 0_+}\frac{T(h)x-x}{h}$ exists and is equal to $-Ax$, that is,

$$D(A)\subset D(B) \quad \text{and} \quad Bx=-Ax, \quad \text{for all } x\in D(A). \tag{1.6.204}$$

On the other hand, if $x\in D(B)$, we have

$$\begin{aligned}\lim_{h\to 0_+}\left\|\frac{S(h)x-x}{h}+Bx\right\| &= \lim_{h\to 0_+}\left\|-S(h)\left(\frac{T(h)x-x}{h}\right)+Bx\right\| \\ &= \lim_{h\to 0_+}\left\|-S(h)\left(\frac{T(h)x-x}{h}-Bx+Bx\right)+Bx\right\| \\ &= \lim_{h\to 0_+}\left\|-S(h)\left(\frac{T(h)x-x}{h}-Bx\right)-S(h)Bx+Bx\right\| \\ &\leq \lim_{h\to 0_+}\left[\| S(h)x \|_{\mathfrak{L}(X)}\left\|\frac{T(h)x-x}{h}-Bx\right\|+\| S(h)Bx-Bx \|\right] \\ &= \lim_{h\to 0_+}\| S(h)x \|_{\mathfrak{L}(X)}\left\|\frac{T(h)x-x}{h}-Bx\right\|+\lim_{h\to 0_+}\| S(h)Bx-Bx \| \\ &= 0,\end{aligned}$$

because $\| S(h) \|$ is bounded on bounded intervals and the limit $\lim_{h\to 0_+}\dfrac{T(h)x-x}{h}$ exists. Thus we conclude that the limit $\lim_{h\to 0_+}\dfrac{S(h)x-x}{h}$ exists and hence

$$D(B)\subset D(A). \tag{1.6.205}$$

From (1.6.204) and (1.6.205) we deduce that $D(A) = D(B)$ and $Bx = -Ax$ for all $x \in D(B)$, that is, $B = -A$. Therefore $-A$ is the infinitesimal generator of a C_0–semigroup, which completes the proof. $\square$

Remark 1.53 *Under the hypotheses of Proposition 1.52, that is, with X a Banach space, S a C_0–semigroup whose infinitesimal generator is A and $S(t_0)^{-1} \in \mathcal{L}(X)$ for some $t_0 > 0$, the operator $-A$ generates a C_0–semigroup*

$$\begin{aligned} T : \mathbb{R}_+ &\to \mathcal{L}(X) \\ t &\mapsto T(t) = (S(t))^{-1}. \end{aligned} \tag{1.6.206}$$

Now, according to Proposition 1.50, A generates the group $U : \mathbb{R} \to \mathcal{L}(X)$ defined by

$$U(t) = \begin{cases} S(t), & t \geq 0, \\ T(-t) = (S(-t))^{-1}, & t < 0. \end{cases} \tag{1.6.207}$$

Proposition 1.54 *Let S_1 and S_2 be groups generated by A. Then $S_1 = S_2$.*

Proof: From the proof of Proposition 1.50 we know that A is the infinitesimal generator of the semigroups $S_{1,+}$ and $S_{2,+}$ and $-A$ is the infinitesinal generator of the semigroups $S_{1,-}$ and $S_{2,-}$. However, by Exercise 1.3.1 we have uniqueness of the semigroup generated by an operator. Hence

$$S_{1,+} = S_{2,+} \quad \text{and} \quad S_{1,-} = S_{2,-},$$

that is, $S_1(t) = S_2(t)$ as well as $S_1(-t) = S_2(-t)$ for all $t \geq 0$, since $S_{i,+}(t) = S_i(t)$ and $S_{i,-}(t) = S_i(-t)$ for all $t \geq 0$ and $i = 1, 2$. Let $t \in \mathbb{R}$. If $t \geq 0$ we have $S_1(t) = S_2(t)$, and if $t < 0$ then $t = -\xi$ for some $\xi > 0$ and hence $S_1(-\xi) = S_2(-\xi)$, that is, $S_1(t) = S_2(t)$, which implies $S_1 = S_2$. $\square$

0.5 cm

Definition 1.55 *We say that an operator $T \in \mathcal{L}(H)$, where H is a Hilbert space, is unitary if T is invertible and $T^* = T^{-1}$.*

Remark 1.56 *Note that if $D(T) = H$ and T is continuous, then T^* exists and $T^* \in \mathcal{L}(H)$. Moreover, if T is unitary, then $T^{-1} = T^* \in \mathcal{L}(H)$. Furthermore,*

$$\begin{aligned} \|Tx\|^2 &= (Tx, Tx) \\ &= (x, T^*Tx) \\ &= (x, T^{-1}Tx) = \|x\|^2, \end{aligned}$$

that is,

$$\|Tx\| = \|x\|, \quad \textit{for all } x \in H. \tag{1.6.208}$$

Thus unitary operators are isometries. Also, since $x = TT^{-1}x$, from (1.6.208) it follows that

$$\|x\| = \|TT^{-1}x\| = \|T^{-1}x\|,$$

and therefore

$$\|T^{-1}x\| = \|x\|, \quad \textit{for all } x \in H, \tag{1.6.209}$$

or equivalently

$$\|T^*x\| = \|x\|, \quad \textit{for all } x \in H. \tag{1.6.210}$$

Hence, if $T : H \to H$ is unitary, then T, T^{-1} and T^* are isometries.

Definition 1.57 *We say that a group S of bounded linear operators on a Hilbert space H is a unitary group if, for each $t \geq 0$, $S(t)$ is a unitary operator, that is, $S(t)^* = S(t)^{-1}$ for all $t \geq 0$.*

Theorem 1.58 *[Stone's Theorem] A linear operator A on a Hilbert space H is the infinitesimal generator of a unitary C_0–group if and only if $A^* = -A$.*

Proof: Let A be the infinitesimal generator of a unitary C_0–group S. By Proposition 1.50, A and $-A$ generate, respectively, C_0–semigroups S_+ and S_-. Since S is a unitary group, $S(t)^{-1}$ exists and $S(t)^{-1} \in \mathcal{L}(H)$. Then, by Proposition 1.49, A^* is the infinitesimal generator of S_+^*, where $S_+^* = (S_+(t))^*$ for all $t \geq 0$. It follows from this and from the fact that S is unitary that, if $h > 0$, we have

$$S_+^*(h) = (S_+(h))^* = (S(h))^* = (S(h))^{-1} = S(-h) = S_-(h),$$

because $I = S(t)S(-t) = S(-t)S(t)$ for all $t \geq 0$. Hence

$$\frac{S_+^*(h)x - x}{h} = \frac{S_-(h)x - x}{h}, \quad \text{for all } x \in H, \tag{1.6.211}$$

which implies $D(A^*) = D(-A)$ and $A^*x = -Ax$, proving the necessity.

We now prove the sufficiency. Let A be a linear operator on H such that A^* exists and satisfies

$$A^* = -A. \tag{1.6.212}$$

We will show that A and $-A$ are infinitesimal generators of C_0– semigroups. To that end we use the Lumer–Phillips Theorem. From the existence of A^* it follows that A and $-A$ are densely defined. We will prove that

$$\mathrm{Re}(\pm Ax, x) = 0, \quad \text{for all } x \in D(A). \tag{1.6.213}$$

Indeed, let $x \in D(A)$. From (1.6.212) we have

$$(Ax, x) = (x, A^*x) = (x, -Ax) = -(x, Ax) = -\overline{(Ax, x)},$$

hence

$$\mathrm{Re}(Ax, x) = 0, \quad \text{for all } x \in D(A),$$

which proves (1.6.213) and therefore shows that A and $-A$ are dissipative with respect to the duality mapping $j = I$.

It remains to show that there exists $\lambda_0 > 0$ such that $\mathrm{Im}(\lambda_0 I \pm A) = H$. Indeed, let $x \in D(A)$. Then

$$((I \pm A)x, x) = \|x\|^2 \pm (Ax, x).$$

From this identity and (1.6.213) we obtain

$$\|x\|^2 = \mathrm{Re}((I \pm A)x, x) \leq \|x \pm Ax\| \, \|x\|,$$

which implies

$$\|x\| \leq \|x \pm Ax\|, \quad \text{for all } x \in D(A). \tag{1.6.214}$$

As we know, A^* is a closed operator. From (1.6.212) it follows that $\pm A$ are also closed, and consequently $(I \pm A)$ are closed as well. We claim that

$$\operatorname{Im}(I \pm A) \text{ are closed subsets of } H. \tag{1.6.215}$$

Indeed, let $(y_\nu)_\nu \subset \operatorname{Im}(I \pm A)$ be such that

$$y_\nu \to y \quad \text{in } H \text{ as } \nu \to \infty. \tag{1.6.216}$$

For each $\nu \in \mathbb{N}$, there exist $x_\nu, \omega_\nu \in D(A)$ such that

$$y_\nu = (I + A)x_\nu \quad \text{and} \quad y_\nu = (I - A)\omega_\nu.$$

We will prove the claim for the operator $I + A$; the proof for $I - A$ is analogous. From (1.6.214) it follows that, if $\nu, \mu \in \mathbb{N}$, then

$$\|x_\nu - x_\mu\| \le \|(I + A)x_\nu - (I + A)x_\mu\| = \|y_\nu - y_\mu\|.$$

From (1.6.216) the expression on the right-hand side of the last inequality converges to zero as $\nu, \mu \to \infty$. Hence, $(x_\nu)_\nu$ is a Cauchy sequence in H and therefore there exists $x \in H$ such that

$$x_\nu \to x \quad \text{in } H \text{ as } \nu \to \infty. \tag{1.6.217}$$

From (1.6.216) we also have

$$(I + A)x_\nu \to y \quad \text{in } H \text{ as } \nu \to \infty. \tag{1.6.218}$$

Since $(I + A)$ is closed, from (1.6.217) and (1.6.218) we conclude that $x \in D(A)$ and $y = (I + A)x$, which proves that $y \in \operatorname{Im}(I + A)$ and hence (1.6.215). It follows from this and from the fact that H is a Hilbert space that

$$H = \operatorname{Im}(I + A) \oplus [\operatorname{Im}(I + A)]^\perp. \tag{1.6.219}$$

We claim that

$$[\operatorname{Im}(I + A)]^\perp = \{0\}. \tag{1.6.220}$$

Indeed, recall that

$$[\operatorname{Im}(I + A)]^\perp = \{y \in H;\ (y, x + Ax) = 0, \text{ for all } x \in D(A)\}.$$

Let $y \in [\operatorname{Im}(I + A)]^\perp$. Then

$$(y, x) = (y, Ax), \quad \text{for all } x \in D(A). \tag{1.6.221}$$

From identity (1.6.221) it follows that $\pm y \in D(A^*)$, and since $D(A) = D(A^*)$, from (1.6.213) and (1.6.221), taking $y = x$, we obtain

$$\|y\|^2 = \operatorname{Re}(y, Ay) = 0,$$

that is, $y = 0$, which proves (1.6.220), since trivially $0 \in [\operatorname{Im}(I + A)]^\perp$. It follows from (1.6.219) and

(1.6.220) that

$$H = \mathrm{Im}(I + A). \tag{1.6.222}$$

From (1.6.213) and (1.6.222) and the fact that $D(A) = D(-A)$ is dense in H, it follows, by the Lumer–Phillips Theorem, that A and $-A$ are, respectively, the infinitesimal generators of the contraction C_0–semigroups S_+ and S_-. By Proposition 1.50 it follows that A generates the group S given by

$$S(t) = \begin{cases} S_+(t), & t \geq 0, \\ S_-(-t), & t < 0. \end{cases}$$

It remains to show that S is a unitary group. Indeed, since A^* is the infinitesimal generator of $S_+^*(t)$ for all $t \geq 0$, where

$$S_+^*(t) = (S_+(t))^*, \quad \forall t \geq 0, \tag{1.6.223}$$

it follows from (1.6.212) and from the uniqueness of the semigroup that

$$S_+^*(t) = S_-(t), \quad \forall t \geq 0. \tag{1.6.224}$$

From (1.6.223) and (1.6.224) we obtain

$$(S_+(t))^* = S_-(t), \quad \forall t \geq 0.$$

But since

$$S_+(t) = S(t) \quad \text{and} \quad S_-(t) = S_-(-(-t)) = S(-t), \quad \forall t \geq 0,$$

we get

$$(S(t))^* = S(-t), \quad \forall t \geq 0. \tag{1.6.225}$$

Since $I = S(t)S(-t) = S(-t)S(t)$, we have

$$(S(t))^{-1} = S(-t), \quad \forall t \in \mathbb{R},$$

and therefore, from (1.6.225), we obtain

$$(S(t))^* = (S(t))^{-1}, \quad \forall t \geq 0,$$

which completes the proof. □

1.6.1 Exercises

1.6.1) Let A be the infinitesimal generator of a C_0–semigroup S_+ and $-A$ the infinitesimal generator of a C_0–semigroup S_-. Define

$$B_\lambda = \lambda^2 R(\lambda, A) - \lambda I \quad \text{and} \quad \tilde{B}_\lambda = \lambda^2 R(\lambda, -A) - \lambda I; \quad \lambda > \omega > M_0 = \max\{\omega_0, \tilde{\omega}_0\},$$

where

$$\omega_0 = \lim_{t \to +\infty} \frac{\log \|S_+(t)\|}{t} \quad \text{and} \quad \tilde{\omega}_0 = \lim_{t \to +\infty} \frac{\log \|S_-(t)\|}{t}.$$

Given $\lambda, \mu > \omega > M_0$, prove that

$$R(\mu, -A)\, R(\lambda, A) = R(\lambda, A)\, R(\mu, -A).$$

1.6.2) Prove that, for A to be the infinitesimal generator of a C_0–group, it is necessary and sufficient that A be closed, densely defined, and that there exist real numbers ω and M such that, if $\lambda \in \mathbb{R}$ and $|\lambda| > \omega$, then $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(|\lambda| - \omega)^n}.$$

1.6.3) Let A be the infinitesimal generator of a C_0–semigroup $T(t)$ satisfying $\|T(t)\| \leq M e^{\omega t}$. Show that, for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$, one has $\lambda \in \rho(A)$ and

$$\|R(\lambda; A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}, \quad n = 1, 2, 3, \ldots$$

Solution:

Define

$$R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} T(t)x\, dt.$$

Since $\|T(t)\| \leq M e^{\omega t}$, the operator $R(\lambda)$ is well defined for all λ with $\operatorname{Re}\lambda > \omega$. Using arguments similar to those employed in the proof of the Hille–Yosida Theorem, it follows that $R(\lambda) = R(\lambda, A) \Rightarrow \lambda \in \rho(A)$. Hence, if $\operatorname{Re}\lambda > \omega$, then

$$\frac{d}{d\lambda} R(\lambda, A)x = \frac{d}{d\lambda} \int_0^{+\infty} e^{-\lambda t} T(t)x\, dt = \int_0^{+\infty} t\, e^{-\lambda t} T(t)x\, dt.$$

Proceeding by induction, we can show that

$$\frac{d}{d\lambda} R(\lambda, A)x = \frac{d}{d\lambda} \int_0^{+\infty} e^{-\lambda t} T(t)x\, dt = \int_0^{+\infty} t\, e^{-\lambda t} T(t)x\, dt. \tag{1.6.226}$$

On the other hand,

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A),$$

and from the fact that, for $\lambda \in \rho(A)$, the map $\lambda \mapsto R(\lambda, A)$ is holomorphic, we obtain

$$\frac{d}{d\lambda} R(\lambda, A)x = -R(\lambda, A)^2 x. \tag{1.6.227}$$

Again, by induction, we have

$$\frac{d^n}{d\lambda^n} R(\lambda, A)x = (-1)^n n!\, R(\lambda, A)^{n+1} x. \tag{1.6.228}$$

From (1.6.226) and (1.6.228) we obtain

$$R(\lambda, A)^{n+1} x = \frac{1}{n!} \int_0^{+\infty} t^n e^{-\lambda t} T(t)x\, dt.$$

Thus

$$\|R(\lambda; A)^n\| \leq \frac{M}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{(\omega - \operatorname{Re}\lambda)t}\, dt = \frac{M}{(\operatorname{Re}\lambda - \omega)^n},$$

as claimed.

1.6.4) Let B be a bounded operator. If $\gamma > \|B\|$, prove that

$$e^{tB} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}R(\lambda; B)\,d\lambda.$$

(Suggestion: Choose $\gamma > r > \|B\|$ and consider C_r, the circle of radius r centred at the origin. Observe that, for $|\lambda| > r$, we have

$$R(\lambda; B) = \sum_{k=0}^{\infty}\frac{B^k}{\lambda^{k+1}}.$$

Multiply the last identity by $(1/2\pi i)e^{\lambda t}$, integrate over C_r and conclude by applying Cauchy's Theorem.)

The convergence above is in the sense of the uniform topology in t on bounded intervals.

1.6.5) Let A be the infinitesimal generator of a C_0–semigroup $T(t)$ satisfying $\|T(t)\| \le Me^{\omega t}$. Let μ be a real number, $\mu > \omega \ge 0$, and consider

$$A_\mu = \mu A R(\mu; A) = \mu^2 R(\mu; A) - \mu I,$$

the Yosida approximation of A. Prove that:

(i) For $\operatorname{Re}\lambda > \omega\mu/(\mu-\omega)$ we have

$$R(\lambda; A_\mu) = (\lambda+\mu)^{-1}(\mu I - A)R\Big(\frac{\mu\lambda}{\mu+\lambda}; A\Big), \quad \text{and}$$

$$\|R(\lambda; A_\mu)\| \le M\Big(\operatorname{Re}\lambda - \frac{\omega\mu}{\mu-\omega}\Big)^{-1}.$$

(Suggestion: Multiply the identity above by $\lambda I - A_\mu$ and use the commutativity of A with its resolvent to obtain the desired expression. To prove the inequality, note that A_μ is the infinitesimal generator of e^{tA_μ} and use Corollary 1.37.)

(ii) For $\operatorname{Re}\lambda > \varepsilon + \omega\mu/(\mu-\omega)$ and $\mu > 2\omega$, there exists a constant C, depending only on M and ε, such that for all $x \in D(A)$,

$$\|R(\lambda; A_\mu)x\| \le \frac{C}{|\lambda|}(\|x\| + \|Ax\|).$$

1.6.6) Let A be as in Exercise 1.6.4, and let $\lambda = \gamma + i\eta$ with fixed $\gamma > \omega + \varepsilon$. Prove that for every $x \in X$ we have

$$\lim_{\mu\to\infty} R(\lambda; A_\mu)x = R(\lambda; A)x,$$

and for every $Y > 0$ the limit is uniform in η for $|\eta| \le Y$. (Suggestion: Let $\nu = \mu\lambda/(\mu+\lambda)$. Use item (i) of Exercise 1.6.5 to conclude that $R(\lambda; A_\mu) - R(\lambda; A) = (\mu+\lambda)^{-1}A^2R(\nu; A)R(\lambda; A)$. For $\gamma > \omega + \varepsilon$ use the Hille–Yosida Theorem to deduce $\|R(\lambda; A)\| \le M\varepsilon^{-1}$. From this, first obtain the desired convergence for elements of $D(A^2)$ and then, by density, for all $x \in X$.)

1.6.7) Let A be the infinitesimal generator of a C_0–semigroup $T(t)$ satisfying $\|T(t)\| \le Me^{\omega t}$ and consider $\gamma > \max(0, \omega)$. If $x \in D(A)$, prove that

$$\int_0^t T(s)x\,ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}R(\lambda; A)x\frac{d\lambda}{\lambda},$$

and that the integral on the right-hand side converges uniformly in t on bounded intervals. (Suggestion:

Take $\mu > 0$ and, for $\delta > \|A_\mu\|$, consider

$$\rho_k(s) = \frac{1}{2\pi i}\int_{\delta-ik}^{\delta+ik} e^{\lambda s} R(\lambda; A_\mu)x\, d\lambda.$$

Using Exercise 1.6.3, conclude that $\rho_k(t) \to e^{tA_\mu}x$ uniformly on $[0,T]$ and that $\lim_{k\to\infty}\int_{\delta-ik}^{\delta+ik} R(\lambda; A_\mu)x\,\frac{d\lambda}{\lambda} = 0$. Conclude then that

$$\int_0^t e^{sA_\mu}x\, ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}R(\lambda; A)x\frac{d\lambda}{\lambda}.$$

To complete the exercise, use Exercises 1.6.4(ii) and 1.6.5 together with the Hille–Yosida Theorem.)

1.6.8) Let A be the infinitesimal generator of a C_0–semigroup $T(t)$ satisfying $\|T(t)\| \le Me^{\omega t}$. Consider $\gamma > \max(0,\omega)$. If $x \in D(A^2)$, prove that

$$T(t)x = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}R(\lambda; A)x\, d\lambda.$$

(Suggestion: use Exercise 1.6.7.)

1.7 Differentiable Semigroups

Let A be the infinitesimal generator of a C_0–semigroup S. As proved in Proposition 1.30, if $x \in D(A)$ then $S(t)x \in D(A)$ for all $t \ge 0$, and therefore $S(t)D(A) \subset D(A)$ for all $t \ge 0$. This property does not, in general, hold for all $x \in X$, because if $S(t)X \subset D(A)$ for all $t \ge 0$, then $X = IX = S(0)X \subset D(A)$, so that $D(A) = X$ and hence, by the Closed Graph Theorem, A is a bounded linear operator. We then fall back into the particular case of uniform convergence, already studied earlier (see Theorem 1.19). This does not happen, however, if $S(t)X \subset D(A)$ only for $t > 0$ and, more generally, only for $t > t_0 \ge 0$. It is precisely this particular case that we now consider.

Definition 1.59 *A C_0–semigroup S with infinitesimal generator A is said to be differentiable for $t > t_0 \ge 0$ if $S(t)X \subset D(A)$ for all $t > t_0$. The semigroup S is said to be differentiable if S is differentiable for $t > 0$.*

Theorem 1.60 *Let S be a semigroup that is differentiable for $t > t_0 \ge 0$. Then:*
(i) The operator $A \circ S(t)$ is continuous in x for $t > t_0$.
(ii) The function $S(t)x$ is continuously differentiable for all $t > t_0$ and all $x \in X$. Moreover,

$$\frac{d}{dt}S(t)x = AS(t)x.$$

(iii) For every $t > nt_0$, $n = 1, 2, \ldots$, we have $S(t) : X \to D(A^n)$ and, defining $S^{(n)}$ by $S^{(n)}(t) = A^n \circ S(t)$, we have that $S^{(n)}$ is a linear and continuous operator, $S^{(n)}(t)x = \frac{d^n}{dt^n}S(t)x$ for all $x \in X$, and the mapping $t \mapsto S(t)x$ is n times continuously differentiable.
(iv) For $t > nt_0$, $n = 1, 2, \ldots$, $S^{(n-1)}(t)$ is continuous in the uniform topology, where $S^{(0)}(t) = S(t)$.

Proof:

(i) First note that, since $S(t)X \subset D(A)$, the composition $A \circ S(t)$ is well defined for $t > t_0$. As $S(t)$ is bounded and A is closed (see Proposition 1.31), it follows that $A \circ S(t)$ is closed for $t > t_0$, and by the Closed Graph Theorem, $A \circ S(t)$ is continuous for $t > t_0$.

(ii) By hypothesis, $S(t)x \in D(A)$ for all $t > t_0$ and all $x \in X$. Thus, let $x \in X$. For every $t > t_0$

the limit

$$\lim_{h\to 0_+}\frac{S(h)S(t)x-S(t)x}{h}=\lim_{h\to 0_+}\frac{S(t+h)x-S(t)x}{h}=AS(t)x$$

exists, that is, $S(t)x$ is right–differentiable for all $t>t_0$ and

$$\frac{d^+}{dt}S(t)x=AS(t)x.$$

Our goal is to use Dini's Lemma (Exercise 1.3.4), and therefore it remains to show that $AS(\cdot)x$ is continuous for all $t>t_0$. Indeed, let $t>t_0$ and choose s with $t>s>t_0$. Then, by item (i), $AS(s)$ is a bounded operator and, since

$$AS(t)=AS(s)S(t-s),$$

we have, for $0<h<t-s$,

$$\begin{aligned}\|AS(t+h)x-AS(t)x\| &= \|AS(s)S(t+h-s)x-AS(s)S(t-s)x\|\\ &\le \|AS(s)\|\,\|S(t+h-s)x-S(t-s)x\|\longrightarrow 0\end{aligned}$$

as $h\to 0_+$, because S is strongly continuous. Hence, $AS(t)x$ is continuous for all $t>t_0$. From the above and Dini's Lemma (see Exercise 1.3.4) it follows that $S(t)x$ is continuously differentiable for all $t>t_0$ and

$$\frac{d}{dt}S(t)x=AS(t)x.$$

(iii) We use induction on n to prove this item. From item (i) we know that $S^{(1)}(t)=A\circ S(t)$ is continuous in x for $t>t_0$. Moreover, from item (ii) we know that the mapping

$$t\longmapsto S(t)x$$

is differentiable in t for every $t>t_0$ and every $x\in X$, and in fact it is $C^1(t_0,+\infty)$. Furthermore,

$$\frac{d}{dt}S(t)x=AS(t)x:=S^{(1)}(t)x,\quad t>t_0.$$

Therefore, we have proved item (iii) for $n=1$.

Assume now that (iii) holds for some n and prove it for $n+1$. Let $t>(n+1)t_0$ and choose $s>nt_0$ such that $t-s>t_0$. Then

$$S^{(n)}(t)x=A^nS(t)x=A^nS(t-s)S(s)x=S(t-s)A^nS(s)x,\quad \forall x\in X$$

(see that $S(s)x\in D(A^n)$). By item (ii) the right-hand side is continuously differentiable, and thus $S(t)x$ is $(n+1)$ times differentiable. Moreover,

$$\frac{d}{dt}S^{(n)}(t)x=\frac{d}{dt}\big(S(t-s)A^nS(s)x\big)=AS(t-s)A^nS(s)x,\quad \forall x\in X.$$

Since $S(t-s)A^nS(s)x=A^nS(t)x$, it follows that

$$\frac{d}{dt}S^{(n)}(t)x=A^{n+1}S(t)x=S^{(n+1)}(t)x.$$

Hence $S^{(n+1)}(t)x=\frac{d^{n+1}}{dt^{n+1}}S(t)x$ for all $x\in X$. Also, $S^{(n)}(t)x\in D(A)$ for all $t>(n+1)t_0$, and since $S^{(n)}(t)$ is a bounded operator and A is closed, it follows that $S^{(n+1)}(t)=AS^{(n)}(t)$ is closed. By the Closed Graph Theorem, $S^{(n+1)}(t):X\to D(A^{n+1})$ is continuous, which completes the proof of item (iii).

(iv) We first show that, for $t > t_0$, the operator $S(t)$ is continuous in the uniform topology. Indeed, since $\|S(t)\|_{\mathfrak{L}(X)}$ is bounded on bounded intervals, there exists $M_1 \geq 0$ such that

$$\|S(t)\|_{\mathfrak{L}(X)} \leq M_1, \quad \forall t \in [0,1].$$

Let $t_0 < t_1 \leq t_2 \leq t_1 + 1$. Then

$$\begin{aligned} S(t_2)x - S(t_1)x &= \int_{t_1}^{t_2} AS(s)x\,ds \\ &= \int_{t_1}^{t_2} AS(s-t_1)S(t_1)x\,ds \\ &= \int_{t_1}^{t_2} S(s-t_1)AS(t_1)x\,ds. \end{aligned}$$

Hence

$$\begin{aligned} \|S(t_2)x - S(t_1)x\| &\leq \int_{t_1}^{t_2} \|S(s-t_1)AS(t_1)x\|\,ds \\ &\leq \int_{t_1}^{t_2} \|S(s-t_1)\|_{\mathfrak{L}(X)}\|AS(t_1)x\|\,ds \\ &\leq M_1 \int_{t_1}^{t_2} \|AS(t_1)x\|\,ds \\ &\leq M_1 \int_{t_1}^{t_2} \|AS(t_1)\|_{\mathfrak{L}(X)}\|x\|\,ds \\ &= M_1\|AS(t_1)\|_{\mathfrak{L}(X)}\|x\|\,|t_2-t_1|. \end{aligned}$$

Therefore

$$\|S(t_2) - S(t_1)\|_{\mathfrak{L}(X)} \leq M_1\|AS(t_1)\|_{\mathfrak{L}(X)}|t_2-t_1|.$$

Thus $S(t)$ is continuous in the uniform topology, and we have proved item (iv) for $n = 1$.

Using induction on n one checks easily that

$$S^{(n)}(t)x \in D(A), \quad \forall x \in X \text{ and } \forall t > (n+1)t_0. \tag{1.7.229}$$

We also have that, for all $t > nt_0$ and all s such that $t - t_0 > s > (n-1)t_0$,

$$S^{(n-1)}(t)x = S(t-s)S^{(n-1)}(s)x, \quad \forall x \in X,\ n = 1,2,\ldots$$

To prove this assertion we again use induction on n. Let $t > t_0$ and $t - t_0 > s > 0$. Then $t - s > t_0 > 0$, and hence

$$S^{(0)}(t)x := S(t)x = S(t-s)S(s)x = S(t-s)S^{(0)}(s)x, \quad \forall x \in X.$$

Thus the assertion holds for $n = 1$. Assume it holds for $n - 1$ and we show it for n.

Let $t > (n+1)t_0$ and $t - t_0 > s > nt_0$. Since $s > nt_0$, it follows from (1.7.229) that

$$S^{(n-1)}(s)x \in D(A), \quad \forall x \in X.$$

By Proposition 1.30,

$$S(t-s)S^{(n-1)}(s)x \in D(A)$$

and

$$AS(t-s)S^{(n-1)}(s)x = S(t-s)AS^{(n-1)}(s)x, \quad \forall x \in X.$$

But $t > (n+1)t_0 > nt_0$ and $t - t_0 > s > nt_0 > (n-1)t_0$. Hence, by the induction hypothesis, we have

$$S^{(n-1)}(t)x = S(t-s)S^{(n-1)}(s)x, \quad \forall x \in X,$$

and so

$$\begin{aligned} S^{(n)}(t)x &:= A^n S(t)x = AA^{n-1}S(t)x := AS^{(n-1)}(t)x \\ &= AS(t-s)S^{(n-1)}(s)x = S(t-s)AS^{(n-1)}(s)x \\ &:= S(t-s)AA^{n-1}S(s)x = S(t-s)S^{(n)}(s)x, \quad \forall x \in X, \end{aligned}$$

which completes the proof of the assertion.

We now prove item (iv) in the general case (we have only proved (iv) for $n = 1$). Let $t > nt_0$. We will show that $S^{(n-1)}(t)$ is continuous in the uniform topology. Take $t - t_0 > s > (n-1)t_0$. Then $t - s > t_0$, and if $|h| < t - s - t_0$, we have

$$s + t_0 - t < h < t - s - t_0, \quad t + h - t_0 > s > (n-1)t_0$$

and $t + h > nt_0$. Therefore, by the assertion proved above, we obtain

$$S^{(n-1)}(t) = S(t-s)S^{(n-1)}(s)$$

and

$$S^{(n-1)}(t+h) = S(t+h-s)S^{(n-1)}(s).$$

Thus

$$\begin{aligned} \|S^{(n-1)}(t+h) - S^{(n-1)}(t)\|_{\mathfrak{L}(X)} &= \|S(t+h-s)S^{(n-1)}(s) - S(t-s)S^{(n-1)}(s)\|_{\mathfrak{L}(X)} \\ &\le \|S(t+h-s) - S(t-s)\|_{\mathfrak{L}(X)} \, \|S^{(n-1)}(s)\|_{\mathfrak{L}(X)} \longrightarrow 0 \end{aligned}$$

as $h \longrightarrow 0$. Therefore, $S^{(n-1)}(t)$ is continuous in the uniform topology for $t > nt_0$. □

1.7.1 Exercises

1.7.1) Let $T(t)$ be a differentiable C_0–semigroup and let A be its infinitesimal generator. Prove that

$$T^n(t) = \left(AT\left(\frac{t}{n}\right)\right)^n = \left(T'\left(\frac{t}{n}\right)\right)^n, \quad n = 1, 2, \ldots$$

1.8 Analytic Semigroups

Definition 1.61 *Let $\Delta = \{z : \varphi_1 < \arg z < \varphi_2,\ \varphi_1 < 0 < \varphi_2\}$ and, for $z \in \Delta$, let $T(z)$ be a bounded linear operator. The family $T(z)$, $z \in \Delta$, is called an analytic semigroup in Δ if*
(i) $z \mapsto T(z)$ is analytic in z.
(ii) $T(0) = I$ and $\lim_{z\in\Delta,\ z\to 0} T(z)x = x$ for every $x \in X$.
(iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.
The semigroup $T(t)$ is said to be analytic if it is analytic in some sector Δ containing the non-negative real axis.

Remark 1.62 *In the definition above we have $0 < |\varphi_1|,\ \varphi_2 \le \pi$.*

Clearly, the restriction of an analytic semigroup to the non-negative real axis is a C_0-semigroup. In what follows, we are interested in the possibility of extending a given C_0-semigroup to an analytic semigroup in some sector Δ around the non-negative real axis.

Since multiplication of a C_0-semigroup $T(t)$ by $e^{\omega t}$ does not affect the possibility (or impossibility) of extending it to an analytic semigroup in some sector Δ, we may restrict ourselves to the case of uniformly bounded C_0-semigroups. The results for general C_0-semigroups follow from the corresponding results for uniformly bounded C_0-semigroups. Without loss of generality, we shall assume that $0 \in \rho(A)$, where A is the infinitesimal generator of a semigroup $T(t)$. This can always be achieved by multiplying the uniformly bounded semigroup $T(t)$ by $e^{-\varepsilon t}$, for $\varepsilon > 0$, and using Proposition 1.34.

Let A be a densely defined operator on a Banach space X satisfying the following conditions:

$$\text{For some } 0 < \delta < \pi/2,\ \rho(A) \supset \Sigma_\delta = \{\lambda : |\arg \lambda| < \pi/2 + \delta\} \cup \{0\}. \tag{1.8.230}$$

There exists a constant M such that

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}, \quad \text{for } \lambda \in \Sigma_\delta,\ \lambda \neq 0. \tag{1.8.231}$$

We have the following result:

Theorem 1.63 *Let A be a closed, densely defined operator on a Banach space X satisfying conditions (1.8.230) and (1.8.231). Then A is the infinitesimal generator of a C_0-semigroup $S(t)$ satisfying $\|S(t)\| \leq C$ for some constant C. Moreover,*

$$S(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda, A)\, d\lambda,$$

where Γ is a regular curve in Σ_δ going from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, with $\pi/2 < \theta < \pi/2 + \delta$. The integral above converges for $t > 0$ in the uniform operator topology.

Before proving Theorem 1.63 we need some auxiliary results, which we now establish. Let $0 < \delta < \frac{\pi}{2}$ and let A satisfy conditions (1.8.230) and (1.8.231). Note that for $0 < \delta' < \delta$, conditions (1.8.230) and (1.8.231) are also satisfied. For each $r > 0$ and $0 < \delta' < \delta$, we define a family of operators $(S(t))_{t \geq 0}$ by

$$S(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma(r,\delta')} e^{t\lambda} R(\lambda, A)\, d\lambda, & t > 0, \\ I \qquad\qquad\qquad\qquad\qquad , & t = 0, \end{cases} \tag{1.8.232}$$

where $\gamma(r, \delta') = \gamma_1(r, \delta') \cup \gamma_2(r, \delta') \cup \gamma_3(r, \delta')$ is the piecewise C^1 curve defined by

$$\begin{aligned} \gamma_1(r, \delta') &= \{\rho e^{i(\pi/2 + \delta')};\ \rho \in [r, +\infty)\}, \\ \gamma_2(r, \delta') &= \{r e^{i\beta};\ -\tfrac{\pi}{2} - \delta' \leq \beta \leq \tfrac{\pi}{2} + \delta'\}, \\ \gamma_3(r, \delta') &= \{-\rho e^{-i(\pi/2 + \delta')};\ \rho \in [r, +\infty)\}, \end{aligned} \tag{1.8.233}$$

oriented counterclockwise, as in Figure 1.1:

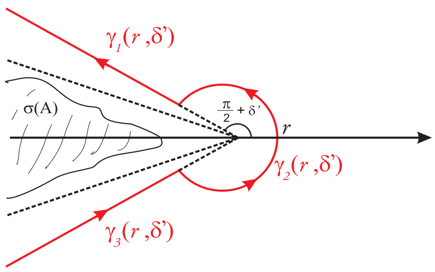

Figure 1.1:

Lemma 1.64 *If A satisfies (1.8.230) and (1.8.231), then the operator $S(t)$ given by (1.8.232) is well defined and is independent of $r > 0$ and of $0 < \delta' < \delta$.*

Proof: Let $t > 0$, $r > 0$ and $\delta' > 0$ be such that $0 < \delta' < \delta$ and $0 < \delta < \frac{\pi}{2}$. We first show the convergence of the integral in (1.8.232) over the curve $\gamma(r, \delta)$. If $\lambda \in \gamma_1(r, \delta)$, then $\lambda = \rho e^{i\theta}$, with $\theta = \frac{\pi}{2} + \delta$ and $\rho \in [r, +\infty)$. Let $\eta = \arg \lambda$. Define

$$f(\lambda) = e^{t\lambda} R(\lambda, A),$$

and set

$$x = \rho \cos \eta = \zeta(\rho) \Rightarrow \zeta'(\rho) = \cos \eta,$$
$$y = \rho \sin \eta = \xi(\rho) \Rightarrow \xi'(\rho) = \sin \eta.$$

Then, by (1.8.231),

$$\begin{aligned} \left\| \int_{\gamma_1(r,\delta)} e^{t\lambda} R(\lambda, A)\, d\lambda \right\| &\leq \int_r^\infty \left\| f(\zeta(\rho) + i\xi(\rho))[\zeta'(\rho) + i\xi'(\rho)] \right\| d\rho \\ &= \int_r^\infty \left\| e^{t\rho e^{i\eta}} R(\rho e^{i\eta}, A) e^{i\eta} \right\| d\rho \\ &\leq \int_r^\infty \left| e^{t\rho e^{i\eta}} \right| \frac{M}{|\rho e^{i\eta}|} \left| e^{i\eta} \right| d\rho \\ &= \int_r^\infty e^{t\rho \cos \eta} \frac{M}{\rho}\, d\rho. \end{aligned} \tag{1.8.234}$$

Since $\pi/2 < \eta < 3\pi/2$, there exists a positive constant C such that $\cos \eta = -C$. Hence

$$\int_r^\infty \frac{e^{-\rho t C}}{\rho}\, d\rho \leq \frac{1}{r} \int_r^\infty e^{-\rho t C}\, d\rho = \frac{1}{rCt} e^{-rCt}. \tag{1.8.235}$$

From (1.8.234) and (1.8.235) we obtain

$$\left\| \int_{\gamma_1(r,\delta)} e^{t\lambda} R(\lambda, A)\, d\lambda \right\| \leq M \int_r^\infty \frac{e^{-\rho t C}}{\rho}\, d\rho \leq C_1, \tag{1.8.236}$$

where $C_1 = M \frac{e^{-rCt}}{rCt}$, for $t > 0$.

Similarly, for $\lambda \in \gamma_3(r, \delta)$, if $x \neq 0$ we obtain

$$\left\| \int_{\gamma_3(r,\delta)} e^{t\lambda} R(\lambda, A)\, d\lambda \right\| \leq C_2, \quad \text{for } t > 0. \tag{1.8.237}$$

For the case $\lambda \in \gamma_2(r, \delta)$, we note that

$$\left\| \int_{\gamma_2(r,\delta)} e^{t\lambda} R(\lambda, A)\, d\lambda \right\| \leq M \int_0^{2\pi} e^{tr \cos \beta}\, d\beta \leq M 2\pi e^{tr}. \tag{1.8.238}$$

Thus, from (1.8.236), (1.8.237) and (1.8.238) it follows that the integral defined in (1.8.232) converges in $\mathcal{L}(X)$ for each $t > 0$.

We now show that the definition is independent of the curve. Let $r, r', \delta', m' > 0$ and let D_ρ be the region bounded by the curves $\Gamma, R_\rho, \Lambda, S_\rho$, given by

$$\begin{aligned}\Gamma &= \Gamma(\rho,r,\delta') = \cup_{j=1}^{3}\Gamma_j(\rho,r,\delta'),\\ \Lambda &= \Lambda(\rho,r',m') = \cup_{j=1}^{3}\Lambda_j(\rho,r',m'),\end{aligned}$$

where,

$$\begin{aligned}\Gamma_1(\rho,r,\delta') &= \{se^{i(\pi/2+\delta')};\ s\in[r,\rho]\},\\ \Gamma_2(\rho,r,\delta') &= \{re^{i\nu};\ -\pi/2-\delta'\le\nu\le\pi/2+\delta'\},\\ \Gamma_3(\rho,r,\delta') &= \{se^{-i(\pi/2+\delta')};\ s\in[r,\rho]\},\\ \Lambda_1(\rho,r',m') &= \{se^{i(\pi/2+m')};\ s\in[r',\rho]\},\\ \Lambda_2(\rho,r',m') &= \{r'e^{i\nu};\ -\pi/2-m'\le\nu\le\pi/2+m'\},\\ \Lambda_3(\rho,r',m') &= \{se^{i(\pi/2+m')};\ s\in[r',\rho]\},\\ R_\rho &= \{\rho e^{i\eta}:\eta\in(\pi/2+m',\pi/2+\delta')\},\\ S_\rho &= \{\rho e^{i\eta}:\eta\in(-\pi/2+\delta',-\pi/2+m')\}.\end{aligned}$$

The boundary of D_ρ is oriented counterclockwise, as in Figure 1.2.

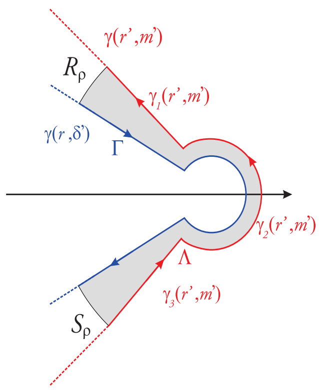

Figure 1.2:

By analyticity of the function $\lambda\mapsto e^{t\lambda}R(\lambda,A)$ in $\Sigma_{\delta'}$ and by Cauchy's Theorem, we have

$$\int_{\partial D_\rho} e^{\lambda t}R(\lambda,A)\,d\lambda = 0,$$

that is,

$$\int_\Gamma e^{\lambda t}R(\lambda,A)\,d\lambda + \int_\Lambda e^{\lambda t}R(\lambda,A)\,d\lambda + \int_{R_\rho} e^{\lambda t}R(\lambda,A)\,d\lambda + \int_{S_\rho} e^{\lambda t}R(\lambda,A)\,d\lambda = 0. \tag{1.8.239}$$

Moreover, the integrals over the two arcs R_ρ, S_ρ tend to zero as $\rho\to\infty$. Indeed, if $\lambda\in R_\rho$ then $\lambda=\rho e^{i\eta}$ with $-K_0\le\cos\eta=-K$, where K_0 and K are positive constants. In this case,

$$\left\|\int_{R_\rho} e^{\lambda t}R(\lambda,A)\,d\lambda\right\| \le M\int_{\pi/2+m'}^{\pi/2+\delta'} e^{-\rho K}\,d\eta = K_1e^{-\rho K}\to 0 \quad \text{as } \rho\to\infty,$$

with $K_1 = M(\delta'-m')$. For $\lambda\in S_\rho$, the estimate is analogous. Therefore, passing to the limit in (1.8.239) as $\rho\to\infty$, we conclude that

$$\int_{\gamma(r',m')} e^{t\lambda}R(\lambda,A)\,d\lambda = \lim_{\rho\to\infty}\int_\Lambda e^{\lambda t}R(\lambda,A)\,d\lambda = -\lim_{\rho\to\infty}\int_\Gamma e^{t\lambda}R(\lambda,A)\,d\lambda = \int_{\gamma(r,\delta')} e^{t\lambda}R(\lambda,A)\,d\lambda,$$

which proves the claim. □

Proposition 1.65 *Assume that A satisfies (1.8.230) and (1.8.231). If $\{S(t)\}_{t\geq 0}$ is the family of operators defined in (1.8.232), then the following properties hold:*

(i) The operator $S(t)$, $t>0$, is linear and continuous on X. There exists $C>0$ such that $\|S(t)\|\leq C$, for all $t\geq 0$.

(ii) $S(0)=I$.

(iii) $S(t+s)=S(t)S(s)$, for all $t,s\geq 0$.

(iv) For each $x\in X$, $S(t)x\to x$ as $t\to 0_+$.

Proof:

(i) Linearity follows from the linearity of $R(\lambda;A)$ and of the integral operator. Continuity follows directly from (1.8.236), (1.8.237) and (1.8.238), and from the fact that for $t>0$,

$$\|S(t)x\|\leq \frac{1}{2\pi}\sum_{i=1}^{3}\left\|\int_{\Gamma_i} e^{t\lambda}R(\lambda,A)x\,d\lambda\right\|\leq \tilde{C}\|x\|,$$

where $\tilde{C}=\tilde{C}(t)$.

We now show uniform boundedness. If $t>0$, then, from (1.8.232),

$$S(t)=\frac{1}{2\pi i}\int_{\gamma(r,\delta)} e^{\lambda t}R(\lambda,A)\,d\lambda. \tag{1.8.240}$$

Performing the change of variables $\xi=\lambda t$ and using Lemma 1.64, we obtain

$$S(t)=\frac{1}{2\pi i}\int_{\gamma(r,\delta')} e^{\xi}R(\xi/t,A)\frac{d\xi}{t}.$$

Let $\xi\in\gamma_1(r,\delta')$ and set $\eta=\arg\xi$. Defining $f(\xi)=e^{\xi}R(\xi/t;A)\frac{1}{t}$ and taking

$$\begin{aligned} x&=\rho\cos\eta=\varphi(\rho)\Rightarrow\varphi'(\rho)=\cos\eta,\\ y&=\rho\sin\eta=\phi(\rho)\Rightarrow\phi'(\rho)=\sin\eta,\end{aligned}$$

we have

$$\begin{aligned}\left\|\int_{\gamma_1(r,\delta')} e^{\xi}R(\xi/t,A)\frac{d\xi}{t}\right\| &\leq \int_r^{\infty}\|f(\varphi(\rho)+i\phi(\rho))[\varphi'(\rho)+i\phi'(\rho)]\|\,d\rho\\ &\leq \int_r^{\infty}\|e^{\rho e^{i\eta}}R(\rho e^{i\eta}/t,A)e^{i\eta}/t\|\,d\rho\\ &\leq \int_r^{\infty}|e^{\rho e^{i\eta}}|\frac{Mt}{|\rho e^{i\eta}|}\frac{|e^{i\eta}|}{t}\,d\rho=M\int_r^{\infty}e^{\rho\cos\eta}\frac{d\rho}{\rho}.\end{aligned}$$

Since $\frac{\pi}{2}<\eta<\frac{3\pi}{2}$, there exists a constant $C>0$ such that $\cos\eta=-C$. Hence,

$$M\int_r^{\infty}e^{\rho\cos\eta}\frac{d\rho}{\rho}\leq\frac{M}{r}\int_r^{\infty}e^{-C\rho}\,d\rho=\frac{M}{rC}e^{-rC}.$$

Thus,

$$\left\| \int_{\gamma_1(r,\delta')} e^{\xi} R(\xi/t, A) \frac{d\xi}{t} \right\| \leq M \int_r^{\infty} e^{\rho \cos \eta} \frac{d\rho}{\rho} \leq \frac{M}{rC} e^{-rC} := C_2. \tag{1.8.241}$$

Note that C_2 does not depend on t.

Similarly, if $\xi \in \gamma_3(r, \delta')$, we obtain

$$\left\| \int_{\gamma_3(r,\delta')} e^{\xi} R(\xi/t, A) \frac{d\xi}{t} \right\| \leq C_2. \tag{1.8.242}$$

Now, if $\xi \in \gamma_2(r, \delta')$, then $\xi = re^{i\nu}$ and, using the parametrisation

$$x = r \cos \nu = \varphi(\nu) \Rightarrow \varphi'(\nu) = -r \sin \nu,$$
$$y = r \sin \nu = \phi(\nu) \Rightarrow \phi'(\nu) = r \cos \nu,$$

we have

$$\begin{aligned} \left\| \int_{\gamma_2(r,\delta')} e^{\xi} R(\xi/t, A) \, \frac{d\xi}{t} \right\| &\leq \int_{-\delta' - \frac{\pi}{2}}^{\delta' + \frac{\pi}{2}} \| e^{re^{i\nu}} R(re^{i\nu}/t, A) \, ire^{i\nu} \frac{1}{t} \| \, d\nu \\ &\leq \int_{-\delta' - \frac{\pi}{2}}^{\delta' + \frac{\pi}{2}} |e^{re^{i\nu}}| \frac{M}{|re^{i\nu}|} t \, |ire^{i\nu}| \frac{1}{t} \, d\nu \\ &\leq \int_{-\delta' - \frac{\pi}{2}}^{\delta' + \frac{\pi}{2}} e^{r \cos \nu} \, d\nu \leq C_3, \end{aligned} \tag{1.8.243}$$

where C_3 is independent of t.

Thus, from (1.8.241), (1.8.242) and (1.8.243), we obtain

$$\|S(t)\| \leq \frac{1}{2\pi} \sum_{i=1}^{3} \left\| \int_{\gamma_i} e^{\xi} R(\xi/t, A) \frac{d\xi}{t} \right\| \leq C_4, \quad \text{for all } t \geq 0.$$

(ii) This follows immediately from the definition.

(iii) Let $t_1, t_2 > 0$, and let $\gamma(r, \delta)$ and $\gamma(r+c, \delta')$, $c > 0$, with $\delta' < \delta$ and $\frac{\pi}{2} < \delta', \delta < \pi$, be piecewise C^1 curves defined as in (1.8.233). For $\mu \in \gamma(r, \delta)$ and $\lambda \in \gamma(r + c, \delta')$, define

$$f(\mu) = \frac{e^{\mu t_1}}{\lambda - \mu} \quad \text{and} \quad g(\lambda) = e^{\lambda t_2}.$$

Consider the regions Ξ and Θ, bounded respectively by $\overline{\Gamma} \cup \Lambda_{\rho,\delta}$ and $\Upsilon \cup \Lambda_{\rho,\delta'}$, oriented positively, where

$$\overline{\Gamma} = \overline{\Gamma}(r, \delta) = \cup_{l=1}^{3} \overline{\Gamma_l}(r, \delta),$$

$$\overline{\Gamma_1}(r, \delta) = \{se^{i(\pi/2+\delta)}; s \in [r, \rho]\},$$
$$\overline{\Gamma_2}(r, \delta) = \{re^{i\nu}; -\pi/2 - \delta \leq \nu \leq \pi/2 + \delta\},$$
$$\overline{\Gamma_3}(r, \delta) = \{-se^{-i(\pi/2+\delta)}; s \in [-\rho, -r]\},$$
$$\Lambda_{\rho,\delta} = \{\rho e^{i\eta}; \eta \in (\pi/2 + \delta, 3\pi/2 - \delta)\};$$

as in Figure 1.3, and

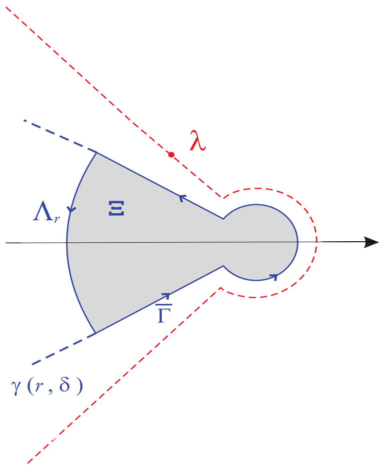

Figure 1.3:

$$\Upsilon = \Upsilon(r+c,\delta') = \cup_{l=1}^{3}\Upsilon_l(r+c,\delta')$$

$$\begin{aligned}
&\Upsilon_1(r+c,\delta') = \{se^{i(\pi/2+\delta')}; s\in[r+c,\rho]\},\\
&\Upsilon_2(r+c,\delta') = \{\rho e^{i\nu}; -\pi/2-\delta' \le \nu \le \pi/2+\delta'\},\\
&\Upsilon_3(r+c,\delta') = \{-se^{-i(\pi/2+\delta')}; s\in[-\rho,-(r+c)]\},\\
&\Lambda_{\rho,\delta'} = \{\rho e^{i\theta}; \theta\in(\pi/2+\delta', 3\pi/2-\delta')\};
\end{aligned}$$

as in Figure 1.4.

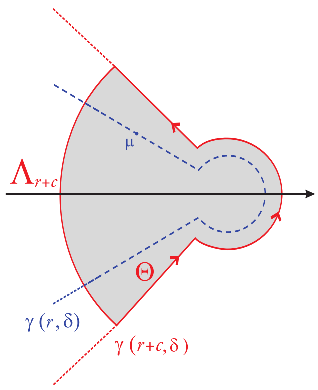

Figure 1.4:

Observe that $f(\cdot)$ is analytic in Ξ and $g(\cdot)$ is analytic in Θ. Thus, by Cauchy's theorem,

$$\int_{\partial\Xi} f(\mu)\,d\mu = 0,$$

that is,

$$\int_{\overline{\Gamma}} f(\mu)\,d\mu + \int_{\Lambda_{\rho,\delta}} f(\mu)\,d\mu = 0. \tag{1.8.244}$$

Since $|\mu-\lambda| \geq |\mu| - |\lambda| = \rho - |\lambda| > 0$ for ρ sufficiently large, we obtain

$$\frac{\rho}{\rho-|\lambda|} = \frac{1}{1-\frac{|\lambda|}{\rho}} \to 1, \quad \text{as } \rho \to \infty.$$

Hence there exists $M > 0$ such that $\frac{\rho}{\rho-|\lambda|} \leq M$ for ρ sufficiently large, and therefore

$$\begin{aligned} \left\| \int_{\Lambda_{\rho,\delta}} f(\mu)\, d\mu \right\| &\leq \int_{\pi/2+\delta}^{3\pi/2-\delta} \frac{|e^{t_1 \rho e^{i\eta}}|}{\rho - |\lambda|} |\rho i|\, d\eta \\ &\leq M \int_{\pi/2+\delta}^{3\pi/2-\delta} e^{\rho t_1 \cos\eta}\, d\eta \\ &\leq Me^{-\rho t_1 k}(2(\pi/2+\delta)) \to 0, \quad \text{as } \rho \to \infty, \end{aligned}$$

where $k = -k_0$, $k_0 < 0$ is such that $k_0 = \max \cos\eta$, $\eta \in [\frac{\pi}{2}+\delta, \frac{3\pi}{2}-\delta]$.

Hence

$$\int_{\Lambda_{\rho,\delta}} f(\mu)\, d\mu \to 0, \quad \text{as } \rho \to \infty. \tag{1.8.245}$$

From (1.8.232) (with $\delta' = \delta$), (1.8.244) and (1.8.245) we deduce that

$$\frac{1}{2\pi i} \int_{\gamma(r,\delta)} \frac{e^{\mu t_1}}{\lambda - \mu}\, d\mu = 0, \quad \text{for all } \lambda \in \gamma(r+c, \delta'). \tag{1.8.246}$$

Moreover, by Cauchy's integral formula, we obtain

$$\int_{\Upsilon} \frac{g(\lambda)}{\lambda-\mu}\, d\lambda + \int_{\Lambda_{\rho,\delta'}} \frac{g(\lambda)}{\lambda-\mu}\, d\lambda = 2\pi i e^{\mu t_2}. \tag{1.8.247}$$

Using the same reasoning as for (1.8.245), we see that

$$\int_{\Lambda_{\rho,\delta'}} \frac{g(\lambda)}{\lambda-\mu}\, d\lambda \to 0, \quad \text{as } \rho \to \infty.$$

Thus

$$\frac{1}{2\pi i} \int_{\gamma(r+c,\delta')} \frac{e^{\lambda t_2}}{\lambda-\mu}\, d\lambda = e^{\mu t_2}, \quad \text{for all } \mu \in \gamma(r,\delta). \tag{1.8.248}$$

On the other hand,

$$S(t_1)S(t_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma(r,\delta)} e^{\mu t_1} R(\mu, A) \int_{\gamma(r+c,\delta')} e^{\lambda t_2} R(\lambda, A)\, d\lambda\, d\mu.$$

Using the identity

$$R(\mu, A)R(\lambda, A) = \frac{R(\mu, A) - R(\lambda, A)}{\lambda - \mu},$$

it follows that

$$
\begin{aligned}
S(t_1)S(t_2) &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma(r,\delta)} \int_{\gamma(r+c,\delta')} e^{\mu t_1} e^{\lambda t_2} \frac{R(\mu, A) - R(\lambda, A)}{\lambda - \mu} \, d\lambda \, d\mu \\
&= \frac{1}{2\pi i} \int_{\gamma(r,\delta)} e^{\mu t_1} R(\mu, A) \left(\frac{1}{2\pi i} \int_{\gamma(r+c,\delta')} \frac{e^{\lambda t_2}}{\lambda - \mu} \, d\lambda \right) d\mu \\
&- \frac{1}{2\pi i} \int_{\gamma(r+c,\delta')} e^{\lambda t_2} R(\lambda, A) \left(\frac{1}{2\pi i} \int_{\gamma(r,\delta)} \frac{e^{\mu t_1}}{\lambda - \mu} \, d\mu \right) d\lambda.
\end{aligned}
$$

Therefore, from (1.8.246), (1.8.248) and the last identity we obtain

$$
S(t_1)S(t_2) = \frac{1}{2\pi i} \int_{\gamma(r,\delta)} e^{\mu(t_1+t_2)} R(\mu, A) \, d\mu = S(t_1 + t_2),
$$

which proves item (iii).

(iv) Let $x \in D(A)$. From the representation of $S(t)$ we can write

$$
S(t)x - x = \frac{1}{2\pi i} \int_{\gamma(r,\delta)} e^{t\lambda} R(\lambda, A)x \, d\lambda - x.
$$

Now consider $\Upsilon = \overline{\Gamma} \cup \Lambda_{\delta,r}$, where $\overline{\Gamma} = \cup_{i=1}^{3} \overline{\Gamma_i}$ and C_r is the circle of radius $\frac{r}{2}$ centred at the origin, as in Figure 1.5.

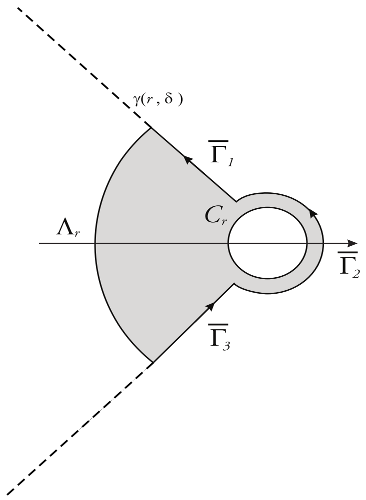

Figure 1.5:

By Cauchy's theorem,

$$
\int_{\Upsilon} \frac{e^{\lambda t}}{\lambda} \, d\lambda = \int_{C_r} \frac{e^{\lambda t}}{\lambda - 0} \, d\lambda = 2\pi i e^{0t},
$$

where the last identity holds by Cauchy's integral formula. Thus

$$
\int_{\Upsilon} \frac{e^{\lambda t}}{\lambda} \, d\lambda = 2\pi i,
$$

and therefore, using an argument analogous to the previous item, we conclude that

$$\frac{1}{2\pi i}\int_{\gamma(r,\delta)}\frac{e^{\lambda t}}{\lambda}\,d\lambda = 1.$$

Using the identity $R(\lambda, A)Ax = \lambda R(\lambda, A)x - x$ for all $x \in D(A)$ (see (1.4.101)), we obtain

$$\begin{aligned} S(t)x - x &= \frac{1}{2\pi i}\int_{\gamma(r,\delta)} e^{t\lambda}\left(R(\lambda, A) - \frac{1}{\lambda}\right)x\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma(r,\delta)}\frac{e^{t\lambda}}{\lambda}R(\lambda, A)Ax\,d\lambda. \end{aligned} \tag{1.8.249}$$

We now show that for each $x \in X$ we have $S(t)x \to x$ as $t \to 0^+$.

Let $x \in D(A)$. Then, from (1.8.249), we have

$$S(t)x - x = \frac{1}{2\pi i}\int_{\gamma(r,\delta)}\frac{e^{t\lambda}}{\lambda}R(\lambda, A)Ax\,d\lambda.$$

We estimate the integral above. Let

$$f_t(\lambda) = \frac{e^{t\lambda}}{\lambda}R(\lambda, A)Ax$$

be a net of functions. We wish to apply the Lebesgue Dominated Convergence Theorem (see [92, p. 1015]) to $\{f_t(\lambda)\}_{t>0}$. To this end, we verify the hypotheses of that theorem.

Note that

$$\|f_t(\lambda)\| = \left\|\frac{e^{t\lambda}}{\lambda}R(\lambda, A)Ax\right\| \leq \frac{|e^{t\lambda}|}{|\lambda|}\|R(\lambda, A)\|\,\|Ax\| \leq \frac{e^{t\,\mathrm{Re}(\lambda)}}{|\lambda|^2}M\|Ax\|. \tag{1.8.250}$$

Claim. There exists $\bar{\delta} > 0$ such that $e^{t\mathrm{Re}(\lambda)} \leq e^t + 1$, for all $t \in (0, \bar{\delta})$.

Indeed, write $\mathrm{Re}(\lambda) = a \in \mathbb{R}$ $(t \geq 0)$ and consider two cases:

(i) If $a \leq 1$, then $ta \leq t$ and thus $e^{ta} \leq e^t \leq e^t + 1$.

(ii) If $a > 1$, define $f(t) = (1 + e^t) - e^{at}$. From real analysis we know that if $f(x_0) > 0$ and f is continuous, there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. For this f, taking $x_0 = 0$, there exists $\bar{\delta} > 0$ such that $f(t) > 0$ for all $t \in (0, \bar{\delta})$. That is, for all $t \in (0, \bar{\delta})$, we have $e^{ta} \leq e^t + 1$.

Therefore,

$$\frac{e^{t\,\mathrm{Re}(\lambda)}}{|\lambda|^2}M\|Ax\| \leq \frac{e^t + 1}{|\lambda|^2}M\|Ax\| < \frac{e^{\bar{\delta}} + 1}{|\lambda|^2}M\|Ax\| = \frac{K}{|\lambda|^2} := g(\lambda), \tag{1.8.251}$$

where the constant K is given by $(e^{\bar{\delta}} + 1)M\|Ax\|$. From (1.8.250) and (1.8.251) we see that the net $\{f_t(\lambda)\}_{t>0}$ is dominated (for each $t > 0$) by the function $g(\lambda)$ defined above. We now show that $g(\lambda)$ is integrable over $\gamma(r,\delta)$. Indeed,

$$\begin{aligned}
\int_{\gamma(r,\delta)} \frac{K}{|\lambda|^2}\,d\lambda &= K\left(\int_{\gamma_1(r,\delta)} \frac{1}{|\lambda|^2}\,d\lambda + \int_{\gamma_2(r,\delta)} \frac{1}{|\lambda|^2}\,d\lambda + \int_{\gamma_3(r,\delta)} \frac{1}{|\lambda|^2}\,d\lambda\right)\\
&= K\left(e^{i\left(\frac{\pi}{2}+\delta\right)} \lim_{\rho\to+\infty}\int_r^{\rho} \frac{1}{s^2}\,ds + \frac{i}{r}\int_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} e^{i\nu}\,d\nu - ie^{-i\left(\frac{\pi}{2}+\delta\right)} \lim_{\rho\to+\infty}\int_{-\rho}^{-r} \frac{1}{s^2}\,ds\right)\\
&= K\left(e^{i\left(\frac{\pi}{2}+\delta\right)} \lim_{\rho\to+\infty}\left(-\frac{1}{\rho}+\frac{1}{r}\right) + \frac{1}{r}\left(e^{i\left(\frac{\pi}{2}+\delta\right)} - e^{i\left(-\frac{\pi}{2}-\delta\right)}\right) - ie^{-i\left(\frac{\pi}{2}+\delta\right)} \lim_{\rho\to+\infty}\left(\frac{1}{r}-\frac{1}{\rho}\right)\right)\\
&= K\left[\left(\frac{1}{r}e^{i\left(\frac{\pi}{2}+\delta\right)}\right) + \left(\frac{i}{r}e^{i\left(\frac{\pi}{2}+\delta\right)}\right) - \left(\frac{i}{r}e^{i\left(-\frac{\pi}{2}-\delta\right)}\right) - \left(\frac{i}{r}e^{-i\left(\frac{\pi}{2}+\delta\right)}\right)\right],
\end{aligned}$$

which is finite.

Moreover,

$$\lim_{t\to0^+} f_t(\lambda) = \lim_{t\to0^+} \frac{e^{t\lambda}}{\lambda}R(\lambda,A)Ax = \frac{1}{\lambda}R(\lambda,A)Ax.$$

Hence, all the hypotheses of the Lebesgue Dominated Convergence Theorem are satisfied and we obtain

$$\begin{aligned}
\lim_{t\to0^+}(S(t)x-x) &= \frac{1}{2\pi i}\lim_{t\to0^+}\int_{\gamma(r,\delta)} \frac{e^{t\lambda}}{\lambda}R(\lambda,A)Ax\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\gamma(r,\delta)} \lim_{t\to0^+}\frac{e^{t\lambda}}{\lambda}R(\lambda,A)Ax\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\gamma(r,\delta)} \frac{1}{\lambda}R(\lambda,A)Ax\,d\lambda. \qquad (1.8.252)
\end{aligned}$$

It remains to show that the integral in (1.8.252) is zero. In order to use the Cauchy–Goursat Theorem, consider the closed curve $\bar\Gamma\cup\bar\Lambda_{\rho,\delta}$ where $\bar\Gamma=\cup_{i=1}^{3}\bar\Gamma_i$ and

$$\bar\Lambda_{\rho,\delta} = \left\{\rho e^{-i\theta}\,;\,-\frac{\pi}{2}-\delta\le\theta\le\frac{\pi}{2}+\delta\right\},$$

as in the figure below.

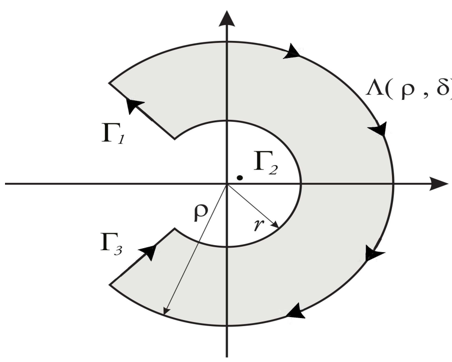

Figure 1.6:

The function $\frac{1}{\lambda}R(\lambda,A)Ax$ is analytic in the whole region R bounded by $\bar\Gamma\cup\bar\Lambda_{\rho,\delta}$. Thus, by the Cauchy–Goursat Theorem,

$$\int_{\bar\Gamma\cup\bar\Lambda_{\rho,\delta}} \frac{1}{\lambda}R(\lambda,A)Ax\,d\lambda = 0,$$

that is,

$$\int_{\bar{\Gamma}} \frac{1}{\lambda} R(\lambda, A) Ax \, d\lambda + \int_{\bar{\Lambda}_{\rho,\delta}} \frac{1}{\lambda} R(\lambda, A) Ax \, d\lambda = 0. \tag{1.8.253}$$

We claim that

$$\int_{\bar{\Lambda}_{\rho,\delta}} \frac{1}{\lambda} R(\lambda, A) Ax \, d\lambda \to 0 \quad \text{as } \rho \to \infty. \tag{1.8.254}$$

Indeed,

$$\begin{aligned}
\left\| \int_{\bar{\Lambda}_{\rho,\delta}} \frac{1}{\lambda} \, R(\lambda, A) \, Ax \, d\lambda \right\| &= \left\| \int_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \frac{1}{\rho e^{-i\theta}} \, R(\rho e^{-i\theta}, A) \, Ax \, (-i\rho e^{-i\theta}) \, d\theta \right\| \\
&\leq \int_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \frac{1}{|\rho e^{-i\theta}|} \, \frac{M}{|\rho e^{-i\theta}|} \, \|Ax\| \, |-i\rho e^{-i\theta}| \, d\theta \\
&\leq \frac{M \, \|Ax\|}{\rho} \int_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} d\theta \\
&= \frac{M \, \|Ax\|}{\rho} (\pi + 2\delta) \\
&= \frac{\tilde{C}}{\rho},
\end{aligned}$$

and the last expression tends to zero as $\rho \to +\infty$, which proves the claim.

From (1.8.252), (1.8.253) and (1.8.254) it follows that

$$\begin{aligned}
0 &= \frac{1}{2\pi i} \left(\lim_{\rho \to +\infty} \left[\int_{\bar{\Gamma}} \frac{1}{\lambda} \, R(\lambda, A) \, Ax \, d\lambda + \int_{\bar{\Lambda}_{\rho,\delta}} \frac{1}{\lambda} \, R(\lambda, A) \, Ax \, d\lambda \right] \right) \\
&= \frac{1}{2\pi i} \int_{\gamma(r,\delta)} \frac{1}{\lambda} \, R(\lambda, A) \, Ax \, d\lambda \\
&= \lim_{t \to 0^+} (S(t)x - x).
\end{aligned}$$

So far we have shown that $\lim_{t\to 0^+}(S(t)x - x) = 0$ for every $x \in D(A)$. To conclude the proof, we must extend this to all $x \in X$.

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$ with $t_n \to 0$ as $n \to +\infty$. Then, by what we have just proved,

$$S(t_n)x \to S(0)x = x, \quad \forall\, x \in D(A),$$

that is, given $\varepsilon > 0$, we have

$$\|S(t_n)\, x - x\| < \varepsilon, \quad \forall\, x \in D(A). \tag{1.8.255}$$

Moreover, from item (i) of this proposition, we have

$$\|S(t_n)\| \leq C. \tag{1.8.256}$$

Thus, for the same given $\varepsilon > 0$ and any $y \in X$, since $D(A)$ is dense in X, there exists $y_0 \in D(A)$ such that

$$\|y - y_0\| < \varepsilon. \tag{1.8.257}$$

Hence, from (1.8.255), (1.8.256) and (1.8.257), it follows that

$$\begin{aligned}\|S(t_n)y - y\| &= \|S(t_n)y - S(t_n)y_0 + S(t_n)y_0 - y_0 + y_0 - y\| \\ &\le \|S(t_n)y - S(t_n)y_0\| + \|S(t_n)y_0 - y_0\| + \|y_0 - y\| \\ &\le \|S(t_n)\|\,\|y - y_0\| + \|S(t_n)y_0 - y_0\| + \|y_0 - y\| \\ &< C\,\varepsilon + \varepsilon + \varepsilon = (C+2)\varepsilon,\end{aligned}$$

which shows that $S(t_n)y \to y$ as $n \to \infty$ for all $y \in X$. Since (t_n) was arbitrary with $t_n \to 0$, we conclude that $S(t)x \to x$ as $t \to 0_+$ for every $x \in X$, which completes the proof. □

Lemma 1.66 *Let A be the infinitesimal generator of a C_0–semigroup $T(t)$ satisfying $\|T(t)\| \le Me^{\omega t}$. Let $\gamma > \max(0, \omega)$. If $x \in D(A^2)$, then*

$$T(t)x = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} R(\lambda; A)x\, d\lambda,$$

and for each $\delta > 0$, the integral converges uniformly in t for $t \in [\delta, 1/\delta]$.

Proof: See [83, p. 29]. □

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We now proceed to the proof of Theorem 1.63.

Proof: Define

$$U(t) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma(r,\delta)} e^{\lambda t} R(\lambda; A)\, d\lambda, & \text{if } t > 0, \\ I, & \text{if } t = 0. \end{cases}$$

Let $\lambda \in \mathbb{R}_+$ be such that $\lambda > \omega_0 = \displaystyle\lim_{t\to\infty} \frac{\ln \|U(t)\|}{t}$. By hypothesis, A is a closed, densely defined linear operator on a Banach space X and $\lambda \in \rho(A)$. Hence A satisfies the hypotheses of Proposition 1.34. By Corollary 1.35, we have

$$R(\lambda; A)^{n+1}x = \frac{1}{n!}\int_0^{+\infty} t^n e^{-\lambda t} U(t)x\, dt.$$

Therefore,

$$\begin{aligned}\|R(\lambda; A)^n x\| &= \left\| \frac{1}{(n+1)!}\int_0^{+\infty} t^{n-1} e^{-\lambda t} U(t)x\, dt \right\| \\ &\le C\frac{1}{(n+1)!}\int_0^{+\infty} t^{n-1} e^{-\lambda t}\|x\|\, dt \\ &= \frac{C}{\lambda^n}\|x\| \\ &\le \frac{C}{(\lambda - \omega_0)^n}.\end{aligned}$$

By the Hille–Yosida Theorem, A is the infinitesimal generator of a C_0–semigroup $T(t)$ satisfying $\|T(t)\| \le Ce^{\omega t}$, $t > 0$. Also, $\lambda > \max\{0, \omega_0\}$. If $x \in D(A^2)$, then

$$T(t)x = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} e^{\lambda t} R(\lambda; A)x\, d\lambda.$$

It remains to prove that $T(t) = U(t)$ for every $t \ge 0$. Let $k > r$.

Now consider the path Λ_k given by

$$\Lambda_k = \bigcup_{l=1}^{4} \Lambda_k^l,$$

where

$$\begin{aligned}
\Lambda_k^1 &= \{\alpha : \alpha = \lambda + is,\ -k \le s \le k\},\\
\Lambda_k^2 &= \{\alpha : \alpha = s - ik,\ -k \le s \le \lambda\},\\
\Lambda_k^3 &= \bigcup_{i=1}^{3} \Gamma_i(k,r,\delta),
\end{aligned}$$

with

$$\begin{aligned}
\Gamma_1(k,r,\delta) &= \{-se^{i(\pi/2+\delta)};\ s \in [-k\sqrt{2}, -r]\},\\
\Gamma_2(k,r,\delta) &= \{-re^{i\mu};\ -\tfrac{\pi}{2} - \delta \le \mu \le \tfrac{\pi}{2} + \delta\},\\
\Gamma_3(k,r,\delta) &= \{se^{-i(\pi/2+\delta)};\ s \in [r, k\sqrt{2}]\},\\
\Lambda_k^4 &= \{\alpha : \alpha = s + ik;\ s \in [-\lambda, k]\},
\end{aligned}$$

oriented counter–clockwise, as in Figure 1.7. Note that $0 < \theta < 1$.

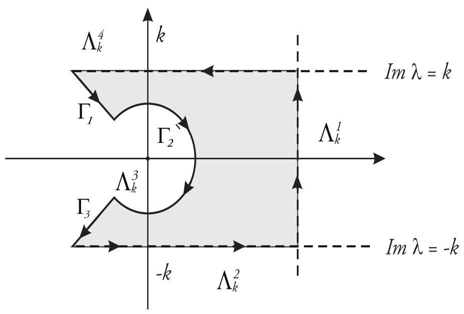

Figure 1.7:

We denote

$$\lim_{k\to\infty} \int_{\Lambda_k^1} e^{\lambda t} R(\lambda, A)x\, d\lambda = \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\lambda t} R(\lambda, A)x\, d\lambda.$$

We have

$$\lim_{k\to\infty} \int_{\Lambda_k^j} e^{\lambda t} R(\lambda, A)\, d\lambda = 0, \quad j = 2, 4.$$

Indeed, we shall treat the case $j = 2$, since the case $j = 4$ is analogous.

Note that

$$\int_{\Lambda_k^2} e^{\lambda t} R(\lambda, A)\, d\lambda = \int_{-k}^{\lambda} e^{(s-ik)t} R(s - ik, A)\, ds.$$

Hence

$$\begin{aligned}\left\|\int_{\Lambda_k^2} e^{\lambda t} R(\lambda, A)\, d\lambda\right\| &\leq \int_{-k}^{\lambda} \|e^{st-ikt} R(s-ik, A)\|\, ds \\ &= \int_{-k}^{\lambda} |\cos(kt) - i\sin(kt)| \frac{C}{|s-ik|}\, ds \\ &\leq \frac{C}{k} \int_{-k}^{\lambda} e^{st}\, ds \\ &= \frac{C}{k} \left[\frac{e^{st}}{t}\right]_{-k}^{\lambda} \\ &= \frac{C}{k} \left[\frac{e^{\lambda t}}{t} - \frac{e^{-kt}}{t}\right] \xrightarrow[k\to+\infty]{} 0.\end{aligned}$$

Moreover, $\int_{\Lambda_k} e^{\lambda t} R(\lambda, A)\, d\lambda = 0$, that is,

$$\sum_{i=1}^{4} \int_{\Lambda_k^i} e^{\lambda t} R(\lambda, A)\, d\lambda = 0,$$

and therefore

$$\lim_{k\to\infty} \sum_{i=1}^{4} \int_{\Lambda_k^i} e^{\lambda t} R(\lambda, A)\, d\lambda = 0,$$

that is,

$$\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t} R(\lambda, A)\, d\lambda - \int_{\Gamma(r,\delta)} e^{\lambda t} R(\lambda, A)\, d\lambda = 0.$$

Thus,

$$T(t)x = \frac{1}{2\pi i} \int_{\gamma(r,\delta)} e^{\lambda t} R(\lambda, A) x\, d\lambda = U(t)x, \tag{1.8.258}$$

for every $x \in D(A^2)$. Since $D(A^2)$ is dense in X, it follows that (1.8.258) holds for all $x \in X$, completing the proof. □

We now turn to the most important result of this section.

Theorem 1.67 *Let $T(t)$ be a uniformly bounded C_0–semigroup and let A be the infinitesimal generator of $T(t)$, assuming that $0 \in \rho(A)$. The following assertions are equivalent:*
(a) $T(t)$ can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z : |\arg(z)| < \delta\}$ and $\|T(t)\|$ is uniformly bounded in any closed subsector $\overline{\Delta}_{\delta'}$, $\delta' < \delta$;
(b) There exists a constant C such that, for every $\sigma > 0$, $\tau \neq 0$,

$$\|R(\sigma + i\tau, A)\| \le \frac{C}{|\tau|};$$

(c) There exist $0 < \delta < \pi/2$ and $M > 0$ such that

$$\rho(A) \supset \Sigma = \left\{\lambda : |\arg \lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\}$$

and

$$\|R(\lambda, A)\| \le \frac{M}{|\lambda|}, \quad \text{for } \lambda \in \Sigma,\ \lambda \neq 0;$$

(d) $T(t)$ is differentiable for $t > 0$ and there exists a constant C such that

$$\|AT(t)\| \le \frac{C}{t}.$$

Proof:

$(a) \Rightarrow (b)$

By hypothesis, there exists $\delta > 0$ such that $T(t)$ can be extended to an analytic semigroup in

$$\Delta_\delta = \{z \in \mathbb{C};\ |\arg(z)| < \delta\}. \tag{1.8.259}$$

Moreover, $\|T(z)\|$ is uniformly bounded in any closed subsector $\overline{\Delta_{\delta'}} \subset \Delta_\delta \cup \{0\}$, $0 < \delta' < \delta$. Fix $0 < \delta' < \delta$. Then there exists $M > 0$ such that

$$\|T(z)\| \le M, \quad \forall z \in \overline{\Delta_{\delta'}} = \{z \in \mathbb{C};\ |\arg(z)| \le \delta'\}. \tag{1.8.260}$$

Observe that, since $T(t)$ is a uniformly bounded C_0–semigroup, we have $w_0 \le 0$. By Proposition 1.34 we obtain, for $\sigma > 0$ and $\tau \in \mathbb{R}$, that $\sigma + i\tau \in \rho(A)$, where A is the infinitesimal generator of T, and, moreover,

$$R(\sigma + i\tau, A)x = \int_0^\infty e^{-(\sigma+i\tau)t}T(t)x\,dt, \quad \forall x \in X. \tag{1.8.261}$$

First assume that $\tau > 0$. For each $R > 0$, define the piecewise C^1 curve C_R by

$$\begin{aligned} C_R = \bigcup_{i=1}^{4} C_{R,i}, \text{ where } C_{R,1} &= \{\rho e^{-i\delta'};\ \rho \in [1/R, R]\}, \\ C_{R,2} &= \{Re^{i\rho};\ \rho \in [-\delta', 0]\}, \\ C_{R,3} &= \{-\rho;\ \rho \in [-R, -1/R]\}, \\ C_{R,4} &= \{\tfrac{1}{R}e^{-i\rho};\ \rho \in [0, \delta']\}, \text{ with } 0 < \delta' < \tfrac{\pi}{2}, \end{aligned} \tag{1.8.262}$$

and oriented as in Figure 1.8.

The mapping

$$z \in \mathbb{C} \mapsto e^{-(\sigma+i\tau)z} \in \mathbb{C} \tag{1.8.263}$$

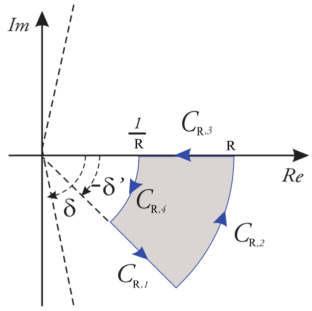

Figure 1.8:

is an analytic (indeed, entire) function, and since T is an analytic semigroup in Δ_δ, it follows that

$$z \in \Delta_\delta \mapsto e^{-(\sigma+i\tau)z}T(z) \in X \tag{1.8.264}$$

is also analytic.

Hence

$$0 = \int_{C_R} e^{-(\sigma+i\tau)z}T(z)\,dz = \sum_{i=1}^{4}\int_{C_{R,i}} e^{-(\sigma+i\tau)z}T(z)\,dz. \tag{1.8.265}$$

For $C_{R,4}$ we have

$$\begin{aligned}
\int_{C_{R,4}} e^{-(\sigma+i\tau)z}T(z)\,dz &= \int_0^{\delta'} e^{-(\sigma+i\tau)\frac{1}{R}e^{-i\rho}}\frac{1}{R}(-i)e^{-i\rho}T\left(\frac{1}{R}e^{-i\rho}\right)d\rho \\
&= \int_0^{\delta'} e^{-\frac{1}{R}\sigma(\cos(-\rho)+i\sin(-\rho))-\frac{i}{R}\tau(\cos(-\rho)+i\sin(-\rho))}\frac{1}{R}ie^{-i\rho}T\left(\frac{1}{R}e^{-i\rho}\right)d\rho \\
&= \int_0^{\delta'} e^{-\frac{1}{R}\sigma(\cos\rho+\tau\sin\rho)-\frac{i}{R}\tau(\cos\rho-i\sigma\sin\rho)}\frac{1}{R}ie^{-i\rho}T\left(\frac{1}{R}e^{-i\rho}\right)d\rho.
\end{aligned}$$

Thus

$$\begin{aligned}
\left\|\int_{C_{R,4}} e^{-(\sigma+i\tau)z}T(z)\,dz\right\| &\le \frac{M}{R}\int_0^{\delta'}\left|e^{-\frac{1}{R}(\sigma\cos\rho+\tau\sin\rho)-\frac{i}{R}(\tau\cos\rho-\sigma\sin\rho)}\right|d\rho \\
&= \frac{M}{R}\int_0^{\delta'} e^{-\frac{1}{R}(\sigma\cos\rho+\tau\sin\rho)}\,d\rho \le \frac{M}{R}\delta',
\end{aligned}$$

since $\sigma, \tau > 0$ and $0 < \rho < \frac{\pi}{2}$ imply $e^{-\frac{1}{R}(\sigma\cos\rho+\tau\sin\rho)} \le 1$. We conclude that

$$\lim_{R\to+\infty}\left\|\int_{C_{R,4}} e^{-(\sigma+i\tau)z}T(z)\,dz\right\| = 0. \tag{1.8.266}$$

Note that

$$\begin{aligned}
\int_{C_{R,2}} e^{-(\sigma+i\tau)z}T(z)\,dz &= \int_{-\delta'}^{0} e^{-(\sigma+i\tau)Re^{i\rho}}T(Re^{i\rho})Re^{i\rho}\,d\rho \\
&= \int_{-\delta'}^{0} e^{-(\sigma+i\tau)R(\cos\rho+i\sin\rho)}T(Re^{i\rho})Re^{i\rho}\,d\rho \\
&= \int_{-\delta'}^{0} e^{-R(\sigma\cos\rho-\tau\sin\rho)}T(Re^{i\rho})Re^{i\rho}\,d\rho.
\end{aligned}$$

Hence

$$\left\|\int_{C_{R,2}} e^{-(\sigma+i\tau)z}T(z)\,dz\right\| \leq RM\int_{-\delta'}^{0} e^{-R\underbrace{(\sigma\cos\rho-\tau\sin\rho)}_{\geq 0}}\,d\rho.$$

The right–hand side tends to zero as $R\to+\infty$, since $\sin\rho\leq 0$. Thus

$$\lim_{R\to+\infty}\left\|\int_{C_{R,2}} e^{-(\sigma+i\tau)z}T(z)\,dz\right\| = 0. \tag{1.8.267}$$

From (1.8.265), (1.8.266) and (1.8.267) we obtain

$$\lim_{R\to+\infty}\left\|\int_{C_{R,1}} e^{-(\sigma+i\tau)z}T(z)\,dz + \int_{C_{R,3}} e^{-(\sigma+i\tau)z}T(z)\,dz\right\| = 0. \tag{1.8.268}$$

We also note that

$$\lim_{R\to+\infty}\int_{C_{R,1}} e^{-(\sigma+i\tau)z}T(z)\,dz$$

exists, because

$$\begin{aligned}\lim_{R\to+\infty}\left|\int_{1/R}^{R} e^{-(\sigma+i\tau)\rho e^{-i\delta'}}T(\rho e^{-i\delta'})e^{-i\delta'}\,d\rho\right| &\leq \lim_{R\to+\infty}\int_{1/R}^{R}\left|e^{-(\sigma+i\tau)\rho e^{-i\delta'}}\right|\,\|T(\rho e^{-i\delta'})\|\,d\rho\\ &\leq M\lim_{R\to+\infty}\int_{1/R}^{R} e^{-\rho(\sigma\cos\delta'+\tau\sin\delta')}\,d\rho\\ &= \frac{M}{\sigma\cos\delta'+\tau\sin\delta'} < +\infty.\end{aligned}$$

Therefore, by (1.8.268), the limit

$$\lim_{R\to+\infty}\int_{C_{R,3}} e^{-(\sigma+i\tau)z}T(z)\,dz$$

also exists. Thus

$$\begin{aligned}\int_0^{\infty} e^{-(\sigma+i\tau)\rho}T(\rho)\,d\rho &= \lim_{R\to+\infty}\int_{1/R}^{R} e^{-(\sigma+i\tau)\rho}T(\rho)\,d\rho\\ &= \lim_{R\to+\infty}\int_{-R}^{-1/R} e^{-(\sigma+i\tau)(-\rho)}T(-\rho)\,d\rho\\ &= -\lim_{R\to+\infty}\int_{C_{R,3}} e^{-(\sigma+i\tau)z}T(z)\,dz\\ &= \lim_{R\to+\infty}\int_{C_{R,1}} e^{-(\sigma+i\tau)z}T(z)\,dz\\ &= \lim_{R\to+\infty}\int_{1/R}^{R} e^{-(\sigma+i\tau)\rho e^{-i\delta'}}T(\rho e^{-i\delta'})e^{-i\delta'}\,d\rho\\ &= \int_0^{\infty} e^{-(\sigma+i\tau)\rho e^{-i\delta'}}T(\rho e^{-i\delta'})e^{-i\delta'}\,d\rho.\end{aligned} \tag{1.8.269}$$

From (1.8.261) and (1.8.269) we conclude that

$$R(\sigma+i\tau,A)x = \int_0^{\infty} e^{-i\delta'}e^{-(\sigma+i\tau)\rho e^{-i\delta'}}T(\rho e^{-i\delta'})x\,d\rho, \quad \text{for all } x\in X. \tag{1.8.270}$$

Estimating (1.8.270) we obtain

$$\begin{aligned}\|R(\sigma+i\tau,A)x\| &\le \int_0^\infty |e^{-i\delta'}|\left|e^{-(\sigma+i\tau)\rho e^{-i\delta'}}\right| \|T(\rho e^{-i\delta'})\|\,\|x\|\,d\rho \\ &\le M\int_0^\infty e^{-\rho(\sigma\cos\delta'+\tau\sin\delta')}\|x\|\,d\rho \\ &\le \frac{M}{\sigma\cos\delta'+\tau\sin\delta'}\|x\| \\ &\le \frac{M}{\tau\sin\delta'}\|x\|, \quad \forall x\in X,\end{aligned} \tag{1.8.271}$$

whence

$$\|R(\sigma+i\tau,A)\| \le \frac{C}{\tau}, \quad \text{with } C=\frac{M}{\sin\delta'}>0. \tag{1.8.272}$$

Now assume $\tau<0$. For each $R>0$, consider the piecewise C^1 curve γ_R in Δ_δ given by

$$\gamma_R=\bigcup_{i=1}^4 \gamma_{R,i}, \quad \text{where} \quad \begin{aligned}\gamma_{R,1}&=\{-\rho e^{i\delta'};\ \rho\in[-R,-1/R]\},\\ \gamma_{R,2}&=\{Re^{i\rho};\ \rho\in[0,\delta']\},\\ \gamma_{R,3}&=\{\rho;\ \rho\in[1/R,R]\},\\ \gamma_{R,4}&=\{1/Re^{-i\rho};\ \rho\in[-\delta',0]\},\end{aligned} \tag{1.8.273}$$

as shown in Figure 1.9.

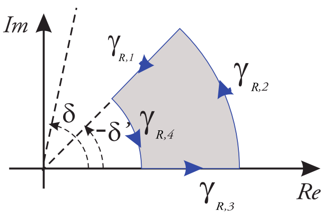

Figure 1.9:

Again, by Cauchy's Theorem,

$$\int_{\gamma_R} e^{-(\sigma+i\tau)z}T(z)\,dz=0. \tag{1.8.274}$$

Proceeding as in the previous case ($\tau>0$) we deduce that

$$\lim_{R\to+\infty}\left\|\int_{\gamma_{R,2}} e^{-(\sigma+i\tau)z}T(z)\,dz\right\|=0,$$
$$\lim_{R\to+\infty}\left\|\int_{\gamma_{R,4}} e^{-(\sigma+i\tau)z}T(z)\,dz\right\|=0.$$

Hence

$$\begin{aligned}R(\sigma+i\tau,A)x &= \lim_{R\to\infty}\int_{\gamma_{R,1}} e^{-(\sigma+i\tau)z}T(z)x\,dz \\ &= \int_0^\infty e^{i\delta'}e^{-(\sigma+i\tau)\rho e^{i\delta'}}T(\rho e^{i\delta'})x\,d\rho, \quad \forall x\in X.\end{aligned} \tag{1.8.275}$$

Estimating (1.8.275), we obtain

$$\begin{aligned}\|R(\sigma+i\tau,A)x\| &\le M\|x\|\int_0^\infty e^{-\rho(\sigma\cos\delta'-\tau\sin\delta')}\,d\rho \\ &\le \frac{M\,\|x\|}{\sigma\cos\delta'-\tau\sin\delta'} \le \frac{-M}{\tau\sin\delta'}\|x\|, \quad \forall x\in X,\end{aligned} \tag{1.8.276}$$

whence

$$\|R(\sigma+i\tau,A)\| \le \frac{C}{-\tau}, \quad C=\frac{M}{\sin\delta'}>0. \tag{1.8.277}$$

From (1.8.272) and (1.8.277) we conclude that there exists $C>0$ such that

$$\|R(\sigma+i\tau,A)\| \le \frac{C}{|\tau|}, \quad \forall \sigma>0 \text{ and } \tau\ne 0, \tag{1.8.278}$$

which proves (b).

(b) $\Rightarrow$ *(c)*

Since A is the infinitesimal generator of a uniformly bounded C_0–semigroup T, by Proposition 1.34 we have

$$\{\lambda\in\mathbb{C};\ \Re\lambda>0\}\subset\rho(A), \tag{1.8.279}$$

and, moreover, by Exercise (1.6.3), for every $\lambda\in\mathbb{C}$ with $\Re\lambda>0$ we obtain

$$\|R(\lambda,A)\| \le \frac{M}{\Re\lambda}, \quad \text{where} \quad \|T(t)\|\le M,\ \forall t\ge 0. \tag{1.8.280}$$

From (b) we infer

$$\|R(\lambda,A)\| \le \frac{C}{|\Im\lambda|}, \quad \forall\lambda\in\mathbb{C} \text{ with } \Re\lambda>0 \text{ and } \Im\lambda\ne 0. \tag{1.8.281}$$

We claim that there exists $C_1>0$ such that

$$\|R(\lambda,A)\| < \frac{C_1}{|\lambda|}, \quad \forall\lambda \text{ with } \Re\lambda>0. \tag{1.8.282}$$

Indeed, let λ be such that $\Re\lambda>0$. There are two cases:

If $|\Im\lambda|\ge\Re\lambda$, then

$$|\lambda|^2=(\Re\lambda)^2+(\Im\lambda)^2\le 2(\Im\lambda)^2,$$

whence

$$|\lambda|\le\sqrt{2}\,|\Im\lambda| \quad \Longleftrightarrow \quad \frac{1}{|\Im\lambda|}\le\frac{\sqrt{2}}{|\lambda|}. \tag{1.8.283}$$

From (1.8.281) and (1.8.283) we deduce

$$\|R(\lambda,A)\| \le \frac{\sqrt{2}C}{|\lambda|}, \tag{1.8.284}$$

and (1.8.282) follows by taking $C_1>\sqrt{2}C$.

If $|\Im\lambda| < \Re\lambda$, then

$$|\lambda|^2 = (\Re\lambda)^2 + (\Im\lambda)^2 < 2(\Re\lambda)^2,$$

which implies, similarly,

$$|\lambda| < \sqrt{2}\,\Re\lambda \quad \iff \quad \frac{1}{\Re\lambda} < \frac{\sqrt{2}}{|\lambda|}. \tag{1.8.285}$$

From (1.8.280) and (1.8.285) we obtain

$$\|R(\lambda, A)\| < \frac{M\sqrt{2}}{|\lambda|}, \tag{1.8.286}$$

which also yields (1.8.282).

Fix $\sigma + i\tau \in \rho(A)$ with $\sigma > 0$ and $\tau \neq 0$. By Corollary 1.35, we have the following Taylor expansion of $R(\lambda, A)$ around $\sigma + i\tau$:

$$\begin{aligned} R(\lambda, A) &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{d\lambda^n} R(\sigma + i\tau, A)(\lambda - (\sigma + i\tau))^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} R(\sigma + i\tau, A)^{n+1} (\lambda - (\sigma + i\tau))^n \\ &= \sum_{n=0}^{\infty} R(\sigma + i\tau, A)^{n+1} (\sigma + i\tau - \lambda)^n. \end{aligned} \tag{1.8.287}$$

Let $0 < k < 1$. Then, for $\lambda \in \rho(A)$ such that

$$\frac{C}{|\tau|}\,|\sigma + i\tau - \lambda| \leq k, \tag{1.8.288}$$

the series in (1.8.287) converges absolutely and, since $\tau \neq 0$, we have

$$|\sigma + i\tau - \lambda| \leq \frac{k|\tau|}{C}. \tag{1.8.289}$$

We claim that

$$\rho(A) \supset A_1 = \{\lambda \in \mathbb{C};\ \Im\lambda \neq 0 \text{ and } 0 \leq |\Re\lambda| < \tfrac{|\Im\lambda|}{C}\} \cup \{0\}. \tag{1.8.290}$$

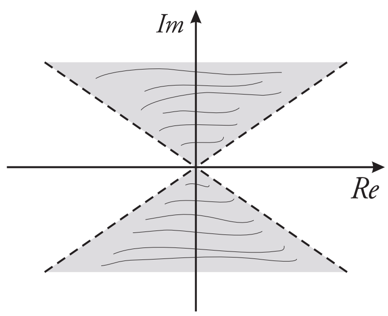

Figure 1.10:

Indeed, let $\lambda \in \mathbb{C}$ be such that $\Im\lambda \neq 0$ and $|\Re\lambda| < \frac{|\Im\lambda|}{C}$ (see Figure 1.10). Then

$$|\Re\lambda| < k\frac{|\Im\lambda|}{C}, \tag{1.8.291}$$

for some $0 < k < 1$ (note that this k depends on λ). In fact, since $0 \leq |\Re\lambda| < \frac{|\Im\lambda|}{C}$, we have $0 < \frac{|\Im\lambda|}{C} - |\Re\lambda|$.

Hence there exists $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{|\Im\lambda|}{C} - |\Re\lambda| \leq \frac{|\Im\lambda|}{C}, \tag{1.8.292}$$

and therefore

$$|\Re\lambda| < \frac{|\Im\lambda|}{C} - \varepsilon = \frac{|\Im\lambda|}{C}\Big(1 - \varepsilon\frac{C}{|\Im\lambda|}\Big). \tag{1.8.293}$$

Setting $k = 1 - \varepsilon\frac{C}{|\Im\lambda|}$, we obtain (1.8.291).

It then follows that there exists $\sigma > 0$, also depending on λ, such that

$$|\Re\lambda| + \sigma < \frac{k|\Im\lambda|}{C}. \tag{1.8.294}$$

Consider $\sigma + i\,\Im\lambda \in \mathbb{C}$. Since $\sigma > 0$ and $\Im\lambda \neq 0$, we have $\sigma + i\,\Im\lambda \in \rho(A)$ and

$$|\sigma + i\,\Im\lambda - \lambda| = |\Re\lambda - \sigma| \leq |\Re\lambda| + \sigma < k\frac{|\Im\lambda|}{C}. \tag{1.8.295}$$

Therefore, the series in (1.8.287) converges and hence $\lambda \in \rho(A)$, proving (1.8.290).

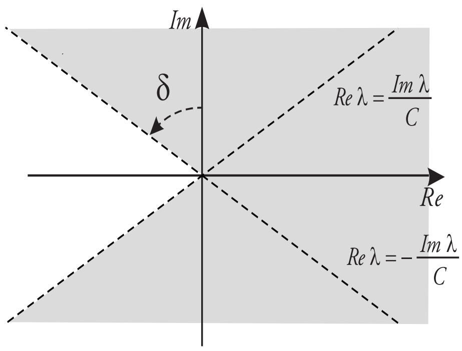

Figure 1.11:

Thus

$$\begin{aligned}\|R(\lambda, A)\| &\leq \sum_{n=0}^{\infty} \|R(\sigma + i\,\Im\lambda, A)\|^{n+1}|\sigma + i\,\Im\lambda - \lambda|^n \\ &< \|R(\sigma + i\,\Im\lambda, A)\|\frac{1}{1-k}.\end{aligned} \tag{1.8.296}$$

Since, for the complex number $\sigma + i\,\Im\lambda \in \rho(A)$, inequality (1.8.281) holds, (1.8.296) becomes

$$\|R(\lambda, A)\| < \frac{C}{|\Im\lambda|(1-k)}. \tag{1.8.297}$$

Moreover, for λ in the region $|\Re\lambda| < \frac{|\Im\lambda|}{C}$, we have

$$\frac{C}{|\Im\lambda|} < \frac{(C^2+1)^{1/2}}{|\lambda|}. \tag{1.8.298}$$

On the other hand, putting $\delta = \arctan(\frac{1}{C})$ and defining $A_2 = \{\lambda \in \mathbb{C};\ |\arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}$ as in Figure 1.11, we see that if $\lambda \in A_2 \setminus A_1$, then (1.8.282) yields

$$\|R(\lambda, A)\| < \frac{M}{|\lambda|}, \tag{1.8.299}$$

where $M = \max\left\{\frac{(C^2+1)^{1/2}}{(1-k)}, C_1\right\} > 0$, which proves (c).

(*c*) ⇒ (*d*)

From the hypotheses in (c) and Theorem 1.63 we have

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda, A)\, d\lambda, \quad \forall t > 0, \tag{1.8.300}$$

where $\Gamma = \{\rho e^{i(\theta+\frac{\pi}{2})};\ 0 < \rho < +\infty\} \cup \{-\rho e^{-i(\theta+\frac{\pi}{2})};\ -\infty < \rho < 0\} \subset \Sigma$, with $0 < \theta < \delta$ (see Figure 1.12).

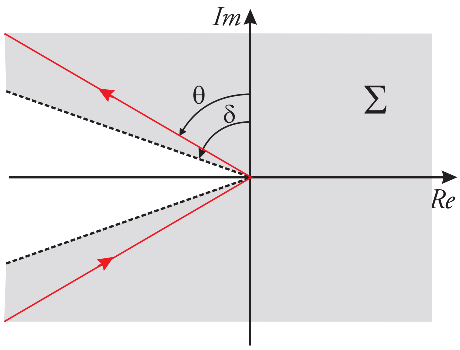

Figure 1.12:

Let $0 < r < +\infty$ be fixed but arbitrary. Define

$$T_r(t) = \frac{1}{2\pi i}\int_{\{\rho e^{i(\theta+\frac{\pi}{2})};\ 0<\rho<r\}\cup\{-\rho e^{-i(\theta+\frac{\pi}{2})};\ -r<\rho<0\}} R(\lambda, A)e^{\lambda t}\, d\lambda, \quad t > 0. \tag{1.8.301}$$

Then $T_r(t)$ is differentiable and

$$T_r'(t) = \frac{1}{2\pi i}\int_{\{\rho e^{i(\theta+\frac{\pi}{2})};\ 0<\rho<r\}\cup\{-\rho e^{-i(\theta+\frac{\pi}{2})};\ -r<\rho<0\}} \lambda R(\lambda, A)e^{\lambda t}\, d\lambda, \quad t > 0. \tag{1.8.302}$$

Furthermore,

$$\|T_r'(t)\| \le \frac{M}{2\pi}\int_{\{\rho e^{i(\theta+\frac{\pi}{2})};\ 0<\rho<r\}\cup\{-\rho e^{-i(\theta+\frac{\pi}{2})};\ -r<\rho<0\}} |e^{\lambda t}|\, d\lambda. \tag{1.8.303}$$

Computing the integral above on each of the sets and using that the cosine function is even, we obtain

$$\begin{aligned}\int_{\{\rho e^{i(\theta+\frac{\pi}{2})};\ 0<\rho<r\}\cup\{-\rho e^{-i(\theta+\frac{\pi}{2})};\ -r<\rho<0\}} |e^{\lambda t}|\,d\lambda &= 2\int_0^r e^{\rho t\cos(\frac{\pi}{2}+\theta)}\,d\rho \\ &= \frac{2}{t\cos(\frac{\pi}{2}+\theta)}\big(e^{rt\cos(\frac{\pi}{2}+\theta)}-1\big). \end{aligned} \tag{1.8.304}$$

From (1.8.303) and (1.8.304) we conclude that

$$\|T_r'(t)\| \le \frac{M}{\pi t\cos(\frac{\pi}{2}+\theta)}\big(e^{rt\cos(\frac{\pi}{2}+\theta)}-1\big). \tag{1.8.305}$$

Thus the integral $\int_\Gamma \lambda e^{\lambda t}R(\lambda,A)\,d\lambda$ converges uniformly, for every $t>0$. Hence (1.8.300) is differentiable for all $t>0$ and

$$T'(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}R(\lambda,A)\,d\lambda, \quad \forall t>0. \tag{1.8.306}$$

From (1.8.305) we obtain

$$\|AT(t)\| = \|T'(t)\| \le \frac{C}{t}, \quad \forall t>0, \tag{1.8.307}$$

with $C=\frac{M}{\pi\cos\delta}>0$, which proves (d).

$(d) \Rightarrow (a)$

Since T is differentiable, it follows from Exercise 1.7.1 that

$$T^{(n)}(t) = \big(T'(t/n)\big)^n, \quad \forall n\in\mathbb{N}^*,\ \forall t>0. \tag{1.8.308}$$

Hence, by the hypothesis in (d) and by (1.8.308), we have

$$\|T^{(n)}(t)\| \le \|T'(t/n)\|^n \le \left(\frac{C}{t}\right)^n, \quad \forall t>0. \tag{1.8.309}$$

We claim that

$$e^n n! \ge n^n, \quad \forall n\in\mathbb{N}^*. \tag{1.8.310}$$

Indeed, we prove this by induction on n. Clearly, for $n=1$, (1.8.310) holds because $e>1$. Assume now that (1.8.310) is valid for some $n>1$, that is,

$$n!e^n \ge n^n. \tag{1.8.311}$$

We prove that (1.8.311) is valid for $n+1$. Equivalently, we shall prove that $(n+1)+\ln((n+1)!) \ge$

$(n+1)\ln(n+1)$. In fact,

$$\begin{aligned}
(n+1)+\ln((n+1)!) &= n+1+\ln((n+1)n!) \\
&= n+1+\ln(n+1)+\ln n! \\
&= (n+\ln n!)+1+\ln(n+1) \\
&\geq n\ln n+1+\ln(n+1) \\
&= \ln(n+1)+1+n\int_1^n \frac{1}{x}\,dx \\
&\geq \ln(n+1)+n\int_1^n \frac{1}{x}\,dx+n\int_n^{n+1}\frac{1}{x}\,dx \\
&= \ln(n+1)+n\int_1^{n+1}\frac{1}{n+1}\frac{1}{x}\,dx \\
&= \ln(n+1)+n(\ln(n+1)-\ln(1)) \\
&= (n+1)\ln(n+1),
\end{aligned}$$

which proves the claim.

From (1.8.309) and (1.8.310) we conclude that

$$\frac{1}{n!}\|T^{(n)}(t)\| \leq \left(\frac{Ce}{t}\right)^n. \tag{1.8.312}$$

Now consider the power series

$$T(z)=T(t)+\sum_{n=1}^{\infty}\frac{T^{(n)}(t)}{n!}(z-t)^n, \tag{1.8.313}$$

with $t>0$ and $z\in\mathbb{C}$. Proceeding formally, from (1.8.312) we infer

$$\left\|\frac{T^{(n)}(t)}{n!}(z-t)^n\right\| \leq \frac{\|T^{(n)}(t)\|}{n!}|z-t|^n \leq \left(\frac{Ce|z-t|}{t}\right)^n. \tag{1.8.314}$$

Thus, from (1.8.314) we conclude that the series in (1.8.313) converges uniformly in $\mathcal{L}(X)$ for every $0<k<1$ and all $z\in\mathbb{C}$ such that

$$\frac{Ce|z-t|}{t}<k \quad\Longleftrightarrow\quad |z-t|<\frac{tk}{Ce}. \tag{1.8.315}$$

Now, setting $\delta=\arctan(1/Ce)$ we obtain $0<\delta<\pi/2$, and defining

$$\Delta=\{z\in\mathbb{C};\ |\arg(z)|<\delta\}, \tag{1.8.316}$$

we see that the series in (1.8.313) converges uniformly in Δ (see Figure 1.13).

Indeed, let $z\in\Delta$. Then

$$\begin{aligned}
|\arg(z)|<\arctan\left(\frac{1}{Ce}\right) &\Longleftrightarrow |\tan(\arg z)|<\frac{1}{Ce} \\
&\Longleftrightarrow \frac{|\Im z|}{\Re z}<\frac{1}{Ce} \\
&\Longleftrightarrow |\Im z|<\frac{\Re z}{Ce}.
\end{aligned} \tag{1.8.317}$$

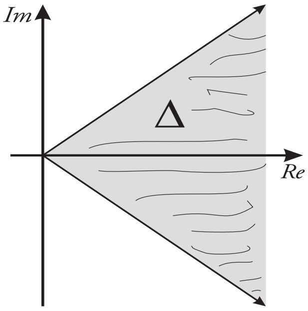

Figure 1.13:

From (1.8.317) it follows that there exists $k \in (0,1)$ such that

$$|\Im z| < k\frac{\Re z}{Ce}. \tag{1.8.318}$$

Choosing $t = \Re z > 0$, we deduce

$$|z - t| = |\Im z| < k\frac{\Re z}{Ce} = \frac{kt}{Ce}, \tag{1.8.319}$$

which proves the assertion. Moreover, if $0 < \delta' < \delta$, then

$$\overline{\Delta}_{\delta'} = \{z \in \mathbb{C};\ |\arg z| \leq \delta'\} \subset \Delta, \tag{1.8.320}$$

and hence there exists $0 < k_0 < 1$ such that

$$\delta' < \arctan\Big(\frac{k_0}{Ce}\Big) < \delta, \tag{1.8.321}$$

and, therefore, for $z \in \overline{\Delta}_{\delta'}$, we infer

$$|z - t| \leq \frac{tk_0}{Ce} \quad \Longleftrightarrow \quad \frac{Ce|z-t|}{t} < k_0, \tag{1.8.322}$$

with $t = \Re z > 0$. Thus, by (1.8.314) and (1.8.322),

$$\begin{aligned} \|T(z)\| &\leq \sum_{n=0}^{\infty} \Big\| \frac{T^{(n)}(t)}{n!}(z-t)^n \Big\| \\ &\leq \sum_{n=0}^{\infty} \Big(\frac{Ce|z-t|}{t} \Big)^n \\ &\leq \sum_{n=0}^{\infty} k_0^n = \frac{1}{1-k_0}, \quad \forall z \in \overline{\Delta}_{\delta'}, \end{aligned}$$

that is, $\{T(z)\}_{z\in\Delta}$ is uniformly bounded in every closed subsector of Δ, which completes the proof. □

1.8.1 Exercises

1.8.1) Let $T(t)$ be a C_0–semigroup which is differentiable for $t > 0$, and let A be the infinitesimal generator of $T(t)$. If

$$\limsup_{t\to 0} t\|AT(t)\| < \frac{1}{e},$$

prove that A is a bounded operator and that $T(t)$ can be extended analytically to the whole complex plane.

1.8.2) Let A be the infinitesimal generator of a C_0–semigroup $T(t)$ satisfying $\|T(t)\| \leq Me^{\omega t}$. Prove that $T(t)$ is analytic if and only if there exist constants $C > 0$ and $\Gamma > 0$ such that

$$\|AR(\lambda, A)^{n+1}\| \leq \frac{C}{n\lambda^n} \quad \text{for } \lambda > n\Gamma,\ n = 1, 2, \ldots$$

1.9 Spectral Properties

Let $T(t)$ be a C_0–semigroup on a Banach space X and let A be its infinitesimal generator. In what follows we are interested in the relationship between the spectrum of A, $\sigma(A) = \mathbb{C}\backslash\rho(A)$, and the spectrum of each operator $T(t)$, $t \geq 0$. From a purely formal point of view, one might expect the relation $\sigma(T(t)) = e^{t\sigma(A)}$. However, this is not true in general. There are counterexamples (see Exercise 1.9.1, and also Pazy [83], p. 44) which justify this assertion.

Proposition 1.68 *Let $T(t)$ be a C_0–semigroup and A its infinitesimal generator. Define*

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)x\,ds.$$

Then

$$(i) \quad (\lambda I - A)B_\lambda(t)x = e^{\lambda t}x - T(t)x, \quad \textit{for all } x \in X.$$

$$(ii) \quad B_\lambda(t)(\lambda I - A)x = e^{\lambda t}x - T(t)x, \quad \textit{for all } x \in D(A).$$

Proof: First observe that $B_\lambda(t)$, for each fixed λ and t, is a bounded linear operator on X. Linearity is immediate, so it remains to show boundedness. In fact, we have

$$\begin{aligned}\|B_\lambda(t)\|_{\mathcal{L}(X)} &= \sup_{\substack{x\in X\\ \|x\|=1}} \left\|\int_0^t e^{\lambda(t-s)}T(s)x\,ds\right\| \\ &\leq \sup_{\substack{x\in X\\ \|x\|=1}} \int_0^t |e^{\lambda(t-s)}|\,\|T(s)\|\,ds \leq M,\end{aligned}$$

where $M = M(\lambda, t)$, which proves the claim. Moreover, for every $x \in X$ we have

$$\begin{aligned}\left(\frac{T(h)-I}{h}\right)B_\lambda(t)x &= \frac{1}{h}\int_0^t e^{\lambda(t-s)}T(h+s)x\,ds \\ &\quad -\frac{1}{h}\int_0^t e^{\lambda(t-s)}T(s)x\,ds.\end{aligned} \tag{1.9.323}$$

From the first integral on the right-hand side of (1.9.323), we obtain

$$\begin{aligned}\frac{1}{h}\int_0^t e^{\lambda(t-s)}T(h+s)x\,ds &\underbrace{=}_{\tilde{s}=h+s} \frac{1}{h}\int_h^{h+t} e^{\lambda(t-\tilde{s}+h)}T(\tilde{s})x\,d\tilde{s} \\ = \frac{e^{\lambda h}}{h}\int_h^{h+t} e^{\lambda(t-\tilde{s})}T(\tilde{s})x\,d\tilde{s}.\end{aligned} \tag{1.9.324}$$

Thus, from (1.9.323) and (1.9.324) we deduce

$$\begin{aligned}\left(\frac{T(h)-I}{h}\right)B_\lambda(t)x &= \left(\frac{e^{\lambda h}-1}{h}\right)\int_h^{t+h} e^{\lambda(t-s)}T(s)x\,ds\\ &+ \frac{1}{h}\int_t^{t+h} e^{\lambda(t-s)}T(s)x\,ds - \frac{1}{h}\int_0^h e^{\lambda(t-s)}T(s)x\,ds.\end{aligned}$$

Taking the limit as $h \to 0^+$ in the last equality, and using the Mean Value Theorem (see Exercise 1.1.5 (v)), we obtain

$$\lim_{h\to 0_+}\left(\frac{T(h)-I}{h}\right)B_\lambda(t)x = \lambda B_\lambda(t)x + T(t)x - e^{\lambda t}x. \tag{1.9.325}$$

From (1.9.325) it follows that $B_\lambda(t)x \in D(A)$ and that

$$AB_\lambda(t)x = \lambda B_\lambda(t)x + T(t)x - e^{\lambda t}x,$$

or equivalently,

$$(\lambda I - A)B_\lambda(t)x = \big(e^{\lambda t} - T(t)\big)x, \quad \text{for all } x \in X, \tag{1.9.326}$$

which proves item (i).

Now let $x \in D(A)$. Then $\lim_{h\to 0}\left(\frac{T(h)-I}{h}\right)x$ exists and $\lim_{h\to 0}\left(\frac{T(h)-I}{h}\right)x = Ax$. Proceeding with $B_\lambda(t)\left(\frac{T(h)-I}{h}\right)x$ in a way analogous to the previous argument, we obtain

$$B_\lambda(t)Ax = \lambda B_\lambda(t)x + T(t)x - e^{\lambda t}x,$$

or equivalently,

$$B_\lambda(t)(\lambda I - A)x = e^{\lambda t}x - T(t)x.$$

This establishes (ii).

From what we have seen above, we also obtain

$$B_\lambda(t)Ax = AB_\lambda(t)x, \quad \text{for all } x \in D(A), \tag{1.9.327}$$

that is, the operators B_λ and A commute on $D(A)$. □

Proposition 1.69 *Let $T(t)$ be a C_0–semigroup and A its infinitesimal generator. Then*

$$\sigma(T(t)) \supset e^{t\sigma(A)}, \quad \textit{for } t \geq 0.$$

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Proof: Let $t \geq 0$. If $t = 0$, then

$$e^{t\sigma(A)} = \{\beta \in \mathbb{C};\ \beta = e^{t\lambda};\ \lambda \in \sigma(A) \text{ and } t = 0\} = \{1\}.$$

Moreover, note that 1 is an eigenvalue of $T(0) = I$, hence $1 \in \sigma(T(t))$, and therefore $e^{t\sigma(A)} \subset \sigma(T(t))$.

If $t \neq 0$, we have two cases to consider: $\rho(T(t)) \neq \emptyset$ or $\rho(T(t)) = \emptyset$. If $\rho(T(t)) = \emptyset$, then $\sigma(T(t)) = \mathbb{C}$ and the inclusion $e^{t\sigma(A)} \subset \sigma(T(t))$ is trivial. Now assume that $t \neq 0$ and $\rho(T(t)) \neq \emptyset$. Then there exists $\beta \in \rho(T(t))$, and we write β in the form $\beta = e^{\lambda t}$. Let $e^{\lambda t} \in \rho(T(t))$ and set $Q = (e^{\lambda t}I - T(t))^{-1}$. First note that the operator $B_\lambda(t)$ defined in Proposition 1.68 and the operator Q defined above commute (we leave this verification to the reader; see Exercise 1.9.2). It follows from Proposition 1.68 that

$$(\lambda I - A)B_\lambda(t)Qx = x, \quad \text{for all } x \in X, \tag{1.9.328}$$

and

$$QB_\lambda(t)(\lambda I - A)x = x, \quad \text{for all } x \in D(A). \tag{1.9.329}$$

Since $B_\lambda(t)$ and Q commute, we have

$$B_\lambda(t)Q(\lambda I - A)x = (\lambda I - A)B_\lambda(t)Qx = x, \quad \text{for all } x \in D(A). \tag{1.9.330}$$

Therefore, as $B_\lambda(t)Q \in \mathcal{L}(X)$ and $B_\lambda(t)Q = (\lambda I - A)^{-1}$, we conclude that $\lambda \in \rho(A)$ and hence $e^{\lambda t} \in e^{\rho(A)t}$. Thus $\rho(T(t)) \subset e^{\rho(A)t}$, and consequently

$$e^{\sigma(A)t} \subset \sigma(T(t)), \quad \text{for all } t \geq 0,$$

which completes the proof. □

Recall that the spectrum of A consists of three mutually disjoint parts: the point (or discrete) spectrum $\sigma_p(A)$, the continuous spectrum $\sigma_c(A)$, and the residual spectrum $\sigma_r(A)$, which are defined as follows: $\lambda \in \sigma_p(A)$ if $\lambda I - A$ is not injective; $\lambda \in \sigma_c(A)$ if $\lambda I - A$ is injective, $\lambda I - A$ is not surjective and its range is dense in X; and finally $\lambda \in \sigma_r(A)$ if $(\lambda I - A)$ is injective but its range is not dense in X. From these definitions it is clear that $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$ are mutually disjoint and their union is $\sigma(A)$. In summary:

$$\begin{aligned}
\sigma_p(A) &= \{\lambda \in \mathbb{C};\ (\lambda I - A) \text{ is not injective}\},\\
\sigma_c(A) &= \{\lambda \in \mathbb{C};\ (\lambda I - A) \text{ is injective, not surjective, but its range is dense}\},\\
\sigma_r(A) &= \{\lambda \in \mathbb{C};\ (\lambda I - A) \text{ is injective but its range is not dense}\}.
\end{aligned}$$

Theorem 1.70 *Let $T(t)$ be a C_0–semigroup and A its infinitesimal generator. Then*

$$e^{t\sigma_p(A)} \subset \sigma_p(T(t)) \subset e^{t\sigma_p(A)} \cup \{0\},$$

or more precisely, if $\lambda \in \sigma_p(A)$, then $e^{\lambda t} \in \sigma_p(T(t))$; and if $e^{\lambda t} \in \sigma_p(T(t))$, there exists $k \in \mathbb{N}$ such that $\lambda_k = \lambda + 2\pi i k/t \in \sigma_p(A)$.

Proof: We first show that

$$e^{t\sigma_p(A)} \subset \sigma_p(T(t)). \tag{1.9.331}$$

Indeed, if $\lambda \in \sigma_p(A)$, then there exists $x_0 \in D(A)$, $x_0 \neq 0$, such that $(\lambda I - A)x_0 = 0$. From the identity

$$B_\lambda(t)(\lambda I - A)x = \big(e^{\lambda t} I - T(t)\big)x, \quad \text{for all } x \in D(A) \text{ (see Proposition 1.69)},$$

we obtain

$$\big(e^{\lambda t} I - T(t)\big)x_0 = 0,$$

and hence $e^{\lambda t} \in \sigma_p(T(t))$, which proves (1.9.331).

We now prove that

$$\sigma_p(T(t)) \subset e^{t\sigma_p(A)} \cup \{0\}. \tag{1.9.332}$$

Indeed, let $\beta \in \sigma_p(T(t))$. If $\beta = 0$, the inclusion is trivial. If $\beta \neq 0$, we may write β as $\beta = e^{\lambda t} \in \sigma_p(T(t))$ and choose $x_0 \neq 0$ such that $\big(e^{\lambda t} I - T(t)\big)x_0 = 0$. We claim that the continuous function f defined by $f(s) = e^{-\lambda s}T(s)x_0$ is periodic with period t, i.e., $f(s+t) = f(s)$. Indeed, first observe that

$$\big(e^{\lambda t} I - T(t)\big)x_0 = 0 \Longleftrightarrow T(t)x_0 = e^{\lambda t}x_0.$$

Then

$$\begin{aligned} f(s+t) &= e^{-\lambda(s+t)}T(s+t)x_0 = e^{-\lambda s}e^{-\lambda t}T(s)T(t)x_0 \\ &= e^{-\lambda s}T(s)e^{-\lambda t}T(t)x_0 \\ &= e^{-\lambda s}T(s)Ix_0 = e^{-\lambda s}T(s)x_0 = f(s), \end{aligned}$$

which proves the claim. Since this function is not identically zero, at least one of its Fourier coefficients must be nonzero. Hence there exists $k \in \mathbb{N}$ such that

$$x_k = \int_0^t e^{-(2k\pi i/t)s}(e^{-\lambda s}T(s)x_0)\, ds \neq 0. \tag{1.9.333}$$

We shall show that $\lambda_k = \lambda + \frac{2\pi i k}{t}$ is an eigenvalue of A. Indeed, since $T(t)$ is a C_0–semigroup, there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t}.$$

By Proposition 1.34, for every $\mu \in \mathbb{C}$ with $\Re\mu > \omega$, we have $\mu \in \rho(A)$ and, moreover,

$$\begin{aligned} R(\mu, A)x_0 &= \int_0^\infty e^{-\mu s}T(s)x_0\, ds = \sum_{n=0}^\infty \int_{nt}^{(n+1)t} e^{-\mu s}T(s)x_0\, ds && (1.9.334)\\ &= \sum_{n=0}^\infty \int_0^t e^{-\mu(r+nt)}T(r+nt)x_0\, dr && (1.9.335)\\ &= \sum_{n=0}^\infty e^{n(\lambda-\mu)t}\int_0^t e^{-\mu r}e^{-\lambda nt}T(r+nt)x_0\, dr \\ &= \sum_{n=0}^\infty e^{n(\lambda-\mu)t}\int_0^t e^{-\mu r}e^{\lambda r}e^{-\lambda(r+nt)}T(r+nt)x_0\, dr \\ &= \sum_{n=0}^\infty e^{n(\lambda-\mu)t}\int_0^t e^{-\mu r}e^{\lambda r}e^{-\lambda r}T(r)x_0\, dr \\ &= \sum_{n=0}^\infty e^{n(\lambda-\mu)t}\int_0^t e^{-\mu s}T(s)x_0\, ds \\ &= \big(1 - e^{(\lambda-\mu)t}\big)^{-1}\int_0^t e^{-\mu s}T(s)x_0\, ds. \end{aligned}$$

The last integral on the right-hand side of (1.9.334) is an entire function, and therefore $R(\mu, A)x_0$ can be extended to a meromorphic function with poles at $\lambda_n = \lambda + \frac{2\pi i n}{t}$, $n \in \mathbb{N}$ (see [44], pp. 169, 184).

We claim that

$$\lim_{\mu \to \lambda_k} (\mu - \lambda_k)R(\mu, A)x_0 = x_k. \tag{1.9.336}$$

Indeed, from (1.9.334) we can write

$$(\mu-\lambda_k)R(\mu,A)x_0=(\mu-\lambda_k)\big(1-e^{(\lambda-\mu)t}\big)^{-1}\int_0^t e^{-\mu s}T(s)x_0\,ds.$$

Passing to the limit as $\mu\to\lambda_k$ in the last identity and using L'Hôpital's Rule, we obtain

$$\lim_{\mu\to\lambda_k}(\mu-\lambda_k)R(\mu,A)x_0=\frac{1}{t}\int_0^t e^{-\lambda_k s}T(s)x_0\,ds,$$

and, recalling that $\lambda_k=\lambda+\frac{2\pi ik}{t}$, we have

$$\lim_{\mu\to\lambda_k}(\mu-\lambda_k)R(\mu,A)x_0=\frac{1}{t}\int_0^t e^{-(2\pi ik/t)s}(e^{-\lambda s}T(s)x_0)\,ds=x_k,$$

which proves (1.9.336).

From the proof of Proposition 1.34 we know that, for every $x\in D(A)$, $AR(\mu,A)x=R(\mu,A)Ax=\mu R(\mu,A)x-x$. Thus

$$(\lambda_kI-A)\big[(\mu-\lambda_k)R(\mu,A)x_0\big]=\lambda_k(\mu-\lambda_k)R(\mu,A)x_0-\mu(\mu-\lambda_k)R(\mu,A)x_0+(\mu-\lambda_k)x_0.$$

Letting $\mu\to\lambda_k$ and using (1.9.336), we obtain

$$\lim_{\mu\to\lambda_k}(\lambda_kI-A)\big[(\mu-\lambda_k)R(\mu,A)x_0\big]=\lambda_kx_k-\lambda_kx_k+0=0.$$

Since A is closed and

$$\begin{aligned}\{(\mu-\lambda_k)R(\mu,A)x_0\}&\subset D(A),\\ \lim_{\mu\to\lambda_k}(\mu-\lambda_k)R(\mu,A)x_0&=x_k,\\ \lim_{\mu\to\lambda_k}A(\mu-\lambda_k)R(\mu,A)x_0&=\lambda_kx_k,\end{aligned}$$

it follows that $x_k\in D(A)$ and $Ax_k=\lambda_kx_k$. Hence λ_k is an eigenvalue of A, that is, $\lambda_k\in\sigma_p(A)$, and therefore $e^{\lambda_kt}\in e^{t\sigma_p(A)}$. But

$$e^{\lambda t}=e^{\lambda_kt}e^{2\pi ki}=e^{\lambda_kt}.$$

Thus $e^{\lambda t}\in e^{t\sigma_p(A)}$, which proves the result. □

1.9.1 Exercises

1.9.1) Prove that, in general, the relation $\sigma(T(t))=e^{t\sigma(A)}$ does not hold by constructing a counterexample.

1.9.2) Prove that the operators B_λ and $Q=(e^{\lambda t}I-T(t))^{-1}$ given in Propositions 1.68 and 1.69 commute.

1.9.3) Let $T(t)$ be a C_0–semigroup and A its infinitesimal generator. Prove that:

(i) If $\lambda\in\sigma_r(A)$ and none of the $\lambda_n=\lambda+2\pi in/t$, $n\in\mathbb{N}$, belongs to $\sigma_p(A)$, then $e^{\lambda t}\in\sigma_r(T(t))$.

(ii) If $e^{\lambda t}\in\sigma_r(T(t))$, then none of the $\lambda_n=\lambda+2\pi in/t$, $n\in\mathbb{N}$, belongs to $\sigma_p(A)$ and there exists $k\in\mathbb{N}$ such that $\lambda_k\in\sigma_r(A)$.

1.9.4) Let $T(t)$ be a C_0–semigroup and A its infinitesimal generator. If $\lambda\in\sigma_c(A)$ and none of the

$\lambda_n = \lambda + 2\pi i n/t$ belongs to $\sigma_p(A) \cup \sigma_r(A)$, prove that $e^{\lambda t} \in \sigma_c(T(t))$.

Chapter 2

The Abstract Cauchy Problem

2.1 The Homogeneous Problem

Let $(X, \|\cdot\|)$ be a Banach space, $A : D(A) \subset X \to X$ a linear operator on X, and for each $u_0 \in X$ consider the *Abstract Cauchy Problem*:

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{2.1.1}$$

Definition 2.1 *A function $u : \mathbb{R}_+ \to X$ is called*

(a) a classical solution (or strong solution) of (2.1.1) if:

i) u is continuous for all $t \geq 0$;

ii) u is continuously differentiable for $t > 0$;

iii) $u(t) \in D(A)$ for all $t > 0$;

iv) u satisfies (2.1.1).

(b) a mild solution (or generalised solution) of (2.1.1) if:

i) u is continuous for all $t \geq 0$;

ii) $\int_0^t u(s)\,ds \in D(A)$ for all $t \geq 0$;

iii) $u(t) = A\int_0^t u(s)\,ds + u_0$.

The second condition appearing in (2.1.1) is called the *initial condition* of the problem, and u_0 its initial value. Note that, since $u(t) \in D(A)$ for all $t > 0$ and u is continuous at $t = 0$, problem (2.1.1) cannot admit a classical solution if $u_0 \notin \overline{D(A)}$.

Lemma 2.2 *Let A be a closed operator. For each $x \in D(A)$, set $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$. Then $\|\cdot\|_{D(A)}$ is a norm on $D(A)$ and $\big(D(A), \|\cdot\|_{D(A)}\big)$ is a Banach space. The norm $\|\cdot\|_{D(A)}$ is called the graph norm.*

Proof: The verification that $\|\cdot\|_{D(A)}$ is a norm on $D(A)$ is left to the reader. We shall prove that $\big(D(A), \|\cdot\|_{D(A)}\big)$ is a Banach space. To this end, let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\big(D(A), \|\cdot\|_{D(A)}\big)$, where $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$. Then

$$\|x_n - x_m\|_{D(A)} \to 0 \quad \text{as } m, n \to +\infty.$$

Hence

$$\|x_n - x_m\|_{D(A)} = \|x_n - x_m\|_X + \|Ax_n - Ax_m\|_X \to 0 \quad \text{as } m,n \to +\infty.$$

Thus $\|x_n - x_m\|_X \to 0$ and $\|Ax_n - Ax_m\|_X \to 0$ as $m,n \to +\infty$. Since X is Banach, there exist $x, y \in X$ such that $x_n \to x$ and $Ax_n \to y$. However, as $(x_n, Ax_n) \in G(A)$ and $G(A)$ is closed, we get $(x,y) \in G(A)$, that is, $y = Ax$. Therefore $x_n \to x$ in $\big(D(A), \|\cdot\|_{D(A)}\big)$. □

Theorem 2.3 *Let A be the infinitesimal generator of a C_0–semigroup S. Then:*

(a) for each $u_0 \in D(A)$ there exists a unique function

$$u \in C^0([0,+\infty); D(A)) \cap C^1([0,+\infty); X),$$

called the regular solution of the Cauchy problem in (2.1.1). Moreover, if S is a contraction semigroup, then

$$\|u(t)\| \le \|u_0\| \quad \textit{and} \quad \left\|\frac{du}{dt}(t)\right\| = \|Au(t)\| \le \|Au_0\| \quad \forall t \ge 0.$$

(b) If $u_0 \in X$, there exists a unique mild solution of the Cauchy problem in (2.1.1).

Proof: (a) Let $u_0 \in D(A)$ be given and set

$$u(t) = S(t)u_0, \quad t \ge 0. \tag{2.1.2}$$

By Proposition 1.30 we have $u(t) = S(t)u_0 \in D(A)$ for all $t \ge 0$. Moreover,

$$\frac{du}{dt}(t) = AS(t)u_0 = S(t)Au_0, \quad \forall t \ge 0. \tag{2.1.3}$$

From (2.1.2) we obtain, in particular,

$$u(0) = S(0)u_0 = u_0. \tag{2.1.4}$$

From (2.1.3) and (2.1.4) we conclude that the map u defined in (2.1.2) indeed satisfies (2.1.1). Now, since S is a strongly continuous semigroup, if $t_0 \in [0,+\infty)$ and $t_n \to t_0$ in $[0,+\infty)$, then by (2.1.2) and (2.1.3) we have

$$\|u(t_n) - u(t_0)\| \to 0 \quad \text{as } n \to +\infty, \tag{2.1.5}$$

and

$$\|Au(t_n) - Au(t_0)\| \to 0 \quad \text{as } n \to +\infty, \tag{2.1.6}$$

that is,

$$\|u(t_n) - u(t_0)\|_{D(A)} = \|u(t_n) - u(t_0)\|_X + \|Au(t_n) - Au(t_0)\|_X \to 0$$

as $n \to +\infty$, which shows that

$$u \in C^0\big([0,+\infty); D(A)\big).$$

Also, by (2.1.3), (2.1.5) and (2.1.6), we obtain

$$u \in C^0\big([0,+\infty); X\big) \quad \text{and} \quad \frac{du}{dt} \in C^0\big([0,+\infty); X\big),$$

that is,

$$u \in C^0\big([0,+\infty); D(A)\big) \cap C^1\big([0,+\infty); X\big),$$

which proves that the map u defined in (2.1.2) is indeed a strong solution of (2.1.1).

We now prove uniqueness. Let u and v be two solutions of (2.1.1). Then $w = u - v$ satisfies the

Cauchy problem

$$\begin{cases} \dfrac{dw}{dt}(t) = Aw(t), \quad t > 0, \\ w(0) = 0. \end{cases} \tag{2.1.7}$$

Let $t > 0$ and $0 \le s < t < +\infty$. If $|h| < t - s$, then

$$\begin{aligned} \frac{d}{ds}[S(t-s)w(s)] = \lim_{h\to 0} \frac{S(t-s-h)w(s+h) - S(t-s)w(s)}{h} = \\ = \lim_{h\to 0} \left\{ \frac{S(t-s-h)w(s+h) - S(t-s-h)w(s) + S(t-s-h)w(s) - S(t-s)w(s)}{h} \right\}. \end{aligned} \tag{2.1.8}$$

We claim that

$$\lim_{h\to 0} S(t-s-h)\left(\frac{w(s+h)-w(s)}{h}\right) = S(t-s)w'(s) = S(t-s)Aw(s), \tag{2.1.9}$$

where the last equality follows from the fact that $w(t)$ is a solution of (2.1.7). Indeed,

$$\begin{aligned} &\left\| S(t-s-h)\left(\tfrac{w(s+h)-w(s)}{h}\right) - S(t-s)w'(s) \right\| = \\ &= \left\| S(t-s-h)\left(\tfrac{w(s+h)-w(s)}{h}\right) - S(t-s-h)w'(s) + S(t-s-h)w'(s) - S(t-s)w'(s) \right\| \\ &\le \left\| S(t-s-h)\left(\tfrac{w(s+h)-w(s)}{h} - w'(s)\right) \right\| + \| S(t-s-h)w'(s) - S(t-s)w'(s) \| \to 0 \end{aligned}$$

as $h \to 0$, since on bounded intervals $\|S(t-s-h)\| \le M$,

$$\lim_{h\to 0} \frac{w(s+h)-w(s)}{h} = w'(s),$$

and S is strongly continuous. Hence (2.1.9) is proved.

Next, we prove that

$$\lim_{h\to 0} \left(\frac{S(t-s-h) - S(t-s)}{h} \right) w(s) = -S(t-s)Aw(s). \tag{2.1.10}$$

Indeed, we distinguish two cases: $h < 0$ and $h > 0$.

<u>(i) $h < 0$.</u> In this case $-h > 0$ and thus

$$\begin{aligned} \left(\frac{S(t-s-h) - S(t-s)}{h} \right) w(s) &= \left(\frac{S(t-s)S(-h) - S(t-s)}{h} \right) w(s) \\ &= -S(t-s) \left(\frac{S(-h) - I}{-h} \right) w(s) \to -S(t-s)Aw(s) \end{aligned}$$

as $h \to 0_-$, since $S(t-s) \in \mathcal{L}(X)$ and $\displaystyle\lim_{h\to 0_-} \left(\frac{S(-h)-I}{-h} \right) w(s) = Aw(s)$.

$\underline{(ii)\ h > 0.}$ In this case,

$$\left(\frac{S(t-s-h)-S(t-s)}{h}\right)w(s) = S(t-s-h)\left(\frac{I-S(h)}{h}\right)w(s)$$
$$= -S(t-s-h)\left(\frac{I-S(h)}{-h}\right)w(s)$$
$$= -S(t-s-h)\left(\frac{S(h)-I}{h}\right)w(s)$$
$$= -S(t-s-h)\left[\left(\frac{S(h)-I}{h}\right)w(s) - Aw(s) + Aw(s)\right]$$
$$= -S(t-s-h)\left[\left(\frac{S(h)-I}{h}\right)w(s) - Aw(s)\right] - S(t-s-h)Aw(s),$$

and, since S is strongly continuous, $\|S(t-s-h)\|$ is bounded on bounded intervals, and

$$\lim_{h\to 0_+}\left(\frac{S(h)-I}{h}\right)w(s) = Aw(s),$$

we obtain

$$\lim_{h\to 0_+}\left(\frac{S(t-s-h)-S(t-s)}{h}\right)w(s) = -S(t-s)Aw(s).$$

Thus (2.1.10) is proved.

Therefore, from (2.1.8), (2.1.9) and (2.1.10), for $0 \le s \le t < +\infty$ we obtain

$$\frac{d}{ds}\big[S(t-s)w(s)\big] = S(t-s)Aw(s) - S(t-s)Aw(s) = 0,$$

which implies that

$$S(t-s)w(s) = c(t), \tag{2.1.11}$$

where $c(t)$ is a constant with respect to s. If $s_n \to t$, then $S(t-s_n)w(s_n) \to S(0)w(t)$, hence

$$S(0)w(t) = c(t) \quad \forall t \ge 0,$$

and since $S(0) = I$ we get

$$w(t) = c(t) \quad \forall t \ge 0. \tag{2.1.12}$$

On the other hand, taking $s = 0$ in (2.1.11) we obtain

$$S(t)w(0) = c(t) \quad \forall t \ge 0,$$

and since $w(0) = 0$ it follows that $c(t) = 0$ for all $t \ge 0$. Returning to (2.1.12) we conclude that $w(t) = 0$ for all $t \ge 0$, that is, $u(t) = v(t)$ for all $t \ge 0$, which proves uniqueness.

Finally, if S is a contraction semigroup, then from (2.1.2) and (2.1.3) we have

$$\|u(t)\| = \|S(t)u_0\| \le \|S(t)\|\,\|u_0\| \le \|u_0\| \quad \forall t \ge 0,$$

and

$$\left\|\frac{du}{dt}(t)\right\| = \|AS(t)u_0\| = \|S(t)Au_0\| \le \|S(t)\|\,\|Au_0\| \le \|Au_0\|, \quad \forall t \ge 0,$$

which completes the proof of (a).

(b) By Proposition 1.30, item (iii), if S is a C_0–semigroup with infinitesimal generator A, then, given $x \in X$,

$$\int_0^t S(\xi)x\,d\xi \in D(A) \quad \text{and} \quad A\int_0^t S(\xi)x\,d\xi = S(t)x - x,$$

that is,

$$S(t)x \;=\; A\int_0^t S(\xi)\,x\,d\xi \;+\; x, \quad \text{for } x \in X. \tag{2.1.13}$$

Let $u(t) = S(t)u_0$. From (2.1.13) we have $u(t) = A\int_0^t u(s)\,ds + u_0$, that is, u satisfies condition (iii) in Definition 2.1 (b). Moreover, by item (iii) of Proposition 1.30, we have $\int_0^t u(s)\,ds \in D(A)$, so condition (ii) in the definition of mild solution is satisfied. Finally, we check condition (i), i.e., the continuity of $u(t)$ for all $t \geq 0$.

Let $t_n \to t_0$ in $\mathbb{R}^+$. Then

$$\|u(t_n) - u(t_0)\| = \|S(t_n)u_0 - S(t_0)u_0\| \to 0,$$

as $n \to +\infty$, since S is strongly continuous. Hence $u(t_n) \to u(t_0)$, which implies that u is continuous for all $t \geq 0$, i.e., $u \in C^0([0,+\infty);X)$. Thus $u(t) = S(t)u_0$ is a mild solution of the Cauchy problem in (2.1.1).

To complete the proof, we show uniqueness of this mild solution, in two steps.

Step 1: $u_0 \equiv 0$.

In this case, it is immediate that $u(t) \equiv 0$ is a mild solution of (2.1.1). Suppose that u is a mild solution of (2.1.1) for $u_0 \equiv 0$ and fix $t > 0$. Then, for each $s \in (0,t)$,

$$\frac{d}{ds}\left(S(t-s)\int_0^s u(r)\,dr\right) = S(t-s)u(s) - S(t-s)\,A\int_0^s u(r)\,dr = 0. \tag{2.1.14}$$

Integrating (2.1.14) over $(0,t)$ with respect to s we obtain

$$\int_0^t u(r)\,dr = 0,$$

and applying A to both sides we conclude that $u(t) \equiv 0$ on $(0,t)$ and, since $u(0) = 0$, we have $u \equiv 0$. Hence, if u is a mild solution of (2.1.1) with $u_0 \equiv 0$, then $u \equiv 0$.

Step 2: Let v, w be mild solutions of (2.1.1) and consider $u(t) = v(t) - w(t)$ for $t \in \mathbb{R}^+$. Note that

$$A\int_0^t u(s)\,ds = A\int_0^t v(s)\,ds - A\int_0^t w(s)\,ds = v(t) - w(t) = u(t), \quad \text{for } u_0 \equiv 0,$$

that is, u satisfies condition (iii) in the definition of mild solution. In addition, u is continuous (for all $t \geq 0$) since v and w are, and also $\int_0^t u(s)\,ds \in D(A)$. Thus u is a mild solution of (2.1.1) with $u_0 \equiv 0$. By Step 1 we conclude that $u \equiv 0$, i.e., $v(t) = w(t)$ for all $t \geq 0$. This proves uniqueness of the mild solution and completes the proof of the theorem. □

0.4 cm We now complete Theorem 2.3 by showing that the solution u of (2.1.1) is more regular under additional assumptions on the initial data. Recall that, since A is the infinitesimal generator of a C_0–semigroup, it is a closed operator. For each $k \in \mathbb{N}$, $k \geq 2$, we define

$$D(A^k) = \{v \in D(A^{k-1});\; A^{k-1}v \in D(A)\}, \quad k \in \mathbb{N},\; k \geq 2. \tag{2.1.15}$$

We shall prove that $D(A^k)$ is a Banach space with norm

$$\|u\|_{D(A^k)} = \left(\sum_{j=0}^{k} \|A^j u\|_X^2\right)^{1/2}. \tag{2.1.16}$$

Indeed, let $\{u_\nu\}_{\nu\in\mathbb{N}}$ be a Cauchy sequence in $D(A^k)$ and let $\mu, \nu \in \mathbb{N}$ with $\nu > \mu$. Then, as $\nu, \mu \to +\infty$,

$$\|u_\nu - u_\mu\|_{D(A^k)}^2 = \sum_{j=0}^{k} \|A^j u_\nu - A^j u_\mu\|_X^2 \to 0.$$

From this convergence it follows that for each $j = 0, 1, \dots, k$ the sequence $\{A^j u_\nu\}_{\nu\in\mathbb{N}}$ is Cauchy in X. Hence, for each $j \in \{0, 1, \dots, k\}$ there exists $u_j \in X$ such that, as $\nu \to +\infty$,

$$A^j u_\nu \to u_j \quad \text{in } X.$$

We now show that $u_0 \in D(A^k)$. Since A is closed, we have

$$u_0 \in D(A) \quad \text{and} \quad Au_0 = u_1. \tag{2.1.17}$$

Similarly,

$$u_1 = Au_0 \quad \text{and} \quad A^2 u_0 = u_2. \tag{2.1.18}$$

Proceeding by induction, we obtain

$$u_{j-1} = A^{j-1} u_0 \in D(A) \quad \text{and} \quad A^j u_0 = u_j, \quad j = 1, \dots, k,$$

which proves the claim.

Theorem 2.4 *Let A be the infinitesimal generator of a C_0–semigroup S such that $A \in G(1, w)$. Suppose that $u_0 \in D(A^k)$, with $k \in \mathbb{N}$, $k \geq 2$. Then the solution of problem (2.1.1) also satisfies*

$$u \in C^{k-j}([0,\infty); D(A^j)) \quad \textit{for } j = 0, 1, \dots, k.$$

Proof: We start with the case $k = 2$ and let $u_0 \in D(A^2)$. Consider the initial value problem

$$\begin{cases} \dfrac{dv}{dt}(t) &= Av(t), \\ v(0) &= Au_0. \end{cases} \tag{2.1.19}$$

By Theorem 2.3, $v(t) = S(t)Au_0$ is the unique solution of (2.1.19) and satisfies

$$v \in C([0,\infty); D(A)) \cap C^1([0,\infty); X).$$

Since u is the solution of (2.1.1) with initial data u_0, we have

$$v(t) = S(t)Au_0 = AS(t)u_0 = Au(t) = \frac{du}{dt}(t), \quad \forall t \geq 0, \tag{2.1.20}$$

that is,

$$\frac{du}{dt} \in C([0,\infty); D(A)) \cap C^1([0,\infty); X),$$

and therefore

$$u \in C^1([0,\infty); D(A)) \cap C^2([0,\infty); X). \tag{2.1.21}$$

It remains to show that

$$u \in C([0,\infty); D(A^2)). \tag{2.1.22}$$

Indeed, since u is the solution of (2.1.1), we have

$$u = S(\cdot)u_0 \ \in \ C([0,\infty); D(A)).$$

Thus, if $\{t_n\} \subset [0,\infty)$ and $t_0 \in [0,\infty)$ are such that $t_n \to t_0$, then

$$\|u(t_n) - u(t_0)\|_{D(A)} = \|u(t_n) - u(t_0)\|_X + \|Au(t_n) - Au(t_0)\|_X \longrightarrow 0. \tag{2.1.23}$$

Moreover, since $v \in C([0,\infty); D(A))$, we have

$$\|v(t_n) - v(t_0)\|_{D(A)} = \|v(t_n) - v(t_0)\|_X + \|Av(t_n) - Av(t_0)\|_X \longrightarrow 0,$$

and, as $v = Au$, it follows that

$$\|A^2u(t_n) - A^2u(t_0)\|_X = \|Av(t_n) - Av(t_0)\|_X \longrightarrow 0.$$

Therefore

$$\begin{aligned} \|u(t_n) - u(t_0)\|_{D(A^2)} &= \|u(t_n) - u(t_0)\|_X + \|Au(t_n) - Au(t_0)\|_X \\ &\quad + \|A^2u(t_n) - A^2u(t_0)\|_X \longrightarrow 0, \end{aligned} \tag{2.1.24}$$

and hence $u \in C([0,\infty); D(A^2))$. This proves the claim for $k = 2$.

Now suppose that the result holds for $k-1$ and we prove it for k. That is, if $u_0 \in D(A^k)$, then

$$u \in \bigcap_{j=0}^{k} C^{k-j}([0,\infty); D(A^j)).$$

Indeed, we have $Au_0 \in D(A^{k-1})$ and, by the induction hypothesis,

$$v = S(\cdot)Au_0 \in \bigcap_{j=0}^{k-1} C^{k-j-1}([0,\infty); D(A^j)). \tag{2.1.25}$$

But

$$v(t) = S(t)Au_0 = \frac{du}{dt}(t),$$

where $u(t) = S(t)u_0$, and hence

$$u \in \bigcap_{j=0}^{k-1} C^{k-j}([0,\infty); D(A^j)).$$

It remains to show that

$$u \in C([0,\infty); D(A^k)). \tag{2.1.26}$$

From (2.1.25), with $j = k-1$, we have

$$Au = v \in C([0,\infty); D(A^{k-1})),$$

so $u(t) \in D(A^k)$ for all $t \geq 0$, and moreover

$$\begin{aligned}\|Au(t_n) - Au(t_0)\|_{D(A^{k-1})} &= \|Au(t_n) - Au(t_0)\|_X + \cdots \\ &\quad + \|A^{k-1}(Au(t_n)) - A^{k-1}(Au(t_0))\|_X \\ &= \|A^u(t_n) - A^u(t_0)\|_X + \cdots + \|A^k u(t_n) - A^k u(t_0)\|_X \\ &\longrightarrow 0, \end{aligned} \tag{2.1.27}$$

as $t_n \to t_0$ in $[0, \infty)$. By Theorem 2.3, $u \in C([0, \infty), D(A))$, hence

$$\|u(t_n) - u(t_0)\|_X \longrightarrow 0.$$

Thus

$$\|u(t_n) - u(t_0)\|_{D(A^k)} \longrightarrow 0$$

as $t_n \to t_0$, which proves (2.1.26) and completes the proof of the theorem. □

Theorem 2.5 *If A is the infinitesimal generator of a differentiable C_0–semigroup, then for each $u_0 \in X$ there exists a unique function*

$$u \in C^0((0, +\infty); D(A)) \cap C^0([0, +\infty); X) \cap C^1((0, +\infty); X)$$

which satisfies

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t), & \text{in } (0, +\infty), \\ u(0) = u_0, \end{cases}$$

and

$$u \in C^k((0, +\infty); D(A^l)) \quad \forall k, l \in \mathbb{N}.$$

Proof: Defining $u(t) = S(t)u_0$ for all $t > 0$, we obtain

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t), & \text{in } (0, +\infty), \\ u(0) = u_0, \end{cases}$$

since S is a differentiable semigroup. Moreover, by Theorem 1.60, the map

$$t \in (0, \infty) \longmapsto AS(t)u_0 = \frac{du}{dt}(t)$$

is continuous, hence

$$\left\| \frac{du}{dt}(t_n) - \frac{du}{dt}(t_0) \right\|_X = \|AS(t_n)u_0 - AS(t_0)u_0\|_X \longrightarrow 0,$$

as $t_n \to t_0$ in $(0, \infty)$. Also,

$$\|u(t_n) - u(t_0)\|_X = \|S(t_n)u_0 - S(t_0)u_0\|_X \longrightarrow 0,$$

and we conclude that

$$u \in C^0((0, \infty); D(A)) \cap C^1((0, \infty); X).$$

Since S is a C_0–semigroup, we also have

$$u \in C^0([0, \infty); X),$$

hence

$$u \in C^0((0,+\infty); D(A)) \cap C^0([0,+\infty); X) \cap C^1((0,+\infty); X).$$

Uniqueness follows from the same arguments used in Theorem 2.3.

Finally, let $k, l \in \mathbb{N}$ be given. Since S is a differentiable semigroup, $u(t) = S(t)u_0 \in D(A^l)$. Moreover, the map $t \in (0,\infty) \mapsto S(t)u_0 = u(t) \in X$ is k–times continuously differentiable by Theorem 1.60, and thus

$$u \in C^k((0,+\infty); D(A^l)).$$

□

Lemma 2.6 *Let A be a dissipative operator on a Hilbert space H and $u : [0,\infty) \to H$ a continuously differentiable function satisfying*

$$\frac{du}{dt}(t) = Au(t), \quad \forall t \geq 0.$$

Then $\|u\|$ is a decreasing function.

Proof: Since

$$\frac{du}{dt}(t) = Au(t),$$

we have

$$(u(t), \frac{du}{dt}(t)) = (u(t), Au(t)), \ \ \forall t \geq 0.$$

Moreover,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= \frac{1}{2}\frac{d}{dt}(u(t), u(t)) = \frac{1}{2}\Big[\Big(\frac{du}{dt}(t), u(t)\Big) + \Big(u(t), \frac{du}{dt}(t)\Big)\Big] \\ &= \frac{1}{2}\, 2\,\mathrm{Re}\Big(u(t), \frac{du}{dt}(t)\Big) = \mathrm{Re}\Big(u(t), \frac{du}{dt}(t)\Big), \end{aligned}$$

that is,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = \mathrm{Re}\big(u(t), Au(t)\big).$$

Integrating the last identity from s to t, with $0 \leq s \leq t$, we obtain

$$\frac{1}{2}\|u(t)\|^2 - \frac{1}{2}\|u(s)\|^2 = \int_s^t \mathrm{Re}\big(u(\tau), Au(\tau)\big)\, d\tau \leq 0,$$

since A is dissipative. Hence

$$\|u(t)\| \leq \|u(s)\| \quad \text{for } 0 \leq s \leq t.$$

□

Lemma 2.7 *Let A be a dissipative and self-adjoint operator on a Hilbert space H and let $u \in C^2([0,\infty); H)$ satisfy*

$$\frac{du}{dt} = Au \quad \text{and} \quad \frac{d^2u}{dt^2} = A^2 u.$$

Then

$$\left\|\frac{du}{dt}(t)\right\| \leq \frac{1}{t}\|u(0)\|.$$

Proof: Notice that

$$\frac{du}{dt} : [0,\infty) \longrightarrow H$$

is a continuously differentiable function satisfying

$$\frac{d}{dt}\left(\frac{du}{dt}\right) = A^2 u = A(Au) = A\left(\frac{du}{dt}\right),$$

and since A is dissipative, Lemma 2.6 implies that $\|\frac{du}{dt}\|$ is a decreasing function. Thus, for $T>0$,

$$\int_0^T \Big(Au(t), \frac{du}{dt}(t)\Big) t\,dt = \int_0^T \left\|\frac{du}{dt}(t)\right\|^2 t\,dt \geq \left\|\frac{du}{dt}(T)\right\|^2 \frac{T^2}{2}. \tag{2.1.28}$$

On the other hand, since A is self-adjoint and

$$\big(Au, \frac{du}{dt}\big) = (Au, Au) = \|Au\|^2 \in \mathbb{R},$$

we obtain

$$\begin{aligned} \frac{d}{dt}(u(t), Au(t)) &= \Big(\frac{du}{dt}(t), Au(t)\Big) + \Big(u(t), \frac{d}{dt}Au(t)\Big) \\ &= \Big(\frac{du}{dt}(t), Au(t)\Big) + \big(u(t), A^2u(t)\big) \\ &= 2\Big(\frac{du}{dt}(t), Au(t)\Big), \end{aligned} \tag{2.1.29}$$

and, integrating by parts,

$$\begin{aligned} \int_0^T \Big(\frac{du}{dt}(t), Au(t)\Big) t\,dt &= \frac{1}{2}\int_0^T \frac{d}{dt}(u(t), Au(t)) t\,dt \\ &= \frac{1}{2}(u(T), Au(T))T - \frac{1}{2}\int_0^T (u(t), Au(t))\,dt. \end{aligned} \tag{2.1.30}$$

Moreover,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = \mathrm{Re}(u(t), Au(t)),$$

and since

$$(u(t), Au(t)) = (Au(t), u(t)) = \overline{(u(t), Au(t))},$$

we have

$$(u(t), Au(t)) \in \mathbb{R},$$

hence

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = (u(t), Au(t)),$$

and consequently

$$\frac{1}{2}\|u(T)\|^2 - \frac{1}{2}\|u(0)\|^2 = \int_0^T (u(t), Au(t))\,dt.$$

Thus

$$\begin{aligned} \left\|\frac{du}{dt}(T)\right\|^2 \frac{T^2}{2} &\leq \int_0^T \Big(Au(t), \frac{du}{dt}(t)\Big) t\,dt = \int_0^T \Big(\frac{du}{dt}(t), Au(t)\Big) t\,dt \\ &= \frac{1}{2}(u(T), Au(T))T - \frac{1}{4}\|u(T)\|^2 + \frac{1}{4}\|u(0)\|^2, \end{aligned}$$

and since

$$\frac{1}{2}(u(T), Au(T))T, \quad -\frac{1}{4}\|u(T)\|^2 \leq 0,$$

we obtain

$$\left\|\frac{du}{dt}(T)\right\| \frac{T}{\sqrt{2}} \leq \frac{1}{2}\|u(0)\|,$$

or equivalently

$$\left\|\frac{du}{dt}(T)\right\| \leq \frac{\sqrt{2}}{2T}\|u(0)\| \leq \frac{1}{T}\|u(0)\|.$$

□

Proposition 2.8 *If A is an m–dissipative and self-adjoint operator on a Hilbert space H, then the C_0–semigroup S generated by A is differentiable.*

Proof: Let $x \in H$ and $t > 0$. We prove that $S(t)x \in D(A)$.

Since $D(A^2)$ is dense in H, there exists a sequence $\{x_n\} \subset D(A^2)$ such that $x_n \to x$ in H. Now,

$$\|S(t)x_n - S(t)x\|_H \leq \|S(t)\|_{\mathcal{L}(H)}\|x_n - x\|_H,$$

and since $x_n \in D(A^2)$ for all $n \in \mathbb{N}$, Theorem 2.4 gives $S(\cdot)x_n \in C^2([0,\infty); H)$ and, moreover,

$$\frac{d}{dt}S(\cdot)x_n = AS(\cdot)x_n \quad \text{and} \quad \frac{d^2}{dt^2}S(\cdot)x_n = A^2S(\cdot)x_n.$$

Thus,

$$\|AS(t)x_n - AS(t)x_m\|_H = \|AS(t)(x_n - x_m)\|_H \leq \frac{1}{t}\|x_n - x_m\|_H,$$

by Lemma 2.7. Hence, as $n \to \infty$ we have $S(t)x_n \to S(t)x$ in H and $AS(t)x_n \to y$ in H, for some $y \in H$, since H is complete. As A is closed, it follows that $S(t)x \in D(A)$ and $y = AS(t)x$. □

2.2 Sesquilinear Forms and Semigroups

Let V and H be complex Hilbert spaces, whose inner products and norms we denote, respectively, by $((\cdot,\cdot)), \|\cdot\|$ on V and $(\cdot,\cdot), |\cdot|$ on H, such that

$$V \hookrightarrow H, \tag{2.2.31}$$

where $\hookrightarrow$ denotes the continuous embedding of one space into the other. We also assume that

$$V \text{ is dense in } H. \tag{2.2.32}$$

Let

$$\begin{aligned} a : V \times V &\longrightarrow \mathbb{C} \\ (u,v) &\longmapsto a(u,v) \end{aligned} \tag{2.2.33}$$

be a continuous sesquilinear form. We define

$$D(A) = \Big\{u \in V;\ v \in V \mapsto a(u,v) \text{ is continuous with respect to the topology induced by } H\Big\}. \tag{2.2.34}$$

In other words, in $D(A)$ we collect those elements $u \in V$ for which the antilinear form

$$\begin{aligned} g_u : V &\to \mathbb{C} \\ v &\longmapsto g_u(v) = a(u,v) \end{aligned} \tag{2.2.35}$$

is continuous when V is endowed with the topology of H.
Consider the set

$$M := \{u \in V;\ \text{there exists } f \in H \text{ such that } a(u,v) = (f,v),\ \text{for all } v \in V\}. \tag{2.2.36}$$

Claim 1. $D(A) = M$.

Indeed, note that $D(A) \neq \emptyset$ because $0 \in D(A)$.

Let $u \in D(A)$. Since V is dense in H, we can extend the continuous antilinear map g_u in (2.2.35) to a map

$$\tilde{g_u} : H \to \mathbb{C},$$

which is antilinear and continuous, and satisfies

$$\tilde{g_u}(v) = g_u(v), \quad \text{for all } v \in V. \tag{2.2.37}$$

By the Riesz representation theorem, there exists a unique $f_u \in H$ such that

$$\tilde{g_u}(v) = (f_u, v), \quad \text{for all } v \in H. \tag{2.2.38}$$

In particular, from (2.2.35), (2.2.37) and (2.2.38) we obtain

$$a(u, v) = g_u(v) = \tilde{g_u}(v) = (f_u, v), \quad \text{for all } v \in V. \tag{2.2.39}$$

Thus $u \in M$ and so $D(A) \subset M$.

Conversely, let $u \in M$. Then there exists $f \in H$ such that $a(u, v) = (f, v)$ for all $v \in V$. We show that the map in (2.2.35) is continuous when V is endowed with the topology of H. Indeed,

$$|g_u(v)| = |a(u, v)| = |(f, v)| \leq |f|\,|v|, \quad \text{for all } v \in V,$$

which shows that $M \subset D(A)$.

Remark 2.9 *From the above discussion we obtain a new characterisation of $D(A)$, namely*

$$D(A) = \{u \in V;\ \textit{there exists } f \in H \textit{ such that } a(u, v) = (f, v),\ \textit{for all } v \in V\}. \tag{2.2.40}$$

Claim 2. $D(A)$ is a vector subspace of H.

Indeed, note first that $0 \in D(A)$. Moreover, let $u_1, u_2 \in D(A)$ and $\alpha \in \mathbb{C}$. By the characterisation in (2.2.40), there exist $f_1, f_2 \in H$ such that

$$a(u_1, v) = (f_1, v), \qquad a(u_2, v) = (f_2, v), \qquad \forall v \in V.$$

Thus

$$\begin{aligned} a(u_1 + \alpha\, u_2, v) &= a(u_1, v) + \alpha\, a(u_2, v) \\ &= (f_1, v) + \alpha\,(f_2, v) \\ &= (f_1 + \alpha\, f_2, v) = (\tilde{f}, v), \end{aligned}$$

where $\tilde{f} = f_1 + \alpha\, f_2 \in H$. Hence $u_1 + \alpha\, u_2 \in D(A)$, which proves the claim.

In this setting, we can define a linear operator

$$\begin{aligned} A : D(A) \subset H &\longrightarrow H \\ u &\longmapsto Au, \end{aligned}$$

by

$$(Au, v) = a(u, v) \quad \text{for all } u \in D(A) \text{ and all } v \in V. \tag{2.2.41}$$

We say that the operator A is defined by the triple $\{V, H; a(u,v)\}$ and we write

$$A \longleftrightarrow \{V, H; a(u,v)\}. \tag{2.2.42}$$

Now consider

$$\begin{aligned} a : V \times V &\longrightarrow \mathbb{C} \\ (u,v) &\longmapsto a(u,v) \end{aligned} \tag{2.2.43}$$

a sesquilinear, Hermitian and continuous map, and suppose that there exist $\lambda_0 \in \mathbb{R}$ and $\alpha > 0$ such that

$$\operatorname{Re} a(v,v) + \lambda_0 |v|^2 \geq \alpha \|v\|^2, \quad \forall v \in V. \tag{2.2.44}$$

Let $A : D(A) \subset H \to H$ be the operator defined by the triple $\{V, H, a(u,v)\}$ and $B : D(B) \subset H \to H$ the operator defined by the triple $\{V, H, b(u,v)\}$, where

$$b(u,v) = a(u,v) + \lambda_0 (u,v).$$

Observe that b is clearly a sesquilinear, Hermitian form, and we now show that b is continuous. Indeed, for $u, v \in V$ we have

$$\begin{aligned} |b(u,v)| &= |a(u,v) + \lambda_0\,(u,v)| \\ &\leqslant |a(u,v)| + |\lambda_0|\ |(u,v)| \\ &\leqslant C\|u\|\,\|v\| + |\lambda_0|\ |u|_H\ |v|_H \\ &\leqslant C\|u\|\,\|v\| + |\lambda_0|\ C'\,\|u\|\,\|v\| \\ &= (C + |\lambda_0|\ C')\,\|u\|\,\|v\| \\ &= K\,\|u\|\,\|v\|, \end{aligned}$$

where C' is a constant given by the embedding $V \hookrightarrow H$.

Besides continuity, b satisfies the coercivity condition, namely

$$\begin{aligned} &\text{There exists a constant } \alpha > 0 \text{ such that} \\ &|b(v,v)| \geq \alpha \|v\|^2, \quad \text{for all } v \in V. \end{aligned} \tag{2.2.45}$$

Indeed,

$$|b(v,v)| \geqslant \operatorname{Re} b(v,v) = \operatorname{Re} a(v,v) + \lambda_0\,|v|^2 \geqslant \alpha \|v\|^2, \qquad \forall v \in V.$$

Therefore, the operator B defined by the triple $\{V, H; b(u,v)\}$ satisfies the assumptions of Proposition 5.129 in [23] and hence:

(i) $D(B)$ is dense in H;

(ii) B is a closed operator.

Claim 3. $D(A) = D(B)$ and $B = A + \lambda_0 I$.

Indeed, let $u \in D(B)$. Then

$$b(u,v) = (Bu, v), \qquad \forall v \in V,$$

or equivalently

$$a(u,v) + \lambda_0\,(u,v) = (Bu,v), \qquad \forall v \in V.$$

Hence

$$(Au,v) = a(u,v) = (Bu,v) - \lambda_0\,(u,v) = \big(\underbrace{(B-\lambda_0 I)u}_{=f_u \in H}, v\big) = (f_u, v).$$

Thus $u \in D(A)$ and we obtain the inclusion $D(B) \subset D(A)$.

Conversely, let $u \in D(A)$. Then

$$a(u,v) = (Au,v), \qquad \forall v \in V.$$

But

$$(Bu,v) = b(u,v) = (Au,v) + \lambda_0\,(u,v) = \big(\underbrace{(A+\lambda_0 I)u}_{=f_u \in H}, v\big) = (f_u, v).$$

Hence $u \in D(B)$ and, consequently, $D(A) \subset D(B)$.
Thus $D(A) = D(B)$.

Moreover, for all $u \in D(A) = D(B)$ and all $v \in V$ we have

$$(Bu,v) = b(u,v) = a(u,v) + \lambda_0\,(u,v) = (Au,v) + \lambda_0\,(u,v) = \big((A+\lambda_0 I)u, v\big).$$

Therefore $B = A + \lambda_0\, I$, which proves the claim.

From the previous claim we deduce that $D(A)$ is dense in H and, since B is closed, it follows that A is closed.

We now wish to prove that $D(A)$ is dense in V.

Claim 4. $D(B)$ is dense in V.

We use the following corollary of the Hahn–Banach theorem: "Let E be a normed vector space and F a vector subspace of E. If for every functional $f \in E'$ such that $\langle f, x\rangle = 0$ for all $x \in F$ we have $f \equiv 0$ (i.e. $\langle f, x\rangle = 0$ for all $x \in E$), then F is dense in E (that is, $\overline{F} = E$)."

We take $E = V$ and $F = D(B)$. Let $f \in V'$ be such that

$$\langle f, u\rangle = 0, \qquad \forall\, u \in D(B).$$

We want to show that

$$\langle f, v\rangle = 0, \qquad \forall\, v \in V.$$

Since $f \in V'$ and $b(\cdot,\cdot)$ is a sesquilinear, continuous and coercive form on V, the Lax–Milgram lemma yields an element $u_f \in V$ such that

$$\langle f, v\rangle = b(u_f, v), \qquad \forall\, v \in V. \tag{2.2.46}$$

In particular, for all $u \in D(B)$ we have

$$0 = \langle f, u\rangle = b(u_f, u) = \overline{b(u, u_f)} = \overline{(Bu, u_f)}, \tag{2.2.47}$$

that is,

$$0 = (Bu, u_f), \qquad \forall u \in D(B). \tag{2.2.48}$$

Let $w \in H$ be arbitrary. Since $b(\cdot,\cdot)$ is sesquilinear, continuous and coercive on V, Proposition 5.126 in [23] guarantees the existence of a unique $u_0 \in D(B)$ such that $w = Bu_0$.

Taking $u = u_0$ in (2.2.48) we obtain

$$0 = (Bu_0, u_f) = (w, u_f), \qquad \forall w \in H. \tag{2.2.49}$$

In particular, for $w = u_f \in V \subset H$ we get

$$0 = |u_f|^2 \qquad \Rightarrow \qquad u_f = 0.$$

Therefore, going back to (2.2.46), we find

$$\langle f, v\rangle = b(0, v) = 0, \qquad \forall\, v \in V.$$

This proves the claim and shows that $D(A) = D(B)$ is dense in V.

Theorem 2.10 *Under the preceding assumptions, we have:*
(a) $-A$ is the infinitesimal generator of a C_0–semigroup S on H satisfying

$$\|S(t)\|_{\mathcal{L}(H)} \le e^{\lambda_0 t}, \quad \forall t \ge 0. \tag{2.2.50}$$

(b) $-A$ is the infinitesimal generator of an analytic semigroup.

Proof: (a) We shall use the Hille–Yosida Theorem. From spectral theory we know that

$$D(A) = D(-A) \text{ is dense in } H \text{ and } -A \text{ is a closed operator.} \tag{2.2.51}$$

We shall prove that, for each $\lambda > \lambda_0$,

$$\lambda \in \rho(-A) \quad \text{ and } \quad \|R(\lambda, -A)\|_{\mathcal{L}(H)} \le \frac{1}{\lambda - \lambda_0}. \tag{2.2.52}$$

Indeed, let $\lambda > \lambda_0$. We first show that $(\lambda I + A)$ is invertible, which is equivalent to showing that

$$\text{Given } f \in H;\ \text{there exists a unique } u \in D(A) \text{ such that } (\lambda I + A)u = f, \tag{2.2.53}$$

or, equivalently,

$$\text{Given } f \in H;\ \text{there exists a unique } u \in D(A) \text{ such that } \lambda(u, v) + a(u, v) = (f, v),\ \forall v \in V, \tag{2.2.54}$$

since $(Au, v) = a(u, v)$ for all $u \in D(A)$ and all $v \in V$.

By (2.2.44) we have, for every $v \in V$,

$$\begin{aligned}\operatorname{Re}\big[a(v, v) + \lambda(v, v)\big] &= \operatorname{Re}\big(a(v, v)\big) + \lambda|v|^2 - \lambda_0|v|^2 + \lambda_0|v|^2 \\ &\ge \alpha\|v\|^2 + (\lambda - \lambda_0)|v|^2 > \alpha\|v\|^2.\end{aligned}$$

Hence

$$b_\lambda(u,v) = a(u,v) + \lambda(u,v)$$

is, for $\lambda > \lambda_0$, a continuous, coercive sesquilinear form on V. Identifying $H \equiv H'$, we have $f \in V'$, and by the Lax–Milgram lemma there exists a unique $u \in V$ such that

$$b_\lambda(u,v) = (f,v), \quad \forall v \in V.$$

It follows that

$$a(u,v) = (f - \lambda u, v), \quad \forall v \in V \quad \text{and} \quad \forall \lambda > \lambda_0,$$

and since $f - \lambda u \in H$ we obtain $u \in D(A)$, which proves (2.2.54) and consequently (2.2.53). Therefore, for each $\lambda > \lambda_0$ there exists $(\lambda I + A)^{-1} : H \to D(A)$. We now show that

$$(\lambda I + A)^{-1} \text{ is closed in } \ H. \tag{2.2.55}$$

Indeed, suppose

$$y_n \to y \quad \text{in} \ \ H \qquad \text{and} \qquad (\lambda I + A)^{-1} y_n \to x \quad \text{in} \ H. \tag{2.2.56}$$

We must prove that $y \in H$ and $x = (\lambda I + A)^{-1} y$. Since it is clear that $y \in H$, it suffices to show that

$$(\lambda I + A)x = y. \tag{2.2.57}$$

In fact, setting $x_n = (\lambda I + A)^{-1} y_n$, from (2.2.56) we have $x_n \to x$ in H and $(\lambda I + A)x_n = y_n \to y$ in H.

Since $(\lambda I + A)$ is closed, it follows that $x \in D(A)$ and $y = (\lambda I + A)x$, which proves (2.2.57) and consequently (2.2.55). Hence $(\lambda I + A)^{-1} \in \mathcal{L}(H)$ for all $\lambda > \lambda_0$ and therefore $\lambda \in \rho(-A)$ for $\lambda > \lambda_0$. It remains to prove that

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(H)} \leq \frac{1}{\lambda - \lambda_0}. \tag{2.2.58}$$

Indeed, taking $v = u$ in (2.2.54) we obtain

$$\lambda |u|^2 + a(u,u) = (f,u),$$

which implies

$$\lambda |u|^2 + \operatorname{Re}\big(a(u,u)\big) = \operatorname{Re}\,(f,u). \tag{2.2.59}$$

Combining (2.2.44) and (2.2.59) we get

$$\begin{aligned} |f||u| \geq \operatorname{Re}\,(f,u) &= \lambda |u|^2 + \operatorname{Re} a(u,u) \\ &\geq \lambda |u|^2 + \alpha \|u\|^2 - \lambda_0 |u|^2 \\ &= (\lambda - \lambda_0)|u|^2 + \alpha \|u\|^2 \\ &\geq (\lambda - \lambda_0)|u|^2, \end{aligned}$$

and for $\lambda > \lambda_0$ it follows that

$$|u| \leq \frac{1}{\lambda - \lambda_0} |f|. \tag{2.2.60}$$

Combining (2.2.53) and (2.2.60) we obtain

$$|(\lambda I + A)^{-1} f| \leq \frac{1}{\lambda - \lambda_0} |f|, \quad \forall \lambda > \lambda_0.$$

Consequently,

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(H)} \leq \frac{1}{\lambda - \lambda_0},$$

which proves item (a).

(b) Set $B = A + \lambda_0 I$.

By (2.2.44) and the continuity of $b(u,v)$ we have

$$\text{Re}\,(Bu,u) = \text{Re}\big(a(u,u)\big) + \lambda_0 |u|^2 \geq \alpha\|u\|^2 \geq 0, \quad \forall u \in D(B), \tag{2.2.61}$$

and

$$|\,\text{Im}\,(Bu,u)| \leq |(Bu,u)| \leq |b(u,u)| \leq c\|u\|^2, \quad \forall u \in D(B). \tag{2.2.62}$$

Introduce the set in the complex plane

$$S(B) = \left\{ \frac{(Bu,u)}{|u|^2} \;;\; u \in D(B) \text{ and } u \neq 0 \right\} \tag{2.2.63}$$

and consider $z \in S(B)$. Then $z = \dfrac{(Bu,u)}{|u|^2}$ for some $u \in D(B)$, $u \neq 0$. We have

$$\tan(\arg z) = \frac{\sin(\arg z)}{\cos(\arg z)} = \frac{\frac{\text{Im}\, z}{|z|}}{\frac{\text{Re}\, z}{|z|}} = \frac{\text{Im}\, z}{\text{Re}\, z} = \frac{\text{Im}\big(\frac{(Bu,u)}{|u|^2}\big)}{\text{Re}\big(\frac{(Bu,u)}{|u|^2}\big)} = \frac{\text{Im}\,(Bu,u)}{\text{Re}\,(Bu,u)}. \tag{2.2.64}$$

From (2.2.61) and (2.2.62) we may write

$$\frac{-c}{\alpha} \leq \frac{-c\|u\|^2}{\text{Re}\,(Bu,u)} \leq \frac{\text{Im}\,(Bu,u)}{\text{Re}\,(Bu,u)} \leq \frac{c\|u\|^2}{\text{Re}\,(Bu,u)} \leq \frac{c}{\alpha}.$$

From this inequality and (2.2.64) we infer the existence of a constant c_1 such that

$$\frac{-c_1}{\alpha} < \frac{-c}{\alpha} \leq \tan(\arg z) \leq \frac{c}{\alpha} < \frac{c_1}{\alpha}.$$

From the last inequality, together with the arbitrariness of z and the independence of the constants with respect to z, we conclude that

$$S(B) \subsetneq \sum_{\theta_1} = \Big\{ z \in \mathbb{C};\; -\theta_1 < \arg z < \theta_1;\; \theta_1 = \arctan\left(\frac{c_1}{\alpha}\right) < \frac{\pi}{2} \Big\}.$$

Therefore

$$S(B) \subsetneq \sum_{\theta_1} \subsetneq \sum_{\theta} = \Big\{ z \in \mathbb{C};\; -\theta < \arg z < \theta;\; \theta_1 < \theta < \frac{\pi}{2} \Big\}. \tag{2.2.65}$$

We claim that there exists a constant c_θ such that, if $d(p, S(B))$ denotes the distance from p to $S(B)$, then

$$d(p, S(B)) \geq d(p, \sum_{\theta_1}) \geq c_\theta |p|, \quad \forall p \notin \sum_{\theta}. \tag{2.2.66}$$

For the sake of clarity, consider Figure 2.1.

The first inequality in (2.2.66) is evident from the inclusions in (2.2.65). We now prove the second one. Consider $p \notin \sum_\theta$ and set $\varphi = \arg p$. There are three cases:

(i) $\theta < \varphi < \dfrac{\pi}{2} + \theta_1$.

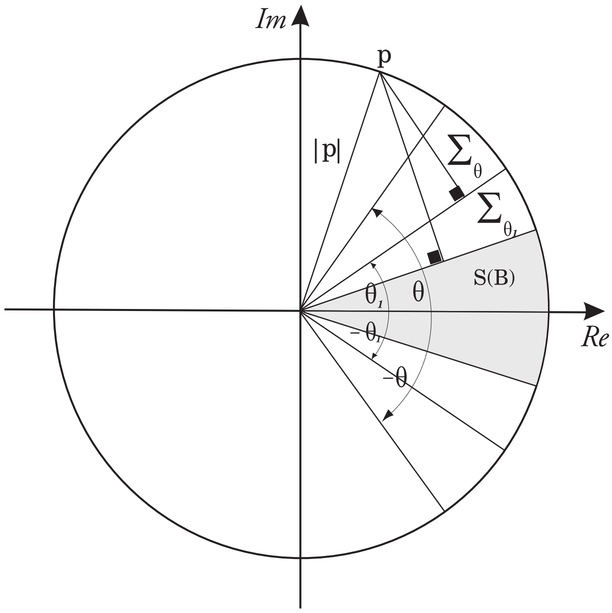

Figure 2.1:

In this case $0 < \theta - \theta_1 < \varphi - \theta_1 \leq \dfrac{\pi}{2}$. Since the sine function is increasing on $[0, \frac{\pi}{2}]$, it follows that $0 < \sin(\theta - \theta_1) < \sin(\varphi - \theta_1)$, whence

$$d(p, \sum_{\theta_1}) = |p| \sin(\varphi - \theta_1) \geq |p| \sin(\theta - \theta_1).$$

Note that the constant $c_\theta = \sin(\theta - \theta_1) > 0$ is independent of p such that $\theta < \varphi \leq \dfrac{\pi}{2} + \theta_1$.

(ii) $\dfrac{\pi}{2} + \theta_1 \leq \varphi \leq \dfrac{3\pi}{2} - \theta_1$.

In this range we always have

$$d(p, \sum_{\theta_1}) = |p|,$$

as illustrated in Figure 2.2.

(iii) $\dfrac{3\pi}{2} - \theta_1 < \varphi < 2\pi - \theta$.

In this case the argument is analogous to that in (ii).

From (i), (ii) and (iii) we obtain the desired estimate in (2.2.66). Now let $u \in D(B)$, $u \neq 0$, set $v = \dfrac{u}{|u|}$ and let $p \in \mathbb{C}$. Observe that $(Bv, v) \in S(B)$ since

$$(Bv, v) = \Big(B\Big(\frac{u}{|u|}\Big), \frac{u}{|u|}\Big) = \frac{(Bu, u)}{|u|^2}.$$

Hence

$$\begin{aligned} d(p, S(B)) \leq |(Bv, v) - p| &= |(Bv, v) - p|v|^2| = |(Bv, v) - p(v, v)| \\ &= |(Bv - pv, v)| = \frac{1}{|u|^2}(Bu - pu, u). \end{aligned}$$

Thus

$$d(p, S(B)) \leq \frac{1}{|u|}\,|Bu - pu|. \tag{2.2.67}$$

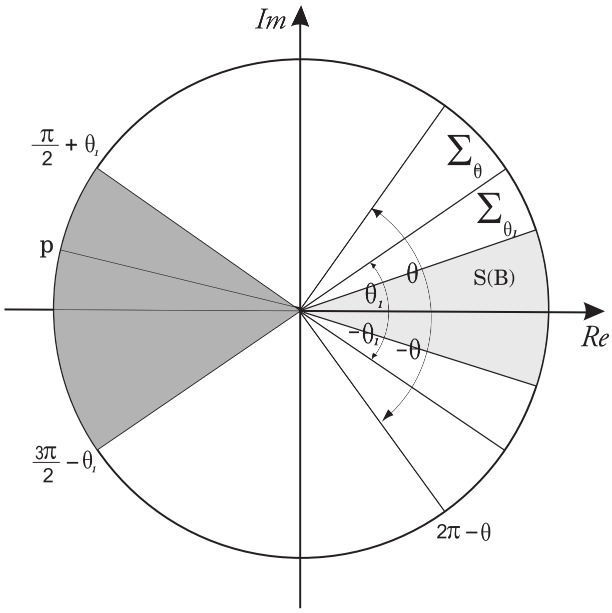

Figure 2.2:

Note that if $\mathrm{Re}\, p \geq 0$, then $(pI+B)^{-1}$ exists. Indeed, it suffices to show that

$$\begin{cases} \text{Given } f \in H;\ \exists!\, u \in D(B) = D(A) \text{ such that} \\ [(p+\lambda_0)I + A]u = f, \end{cases}$$

or equivalently,

$$\begin{cases} \text{Given } f \in H;\ \exists!\, u \in D(B) \text{ such that} \\ p(u,v) + \lambda_0(u,v) + a(u,v) = (f,v),\ \ \forall v \in V. \end{cases}$$

This holds because the continuous sesquilinear form

$$b^*(u,v) = p(u,v) + \lambda_0(u,v) + a(u,v)$$

is coercive whenever $\mathrm{Re}\, p \geq 0$. Similarly, if $\mathrm{Re}\, p \leq 0$ then $\mathrm{Re}(-p) \geq 0$, and the same argument shows that $(-pI+B)^{-1}$ exists.

Now, if $f \in H$ and $p \in \mathbb{C}$ is such that $\mathrm{Re}\, p < 0$, then there exists $u \in D(B)$ such that

$$(-pI+B)^{-1} f = u, \tag{2.2.68}$$

and hence

$$\frac{|(-pI+B)^{-1}f|}{|f|} = \frac{|u|}{|f|} \leq \frac{|Bu - pu|}{d(p, S(B))} \cdot \frac{1}{|f|} = \frac{1}{d(p, S(B))}, \tag{2.2.69}$$

so that

$$\sup_{f \in H,\ f \neq 0} \frac{|(-pI+B)^{-1}f|}{|f|} < \infty, \tag{2.2.70}$$

and therefore $(-pI+B)^{-1} \in \mathcal{L}(H)$, that is, $R(-p,-B) = (-pI+B)^{-1}$.

From (2.2.66) and (2.2.69), for $p \in \mathbb{C} \setminus \Sigma_\theta$ with $\mathrm{Re}\, p < 0$ we also have

$$\|R(-p,-B)\|_{\mathcal{L}(H)} \leq \frac{1}{d(p, S(B))} \leq \frac{1}{|p|} \cdot \frac{1}{c_\theta}.$$

Finally, let $\lambda, \mu \in \mathbb{R}$ with $\lambda > 0$ and $\mu \neq 0$. Writing $p = -\lambda - i\mu \in \mathbb{C}$, we have $-\lambda < 0$ and therefore $\mathrm{Re}\, p < 0$ with $p \in \mathbb{C} \setminus \Sigma_\theta$. Thus

$$\|R(\lambda + i\mu, -B)\|_{\mathcal{L}(H)} \leq \frac{1}{c_\theta} \cdot \frac{1}{\sqrt{\lambda^2 + \mu^2}} \leq \frac{1}{c_\theta} \cdot \frac{1}{|\mu|},$$

that is,

$$\|R(\lambda + i\mu, -B)\|_{\mathcal{L}(H)} \leq \frac{C}{|\mu|}, \quad \forall \lambda > 0,\ \forall \mu \in \mathbb{R}^*,\ \text{where } C = \frac{1}{c_\theta}. \tag{2.2.71}$$

On the other hand, note that $-A \in G(1, \lambda_0)$ and, by Proposition 1.39, $-A - \lambda_0 I = -B \in G(1, 0)$. Moreover, $(0I + B)^{-1} = (\lambda_0 I + A)^{-1}$ exists and $B^{-1} = (\lambda_0 I + A)^{-1} \in \mathcal{L}(H)$, and therefore $0 \in \rho(-B)$. By Theorem 1.67 and (2.2.71), $-B$ is the infinitesimal generator of an analytic semigroup, say $\tilde{S}$.

Define $S(z) = e^{\lambda_0 z} \tilde{S}(z)$. Then $S(z)$ is an analytic semigroup whose infinitesimal generator is $-A$.
□

2.2.1 Applications

(A) Parabolic case:
Under the assumptions of Theorem 2.10 we have, by virtue of Theorem 2.3, that, given $u_0 \in D(A) = D(B)$, the problems

$$\begin{cases} \dfrac{du}{dt} = -Bu \\ u(0) = u_0 \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{du}{dt} = -Au \\ u(0) = u_0 \end{cases} \tag{2.2.72}$$

admit unique solutions in the class

$$u \in C^0([0, \infty); D(A)) \cap C^1([0, \infty); H). \tag{2.2.73}$$

Moreover, if $u_0 \in D(A^k)$ with $k \geq 2$, it follows from Theorem 2.4 that the solutions of problems (2.2.72) satisfy the condition

$$u \in C^{k-j}([0, \infty); D(A^j)) \quad \text{for} \quad j = 0, 1, \ldots, k. \tag{2.2.74}$$

Now, given $u_0 \in H$, the problems (2.2.72) admit unique solutions in the class

$$u \in C^0((0, \infty); D(A)) \cap C^0([0, \infty); H) \cap C^1((0, \infty); H), \tag{2.2.75}$$

by Theorem 2.5 and the fact that $-A$ and $-B$ generate analytic semigroups.

Remark 2.11 *Another way of obtaining regular solutions for problems (2.2.72) in the class (2.2.73) is to make use of the Lumer–Phillips theorem.*

Indeed, for each $\lambda > 0$ we define

$$b_\lambda(u, v) = a(u, v) + (\lambda + \lambda_0)(u, v) = b(u, v) + \lambda(u, v), \quad u, v \in V.$$

Then, for each $\lambda > 0$, $b_\lambda(u, v)$ is a bilinear form and, from (2.2.44), we also obtain the coercivity of $b_\lambda(u, v)$. Therefore, the operator

$$B_\lambda = B + \lambda I \leftrightarrow \{V, H; b_\lambda(u, v)\}$$

is a bijection from $D(B)$ onto H. Consequently,

$$\mathrm{Im}[\lambda I - (-B)] = H, \quad \forall \lambda > 0. \tag{2.2.76}$$

On the other hand, observe that if $u \in D(B)$, then from the fact that $B \leftrightarrow \{V, H, b(u,v)\}$ we obtain

$$\mathrm{Re}(-Bu, u) = -\mathrm{Re}(b(u,u)) = -\mathrm{Re}[a(u,u) + \lambda_0 |u|^2] \leq -\alpha \|u\|^2 \leq 0, \quad \forall u \in D(B). \tag{2.2.77}$$

Thus, since $D(B)$ is dense in H and in view of (2.2.76) and (2.2.77), we conclude, by the Lumer–Phillips theorem, that

$$-B \in G(1,0), \tag{2.2.78}$$

that is, $-B$ is the infinitesimal generator of a contraction semigroup. But since $D(A) = D(B)$ and $B = A + \lambda_0 I$, Proposition 1.39 yields

$$-A \in G(1, \lambda_0), \tag{2.2.79}$$

that is, $-A$ is the infinitesimal generator of a C_0–semigroup satisfying

$$\|S(t)\| \leq e^{\lambda_0 t}, \quad \forall t \geq 0. \tag{2.2.80}$$

In this way, given $u_0 \in D(A) = D(B)$, the Cauchy problems in (2.2.72) admit, in view of Theorem 2.3, unique solutions in the class (2.2.73), as mentioned at the beginning of the Remark. Moreover, the solution associated with the operator B satisfies

$$\|u\| \leq \|u_0\| \quad \text{and} \quad \left\| \frac{du}{dt}(t) \right\| \leq \|Bu_0\|.$$

(B) Hyperbolic case

Let V and H be Hilbert spaces such that

$$V \hookrightarrow H \quad \text{and} \quad V \text{ is dense in } H. \tag{2.2.81}$$

Let $a(u,v)$ be a continuous, coercive, hermitian sesquilinear form on V. Hence $a(u,v)$ defines an inner product on V, denoted by $((\cdot,\cdot))_1$. From the continuity and coercivity of $a(\cdot,\cdot)$ one can show that the norm $\|\cdot\|$ on V arising from the inner product $((\cdot,\cdot))$ is equivalent to the norm $\|\cdot\|_1$ coming from the inner product $((\cdot,\cdot))_1$ defined by $a(\cdot,\cdot)$. Thus $(V, \|\cdot\|_1)$ is complete.

Moreover, since $D(A)$ is dense in V with respect to the norm $\|\cdot\|$, we obtain, by the equivalence of norms, that $D(A)$ is also dense in V with respect to the norm $\|\cdot\|_1$.

Let

$$A \leftrightarrow \{V, H, a(u,v)\}.$$

As is well known, $D(A)$ is dense in V and A is a closed, self-adjoint, non-limited[1] and bijective operator. Consider the problem

$$\begin{cases} \dfrac{d^2u}{dt^2} + Au = 0 \\ u(0) = u_0, \quad u_t(0) = u_1 \end{cases}. \tag{2.2.82}$$

We shall prove that if $u_0 \in D(A)$ and $u_1 \in V$, then problem (2.2.82) admits a unique regular solution. Indeed, consider the change of variables

$$v = \frac{du}{dt}, \tag{2.2.83}$$

[1]By a non-limited operator we mean an operator which may or may not be bounded.

and set

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2.2.84}$$

then, in view of (2.2.82) and (2.2.83), we obtain

$$\frac{dU}{dt} = \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} v \\ -Au \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{2.2.85}$$

Define $D(\mathcal{B}) = D(A) \times V$ and

$$\begin{aligned} \mathcal{B}: \quad & D(\mathcal{B}) \to V \times H \\ & [u, v] \longmapsto \mathcal{B}([u, v]) = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \end{aligned} \tag{2.2.86}$$

From (2.2.82)–(2.2.86) we obtain

$$\begin{cases} \dfrac{dU}{dt} = \mathcal{B}U; \\ U(0) = U_0, \end{cases} \tag{2.2.87}$$

where

$$U_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}.$$

Notice that, through the change of variables given in (2.2.83), problems (2.2.82) and (2.2.87) are equivalent.

Set

$$\mathcal{H} = V \times H. \tag{2.2.88}$$

Observe that $D(\mathcal{B})$ is dense in $\mathcal{H}$, since $D(A)$ is dense in V and the latter is dense in H. Thus, we may consider the adjoint of $\mathcal{B}$. Recall that $\mathcal{B}^*$ is an operator on $\mathcal{H}$ whose domain is given by

$$D(\mathcal{B}^*) = \{v \in \mathcal{H}; \exists v^* \in \mathcal{H} \text{ such that } (\mathcal{B}u, v)_{\mathcal{H}} = (u, v^*)_{\mathcal{H}}, \ \forall u \in D(\mathcal{B})\}$$

and $\mathcal{B}^* v = v^*$ for all $v \in D(\mathcal{B}^*)$. We claim that

$$\mathcal{B}^* = -\mathcal{B}. \tag{2.2.89}$$

Indeed, let $v \in D(\mathcal{B}^*)$. Then $v = [v_1, v_2] \in V \times H$ and there exists $v^* = [v_1^*, v_2^*] \in V \times H$ such that

$$(\mathcal{B}[u_1, u_2], [v_1, v_2])_{\mathcal{H}} = ([u_1, u_2], [v_1^*, v_2^*])_{\mathcal{H}}$$

for all $[u_1, u_2] \in D(A) \times V$, that is,

$$([\ u_2, -Au_1\ ], [v_1, v_2])_{\mathcal{H}} = ([u_1, u_2], [v_1^*, v_2^*])_{\mathcal{H}}$$

for all $[u_1, u_2] \in D(A) \times V$, which implies

$$((u_2, v_1))_1 + (-Au_1, v_2) = ((u_1, v_1^*))_1 + (u_2, v_2^*), \tag{2.2.90}$$

for all $u_1 \in D(A)$ and all $u_2 \in V$.

Taking in particular $u_1 = 0$ in (2.2.90) we obtain

$$((u_2, v_1))_1 = (u_2, v_2^*), \quad \forall u_2 \in V. \tag{2.2.91}$$

From (2.2.91) we deduce that

$$v_1 \in D(A) \quad \text{and} \quad v_2^* = Av_1, \tag{2.2.92}$$

since

$$((u_2, v_1))_1 = (u_2, v_2^*) \;\Rightarrow\; \overline{((u_2, v_1))_1} = \overline{(u_2, v_2^*)} \;\Rightarrow\; ((v_1, u_2))_1 = (v_2^*, u_2).$$

Thus, from (2.2.91) there exists $v_2^* \in H$ such that

$$((v_1, u_2))_1 = (v_2^*, u_2), \quad \forall u_2 \in V,$$

which implies $v_1 \in D(A)$. Moreover,

$$(v_2^*, u_2) = ((v_1, u_2))_1 = a(v_1, u_2) = (Av_1, u_2), \quad \forall u_2 \in V.$$

Hence

$$(Av_1 - v_2^*, u_2) = 0 \quad \forall u_2 \in V. \tag{2.2.93}$$

Since V is dense in H, (2.2.93) holds for all $u_2 \in H$. Thus, in particular, for $u_2 = Av_1 - v_2^*$ we get

$$(Av_1 - v_2^*, Av_1 - v_2^*) = 0 \;\Rightarrow\; Av_1 - v_2^* = 0 \;\Rightarrow\; Av_1 = v_2^*.$$

Substituting (2.2.92) into (2.2.90), we obtain

$$((u_2, v_1))_1 + (-Au_1, v_2) = ((u_1, v_1^*))_1 + (u_2, Av_1), \quad \forall u_1 \in D(A),\ \forall u_2 \in V.$$

In particular, for $u_2 = 0$ we obtain

$$(-Au_1, v_2) = ((u_1, v_1^*))_1, \qquad \forall u_1 \in D(A),$$

which implies, in view of the bijectivity of $A : D(A) \to H$, that

$$v_1^* = -v_2. \tag{2.2.94}$$

From (2.2.92) and (2.2.94) it follows that

$$[v_1, v_2] \in D(A) \times V = D(\mathcal{B}),$$

that is,

$$D(\mathcal{B}^*) \subset D(\mathcal{B}). \tag{2.2.95}$$

Furthermore,

$$\mathcal{B}^* v = v^* = [v_1^*, v_2^*] = [-v_2, Av_1] = -[v_2, -Av_1] = -\mathcal{B}v. \tag{2.2.96}$$

Conversely, let $[v_1, v_2] \in D(\mathcal{B}) = D(A) \times V$. We show that there exists $[v_1^*, v_2^*] \in V \times H$ such that (2.2.90) holds. Indeed, let $[u_1, u_2] \in D(A) \times V$ and set $v_1^* = -v_2$ and $v_2^* = Av_1$. Then

$$\begin{aligned} ((u_1, v_1^*))_1 + (u_2, v_2^*) &= ((u_1, -v_2))_1 + (u_2, Av_1) \\ &= (-Au_1, v_2) + ((u_2, v_1))_1 \\ &= ((u_2, v_1))_1 + (-Au_1, v_2), \end{aligned}$$

which proves (2.2.90) and consequently that $[v_1, v_2] \in D(\mathcal{B}^*)$, that is,

$$D(\mathcal{B}) \subset D(\mathcal{B}^*). \tag{2.2.97}$$

From (2.2.95), (2.2.96) and (2.2.97) we obtain (2.2.89). It then follows, by Stone's theorem, that $\mathcal{B}$ is the infinitesimal generator of a unitary C_0–group and, consequently, of a C_0–semigroup S with the property

$$(S(t))^* = (S(t))^{-1}, \qquad \forall t \geq 0. \tag{2.2.98}$$

Thus, setting

$$U(t) = S(t)U_0, \qquad \forall t \geq 0, \tag{2.2.99}$$

we deduce, by Theorem 2.3, that U is the unique regular solution of the Cauchy problem (2.2.87), and it belongs to the class

$$U \in C^0([0,+\infty); D(\mathcal{B})) \cap C^1([0,+\infty); \mathcal{H}). \tag{2.2.100}$$

Now, from (2.2.100) and using the change of variables (2.2.83), we conclude that problem (2.2.82) admits a unique solution u in the class

$$u \in C^0([0,+\infty); D(A)) \cap C^1([0,+\infty); V)$$

with

$$u_t \in C^0([0,+\infty); V) \cap C^1([0,+\infty); H),$$

and hence

$$u \in C^0([0,+\infty); D(A)) \cap C^1([0,+\infty); V) \cap C^2([0,+\infty); H). \tag{2.2.101}$$

We also note that, from (2.2.88) and (2.2.99), we have

$$\|S(t)U_0\|_{\mathcal{H}} = \|U_0\|_{\mathcal{H}}, \quad \forall t \geq 0,$$

that is,

$$\|U(t)\|_{V\times H} = \|[u_0,u_1]\|_{\mathcal{H}}, \quad \forall t \geq 0,$$

or equivalently,

$$\|u(t)\|^2 + |u'(t)|^2 = \|u_0\|^2 + |u_1|^2, \quad \forall t \geq 0. \tag{2.2.102}$$

Identity (2.2.102) is known as the energy identity.

Now define

$$\widetilde{A} : V \longrightarrow V'$$

by

$$\langle \widetilde{A}u, v\rangle_{V'\times V} = a(u,v), \qquad \forall u,v \in V.$$

Then $\widetilde{A}$ is a linear isometry and induces on V' the following inner product

$$(u,v)_{V'} = ((\widetilde{A}^{-1}u, \widetilde{A}^{-1}v))_1, \qquad \forall u,v \in V',$$

and moreover

$$\widetilde{A}u = Au, \qquad \forall u \in D(A).$$

Consider the problem

$$\begin{cases} \dfrac{d^2u}{dt^2} + \widetilde{A}u = 0 \\ u(0) = u_0, \quad u_t(0) = u_1 \end{cases}. \tag{2.2.103}$$

We shall prove that, if $u_0 \in V$ and $u_1 \in H$, then problem (2.2.103) admits a unique regular solution.

As before, consider the change of variables

$$v = \frac{du}{dt}, \tag{2.2.104}$$

and define

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2.2.105}$$

then, in view of (2.2.103) and (2.2.104), we obtain

$$\frac{dU}{dt} = \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} v \\ -\widetilde{A}u \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\widetilde{A} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{2.2.106}$$

Define

$$\begin{array}{rcl} B: \quad V \times H & \to & H \times V' \\ [u,v] & \mapsto & B([u,v]) = \begin{bmatrix} 0 & I \\ -\widetilde{A} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \end{array} \tag{2.2.107}$$

and from (2.2.103)–(2.2.107) we arrive at

$$\begin{cases} \dfrac{dU}{dt} = BU \\ U(0) = U_0, \end{cases} \tag{2.2.108}$$

where

$$U_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}.$$

Observe that, via the change of variables given in (2.2.104), problems (2.2.103) and (2.2.108) are equivalent.

As before, our goal is to show that B is the infinitesimal generator of a C_0–semigroup, and for this we shall use the Hille–Yosida theorem. We have already seen that $D(B) = V \times H$ is dense in $H \times V'$. Thus, it remains to verify that:

(i) B is closed.

(ii) There exist real numbers M and ω such that, for each real $\lambda > \omega$, we have $\lambda \in \rho(B)$ and

$$\|R(\lambda, B)^n\|_{\mathfrak{L}(H \times V')} \leq \frac{M}{(\lambda - \omega)^n}, \quad \forall n \in \mathbb{N}.$$

Indeed, let $([u_n, v_n])_n \subset V \times H = D(B)$ be such that

$$[u_n, v_n] \longrightarrow [\tilde{u}, \tilde{v}] \quad \text{in } H \times V'$$

and

$$B([u_n, v_n]) \longrightarrow [f, g] \quad \text{in } H \times V'.$$

We shall show that $[\tilde{u}, \tilde{v}] \in D(B)$ and that $B([\tilde{u}, \tilde{v}]) = [f, g]$. From the convergences above we have

$$\begin{aligned} u_n &\longrightarrow \tilde{u} \quad \text{in } H, \\ v_n &\longrightarrow \tilde{v} \quad \text{in } V', \\ v_n &\longrightarrow f \quad \text{in } H, \\ -\widetilde{A}u_n &\longrightarrow g \quad \text{in } V'. \end{aligned}$$

Thus, $u_n \longrightarrow -\widetilde{A}^{-1}g$ in V. Since V is continuously embedded into H, we also have $u_n \longrightarrow -\widetilde{A}^{-1}g$ in H. By uniqueness of limits, we obtain

$$-\widetilde{A}^{-1}g = \tilde{u}.$$

Moreover, $f = \tilde{v}$. Therefore, $[\tilde{u}, \tilde{v}] \in D(B)$ and $B([\tilde{u}, \tilde{v}]) = [\tilde{v}, -\widetilde{A}\tilde{u}] = [f, g]$. That is, B is closed.

We now show that, for each real $\lambda > 0$, we have $\lambda \in \rho(B)$ and

$$\|R(\lambda, B)^n\|_{\mathfrak{L}(H\times V')} \leq \frac{1}{\lambda^n}, \quad \forall n \in \mathbb{N}.$$

Since $\widetilde{A}$ is bijective, it follows that $\lambda I - B$ is bijective for every $\lambda > 0$, i.e., $\lambda \in \rho(B)$ for all $\lambda > 0$.

Given $u, v \in D(A)$, we have

$$(\lambda I - B)[u, v] = [f, g], \quad f \in H,\ g \in V'.$$

Hence

$$[\lambda u - v, \lambda v + \widetilde{A}u] = [f, g],$$

which is equivalent to

$$\begin{cases} \lambda u - v = f \quad \text{in } H, \\ \lambda v + \widetilde{A}u = g \quad \text{in } V'. \end{cases}$$

Thus

$$\begin{cases} \lambda(u, u) - (v, u) = (f, u), \\ \lambda(v, v)_{V'} + (\widetilde{A}u, v)_{V'} = (g, v)_{V'}. \end{cases}$$

Adding the last two equalities we get

$$\lambda|u|^2 + \lambda|v|^2_{V'} + (\widetilde{A}u, v)_{V'} - (v, u) = (f, u) + (g, v)_{V'}. \tag{2.2.109}$$

However,

$$\begin{aligned} (\widetilde{A}u, v)_{V'} &= ((\widetilde{A}^{-1}\widetilde{A}u, \widetilde{A}^{-1}v))_1 \\ &= ((u, \widetilde{A}^{-1}v))_1 \\ &= a(u, \widetilde{A}^{-1}v) \\ &= (Au, \widetilde{A}^{-1}v) \\ &= (u, A\widetilde{A}^{-1}v) \\ &= (u, v), \quad \forall u, v \in D(A). \end{aligned} \tag{2.2.110}$$

From (2.2.109) and (2.2.110) we obtain

$$\lambda(|u|^2 + |v|^2_{V'}) + (u, v) - (v, u) = (f, u) + (g, v)_{V'}.$$

Hence

$$\lambda(|u|^2 + |v|^2_{V'}) + \mathrm{Re}\big[(u, v) - (v, u)\big] = \mathrm{Re}\big[(f, u) + (g, v)_{V'}\big],$$

that is,

$$\begin{aligned} \lambda(|u|^2 + |v|^2_{V'}) &= \mathrm{Re}\big[(f, u) + (g, v)_{V'}\big] \\ &\leq |(f, u) + (g, v)_{V'}| \\ &\leq |f|\,|u| + |g|_{V'}\,|v|_{V'} \\ &\leq (|f|^2 + |g|^2_{V'})^{\frac{1}{2}}(|u|^2 + |v|^2_{V'})^{\frac{1}{2}}, \end{aligned}$$

and thus

$$\lambda(|u|^2 + |v|_{V'}^2)^{\frac{1}{2}} \leq (|f|^2 + |g|_{V'}^2)^{\frac{1}{2}}.$$

Therefore,

$$\|[u,v]\|_{H\times V'} \leq \frac{1}{\lambda}\|[f,g]\|_{H\times V'}, \quad \forall u,v \in D(A).$$

Since $D(A)\times D(A)$ is dense in $V\times H$, it follows that

$$\|[u,v]\|_{H\times V'} \leq \frac{1}{\lambda}\|[f,g]\|_{H\times V'}, \quad \forall [u,v] \in V\times H. \tag{2.2.111}$$

Moreover,

$$[u,v] = R(\lambda,B)[f,g], \quad \forall [u,v] \in V\times H. \tag{2.2.112}$$

From (2.2.111) and (2.2.112) we conclude that

$$\|R(\lambda,B)[f,g]\|_{V\times H} \leq \frac{1}{\lambda}\|[f,g]\|_{H\times V'}, \quad \forall [f,g] \in H\times V'.$$

Hence

$$\|R(\lambda,B)\|_{\mathfrak{L}(H\times V')} \leq \frac{1}{\lambda}.$$

Furthermore,

$$\|R(\lambda,B)^n\|_{\mathfrak{L}(H\times V')} \leq \|R(\lambda,B)\|_{\mathfrak{L}(H\times V')}\cdots\|R(\lambda,B)\|_{\mathfrak{L}(H\times V')} \leq \frac{1}{\lambda^n}, \quad \forall n \in \mathbb{N}.$$

Thus B satisfies the hypotheses of the Hille–Yosida theorem, from which we obtain that B is the infinitesimal generator of a C_0–semigroup $S:\mathbb{R}_+ \longrightarrow \mathfrak{L}(H\times V')$. The conclusion is analogous to the case $[u_0,u_1] \in D(A)\times V$.

2.3 The Non-homogeneous Problem

Let A be the infinitesimal generator of a C_0–semigroup S, $f:\mathbb{R}_+ \to X$ a continuous function with values in a Banach space X, and consider the Abstract Cauchy Problem

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t) + f(t), \quad t>0, \\ u(0) = u_0. \end{cases} \tag{2.3.113}$$

Definition 2.12 *A function $u:\mathbb{R}_+ \to X$ is said to be a classical solution of (2.3.113) if:*

i) u is continuous for $t \geq 0$;

ii) u is continuously differentiable for $t>0$;

iii) $u(t) \in D(A)$ for $t>0$;

iv) u satisfies (2.3.113).

Let u be a classical solution of (2.3.113) and set

$$g(s) = S(t-s)u(s), \quad 0 \leq s \leq t.$$

We have, as in the proof of Theorem 2.3, that

$$\frac{dg}{ds}(s) = S(t-s)\frac{du}{ds}(s) - S(t-s)Au(s). \tag{2.3.114}$$

Hence, from (2.3.114), Proposition 1.30 and (2.3.113), it follows that

$$\frac{dg}{ds}(s) = S(t-s)[Au(s) + f(s)] - S(t-s)Au(s),$$

that is,

$$\frac{dg}{ds}(s) = S(t-s)f(s).$$

Integrating this last identity from 0 to t we obtain

$$g(t) - g(0) = \int_0^t S(t-s)f(s)\,ds,$$

or equivalently,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds, \tag{2.3.115}$$

which is a necessary condition for u to be a classical solution of (2.3.113).

Under the hypotheses stated above, the formula (2.3.115) makes sense whether or not u is a classical solution of (2.3.113). For this reason we introduce the following definition.

Definition 2.13 *Let A be the infinitesimal generator of a C_0–semigroup, and let $u_0 \in X$ and $f \in L^1(0,T;X)$. The function $u \in C^0([0,T];X)$ given by*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds, \qquad 0 \le t \le T,$$

is called a generalized (mild) solution of problem (2.3.113) on $[0,T]$.

Note that generalized solutions of (2.3.113) are not necessarily classical solutions, even when f is continuous, as can be seen by taking

$$f(t) = S(t)v \notin D(A), \quad \forall t \ge 0, \; v \in X.$$

In this case,

$$\begin{aligned} u(t) &= S(t)u_0 + \int_0^t S(t-s)S(s)v\,ds \\ &= S(t)u_0 + \int_0^t S(t)v\,ds \\ &= S(t)u_0 + tS(t)v \end{aligned}$$

is a generalized solution which is not a classical solution, because this function is not differentiable for $t \ge 0$. Furthermore, this example shows that the mere continuity of f does not ensure the existence of a classical solution. Thus, for a generalized solution to be classical, it is necessary that A or f satisfy additional conditions, as we shall see below.

As an immediate consequence of (2.3.115) and Theorem 2.3 we have:

Proposition 2.14 *System (2.3.113) has at most one classical solution.*

Theorem 2.15 *System (2.3.113) has a classical solution for each $u_0 \in D(A)$ if and only if the function v defined by*

$$v(t) = \int_0^t S(t-s)f(s)\,ds \tag{2.3.116}$$

is continuously differentiable for $t > 0$.

Proof: Let u be a classical solution of (2.3.113) for $u_0 \in D(A)$. From (2.3.115) and (2.3.116) we may write

$$v(t) = u(t) - S(t)u_0.$$

Since u is a classical solution of (2.3.113), by definition u is continuous for $t \geq 0$, $u(t) \in D(A)$ for $t > 0$, and u is continuously differentiable for $t > 0$. Moreover, since $u_0 \in D(A)$, it follows from Proposition 1.30 that $S(t)u_0 \in D(A)$ for all $t \geq 0$ and, in addition,

$$v'(t) = u'(t) - S(t)Au_0,$$

which is continuous for $t > 0$. Hence v is continuously differentiable.

Conversely, suppose that $v(t)$ given by (2.3.116) is continuously differentiable for $t > 0$. For $h > 0$, define

$$A_h v(t) = \frac{S(h)v(t) - v(t)}{h}. \tag{2.3.117}$$

Then, from (2.3.116) and (2.3.117), we obtain

$$\begin{aligned} A_h v(t) &= \frac{1}{h}\left[\int_0^t S(t-s+h)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds\right] \\ &= \frac{1}{h}\left[\int_0^{t+h} S(t-s+h)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds - \int_t^{t+h} S(t-s+h)f(s)\,ds\right] \\ &= \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} S(t-s+h)f(s)\,ds. \end{aligned} \tag{2.3.118}$$

Since f is continuous on $\mathbb{R}_+$, the second term on the right-hand side of (2.3.118) has limit $f(t)$ as $h \to 0_+$, and the first term has limit $v'(t)$ as $h \to 0_+$. Hence, in the limit $h \to 0_+$ in (2.3.118) we obtain

$$v(t) \in D(A) \quad \text{and} \quad Av(t) = v'(t) - f(t).$$

Moreover, from (2.3.116) we have $v(0) = 0$. Thus, the function

$$u(t) = S(t)u_0 + v(t)$$

is a classical solution of (2.3.113). □

Corollary 2.16 *If $v(t) \in D(A)$ for all $t > 0$ and Av is continuous, then problem (2.3.113) has a classical solution for every $u_0 \in D(A)$.*

Proof: From (2.3.118) we obtain

$$\frac{v(t+h)-v(t)}{h} = A_h v(t) + \frac{1}{h}\int_t^{t+h} S(t-s+h)f(s)\,ds. \tag{2.3.119}$$

Since $v(t) \in D(A)$, we have

$$A_h v(t) \to Av(t), \quad \forall t > 0 \quad \text{as } h \to 0_+. \tag{2.3.120}$$

Also,

$$\frac{1}{h}\int_t^{t+h} S(t-s+h)f(s)\,ds \to f(t) \quad \text{as } h\to 0_+, \quad \text{since } f \text{ is continuous for all } t>0. \tag{2.3.121}$$

Therefore, from (2.3.119), (2.3.120) and (2.3.121) it follows that v is right differentiable at every $t > 0$ and

$$\frac{d^+v}{dt}(t) = Av(t) + f(t).$$

By the continuity of Av and f, by hypothesis, we have that $\frac{d^+v}{dt}(t)$ is continuous. Hence, by Dini's lemma, v is continuously differentiable for $t > 0$, and from Theorem 2.15 it follows that problem (2.3.113) has a classical solution for all $u_0 \in D(A)$, which is given by (2.3.115). □

Proposition 2.17 *Let A be the infinitesimal generator of a C_0–semigroup S, and $f : \mathbb{R}_+ \to X$ a continuous function. Suppose that f satisfies one of the following conditions:*

i) f is continuously differentiable for all $t \geq 0$;

ii) $f(t) \in D(A)$ for all $t \geq 0$ and Af is integrable (in $L^1_{\text{loc}}(0,\infty;X)$).

Then, for every $u_0 \in D(A)$, (2.3.113) has a unique classical solution.

Proof: Assume that (i) holds. Let $v(t)$ be given by (2.3.116), that is,

$$v(t) = \int_0^t S(t-s)f(s)\,ds = \int_0^t S(s)f(t-s)\,ds.$$

Then, for $h > 0$,

$$\begin{aligned}
\frac{v(t+h)-v(t)}{h} &= \tfrac{1}{h}\int_0^{t+h} S(s)f(t+h-s)\,ds - \tfrac{1}{h}\int_0^t S(s)f(t-s)\,ds \\
&= \tfrac{1}{h}\int_0^{t+h} S(s)(f(t+h-s)-f(t-s))\,ds + \tfrac{1}{h}\int_t^{t+h} S(s)f(t-s)\,ds \\
&= \tfrac{1}{h}\int_0^{t} S(s)(f(t+h-s)-f(t-s))\,ds \\
&\quad +\tfrac{1}{h}\int_t^{t+h} S(s)(f(t+h-s)-f(t-s))\,ds \\
&\quad +\tfrac{1}{h}\int_t^{t+h} S(s)(f(t+h-s)-f(t-s))\,ds + \tfrac{1}{h}\int_t^{t+h} S(s)f(t-s)\,ds \\
&= \tfrac{1}{h}\int_0^{t} S(s)(f(t+h-s)-f(t-s))\,ds + \int_t^{t+h} S(s)f'(\gamma)\,ds \\
&\quad +\tfrac{1}{h}\int_t^{t+h} S(s)f(t-s)\,ds,
\end{aligned} \tag{2.3.122}$$

where $\gamma \in (t-s, t-s+h)$, by the Mean Value Theorem, since $f(t-s+h) - f(t-s) = f'(\gamma)h$ for some $\gamma \in (t-s, t-s+h)$. The right-hand side of (2.3.122) converges to

$$\int_0^t S(s)f'(t-s)\,ds + S(t)f(0) \tag{2.3.123}$$

as $h \to 0^+$. From the hypothesis that f is continuously differentiable for $t \geq 0$, it follows from (2.3.122) and (2.3.123) in the limit that

$$\frac{dv^+}{dt}(t) = S(t)f(0) + \int_0^t S(s)f'(t-s)\,ds$$

is continuous for $t > 0$. Hence, by Dini's lemma, v is continuously differentiable for $t > 0$. Therefore, by Theorem 2.15, system (2.3.113) has a classical solution for all $u_0 \in D(A)$.

Now assume that (ii) holds. Since $f(s) \in D(A)$, by Proposition 1.29 we have

$$S(t-s)f(s) \in D(A) \quad \text{and} \quad AS(t-s)f(s) = S(t-s)Af(s).$$

From this last identity, since Af is integrable and A is closed, we obtain

$$\int_0^t S(t-s)Af(s)\,ds = \int_0^t AS(t-s)f(s)\,ds = A\int_0^t S(t-s)f(s)\,ds = Av(t),$$

that is, $v(t) \in D(A)$ for $t > 0$ and Av is continuous. Hence, by Corollary 2.3.114, problem (2.3.113) has a unique classical solution for all $u_0 \in D(A)$. Uniqueness follows as in the homogeneous case. □

0.5 cm

We conclude this section with some results concerning a notion of solution that we now introduce.

Definition 2.18 *Let A be the infinitesimal generator of a C_0–semigroup S. A function u which is differentiable almost everywhere on $[0,T]$ and such that $u' \in L^1(0,T;X)$ is called a strong solution of the initial value problem (2.3.113) if $u(0) = u_0$ and $u'(t) = Au(t) + f(t)$ almost everywhere on $[0,T]$.*

Note that if $A \equiv 0$ and $f \in L^1(0,T;X)$, then the initial value problem (2.3.113) does not, in general, admit a classical solution unless f is continuous. However, it always admits a strong solution given by

$$u(t) = u_0 + \int_0^t f(s)\,ds.$$

As in the classical case, a natural question is to determine when a generalized (mild) solution of (2.3.113) is a strong solution.

Theorem 2.19 *Let A be the infinitesimal generator of a C_0–semigroup S and let $f \in L^1(0,T;X)$. Define*

$$v(t) = \int_0^t S(t-s)f(s)\,ds, \qquad 0 \le t \le T,$$

and suppose that $v(t)$ satisfies one of the following conditions:

(i) $v(t)$ is differentiable almost everywhere on $[0,T]$ and $v'(t) \in L^1(0,T;X)$;

(ii) $v(t) \in D(A)$ almost everywhere on $[0,T]$ and $Av(t) \in L^1(0,T;X)$.

Then (2.3.113) admits a strong solution u on $[0,T]$ for some $u_0 \in D(A)$.
Conversely, if (2.3.113) admits a strong solution u on $[0,T]$ for some $u_0 \in D(A)$, then v satisfies (i) and (ii).

Proof: First observe that items (i) and (ii) are equivalent.

$(i) \Rightarrow (ii)$. Note that

$$\begin{aligned}
\left(\frac{S(h)-I}{h}\right)v(t) &= \frac{1}{h}\left[S(h)\int_0^t S(t-s)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds\right] \\
&= \frac{1}{h}\left[\int_0^t S(t-s+h)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds\right] \\
&= \frac{1}{h}\left[\int_0^{t+h} S(t-s+h)f(s)\,ds - \int_t^{t+h} S(t-s+h)f(s)\,ds\right. \\
&\qquad \left. - \int_0^t S(t-s)f(s)\,ds\right] \\
&= \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} S(h)S(t-s)f(s)\,ds.
\end{aligned} \tag{2.3.124}$$

The first equality is justified because $S(t-s)f(s) \in L^1(0,T;X)$ and $S(h)$ is bounded on X. Thus

$$\left(\frac{S(h)-I}{h}\right)v(t) = \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} S(h)S(t-s)f(s)\,ds.$$

Now, since v is differentiable almost everywhere, the limit of the first term on the right-hand side exists almost everywhere as $h \to 0^+$. Moreover, by standard results on Bochner integration (see, for instance, [24], p. 10), the second term also has a limit almost everywhere as $h \to 0^+$. Furthermore, as $h \to 0^+$ we have

$$\frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} S(h)S(t-s)f(s)\,ds \longrightarrow v'(t) - f(t) \quad \text{a.e. in } [0,T].$$

Therefore

$$v(t) \in D(A) \quad \text{and} \quad Av(t) = v'(t) - f(t) \in L^1(0,T;X),$$

which proves (ii).

$(ii) \Rightarrow (i)$. Since $v(t) \in D(A)$, we have $\lim_{h\to 0^+}\left(\frac{S(h)-I}{h}\right)v(t) = Av(t)$. Now, from (2.3.124) it follows that

$$\frac{v(t+h)-v(t)}{h} = \left(\frac{S(h)-I}{h}\right)v(t) + \frac{1}{h}\int_t^{t+h} S(h)S(t-s)f(s)\,ds.$$

Thus, as $h \to 0^+$,

$$\frac{d^+v}{dt}(t) = Av(t) - f(t).$$

Considering $h < 0$ and replacing h by $-h$ in (2.3.124), we obtain

$$\frac{d^-v}{dt}(t) = Av(t) - f(t).$$

Hence $v(t)$ is differentiable almost everywhere on $[0,T]$ and

$$v'(t) = Av(t) - f(t) \in L^1(0,T;X),$$

which proves (i) (and therefore the equivalence).

We now prove Theorem 2.19. Suppose that (i) holds (and hence (ii) also holds).

Since

$$u(t) = S(t)u_0 + v(t),$$

$S(t)u_0$ is differentiable (because $u_0 \in D(A)$) and, by hypothesis, $v(t)$ is differentiable almost everywhere, it follows that u is differentiable almost everywhere on $[0,T]$.

Moreover,

$$\frac{du}{dt}(t) = \frac{d}{dt}(S(t)u_0 + v(t)) = S(t)Au_0 + v'(t) \in L^1(0, T; X).$$

Finally, to show that u satisfies (2.3.113) almost everywhere, it suffices to follow the same arguments used in the proof of Theorem 2.15.

Conversely, if (2.3.113) admits a strong solution u, then

$$u(t) = S(t)u_0 + v(t),$$

whence v is differentiable almost everywhere, since u and $S(\cdot)u_0$ are. Furthermore, as $S(t)Au_0, u'(t) \in L^1(0, T; X)$ and

$$v'(t) = u'(t) - S(t)Au_0,$$

we have $v'(t) \in L^1(0, T; X)$. This proves (i) (and thus (ii)) and completes the proof of the theorem. □

As a consequence of Theorem 2.19 we obtain:

Corollary 2.20 *Let A be the infinitesimal generator of a C_0–semigroup S. If f is differentiable almost everywhere on $[0, T]$ and $f' \in L^1(0, T; X)$, then for every $u_0 \in D(A)$ the problem (2.3.113) admits a unique strong solution on $[0, T]$.*

Proof: Observe that

$$v(t) = \int_0^t S(t-s)f(s)\,ds = \int_0^t S(s)f(t-s)\,ds,$$

and hence

$$\begin{aligned}
\frac{v(t+h) - v(t)}{h} &= \frac{1}{h}\left[\int_0^{t+h} S(s)f(t+h-s)\,ds - \int_0^{t+h} S(s)f(t-s)\,ds \right.\\
&\quad \left. + \int_t^{t+h} S(s)f(t-s)\,ds\right]\\
&= \int_0^t S(s)\left[\frac{f(t+h-s) - f(t-s)}{h}\right] ds + \frac{1}{h}\int_t^{t+h} S(s)f(t-s)\,ds\\
&\quad + \frac{1}{h}\int_t^{t+h} S(s)\big(f(t+h-s) - f(t-s)\big)\,ds.
\end{aligned}$$

Thus v is differentiable almost everywhere and

$$\frac{dv}{dt}(t) = \int_0^t S(s)f'(t-s)\,ds + S(t)f(0).$$

Since $f(0) \in X$, the function $S(\cdot)\,f(0)$ belongs to $L^1(0, T; X)$. Note also that

$$\varphi(t) = \int_0^t S(s)f'(t-s)\,ds = \int_0^t S(t-s)f'(s)\,ds,$$

and therefore

$$\begin{aligned}\int_0^T \|\varphi(t)\|\,dt &\leq \int_0^T\int_0^t \|S(t-s)f'(s)\|\,ds\,dt\\ &\leq \int_0^T\int_0^T \|S(t-s)f'(s)\|\,ds\,dt\\ &\leq \int_0^T\int_0^T Me^{\omega T}\|f'(s)\|\,ds\,dt\\ &= Me^{\omega T}\int_0^T \|f'(s)\|\,ds,\end{aligned}$$

so that $\int_0^t S(t-s)f'(s)\,ds \in L^1(0,T;X)$.

Hence, by Theorem 2.19, the result follows. □

Before we state the next corollary of Theorem 2.19, let us introduce the following notion.

Definition 2.21 *A function $f:\mathbb{R}_+\to X$ is said to be Hölder continuous for $t\geq 0$ if*

$$\|f(t)-f(s)\|\leq L(t-s)^k,\quad 0\leq s\leq t,$$

where L and k are constants with $0\leq k\leq 1$. When $k=1$ we say that f is Lipschitz continuous.

In general, the Lipschitz continuity of f on $[0,T]$ is not sufficient to ensure the existence of a strong solution of (2.3.113) for $u_0\in D(A)$. However, if X is reflexive and f is Lipschitz continuous on $[0,T]$, then by classical results (see [24], p. 17) f is differentiable almost everywhere and $f'\in L^1(0,T;X)$. In view of this, Corollary 2.20 implies:

Corollary 2.22 *Let X be a reflexive Banach space and let A be the infinitesimal generator of a C_0–semigroup S on X. If f is Lipschitz continuous on $[0,T]$, then for every $u_0\in D(A)$ the initial value problem (2.3.113) admits a unique strong solution u on $[0,T]$, given by*

$$u(t)=S(t)u_0+\int_0^t S(t-s)f(s)\,ds.$$

2.4 The Nonlinear Problem

Let X be a reflexive Banach space. Consider the initial value problem

$$\begin{cases}\dfrac{du}{dt}(t)=Au(t)+F(u(t)), & t>0,\\ u(0)=u_0,\end{cases}\tag{2.4.125}$$

where $F:X\to X$ is a continuous function and A is the infinitesimal generator of a C_0–semigroup $S(t)$ such that $\|S(t)\|\leq M$ for all $t\geq 0$. If u is a classical or strong solution of (2.4.125), then, as in the previous section, it is not difficult to verify that u satisfies the integral equation

$$u(t)=S(t)u_0+\int_0^t S(t-s)F(u(s))\,ds.\tag{2.4.126}$$

We have the following result.

Theorem 2.23 *Let $F : X \to X$ be a Lipschitz function, that is,*

$$\|F(u) - F(v)\|_X \leq L\|u - v\|_X, \quad \forall u, v \in X.$$

Then, for every $u_0 \in X$, there exists a unique function $u \in C^0([0, +\infty); X)$ which is a generalized solution.
i) If $u_0, v_0 \in X$ are initial data for (2.4.125), then the corresponding generalized solutions u and v satisfy

$$\|u(t) - v(t)\|_X \leq Me^{LMt}\|u_0 - v_0\|_X. \tag{2.4.127}$$

ii) If $u_0 \in D(A)$, then the solution is strong on the interval $[0, T]$, for every $T > 0$.

Proof: (i) Let $u_0 \in X$. For each $k > 0$, define

$$X_k = \big\{u \in C^0([0, +\infty); X) \; ; \; \|u(t)\|_X \leq Ce^{kt} \text{ for some } C > 0 \text{ and all } t \geq 0\big\}.$$

By Proposition 1.17, X_k is a Banach space endowed with the norm

$$\|u\|_{X_k} = \sup_{t \geq 0} e^{-kt}\|u(t)\|_X.$$

Define $\phi : X_k \to X_k$ by

$$\phi u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\, ds.$$

We claim that $\phi(X_k) \subset X_k$. Indeed, we first show that, for each $u \in X_k$, the function ϕu is continuous. We already know that $S(t)u_0$ is continuous (see Corollary 1.23), so it remains to show that

$$g(t) = \int_0^t S(t-s)F(u(s))\, ds = \int_0^t S(s)F(u(t-s))\, ds$$

is continuous on $[0, \infty)$. Let $(t_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+$ such that $t_n \to t_0$ as $n \to \infty$ and $t_n \geqslant t_0$ for all $n \in \mathbb{N}$. Then

$$\|g(t_n) - g(t_0)\| \;=\; \left\|\int_0^{t_0} S(s)\big(F(u(t_n - s)) - F(u(t_0 - s))\big)\, ds + \int_{t_0}^{t_n} S(s)F(u(t_n - s))\, ds\right\|.$$

Using the triangle inequality, the boundedness of the semigroup and the fact that F is Lipschitz, we obtain

$$\|g(t_n) - g(t_0)\| \;\leq\; ML\int_0^{t_0} \|u(t_n - s) - u(t_0 - s)\|\, ds + M\int_{t_0}^{t_n} \|F(u(t_n - s))\|\, ds.$$

Choose $T > 0$ so that $t_0, t_n \in (0, T)$ and, consequently, $t_n - s, t_0 - s \in (0, T)$. Then

$$\|g(t_n) - g(t_0)\| \leq M\left(L\int_0^{t_0} \|u(t_n - s) - u(t_0 - s)\|\, ds + \int_{t_0}^{T} \chi_{[t_0,t_n]}(s)\|F(u(t_n - s))\|\, ds\right), \tag{2.4.128}$$

where $\chi_{[t_0,t_n]}$ denotes the characteristic function of the interval $[t_0, t_n]$.

Now define, for $s \in (t_0, T)$,

$$f_n(s) = \chi_{[t_0,t_n]}(s)\|F(u(t_n - s))\| \quad \text{and} \quad f(s) = \chi_{[t_0,t_0]}(s)\|F(u(t_0 - s))\|.$$

Since $\chi_{[t_0,t_0]}(s) = 0$ almost everywhere and $\|F(u(t_n - s))\|$ is bounded, it follows that $f_n(s) \to 0$ almost everywhere as $n \to \infty$. Hence, by the Dominated Convergence Theorem,

$$\int_{t_0}^{T} f_n(s)\, ds \to 0,$$

that is,

$$\int_{t_0}^{T} \chi_{[t_0,t_n]}(s) \|F(u(t_n - s))\|\, ds \to 0.$$

Moreover, since u is continuous, as $n \to \infty$ we have

$$\|u(t_n - s) - u(t_0 - s)\| \to 0.$$

Furthermore, because $u \in X_k$, we have

$$\|u(t_n - s) - u(t_0 - s)\| \leq 2Ce^{kt}.$$

Thus, again by the Dominated Convergence Theorem,

$$\int_{0}^{t_0} \|u(t_n - s) - u(t_0 - s)\|\, ds \to 0,$$

as $n \to \infty$.

Therefore, the right-hand side of (2.4.128) converges to zero as $t_n \to t_0$ with $t_n \geqslant t_0$ for all $n \in \mathbb{N}$, which implies that g is right-continuous on $[0, T]$. Since $T > 0$ is arbitrary, it follows that g is right-continuous on $[0, \infty)$ and, consequently, that ϕu is right-continuous on $[0, \infty)$. A similar argument shows that ϕu is left-continuous on $[0, \infty)$. Hence ϕu is continuous, that is, $\phi u \in C([0, \infty), X)$.

It remains to show that $\phi u \in X_k$. For $u \in X_k$ we have

$$\begin{aligned}
\|\phi u(t)\| &\leq \|S(t)u_0\| + \int_0^t \|S(t-s)\| \|F(u(s))\|\, ds \\
&\leq M\|u_0\| + M \int_0^t \|F(u(s)) - F(0)\|\, ds + M \int_0^t \|F(0)\|\, ds \\
&\leq M\|u_0\| + ML \int_0^t \|u(s)\|\, ds + M\|F(0)\|t \\
&\leq M\|u_0\| + MLC \int_0^t e^{ks}\, ds + M\|F(0)\|t \\
&\leq M\|u_0\| + MLC\frac{e^{kt} - 1}{k} + M\|F(0)\|t, \quad \forall t \geq 0,
\end{aligned}$$

and, for $k > 0$,

$$\begin{aligned}
e^{-kt}\|\phi u(t)\| &\leq Me^{-kt}\|u_0\| + MLC\frac{1 - e^{-kt}}{k} + M\|F(0)\|\frac{t}{e^{kt}} \\
&\leq M\|u_0\| + \frac{MLC}{k} + MM'\|F(0)\| < \infty, \quad \forall t \geq 0,
\end{aligned}$$

where $C > 0$ is a constant depending on u and $M' > 0$ is such that $\left|\frac{t}{e^{kt}}\right| \leq M'$ for all $t \in \mathbb{R}$. Thus $\sup_{t\geq 0} e^{-kt}\|\phi u(t)\| < \infty$, and therefore $\phi u \in X_k$, as claimed.

We now show that $\phi : X_k \to X_k$ is $\dfrac{ML}{k}$–Lipschitz continuous. Indeed, if $u, v \in X_k$, then

$$\begin{aligned} e^{-kt}\|\phi u(t) - \phi v(t)\| &\leq Me^{-kt}\int_0^t \|F(u(s)) - F(v(s))\|\,ds \\ &\leq MLe^{-kt}\int_0^t \|u(s) - v(s)\|\,ds \\ &\leq MLe^{-kt}\int_0^t \|u(s) - v(s)\|e^{-ks}e^{ks}\,ds \\ &\leq \frac{ML}{k}e^{-kt}(e^{kt}-1)\|u-v\|_{X_k} \\ &\leq \frac{ML}{k}\|u-v\|_{X_k}, \quad \forall t \geq 0, \end{aligned}$$

and hence

$$\|\phi u - \phi v\|_{X_k} = \sup_{t\geq 0} e^{-kt}\|\phi u(t) - \phi v(t)\| \leq \frac{ML}{k}\|u - v\|_{X_k}.$$

Thus, when $k = 2ML$ we have that $\phi : X_k \to X_k$ is a contraction, that is,

$$\|\phi u - \phi v\|_{X_k} \leq c\|u - v\|_{X_k},$$

with $c = \frac{1}{2} < 1$. Hence, by the Banach fixed point theorem there exists a unique fixed point of ϕ, i.e., there exists $u \in X_k$ such that

$$u(t) = S(t)u_0 - \int_0^t S(t-s)F(u(s))\,ds,$$

which proves the existence of a generalized solution of (2.4.125).

Let u and v be generalized solutions of (2.4.125) corresponding to the initial data u_0 and v_0, respectively. Then, from (2.4.126) we obtain

$$\begin{aligned} \|u(t) - v(t)\|_X &= \left\| S(t)u_0 + \int_0^t S(t-s)F(u(s))\,ds - \left(S(t)v_0 + \int_0^t S(t-s)F(v(s))\,ds\right)\right\| \\ &\leq \|S(t)(u_0 - v_0)\| + \int_0^t \|S(t-s)\|\|F(u(s)) - F(v(s))\|\,ds \\ &\leq M\|u_0 - v_0\| + ML\int_0^t \|u(s) - v(s)\|\,ds, \end{aligned}$$

and, by Gronwall's lemma,

$$\|u(t) - v(t)\| \leq Me^{MLt}\|u_0 - v_0\| \quad \text{for all } t \in [0, T],$$

for every given $T > 0$, which proves (2.4.127) as well as the uniqueness of generalized solutions.

(ii) Now suppose that $u_0 \in D(A)$. We shall prove that u is Lipschitz continuous on $[0, T]$ for every $T > 0$, which implies that $F(u(t))$ is also Lipschitz continuous. Then, by Corollary 2.22, we conclude that u is a strong solution. Indeed, let $h > 0$ and define

$$v(t) = u(t+h), \quad \forall t \geq 0. \tag{2.4.129}$$

Note that v is a generalized solution of (2.4.125) with initial data $v_0 = u(h)$. From (2.4.127) and (2.4.129) we have

$$\|u(t+h) - u(t)\| \leq Me^{LMt}\|u(h) - u(0)\|. \tag{2.4.130}$$

On the other hand, from (2.4.126) we can write

$$u(h) = S(h)u_0 + \int_0^h S(h-s)F(u(s))\,ds. \tag{2.4.131}$$

From (2.4.131) we obtain

$$\begin{aligned}
\|u(h)-u(0)\| &= \left\| S(h)u_0 - u_0 + \int_0^h S(h-s)Fu(s)ds \right\| \\
&= \left\| S(h)u_0 - u_0 + \int_0^h [S(h-s)Fu(s) - S(h-s)Fu(0) + S(h-s)Fu(0)]\,ds \right\| \\
&\le \|S(h)u_0 - u_0\| + \| \int_0^h [S(h-s)\,(Fu(s)-Fu(0)) + S(h-s)Fu(0)]\,ds\| \\
&\le \|S(h)u_0 - u_0\| + \int_0^h \|S(h-s)\|\|Fu(s)-Fu(0)\|ds \\
&\quad + \int_0^h \|S(h-s)\|\|Fu(0)\|ds \\
&\le \|S(h)u_0 - u_0\| + ML\int_0^h \|u(s)-u(0)\|ds + Mh\|F(u(0))\|
\end{aligned} \tag{2.4.132}$$

Since $u_0 \in D(A)$, we have

$$S(h)u_0 - u_0 = A\int_0^h S(s)u_0\,ds = \int_0^h S(s)Au_0\,ds,$$

and hence

$$\|S(h)u_0 - u_0\| \le \int_0^h \|S(s)\|\|Au_0\|\,ds \le M\|Au_0\|\,h. \tag{2.4.133}$$

Combining (2.4.132) and (2.4.133) we obtain

$$\|u(h)-u_0\| \le Mh\|Au_0\| + Mh\|F(u_0)\| + ML\int_0^h \|u(s)-u(0)\|\,ds,$$

and by Gronwall's lemma

$$\|u(h)-u_0\| \le Mh(\|Au_0\| + \|F(u_0)\|)e^{MLh}. \tag{2.4.134}$$

Thus, from (2.4.130) and (2.4.134) we conclude that

$$\|u(t+h)-u(t)\| \le Me^{MLt}\,Me^{MLh}(\|Au_0\| + \|F(u_0)\|)\,h, \quad \forall t \ge 0,\ \forall h > 0. \tag{2.4.135}$$

Now let $T > 0$ be given and take $t, t' \in [0,T]$. From (2.4.135) it follows that

$$\|u(t)-u(t')\| \le Me^{2MT}(\|Au_0\| + \|F(u_0)\|)\,|t-t'|,$$

which proves that u is Lipschitz continuous on $[0,T]$, and since $T > 0$ is arbitrary, on any bounded interval. This implies that $F(u(t))$ is also Lipschitz continuous and, in view of Corollary 2.22, u is a strong solution of (2.4.125) on $[0,T]$, which completes the proof. □

Theorem 2.24 *Let $F : D(A) \to D(A)$ be a Lipschitz continuous function. Then, for every $u_0 \in D(A)$ there exists a classical solution of (2.4.125) on $[0,T]$, for every given $T > 0$.*

Proof: Set

$$X_1 = D(A),$$

and

$$A_1 = A|_{D(A^2)} : D(A_1) = D(A^2) \subset X_1 \to X_1.$$

Then $S_1(t)$, the semigroup generated by A_1, is the restriction of $S(t)$ to $D(A)$. Hence, by Theorem 2.4, there exists a generalized solution $u \in C^0([0, +\infty); X_1)$ such that

$$u(t) = S_1(t)u_0 + \int_0^t S_1(t-s)F(u(s))\, ds. \tag{2.4.136}$$

Since $u_0 \in D(A)$ and $F(u(s)) \in D(A)$, we may replace $S_1(t)$ by $S(t)$ in (2.4.136), so that $u \in C^0([0, +\infty); D(A))$. Moreover, since $F : D(A) \to D(A)$ is Lipschitz continuous, we have

$$g(\cdot) = F(u(\cdot)) \in C^0([0, +\infty); D(A)) \hookrightarrow L^1(0, T; D(A)), \tag{2.4.137}$$

for any fixed $T > 0$. In particular,

$$g(s) \in D(A) \quad \forall s \in [0, T], \qquad \text{and} \qquad Ag \in L^1(0, T; X). \tag{2.4.138}$$

Taking into account (2.4.136), (2.4.137), (2.4.138) and Proposition 2.17, we conclude that u is a classical solution of (2.4.125). □

Theorem 2.25 *Let $F : X \to X$ be a locally Lipschitz function, that is, for every $R > 0$ there exists $L_R \geq 0$ such that $\|u\| \leq R$ and $\|v\| \leq R$ imply*

$$\|F(u) - F(v)\| \leq L_R \|u - v\|.$$

(i) Then, for every $u_0 \in X$ there exists a function $u \in C^0([0, +\infty); X)$ which is a generalized solution of (2.4.125) on $[0, T]$, and which can be extended to a maximal solution on $[0, T_{max})$, where either $T_{max} = +\infty$ or $T_{max} < +\infty$ and $\lim_{t \to T_{max}^-} \|u(t)\| = +\infty$.
(ii) If $u_0 \in D(A)$, then the solution is strong.

Proof: (i) For each $T > 0$, define

$$K_T = \{u \in C^0([0, T]; X) \; ; \; \|u(t)\| \leq M\|u_0\| + 1 \; \forall t \in [0, T]\}. \tag{2.4.139}$$

Note that K_T is closed, since it is a closed ball in $C([0, T], X)$ and, by Proposition 1.8, $C([0, T], X)$ is a Banach space. Hence K_T is also a Banach space.

Define further the map $\Phi : K_T \to C^0([0, T]; X)$ by

$$\Phi u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\, ds. \tag{2.4.140}$$

Let $R = M\|u_0\| + 1$ and $u \in K_T$. Then, by hypothesis, there exists $L = L(\|u_0\|) > 0$ such that

$$\|F(u(t)) - F(u_0)\| \leq L\|u(t) - u_0\|, \quad \forall t \in [0, T], \tag{2.4.141}$$

since $\|u(t)\| \leq M\|u_0\| + 1$. Hence

$$\begin{aligned}
\|\Phi u(t)\| &\leq \|S(t)u_0\| + \int_0^t \|S(t-s)\|\|F(u(s))\|\,ds \\
&\leq M\|u_0\| + M\int_0^t \|F(u(s)) - F(u_0) + F(u_0)\|\,ds \\
&\leq M\|u_0\| + M\int_0^t \|F(u(s)) - F(u_0)\|\,ds + M\int_0^t \|F(u_0)\|\,ds \\
&\leq M\|u_0\| + ML\int_0^t \|u(s) - u_0\|\,ds + MT\|F(u_0)\| \\
&\leq M\|u_0\| + MLT(M\|u_0\| + 1 + \|u_0\|) + MT\|F(u_0)\|.
\end{aligned}$$

Choosing

$$0 < T^* < \frac{1}{ML(M\|u_0\| + 1 + \|u_0\|) + M\|F(u_0)\|},$$

we see that $\|\Phi u(t)\| \leq M\|u_0\| + 1$ for every $t \in [0, T^*]$. Thus $\Phi(K_{T^*}) \subset K_{T^*}$.

Next, we show that for T sufficiently small, Φ is a contraction. Indeed, for $u, v \in K_{T^*}$ and $R = M\|u_0\| + 1$ there exists $L = L(\|u_0\|) > 0$ such that

$$\|F(u(t)) - F(v(t))\| \leq L\|u(t) - v(t)\|, \quad \forall u, v \in K_{T^*},\ \forall t \in [0, T^*]. \tag{2.4.142}$$

Hence, for $0 < T \leq T^*$,

$$\begin{aligned}
\|\Phi u(t) - \Phi v(t)\| &= \left\| \int_0^t S(t-s)\big(F(u(s)) - F(v(s))\big)\,ds \right\| \\
&\leq M\int_0^T \|F(u(s)) - F(v(s))\|\,ds \\
&\leq MLT\|u - v\|_{C^0([0,T];X)}, \quad \forall u, v \in K_{T^*},\ \forall t \in [0, T].
\end{aligned}$$

Thus, if we choose $0 < T < \dfrac{1}{2ML}$, then Φ is a contraction. Setting

$$T_0 = \min\left\{\frac{T^*}{2}, \frac{1}{2ML}\right\},$$

we conclude that Φ has a unique fixed point, which is a generalized solution of (2.4.125) on $[0, T_0]$.

Let u_1 be the generalized solution of

$$\begin{cases} \dfrac{du_1}{dt}(t) = Au_1(t) + F(u_1(t)) \quad \text{on } [0, T_0], \\ u_1(0) = u_0, \end{cases} \tag{2.4.143}$$

mentioned above. Since $u_1 \in K_{T_0}$, it follows that $u_1 \in C^0([0, T_0]; X)$ and $\|u_1(t)\| \leq M\|u_0\| + 1$ for all $t \in [0, T_0]$.

Now consider the problem

$$\begin{cases} \dfrac{dv_1}{dt}(t) = Av_1(t) + F(v_1(t)) \quad \text{on } [0, T], \\ v_1(0) = u_1(T_0), \end{cases} \tag{2.4.144}$$

and arguing in the same way as for (2.4.143), we find $T_1 > 0$ such that (2.4.144) admits a generalized solution v_1 on $[0, T_1]$.

Observe that, by Theorem 2.4, if $u_0 \in D(A)$ then u_1 is a strong solution of (2.4.125) on $[0, T_0]$. Likewise, v_1 is a strong solution of the same problem on $[0, T_1]$.

Define

$$u_2(t) = \begin{cases} u_1(t), & t \in [0, T_0], \\ v_1(t - T_0), & t \in [T_0, T_0 + T_1]. \end{cases}$$

Set $T_0^* = T_0$ and $T_1^* = T_0 + T_1$. We show that u_2 is a generalized solution of (2.4.125) on $[0, T_1^*]$.

Note that

$$u_1(t) = S(t)u_0 + \int_0^t S(t-s)F(u_1(s))\,ds, \quad \forall t \in [0, T_0],$$

and

$$v_1(t) = S(t)u_1(T_0) + \int_0^t S(t-s)F(v_1(s))\,ds, \quad \forall t \in [0, T_1].$$

If $0 \le t \le T_0$, then

$$\begin{aligned} u_2(t) &= u_1(t) = S(t)u_0 + \int_0^t S(t-s)F(u_1(s))\,ds \\ &= S(t)u_0 + \int_0^t S(t-s)F(u_2(s))\,ds. \end{aligned}$$

Thus u_2 is a generalized solution of (2.4.125) on $[0, T_0]$. Now, if $T_0 \le t \le T_0 + T_1$, then

$$\begin{aligned} u_2(t) &= v_1(t-T_0) = S(t-T_0)u_1(T_0) + \int_0^{t-T_0} S(t-T_0-s)F(v_1(s))\,ds \\ &= S(t-T_0)\left[S(T_0)u_0 + \int_0^{T_0} S(T_0-s)F(u_1(s))\,ds\right] \\ &\quad + \int_0^{t-T_0} S(t-T_0-s)F(v_1(s))\,ds \\ &= S(t-T_0)S(T_0)u_0 + \int_0^{T_0} S(t-T_0)S(T_0-s)F(u_1(s))\,ds \\ &\quad + \int_0^{t-T_0} S(t-T_0-s)F(v_1(s))\,ds \\ &= S(t)u_0 + \int_0^{T_0} S(t-s)F(u_1(s))\,ds \\ &\quad + \int_{T_0}^{t} S(t-T_0-w+T_0)F(v_1(w-T_0))\,dw \\ &= S(t)u_0 + \int_0^{T_0} S(t-s)F(u_1(s))\,ds + \int_{T_0}^{t} S(t-s)F(v_1(s-T_0))\,ds \\ &= S(t)u_0 + \int_0^{T_0} S(t-s)F(u_2(s))\,ds + \int_{T_0}^{t} S(t-s)F(u_2(s))\,ds \\ &= S(t)u_0 + \int_0^{t} S(t-s)F(u_2(s))\,ds, \end{aligned} \tag{2.4.145}$$

and the claim follows. Thus, for the problem (2.4.125) with initial data u_0, we have

u_1 is a generalized solution of (2.4.125) on $[0, T_0^*]$,
u_2 is a generalized solution of (2.4.125) on $[0, T_1^*]$.

Proceeding in this way, we obtain a family of functions $\{u_i\}_{i \in I}$ and a collection of numbers $\{T_{i-1}^*\}_{i \in I}$ such that

$$u_i \text{ is a generalized solution of (2.4.125) on } [0, T^*_{i-1}],$$

where I is a subset of the natural numbers.

Set

$$[0, T_{max}) = \bigcup_{i \in I} [0, T^*_{i-1}[.$$

We now define a function u with values in X and domain $[0, T_{max}[$ as follows: given $i \in I$, we define u on $[0, T^*_{i-1}[$ as the restriction of u_j to $[0, T^*_{i-1}[$ for any $j \geq i$. This makes sense because if $k, j \geq i$, then u_k and u_j coincide on $[0, T^*_{i-1}[$.

We now prove that u is the unique generalized solution of (2.4.125) on $[0, T_{max})$. By definition, $u \in C^0([0, T_{max}[; X)$. Given $t \in [0, T_{max}[$, there exists $i \in I$ such that $T^*_{i-1} \leq t \leq T^*_i$ and, by induction, by arguments analogous to those used in (2.4.145), we obtain

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\, ds,$$

which shows that u is a generalized solution of (2.4.125) on $[0, T_{max}[$. To prove uniqueness, suppose there exists another function v which is a generalized solution of (2.4.125) on $[0, T_{max}[$. In particular, for each $i \in I$ we have that v satisfies

$$v(t) = S(t)u_0 + \int_0^t S(t-s)F(v(s))\, ds, \quad \forall t \in [T^*_{i-1}, T^*_i].$$

But, by Theorem 2.4, u_i is the unique generalized solution of (2.4.125) on $[T^*_{i-1}, T^*_i]$, hence $v = u_i$ on $[T^*_{i-1}, T^*_i]$ for each $i \in I$. Thus $u = v$.

It remains to show that

$$T_{max} = +\infty \text{ or, if } T_{max} < +\infty, \text{ then } \lim_{t \to T^-_{max}} \|u(t)\| = +\infty.$$

Indeed, suppose, by contradiction, that

$$T_{max} < \infty \quad \text{and} \quad \lim_{t \to T^-_{max}} \|u(t)\| < \infty.$$

Then

$$\|u(t)\| \leq C, \quad \forall t \in [0, T_{max}], \tag{2.4.146}$$

for some $C > 0$.

Consider, in view of (2.4.146), the solution v of the problem

$$\begin{cases} \dfrac{dv}{dt}(t) = Av(t) + F(v(t)), \\ v(0) = u(T_{max}) = \lim\limits_{t \to T^-_{max}} u(t), \end{cases}$$

and set

$$w(t) = \begin{cases} u(t), & t \in [0, T_{max}], \\ v(t - T_{max}), & t \in [T_{max}, T_{max} + \delta], \end{cases} \qquad \delta > 0.$$

Then w is a generalized solution of (2.4.125) which extends the maximal solution u, a contradiction.

(ii) Let u be the generalized solution of (2.4.125) on $[0, T_{max}[$ and write $u = u_i$ for its restriction to $[T^*_{i-1}, T^*_i]$. As in part (i), if $u_0 \in D(A)$ then each u_i is a strong solution of (2.4.125) on $[T^*_{i-1}, T^*_i]$. By the arbitrariness of $i \in I$, it follows that u is a strong solution of (2.4.125). □

Theorem 2.26 *Assume that $\|S(t)\| \leq 1$. Let $F : D(A) \to D(A)$ be a locally Lipschitz function. Given $u_0 \in D(A)$, there exists $u \in C^1([0, T_{max}), X) \cap C^0([0, T_{max}), D(A))$; moreover, u is a classical solution on $[0, T_{max})$, and either $T_{max} = +\infty$ or $T_{max} < +\infty$ and $\lim_{t \to T_{max}} (\|u(t)\| + \|Au(t)\|) = +\infty$.*

Proof: In Theorem 2.25, consider $X = D(A)$ endowed with the graph norm. Then, for $u_0 \in D(A)$, we have that

$$u(t) = \begin{cases} u_1(t), & t \in [0, T_1), \\ u_2(t - T_1), & t \in [T_1, T_2), \\ \vdots \\ u_n(t - T_{n-1}), & t \in [T_{n-1}, T_n), \\ \vdots \end{cases} \tag{2.4.147}$$

is a strong solution of (2.4.125), with $u \in C^0([0, T_{max}), D(A))$ and $T_{max} = +\infty$ or $T_{max} < +\infty$ and $\lim_{t \to T_{max}} (\|u(t)\| + \|Au(t)\|) = +\infty$. Recall that u_i is a classical solution of (2.4.125) on $[0, T_i]$ with $u_i(0) = u_{i-1}(T_{i-1})$ and $T_0 = 0$.

It remains to show that $u \in C^1([0, T_{max}), X)$. Since $u(t) \in D(A)$ for every $t \in [0, T_{max})$ and $u \in C^0([0, T_{max}), D(A))$, we have $u \in C^0([0, T_{max}), X)$. Now observe that, for each interval $[T_{i-1}, T_i)$, the derivative $\frac{du}{dt}$ is continuous. Thus, it suffices to prove continuity at T_i.

Indeed, note that

$$\begin{aligned} \frac{du}{dt}^+(T_i) &= \lim_{h \to 0^+} \frac{u(T_i + h) - u(T_i)}{h} \\ &= \lim_{h \to 0^+} \frac{u_{i+1}(h) - u_{i+1}(0)}{h} \\ &= \frac{du_{i+1}}{dt}^+(0), \end{aligned}$$

and

$$\begin{aligned} \frac{du}{dt}^-(T_i) &= \lim_{h \to 0^-} \frac{u(T_i + h) - u(T_i)}{h} \\ &= \lim_{h \to 0^-} \frac{u_i(T_i + h) - u_i(T_i)}{h} \\ &= \frac{du_i}{dt}^-(T_i). \end{aligned}$$

Hence we must show that

$$\frac{du_{i+1}}{dt}^+(0) = \frac{du_i}{dt}^-(T_i).$$

Since u_i is a classical solution of (2.4.125) on $[0, T_i]$, we have

$$\begin{cases} \frac{du_i}{dt}(t) = Au_i(t) + F(u_i(t)), & t \in [0, T_i], \\ u_i(0) = u_{i-1}(T_{i-1}), \end{cases} \tag{2.4.148}$$

and

$$\begin{cases} \frac{du_{i+1}}{dt}(t) = Au_{i+1}(t) + F(u_{i+1}(t)), & t \in [0, T_{i+1}], \\ u_{i+1}(0) = u_i(T_i). \end{cases} \tag{2.4.149}$$

Thus

$$\frac{du_i}{dt}^-(T_i) = Au_i(T_i) + F(u_i(T_i))$$

and

$$\begin{aligned}\frac{du_{i+1}}{dt}^{+}(0) &= Au_{i+1}(0) + F(u_{i+1}(0)) \\ &= Au_i(T_i) + F(u_i(T_i)) \\ &= \frac{du_i}{dt}^{-}(T_i),\end{aligned}$$

which proves the claim. □

Chapter 3

Evolution Equations

3.1 The Heat Equation

In this section we consider Ω a bounded open subset of $\mathbb{R}^n$ with sufficiently smooth boundary Γ.

3.1.1 Dirichlet boundary condition

Consider the following problem with Dirichlet boundary condition:

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ u = 0 & \text{on} \quad (0,\infty)\times\Gamma, \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{3.1.1}$$

We shall study existence and uniqueness of solutions of (3.1.1), taking the initial datum u_0 in each of the following sets: $H_0^1(\Omega)\cap H^2(\Omega)$, $L^2(\Omega)$, $H_0^1(\Omega)$, and $H^{-1}(\Omega)$.

First case $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$

We rewrite (3.1.1) in the abstract form

$$\begin{cases} u_t = \Delta u, \\ u(0) = u_0, \end{cases} \tag{3.1.2}$$

where

$$\Delta : H_0^1(\Omega)\cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega).$$

We first use the Lumer–Phillips theorem to prove that $\Delta \in G(1,0)$. Indeed:

i) We know that $H_0^1(\Omega)\cap H^2(\Omega)$ is dense in $L^2(\Omega)$;

ii) Δ is dissipative, since

$$(\Delta u, u)_{L^2(\Omega)} = \int_\Omega \Delta u\, u\, dx = -\int_\Omega \nabla u\cdot\nabla u\, dx + \underbrace{\int_\Gamma \frac{\partial u}{\partial \nu} u\, d\Gamma}_{=0} \le 0,$$

for every $u \in H_0^1(\Omega)\cap H^2(\Omega)$;

iii) $Im(I-\Delta) = L^2(\Omega)$. In fact, proving $Im(I-\Delta) = L^2(\Omega)$ is equivalent to proving that, for each $f \in L^2(\Omega)$, the problem $u - \Delta u = f$ has a solution $u \in H_0^1(\Omega)\cap H^2(\Omega)$. To prove this, we use the

Lax–Milgram lemma. Define

$$a(u,v)=\int_\Omega (\nabla u\cdot\nabla v+uv)\,dx,\quad \forall u,v\in H_0^1(\Omega),$$

which is clearly bilinear. This bilinear form is continuous, since

$$\begin{aligned}
|a(u,v)| &= \left|\int_\Omega (\nabla u\cdot\nabla v+uv)\,dx\right| \\
&\le \int_\Omega |\nabla u||\nabla v|\,dx+\int_\Omega |u||v|\,dx \\
&\le \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}+\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\le \left(\|u\|^2_{L^2(\Omega)}+\|\nabla u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\left(\|v\|^2_{L^2(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\
&= \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \\
&\le c\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}.
\end{aligned}$$

Moreover, it is coercive, since

$$a(u,u)=\int_\Omega |\nabla u|^2\,dx+\int_\Omega |u|^2\,dx\ge\int_\Omega |\nabla u|^2\,dx=\|u\|^2_{H_0^1(\Omega)}.$$

By the Lax–Milgram lemma, there exists a unique $u\in H_0^1(\Omega)$ such that $a(u,v)=\langle f,v\rangle$ for all $v\in H_0^1(\Omega)$, that is

$$\int_\Omega (\nabla u\cdot\nabla v+uv)\,dx=\int_\Omega fv\,dx,\quad \forall v\in H_0^1(\Omega).$$

From this and the regularity theory for the associated elliptic problem, we obtain $u\in H^2(\Omega)$. Hence, using Green's identity, we see that there exists a unique $u\in H_0^1(\Omega)\cap H^2(\Omega)$ satisfying $u-\Delta u=f$, which proves the claim.

By (i), (ii) and (iii), the operator Δ is m–dissipative with dense domain and, by the Lumer–Phillips theorem, it follows that $\Delta\in G(1,0)$. Thus, if $u_0\in H_0^1(\Omega)\cap H^2(\Omega)$, then by Theorem 2.3, the problem (3.1.2) admits a unique solution

$$u\in C([0,\infty);H_0^1(\Omega)\cap H^2(\Omega))\cap C^1([0,\infty);L^2(\Omega)).$$

Second case $u_0\in L^2(\Omega)$

We first show that $\Delta: H_0^1(\Omega)\cap H^2(\Omega)\subset L^2(\Omega)\to L^2(\Omega)$ is self-adjoint. Indeed, since Δ is m–dissipative, $-\Delta$ is maximal monotone. Moreover, $-\Delta$ is symmetric, because

$$(-\Delta u,v)_{L^2(\Omega)}=(\nabla u,\nabla v)_{L^2(\Omega)}=(u,-\Delta v)_{L^2(\Omega)},\quad \forall u,v\in H_0^1(\Omega)\cap H^2(\Omega),$$

so that $-\Delta$ is self-adjoint and hence $\Delta=\Delta^*$.

Since Δ is m–dissipative and self-adjoint, it follows that Δ generates a differentiable semigroup, by Proposition 2.8. Then, by Theorem 2.5, if $u_0\in L^2(\Omega)$ the problem (3.1.2) has a unique solution in the class

$$u\in C((0,\infty);H_0^1(\Omega)\cap H^2(\Omega))\cap C([0,\infty);L^2(\Omega))\cap C^1((0,\infty);L^2(\Omega)).$$

Third case $u_0\in H_0^1(\Omega)$

We first show that the operator $-\Delta: H_0^1(\Omega)\cap H^2(\Omega)\subset L^2(\Omega)\to L^2(\Omega)$ is defined by the triple $\{H_0^1(\Omega),L^2(\Omega),b(u,v)\}$, where $b(u,v)=(\nabla u,\nabla v)_{L^2(\Omega)}=(u,v)_{H_0^1(\Omega)}$ is a sesquilinear form, continuous and coercive on $H_0^1(\Omega)$.

Let A be the operator defined by the triple $\{H_0^1(\Omega), L^2(\Omega), b(u,v)\}$. We shall prove that

$$D(-\Delta) = D(A) \quad \text{and} \quad Au = -\Delta u, \quad \forall u \in D(-\Delta).$$

Let $u \in D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$. Since

$$D(A) = \{u \in H_0^1(\Omega);\ \exists f \in L^2(\Omega) \text{ such that } b(u,v) = (f,v)_{L^2(\Omega)},\ \forall v \in H_0^1(\Omega)\},$$

we must exhibit $f \in L^2(\Omega)$ such that $b(u,v) = (f,v)_{L^2(\Omega)}$ for all $v \in H_0^1(\Omega)$. Taking $f = -\Delta u \in L^2(\Omega)$, we obtain the desired identity, since

$$b(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)} = (-\Delta u, v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

Thus $u \in D(A)$ and $-\Delta u = Au$.

Conversely, if $u \in D(A)$, there exists $f \in L^2(\Omega)$ such that

$$(f,v)_{L^2(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

In particular,

$$(f,\varphi)_{L^2(\Omega)} = (\nabla u, \nabla \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

and, using Green's identity, we deduce that $f = -\Delta u$ in $\mathcal{D}'(\Omega)$. As $f \in L^2(\Omega)$, it follows that $-\Delta u \in L^2(\Omega)$. Therefore, u satisfies the problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \end{cases} \tag{3.1.3}$$

and hence $u \in H^2(\Omega)$. Thus $u \in H_0^1(\Omega) \cap H^2(\Omega) = D(-\Delta)$. We conclude that the operator $-\Delta : H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is defined by the triple $\{H_0^1(\Omega), L^2(\Omega), b(u,v)\}$.

Consider the following chain of continuous and dense embeddings:

$$H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow (H_0^1(\Omega) \cap H^2(\Omega))',$$

where we identify $L^2(\Omega)$ with its topological dual.

Since the bilinear form $b(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)}$ is coercive, the operator $-\Delta$, defined by the triple $\{H_0^1(\Omega), L^2(\Omega), b(u,v)\}$, admits an extension

$$\begin{array}{rccc} -\tilde{\Delta}\ : & H_0^1(\Omega) & \longrightarrow & H^{-1}(\Omega) \\ & u & \longmapsto & -\tilde{\Delta} u, \end{array}$$

where $-\tilde{\Delta} u : H_0^1(\Omega) \to \mathbb{C}$ is defined by

$$\langle -\tilde{\Delta} u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega)} = (u,v)_{H_0^1(\Omega)}.$$

This extension is a bijection and, endowing $H^{-1}(\Omega)$ with the inner product

$$(x,y)_{H^{-1}(\Omega)} = (-\tilde{\Delta}^{-1} x, -\tilde{\Delta}^{-1} y)_{H_0^1(\Omega)} = (\tilde{\Delta}^{-1} x, \tilde{\Delta}^{-1} y)_{H_0^1(\Omega)},$$

we obtain $\|\tilde{\Delta} u\|_{H^{-1}(\Omega)} = \|u\|_{H_0^1(\Omega)}$ for all $u \in H_0^1(\Omega)$.

Now consider the problem

$$\begin{cases} u_t - \tilde{\Delta} u = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ u = 0 & \text{on} \quad (0,\infty) \times \Gamma, \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{3.1.4}$$

We now prove that $\tilde{\Delta} \in G(1,0)$.

i) We know that $H_0^1(\Omega)$ is dense in $H^{-1}(\Omega)$;

ii) We prove that $\tilde{\Delta}$ is dissipative. Let $u \in D(\Delta)$. Then

$$\begin{aligned}
(\tilde{\Delta}u, u)_{H^{-1}(\Omega)} &= (\tilde{\Delta}^{-1}\tilde{\Delta}u, \tilde{\Delta}^{-1}u)_{H_0^1(\Omega)} = (u, \tilde{\Delta}^{-1}u)_{H_0^1(\Omega)} \\
&= (\nabla u, \nabla\tilde{\Delta}^{-1}u)_{L^2(\Omega)} \\
&= \int_\Omega \nabla u \cdot \nabla\tilde{\Delta}^{-1}u\,dx = -\int_\Omega u\,\Delta\tilde{\Delta}^{-1}u\,dx \\
&= -\int_\Omega u^2\,dx \le 0.
\end{aligned}$$

Now let $u \in D(\tilde{\Delta})$. Then there exists $\{u_n\} \subset D(\Delta)$ such that $u_n \to u$ in $D(\tilde{\Delta})$. From the previous computation we have

$$(\tilde{\Delta}u_n, u_n)_{H^{-1}(\Omega)} \le 0 \quad \forall n \in \mathbb{N}.$$

Observe that

$$\begin{aligned}
|(\tilde{\Delta}u_n, u_n)_{H^{-1}(\Omega)} &- (\tilde{\Delta}u, u)_{H^{-1}(\Omega)}| \\
&= |(\tilde{\Delta}u_n, u_n)_{H^{-1}(\Omega)} - (\tilde{\Delta}u, u_n)_{H^{-1}(\Omega)} + (\tilde{\Delta}u, u_n)_{H^{-1}(\Omega)} - (\tilde{\Delta}u, u)_{H^{-1}(\Omega)}| \\
&\le |(\tilde{\Delta}u_n - \tilde{\Delta}u, u_n)_{H^{-1}(\Omega)}| + |(\tilde{\Delta}u, u_n - u)_{H^{-1}(\Omega)}| \\
&\le \|\tilde{\Delta}u_n - \tilde{\Delta}u\|_{H^{-1}(\Omega)}\,\|u_n\|_{H^{-1}(\Omega)} + \|\tilde{\Delta}u\|_{H^{-1}(\Omega)}\,\|u_n - u\|_{H^{-1}(\Omega)} \\
&\le c\|\tilde{\Delta}u_n - \tilde{\Delta}u\|_{H^{-1}(\Omega)}\,\|u_n\|_{H_0^1(\Omega)} + c\|\tilde{\Delta}u\|_{H^{-1}(\Omega)}\,\|u_n - u\|_{H_0^1(\Omega)} \to 0,
\end{aligned}$$

where $c > 0$ is the constant in the embedding $H_0^1(\Omega) \hookrightarrow H^{-1}(\Omega)$ and we have used that $\tilde{\Delta}$ is a bijective isometry. From this convergence we conclude that $(\tilde{\Delta}u, u)_{H^{-1}(\Omega)} \le 0$.

iii) We prove that $Im(I - \tilde{\Delta}) = H^{-1}(\Omega)$. Let $f \in H^{-1}(\Omega)$ be given. Consider again

$$a(u,v) = \int_\Omega (uv + \nabla u \cdot \nabla v)\,dx, \quad \forall u, v \in H_0^1(\Omega),$$

which is a bilinear, continuous and coercive form. By the Lax–Milgram lemma, there exists a unique $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in H_0^1(\Omega).$$

Using

$$\langle -\tilde{\Delta}u, v\rangle = (\nabla u, \nabla v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega),$$

we obtain

$$\langle u - \tilde{\Delta}u, v\rangle = \langle f, v\rangle, \quad \forall v \in H_0^1(\Omega),$$

which yields the desired conclusion.

Therefore, $\tilde{\Delta}$ is m–dissipative with dense domain and, by the Lumer–Phillips theorem, $\tilde{\Delta} \in G(1,0)$. Thus, when $u_0 \in H_0^1(\Omega)$, Theorem 2.3 implies that the problem (3.1.4) has a unique solution

$$u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); H^{-1}(\Omega)).$$

Fourth case $u_0 \in H^{-1}(\Omega)$

First note that, if $u, v \in D(\Delta)$, then

$$\begin{aligned}(u, \tilde{\Delta}v)_{H^{-1}(\Omega)} &= (\tilde{\Delta}^{-1}u, \tilde{\Delta}^{-1}\tilde{\Delta}v)_{H_0^1(\Omega)} = (\tilde{\Delta}^{-1}u, v)_{H_0^1(\Omega)} \\ &= (\nabla\tilde{\Delta}^{-1}u, \nabla v)_{L^2(\Omega)} \\ &= -(\Delta\tilde{\Delta}^{-1}u, v)_{L^2(\Omega)} \\ &= -(u, v)_{L^2(\Omega)},\end{aligned}$$

and also

$$\begin{aligned}(\tilde{\Delta}u, v)_{H^{-1}(\Omega)} &= (\tilde{\Delta}^{-1}\tilde{\Delta}u, \tilde{\Delta}^{-1}v)_{H_0^1(\Omega)} = (u, \tilde{\Delta}^{-1}v)_{H_0^1(\Omega)} \\ &= (\nabla u, \nabla\tilde{\Delta}^{-1}v)_{L^2(\Omega)} \\ &= -(u, \tilde{\Delta}\tilde{\Delta}^{-1}v)_{L^2(\Omega)} \\ &= -(u, v)_{L^2(\Omega)},\end{aligned}$$

so that

$$(u, \tilde{\Delta}v)_{H^{-1}(\Omega)} = (\tilde{\Delta}u, v)_{H^{-1}(\Omega)}, \quad \forall u, v \in D(\Delta),$$

and, by density, the same equality holds for all $u, v \in D(\tilde{\Delta})$, which shows that $\tilde{\Delta}$ is symmetric.

Since $\tilde{\Delta}$ is m–dissipative, $-\tilde{\Delta}$ is maximal monotone and, being symmetric, it follows that it is also self-adjoint. Hence $\tilde{\Delta}$ is self-adjoint as well. As $\tilde{\Delta}$ is m–dissipative and self-adjoint, Proposition 2.8 implies that this operator generates a differentiable semigroup. Therefore, by Theorem 2.5, the problem (3.1.4) with initial data $u_0 \in H^{-1}(\Omega)$ has a unique solution

$$u \in C((0,\infty); H_0^1(\Omega)) \cap C([0,\infty); H^{-1}(\Omega)) \cap C^1((0,\infty); H^{-1}(\Omega)).$$

3.1.2 Neumann boundary condition

Let Ω be as at the beginning of the chapter. We now consider the heat equation with Neumann boundary condition

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ \dfrac{\partial u}{\partial\nu} = 0 & \text{on} \quad (0,\infty)\times\Gamma, \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{3.1.5}$$

We shall study existence and uniqueness of solutions to (3.1.5), taking the initial datum u_0 in each of the following sets: $L^2(\Omega)$, $H^1(\Omega)$, $(H^1(\Omega))'$.

First case: $u_0 \in L^2(\Omega)$ We first consider the Laplace operator $\Delta : D(\Delta) \subset L^2(\Omega) \to L^2(\Omega)$ with domain

$$D(\Delta) = \Big\{u \in H^2(\Omega);\ \frac{\partial u}{\partial\nu} = 0 \text{ on } \Gamma\Big\},$$

and we prove that the operator $I - \Delta : D(I-\Delta) \subset L^2(\Omega) \to L^2(\Omega)$, with $D(I-\Delta) = D(\Delta)$, is defined by the triple $\{H^1(\Omega), L^2(\Omega), a(u,v)\}$, where

$$a(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)} + (u,v)_{L^2(\Omega)}, \quad \forall u, v \in H^1(\Omega).$$

Since a is a bilinear and continuous mapping, we know that the triple $\{H^1(\Omega), L^2(\Omega), a(u,v)\}$ defines an operator A. We shall show that

$$D(\Delta) = D(A) \quad \text{and} \quad Au = (I-\Delta)u, \quad \forall u \in D(\Delta).$$

Indeed, let

$$u \in D(A) = \Big\{u \in H^1(\Omega);\ \exists f \in L^2(\Omega) \text{ such that } a(u,v) = (f,v)_{L^2(\Omega)},\ \forall v \in H^1(\Omega)\Big\}.$$

Then there exists $f \in L^2(\Omega)$ such that

$$(\nabla u, \nabla v)_{L^2(\Omega)} + (u,v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)}, \quad \forall v \in H^1(\Omega). \tag{3.1.6}$$

In particular, for $\varphi \in \mathcal{D}(\Omega)$ we have

$$(\nabla u, \nabla \varphi)_{L^2(\Omega)} + (u,\varphi)_{L^2(\Omega)} = (f,\varphi)_{L^2(\Omega)},$$

whence

$$f = -\Delta u + u \quad \text{in } \mathcal{D}'(\Omega).$$

Since $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$, we get $-\Delta u \in L^2(\Omega)$. Then, from (3.1.6) we obtain

$$(\nabla u, \nabla v)_{L^2(\Omega)} + (\Delta u, v)_{L^2(\Omega)} = 0, \quad \forall v \in H^1(\Omega).$$

On the other hand, by the second Green's formula in its general form,

$$(\nabla u, \nabla v)_{L^2(\Omega)} + (\Delta u, v)_{L^2(\Omega)} = \Big\langle \frac{\partial u}{\partial \nu}, v\Big\rangle_{H^{-1/2},H^{1/2}},$$

and thus

$$\Big\langle \frac{\partial u}{\partial \nu}, v\Big\rangle_{H^{-1/2},H^{1/2}} = 0, \quad \forall v \in H^1(\Omega).$$

Since the trace operator is surjective, we deduce $\dfrac{\partial u}{\partial \nu} = 0$ on Γ. Moreover, by the regularity theory for the Neumann problem, we have $u \in H^2(\Omega)$, and therefore $u \in D(\Delta)$.

Conversely, let $u \in D(\Delta)$. We must exhibit $f \in L^2(\Omega)$ such that $a(u,v) = (f,v)_{L^2(\Omega)}$ for all $v \in H^1(\Omega)$. Taking $f = -\Delta u + u \in L^2(\Omega)$ we obtain the desired identity, so $u \in D(A)$. Now, since $u \in D(A)$ we have $a(u,v) = (Au,v)_{L^2(\Omega)}$, and using the second Green's formula again we conclude that

$$u - \Delta u = Au, \quad \forall u \in D(\Delta).$$

Thus, the operator $I - \Delta$ is defined by the triple $\{H^1(\Omega), L^2(\Omega), a(u,v)\}$. Therefore, applying the parabolic case, when $z_0 \in L^2(\Omega)$, the problem

$$\begin{cases} z_t + (I-\Delta)z = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ z(0) = z_0 & \text{in} \quad \Omega \end{cases} \tag{3.1.7}$$

has a unique solution

$$z \in C(]0,\infty[; D(\Delta)) \cap C^0([0,\infty[; L^2(\Omega)) \cap C^1(]0,\infty[; L^2(\Omega)).$$

Setting $u(t) = e^t z(t)$, we see that u is the unique solution of (3.1.5) with $u_0 = u(0) \in L^2(\Omega)$ in the class

$$u \in C(]0,\infty[; D(\Delta)) \cap C^0([0,\infty[; L^2(\Omega)) \cap C^1(]0,\infty[; L^2(\Omega)).$$

Second case $u_0 \in (H^1(\Omega))'$

Since $I - \Delta$ is defined by a triple, we may consider its extension

$$\begin{array}{rcccl} \widetilde{I-\Delta} & : & H^1(\Omega) & \longrightarrow & (H^1(\Omega))' \\ & & u & \longmapsto & \widetilde{I-\Delta}u : H^1(\Omega) \to \mathbb{C}, \end{array}$$

where

$$\langle \widetilde{I-\Delta}u, v\rangle_{(H^1(\Omega))', H^1(\Omega)} = a(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)} + (u,v)_{L^2(\Omega)}.$$

This extension is a bijective isometry, and via it we define an inner product on $(H^1(\Omega))'$ by

$$(u,v)_{(H^1(\Omega))'} = ((\widetilde{I-\Delta})^{-1}u, (\widetilde{I-\Delta})^{-1}v)_{H^1(\Omega)}.$$

We first show that $\widetilde{I-\Delta}$ is maximal monotone. Indeed, let $u \in D(I-\Delta) = D(\Delta)$. Then

$$\begin{aligned}
((\widetilde{I-\Delta})u, u)_{(H^1(\Omega))'} &= ((\widetilde{I-\Delta})^{-1}(\widetilde{I-\Delta})u, (\widetilde{I-\Delta})^{-1}u)_{H^1(\Omega)} \\
&= (u, (\widetilde{I-\Delta})^{-1}u)_{H^1(\Omega)} \\
&= (\nabla u, \nabla(\widetilde{I-\Delta})^{-1}u)_{L^2(\Omega)} + (u, (\widetilde{I-\Delta})^{-1}u)_{L^2(\Omega)} \\
&= (u, -\Delta(\widetilde{I-\Delta})^{-1}u)_{L^2(\Omega)} + \langle \gamma_1 u, \gamma_0(\widetilde{I-\Delta})^{-1}u\rangle_{H^{-1/2}, H^{1/2}} \\
&\quad + (u, (\widetilde{I-\Delta})^{-1}u)_{L^2(\Omega)} \\
&= (u, (I-\Delta)(\widetilde{I-\Delta})^{-1}u)_{L^2(\Omega)} \\
&= \|u\|^2_{L^2(\Omega)} \geq 0, \quad \forall u \in D(\Delta).
\end{aligned}$$

Now let $u \in H^1(\Omega)$; then there exists $\{u_n\} \subset D(\Delta)$ such that $u_n \to u$ in $H^1(\Omega)$. Since $\widetilde{I-\Delta}$ is continuous, we have $(\widetilde{I-\Delta})u_n \to (\widetilde{I-\Delta})u$ in $(H^1(\Omega))'$. As $((\widetilde{I-\Delta})u_n, u_n)_{(H^1(\Omega))'} \to ((\widetilde{I-\Delta})u, u)_{(H^1(\Omega))'}$, we obtain $((\widetilde{I-\Delta})u, u)_{(H^1(\Omega))'} \geq 0$. Hence $\widetilde{I-\Delta}$ is monotone.

Next we prove that $Im(I + (\widetilde{I-\Delta})) = (H^1(\Omega))'$, that is, given $f \in (H^1(\Omega))'$ we must find $u \in H^1(\Omega)$ such that $u + (\widetilde{I-\Delta})u = f$. Consider

$$b(u,v) = (u,v)_{L^2(\Omega)} + a(u,v).$$

Then, by the Lax–Milgram lemma, there exists a unique $u \in H^1(\Omega)$ such that

$$b(u,v) = \langle f, v\rangle_{(H^1(\Omega))', H^1(\Omega)}, \quad \forall v \in H^1(\Omega).$$

It follows that

$$\langle u, v\rangle + \langle (\widetilde{I-\Delta})u, v\rangle = \langle f, v\rangle, \quad \forall v \in H^1(\Omega),$$

which yields the desired conclusion. Therefore $\widetilde{I-\Delta} \in G(1,0)$.

Moreover,

$$\begin{aligned}
((\widetilde{I-\Delta})u, v)_{(H^1(\Omega))'} &= ((\widetilde{I-\Delta})^{-1}(\widetilde{I-\Delta})u, (\widetilde{I-\Delta})^{-1}v)_{H^1(\Omega)} \\
&= (u, (\widetilde{I-\Delta})^{-1}v)_{H^1(\Omega)} \\
&= (\nabla u, \nabla(\widetilde{I-\Delta})^{-1}v)_{L^2(\Omega)} + (u, (\widetilde{I-\Delta})^{-1}v)_{L^2(\Omega)} \\
&= (u, -\Delta(\widetilde{I-\Delta})^{-1}v)_{L^2(\Omega)} + (u, (\widetilde{I-\Delta})^{-1}v)_{L^2(\Omega)} \\
&= (u, (\widetilde{I-\Delta})(\widetilde{I-\Delta})^{-1}v)_{L^2(\Omega)} \\
&= (u, v)_{L^2(\Omega)} \\
&= ((\widetilde{I-\Delta})(\widetilde{I-\Delta})^{-1}u, v)_{L^2(\Omega)} \\
&= ((\widetilde{I-\Delta})^{-1}u, v)_{L^2(\Omega)} + (-\Delta(\widetilde{I-\Delta})^{-1}u, v)_{L^2(\Omega)} \\
&= ((\widetilde{I-\Delta})^{-1}u, v)_{L^2(\Omega)} + (\nabla(\widetilde{I-\Delta})^{-1}u, \nabla v)_{L^2(\Omega)} \\
&= ((\widetilde{I-\Delta})^{-1}u, v)_{H^1(\Omega)}.
\end{aligned}$$

Therefore, $\widetilde{I-\Delta}$ is symmetric for all $u, v \in D(\Delta)$. Again, by density we conclude that $\widetilde{I-\Delta}$ is symmetric for all $u, v \in H^1(\Omega)$. Since this operator is maximal monotone, we have $(\widetilde{I-\Delta})^* = \widetilde{I-\Delta}$. Hence $\widetilde{I-\Delta}$ generates a differentiable semigroup.

Thus, the problem

$$\begin{cases} z_t + (\widetilde{I-\Delta})z = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ z(0) = u_0 & \text{in} \quad \Omega \end{cases} \tag{3.1.8}$$

has a unique solution in the class

$$z \in C((0,\infty); H^1(\Omega)) \cap C([0,\infty); (H^1(\Omega))') \cap C^1((0,\infty); (H^1(\Omega))')$$

whenever $u_0 \in (H^1(\Omega))'$.

The operator

$$\begin{aligned} -\widetilde{\Delta} : H^1(\Omega) &\to (H^1(\Omega))' \\ u &\mapsto -\widetilde{\Delta}u, \quad \text{where } -\widetilde{\Delta}u : H^1((\Omega)) \to \mathbb{C} \text{ is given by} \\ &\qquad\qquad\qquad v \quad \mapsto (\nabla u, \nabla v)_{L^2((\Omega))}, \end{aligned} \tag{3.1.9}$$

is a continuous extension of $-\Delta$. Moreover, $\widetilde{I-\Delta} = \widetilde{I} - \widetilde{\Delta} = I - \widetilde{\Delta}$.

Setting $u(t) = e^t z(t)$, we see that u is the unique solution of

$$\begin{cases} u_t - \widetilde{\Delta}u = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ u(0) = u_0 & \text{in} \quad \Omega \end{cases} \tag{3.1.10}$$

in the class

$$u \in C((0,\infty); H^1(\Omega)) \cap C^0([0,\infty); H^1(\Omega)) \cap C^1((0,\infty); H^1(\Omega)),$$

when $u(0) = u_0 \in (H^1(\Omega))'$.

3.2 Wave equation

3.2.1 Dirichlet boundary condition

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary Γ. Consider the problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ u = 0 & \text{on} \quad (0,\infty) \times \Gamma, \\ u(0) = u_0, \; u_t(0) = v_0 & \text{in} \quad \Omega. \end{cases} \tag{3.2.11}$$

We shall study existence and uniqueness of solutions to (3.2.11), considering the pair of initial data (u_0, u_1) in each of the following spaces:

$$H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega), \quad H_0^1(\Omega) \times L^2(\Omega), \quad \text{and} \quad L^2(\Omega) \times H^{-1}(\Omega).$$

First case: $(u_0, u_1) \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

We first observe that, by what was done in the Third Case in the study of the heat equation, the operator $-\Delta : H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is defined by the triple $\{H_0^1(\Omega), L^2(\Omega), b(u,v)\}$, where

$$b(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)} = (u,v)_{H_0^1}.$$

Thus, by the Hyperbolic Case of Section 2.2, problem (3.2.11) has a unique solution u for $(u_0, u_1) \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$, with

$$u \in C^0([0,+\infty); H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0,+\infty); H_0^1(\Omega)) \cap C^2([0,+\infty); L^2(\Omega)),$$

and, moreover, u satisfies the energy identity

$$\|\nabla u(t)\|^2_{L^2(\Omega)} + \|u'(t)\|^2_{L^2(\Omega)} = \|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}, \quad \forall t \geq 0.$$

Again by the Hyperbolic Case of Section 2.2, problem (3.2.11) has a unique solution u for $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, with

$$u \in C^0([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega)) \cap C^2([0,+\infty); H^{-1}(\Omega)),$$

and, moreover, u satisfies the energy identity

$$\|u(t)\|^2_{L^2(\Omega)} + \|u'(t)\|^2_{H^{-1}(\Omega)} = \|u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{H^{-1}(\Omega)}, \quad \forall t \geq 0.$$

Second case: $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

Following the steps of the Hyperbolic Case in Section 2.2, consider the extension

$$\tilde{\tilde{\Delta}} : L^2(\Omega) \longrightarrow (H_0^1(\Omega) \cap H^2(\Omega))'$$

and

$$\begin{array}{rccc} \tilde{\tilde{B}} : & L^2(\Omega) \times H^{-1}(\Omega) & \longrightarrow & \left(H^{-1}(\Omega) \times (H_0^1(\Omega) \cap H^2(\Omega))'\right) := \tilde{\tilde{X}} \\ & (u, v) & \longmapsto & \tilde{\tilde{B}}(u,v) = (v, \tilde{\tilde{\Delta}} u). \end{array}$$

Once again we have $\tilde{\tilde{B}}^* = -\tilde{\tilde{B}}$ and hence $\tilde{\tilde{B}}$ generates a unitary group. Therefore there exists a unique solution

$$U \in C([0,\infty), D(\tilde{\tilde{B}})) \cap C^1([0,\infty), \tilde{\tilde{X}}),$$

that is,

$$u \in C([0,\infty), L^2(\Omega)) \cap C^1([0,\infty), H^{-1}(\Omega)) \cap C^2([0,\infty), (H_0^1(\Omega) \cap H^2(\Omega))').$$

Remark 3.1 *To justify the boundary condition $u = 0$ on $\Gamma \times (0,\infty)$ one must use the results obtained in [69].*

3.2.2 Neumann boundary condition

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary Γ. Consider the problem

$$\left\{ \begin{array}{rcll} u_{tt} - \Delta u & = & 0 \text{ in} & (0,\infty) \times \Omega, \\ \dfrac{\partial u}{\partial \nu} & = & 0 \text{ on} & (0,\infty) \times \Gamma, \\ u(0) = u_0, \ u_t(0) & = & v_0 \text{ in} & \Omega. \end{array} \right. \tag{3.2.12}$$

We shall study existence and uniqueness of solutions to (3.2.12), considering the pair of initial data (u_0, v_0) in each of the following spaces:

$$D(\Delta) \times H^1(\Omega), \quad H^1(\Omega) \times L^2(\Omega),$$

where

$$D(\Delta) = \left\{ u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \right\}.$$

First case: $(u_0, v_0) \in D(\Delta) \times H^1(\Omega)$.

Consider the change of variables

$$v = \frac{du}{dt}, \tag{3.2.13}$$

and set

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{3.2.14}$$

so that we obtain

$$\frac{dU}{dt} = \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} v \\ \Delta u \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{3.2.15}$$

Let $D(B) = D(\Delta) \times H^1(\Omega)$ and

$$\begin{array}{rlcl} B: & D(B) \subset H^1(\Omega) \times L^2(\Omega) & \longrightarrow & H^1(\Omega) \times L^2(\Omega) \\ & [u, v] & \longmapsto & B([u,v]) = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \end{array} \tag{3.2.16}$$

From (3.2.12) we obtain

$$\begin{cases} \dfrac{dU}{dt} = BU, \\ U(0) = U_0, \end{cases} \tag{3.2.17}$$

with

$$U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}.$$

Now, setting $U(t) = e^t Z(t)$ we obtain

$$\begin{cases} \dfrac{dZ}{dt} = (B - I)Z, \\ Z(0) = U_0, \end{cases} \tag{3.2.18}$$

with

$$U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}.$$

We need to show that $B - I$ is the generator of a C_0–semigroup. To this end we use the Lumer–Phillips theorem. We have

$$B - I : D(\Delta) \times H^1(\Omega) \subset H^1(\Omega) \times L^2(\Omega) \longrightarrow H^1(\Omega) \times L^2(\Omega).$$

Note that

i) $D(B-I) = D(\Delta) \times H^1(\Omega)$ is dense in $H^1(\Omega) \times L^2(\Omega)$, since

$$\overline{D(B-I)} = \overline{D(\Delta) \times H^1(\Omega)} = \overline{D(\Delta)} \times \overline{H^1(\Omega)} = H^1(\Omega) \times L^2(\Omega).$$

ii) $B - I$ is dissipative. Indeed, let $(u_1, v_1) \in D(B-I) = D(\Delta) \times H^1(\Omega)$. Then

$$\begin{aligned}
&\big((B-I)(u_1,v_1),(u_1,v_1)\big)_{H^1(\Omega)\times L^2(\Omega)}\\
&= \big((v_1-u_1,\Delta u_1 - v_1),(u_1,v_1)\big)_{H^1(\Omega)\times L^2(\Omega)}\\
&= (v_1-u_1,u_1)_{H^1(\Omega)} + (\Delta u_1 - v_1, v_1)_{L^2(\Omega)}\\
&= (v_1-u_1,u_1)_{L^2(\Omega)} + (\nabla(v_1-u_1),\nabla u_1)_{L^2(\Omega)} + (\Delta u_1 - v_1, v_1)_{L^2(\Omega)}\\
&= \int_\Omega (v_1-u_1)u_1\,dx + \int_\Omega (\nabla v_1 - \nabla u_1)\nabla u_1\,dx + \int_\Omega (\Delta u_1 - v_1)v_1\,dx\\
&= \int_\Omega (v_1u_1 - u_1^2)\,dx + \int_\Omega (\nabla v_1\nabla u_1 - |\nabla u_1|^2)\,dx + \int_\Omega (\Delta u_1)v_1 - v_1^2\,dx\\
&= \int_\Omega v_1u_1 - u_1^2\,dx + \int_\Omega \nabla v_1\nabla u_1\,dx - \int_\Omega |\nabla u_1|^2\,dx - \int_\Omega \nabla u_1\nabla v_1\,dx - \int_\Omega v_1^2\,dx\\
&= \int_\Omega v_1u_1\,dx - \int_\Omega u_1^2\,dx - \int_\Omega |\nabla u_1|^2\,dx - \int_\Omega v_1^2\,dx.
\end{aligned}$$

If $\int_\Omega v_1u_1\,dx < 0$, then the last expression is clearly less than or equal to zero. If $\int_\Omega v_1u_1\,dx \geq 0$, then

$$\begin{aligned}
&\int_\Omega v_1u_1\,dx - \int_\Omega u_1^2\,dx - \int_\Omega |\nabla u_1|^2\,dx - \int_\Omega v_1^2\,dx\\
&\leq 2\int_\Omega v_1u_1\,dx - \int_\Omega u_1^2\,dx - \int_\Omega |\nabla u_1|^2\,dx - \int_\Omega v_1^2\,dx\\
&= -\int_\Omega (u_1-v_1)^2\,dx - \int_\Omega |\nabla u_1|^2\,dx \leq 0.
\end{aligned}$$

Thus, in both cases, we conclude that $B - I$ is dissipative.

iii) $Im(I-(B-I)) = H^1(\Omega)\times L^2(\Omega)$. Equivalently, $Im(2I-B) = H^1(\Omega)\times L^2(\Omega)$. We must show that, given $(w,z) \in H^1(\Omega)\times L^2(\Omega)$, there exists $(u,v) \in D(\Delta)\times H^1(\Omega)$ such that $(2I-B)(u,v) = (w,z)$, that is, such that $(2u-v, 2v-\Delta u) = (w,z)$, or equivalently

$$\begin{cases} 2u - v = w, \\ 2v - \Delta u = z \end{cases} \Rightarrow \begin{cases} 4u - 2v = 2w, \\ 2v - \Delta u = z. \end{cases} \tag{3.2.19}$$

Adding the two equations in the system above, we obtain $4u - \Delta u = 2w + z$. We now show that there exists $u \in D(\Delta)$ such that $(4I - \Delta)u = 2w + z$.

Define

$$a(u,v) = \int_\Omega (\nabla u\nabla v + uv)\,dx, \quad \forall u, v \in H^1(\Omega),$$

which, as we have already seen, is bilinear and satisfies

$$|a(u,v)| \leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},$$

showing that a is continuous. It is also coercive, since

$$a(u,u) = \int_\Omega |\nabla u|^2\,dx + \int_\Omega |u|^2\,dx = \|u\|^2_{H^1(\Omega)}.$$

By the Lax–Milgram lemma, there exists a unique $u \in H^1(\Omega)$ such that $a(u,v) = \langle 2w + z - 3u, v\rangle$ for all $v \in H^1(\Omega)$, that is,

$$\int_\Omega (\nabla u\nabla v + uv)\,dx = \int_\Omega (2w + z - 3u)v\,dx, \quad \forall v \in H^1(\Omega). \tag{3.2.20}$$

From (3.2.20) and the regularity theory for the Neumann problem we obtain $u \in H^2(\Omega)$. Furthermore, from (3.2.20) we have

$$(\nabla u, \nabla v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)} = (2w + z - 3u, v)_{L^2(\Omega)}, \quad \forall v \in H^1(\Omega). \tag{3.2.21}$$

Since (3.2.21) holds for all $v \in H^1(\Omega)$, in particular for $\varphi \in \mathcal{D}(\Omega)$, we get

$$(\nabla u, \nabla \varphi)_{L^2(\Omega)} + (u, \varphi)_{L^2(\Omega)} = (2w + z - 3u, \varphi)_{L^2(\Omega)}.$$

Hence $-\Delta u + u = 2w + z - 3u$ in $\mathcal{D}'(\Omega)$. Since $2w + z - 3u \in L^2(\Omega)$ and $u \in L^2(\Omega)$, it follows that $\Delta u \in L^2(\Omega)$. Thus, from (3.2.21),

$$(\nabla u, \nabla v)_{L^2(\Omega)} + (\Delta u, v)_{L^2(\Omega)} = 0, \quad \forall v \in H^1(\Omega).$$

By the second Green's formula in its general form,

$$(\nabla u, \nabla v)_{L^2(\Omega)} + (\Delta u, v)_{L^2(\Omega)} = \left\langle \frac{\partial u}{\partial \nu}, v \right\rangle_{H^{-1/2}, H^{1/2}},$$

and therefore

$$\left\langle \frac{\partial u}{\partial \nu}, v \right\rangle_{H^{-1/2}, H^{1/2}} = 0, \quad \forall v \in H^1(\Omega).$$

Since the trace operator is surjective, we obtain $\dfrac{\partial u}{\partial \nu} = 0$ on Γ. Hence $u \in D(\Delta)$.

Observe that, from (3.2.19), we have $v = 2u - w$. Since $u, w \in H^1(\Omega)$, it follows that $v \in H^1(\Omega)$. Therefore, we conclude that $(u, v) \in D(\Delta) \times H^1(\Omega)$, as required.

By (i), (ii) and (iii), the operator $B - I$ is m-dissipative with dense domain. Hence, by the Lumer–Phillips theorem, we obtain $B - I \in G(1, 0)$. Thus, when $U_0 \in D(B - I)$, the theorem 2.3 guarantees the existence of a unique function

$$Z \in C([0, \infty); D(B - I)) \cap C^1([0, \infty); H^1(\Omega) \times L^2(\Omega))$$

solving (3.2.18). We have

$$Z(t) = e^{-t} U(t) = e^{-t} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} e^{-t} u(t) \\ e^{-t} v(t) \end{bmatrix} \in C([0, \infty); D(\Delta) \times H^1(\Omega)) \cap C^1([0, \infty); H^1(\Omega) \times L^2(\Omega)).$$

We conclude that

$$u \in C([0, \infty); D(\Delta)) \cap C^1([0, \infty); H^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega)).$$

Second case: $(u_0, v_0) \in H^1(\Omega) \times L^2(\Omega)$.

We know, by Corollary 1.23, that every C_0–semigroup is strongly continuous on $\mathbb{R}_+$, that is, if $t \in \mathbb{R}_+$, then

$$\lim_{s \to t} S(s)(x, y) = S(t)(x, y), \quad \text{for all } (x, y) \in H^1(\Omega) \times L^2(\Omega).$$

Thus, if $(u_0, v_0) \in H^1(\Omega) \times L^2(\Omega)$, the corresponding solution satisfies

$$u \in C([0, \infty); H^1(\Omega) \times L^2(\Omega)).$$

3.3 Schrödinger equation

We now state a result that will be useful in the study of the Schrödinger equation.

Proposition 3.2 *Let H be a Hilbert space and $A : D(A) \subset H \to H$ a symmetric linear operator such* *that $Im(\lambda_0 I - A) = H$ for some $\lambda_0 \in \mathbb{R}$ with $\lambda_0 \in \rho(A)$. Then A is self-adjoint.*

Proof: Since $\lambda_0 \in \rho(A)$ and $Im(\lambda_0 I - A) = H$, the domain of the operator $R(\lambda_0, A)$ is H. Given $x, y \in H$, write $x' = R(\lambda_0, A)x$ and $y' = R(\lambda_0, A)y$. Then

$$x = \lambda_0 x' - Ax' \quad \text{and} \quad y = \lambda_0 y' - Ay'.$$

Since A is symmetric, $(x', Ay') = (Ax', y')$ because $x', y' \in D(A)$. Thus,

$$(x', \lambda_0 y' - y) = (x', Ay') = (Ax', y') = (\lambda_0 x' - x, y'),$$

and hence $(x', y) = (x, y')$, that is, $(R(\lambda_0, A)x, y) = (x, R(\lambda_0, A)y)$. Since $Dom(R(\lambda_0, A)) = H$ and by the equality above, it follows that $R(\lambda_0, A)$ is self-adjoint.

To prove that A is self-adjoint, it suffices to show that $D(A^*) \subset D(A)$, since A is symmetric by hypothesis. Let $x \in D(A^*)$ and set $z = (\lambda_0 I - A)^* x$. Given $y \in H$ with $w = R(\lambda_0, A)y$ we obtain

$$(w, z) = (R(\lambda_0, A)y, z) = (y, R(\lambda_0, A)z)$$

and also

$$(w, z) = (w, (\lambda_0 I - A)^* x) = ((\lambda_0 I - A)w, x) = (y, x),$$

so that

$$(y, R(\lambda_0, A)z) = (y, x).$$

By the arbitrariness of $y \in H$ it follows that $x = R(\lambda_0, A)z \in D(A)$. □

Let H be a Hilbert space and $A : D(A) \subset H \to H$ a linear operator with dense domain in H. We have $A^* = -A$ if and only if iA is self-adjoint. Indeed, if $A^* = -A$, then

$$(iA)^* = \bar{i}A^* = \bar{i}(-A) = (-i)(-A) = iA.$$

Conversely, if iA is self-adjoint, then

$$iA = (iA)^* = \bar{i}A^* = (-i)A^* = i(-A^*),$$

which implies $A = -A^*$, that is, $-A = A^*$. In this way, by Stone's theorem, the operator A generates a unitary C_0–group if and only if iA is self-adjoint.

We now consider the Schrödinger equation

$$\begin{cases} \dfrac{du}{dt}(t) = i\Delta u(t) & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{3.3.22}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary.

Let $A : H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ be the operator defined by $Au = i\Delta u$. We already know that the operator $-iA = \Delta$ is self-adjoint, that is, $(-iA)^* = (-iA)$. On the other hand,

$$(-iA)^* = \overline{-i}A^* = iA^*,$$

so $iA^* = -iA$, and hence iA is self-adjoint. Thus A generates a unitary C_0–group and, in particular, a C_0–semigroup. By Theorem 2.3, problem (3.3.22) admits a unique solution u in the class

$$C^0([0, \infty); H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$$

whenever $u_0 \in H_0^1(\Omega) \cap H^2(\Omega) = D(A)$.

Our aim now is to study the Schrödinger equation in $L^2(\mathbb{R}^n)$:

$$\begin{cases} \dfrac{1}{i}\dfrac{du}{dt}(t) = \Delta u(t) - qu(t) & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(0) = u_0 & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{3.3.23}$$

where Δ is the Laplacian and q is a real-valued measurable function defined on $\mathbb{R}^n$. Before studying this equation, we shall prove that the operators

$$\begin{array}{rcccl} A_1 & : & D(A_1) \subset L^2(\mathbb{R}^n) & \longrightarrow & L^2(\mathbb{R}^n) \\ & & u & \longmapsto & A_1 u = i\Delta u \end{array}$$

and

$$\begin{array}{rcccl} iM_q & : & D(M_q) \subset L^2(\mathbb{R}^n) & \longrightarrow & L^2(\mathbb{R}^n) \\ & & u & \longmapsto & iM_q u = iqu, \end{array}$$

where $D(A_1) = H^2(\mathbb{R}^n)$ and $D(M_q) = \{u \in L^2(\mathbb{R}^n);\, qu \in L^2(\mathbb{R}^n)\}$, generate unitary C_0–groups.

We first prove that $-iA_1$ is self-adjoint. Note that $\mathcal{S}(\mathbb{R}^n) \subset H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and since $\overline{\mathcal{S}(\mathbb{R}^n)}^{L^2(\mathbb{R}^n)} = L^2(\mathbb{R}^n)$, it follows that $\overline{H^2(\mathbb{R}^n)}^{L^2(\mathbb{R}^n)} = L^2(\mathbb{R}^n)$. Hence $D(A_1) = H^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Moreover,

$$\begin{aligned} (\Delta u, v)_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \Delta u(\xi)\overline{v(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^n} \widehat{\Delta u}(\xi)\, \overline{\widehat{v}(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^n} (-4\pi^2)\|\xi\|^2 \widehat{u}(\xi)\, \overline{\widehat{v}(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^n} \widehat{u}(\xi)\, \overline{(-4\pi^2)\|\xi\|^2 \widehat{v}(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^n} \widehat{u}(\xi)\, \overline{\widehat{\Delta v}(\xi)}\, d\xi \\ &= (u, \Delta v)_{L^2(\mathbb{R}^n)}, \quad \forall u, v \in D(A_1), \end{aligned}$$

and therefore $(-iA_1 u, v)_{L^2(\mathbb{R}^n)} = (u, -iA_1 v)_{L^2(\mathbb{R}^n)}$ for all $u, v \in D(A_1)$, so that $-iA_1$ is symmetric. Furthermore,

$$\begin{aligned} (-iA_1 u, u)_{L^2(\mathbb{R}^n)} &= (\Delta u, u)_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} \widehat{\Delta u}(\xi)\, \overline{\widehat{u}(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^n} (-4\pi^2)\|\xi\|^2 \widehat{u}(\xi)\, \overline{\widehat{u}(\xi)}\, d\xi \\ &= -\int_{\mathbb{R}^n} 4\pi^2 \|\xi\|^2 |\widehat{u}(\xi)|^2\, d\xi \le 0, \quad \forall u \in D(A_1), \end{aligned}$$

so that $-iA_1$ is dissipative. By Proposition 1.42,

$$\|(I - (-iA_1))u\| \ge \|u\|, \quad \forall u \in D(A_1). \tag{3.3.24}$$

Thus $I - (-iA_1)$ is injective.

Given $v \in L^2(\mathbb{R}^n)$, we know that there exists a unique $u \in H^2(\mathbb{R}^n)$ such that

$$-\Delta u + u = v \quad \text{in } L^2(\mathbb{R}^n),$$

that is,

$$(I - (-iA_1))u = v \quad \text{in } L^2(\mathbb{R}^n),$$

which shows that $I - (-iA_1)$ is surjective. Hence $(I + iA_1)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ exists. Given $v \in L^2(\mathbb{R}^n)$, let $u = (I + iA_1)^{-1}v \in H^2(\mathbb{R}^n)$. From (3.3.24),

$$\|(I + iA_1)^{-1}v\| = \|u\| \leq \|(I + iA_1)u\| = \|v\|,$$

so $(I + iA_1)^{-1}$ is continuous and therefore $1 \in \rho(-iA_1)$.

We have that $-iA_1$ is symmetric, $Im(I - (-iA_1)) = L^2(\mathbb{R}^n)$ and $1 \in \rho(-iA_1)$. By Theorem 3.2, $-iA_1$ is self-adjoint. Thus iA_1 is self-adjoint and therefore A_1 generates a unitary C_0–group.

The operator

$$\begin{array}{rcccl} M_q & : & D(M_q) \subset L^2(\mathbb{R}^n) & \longrightarrow & L^2(\mathbb{R}^n) \\ & & u & \longmapsto & M_q u = qu, \end{array}$$

is called the multiplication operator.

We now prove that iM_q generates a unitary C_0–group. For this, it suffices to show that M_q is self-adjoint.

For each $n \in \mathbb{N}$, set $E_n = \{x \in \mathbb{R}^n : |q(x)| \leq n\}$. Fix an arbitrary $u \in L^2(\mathbb{R}^n)$. We have

$$\int_{\mathbb{R}^n} |u\chi_{E_n}(x)|^2 \, dx \leq \int_{\mathbb{R}^n} |u(x)|^2 \, dx < \infty,$$

so $u\chi_{E_n} \in L^2(\mathbb{R}^n)$ for all $n \in \mathbb{N}$. Also,

$$|u\chi_{E_n}(x) - u(x)| \to 0 \quad \text{as } n \to \infty \text{ for almost every } x \in \mathbb{R}^n,$$

and since $|u\chi_{E_n}(x) - u(x)|^2 \leq 4|u(x)|^2$ for almost every $x \in \mathbb{R}^n$, the Dominated Convergence Theorem implies that $u\chi_{E_n} \to u$ in $L^2(\mathbb{R}^n)$. Moreover,

$$\int_{\mathbb{R}^n} |q(x)u(x)\chi_{E_n}(x)|^2 \, dx \leq n^2 \int_{\mathbb{R}^n} |u(x)|^2 \, dx < \infty, \quad \forall n \in \mathbb{N},$$

so $u\chi_{E_n} \in D(M_q)$ for all $n \in \mathbb{N}$. Thus $D(M_q)$ is dense in $L^2(\mathbb{R}^n)$.

Given $u, v \in D(M_q)$, since q is real-valued, we have $(M_q u, v) = (u, M_q v)$, and therefore M_q is symmetric. Hence $D(M_q) \subset D(M_q^*)$ and $M_q^* u = M_q u$ for all $u \in D(M_q)$. We now prove that $D(M_q) = D(M_q^*)$. Note that if $u, v \in D(M_q)$, then

$$(\pm iI + M_q)u = (\pm iI + M_q)v$$

implies $u = v$. Thus the operators $\pm iI + M_q$ are injective. Moreover,

$$|\pm i + q(x)| = \sqrt{1^2 + (q(x))^2} \geq 1, \quad \forall x \in \mathbb{R}^n,$$

and therefore, for $u \in L^2(\mathbb{R}^n)$,

$$\left| \frac{\pm iu}{\pm i + q} \right| \leq |\pm iu| = |u|,$$

which shows that

$$\left| \frac{\pm iu}{\pm i + q} \right| \in L^2(\mathbb{R}^n),$$

and hence

$$u - \frac{\pm iu}{\pm i + q} = \frac{\pm iu + qu}{\pm i + q} + \frac{\mp iu}{\pm i + q} = \frac{qu}{\pm i + q} \in L^2(\mathbb{R}^n),$$

so

$$\frac{u}{\pm i + q} \in D(M_q) \quad \text{and} \quad (\pm iI + M_q)\left(\frac{u}{\pm i + q}\right) = u,$$

which shows that the operators $\pm iI + M_q : D(M_q) \to L^2(\mathbb{R}^n)$ are surjective.

Suppose, by contradiction, that $D(M_q^*) \neq D(M_q)$.

Claim: $iI + M_q^* : D(M_q^*) \to L^2(\mathbb{R}^n)$ is not injective.

Indeed, since $D(M_q) \subsetneq D(M_q^*)$, there exists $w \in D(M_q^*) \setminus D(M_q)$. As $iI + M_q$ is surjective and $(iI + M_q^*)w \in L^2(\mathbb{R}^n)$, there exists $v \in D(M_q)$ such that

$$(iI + M_q)v = (iI + M_q^*)w.$$

But $v \in D(M_q) \subset D(M_q^*)$ and $M_q v = M_q^* v$, so $(iI + M_q)v = (iI + M_q^*)w$ with $v \neq w$.

Since $iI + M_q^*$ is not injective, there exists $v \neq 0$ in $D(M_q^*)$ such that $(iI + M_q^*)v = 0$, that is, $M_q^* v = -iv$. Hence

$$(M_q u, v)_{L^2(\mathbb{R}^n)} = (u, M_q^* v)_{L^2(\mathbb{R}^n)} = (u, -iv)_{L^2(\mathbb{R}^n)} = (iu, v)_{L^2(\mathbb{R}^n)}, \quad \forall u \in D(M_q),$$

and therefore

$$((-iI + M_q)u, v)_{L^2(\mathbb{R}^n)} = 0, \quad \forall u \in D(M_q).$$

Since $-iI + M_q$ is surjective, the above equality implies $v = 0$, which is a contradiction. Thus $D(M_q) = D(M_q^*)$ and so M_q is self-adjoint.

Returning to problem

$$\begin{cases} \dfrac{1}{i}\dfrac{du}{dt}(t) = \Delta u(t) - qu(t) & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(0) = u_0 & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{3.3.25}$$

we restrict ourselves to the following three cases:

a) $q(x) = 0$ for almost every $x \in \mathbb{R}^n$;

b) $q \in L^\infty(\mathbb{R}^n)$;

c) (i) $H^2(\mathbb{R}^n) \subset D(M_q)$ and there exist constants $a, b \in \mathbb{R}$ with $0 \leq a < 1$ and $b \geq 0$ such that

$$\|M_q u\|_{L^2(\mathbb{R}^n)} \leq a\|iA_1 u\|_{L^2(\mathbb{R}^n)} + b\|u\|_{L^2(\mathbb{R}^n)}, \quad \forall u \in H^2(\mathbb{R}^n);$$

(ii) $q(x) \geq 0$ for almost every $x \in \mathbb{R}^n$.

a) In this case, $M_q \equiv 0$. We have already seen that A_1 generates a C_0–semigroup. Thus, if $u_0 \in \mathcal{D}(A_1) = H^2(\mathbb{R}^n)$, then there exists a unique solution u of (3.3.25) in the class

$$u \in C([0,\infty); H^2(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n)).$$

b) Since $q \in L^\infty(\mathbb{R}^n)$, we have $D(M_q) = L^2(\mathbb{R}^n)$, because

$$\|M_q u\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |q(x)u(x)|^2 \, dx \right)^{\frac{1}{2}} \leq \|q\|_{L^\infty(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)} < \infty, \quad \forall u \in L^2(\mathbb{R}^n),$$

and hence $M_q \in \mathcal{L}(L^2(\mathbb{R}^n))$ with $\|M_q\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|q\|_{L^\infty(\mathbb{R}^n)}$. Moreover, $Im(I - (-iA_1)) = L^2(\mathbb{R}^n)$, $-iA_1$ has dense domain and is dissipative. By the Lumer–Phillips theorem, $-iA_1 \in G(1,0)$. By Exercise 1.52 we obtain $-iA_1 - M_q \in G(1, \|M_q\|_{\mathcal{L}(L^2(\mathbb{R}^n))})$, and therefore

$$-iA_1 - M_q - \|M_q\|_{\mathcal{L}(L^2(\mathbb{R}^n))} I \in G(1,0),$$

by Proposition 1.37. By the Lumer–Phillips theorem,

$$Im(\lambda - (-iA_1 - M_q - \|M_q\|_{\mathcal{L}(L^2(\mathbb{R}^n))} I)) = L^2(\mathbb{R}^n),$$

for some fixed $\lambda > 0$. Setting $\lambda_0 = \lambda + \|M_q\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$, we have

$$Im(\lambda_0 - (-iA_1 - M_q)) = L^2(\mathbb{R}^n).$$

Moreover, since $-iA_1 - M_q \in G(1, \|M_q\|_{\mathcal{L}(L^2(\mathbb{R}^n))})$ and $\lambda_0 > \|M_q\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$, it follows that $\lambda_0 \in \rho(-iA_1 - M_q)$. As

$$D(-iA_1 - M_q) = D(A_1) \cap D(M_q) = H^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = H^2(\mathbb{R}^n)$$

and $-iA_1$ and M_q are symmetric, we deduce that $-iA_1 - M_q$ is also symmetric. Hence $Im(\lambda_0 - (-iA_1 - M_q)) = L^2(\mathbb{R}^n)$ and $\lambda_0 \in \rho(-iA_1 - M_q)$, and by Theorem 3.2 we conclude that $-iA_1 - M_q$ is self-adjoint. Therefore $iA_1 + M_q$ is also self-adjoint and $A_1 - iM_q$ generates a C_0–semigroup. Arguing as in the previous case, if $u_0 \in H^2(\mathbb{R}^n)$, then there exists a unique solution u of (3.3.25) in the class

$$u \in C([0, \infty); H^2(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n)).$$

c) By (ii) we have

$$(-M_q u, u)_{L^2(\mathbb{R}^n)} = (-qu, u)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} -q(x)u(x)\overline{u(x)}\, dx = -\int_{\mathbb{R}^n} q(x)|u(x)|^2\, dx \le 0, \quad \forall u \in D(M_q),$$

so $-M_q$ is dissipative. We have already seen that $-iA_1 \in G(1, 0)$. By the hypotheses and Exercise 1.5.1 it follows that $-iA_1 - M_q \in G(1, 0)$. Thus, for some λ_0,

$$Im(\lambda_0 - (-iA_1 - M_q)) = L^2(\mathbb{R}^n) \quad \text{and} \quad \lambda_0 \in \rho(-iA_1 - M_q).$$

Since $-iA_1 - M_q$ is symmetric and $D(-iA_1 - M_q) = H^2(\mathbb{R}^n)$, we conclude that $-iA_1 - M_q$ is self-adjoint, so $iA_1 + M_q$ is also self-adjoint and $A_1 - iM_q$ generates a unitary C_0–group. In particular, $A_1 - iM_q$ generates a C_0–semigroup. Hence, if $u_0 \in H^2(\mathbb{R}^n)$, there exists a unique solution u of (3.3.25) in the class

$$u \in C([0, \infty); H^2(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n)).$$

Example 3.3 Consider the problem

$$\begin{cases} u_t = i\Delta u & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{on} \quad \Gamma \times (0, +\infty), \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{3.3.26}$$

where $\Omega \subset \mathbb{R}^n$ is open.

Let $u_0 \in H_0^1(\Omega)$ and consider the operator

$$\begin{aligned} I - \Delta : H_0^1(\Omega) &\to H^{-1}(\Omega) \\ u &\mapsto (I - \Delta)u. \end{aligned}$$

We show that $I - \Delta$ is a bijection; for this we shall use the Lax–Milgram lemma.

Define

$$\begin{aligned} a : H_0^1(\Omega) \times H_0^1(\Omega) &\to \mathbb{R} \\ (u, v) &\mapsto a(u, v) = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}. \end{aligned}$$

We have:

(i) $a(\cdot, \cdot)$ is bilinear.

(ii) $a(\cdot, \cdot)$ is continuous.

Indeed,

$$|a(u,v)| = |(u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}| = |(u,v)_{H_0^1(\Omega)}| \leqslant \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \quad \forall u,v \in H_0^1(\Omega).$$

(iii) $a(\cdot,\cdot)$ is coercive.

Indeed,

$$|a(u,u)| = |(u,u)_{L^2(\Omega)} + (\nabla u, \nabla u)_{L^2(\Omega)}| = \|u\|_{H_0^1(\Omega)}^2.$$

Since $a(\cdot,\cdot)$ is bilinear, continuous and coercive, by the Lax–Milgram lemma, given $f \in H^{-1}(\Omega) = (H_0^1(\Omega))'$, there exists a unique $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

In particular, $a(u,w) = \langle f, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$ for every $w \in C_0^\infty(\Omega)$, that is,

$$(u,w)_{L^2(\Omega)} + (\nabla u, \nabla w)_{L^2(\Omega)} = \langle f, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall w \in C_0^\infty(\Omega). \tag{3.3.27}$$

Hence

$$\langle u, w\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} - \langle \Delta u, w\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle f, w\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)},$$

because

$$\mathcal{D}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \equiv (L^2(\Omega))' \hookrightarrow H^{-1}(\Omega) \hookrightarrow \mathcal{D}'(\Omega). \tag{3.3.28}$$

Thus

$$\langle u - \Delta u, w\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle f, w\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}, \quad \forall w \in C_0^\infty(\Omega).$$

Hence $u - \Delta u = f$ in $\mathcal{D}'(\Omega)$.

Since the equality above holds in $\mathcal{D}'(\Omega)$, it follows from (3.3.28) that there exists a unique $u \in H_0^1(\Omega)$ such that

$$(I - \Delta)u = f \quad \text{in } H^{-1}(\Omega), \tag{3.3.29}$$

as desired.

Thus there exists the operator $(I-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$.

From (3.3.29) and (3.3.27) we get

$$(u,w)_{L^2(\Omega)} + (\nabla u, \nabla w)_{L^2(\Omega)} = \langle (I-\Delta)u, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall w \in H_0^1(\Omega). \tag{3.3.30}$$

Moreover,

$$\begin{aligned} |\langle (I-\Delta)u, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}| &= |(u,w)_{L^2(\Omega)} + (\nabla u, \nabla w)_{L^2(\Omega)}| \\ &\leqslant \|u\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}. \end{aligned} \tag{3.3.31}$$

Using Hölder's inequality for series (with $p = q = 2$), we deduce from (3.3.31) that

$$\begin{aligned} |\langle (I-\Delta)u, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}| &\leqslant (\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2)^{\frac{1}{2}} (\|w\|_{L^2(\Omega)}^2 + \|\nabla w\|_{L^2(\Omega)}^2)^{\frac{1}{2}} \\ &= \|u\|_{H_0^1(\Omega)}\|w\|_{H_0^1(\Omega)}. \end{aligned}$$

Hence

$$\|(I-\Delta)u\|_{H^{-1}(\Omega)} = \sup_{\substack{w\in H_0^1(\Omega)\\ \|w\|\leqslant 1}} |\langle (I-\Delta)u, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}| \leqslant \|u\|_{H_0^1(\Omega)}.$$

Therefore

$$\|(I-\Delta)u\|_{H^{-1}(\Omega)} \leqslant \|u\|_{H_0^1(\Omega)}. \tag{3.3.32}$$

On the other hand, from (3.3.30) we have

$$\|u\|^2_{H_0^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} = |\langle (I-\Delta)u, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}|.$$

For $u \neq 0$ we obtain

$$\begin{aligned}\|u\|_{H_0^1(\Omega)} &= \frac{1}{\|u\|_{H_0^1(\Omega)}} |\langle (I-\Delta)u, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}| \\ &= \left| \left\langle (I-\Delta)u, \frac{u}{\|u\|_{H_0^1(\Omega)}} \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \right| \\ &\leqslant \sup_{\substack{w\in H_0^1(\Omega)\\ \|w\|=1}} |\langle (I-\Delta)u, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}| = \|(I-\Delta)u\|_{H^{-1}(\Omega)}.\end{aligned}$$

Thus

$$\|u\|_{H_0^1(\Omega)} \leqslant \|(I-\Delta)u\|_{H^{-1}(\Omega)}. \tag{3.3.33}$$

From (3.3.32) and (3.3.33) we conclude that $\|u\|_{H_0^1(\Omega)} = \|(I-\Delta)u\|_{H^{-1}(\Omega)}$. Hence $(I-\Delta) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is a surjective isometry, that is, an isometric isomorphism. Consequently, $R(1,\Delta) = (I-\Delta)^{-1}$ exists.

Define on $H^{-1}(\Omega)$ the inner product

$$((u,v))_1 = ((I-\Delta)^{-1}u, (I-\Delta)^{-1}v)_{H_0^1(\Omega)}, \quad \forall u, v \in H^{-1}(\Omega).$$

Moreover, $(I-\Delta)^{-1}$ is also an isometry, hence $\|(I-\Delta)^{-1}u\|_{H_0^1(\Omega)} = \|u\|_{H^{-1}(\Omega)}$ for all $u \in H^{-1}(\Omega)$.

We first show that there exist constants $C_1, C_2 > 0$ such that

$$C_1\|u\|_{H^{-1}(\Omega)} \leqslant \|u\|_{H_0^1(\Omega)} \leqslant C_2\|u\|_{H^{-1}(\Omega)}, \quad \forall u \in H^{-1}(\Omega).$$

Indeed, $\|u\|_{H^{-1}(\Omega)} = \|(I-\Delta)^{-1}u\|_{H_0^1(\Omega)} = \|u\|_1$ for all $u \in H^{-1}(\Omega)$, where the first equality follows from the isometry and the second from the definition.

We now prove the continuity of $R(1,\Delta)$.

Recall that $R(1,\Delta) = (I-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega) \subset H^{-1}(\Omega)$. We wish to show that $(I-\Delta)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, that is, there exists $C > 0$ such that

$$\|(I-\Delta)^{-1}u\|_1 \leqslant C\|u\|_1, \quad \forall u \in H^{-1}(\Omega). \tag{3.3.34}$$

Indeed,

$$\|(I-\Delta)^{-1}u\|_{H^{-1}(\Omega)} \leqslant C\|(I-\Delta)^{-1}u\|_{H_0^1(\Omega)} = C\|u\|_{H^{-1}(\Omega)},$$

where the inequality follows from the chain of embeddings in (3.3.28) and the equality from the definition.

Furthermore,

$$\|(I-\Delta)^{-1}u\|_1 = \|(I-\Delta)^{-1}u\|_{H^{-1}(\Omega)} \leqslant C\|u\|_{H^{-1}(\Omega)} = C\|u\|_1, \quad \forall u \in H^{-1}(\Omega).$$

Thus $R(1,\Delta) \in \mathcal{L}(H^{-1}(\Omega))$, which implies that $1 \in \rho(\Delta)$ and, from the surjectivity of $(I-\Delta)$ already proved, we also have $Im(I-\Delta) = H^{-1}(\Omega)$.

In order to apply Proposition 3.2, it remains to show that $\Delta : H_0^1(\Omega) \subset H^{-1}(\Omega) \to H^{-1}(\Omega)$ is a symmetric operator (with respect to the inner product $((\cdot,\cdot))_1$ defined on $H^{-1}(\Omega)$).

We first observe that $D(\Delta) = H_0^1(\Omega)$ is dense in $H^{-1}(\Omega)$.

Let $u \in H^{-1}(\Omega)$. Then there exists $w \in H_0^1(\Omega)$ such that $u = (I-\Delta)w$. Since $w \in H_0^1(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, there exists $(\varphi_\nu) \subset \mathcal{D}(\Omega)$ such that $\varphi_\nu \to w$ in $H_0^1(\Omega)$. By the continuity of the operator $(I-\Delta)$ we have

$$\psi_\nu := (I-\Delta)\varphi_\nu \to (I-\Delta)w = u \quad \text{in } H^{-1}(\Omega).$$

Since $(\psi_\nu) \subset \mathcal{D}(\Omega)$, this implies that $\mathcal{D}(\Omega)$ is dense in $H^{-1}(\Omega)$. From $\mathcal{D}(\Omega) \subset H_0^1(\Omega) \subset H^{-1}(\Omega)$ we obtain

$$H^{-1}(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^{-1}(\Omega)} \subset \overline{H_0^1(\Omega)}^{H^{-1}(\Omega)} \subset \overline{H^{-1}(\Omega)}^{H^{-1}(\Omega)} = H^{-1}(\Omega).$$

We conclude that

$$\overline{H_0^1(\Omega)}^{H^{-1}(\Omega)} = \overline{H_0^1(\Omega)}^{\|\cdot\|_1} = H^{-1}(\Omega).$$

That is, $H_0^1(\Omega)$ is dense in $H^{-1}(\Omega)$.

We now show that $((\Delta u, v))_1 = ((u, \Delta v))_1$. We carry out this proof in the real case, i.e., $\mathbb{K} = \mathbb{R}$.

Let $u, v \in C_0^\infty(\Omega)$. Then

$$\begin{aligned}
((\Delta u, v))_1 &= ((\Delta u - u + u, v))_1 = ((-(I-\Delta)u + u, v))_1 \\
&= ((-(I-\Delta)u, v))_1 + ((u,v))_1 \\
&= (-u, (I-\Delta)v)_{H_0^1(\Omega)} + ((u,v))_1 \\
&= (-u, (I-\Delta)^{-1}v)_{L^2(\Omega)} + (-\nabla u, \nabla[(I-\Delta)^{-1}v])_{L^2(\Omega)} + ((u,v))_1.
\end{aligned}$$

Note that

$$(-\nabla u, \nabla[(I-\Delta)^{-1}v])_{L^2(\Omega)} = (\nabla[(I-\Delta)^{-1}v], -\nabla u)_{L^2(\Omega)} = \langle \Delta[(I-\Delta)^{-1}v], u\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

Hence

$$\begin{aligned}
((\Delta u, v))_1 &= \langle -(I-\Delta)^{-1}v, u\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\
&\quad + \langle \Delta[(I-\Delta)^{-1}v], u\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + ((u,v))_1 \\
&= \langle -(I-\Delta)^{-1}v + \Delta[(I-\Delta)^{-1}v], u\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + ((u,v))_1 \\
&= \langle -(I-\Delta)[(I-\Delta)^{-1}v], u\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + ((u,v))_1 \\
&= \langle -v, u\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + ((u,v))_1 \\
&= (-v, u)_{L^2(\Omega)} + ((u,v))_1 \\
&= -(u,v)_{L^2(\Omega)} + ((u,v))_1.
\end{aligned}$$

Thus

$$((\Delta u, v))_1 = -(u, v)_{L^2(\Omega)} + ((u, v))_1. \tag{3.3.35}$$

On the other hand,

$$\begin{aligned}
((u, \Delta v))_1 &= ((u, \Delta v - v + v))_1 = ((u, -(I - \Delta)v + v))_1 \\
&= ((u, -(I - \Delta)v))_1 + ((u, v))_1 \\
&= (-(I - \Delta)^{-1}u, v)_{H_0^1(\Omega)} + ((u, v))_1 \\
&= (-(I - \Delta)^{-1}u, v)_{L^2(\Omega)} \\
&\quad + (\nabla[(I - \Delta)^{-1}u], -\nabla v)_{L^2(\Omega)} + ((u, v))_1.
\end{aligned}$$

Note that

$$(\nabla[(I - \Delta)^{-1}u], -\nabla v)_{L^2(\Omega)} = \langle \Delta[(I - \Delta)^{-1}u], v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

Hence

$$\begin{aligned}
((u, \Delta v))_1 &= \langle -(I - \Delta)^{-1}u, v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\
&\quad + \langle \Delta[(I - \Delta)^{-1}u], v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + ((u, v))_1 \\
&= \langle -(I - \Delta)^{-1}u + \Delta[(I - \Delta)^{-1}u], v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + ((u, v))_1 \\
&= \langle -(I - \Delta)[(I - \Delta)^{-1}u], v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + ((u, v))_1 \\
&= \langle -u, v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + ((u, v))_1 \\
&= -(u, v)_{L^2(\Omega)} + ((u, v))_1.
\end{aligned}$$

Thus

$$((u, \Delta v))_1 = -(u, v)_{L^2(\Omega)} + ((u, v))_1. \tag{3.3.36}$$

From (3.3.35) and (3.3.36) we obtain

$$((\Delta u, v))_1 = ((u, \Delta v))_1, \quad \text{for all } u, v \in C_0^\infty(\Omega). \tag{3.3.37}$$

Now let $w, z \in H_0^1(\Omega)$. Since $\overline{C_0^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$, there exist sequences $(\varphi_\nu), (\psi_\nu) \subset C_0^\infty(\Omega)$ such that $\varphi_\nu \to w$ in $H_0^1(\Omega)$ and $\psi_\nu \to z$ in $H_0^1(\Omega)$ as $\nu \to +\infty$. By the continuity of the operator $(I - \Delta)$ we have, as $\nu \to +\infty$,

$$\begin{aligned}
(I - \Delta)\varphi_\nu &\to (I - \Delta)w \quad \text{in } (H^{-1}(\Omega), \|\cdot\|_1), \\
(I - \Delta)\psi_\nu &\to (I - \Delta)z \quad \text{in } (H^{-1}(\Omega), \|\cdot\|_1).
\end{aligned}$$

Moreover, applying (3.3.37) to (φ_ν) and (ψ_ν) we obtain $((\Delta\varphi_\nu, \psi_\nu))_1 = ((\varphi_\nu, \Delta\psi_\nu))_1$. It follows that

$$\begin{aligned}
((\Delta\varphi_\nu - \varphi_\nu + \varphi_\nu, \psi_\nu))_1 &= ((\varphi_\nu, \Delta\psi_\nu - \psi_\nu + \psi_\nu))_1 \\
((-(I - \Delta)\varphi_\nu + \varphi_\nu, \psi_\nu))_1 &= ((\varphi_\nu, -(I - \Delta)\psi_\nu + \psi_\nu))_1 \\
((-(I - \Delta)\varphi_\nu, \psi_\nu))_1 + ((\varphi_\nu, \psi_\nu))_1 &= ((\varphi_\nu, -(I - \Delta)\psi_\nu))_1 + ((\varphi_\nu, \psi_\nu))_1.
\end{aligned} \tag{3.3.38}$$

Since $H_0^1(\Omega) \hookrightarrow H^{-1}(\Omega)$, as $\nu \to +\infty$ we have

$$\begin{aligned} \varphi_\nu \to w &\quad \text{in } (H^{-1}(\Omega), \|\cdot\|_1), \\ \psi_\nu \to z &\quad \text{in } (H^{-1}(\Omega), \|\cdot\|_1). \end{aligned}$$

Taking the limit as $\nu \to +\infty$ in (3.3.38), we deduce

$$\begin{aligned} ((-(I-\Delta)w, z))_1 + ((w,z))_1 &= ((w, -(I-\Delta)z))_1 + ((w,z))_1 \\ ((-w+\Delta w, z))_1 + ((w,z))_1 &= ((w, -z+\Delta z))_1 + ((w,z))_1 \\ ((-w+\Delta w + w, z))_1 &= ((w, -z+\Delta z + z))_1 \\ ((\Delta w, z))_1 &= ((w, \Delta z))_1, \quad \forall z, w \in H_0^1(\Omega). \end{aligned}$$

Therefore Δ is symmetric.

By Proposition 3.2 we conclude that Δ is self-adjoint. Hence $(i\Delta)^* = -i\Delta^* = -i\Delta$. By Stone's theorem, $i\Delta$ generates a unitary C_0–group. In particular, it generates a C_0–semigroup.

Consequently, problem (3.3.26) admits, by Theorem 2.3, a unique solution

$$u \in C^0([0,+\infty), H_0^1(\Omega)) \cap C^1([0,+\infty), H^{-1}(\Omega)),$$

whenever $u_0 \in D(i\Delta) = D(\Delta) = H_0^1(\Omega)$.

3.4 Nonlinear Equations

In this section we restrict ourselves to the study of the nonlinear heat equation and, to this end, let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary Γ, let $f : [0,T) \to \mathbb{R}$ be a function, and consider the problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\Gamma, \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{3.4.39}$$

Theorem 3.4 *If $f \in C^1(\mathbb{R})$ and f' is bounded, then, for every $u_0 \in L^2(\Omega)$, there exists a global solution of problem (3.4.39), that is, $T_{max} = +\infty$, with*

$$u \in C^1((0,\infty); L^2(\Omega)) \cap C^0((0,\infty); H^2(\Omega)) \cap C^0([0,\infty); L^2(\Omega)).$$

Moreover, if $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$u \in C^1([0,\infty); L^2(\Omega)) \cap C^0([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)).$$

Proof: We first observe that f is Lipschitz. Indeed, if $t, s \in \mathbb{R}$ with $t \le s$, then there exists $t_0 \in (t,s)$ such that

$$\frac{|f(t)-f(s)|}{|t-s|} = |f'(t_0)| \le L,$$

hence

$$|f(t)-f(s)| \le L|t-s|.$$

Given $v \in L^2(\Omega)$, define $F : L^2(\Omega) \to L^2(\Omega)$ by $F(v)(x) = f(v(x))$. Note that:

(i) F is well-defined. We need to show that $F(v) \in L^2(\Omega)$.

In fact,

$$
\begin{aligned}
\|F(v)\|^2_{L^2(\Omega)} &= \int_\Omega |f(v(x))|^2\,dx = \int_\Omega |f(v(x)) - f(0) + f(0)|^2\,dx \\
&\leqslant \int_\Omega (|f(v(x)) - f(0)| + |f(0)|)^2\,dx \\
&= \int_\Omega |f(v(x)) - f(0)|^2\,dx + 2\int_\Omega |f(v(x)) - f(0)||f(0)|\,dx + \int_\Omega |f(0)|^2\,dx \\
&\leqslant L^2 \int_\Omega |v(x)|^2\,dx + 2L|f(0)| \int_\Omega |v(x)|\,dx + |f(0)|^2 \int_\Omega dx,
\end{aligned}
$$

which is finite because $v \in L^2(\Omega)$ and, since Ω is a bounded subset of $\mathbb{R}^n$, we have $L^2(\Omega) \hookrightarrow L^1(\Omega)$, so $v \in L^1(\Omega)$; moreover, the measure of Ω is finite. Therefore $F(v) \in L^2(\Omega)$.

(ii) F is Lipschitz.

Indeed, for $v, w \in L^2(\Omega)$, using the definition of F and the fact that f is Lipschitz, we obtain

$$
\|F(v) - F(w)\|_{L^2(\Omega)} = \big\|f(v) - f(w)\big\|_{L^2(\Omega)} \leqslant L\|v - w\|_{L^2(\Omega)}.
$$

Therefore, by Theorem 2.4, given $u_0 \in L^2(\Omega)$ there exists a unique solution

$$
u \in C^0([0,\infty); L^2(\Omega))
$$

which is a mild solution of

$$
\begin{cases}
u_t = \Delta u + f(u(t)) & \text{in} \quad (0,+\infty) \times \Omega, \\
u = 0 & \text{on} \quad (0,+\infty) \times \Gamma, \\
u(0) = u_0 & \text{in} \quad \Omega.
\end{cases}
\tag{3.4.40}
$$

That is,

$$
u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))\,ds, \quad \forall t \geq 0.
$$

We claim that $u(t)$ is continuously differentiable for every $t > 0$, i.e. $u \in C^1((0,\infty); L^2(\Omega))$.

Indeed, since Δ generates a differentiable semigroup $S(t)$ for $t > 0$ (see Section 3.1, second case), we have that, for every $t > 0$, $S(t)u_0$ is continuously differentiable (by Theorem 1.60, item (ii)) and, moreover,

$$
\frac{d}{dt} S(t)u_0 = \Delta S(t)u_0, \quad \forall u_0 \in L^2(\Omega). \tag{3.4.41}
$$

Now, for every $s \in \mathbb{R}$ we have $u(s) \in L^2(\Omega)$, hence $F(u(s)) = f(u(s)) \in L^2(\Omega)$. Thus $S(t-s)f(u(s))$ is continuously differentiable for every $t > s$, and so $\int_0^t S(t-s)f(u(s))\,ds$ is also continuously differentiable, since

$$
\frac{d}{dt}\int_0^t S(t-s)f(u(s))\,ds = \int_0^t \frac{d}{dt} S(t-s)f(u(s))\,ds + f(u(t)). \tag{3.4.42}
$$

This identity is adapted from Example 12A in [61], in the section dealing with the Leibniz rule.

Therefore $u(t)$ is continuously differentiable for $t > 0$ (being the sum of continuously differentiable functions), which proves the claim.

Furthermore, from (3.4.41) and (3.4.42) we obtain

$$\begin{aligned}
\frac{d}{dt}u(t) &= \frac{d}{dt}\left(S(t)u_0 + \int_0^t S(t-s)f(u(s))\,ds\right) \\
&= \frac{d}{dt}(S(t)u_0) + \frac{d}{dt}\left(\int_0^t S(t-s)f(u(s))\,ds\right) \\
&= \Delta S(t)u_0 + \int_0^t \frac{d}{dt}S(t-s)f(u(s))\,ds + f(u(t)) \\
&= \Delta S(t)u_0 + \int_0^t \Delta S(t-s)f(u(s))\,ds + f(u(t)) \\
&= \Delta S(t)u_0 + \Delta \int_0^t S(t-s)f(u(s))\,ds + f(u(t)) \\
&= \Delta\left(S(t)u_0 + \int_0^t S(t-s)f(u(s))\,ds\right) + f(u(t)) \\
&= \Delta u(t) + f(u(t)).
\end{aligned}$$

In the antepenultimate equality we used that the Laplacian is a closed operator (see Proposition 1.31 and the theorem used in the proof of Proposition 1.34 concerning closed operators), and in the penultimate equality we used the linearity of the Laplacian.

Moreover, we have:

(i) $u(t) \in C^0([0,\infty), L^2(\Omega))$, that is, u is continuous for every $t \geqslant 0$;

(ii) $u(t) \in C^1((0,\infty), L^2(\Omega))$, that is, u is continuously differentiable for every $t > 0$;

(iii) $u(t) \in D(\Delta)$, since $D(\Delta)$ is a vector space, $S(t)u_0 \in D(\Delta)$ by Theorem 1.60 (because the semigroup generated by Δ is differentiable) and $\int_0^t S(t-s)f(u(s))\,ds \in D(\Delta)$ by Proposition 1.30(iii) (as $f(u(s)) \in L^2(\Omega)$);

(iv) $u(t)$ satisfies $u_t = \Delta u(t) + f(u(t))$.

Hence u is a classical solution of problem (3.4.40).

It remains to show that $u(t) \in C^0((0,\infty), H^2(\Omega))$. To this end, we first prove that $u(t) \in C^0((0,\infty), D(\Delta))$. Let $t \to t_0$ in $\mathbb{R}_+$. Then

$$\|u(t) - u(t_0)\|_{D(\Delta)} = \|u(t) - u(t_0)\|_{L^2(\Omega)} + \|\Delta u(t) - \Delta u(t_0)\|_{L^2(\Omega)} \to 0,$$

because $u(t) \in C^0([0,+\infty), L^2(\Omega))$ and Δ generates a differentiable semigroup. Thus

$$u(t) \in C^0((0,\infty), D(\Delta)).$$

But, under our hypotheses (Ω open in $\mathbb{R}^n$ with sufficiently smooth boundary Γ), the norm $\|\cdot\|_{D(\Delta)}$ is equivalent to the norm $\|\cdot\|_{H^2(\Omega)}$. Hence

$$u(t) \in C^0((0,\infty), H^2(\Omega)).$$

The second part of the theorem follows from Theorem 2.4 (ii) (since f is Lipschitz). □

Lemma 3.5 *$D(\Delta^2)$ is dense in $H_0^1(\Omega) \cap H^2(\Omega)$.*

Proof: We know that $H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow L^2(\Omega)$. Moreover,

$$L^2(\Omega) = \overline{C_0^\infty(\Omega)}^{L^2(\Omega)} \subset \overline{H_0^1(\Omega) \cap H^2(\Omega)}^{L^2(\Omega)} \subset L^2(\Omega),$$

that is, $H_0^1(\Omega) \cap H^2(\Omega)$ is dense in $L^2(\Omega)$.

Consider the bilinear form $a : (H_0^1(\Omega) \cap H^2(\Omega))^2 \to \mathbb{R}$, defined by

$$a(u, v) = (\Delta u, \Delta v)_{L^2(\Omega)}, \quad \forall u, v \in H_0^1(\Omega) \cap H^2(\Omega).$$

We have that a is continuous, because for $u, v \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$\begin{aligned} |a(u,v)| &= \left| \int_\Omega \Delta u(x)\, \Delta v(x)\, dx \right| \\ &\leq \left(\int_\Omega |\Delta u(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |\Delta v(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \|\Delta u\|_{L^2(\Omega)} \|\Delta v\|_{L^2(\Omega)} \\ &= \|u\|_{H_0^1(\Omega)\cap H^2(\Omega)} \|v\|_{H_0^1(\Omega)\cap H^2(\Omega)}. \end{aligned}$$

Moreover, a is coercive, since

$$a(u, u) = (\Delta u, \Delta u)_{L^2(\Omega)} = \|\Delta u\|^2_{L^2(\Omega)} = \|u\|^2_{H_0^1(\Omega)\cap H^2(\Omega)}, \quad \forall u \in H_0^1(\Omega) \cap H^2(\Omega).$$

Under these conditions, by the Lax–Milgram Lemma, given $f \in L^2(\Omega)$ there exists a unique $u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$(\Delta u, \Delta v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega) \cap H^2(\Omega).$$

Now, observe that for every $\varphi \in \mathcal{D}(\Omega)$, we get from the equality above

$$\langle \Delta u, \Delta \varphi \rangle_{\mathcal{D}', \mathcal{D}} = \langle f, \varphi \rangle_{\mathcal{D}', \mathcal{D}},$$

or equivalently,

$$\langle \Delta(\Delta u), \varphi \rangle_{\mathcal{D}', \mathcal{D}} = \langle f, \varphi \rangle_{\mathcal{D}', \mathcal{D}},$$

whence

$$\langle \Delta^2 u, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = \langle f, \varphi \rangle_{\mathcal{D}', \mathcal{D}}, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Thus $\Delta^2 u = f$ in $\mathcal{D}'(\Omega)$ and, since $f \in L^2(\Omega)$, it follows that $\Delta^2 u \in L^2(\Omega)$.

Because $u \in H_0^1(\Omega) \cap H^2(\Omega)$, we have $u = 0$ on Γ. In this way, u satisfies

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma. \end{cases} \tag{3.4.43}$$

Moreover, for every $v \in H_0^1(\Omega) \cap H^2(\Omega)$, we have

$$\begin{aligned}
(f, v)_{L^2(\Omega)} &= (\Delta^2 u, v)_{L^2(\Omega)} \\
&= \int_\Omega \Delta^2 u \, v \, dx \\
&= \int_\Omega \Delta(\Delta u) \, v \, dx \\
&= -\int_\Omega \nabla(\Delta u) \cdot \nabla v \, dx + \int_\Gamma \frac{\partial \Delta u}{\partial \nu} v \, d\Gamma \\
&= \int_\Omega \nabla v \cdot \nabla(\Delta u) \, dx + \int_\Gamma \frac{\partial \Delta u}{\partial \nu} v \, d\Gamma \\
&= \int_\Omega \Delta v \, \Delta u \, dx - \int_\Gamma \frac{\partial v}{\partial \nu} \Delta u \, d\Gamma \\
&= (\Delta u, \Delta v)_{L^2(\Omega)} - \int_\Gamma \frac{\partial v}{\partial \nu} \Delta u \, d\Gamma \\
&= a(u, v) - \int_\Gamma \frac{\partial v}{\partial \nu} \Delta u \, d\Gamma.
\end{aligned}$$

Hence

$$(f, v)_{L^2(\Omega)} = a(u, v) - \int_\Gamma \frac{\partial v}{\partial \nu} \Delta u \, d\Gamma = (f, v)_{L^2(\Omega)} - \int_\Gamma \frac{\partial v}{\partial \nu} \Delta u \, d\Gamma,$$

and therefore

$$\int_\Gamma \frac{\partial v}{\partial \nu} \Delta u \, d\Gamma = 0, \quad \forall v \in H_0^1(\Omega) \cap H^2(\Omega),$$

that is, $\Delta u = 0$ on Γ. Thus we have found a solution of

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \\ \Delta u = 0 & \text{on} \quad \Gamma. \end{cases} \tag{3.4.44}$$

Furthermore, by elliptic regularity theory we obtain $u \in H^4(\Omega)$.

Now, since the following conditions hold:

(i) $H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow L^2(\Omega)$;

(ii) $H_0^1(\Omega) \cap H^2(\Omega)$ is dense in $L^2(\Omega)$;

(iii) $a(u, v) = (\Delta u, \Delta v)_{L^2(\Omega)}$ is bilinear, continuous and coercive,

we obtain that the triple $\{H_0^1(\Omega) \cap H^2(\Omega), L^2(\Omega), a(u, v)\}$ defines an operator A, whose domain is characterised by

$$D(A) = \{u \in H_0^1(\Omega) \cap H^2(\Omega);\, \Delta^2 u \in L^2(\Omega) \text{ and } \Delta u = 0 \text{ on } \Gamma\} =: Y, \quad A = \Delta^2.$$

Indeed, let $u \in D(A)$. Then there exists $f \in L^2(\Omega)$ such that $a(u, v) = (f, v)_{L^2(\Omega)}$ for every $v \in H_0^1(\Omega) \cap H^2(\Omega)$. Taking $\varphi \in \mathcal{D}(\Omega)$, we obtain

$$\langle \Delta^2 u, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = \langle \Delta u, \Delta \varphi \rangle_{\mathcal{D}', \mathcal{D}} = \langle f, \varphi \rangle_{\mathcal{D}', \mathcal{D}},$$

which implies $\Delta^2 u = f \in L^2(\Omega)$ and, as above, $\Delta u = 0$ on Γ. Hence $u \in Y$.

Conversely, let $u \in Y$. Then $u \in H_0^1(\Omega) \cap H^2(\Omega)$, $\Delta^2 u \in L^2(\Omega)$ and $\Delta u = 0$ on Γ. Thus, for every

$v \in H_0^1(\Omega) \cap H^2(\Omega)$, applying the generalised Green formula, we get

$$\begin{aligned}(\Delta^2 u, v)_{L^2(\Omega)} &= (\Delta(\Delta u), v)_{L^2(\Omega)} \\ &= -\sum_{i=1}^n \int_\Omega \frac{\partial \Delta u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx + \langle \gamma_1(\Delta u), \gamma_0 v\rangle_{H^{-1/2},H^{1/2}} \\ &= -\sum_{i=1}^n \int_\Omega \frac{\partial \Delta u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx \\ &= \int_\Omega \Delta u\, \Delta v\, dx \\ &= (\Delta u, \Delta v)_{L^2(\Omega)}.\end{aligned}$$

Therefore $u \in D(A)$. Thus we have shown that $D(A) = Y$. Moreover,

$$D(A) = \{u \in H^4(\Omega);\ \Delta u = u = 0 \text{ on } \Gamma\},$$

where the right-hand side is Y rewritten using the regularity already obtained. Also, since

$$(\Delta^2 u, v)_{L^2(\Omega)} = a(u, v) = (Au, v)_{L^2(\Omega)},$$

for every $v \in H_0^1(\Omega) \cap H^2(\Omega)$, we have $Au = \Delta^2 u$ for every $u \in H_0^1(\Omega) \cap H^2(\Omega)$. Therefore

$$\Delta^2 \leftrightarrow \{H_0^1(\Omega) \cap H^2(\Omega), L^2(\Omega), a(u, v)\}$$

and we conclude that $D(\Delta^2) = D(A)$ is dense in $H_0^1(\Omega) \cap H^2(\Omega)$, as claimed. □

Theorem 3.6 *If $f \in C^3(\mathbb{R})$, $f(0) = 0$ and $n \le 3$, then, for each $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, there exists a classical solution of (3.4.39) on $[0, T_{max})$, with*

$$u \in C^1([0, T_{\max}); L^2(\Omega)) \cap C([0, T_{\max}); H^2(\Omega))$$

and either $T_{\max} = +\infty$ or $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} \|u(t)\|_{H^2(\Omega)} = \infty$.

Proof: We must show that $F : D(\Delta) \to D(\Delta)$ is locally Lipschitz, where $F(u)(x) = f(u(x))$, so that we can apply Theorem 2.24.

First, we prove that if $u \in H_0^1(\Omega) \cap H^2(\Omega)$, then

$$f(u) \in H_0^1(\Omega) \cap H^2(\Omega).$$

To see this, note that $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, so there exists $c_1 > 0$ such that

$$\|u\|_\infty \le c_1 \|u\|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega).$$

Now, given $M > 0$ and $f \in C^3(\mathbb{R})$, there exist constants $L_1, L_2, L_3 > 0$ such that

$$|f(t)| \le L_1, \quad |f'(t)| \le L_2 \ \text{ and } \ |f''(t)| \le L_3, \quad \forall t \in [0, M].$$

Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and set $M = c_1 \|u\|_{H^2(\Omega)}$. Since

$$|u(x)| \le \|u\|_\infty \le M \quad \text{for a.e. } x \in \Omega,$$

we obtain

$$|f(u(x))| \le L_1, \quad |f'(u(x))| \le L_2 \ \text{ and } \ |f''(u(x))| \le L_3 \quad \text{for a.e. } x \in \Omega,$$

so $f(u), f'(u)$ and $f''(u)$ belong to $L^\infty(\Omega)$.

Moreover,

$$\begin{aligned}
\frac{\partial}{\partial x_i} f(u) &= f'(u)\frac{\partial u}{\partial x_i} \in L^2(\Omega),\\
\frac{\partial^2}{\partial x_i^2} f(u) &= f'(u)\frac{\partial^2 u}{\partial x_i^2} + f''(u)\left(\frac{\partial u}{\partial x_i}\right)^2 \in L^2(\Omega),\\
\frac{\partial^2}{\partial x_i \partial x_j} f(u) &= f'(u)\frac{\partial^2 u}{\partial x_i \partial x_j} + f''(u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \in L^2(\Omega),
\end{aligned}$$

because $f'(u), f''(u) \in L^\infty(\Omega)$ and, since $\dfrac{\partial u}{\partial x_i} \in H^1(\Omega)$ and $n \leq 3$, we have $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega)$; hence $\left(\dfrac{\partial u}{\partial x_i}\right)^2 \in L^2(\Omega)$ and, for the same reason, $\dfrac{\partial u}{\partial x_i}\dfrac{\partial u}{\partial x_j} \in L^2(\Omega)$. Therefore

$$f(u) \in H^2(\Omega).$$

Since $u \in H_0^1(\Omega)$, there exists a sequence $\{\varphi_n\} \subset C_0^\infty(\Omega)$ such that

$$\varphi_n \longrightarrow u \quad \text{in } H^1(\Omega),$$

and, because $f \in C^3(\mathbb{R})$, we have

$$f(\varphi_n) \longrightarrow f(u) \quad \text{in } H^1(\Omega).$$

Moreover,

$$\gamma_0(f(\varphi_n)) = f(\varphi_n)\big|_\Gamma = f(0) = 0, \quad \forall n \in \mathbb{N},$$

so

$$0 = \gamma_0(f(\varphi_n)) \longrightarrow \gamma_0(f(u)) \quad \text{in } H^{1/2}(\Gamma),$$

and hence

$$\gamma_0(f(u)) = 0.$$

Consequently,

$$f(u) \in H_0^1(\Omega).$$

Thus $F : H_0^1(\Omega) \cap H^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega)$, defined by $F(u) = f(u)$, is well-defined.

Finally, if $\|u\|_{H^2(\Omega)} \leq M$ and $\|v\|_{H^2(\Omega)} \leq M$, then

$$\|u\|_\infty \leq c_1 M \quad \text{and} \quad \|v\|_\infty \leq c_1 M.$$

Hence

$$\begin{aligned}
\|f(u) - f(v)\|_2 &\leq C_M \|u - v\|_2,\\
\left\|\frac{\partial}{\partial x_i} f(u) - \frac{\partial}{\partial x_i} f(v)\right\|_2 &\leq \left\|(f'(u) - f'(v))\frac{\partial u}{\partial x_i}\right\|_2 + \left\|f'(v)\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right)\right\|_2\\
&\leq C_1 M \|u - v\|_2 + \|f'(v)\|_\infty \left\|\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right\|_2,
\end{aligned}$$

and, in a similar way, we obtain estimates for $\dfrac{\partial^2}{\partial x_i \partial x_j} f(u) - \dfrac{\partial^2}{\partial x_i \partial x_j} f(v)$. Therefore

$$F : D(\Delta) \to D(\Delta)$$

is locally Lipschitz, and the result follows from Theorem 2.24. □

Theorem 3.7 *If $f \in C^1(\mathbb{R})$ and $f(0) = 0$, then, for every $u_0 \in L^\infty(\Omega)$, there exists $u \in L^\infty([0,T], L^\infty(\Omega))$*

for all $T < T_{\max}$ which is a solution of (3.4.39) on every interval $[0,T]$ with $T < T_{\max}$, and

$$T_{\max} = \infty \quad \text{or else, if} \quad T_{\max} < \infty, \quad \text{then} \quad \lim_{t \to T_{\max}} \|u(\cdot,t)\|_\infty = \infty.$$

Proof: Let $M = \|u_0\|_\infty$ and define

$$\widetilde{f} : \mathbb{R} \to \mathbb{R}$$

by

$$\widetilde{f}(t) = \begin{cases} f(t) & \text{if } |t| \leq M+1, \\ f(M+1) & \text{if } t > M+1, \\ f(-M-1) & \text{if } t < -M-1. \end{cases}$$

We claim that $\widetilde{f}$ is Lipschitz. Indeed, if $a, b \in [-M-1, M+1]$, then, by the Mean Value Theorem, by the continuity of f' and since $[-M-1, M+1]$ is compact, there exists $d > 0$ such that

$$|\widetilde{f}(b) - \widetilde{f}(a)| \leq d|b-a|.$$

If $a, b \in (M+1, \infty)$ or $a, b \in (-\infty, -M-1)$, then

$$|\widetilde{f}(b) - \widetilde{f}(a)| = 0 \leq |b-a|.$$

If $a \in [-M-1, M+1]$ and $b \in (M+1, \infty)$, then, by the Mean Value Theorem, by the continuity of f', by the compactness of $[-M-1, M+1]$ and since $|M+1-a| \leq |b-a|$, there exists $d > 0$ such that

$$|\widetilde{f}(b) - \widetilde{f}(a)| = |f(M+1) - f(a)| \leq d|M+1-a| \leq d|b-a|.$$

By an analogous argument, if $a \in [-M-1, M+1]$ and $b \in (-\infty, -M-1)$, we have

$$|\widetilde{f}(b) - \widetilde{f}(a)| \leq d|b-a|.$$

Thus, taking $L_f = \max\{1, d\}$, for all $a, b \in \mathbb{R}$,

$$|\widetilde{f}(b) - \widetilde{f}(a)| \leq L_f|b-a|.$$

Define $F : L^p(\Omega) \to L^p(\Omega)$ by $F(g) = \widetilde{f}(g)$.

We show that F is well-defined, that is, for each $g \in L^p(\Omega)$ we must show that $\widetilde{f}(g) \in L^p(\Omega)$, for every $1 < p < \infty$. Since $\widetilde{f}$ is Lipschitz, we have

$$|\widetilde{f}(g(x)) - \widetilde{f}(0(x))| \leqslant L_f|g(x)|.$$

Hence

$$\begin{aligned} \|\widetilde{f}(g)\|_p^p &= \int_\Omega |\widetilde{f}(g(x))|^p dx = \int_\Omega |\widetilde{f}(g(x)) - \widetilde{f}(0(x))|^p dx \\ &\leqslant \widetilde{L_f} \int_\Omega |g(x)|^p dx < +\infty, \end{aligned}$$

since $\widetilde{f}(0) = 0$ and $g \in L^p(\Omega)$.

We now show that F is Lipschitz, since

$$\begin{aligned} \|F(g_1) - F(g_2)\|_{L^p(\Omega)}^p &= \int_\Omega \left|\widetilde{f}(g_1(x)) - \widetilde{f}(g_2(x))\right|^p dx \\ &\leqslant \widetilde{L_f} \int_\Omega |g_1(x) - g_2(x)|^p dx = \widetilde{L_f}\, \|g_1 - g_2\|_{L^p(\Omega)}^p . \end{aligned}$$

Our goal is to use Theorem 2.4. To do so, we verify the hypotheses of that theorem and of the section:

(i) X is a reflexive Banach space;

(ii) F is continuous;

(iii) A is the infinitesimal generator of a C_0-semigroup S such that $\|S(t)\| \leqslant M$ for all $t \geqslant 0$.

Indeed:

(i) Since $X = L^p(\Omega)$, $1 < p < +\infty$, we know that X is a reflexive Banach space.

(ii) Since F is Lipschitz, F is continuous.

(iii) Taking $A = \Delta$ and $D(A) = D(\Delta) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \subset L^p(\Omega)$, where $W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega)\, ;\, \gamma_0(u) = 0\}$, we shall prove that $\Delta : W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \subset L^p(\Omega)$ generates a C_0-semigroup of contractions. That is, $\Delta \in G(1, 0)$. For this we use the Lumer–Phillips Theorem. We prove that:

(I) $D(\Delta)$ is dense in $X = L^p(\Omega)$.

(II) Δ is a dissipative operator with respect to a duality mapping.

(III) $Im(\lambda_0 I - \Delta) = L^p(\Omega)$ for some $\lambda_0 > 0$.

We now check these conditions.

(I) We know that $\mathcal{D}(\Omega) \subset W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \subset L^p(\Omega)$. Hence,

$$L^p(\Omega) = \overline{\mathcal{D}(\Omega)}^{L^p(\Omega)} \subset \overline{W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)}^{L^p(\Omega)} \subset \overline{L^p(\Omega)}^{L^p(\Omega)} = L^p(\Omega),$$

so

$$\overline{W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)}^{L^p(\Omega)} = L^p(\Omega),$$

that is, $D(\Delta)$ is dense in X.

(II) We show that

$$\langle j(u), \Delta u\rangle \leqslant 0, \quad \forall u \in D(\Delta) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega),$$

where $j : L^p \to L^{p'}$ is such that, for each $u \in L^p(\Omega)$,

$$j(u) \in F(u) = \left\{u^* \in L^{p'}\, ;\, \langle u^*, u\rangle = \|u^*\|_{p'}^2 = \|u\|_p^2\right\}.$$

For this, we divide into three cases:

1. $p > 2$;
2. $p = 2$;
3. $p \in (1, 2)$.

<u>Case 1.</u> Consider $j : L^p \to L^{p'}$ such that, for each $u \in L^p(\Omega)$, we associate

$$j(u) = u|u|^{p-2}\|u\|_p^{2-p}.$$

We show that $u|u|^{p-2}\|u\|_p^{2-p} \in L^{p'}(\Omega)$ and that $j(u) \in F(u)$ (so j is well-defined and is indeed a duality mapping).

In fact,

$$\begin{aligned}\left\|u|u|^{p-2}\|u\|_p^{2-p}\right\|_{p'}^{p'} &= \int_\Omega \left|u|u|^{p-2}\|u\|_p^{2-p}\right|^{p'} dx = \int_\Omega \left(u|u|^{p-2}\|u\|_p^{2-p}\right)^{p'} dx \\ &= \int_\Omega (|u|^{p-1}\|u\|_p^{2-p})^{p'} dx = \int_\Omega (|u|^{p-1}\|u\|_p^{2-p})^{\frac{p}{p-1}} dx \\ &= \|u\|_p^{\frac{(2-p)p}{p-1}} \int_\Omega (|u|^{p-1})^{\frac{p}{p-1}} dx = \|u\|_p^{\frac{(2-p)p}{p-1}} \|u\|_p^p \\ &= \|u\|_p^{p'}.\end{aligned}$$

Hence

$$\left\|u|u|^{p-2}\|u\|_p^{2-p}\right\|_{p'} = \|u\|_p,$$

and in particular $u|u|^{p-2}\|u\|_p^{2-p} \in L^{p'}(\Omega)$ (since $u \in L^p(\Omega)$). Moreover,

$$\left\|u|u|^{p-2}\|u\|_p^{2-p}\right\|_{p'}^2 = \|u\|_p^2. \tag{3.4.45}$$

To guarantee that $u|u|^{p-2}\|u\|_p^{2-p} \in F(u)$, it remains to show that

$$\langle u|u|^{p-2}\|u\|_p^{2-p}, u\rangle = \|u\|_p^2.$$

Indeed,

$$\begin{aligned}\langle u|u|^{p-2}\|u\|_p^{2-p}, u\rangle &= \int_\Omega u|u|^{p-2}\|u\|_p^{2-p} u\, dx = \|u\|_p^{2-p} \int_\Omega u^2|u|^{p-2} dx \\ &= \|u\|_p^{2-p} \int_\Omega u^2(u^2)^{\frac{p-2}{2}} dx = \|u\|_p^{2-p} \int_\Omega (u^2)^{\frac{p-2}{2}+1} dx \\ &= \|u\|_p^{2-p} \int_\Omega (u^2)^{\frac{p}{2}} dx = \|u\|_p^{2-p} \int_\Omega |u|^p dx \\ &= \|u\|_p^{2-p}\|u\|_p^p = \|u\|_p^2.\end{aligned} \tag{3.4.46}$$

From (3.4.45) and (3.4.46) it follows that $j(u) \in F(u)$. Thus, in Case 1 we can use the duality mapping $j(u) = u|u|^{p-2}\|u\|_p^{2-p}$. Then

$$\begin{aligned}\langle j(u), \Delta u\rangle &= \langle u|u|^{p-2}\|u\|_p^{2-p}, \Delta u\rangle = \int_\Omega u|u|^{p-2}\|u\|_p^{2-p} \Delta u\, dx \\ &= \|u\|_p^{2-p} \int_\Omega u|u|^{p-2}\Delta u\, dx = -\|u\|_p^{2-p} \int_\Omega \nabla(u|u|^{p-2})\nabla u\, dx \\ &= -\|u\|_p^{2-p} \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}(u|u|^{p-2}) \frac{\partial u}{\partial x_i} dx \\ &= -\|u\|_p^{2-p} \int_\Omega \sum_{i=1}^n \left((p-1)|u|^{p-2}\frac{\partial u}{\partial x_i}\right) \frac{\partial u}{\partial x_i} dx \\ &= -\|u\|_p^{2-p}(p-1) \int_\Omega \sum_{i=1}^n |u|^{p-2} \left|\frac{\partial u}{\partial x_i}\right|^2 dx \\ &= -\|u\|_p^{2-p}(p-1) \int_\Omega |\nabla u|^2|u|^{p-2}\, dx \leqslant 0,\end{aligned}$$

where in the antepenultimate equality we used

$$\frac{\partial}{\partial x_i}(u|u|^{p-2}) = (p-1)|u|^{p-2}\frac{\partial u}{\partial x_i}.$$

Note also that $u|u|^{p-2}\|u\|_p^{2-p}\Delta u$ is integrable because

$$\int_\Omega u|u|^{p-2}\|u\|_p^{2-p}\Delta u\,dx \leqslant \left(\int_\Omega |u|u|^{p-2}\|u\|_p^{2-p}|^{p'}\right)^{\frac{1}{p'}}\left(\int_\Omega |\Delta u|^p\,dx\right)^{\frac{1}{p}} < +\infty.$$

Thus, in Case 1 we have $\langle j(u),\Delta u\rangle \leqslant 0$ for all $u \in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega) = D(\Delta)$ for the mapping $j : L^p \to L^{p'}$ given by $j(u) = u|u|^{p-2}\|u\|_p^{2-p}$. That is, Δ is dissipative with respect to this duality mapping.

Case 2. Consider the duality mapping $j(u) = u$ (since $L^2(\Omega)$ is a Hilbert space). Then

$$\langle j(u),\Delta u\rangle = \langle u,\Delta u\rangle = \int_\Omega u\Delta u\,dx = -\int_\Omega \nabla u\nabla u\,dx = -\int_\Omega |\nabla u|^2\,dx = -\|\nabla u\|^2_{L^2(\Omega)} \leqslant 0,$$

showing that in Case 2 also we have $\langle j(u),\Delta u\rangle \leqslant 0$ for all $u \in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega) = D(\Delta)$ and $j(u) = u$, i.e. Δ is dissipative for this duality mapping.

Case 3. Here $1 < p < 2$ and Ω is bounded, so $L^2(\Omega) \hookrightarrow L^p(\Omega)$, whence $-C\|\cdot\|_{L^2(\Omega)} \leqslant -\|\cdot\|_{L^p(\Omega)}$. Thus, by the same estimates as in Case 2 (and since $\Delta u \in L^p(\Omega)$), it follows that Δ is a dissipative operator with respect to the same duality mapping as in Case 2.

We conclude that Δ is dissipative with respect to a suitable duality mapping (in each of the three cases).

(III) We show that $Im(\lambda_0 I - \Delta) = L^p(\Omega)$ for some $\lambda_0 > 0$. Take $\lambda_0 = 1$. We want to prove: given $f \in L^p(\Omega)$, there exists $u \in D(\Delta)$ such that

$$u - \Delta u = f.$$

To obtain this, it suffices to use Theorem 9.32 (Agmon–Douglis–Nirenberg) in [18].

Hence, by the Lumer–Phillips Theorem, we deduce that $\Delta \in G(1,0)$, that is, Δ is the infinitesimal generator of a contraction semigroup.

Thus, by Theorem 2.4, given $u_0 \in L^p(\Omega) = X$, there exists a unique function

$$u \in C^0([0,+\infty);L^p(\Omega))$$

which is a mild solution of problem (3.4.40), that is,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\widetilde{f}(u(x,s))\,ds. \tag{3.4.47}$$

Note that for every $u_0 \in L^p(\Omega)$ with $1 < p < +\infty$, we have

$$\|S(t)u_0\|_p \leqslant \|S(t)\|\|u_0\|_p \leqslant 1\cdot\|u_0\|_p. \tag{3.4.48}$$

For $u_0 \in L^\infty(\Omega)$, from (3.4.48) we get

$$\lim_{p\to+\infty}\|S(t)u_0\|_p \leqslant \lim_{p\to+\infty}\|u_0\|_p,$$

that is,

$$\|S(t)u_0\|_\infty \leqslant \|u_0\|_\infty < +\infty, \tag{3.4.49}$$

since $u_0 \in L^\infty(\Omega)$. Hence $S(t)u_0 \in L^\infty(\Omega)$. In addition, since $\widetilde{f}$ is bounded, it follows that $\int_0^t S(t-$

$s)\widetilde{f}(u(s))\,ds \in L^\infty(\Omega)$, because

$$\begin{aligned}\left\|\int_0^t S(t-s)\widetilde{f}(u(s))\,ds\right\|_\infty &= \lim_{p\to+\infty}\left\|\int_0^t S(t-s)\widetilde{f}(u(s))\,ds\right\|_p \\ &\leqslant \lim_{p\to+\infty}\int_0^t \|S(t-s)\|_{\mathcal{L}(X)}\|\widetilde{f}(u(s))\|_p\,ds \leqslant \lim_{p\to+\infty}\int_0^t \|\widetilde{f}(u(s))\|_p\,ds. \quad (3.4.50)\end{aligned}$$

Note that, fixing $T>0$,

$$\|\widetilde{f}(u(s))\|_p = \left(\int_\Omega |\widetilde{f}(u(x,s))|^p\,dx\right)^{\frac{1}{p}} = C\left(\int_\Omega |1|^p\,dx\right)^{\frac{1}{p}} = C\,(\text{meas}(\Omega))^{\frac{1}{p}}. \tag{3.4.51}$$

Using (3.4.51) in (3.4.50), we obtain

$$\begin{aligned}\left\|\int_0^t S(t-s)\widetilde{f}(u(s))\,ds\right\|_\infty &\leqslant \lim_{p\to+\infty}\int_0^t C\cdot(\text{meas}(\Omega))^{\frac{1}{p}}\,ds = C\cdot t\cdot\lim_{p\to+\infty}(\text{meas}(\Omega))^{\frac{1}{p}} \\ &= C\cdot t < C\cdot T < +\infty.\end{aligned}$$

Thus

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\widetilde{f}(u(x,s))\,ds \in L^\infty(\Omega).$$

Moreover, using (3.4.49), the fact that $\widetilde{f}$ is Lipschitz, and the Dominated Convergence Theorem, we have

$$\begin{aligned}\|u(t)\|_\infty &= \left\|S(t)u_0 + \int_0^t S(t-s)\widetilde{f}(u(s))\,ds\right\|_\infty \\ &\leqslant \|S(t)u_0\|_\infty + \lim_{p\to+\infty}\int_0^t \|S(t-s)\|_{\mathcal{L}(X)}\|\widetilde{f}(u(s))\|_p\,ds \\ &\leqslant \|u_0\|_\infty + \lim_{p\to+\infty}\int_0^t \|\widetilde{f}(u(s))\|_p\,ds \\ &\leqslant \|u_0\|_\infty + \widetilde{L_f}\lim_{p\to+\infty}\int_0^t \|u(s)\|_p\,ds \\ &\leqslant \|u_0\|_\infty + \widetilde{L_f}\int_0^t \|u(s)\|_\infty\,ds = M + \widetilde{L_f}\int_0^t \|u(s)\|_\infty\,ds.\end{aligned}$$

By Gronwall's Lemma,

$$\|u(t)\|_\infty \leqslant Me^{\widetilde{L_f}\int_0^t 1ds} = Me^{\widetilde{L_f}t} \leqslant Me^{\widetilde{L_f}T}.$$

Hence

$$u \in L^\infty(0,T;L^\infty(\Omega)).$$

Now, if T is sufficiently small so that

$$Me^{LT} \leq M+1,$$

then u is also a weak solution of problem (3.4.40) with f in place of $\widetilde{f}$, that is,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))\,ds$$

is a weak solution of (3.4.40). Thus, u can be extended to an interval $[0,T_{\max})$, with $T_{\max}=+\infty$ or, if $T_{\max}<+\infty$, then

$$\lim_{t\to T_{\max}^-}\|u(t)\| = +\infty,$$

as in Theorem 2.25. □

In what follows, we present two particular cases of problem (3.4.39) in which, in the first case, one obtains a global solution, and in the second case one obtains blow-up in finite time of the solution.

Example 3.8 Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(t) = -t^3$ and assume $n \leq 3$.

We observe that f satisfies
$$f(t)t < 0 \quad \forall t.$$

Proposition 3.9 *For every $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, we have $T_{\max} = +\infty$.*

Proof: By Theorem 3.6, there exists a classical solution of (3.4.39), that is, there exists a unique u satisfying
$$\begin{cases} u_t - \Delta u = -u^3 & \text{in} \quad [0, T_{\max}) \times \Omega, \\ u = 0 & \text{on} \quad [0, T_{\max}) \times \Gamma, \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{3.4.52}$$
and
$$u \in C^1([0, T_{\max}); L^2(\Omega)) \cap C([0, T_{\max}); H^2(\Omega)).$$

Since $n \leq 3$, we have $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, hence $u_0 \in L^\infty(\Omega)$ and, therefore, by Theorem 3.7,
$$u \in L^\infty([0, T]; L^\infty(\Omega)), \quad \forall T < T_{\max}.$$

Multiplying $(3.4.52)_1$ by $|u(t)|^{p-2}u(t)$, we obtain
$$\int_\Omega u'(t)|u(t)|^{p-2}u(t)\, dx = \int_\Omega \Delta u(t)|u(t)|^{p-2}u(t)\, dx - \int_\Omega |u(t)|^{p-2}u^4(t)\, dx.$$
Hence,
$$\frac{1}{p}\frac{d}{dt}\int_\Omega |u(t)|^p dx \leq 0,$$
which, integrating from 0 to t, yields
$$\|u(t)\|_p \leq \|u_0\|_p \quad \forall p.$$
Passing to the limit as $p \to \infty$ we get
$$\|u(t)\|_\infty \leq \|u_0\|_\infty,$$
and thus
$$\lim_{t \to T_{\max}} \|u(t)\|_\infty < \infty.$$
It follows that $T_{\max} = +\infty$. □

Example 3.10 Consider the particular case of problem (3.4.39) in which $f : \mathbb{R} \to \mathbb{R}$ is given by $f(t) = t^3$ and $n \leq 3$, that is,
$$\begin{cases} u_t - \Delta u = u^3 & \text{in} \quad [0, T_{\max}) \times \Omega, \\ u = 0 & \text{on} \quad [0, T_{\max}) \times \Gamma, \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{3.4.53}$$
where $\Omega \subset \mathbb{R}^n$, $n \leq 3$, is a bounded open set with regular boundary Γ, as fixed at the beginning of this section. Thus, if $u_0 \neq 0$ with $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and
$$E(0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{4}\int_\Omega u_0^4 dx \leq 0,$$

where $E = E(t)$ is defined by

$$E(t) = \frac{1}{2}\int_{\Omega} |\nabla u(t)|^2 dx - \frac{1}{4}\int_{\Omega} u^4(t) dx, \tag{3.4.54}$$

then $T_{\max} < \infty$.

Proof: To prove that $T_{\max} < \infty$ we first show that (3.4.54) is decreasing, and then, assuming that $T_{\max} = \infty$, we derive a contradiction.

To prove that (3.4.54) is decreasing we shall use multiplicative methods to obtain $\frac{dE(t)}{dt} \leqslant 0$. Before that, note that, by Theorem 3.6, problem (3.4.53) admits a classical solution u such that

$$u \in C^1\big([0, T_{\max}); L^2(\Omega)\big) \cap C\big([0, T_{\max}); H_0^1(\Omega) \cap H^2(\Omega)\big). \tag{3.4.55}$$

We now construct a sequence of functions $\varphi_{n,k} \in C\big([0,T]; H_0^1(\Omega) \cap H^1(\Omega)\big)$ which, when used in $(3.4.53)_1$, will allow us to obtain $\frac{dE(t)}{dt} \leqslant 0$.

Regularising sequences. Let $0 < s_0 < t_0 < T < T_{\max}$ and choose $n_0 \in \mathbb{N}$ such that

$$n_0 > \max\left\{\frac{1}{s_0}, \frac{1}{T - t_0}\right\}.$$

For $n \geq n_0$, define $\theta_n : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\theta_n(t) = \begin{cases} 0 & \text{if } t \in \big[0, s_0 - \frac{1}{n}\big), \\ 1 + n(t - s_0) & \text{if } t \in \big[s_0 - \frac{1}{n}, s_0\big), \\ 1 & \text{if } t \in [s_0, t_0], \\ 1 - n(t - t_0) & \text{if } t \in \big(t_0, t_0 + \frac{1}{n}\big], \\ 0 & \text{if } t \in \big(t_0 + \frac{1}{n}, T\big], \end{cases}$$

whose derivative in the sense of distributions is

$$\theta_n'(t) = \begin{cases} 0 & \text{if } t \in \big[0, s_0 - \frac{1}{n}\big), \\ n & \text{if } t \in \big[s_0 - \frac{1}{n}, s_0\big), \\ 0 & \text{if } t \in [s_0, t_0], \\ -n & \text{if } t \in \big(t_0, t_0 + \frac{1}{n}\big], \\ 0 & \text{if } t \in \big(t_0 + \frac{1}{n}, T\big]. \end{cases}$$

Let $(\rho_k)_{k\in\mathbb{N}}$ be an even regularising sequence, that is, a sequence such that, for every $k \in \mathbb{N}$,

$$\rho_k \geqslant 0, \quad \rho_k \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}(\rho_k) \subset \left[-\frac{1}{k}, \frac{1}{k}\right],$$

$$\int_{\mathbb{R}} \rho_k(\xi)\, d\xi = 1, \qquad \rho_k(-\xi) = \rho_k(\xi),$$

and set

$$\varphi_{n,k} = \theta_n\big[(\theta_n u') * \rho_k * \rho_k\big], \tag{3.4.56}$$

where $*$ denotes the convolution in the variable t, defined in general by

$$(f * g)(t) = \int_{\mathbb{R}} f(t - \xi) g(\xi)\, d\xi.$$

The function given in (3.4.56) is well defined because, if $\tilde{\theta}_n$ and $\tilde{u}'$ denote the zero extensions of θ_n and u' outside $[0,T]$, then, for every $t \in [0,T]$,

$$\begin{aligned}\tilde{\theta}_n\big[(\tilde{\theta}_n\tilde{u}') * \rho_k * \rho_k\big](t) &= \tilde{\theta}_n(t)\int_{\mathbb{R}} (\tilde{\theta}_n\tilde{u}')(\xi)(\rho_k * \rho_k)(t-\xi)\,d\xi \\ &= \theta_n(t)\int_0^T (\theta_n u')(\xi)(\rho_k * \rho_k)(t-\xi)\,d\xi \\ &= \varphi_{n,k},\end{aligned}$$

so $\varphi_{n,k}$ is well-defined.

Moreover,

$$\begin{aligned}\operatorname{supp}\big((\theta_n u') * \rho_k * \rho_k\big) &\subset \operatorname{supp}(\theta_n u') + \left[-\frac{1}{k}, \frac{1}{k}\right] \\ &\subset \big(\operatorname{supp}(\theta_n) \cap \operatorname{supp}(u')\big) + \left[-\frac{2}{k}, \frac{2}{k}\right] \\ &\subset \operatorname{supp}(\theta_n) + \left[-\frac{2}{k}, \frac{2}{k}\right].\end{aligned}$$

If $x \in \left[s_0 - \frac{1}{n}, t_0 + \frac{1}{n}\right]$ and $y \in \left[-\frac{2}{k}, \frac{2}{k}\right]$, then

$$s_0 - \frac{1}{n_0} - \frac{2}{n_0} \leqslant x + y \leqslant t_0 + \frac{1}{n_0} + \frac{2}{k}. \tag{3.4.57}$$

Assume that

$$s_0 - \frac{1}{n_0} - \frac{2}{k} > 0 \tag{3.4.58}$$

and

$$t_0 + \frac{1}{n_0} + \frac{2}{k} < T. \tag{3.4.59}$$

From (3.4.58) we must have

$$\frac{1}{k} < \frac{s_0}{2} - \frac{1}{2s_0} \quad \Rightarrow \quad k > \frac{2n_0}{n_0 s_0 - 1},$$

and from (3.4.59) we deduce

$$\frac{1}{k} < \frac{T}{2} - \frac{1}{2n_0} - \frac{t_0}{2} \quad \Rightarrow \quad k > \frac{2n_0}{Tn_0 - t_0 n_0 - 1}.$$

Thus, imposing

$$k > \max\left\{\frac{2n_0}{n_0 s_0 - 1}, \frac{2n_0}{Tn_0 - t_0 n_0 - 1}\right\} =: k_0,$$

we obtain from (3.4.57) that $x + y \in (0,T)$, that is, for $k > k_0$,

$$\left[s_0 - \frac{1}{n_0}, t_0 + \frac{1}{n_0}\right] + \left[-\frac{2}{k}, \frac{2}{k}\right] \subset (0,T),$$

and hence $\operatorname{supp}\big((\theta_n u') * \rho_k * \rho_k\big)$ is compact in $(0,T)$, since

$$\operatorname{supp}\big((\theta_n u') * \rho_k * \rho_k\big) \subset (0,T) \subset [0,T], \quad \forall k > k_0.$$

From now on we consider $(\rho_k)_{k>k_0}$ and $(\theta_n)_{n\geqslant n_0}$.

On the other hand, for each $n \in \mathbb{N}$, $n \geqslant n_0$, both θ_n and θ_n' belong to $L^2(0,T)$, i.e. $\theta_n \in H_0^1(0,T)$.

Moreover, since

$$u \in C^1\big([0,T];L^2(\Omega)\big) \cap C\big([0,T];H_0^1(\Omega)\cap H^2(\Omega)\big),$$

we have $u' \in C\big([0,T];L^2(\Omega)\big)$. By the Leibniz rule,

$$(u\theta_n)' = u'\theta_n + u\theta_n' \quad \Rightarrow \quad u'\theta_n = (u\theta_n)' - u\theta_n',$$

and hence

$$(u'\theta_n) * \rho_k * \rho_k = \big[(u\theta_n)' * \rho_k * \rho_k\big] - \big[(u\theta_n') * \rho_k * \rho_k\big]. \tag{3.4.60}$$

Taking the first term on the right-hand side of (3.4.60) and integrating by parts, we obtain, for all $t \in [0,T]$,

$$\begin{aligned}
\big[(u\theta_n)' * \rho_k * \rho_k\big](t) &= \int_0^T (u\theta_n)'(\xi)(\rho_k * \rho_k)(t-\xi)\,d\xi \\
&= \big[(u\theta_n)(\xi)(\rho_k*\rho_k)(t-\xi)\big]_0^T - \int_0^T (u\theta_n)(\xi)\big(-(\rho_k*\rho_k)'\big)(t-\xi)\,d\xi \\
&= \int_0^T (u\theta_n)(\xi)(\rho_k * \rho_k')(t-\xi)\,d\xi,
\end{aligned}$$

that is,

$$(u\theta_n') * \rho_k * \rho_k = (u\theta_n) * \rho_k * \rho_k',$$

and thus (3.4.60) can be rewritten as

$$(u'\theta_n) * \rho_k * \rho_k = \big[(u\theta_n) * \rho_k * \rho_k'\big] - \big[(u\theta_n') * \rho_k * \rho_k\big].$$

Consequently we may rewrite (3.4.56) as

$$\varphi_{n,k} = \theta_n\big[(u'\theta_n) * \rho_k * \rho_k\big] = \theta_n\Big(\big[(u\theta_n) * \rho_k * \rho_k'\big] - \big[(u\theta_n') * \rho_k * \rho_k\big]\Big),$$

which implies

$$\varphi_{n,k} \in C_0\big([0,T];H_0^1(\Omega)\cap H^2(\Omega)\big).$$

Approximate problem. Since $n \leqslant 3$, we have

$$u(t) \in H^1(\Omega) \quad \Rightarrow \quad u(t) \in L^6(\Omega) \quad \Rightarrow \quad u^3(t) \in L^2(\Omega),$$

and from (3.4.55),

$$u' \in C\big([0,T];L^2(\Omega)\big) \quad \Rightarrow \quad u'(t) \in L^2(\Omega),$$

and

$$u \in C\big([0,T];H_0^1(\Omega)\cap H^2(\Omega)\big) \quad \Rightarrow \quad \Delta u(t) \in L^2(\Omega).$$

It then makes sense to multiply $(3.4.53)_1$ by $\varphi_{n,k}$ and integrate in Ω and in $(0,T)$, obtaining

$$\int_0^T \big(u'(t),\varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt - \int_0^T \big(\Delta u(t),\varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt = \int_0^T \big(u^3(t),\varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt. \tag{3.4.61}$$

The first term on the left-hand side of (3.4.61) can be rewritten as

$$\int_0^T \big(u'(t),\varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt = \int_0^T \big(((u'\theta_n)*\rho_k)(t),((u'\theta_n)*\rho_k)(t)\big)_{L^2(\Omega)}\,dt, \tag{3.4.62}$$

because, extending u' and θ_n by zero (denoted by $\tilde{u}'$ and $\tilde{\theta}_n$), and for notational simplicity letting $h = \rho_k$,

$\check{h}(t) = h(-t)$, $f = \tilde{u}'\tilde{\theta}_n$ and $g = (\tilde{u}'\tilde{\theta}_n) * \rho_k$, and changing the order of integration, we obtain

$$\begin{aligned}\int_0^T \big(u'(t), \varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt &= \int_0^T \big((u'\theta_n)(t), ((u'\theta_n) * \rho_k * \rho_k)(t)\big)_{L^2(\Omega)}\,dt \\ &= \int_{\mathbb{R}}\int_\Omega (\tilde{u}'\tilde{\theta}_n)(t)((\tilde{u}'\tilde{\theta}_n) * \rho_k * \rho_k)(t)\,dx\,dt \\ &= \int_\Omega\int_{\mathbb{R}} f(t)\big(g * h\big)(t)\,dt\,dx.\end{aligned}$$

A standard convolution computation then yields

$$\int_\Omega\int_{\mathbb{R}} f(t)\big(g * h\big)(t)\,dt\,dx = \int_{\mathbb{R}} \big((f * \check{h})(t), g(t)\big)_{L^2(\Omega)}\,dt,$$

and, substituting back h, f and g by ρ_k, $\tilde{u}'\tilde{\theta}_n$ and $(\tilde{u}'\tilde{\theta}_n) * \rho_k$, we obtain (3.4.62).

Since

$$\lim_{k\to\infty} (u'\theta_n) * \rho_k = (u'\theta_n) \quad \text{in } L^2\big([0,T]; L^2(\Omega)\big),$$

from (3.4.62) we conclude that

$$\lim_{k\to\infty} \int_0^T \big(u'(t), \varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt = \int_0^T \theta_n^2(t)\big\|u'(t)\big\|^2_{L^2(\Omega)}\,dt. \tag{3.4.63}$$

For the second term in (3.4.61), we have

$$\begin{aligned}\int_0^T \big(\Delta u(t), \varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt &= -\int_0^T \big(\nabla u(t), \nabla\varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt \\ &= -\int_0^T \big(\nabla u(t), \nabla\big[\theta_n((u'\theta_n) * \rho_k * \rho_k)\big](t)\big)_{L^2(\Omega)}\,dt \\ &= -\int_0^T \big(\nabla\big((u\theta_n) * \rho_k\big)(t), \nabla\big((u'\theta_n) * \rho_k\big)(t)\big)_{L^2(\Omega)}\,dt \\ &= -\int_0^T \big(\nabla\big((u\theta_n) * \rho_k\big)(t), \nabla\big[(u'\theta_n) * \rho_k - (\theta_n' u) * \rho_k\big](t)\big)_{L^2(\Omega)}\,dt \\ &= -\int_0^T \big(\nabla\big((u\theta_n) * \rho_k\big)(t), \nabla\big((\theta_n u)' * \rho_k\big)(t)\big)_{L^2(\Omega)}\,dt \\ &\quad + \int_0^T \big(\nabla\big((u\theta_n) * \rho_k\big)(t), \nabla\big((\theta_n' u) * \rho_k\big)(t)\big)_{L^2(\Omega)}\,dt \\ &= -\int_0^T \big(\nabla\big((u\theta_n) * \rho_k\big)(t), \nabla\big((\theta_n u) * \rho_k\big)'(t)\big)_{L^2(\Omega)}\,dt \\ &\quad + \int_0^T \big(\nabla\big((u\theta_n) * \rho_k\big)(t), \nabla\big((\theta_n' u) * \rho_k\big)(t)\big)_{L^2(\Omega)}\,dt.\end{aligned}$$

Observe that

$$\begin{aligned}\frac{d}{dt}\big(\nabla\big((u\theta_n) * \rho_k\big)(t), \nabla\big((\theta_n u) * \rho_k\big)(t)\big)_{L^2(\Omega)} &= 2\big(\nabla\big((u\theta_n) * \rho_k\big)(t), \nabla\big((u\theta_n) * \rho_k\big)'(t)\big)_{L^2(\Omega)} \\ &= 2\big(\nabla\big((u\theta_n) * \rho_k\big)(t), \nabla\big((u\theta_n)' * \rho_k\big)(t)\big)_{L^2(\Omega)}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2}\int_0^T \frac{d}{dt}\big(\nabla\big((u\theta_n)*\rho_k\big)(t),\nabla\big((\theta_n u)*\rho_k\big)(t)\big)_{L^2(\Omega)}\,dt &= \int_0^T \big(\nabla\big((u\theta_n)*\rho_k\big)(t),\nabla\big((u\theta_n)'*\rho_k\big)(t)\big)_{L^2(\Omega)}\,dt\\ &= \int_0^T \big(\nabla(u\theta_n)(t),\nabla\big((u\theta_n)'*\rho_k*\rho_k\big)(t)\big)_{L^2(\Omega)}\,dt\\ &= 0,\end{aligned}$$

since $\operatorname{supp}\big((\partial_{x_i}u\,\theta_n)*\rho_k\big)$ is compact in $(0,T)$ and therefore contained in $[0,T]$.

It follows that

$$\begin{aligned}\int_0^T \big(\Delta u(t),\varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt &= \int_0^T \big(\nabla\big((u\theta_n)*\rho_k\big)(t),\nabla\big((\theta_n' u)*\rho_k\big)(t)\big)_{L^2(\Omega)}\,dt\\ &= \sum_{i=1}^n \int_0^T \Big(\big((\theta_n\partial_{x_i}u)*\rho_k\big)(t),\big((\theta_n'\partial_{x_i}u)*\rho_k\big)(t)\Big)_{L^2(\Omega)}\,dt.\end{aligned}$$

Since

$$\lim_{k\to\infty}(\theta_n\partial_{x_i}u)*\rho_k = \theta_n\partial_{x_i}u \quad \text{in } L^2\big(0,T;L^2(\Omega)\big)$$

and

$$\lim_{k\to\infty}(\theta_n'\partial_{x_i}u)*\rho_k = \theta_n'\partial_{x_i}u \quad \text{in } L^2\big(0,T;L^2(\Omega)\big),$$

we obtain

$$\lim_{k\to\infty}\sum_{i=1}^n\Big(\big[\theta_n\partial_{x_i}u\big]*\rho_k,\big[\theta_n'\partial_{x_i}u\big]*\rho_k\Big)_{L^2(0,T;L^2(\Omega))} = \sum_{i=1}^n\big(\theta_n\partial_{x_i}u,\theta_n'\partial_{x_i}u\big)_{L^2(0,T;L^2(\Omega))}.$$

Therefore

$$\lim_{k\to\infty}\int_0^T \big(\Delta u(t),\varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt = \int_0^T \big(\theta_n\theta_n'\big)(t)\big\|\nabla u(t)\big\|^2_{L^2(\Omega)}\,dt. \tag{3.4.64}$$

For the right-hand side of (3.4.61), by an argument analogous to that used to obtain (3.4.63) and (3.4.64), we arrive at

$$\lim_{k\to\infty}\int_0^T \big(u^3(t),\varphi_{n,k}(t)\big)_{L^2(\Omega)}\,dt = \int_0^T \theta_n^2(t)\big(u^3(t),u'(t)\big)_{L^2(\Omega)}\,dt. \tag{3.4.65}$$

Passing to the limit as $k\to\infty$ in (3.4.61), and using (3.4.63), (3.4.64) and (3.4.65), we obtain

$$\int_0^T \theta_n^2(t)\big\|u'(t)\big\|^2_{L^2(\Omega)}\,dt - \int_0^T \big(\theta_n\theta_n'\big)(t)\big\|\nabla u(t)\big\|^2_{L^2(\Omega)}\,dt = \int_0^T \theta_n^2(t)\big(u^3(t),u'(t)\big)_{L^2(\Omega)}\,dt. \tag{3.4.66}$$

Letting $n\to\infty$ in (3.4.66) and using the explicit form of θ_n and θ_n', we obtain

$$\int_s^t \big\|u'(\tau)\big\|^2_{L^2(\Omega)}\,d\tau - \frac{1}{2}\Big(\big\|\nabla u(t)\big\|^2_{L^2(\Omega)} - \big\|\nabla u(s)\big\|^2_{L^2(\Omega)}\Big) = \int_s^t \big(u^3(\tau),u'(\tau)\big)_{L^2(\Omega)}\,d\tau. \tag{3.4.67}$$

To justify the passage to the limit in n, we use the regularity of u in (3.4.55), from which we deduce

$$\|u'(\cdot)\|^2_{L^2(\Omega)} \in C([0,T]), \tag{3.4.68}$$

$$\|\nabla u(\cdot)\|^2_{L^2(\Omega)} \in C([0,T]), \tag{3.4.69}$$

$$\big(u^3(\cdot), u'(\cdot)\big)_{L^2(\Omega)} \in C([0,T]). \tag{3.4.70}$$

Taking $h(t) = \|u'(t)\|^2_{L^2(\Omega)}$ in the first term of (3.4.66), it follows that

$$\lim_{n\to\infty} \int_0^T \theta_n(\tau)^2 h(\tau)\, d\tau = \int_s^t h(\tau)\, d\tau, \tag{3.4.71}$$

using the explicit form of θ_n and standard one-dimensional estimates on the shrinking intervals. Similarly, taking $h(\tau) = \big(u^2(\tau), u'(\tau)\big)_{L^2(\Omega)}$, we obtain

$$\lim_{n\to\infty} \int_0^T \theta_n(\tau)^2 \big(u^2(\tau), u'(\tau)\big)_{L^2(\Omega)}\, d\tau = \int_s^t \big(u^2(\tau), u'(\tau)\big)_{L^2(\Omega)}\, d\tau. \tag{3.4.72}$$

Finally, taking $h(\tau) = \|\nabla u(\tau)\|^2_{L^2(\Omega)}$, one shows that

$$\int_0^T (\theta_n \theta_n')(\tau) h(\tau)\, d\tau \overset{n\to\infty}{\longrightarrow} \frac{1}{2}\big[h(t) - h(s)\big], \tag{3.4.73}$$

again by direct computation on the piecewise linear θ_n and the continuity of h.

Thus (3.4.67) follows from (3.4.66) by letting $n \to \infty$ and using (3.4.71), (3.4.72), and (3.4.73).

Now let $(s_\nu)_{\nu\in\mathbb{N}} \subset [0,T]$ with $s_\nu \to 0$. For each $t \in [0,T]$, (3.4.67) gives

$$\int_{s_\nu}^t \|u'(\tau)\|^2_{L^2(\Omega)}\, d\tau - \frac{1}{2}\Big(\|\nabla u(t)\|^2_{L^2(\Omega)} - \|\nabla u(s_\nu)\|^2_{L^2(\Omega)}\Big) = \int_{s_\nu}^t \big(u^3(\tau), u'(\tau)\big)_{L^2(\Omega)}\, d\tau. \tag{3.4.74}$$

From (3.4.55) we have

$$u \in C\big([0,T]; H_0^1(\Omega) \cap H^2(\Omega)\big) \hookrightarrow C\big([0,T]; H_0^1(\Omega)\big),$$

so

$$\frac{\partial u}{\partial x_i} \in C\big([0,T]; L^2(\Omega)\big),$$

and hence

$$\begin{aligned}
\lim_{\nu\to\infty} \|\nabla u(s_\nu)\|^2_{L^2(\Omega)} &= \lim_{\nu\to\infty} \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}(s_\nu), \frac{\partial u}{\partial x_i}(s_\nu)\Big)_{L^2(\Omega)} \\
&= \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}(0), \frac{\partial u}{\partial x_i}(0)\Big)_{L^2(\Omega)} \\
&= \|\nabla u(0)\|^2_{L^2(\Omega)}.
\end{aligned}$$

That is,

$$\lim_{\nu\to\infty} \|\nabla u(s_\nu)\|^2_{L^2(\Omega)} = \|\nabla u(0)\|^2_{L^2(\Omega)}. \tag{3.4.75}$$

Therefore, letting $\nu \to \infty$ in (3.4.74), we obtain

$$\int_0^t \|u'(\tau)\|^2_{L^2(\Omega)}\, d\tau - \frac{1}{2}\Big(\|\nabla u(t)\|^2_{L^2(\Omega)} - \|\nabla u(0)\|^2_{L^2(\Omega)}\Big) = \int_0^t \big(u^3(\tau), u'(\tau)\big)_{L^2(\Omega)}\, d\tau. \tag{3.4.76}$$

Notice that the right-hand side of (3.4.76) can be rewritten as

$$\int_0^t \big(u^3(\tau), u'(\tau)\big)_{L^2(\Omega)}\, d\tau = \int_0^t \frac{1}{4}\frac{d}{d\tau}\int_\Omega u^4(\tau)\, dx\, d\tau, \tag{3.4.77}$$

and hence from (3.4.77) and (3.4.76) we get

$$-\int_0^t \big\|u'(\tau)\big\|^2_{L^2(\Omega)}\, d\tau = \frac{1}{2}\Big[\int_\Omega \big|\nabla u(t)\big|^2 dx - \int_\Omega \big|\nabla u(0)\big|^2 dx\Big] - \frac{1}{4}\Big[\int_\Omega u^4(t)\, dx - \int_\Omega u^4(0)\, dx\Big].$$

Using the definition (3.4.54) of $E(t)$, this is equivalent to

$$-\int_0^t \big\|u'(\tau)\big\|^2_{L^2(\Omega)}\, d\tau = E(t) - E(0). \tag{3.4.78}$$

Thus,

$$\frac{d}{dt}E(t) = -\big\|u'(t)\big\|^2_{L^2(\Omega)} \leqslant 0,$$

i.e. $E(t)$ is decreasing, and, since by hypothesis $E(0) \leqslant 0$, we have

$$E(t) \leqslant E(0) \leqslant 0, \qquad \forall t \in [0, T].$$

We now show that $T_{\max} < \infty$. Suppose, by contradiction, that $T_{\max} = \infty$ and define $G : [0, \infty) \to \mathbb{R}$ by

$$G(t) = \|u(\cdot, t)\|^2_{L^2(\Omega)}.$$

By the regularity in (3.4.55), $G \in C^1([0, +\infty))$ and

$$\begin{aligned}
G'(t) &= \frac{d}{dt}\big\|u(t)\big\|^2_{L^2(\Omega)} = 2\big(u'(t), u(t)\big)_{L^2(\Omega)} \\
&= 2\big(\Delta u(t) + u^3(t), u(t)\big)_{L^2(\Omega)} \\
&= -4\left[\frac{1}{2}\int_\Omega |\nabla u(t)|^2 dx - \frac{1}{4}\int_\Omega u^4(t) dx\right] + \int_\Omega u^4(t) dx \\
&= -4E(t) + \int_\Omega u^4(t) dx.
\end{aligned}$$

Since $E(t) \leq E(0)$, it follows that

$$G'(t) \geq -4E(0) + \int_\Omega u^4(t) dx.$$

Because $L^4(\Omega) \hookrightarrow L^2(\Omega)$, there exists $c_1 > 0$ such that

$$\|u(t)\|^4_{L^2(\Omega)} \leqslant c_1 \|u(t)\|^4_{L^4(\Omega)},$$

and hence

$$G^2(t) \leq c_1 \int_\Omega |u(t)|^4 dx.$$

Writing $c = \frac{1}{c_1} > 0$, we obtain

$$G'(t) \geqslant -4E(0) + cG^2(t) \geqslant cG^2(t).$$

Thus G is increasing and, since $G(0) \neq 0$ (because $u_0 \neq 0$ by hypothesis), we can write

$$\frac{G'(s)}{G^2(s)} \geqslant c,$$

which, upon integration from 0 to t, yields

$$\int_0^t \frac{G'(s)}{G^2(s)}\,ds \geqslant ct, \quad \forall t \geqslant 0.$$

But

$$\int_0^t \frac{G'(s)}{G^2(s)}\,ds = \frac{1}{G(0)} - \frac{1}{G(t)} \leq \frac{1}{G(0)},$$

that is, $ct \leqslant \dfrac{1}{G(0)}$ for all $t \geqslant 0$, which is impossible. Therefore $T_{\max} < \infty$. □

3.5 Some Additional Problems

3.5.1 The Timoshenko System with Dirichlet–Dirichlet Boundary Conditions

In this example we study the existence of solutions for the Timoshenko system with Dirichlet–Dirichlet boundary conditions. Consider the problem

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 & \text{in} \quad (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) = 0 & \text{in} \quad (0,L)\times(0,+\infty), \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, & \text{in} \quad (0,+\infty), \\ \varphi(0) = \varphi^0,\ \varphi_t(0) = \varphi^1,\ \psi(0) = \psi^0,\ \psi_t(0) = \psi^1 & \text{in} \quad (0,L), \end{cases} \tag{3.5.79}$$

where ρ_1, ρ_2, k, b are positive constants.
Proof: The energy functional associated with the problem is

$$E(t) = \frac{1}{2}\left[\rho_1\|\varphi_t\|^2_{L^2(0,L)} + \rho_2\|\psi_t\|^2_{L^2(0,L)} + k\|\varphi_x+\psi\|^2_{L^2(0,L)} + b\|\psi_x\|^2_{L^2(0,L)}\right].$$

Let the phase space be

$$\mathcal{H} = H^1_0(0,L)\times L^2(0,L)\times H^1_0(0,L)\times L^2(0,L).$$

Given $U = (\varphi_1, \Phi_1, \psi_1, \Psi_1)$, $V = (\varphi_2, \Phi_2, \psi_2, \Psi_2) \in \mathcal{H}$, we define the following inner product:

$$(U,V)_{\mathcal{H}} = \rho_1(\Phi_1,\Phi_2)_{L^2} + \rho_2(\Psi_1,\Psi_2)_{L^2} + k((\varphi_1)_x+\psi_1, (\varphi_2)_x+\psi_2)_{L^2} + b((\psi_1)_x, (\psi_2)_x)_{L^2}. \tag{3.5.80}$$

This induces the norm

$$\|U\|^2_{\mathcal{H}} = \rho_1\|\Phi\|^2_{L^2} + \rho_2\|\Psi\|^2_{L^2} + k\|\varphi_x+\psi\|^2_{L^2} + b\|\psi_x\|^2_{L^2}, \tag{3.5.81}$$

where $U = (\varphi, \Phi, \psi, \Psi)$.

We now prove that $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Hilbert space. We already know that $(\mathcal{H}, |\cdot|_{\mathcal{H}})$ is a Hilbert space when equipped with the usual norm

$$|U|^2_{\mathcal{H}} = \|\varphi_x\|^2_{L^2} + \|\Phi\|^2_{L^2} + \|\psi_x\|^2_{L^2} + \|\Psi\|^2_{L^2}. \tag{3.5.82}$$

To show that $\mathcal{H}$ is a Banach space under $\|\cdot\|_{\mathcal{H}}$, it suffices to establish the equivalence of the norms

(3.5.81) and (3.5.82). Using the inequality $|a+b|^2 \leq 2(|a|^2+|b|^2)$ and the Poincaré inequality, we obtain:

$$\|U\|_{\mathcal{H}}^2 \leq \max\{\rho_1, \rho_2, b+2kL^2, 2k\}\,|U|_{\mathcal{H}}^2 = \widetilde{C_1}|U|_{\mathcal{H}}^2.$$

Hence, $\|U\|_{\mathcal{H}} \leq C_1|U|_{\mathcal{H}}$.

Conversely,

$$|U|_{\mathcal{H}}^2 \leq \widetilde{C_2}\|U\|_{\mathcal{H}}^2,$$

thus proving the equivalence of the norms.

Since $(\mathcal{H}, |\cdot|_{\mathcal{H}})$ is Banach and the norms are equivalent, $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is also a Banach space. As an inner product has already been defined, we conclude that $\mathcal{H}$ is a Hilbert space.

Semigroup Formulation. Let $U = (\varphi, \Phi, \psi, \Psi) \in \mathcal{H}$. When $\Phi = \varphi_t$ and $\Psi = \psi_t$, we have

$$\frac{dU}{dt} = \begin{bmatrix} \varphi_t \\ \dfrac{k}{\rho_1}(\varphi_x+\psi)_x \\ \psi_t \\ \dfrac{b}{\rho_2}(\psi_{xx}-\varphi_x+\psi) \end{bmatrix} = AU.$$

The initial condition is

$$U(0) = \begin{bmatrix} \varphi^0 \\ \varphi^1 \\ \psi^0 \\ \psi^1 \end{bmatrix} = U_0.$$

Thus, we obtain the Abstract Cauchy Problem (ACP):

$$\frac{dU}{dt} = AU, \qquad U(0) = U_0,$$

where

$$D(A) = (H_0^1(0,L)\cap H^2(0,L)) \times H_0^1(0,L) \times (H_0^1(0,L)\cap H^2(0,L)) \times H_0^1(0,L).$$

Existence and Uniqueness. If $U_0 \in \mathcal{H}$, then the ACP has a unique mild solution

$$U \in C([0,+\infty), \mathcal{H}), \qquad U(t) = U_0 + A\int_0^t U(s)\,ds.$$

If $U_0 \in D(A)$, then the ACP admits a unique classical solution

$$U \in C([0,+\infty), D(A)) \cap C^1([0,+\infty), \mathcal{H}).$$

By Theorem 2.3, it suffices to show that A is the infinitesimal generator of a contraction semigroup. Using the Lumer–Phillips Theorem, we must verify that:

(i) A is dissipative;

(ii) $\mathrm{Im}(\lambda I - A) = \mathcal{H}$ for some $\lambda > 0$;

(iii) $D(A)$ is dense in $\mathcal{H}$.

A straightforward computation shows that A is dissipative. Since $H_0^1 \cap H^2$ is dense in H_0^1, and H_0^1 is dense in L^2, we conclude that $D(A)$ is dense in $\mathcal{H}$. Thus it remains to show surjectivity of $(\lambda I - A)$. We prove item (ii) for $\lambda = 1$.

Given $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, we seek $U \in D(A)$ such that $(I - A)U = F$. This is equivalent to the system

$$\varphi - \Phi = f_1, \tag{3.5.83}$$
$$\Phi - \tfrac{k}{\rho_1}(\varphi_x + \psi)_x = f_2, \tag{3.5.84}$$
$$\psi - \Psi = f_3, \tag{3.5.85}$$
$$\Psi - \tfrac{b}{\rho_2}\psi_{xx} + \tfrac{k}{\rho_2}(\varphi_x + \psi) = f_4. \tag{3.5.86}$$

From (3.5.83)–(3.5.85) we obtain

$$\Phi = \varphi - f_1, \tag{3.5.87}$$
$$\Psi = \psi - f_3. \tag{3.5.88}$$

Substituting (3.5.87)–(3.5.88) into (3.5.84) and (3.5.86) gives

$$\rho_1\varphi - k(\varphi_x + \psi)_x = g_1, \tag{3.5.89}$$
$$\rho_2\psi - b\psi_{xx} + k(\varphi_x + \psi) = g_2, \tag{3.5.90}$$

where $g_1 = \rho_1(f_1 + f_2)$, $g_2 = \rho_2(f_3 + f_4)$.

To solve (3.5.89)–(3.5.90) we apply the Lax–Milgram Theorem. Define the bilinear form

$$a((\varphi, \psi), (\tilde{\varphi}, \tilde{\psi})) = \rho_1(\varphi, \tilde{\varphi}) + \rho_2(\psi, \tilde{\psi}) + b(\psi_x, \tilde{\psi}_x) + k(\varphi_x + \psi, \tilde{\varphi}_x + \tilde{\psi}).$$

One checks that a is continuous and coercive. Thus, by Lax–Milgram, for each $(g_1, g_2) \in H^{-1} \times H^{-1}$ there exists a unique $(\varphi, \psi) \in H_0^1 \times H_0^1$ such that

$$a((\varphi, \psi), (\tilde{\varphi}, \tilde{\psi})) = (g_1, \tilde{\varphi}) + (g_2, \tilde{\psi}), \qquad \forall (\tilde{\varphi}, \tilde{\psi}) \in H_0^1 \times H_0^1. \tag{3.5.91}$$

Choosing $\tilde{\varphi} = 0$ in (3.5.91) yields

$$\rho_2\psi - b\psi_{xx} + k(\varphi_x + \psi) = g_2 \quad \text{in } H^{-1},$$

so that

$$\psi_{xx} = \frac{1}{b}[-g_2 + \rho_2\psi + k(\varphi_x + \psi)] \in L^2. \tag{??}$$

Choosing $\tilde{\psi} = 0$ in (3.5.91) gives

$$\rho_1\varphi - k(\varphi_x + \psi)_x = g_1 \quad \text{in } H^{-1},$$

so that

$$\varphi_{xx} = \frac{1}{k}[-g_1 + \rho_1\varphi - k\psi_x] \in L^2. \tag{??}$$

Thus $(\varphi, \psi) \in (H^2 \cap H_0^1) \times (H^2 \cap H_0^1)$ solves (3.5.89)–(3.5.90). From (3.5.87)–(3.5.88) we also have

$$\Phi = \varphi - f_1 \in H_0^1, \qquad \Psi = \psi - f_3 \in H_0^1.$$

Therefore $U = (\varphi, \Phi, \psi, \Psi) \in D(A)$ and solves (3.5.83)–(3.5.86). Hence $(I - A)$ is surjective, proving existence and uniqueness of the mild solution for the Timoshenko system. □

3.5.2 Bresse System

In this example, we shall verify the existence and uniqueness of a mild solution for the Bresse system given by

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) = 0 & \text{in} \quad (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) = 0 & \text{in} \quad (0,L)\times(0,+\infty), \\ \rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) = 0 & \text{in} \quad (0,L)\times(0,+\infty), \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = \omega(0,t) = \omega(L,t) = 0, & \text{in} \quad (0,+\infty), \\ \varphi(0) = \varphi^0,\ \varphi_t(0) = \varphi^1,\ \psi(0) = \psi^0,\ \psi_t(0) = \psi^1,\ \omega(0) = \omega^0,\ \omega_t(0) = \omega^1 & \text{in} \quad (0,L), \end{cases} \tag{3.5.92}$$

where $\rho_1, \rho_2, k, k_0, b$ and l are positive constants.
Proof: The energy functional associated with the problem is

$$E(t) = \frac{1}{2}\Big[\rho_1\|\varphi_t\|^2_{L^2(0,L)} + \rho_2\|\psi_t\|^2_{L^2(0,L)} + \rho_1\|\omega_t\|^2_{L^2(0,L)} + k\|\varphi_x + \psi + l\omega\|^2_{L^2(0,L)} + k_0\|\omega_x - l\varphi\|^2_{L^2(0,L)} + b\|\psi_x\|^2_{L^2(0,L)}\Big],$$

and the phase space is

$$\mathcal{H} = H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\times L^2(0,L)\times L^2(0,L)\times L^2(0,L).$$

Let $U = (\varphi, \Phi, \psi, \Psi, \omega, W)$, where $\Phi = \varphi_t$, $\Psi = \psi_t$ and $W = \omega_t$. It is known that $\mathcal{H}$, endowed with the usual norm, is a Hilbert space. Thus, if we define on $\mathcal{H}$ a norm equivalent to the usual one, then $\mathcal{H}$ endowed with this new norm will also be a Hilbert space.

For $U = (\varphi_1, \Phi_1, \psi_1, \Psi_1, \omega_1, W_1)$, $V = (\varphi_2, \Phi_2, \psi_2, \Psi_2, \omega_2, W_2) \in \mathcal{H}$, we define the following inner product:

$$\begin{aligned} (U,V)_{\mathcal{H}} &= \rho_1\int_0^L \Phi_1\Phi_2\,dx + \rho_2\int_0^L \Psi_1\Psi_2\,dx + \rho_1\int_0^L W_1W_2\,dx \\ &\quad + k\int_0^L \big((\varphi_1)_x + \psi_1 + l\omega_1\big)\big((\varphi_2)_x + \psi_2 + l\omega_2\big)\,dx \\ &\quad + k_0\int_0^L \big((\omega_1)_x - l\varphi_1\big)\big((\omega_2)_x - l\omega_2\big)\,dx + b\int_0^L (\psi_1)_x(\psi_2)_x\,dx \\ &= \rho_1(\Phi_1,\Phi_2)_{L^2} + \rho_2(\Psi_1,\Psi_2)_{L^2} + \rho_1(W_1,W_2)_{L^2} \\ &\quad + k\big((\varphi_1)_x + \psi_1 + l\omega_1, (\varphi_2)_x + \psi_2 + l\omega_2\big)_{L^2} \\ &\quad + k_0\big((\omega_1)_x - l\varphi_1, (\omega_2)_x - l\omega_2\big)_{L^2} + b\big((\psi_1)_x, (\psi_2)_x\big)_{L^2}. \end{aligned} \tag{3.5.93}$$

This inner product induces the norm

$$\begin{aligned} \|U\|^2_{\mathcal{H}} &= \rho_1\|\Phi\|^2_{L^2(0,L)} + \rho_2\|\Psi\|^2_{L^2(0,L)} + \rho_1\|W\|^2_{L^2(0,L)} \\ &\quad + k\|\varphi_x + \psi + l\omega\|^2_{L^2(0,L)} + k_0\|\omega_x - l\varphi\|^2_{L^2(0,L)} + b\|\psi_x\|^2_{L^2(0,L)}, \end{aligned} \tag{3.5.94}$$

where $\|\cdot\|_{L^2(0,L)}$ denotes the usual norm in $L^2(0,L)$ and $U = (\varphi, \Phi, \psi, \Psi, \omega, W)$.

We now show that the usual norm and the norm defined above are equivalent. First, we prove that there exists a constant $c_1 > 0$ such that

$$\|U\|_{\mathcal{H}} \le c_1|U|_{\mathcal{H}}, \quad \text{for all } U = (\varphi, \Phi, \psi, \Psi, \omega, W) \in \mathcal{H},$$

where $|\cdot|_{\mathcal{H}}$ denotes the usual Hilbert norm on the Cartesian product.

Given $U = (\varphi, \Phi, \psi, \Psi, \omega, W)$, we have

$$\begin{aligned}
\|U\|_{\mathcal{H}}^2 &= \rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2 + \rho_1\|W\|^2 + k\|\varphi_x + \psi + l\omega\|^2 + k_0\|\omega_x - l\varphi\|^2 + b\|\psi_x\|^2 \\
&\leq \rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2 + \rho_1\|W\|^2 + 2k\|\varphi_x + \psi\|^2 + 2kl^2\|\omega\|^2 + 2k_0\|\omega_x\|^2 + 2k_0 l^2\|\varphi\|^2 + b\|\psi_x\|^2 \\
&\leq \rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2 + \rho_1\|W\|^2 + (b + 4kL^2)\|\psi_x\|^2 + (4k + 2k_0 l^2 L^2)\|\varphi_x\|^2 \\
&\quad + (2kl^2L^2 + 2k_0)\|\omega_x\|^2 \\
&\leq \tilde{c}_1^2\big(\|\varphi_x\|^2 + \|\Phi\|^2 + \|\psi_x\|^2 + \|\Psi\|^2 + \|\omega_x\|^2 + \|W\|^2\big) \\
&= c_1|U|_{\mathcal{H}}^2,
\end{aligned}$$

where

$$\tilde{c}_1 := \max\{\rho_1, \rho_2, b + 4kL^2,\ 4k + 2k_0 l^2 L^2,\ 2kl^2L^2 + 2k_0\} \quad \text{and} \quad c_1 := \tilde{c}_1.$$

Next, we prove that there exists a constant $c_2 > 0$ such that

$$|U|_{\mathcal{H}} \leq c_2\|U\|_{\mathcal{H}}, \quad \text{for all } U = (\varphi, \Phi, \psi, \Psi, \omega, W) \in \mathcal{H}.$$

To this end, it is enough to verify that

$$|(\varphi,\psi,\omega)|_{\mathcal{H}} = \|\varphi_x\|^2 + \|\psi_x\|^2 + \|\omega_x\|^2 \leq c_2\big(k\|\varphi_x + \psi + l\omega\|^2 + k_0\|\omega_x - l\varphi\|^2 + b\|\psi_x\|^2\big) = \|(\varphi,\psi,\omega)\|_{\mathcal{H}}.$$

Suppose, by contradiction, that there exists a sequence $(\varphi_n, \psi_n, \omega_n) \in \mathcal{H}$ such that

$$\frac{|(\varphi_n, \psi_n, \omega_n)|_{\mathcal{H}}}{\|(\varphi_n, \psi_n, \omega_n)\|_{\mathcal{H}}} \longrightarrow \infty.$$

Define

$$\tilde{\varphi}_n := \frac{\varphi_n}{|(\varphi_n, \psi_n, \omega_n)|_{\mathcal{H}}}, \quad \tilde{\psi}_n := \frac{\psi_n}{|(\varphi_n, \psi_n, \omega_n)|_{\mathcal{H}}}, \quad \tilde{\omega}_n := \frac{\omega_n}{|(\varphi_n, \psi_n, \omega_n)|_{\mathcal{H}}}.$$

Then

(1) $\big|(\tilde{\varphi}_n, \tilde{\psi}_n, \tilde{\omega}_n)\big|_{\mathcal{H}} = 1 \Rightarrow \lim_{n\to\infty}\big|(\tilde{\varphi}_n, \tilde{\psi}_n, \tilde{\omega}_n)\big|_{\mathcal{H}} = 1$;

(2) $\|(\tilde{\varphi}_n, \tilde{\psi}_n, \tilde{\omega}_n)\|_{\mathcal{H}} \longrightarrow 0$.

From (2), we have in particular

$$(\tilde{\varphi}_n)_x + \tilde{\psi}_n + l\tilde{\omega}_n \longrightarrow 0, \tag{3.5.95}$$
$$(\tilde{\psi}_n)_x \longrightarrow 0, \tag{3.5.96}$$
$$(\tilde{\omega}_n)_x - l\tilde{\varphi}_n \longrightarrow 0, \tag{3.5.97}$$

in $L^2(0,L)$. Since the derivative operator from H_0^1 to L^2 is linear and continuous, there exist $(f,g,h) \in (H_0^1)^3$ and a (not relabelled) subsequence such that

$$\begin{aligned}
\tilde{\varphi}_n &\rightharpoonup f \quad \text{in } H_0^1, \\
\tilde{\psi}_n &\rightharpoonup g \quad \text{in } H_0^1, \\
\tilde{\omega}_n &\rightharpoonup h \quad \text{in } H_0^1.
\end{aligned} \tag{3.5.98}$$

Hence

$$\begin{aligned}
(\tilde{\varphi}_n)_x &\rightharpoonup f_x \quad \text{in } L^2, \\
(\tilde{\psi}_n)_x &\rightharpoonup g_x \quad \text{in } L^2, \\
(\tilde{\omega}_n)_x &\rightharpoonup h_x \quad \text{in } L^2.
\end{aligned} \tag{3.5.99}$$

Thus

$$\begin{aligned}(\tilde{\varphi}_n)_x + \tilde{\psi}_n + l\tilde{\omega}_n &\rightharpoonup f_x + g + lh,\\ (\tilde{\psi}_n)_x &\rightharpoonup g_x,\\ (\tilde{\omega}_n)_x - l\tilde{\varphi}_n &\rightharpoonup h_x - lf,\end{aligned} \tag{3.5.100}$$

in L^2, and by uniqueness of the weak limit, from (3.5.95)–(3.5.97) and (3.5.100) we obtain

$$\begin{aligned}f_x + g + lh &= 0,\\ g_x &= 0,\\ h_x - lf &= 0.\end{aligned} \tag{3.5.101}$$

By the Poincaré inequality and the second equation in (3.5.101), we have $\|g\| \le L\|g_x\| = 0$, hence $g \equiv 0$. Therefore, (3.5.101) reduces to

$$\begin{aligned}f_x + lh &= 0,\\ h_x - lf &= 0,\end{aligned} \tag{3.5.102}$$

whose only solution is $f = h = 0$.

Since $H_0^1 \hookrightarrow L^2$ compactly, it follows from (3.5.98) that

$$\begin{aligned}\tilde{\varphi}_n &\longrightarrow f = 0 \quad \text{in } L^2,\\ \tilde{\omega}_n &\longrightarrow h = 0 \quad \text{in } L^2.\end{aligned} \tag{3.5.103}$$

Using (3.5.95)–(3.5.97), (3.5.103) and the Poincaré inequality, we then infer

$$\begin{aligned}(\tilde{\varphi}_n)_x &= \big((\tilde{\varphi}_n)_x + \tilde{\psi}_n + l\tilde{\omega}_n\big) - \big(\tilde{\psi}_n + l\tilde{\omega}_n\big) \longrightarrow 0,\\ \tilde{\psi}_n &\longrightarrow 0,\\ (\tilde{\omega}_n)_x &= \big((\tilde{\omega}_n)_x - l\tilde{\varphi}_n\big) + l\tilde{\varphi}_n \longrightarrow 0,\end{aligned} \tag{3.5.104}$$

in L^2. Thus, from (1) and (3.5.104), we obtain the contradiction

$$\lim_{n\to\infty} \big|(\tilde{\varphi}_n, \tilde{\psi}_n, \tilde{\omega}_n)\big|_{\mathcal{H}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \big|(\tilde{\varphi}_n, \tilde{\psi}_n, \tilde{\omega}_n)\big|_{\mathcal{H}} = 0.$$

Hence there exists $c_2 > 0$ such that

$$|(\varphi, \psi, \omega)|_{\mathcal{H}} \le c_2 \|(\varphi, \psi, \omega)\|_{\mathcal{H}}, \quad \text{for all } (\varphi, \psi, \omega) \in \mathcal{H}.$$

Therefore, the norms $|\cdot|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent. □

3.5.3 A Non-homogeneous Timoshenko System

We now study the existence of solutions for the non-homogeneous Timoshenko system

$$\begin{cases} \rho_1(x)\varphi_{tt} - (\kappa(x)\varphi_x + \psi)_x = 0 & \text{in} \quad (0, L) \times (0, +\infty),\\ \rho_2(x)\psi_{tt} - (b(x)\psi_x)_x + \kappa(x)(\varphi_x + \psi) = 0 & \text{in} \quad (0, L) \times (0, +\infty).\end{cases} \tag{3.5.105}$$

with boundary conditions at $x = 0$

$$\varphi(0, t) = \psi(0, t) = 0, \tag{3.5.106}$$

and boundary conditions at $x = L$ given by

$$\begin{cases} m\varphi_{tt}(L,t) - k_0\varphi_t(L,t) + \kappa(L)(\varphi_x(L,t) + \psi(L,t)) = 0, \\ I_m\psi_{tt}(L,t) + k_1\psi(L,t) + b(L)\psi_x(L,t) = 0. \end{cases} \tag{3.5.107}$$

The initial data are

$$\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \tag{3.5.108}$$

where

$$\rho_1, \rho_2 \in L^\infty(0,L) \quad \text{and} \quad b, \kappa \in W^{1,\infty}(0,L) \tag{3.5.109}$$

are such that

$$\rho_1(x) \geqslant m_1 > 0, \qquad \forall x \in (0,L), \tag{3.5.110}$$
$$\rho_2(x) \geqslant m_2 > 0, \qquad \forall x \in (0,L), \tag{3.5.111}$$
$$b(x) \geqslant m_3 > 0, \qquad \forall x \in (0,L), \tag{3.5.112}$$
$$\kappa(x) \geqslant m_4 > 0, \qquad \forall x \in (0,L), \tag{3.5.113}$$

for some $m_i \in \mathbb{R}$, $i \in \{1,2,3,4\}$.

Denoting

$$u(t) := \varphi_t(L,t) \quad \text{and} \quad v(t) := \psi_t(L,t), \tag{3.5.114}$$

we observe that u and v satisfy

$$\begin{cases} mu_t(t) - k_0u(t) + \kappa(L)\big(\varphi_x(L,t) + \psi(L,t)\big) = 0, \\ I_mv_t(t) + k_1v(t) + b(L)\psi_x(L,t) = 0, \end{cases} \tag{3.5.115}$$

with initial conditions

$$u(0) = \varphi_1(L), \quad v(0) = \psi_1(L). \tag{3.5.116}$$

Energy Functional

The energy functional associated with the problem is

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^L \rho_1(x)\big|\varphi_t(x,t)\big|^2 + \rho_2(x)\big|\psi_t(x,t)\big|^2 + \kappa(x)\big|\varphi_x(x,t) + \psi(x,t)\big|^2 \\ + b(x)\big|\psi_x(x,t)\big|^2\,dx + \frac{m}{2}|u(t)|^2 + \frac{I_m}{2}|v(t)|^2. \end{aligned} \tag{3.5.117}$$

Phase Space

Consider the space

$$H^1_*(0,L) = \big\{w \in H^1(0,L)\ ;\ w(0) = 0\big\}$$

endowed with the inner product

$$(f,g)_* = \int_0^L f'(x)g'(x)\,dx,$$

whose induced norm is

$$|f|_*^2 = \int_0^L |f'(x)|^2\,dx = \|f'\|^2_{L^2(0,L)}.$$

Then $(H^1_*, |\cdot|_*)$ is a Hilbert space.

Define

$$\mathcal{H} = H^1_*(0,L) \times L^2(0,L) \times H^1_*(0,L) \times L^2(0,L) \times \mathbb{R} \times \mathbb{R}, \tag{3.5.118}$$

which we endow with the usual inner product $(\cdot,\cdot)_{\mathcal{H}}$ given by

$$(U_1,U_2)_{\mathcal{H}} = (\varphi_{1,x},\varphi_{2,x})_2 + (\Phi_1,\Phi_2)_2 + (\psi_{1,x},\psi_{2,x})_2 + (\Psi_1,\Psi_2)_2 + u_1u_2 + v_1v_2, \tag{3.5.119}$$

for all $U_i = (\varphi_i,\Phi_i,\psi_i,\Psi_i,u_i,v_i) \in \mathcal{H}$, $i=1,2$. The induced norm is

$$|U|^2_{\mathcal{H}} = \|\varphi_x\|_2^2 + \|\Phi\|_2^2 + \|\psi_x\|_2^2 + \|\Psi\|_2^2 + |u|^2 + |v|^2, \quad \forall\, U\in\mathcal{H}, \tag{3.5.120}$$

where $(\cdot,\cdot)_2$ and $\|\cdot\|_2$ respectively denote the inner product and norm in $L^2(0,L)$. Thus $(\mathcal{H},|\cdot|_{\mathcal{H}})$ is a Hilbert space.

We now define on $\mathcal{H}$ the bilinear mapping $((\cdot,\cdot))_{\mathcal{H}}:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ by

$$\begin{aligned}((U_1,U_2))_{\mathcal{H}} &= \int_0^L \rho_1(x)\Phi_1(x)\Phi_2(x)\,dx + \int_0^L \rho_2(x)\Psi_1(x)\Psi_2(x)\,dx\\ &\quad+\int_0^L \kappa(x)\big(\varphi_{1,x}(x)+\psi_1(x)\big)\big(\varphi_{2,x}(x)+\psi_2(x)\big)\,dx\\ &\quad+\int_0^L b(x)\psi_{1,x}(x)\psi_{2,x}(x)\,dx + \frac{m}{2}u_1u_2 + \frac{I_m}{2}v_1v_2,\end{aligned} \tag{3.5.121}$$

for all $U_i = (\varphi_i,\Phi_i,\psi_i,\Psi_i,u_i,v_i) \in \mathcal{H}$, $i=1,2$.

It is straightforward to check that $((\cdot,\cdot))_{\mathcal{H}}$ defines an inner product on $\mathcal{H}$; we denote its induced norm by $\|\cdot\|_{\mathcal{H}}$. One can show, using standard inequalities (Poincaré and Young) and the positivity assumptions on ρ_1,ρ_2,b,κ, that $|\cdot|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent. Therefore, $(\mathcal{H},\|\cdot\|_{\mathcal{H}})$ is also a Hilbert space.

Semigroup Formulation

Our goal is now to write the problem in the form of an abstract Cauchy problem, that is,

$$\begin{cases}\dfrac{dU}{dt} = AU,\\ U(0)=U_0,\end{cases}$$

where $U: D(A)\subset\mathcal{H}\to\mathcal{H}$.

To determine A and $D(A)$, let $U=(\varphi,\Phi,\psi,\Psi,u,v)\in\mathcal{H}$ with $\Phi=\varphi_t$, $\Psi=\psi_t$. Then

$$\frac{dU}{dt} = \begin{bmatrix}\varphi_t\\ \varphi_{tt}\\ \psi_t\\ \psi_{tt}\\ u_t\\ v_t\end{bmatrix} = \begin{bmatrix}\varphi_t\\ \dfrac{1}{\rho_1(x)}\Big[\kappa'(x)\varphi_x+\kappa(x)\varphi_{xx}+\kappa'(x)\psi+\kappa(x)\psi_x\Big]\\ \psi_t\\ \dfrac{1}{\rho_2(x)}\Big[b'(x)\psi_x+b(x)\psi_{xx}-\kappa(x)\varphi_x-\kappa(x)\psi\Big]\\ -\dfrac{1}{m}\Big[k_0u(t)+\kappa(L)\varphi_x(L,t)+\kappa(L)\psi(L,t)\Big]\\ -\dfrac{1}{I_m}\Big[k_1v(t)+b(L)\psi_x(L,t)\Big]\end{bmatrix}.$$

We set

$$D(A) = \{U=(\varphi,\Phi,\psi,\Psi,u,v)\in\mathcal{H}\,;\, AU\in\mathcal{H} \text{ with } u=\Phi(L),\ v=\Psi(L)\},$$

in view of (3.5.114); that is,

$$u(t)=\varphi_t(L,t) \quad\Longleftrightarrow\quad u=\Phi(L), \qquad v(t)=\psi_t(L,t)\quad\Longleftrightarrow\quad v=\Psi(L).$$

More explicitly, we can write

$$D(A) = \Big\{ U = (\varphi, \Phi, \psi, \Psi, u, v) \in \left(H^1_* \cap H^2 \times H^1_*\right)^2 \times \mathbb{R}^2 \,;\, u = \Phi(L),\ v = \Psi(L) \Big\}.$$

Existence and Uniqueness

If $u_0 \in \mathcal{H}$, then the Abstract Cauchy Problem (ACP) admits a unique mild solution

$$u \in C([0, +\infty), \mathcal{H})$$

satisfying

$$u(t) = u_0 + A \int_0^t u(s)\, ds.$$

Moreover, if $u_0 \in D(A)$, then the ACP admits a unique classical solution

$$u \in C^0([0, +\infty), D(A)) \cap C^1([0, +\infty), \mathcal{H}).$$

Proof: In view of Theorem 2.3, it suffices to show that A is the infinitesimal generator of a contraction C_0-semigroup. For this, we apply the Lumer–Phillips Theorem, and thus we must verify:

(i) A is dissipative;

(ii) $\operatorname{Im}(\lambda I - A) = \mathcal{H}$ for some $\lambda > 0$;

(iii) $D(A)$ is dense in $\mathcal{H}$.

A direct computation shows that A is dissipative. We prove (ii) for $\lambda = 1$; that is, given $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$, we seek $U \in D(A)$ such that $(I - A)U = F$.

The equation $(I - A)U = F$ is equivalent to

$$\varphi - \Phi = f_1, \tag{3.5.122}$$
$$\Phi - \frac{1}{\rho_1(x)}\big[\kappa(x)(\varphi_x + \psi)_x\big] = f_2, \tag{3.5.123}$$
$$\psi - \Psi = f_3, \tag{3.5.124}$$
$$\Psi - \frac{1}{\rho_2(x)}\big[(b(x)\psi_x)_x - \kappa(x)(\varphi_x + \psi)\big] = f_4, \tag{3.5.125}$$
$$u + \frac{1}{m}\big[k_0 u + \kappa(L)(\varphi_x(L) + \psi(L))\big] = f_5, \tag{3.5.126}$$
$$v + \frac{1}{I_m}\big[k_1 v + b(L)\psi_x(L)\big] = f_6. \tag{3.5.127}$$

From (3.5.122) and (3.5.124) we obtain

$$\Phi = \varphi - f_1, \tag{3.5.128}$$
$$\Psi = \psi - f_3. \tag{3.5.129}$$

Substituting (3.5.128)–(3.5.129) into (3.5.123)–(3.5.125), we get

$$\rho_1(x)\varphi - \big(\kappa(x)(\varphi_x + \psi)\big)_x = g_1, \tag{3.5.130}$$
$$\rho_2(x)\psi - \big(b(x)\psi_x\big)_x + \kappa(x)(\varphi_x + \psi) = g_2, \tag{3.5.131}$$

where $g_1 = \rho_1(f_1 + f_2)$, $g_2 = \rho_2(f_3 + f_4)$.

To solve (3.5.130)–(3.5.131), we apply the Lax–Milgram Theorem. Define the bilinear form

$$a\colon [H^1_* \times H^1_*] \times [H^1_* \times H^1_*] \to \mathbb{R},$$
$$((\varphi,\psi),(\tilde\varphi,\tilde\psi)) \mapsto a((\varphi,\psi),(\tilde\varphi,\tilde\psi))$$

by

$$\begin{aligned} a((\varphi,\psi),(\tilde\varphi,\tilde\psi)) = &\int_0^L \rho_1(x)\varphi\tilde\varphi\,dx + \int_0^L \rho_2(x)\psi\tilde\psi\,dx \\ &+ \int_0^L \kappa(x)(\varphi_x+\psi)(\tilde\varphi_x+\tilde\psi)\,dx + \int_0^L b(x)\psi_x\tilde\psi_x\,dx \\ &+ \alpha\,\varphi(L)\tilde\varphi(L) + \beta\,\psi(L)\tilde\psi(L), \end{aligned}$$

where $\alpha = 1 + \dfrac{k_0}{m}$ and $\beta = 1 + \dfrac{k_1}{I_m}$.

One verifies that a is bilinear and continuous on $H^1_* \times H^1_*$ by means of the Cauchy–Schwarz and Poincaré inequalities together with the boundedness of the coefficients. Moreover, using the inequalities $(a+b)^2 \le 2(a^2+b^2)$ and the positivity of $\rho_1, \rho_2, b, \kappa, \alpha, \beta$, one shows that a is coercive.

Hence, by the Lax–Milgram Theorem, for each $(g_1, g_2) \in (H^1_*)' \times (H^1_*)'$ there exists a unique $(\varphi,\psi) \in H^1_* \times H^1_*$ such that

$$a((\varphi,\psi),(\tilde\varphi,\tilde\psi)) = (g_1,\tilde\varphi) + (g_2,\tilde\psi), \quad \forall\,(\tilde\varphi,\tilde\psi) \in H^1_* \times H^1_*.$$

This solution (φ,ψ) satisfies (3.5.130)–(3.5.131) and yields, together with (3.5.128)–(3.5.129), a unique $U \in D(A)$ that solves $(I-A)U = F$. Therefore, $\mathrm{Im}(I-A) = \mathcal{H}$, and the proof is complete. □

Chapter 4

Nonlinear Semigroups

4.1 Duality Operator

In what follows, X will denote either a topological vector space (t.v.s.) or a real normed vector space, whose norm will be represented by $\|\cdot\|$. We denote by X' the topological dual of X, or more precisely, the space consisting of all linear and continuous forms $x' : X \longrightarrow \mathbb{R}$, and when X is normed, we endow X' with the norm

$$\|x'\| = \sup_{\|x\|\leq 1} |x'(x)|.$$

The duality between X and X' will be denoted, interchangeably, by $\langle x, x' \rangle$ or $\langle x', x \rangle$, for every $x \in X$ and every $x' \in X'$, so that

$$\langle x', x \rangle = \langle x, x' \rangle = x'(x),$$

that is, the value taken by the functional x' at the point x.

We recall below the Hahn–Banach Theorem, a powerful tool in Functional Analysis, whose proof may be found, for instance, in Brézis [14], in Bachman–Narici [8] and in Horváth [53].

Theorem 4.1 (Hahn–Banach) *Let X be a vector space and p a positively homogeneous and subadditive functional on X. If G is a proper vector subspace of X, $g \in G'$ and $g(x) \leq p(x)$ for every $x \in G$, then there exists an extension h of g to X such that $h(x) \leq p(x)$ for every $x \in X$.*

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As an immediate consequence of the Hahn–Banach Theorem, we have the following results:

Corollary 4.2 *Let X be a normed vector space, $G \subset X$ a subspace of X and $g \in G'$. Then there exists an extension f of g such that $f \in X'$ and $\|f\|_{X'} = \|g\|_{G'}$.*

Proof: Define

$$p(x) = \|g\|_{G'}\|x\|, \quad \forall x \in X,$$

then

$$g(x) \leq |g(x)| \leq \|g\|_{G'}\|x\| = p(x), \quad \forall x \in G.$$

Thus, by Theorem 4.1, there exists an extension f of g to X such that

$$f(x) \leq p(x), \quad x \in X.$$

Observe also that

$$-f(x) = f(-x) \leq p(-x) = \|g\|_{G'}\|x\| = p(x).$$

Consequently,

$$|f(x)| \leq p(x) = \|g\|_{G'}\|x\|, \quad \forall x \in X,$$

which implies

$$\|f\|_{X'} = \sup_{\|x\|\leq 1} |f(x)| \leq \|g\|_{G'},$$

i.e.,

$$\|f\|_{X'} \leq \|g\|_{G'}. \tag{4.1.1}$$

On the other hand, since $f(x) = g(x)$ for every $x \in G$, it follows that

$$\|f\|_{X'} = \sup_{\substack{x\in X\\ \|x\|\leq 1}} |f(x)| \geq \sup_{\substack{x\in G\\ \|x\|\leq 1}} |g(x)| = \|g\|_{G'}. \tag{4.1.2}$$

From inequalities (4.1.1) and (4.1.2), we conclude that $\|f\|_{X'} = \|g\|_{G'}$. □

Corollary 4.3 *Let X be a normed vector space. Then, for each $x_0 \in X$, there exists a form $f_0 \in X'$ such that $\|f_0\|_{X'} = \|x_0\|$ and $\langle f_0, x_0\rangle = \|x_0\|^2$.*

Proof: If $x_0 = 0$, then $f_0 = 0$ satisfies the statement. Suppose now $x_0 \neq 0$. Define

$$G := \mathbb{R}x_0 = \{tx_0;\ t \in \mathbb{R}\},$$

and

$$g(tx_0) = t\|x_0\|^2, \quad \forall t \in \mathbb{R}.$$

Thus,

$$\sup_{\substack{x\in G\\ \|x\|\leq 1}} |g(x)| = \sup_{\substack{t\in\mathbb{R}\\ |t|=\frac{1}{\|x_0\|}}} |t|\|x_0\|^2 = \|x_0\|.$$

Since g is linear, it follows that $g \in G'$ and

$$\|g\|_{G'} = \|x_0\|.$$

By Corollary 4.2, there exists an extension f_0 of g to X such that $f_0 \in X'$ and

$$\|f_0\|_{X'} = \|g\|_{G'} = \|x_0\|.$$

Moreover, since $x_0 \in G$, it follows that

$$\langle f_0, x_0\rangle = \langle g, x_0\rangle = \|x_0\|^2.$$

□

Let X be a normed space. For each $x \in X$, define the set

$$F(x) = \{x' \in X' :\ \langle x', x\rangle = \|x\|^2 = \|x'\|^2\}. \tag{4.1.3}$$

Proposition 4.4 *Let X be a Banach space. Then, for every $x \in X$, the following properties hold:*

(i) $F(x) \neq \emptyset$;

(ii) $F(x)$ *is convex and compact in the weak** *topology of* X';

(iii) $F(\lambda x) = \lambda F(x), \quad \forall \lambda \in \mathbb{R}$.

Proof:

(i) Follows from Corollary 4.3.

(ii) Let $x_1', x_2' \in F(x)$ and $t \in [0,1]$. From (4.1.3),

$$\begin{aligned} \|x\|^2 &= t\|x\|^2 + (1-t)\|x\|^2 \\ &= t\langle x, x_1'\rangle + (1-t)\langle x, x_2'\rangle \\ &= \langle x, tx_1' + (1-t)x_2'\rangle. \end{aligned} \tag{4.1.4}$$

Since

$$\langle x, tx_1' + (1-t)x_2'\rangle \leq \|x\|\|tx_1' + (1-t)x_2'\|,$$

it follows from (4.1.4) that

$$\|x\| \leq \|tx_1' + (1-t)x_2'\|. \tag{4.1.5}$$

On the other hand, from (4.1.3) we also have

$$\begin{aligned} \|tx_1' + (1-t)x_2'\| &\leq t\|x_1'\| + (1-t)\|x_2'\| \\ &= t\|x\| + (1-t)\|x\| \\ &= \|x\|. \end{aligned} \tag{4.1.6}$$

From (4.1.5) and (4.1.6), we conclude that

$$\|tx_1' + (1-t)x_2'\| = \|x\|,$$

hence $tx_1' + (1-t)x_2' \in F(x)$, proving that $F(x)$ is convex.

To show that $F(x)$ is compact in the weak* topology of X', by Alaoglu's Theorem it suffices to show that $F(x)$ is weak* closed.

Let $x_0' \in X'$ be a weak* limit point of $F(x)$. Then for every $\varepsilon > 0$, the neighbourhood

$$\{\xi \in X'; |\langle x_0' - \xi, x\rangle| < \varepsilon\}$$

contains some $x' \in F(x)$, i.e.,

$$|\langle x_0' - x', x\rangle| < \varepsilon.$$

Thus

$$\|x\|^2 - \varepsilon < \langle x_0', x\rangle < \|x\|^2 + \varepsilon,$$

and so

$$\langle x_0', x\rangle = \|x\|^2. \tag{4.1.7}$$

From (4.1.7),

$$\|x\|^2 \leq \|x\|\|x_0'\|,$$

hence

$$\|x\| \leq \|x_0'\|.$$

But $F(x)$ is contained in the weak* compact ball $\{\xi : \|\xi\| \leq \|x\|\}$; therefore $\|x_0'\| \leq \|x\|$ and so $\|x_0'\| = \|x\|$, showing that $x_0' \in F(x)$.

(iii) If $x' \in F(x)$, then for any $\lambda \in \mathbb{R}$,

$$\langle \lambda x, \lambda x' \rangle = \lambda^2 \langle x, x' \rangle = \lambda^2 \|x\|^2 = \|\lambda x\|^2,$$

so $\lambda x' \in F(\lambda x)$ and hence $\lambda F(x) \subset F(\lambda x)$.

Conversely, if $y' \in F(\lambda x)$ with $\lambda \neq 0$, then

$$\langle x, y'/\lambda \rangle = \|x\|^2 = \|y'/\lambda\|^2,$$

so $y'/\lambda \in F(x)$, proving the reverse inclusion.

Definition 4.5 *We call an* _operator_ *with domain in a set X and range in a set Y any relation A from X to Y, or equivalently, any subset of the Cartesian product $X \times Y$.*

Thus, if A is an operator from X to Y, then for every $x \in X$, Ax will be a subset of Y. Observe that an operator A from X to Y defines a map (still denoted by the same letter)

$$A : X \longrightarrow 2^Y, \quad x \longmapsto Ax,$$

where 2^Y denotes the power set of Y.

The set $D(A)$ of those $x \in X$ for which $Ax \neq \emptyset$ is called the _domain_ of A. The set $Im(A)$ of those $y \in Y$ such that $y \in Ax$ for some $x \in D(A)$ is called the _image_ of A.

Thus,

$$Im(A) = \bigcup_{x \in D(A)} Ax.$$

To express that A is an operator with domain X and image Y, we write $A : X \longrightarrow Y$.

The graph of A is the set of points $(x, y) \in X \times Y$ such that $y \in Ax$ for some $x \in D(A)$.

Let Y be a vector space, and let $A, B : X \longrightarrow Y$ be operators. We define $A + B$, λA and A^{-1} respectively by:

$$A + B = \{(x, y + z);\ (x, y) \in A,\ (x, z) \in B\},$$

$$\lambda A = \{(x, \lambda y);\ (x, y) \in A\},$$

$$A^{-1} = \{(y, x);\ (x, y) \in A\},$$

where $D(A + B) = D(A) \cap D(B)$, $D(\lambda A) = D(A)$ and $D(A^{-1}) = Im(A)$.

Definition 4.6 *If for each $x \in D(A)$ the set Ax is a singleton, then we say that A is* _single–valued_.

Definition 4.7 *We say that $B : X \longrightarrow Y$ is an* _extension_ *of $A : X \longrightarrow Y$ if $A \subset B$.*

Note that, in the case of single–valued operators, B is a proper extension of A if and only if $D(B)$ properly contains $D(A)$, but this is not true in the multivalued case. Indeed, consider $X = Y = \{1, 2, 3\}$,

$$A = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1)\},$$

$$B = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3)\}.$$

Then $A \subsetneq B$, but $D(A) = D(B) = X$.

Definition 4.8 *An operator is said to be* _closed_ *if, whenever $\{x_n\} \subset D(A)$ with $x_n \to x$ and $y_n \in Ax_n$ with $y_n \to y$, it follows that $x \in D(A)$ and $y \in Ax$.*

Definition 4.9 *Let X be a Banach space. The duality operator is the operator $F : X \longrightarrow X'$ defined by:*

$$D(F) = X, \qquad F(x) = \{x' \in X';\ \langle x', x\rangle = \|x\|^2 = \|x'\|^2\},$$

for every $x \in X$.

A duality mapping of X is any map $f : X \longrightarrow X'$ such that $f(x) \in F(x)$ for every $x \in X$.

Definition 4.10 *Let $f : X \longrightarrow (-\infty, +\infty]$ be a function. The effective domain of f is the set*

$$D_e(f) = \{x \in X;\ f(x) < +\infty\}.$$

A function is said to be proper if $D_e(f) \neq \emptyset$.

Result 1: Let X be a topological space satisfying the first axiom of countability, i.e., X admits a countable local base at each of its points, and let $f : X \longrightarrow [-\infty, +\infty]$ be sequentially lower semicontinuous (l.s.c.). Then f is l.s.c.

Proof: Fix an arbitrary point $x_0 \in X$. Suppose that f is not l.s.c. at x_0. Then there exists $\varepsilon_0 > 0$ such that for any neighbourhood $V(x_0)$ of x_0 we have

$$f(x) < f(x_0) - \varepsilon_0 < f(x_0) \quad \text{for some } x \in V(x_0). \tag{4.1.8}$$

Since X satisfies the first axiom of countability, there exists a countable neighbourhood base $\{U_n\}_{n\in\mathbb{N}}$ at x_0. We now construct the following sequence:

For $n = 1$, U_1 is a neighbourhood of $x_0 \Rightarrow \exists n_1 \in \mathbb{N}$ such that $U_{n_1} \subset U_1$. Define $V_1 = U_{n_1}$.

For $n = 2$, $U_2 \cap V_1$ is a neighbourhood of $x_0 \Rightarrow \exists n_2 \in \mathbb{N}$ such that $U_{n_2} \subset U_2 \cap V_1$. Define $V_2 = U_{n_2}$.

Proceeding inductively, we obtain a collection $\{V_n\}_{n\in\mathbb{N}}$ of neighbourhoods of x_0 such that $V_{n+1} \subset V_n$ and $V_n \subset U_n$ for all $n \in \mathbb{N}$.

We claim that $\{V_n\}_{n\in\mathbb{N}}$ is a local base at x_0. Indeed, let $V(x_0)$ be any neighbourhood of x_0. Since $\{U_n\}$ is a base at x_0, there exists n such that $U_n \subset V(x_0)$. Then

$$V_n \subset U_n \subset V(x_0),$$

as claimed.

Hence, from assumption (4.1.8), for each n there exists $x_n \in V_n$ such that

$$f(x_n) < f(x_0) - \varepsilon_0. \tag{4.1.9}$$

Thus we obtain a sequence (x_n) with

$$f(x_n) < f(x_0) - \varepsilon_0, \quad \forall n \in \mathbb{N}. \tag{4.1.10}$$

In particular, for each n,

$$f(x_k) < f(x_0) - \varepsilon_0, \quad \forall k \geq n. \tag{4.1.11}$$

From (4.1.11),

$$\inf_{k\geq n} f(x_k) \leq f(x_0) - \varepsilon_0, \quad \forall n. \tag{4.1.12}$$

Hence,

$$\lim_{n\to\infty} \inf_{k\geq n} f(x_k) = \sup_{n\in\mathbb{N}} \inf_{k\geq n} f(x_k) \tag{4.1.13}$$
$$\leq f(x_0) - \varepsilon_0 < f(x_0). \tag{4.1.14}$$

However, $x_n \to x_0$ because for any neighbourhood $V(x_0)$, since $\{V_n\}$ is a base at x_0, there exists k_0 such that

$$V_{k_0} \subset V(x_0). \tag{4.1.15}$$

Because $V_{n+1} \subset V_n$, we have

$$V_k \subset V_{k_0} \subset V(x_0), \quad \forall k \geq k_0. \tag{4.1.16}$$

Thus,

$$x_k \in V_k \subset V(x_0), \quad \forall k \geq k_0, \tag{4.1.17}$$

showing $x_n \to x_0$.

Since f is sequentially l.s.c.,

$$\lim_{n\to\infty} \inf f(x_n) \geq f(x_0). \tag{4.1.18}$$

But (4.1.14) and (4.1.18) together imply

$$f(x_0) \leq \lim_{n\to\infty} \inf f(x_n) < f(x_0),$$

a contradiction. Hence f is l.s.c. at x_0, and by arbitrariness of x_0, f is l.s.c. on X. □

Result 2: Let X be a topological space and $f : X \to [0, +\infty]$. Let $\{u_n\}_{n\in\mathbb{N}} \subset X$ satisfy

$$\lim_{n\to\infty} \inf f(u_n) = \lambda < +\infty.$$

Then there exists a subsequence $\{u_{n_k}\}$ such that $\{f(u_{n_k})\}$ is bounded and

$$\lim_{k\to\infty} \inf f(u_{n_k}) = \lambda. \tag{4.1.19}$$

Proof: From the hypothesis,

$$\sup_{n\in\mathbb{N}} \inf_{k\geq n} f(u_k) = \lambda,$$

i.e.

$$\inf_{k\geq n} f(u_k) \leq \lambda, \quad \forall n. \tag{4.1.20}$$

Assume first that

$$\inf_{k\geq n} f(u_k) < \lambda, \quad \forall n. \tag{4.1.21}$$

Then for each n,

$$\exists k_n > n \text{ such that } f(u_{k_n}) < \lambda. \tag{4.1.22}$$

Indeed, if $f(u_k) \geq \lambda$ for all $k \geq n$, then $\inf_{k\geq n} f(u_k) \geq \lambda$, contradicting (4.1.21).

Thus we obtain a subsequence $\{u_{k_n}\}$ such that

$$0 \leq f(u_{k_n}) < \lambda, \quad \forall n, \tag{4.1.23}$$

showing that $\{f(u_{k_n})\}$ is bounded.

From (4.1.23),

$$j \geq n \Rightarrow f(u_{k_j}) < \lambda \quad \Rightarrow \quad \inf_{j \geq n} f(u_{k_j}) < \lambda. \tag{4.1.24}$$

Taking the supremum over n,

$$\lim_{n\to\infty} \inf f(u_{k_n}) \leq \lambda. \tag{4.1.25}$$

On the other hand, for each n,

$$j \geq n \Rightarrow k_j \geq j \geq n,$$

hence

$$\inf_{j \geq n} f(u_j) \leq \inf_{j \geq n} f(u_{k_j}). \tag{4.1.26}$$

Thus

$$\lambda = \lim_{n\to\infty} \inf f(u_n) \leq \lim_{n\to\infty} \inf f(u_{k_n}). \tag{4.1.27}$$

Combining (4.1.25) and (4.1.27) yields the desired equality.

Now assume instead that

$$\inf_{k \geq n} f(u_k) = \lambda, \quad \forall n. \tag{4.1.28}$$

Then

$$\inf_{k \geq n} f(u_k) < \lambda + \frac{1}{n}, \quad \forall n. \tag{4.1.29}$$

Hence, for each n,

$$\exists k_n > n \text{ such that } f(u_{k_n}) < \lambda + \frac{1}{n}. \tag{4.1.30}$$

Then

$$0 \leq f(u_{k_n}) < \lambda + \frac{1}{n} \leq \lambda + 1, \quad \forall n, \tag{4.1.31}$$

so $\{f(u_{k_n})\}$ is bounded.

From (4.1.30),

$$\inf_{j \geq n} f(u_{k_j}) \leq \lambda, \quad \forall n. \tag{4.1.32}$$

Thus

$$\lim_{n\to\infty} \inf f(u_{k_n}) \leq \lambda. \tag{4.1.33}$$

But again

$$j \geq n \Rightarrow k_j \geq j \Rightarrow \{f(u_{k_j})\}_{j \geq n} \subset \{f(u_j)\}_{j \geq n}, \tag{4.1.34}$$

so

$$\inf_{j \geq n} f(u_j) \leq \inf_{j \geq n} f(u_{k_j}). \tag{4.1.35}$$

Hence

$$\lambda = \lim_{n\to\infty} \inf f(u_n) \leq \lim_{n\to\infty} \inf f(u_{k_n}). \tag{4.1.36}$$

Together with (4.1.33), this proves (4.1.19). □

Example 4.11 Let $\Omega \subset \mathbb{R}^n$ be an open set with regular boundary. Consider the function $f : L^2(\Omega) \to$

$(-\infty, +\infty]$ defined by

$$f(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 \, dx, & \text{if } u \in H^1(\Omega); \\ +\infty, & \text{otherwise.} \end{cases}$$

We have that f is proper and l.s.c. Indeed, note that f is proper since $D_e(f) = H^1(\Omega)$. To show that f is l.s.c. it is enough to prove that f is sequentially l.s.c., since $L^2(\Omega)$ satisfies the first axiom of countability (see Result 1).

Recall that f is sequentially l.s.c. at a point $u \in L^2(\Omega)$ if, for every sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to u$, one has

$$f(u) \leq \lim_{n\to+\infty} \inf f(u_n). \tag{4.1.37}$$

Let $u \in L^2(\Omega)$ and $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ be such that $u_n \to u$ in $L^2(\Omega)$. If $\lim_{n\to+\infty} \inf f(u_n) = +\infty$, then relation (4.1.37) is trivially satisfied. Hence, assume that

$$\lim_{n\to+\infty} \inf f(u_n) = \lambda < +\infty.$$

We now suppose that the sequence $\{f(u_n)\}_{n\in\mathbb{N}}$ is bounded, which is not restrictive, since we may extract a bounded subsequence with the same lower limit λ (see Result 2).

From this hypothesis and the convergence of (u_n) in $L^2(\Omega)$, it follows that (u_n) converges weakly in $H^1(\Omega)$. Hence there exists a subsequence of (u_n) which converges weakly in $H^1(\Omega)$.

Since strong convergence implies weak convergence, we have $u_n \rightharpoonup u$ in $L^2(\Omega)$. As the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is linear and continuous with respect to the strong topologies, it follows that the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is also continuous with respect to the weak topologies.

Therefore $u_n \rightharpoonup u$ in $H^1(\Omega)$ and, by Corollary 3.23 of [23], we have

$$\begin{aligned} \|u\|^2_{L^2(\Omega)} + \int_\Omega |\nabla u|^2 \, dx &= \|u\|^2_{H^1(\Omega)} \leq \lim_{n\to+\infty} \inf \|u_n\|^2_{H^1(\Omega)} && (4.1.38) \\ &= \lim_{n\to+\infty} \inf \left(\|u_n\|^2_{L^2(\Omega)} + \int_\Omega |\nabla u_n|^2 \, dx \right) && (4.1.39) \\ &= \|u\|^2_{L^2(\Omega)} + \lim_{n\to+\infty} \inf \int_\Omega |\nabla u_n|^2 \, dx, && (4.1.40) \end{aligned}$$

and thus

$$\int_\Omega |\nabla u|^2 \, dx \leq \lim_{n\to+\infty} \inf \int_\Omega |\nabla u_n|^2 \, dx,$$

or equivalently,

$$f(u) \leq \lim_{n\to+\infty} \inf f(u_n),$$

which proves (4.1.37). Hence f is sequentially l.s.c. and, consequently, l.s.c.

Example 4.12 Let $\varphi : \mathbb{R} \to (-\infty, +\infty]$ be a proper, l.s.c. and non-negative function. Consider $\Phi : L^p(0,T) \to \mathbb{R}$, $1 \leq p < +\infty$, defined by

$$\Phi(u) = \begin{cases} \displaystyle\int_0^T \varphi(u(t)) \, dt, & \text{if } \varphi(u) \in L^1(0,T); \\ +\infty, & \text{otherwise.} \end{cases}$$

We claim that Φ is proper and l.s.c.

To show that Φ is proper we must verify that

$$D_e(\Phi) = \Big\{u \in L^p(0,T);\ \varphi \circ u \in L^1(0,T)\Big\} \neq \emptyset.$$

By hypothesis, φ is proper, that is, $D_e(\varphi) \neq \emptyset$.

Let $M \in D_e(\varphi)$ and consider $v : (0,T) \to \mathbb{R}$ defined by $v(t) = M$ for all $t \in (0,T)$. Note that $v \in L^\infty(0,T) \hookrightarrow L^p(0,T)$ for $1 \leq p \leq +\infty$.

Thus

$$(\varphi \circ v)(t) = \varphi(M) =: \tilde{M} < +\infty \quad \text{since } M \in D_e(\varphi).$$

Hence $\varphi \circ v \in L^\infty(0,T) \hookrightarrow L^1(0,T)$, i.e.

$$\varphi \circ v \in L^1(0,T),$$

and therefore $v \in D_e(\Phi)$. We conclude that $D_e(\Phi) \neq \emptyset$, so Φ is proper.

It remains to prove that Φ is l.s.c. For this it suffices to prove that all level sets of Φ,

$$N(\lambda, \Phi) = \Big\{u \in L^p(0,T);\ \Phi(u) \leq \lambda\Big\}, \tag{4.1.41}$$

are closed. Fix $\lambda \in \mathbb{R}$ and let $u \in \overline{N(\lambda, \Phi)}$. Then there exists $(u_n)_{n\in\mathbb{N}} \subset N(\lambda, \Phi)$ such that

$$u_n \longrightarrow u \text{ in } L^p(0,T). \tag{4.1.42}$$

Hence

$$\Phi(u_n) \leq \lambda, \quad \forall n \in \mathbb{N},$$

that is,

$$\int_0^T \varphi(u_n(t))\, dt \leq \lambda, \quad \forall n \in \mathbb{N}. \tag{4.1.43}$$

Note that $(\varphi(u_n))_{n\in\mathbb{N}} \subset L^1(0,T)$ and $(\varphi(u_n)) \geq 0$ almost everywhere, for all n.

From (4.1.43),

$$\sup_{n\in\mathbb{N}} \int_0^T \varphi(u_n(t))\, dt \leq \lambda < +\infty.$$

Thus, by Fatou's lemma,

$$\lim_{n\to\infty} \inf_{n\in\mathbb{N}} \varphi(u_n) \in L^1(0,T),$$

and moreover,

$$\int_0^T \lim_{n\to\infty} \inf_{n\in\mathbb{N}} \varphi(u_n(t))\, dt \leq \lim_{n\to\infty} \inf_{n\in\mathbb{N}} \int_0^T \varphi(u_n(t))\, dt \leq \lambda. \tag{4.1.44}$$

On the other hand, since $u_n \to u$ in $L^p(0,T)$, there exists a subsequence of (u_n) (not relabelled) such that

$$u_n(t) \longrightarrow u(t) \quad \text{almost everywhere in } (0,T).$$

Since φ is l.s.c. and hence sequentially l.s.c., we have

$$\varphi(u(t)) \leq \lim_{n\to\infty} \inf_{n\in\mathbb{N}} \varphi(u_n(t)) \quad \text{for a.e. } t \in (0,T). \tag{4.1.45}$$

Therefore, from (4.1.44) and (4.1.45),

$$\Phi(u) = \int_0^T \varphi(u(t))\,dt \le \int_0^T \lim_{n\to\infty} \inf_{n\in\mathbb{N}} \varphi(u_n(t))\,dt \le \lambda,$$

so $u \in N(\lambda, \Phi)$, proving (4.1.41): the level sets are closed and hence Φ is l.s.c.

We shall use this result to prove the next theorem.

Theorem 4.13 (Composition) *Let V and W be vector spaces, $\varphi : W \to (-\infty, +\infty]$ a convex mapping, $\Lambda : V \to W$ a linear mapping, and suppose that $D_e(\varphi) \cap \operatorname{Im}(\Lambda) \neq \emptyset$. If φ is continuous at some point of $\operatorname{Im}(\Lambda) \cap D_e(\varphi)$, then*

$$\partial(\varphi \circ \Lambda) = \Lambda' \cdot \partial\varphi \cdot \Lambda.$$

Proof: First, we show that $\varphi \circ \Lambda : V \longrightarrow (-\infty, +\infty]$ is convex and proper. Indeed, let $u, v \in V$ and $t \in [0,1]$. Then

$$\begin{aligned}(\varphi \circ \Lambda)(tu + (1-t)v) &= \varphi\big(\Lambda(tu + (1-t)v)\big)\\ &= \varphi\big(t\Lambda u + (1-t)\Lambda v\big)\\ &\le t\varphi(\Lambda u) + (1-t)\varphi(\Lambda v)\\ &= t(\varphi\circ\Lambda)(u) + (1-t)(\varphi\circ\Lambda)(v),\end{aligned}$$

which shows that $\varphi \circ \Lambda$ is convex.

We now prove that $\varphi \circ \Lambda$ is proper. By hypothesis, $D_e(\varphi) \cap \operatorname{Im}\Lambda \neq \emptyset$, that is, there exists $u \in D_e(\varphi) \cap \operatorname{Im}\Lambda$. Thus $u = \Lambda v$ for some $v \in V$, and then

$$(\varphi \circ \Lambda)(v) = \varphi(\Lambda v) = \varphi(u) < +\infty,$$

so $v \in D_e(\varphi \circ \Lambda)$, and hence $\varphi \circ \Lambda$ is proper.

Let $u \in V$ be such that $\partial\varphi(\Lambda u) \neq \emptyset$. By the definition of the subdifferential, for each $\omega' \in \partial\varphi(\Lambda u)$, we have

$$\big\langle \omega', \omega - \Lambda u \big\rangle \le \varphi(\omega) - \varphi(\Lambda u), \quad \forall \omega \in D_e(\varphi). \tag{4.1.46}$$

Denote by $\Lambda' : W' \longrightarrow V'$ the adjoint operator of Λ. Then

$$\begin{aligned}\big\langle \Lambda'\omega', v - u \big\rangle &= \big\langle \omega', \Lambda(v-u) \big\rangle\\ &= \big\langle \omega', \Lambda v - \Lambda u \big\rangle, \quad \forall\, v \in V,\end{aligned}$$

and in particular

$$\big\langle \Lambda'\omega', v - u \big\rangle = \big\langle \omega', \Lambda v - \Lambda u \big\rangle, \quad \forall\, v \in V \text{ such that } \Lambda v \in D_e(\varphi). \tag{4.1.47}$$

From (4.1.46) and (4.1.47) we obtain

$$\big\langle \Lambda'\omega', v - u \big\rangle \le \varphi(\Lambda v) - \varphi(\Lambda u) = (\varphi\circ\Lambda)(v) - (\varphi\circ\Lambda)(u), \quad \forall v \in D_e(\varphi\circ\Lambda),$$

whence

$$\Lambda'\omega' \in \partial(\varphi \circ \Lambda)(u).$$

This implies that $\partial(\varphi \circ \Lambda)(u) \neq \emptyset$ and therefore $u \in D(\partial(\varphi \circ \Lambda))$. In other words,

$$\big(\Lambda' \circ \partial\varphi \circ \Lambda\big)(u) \subset \partial\big(\varphi \circ \Lambda\big)(u), \quad \forall u \in D\big(\Lambda' \circ \partial\varphi \circ \Lambda\big),$$

and hence
$$\Lambda' \circ \partial\varphi \circ \Lambda \subset \partial(\varphi \circ \Lambda).$$

Conversely, let $u' \in \partial(\varphi \circ \Lambda)(u)$. Then, by definition,
$$\big\langle u', v-u \big\rangle + (\varphi \circ \Lambda)(u) \le (\varphi \circ \Lambda)(v), \quad \forall v \in V. \tag{4.1.48}$$

Consider the set
$$K = \Big\{ \big(\Lambda v, \big\langle u', v-u \big\rangle + \varphi(\Lambda u)\big);\ v \in V \Big\} \subset W \times \mathbb{R}.$$

We claim that $K \cap \mathrm{epi}(\varphi) \subset \mathrm{bdr}(\mathrm{epi}(\varphi))$. Indeed, let $x \in K \cap \mathrm{epi}(\varphi)$. By the definition of K,
$$x = \big(\Lambda v, \big\langle u', v-u \big\rangle + \varphi(\Lambda u)\big), \quad \text{for some } v \in V,$$
and from (4.1.48) we have
$$\varphi(\Lambda v) \ge \big\langle u', v-u \big\rangle + \varphi(\Lambda u). \tag{4.1.49}$$

On the other hand, since $x \in \mathrm{epi}(\varphi)$, we must have
$$\varphi(\Lambda v) \le \big\langle u', v-u \big\rangle + \varphi(\Lambda u). \tag{4.1.50}$$

From (4.1.49) and (4.1.50), we obtain
$$\varphi(\Lambda v) = \big\langle u', v-u \big\rangle + \varphi(\Lambda u) := \lambda. \tag{4.1.51}$$

Let $\mathbb{V}_x$ be a neighbourhood of $x \in K \cap \mathrm{epi}(\varphi)$, i.e.
$$\mathbb{V}_x = U_{\Lambda v} \times (\lambda - \alpha, \lambda + \alpha),$$
where $U_{\Lambda v}$ is a neighbourhood of Λv in W and $\alpha > 0$. In order to conclude the claim, we need to exhibit a point $y \in \mathbb{V}_x$ such that $y \notin \mathrm{epi}(\varphi)$, since $x \in \mathbb{V}_x \cap \mathrm{epi}(\varphi)$ for every neighbourhood $\mathbb{V}_x$ of x, that is, $\mathbb{V}_x \cap \mathrm{epi}(\varphi) \ne \emptyset$. Choosing β with $\lambda - \alpha < \beta < \lambda$, we have $y = (\Lambda v, \beta) \in \mathbb{V}_x$. But, by (4.1.51), we get
$$\varphi(\Lambda v) = \lambda > \beta,$$
which implies
$$y = (\Lambda v, \beta) \in W \times \mathbb{R} \setminus \mathrm{epi}(\varphi),$$
or equivalently,
$$\mathbb{V}_x \cap \big(W \times \mathbb{R} \setminus \mathrm{epi}(\varphi)\big) \ne \emptyset.$$
Hence $x \in \mathrm{bdr}(\mathrm{epi}(\varphi))$, and the claim follows.

We now prove that $\mathrm{int}(\mathrm{epi}(\varphi)) \ne \emptyset$. For this, we must find an element $x \in \mathrm{epi}(\varphi)$ and a neighbourhood $\mathbb{V}_x$ of x such that $\mathbb{V}_x \subset \mathrm{epi}(\varphi)$.

Let $v \in V$ be such that φ is continuous at Λv. Consequently, $\Lambda v \in D_e(\varphi)$, i.e. $\varphi(\Lambda v) < \infty$. Choose $\beta \in \mathbb{R}$ with $\varphi(\Lambda v) < \beta$.

Set
$$\varepsilon = \frac{\beta - \varphi(\Lambda v)}{4} > 0.$$

By continuity of φ at Λv, there exists a neighbourhood $U_{\Lambda v} \subset D_e(\varphi)$ such that

$$\varphi(U_{\Lambda v}) \subset \big(\varphi(\Lambda v) - \varepsilon, \varphi(\Lambda v) + \varepsilon\big),$$

that is,

$$\varphi(\Lambda v) - \varepsilon \leq \varphi(u) \leq \varphi(\Lambda v) + \varepsilon, \quad \forall u \in U_{\Lambda v}.$$

Note that $x = (\Lambda v, \beta) \in \mathrm{epi}(\varphi)$. Consider the neighbourhood

$$\mathbb{V}_x = U_{\Lambda v} \times (\beta - \varepsilon, \beta + \varepsilon).$$

We assert that $\mathbb{V}_x \subset \mathrm{epi}(\varphi)$. Indeed, let $y \in \mathbb{V}_x$, so that

$$y = (u, \lambda), \quad \text{with } u \in U_{\Lambda v} \text{ and } \lambda \in (\beta - \varepsilon, \beta + \varepsilon).$$

Moreover,

$$\begin{aligned} \varphi(\Lambda v) + 2\varepsilon &= \varphi(\Lambda v) + 2\frac{\beta - \varphi(\Lambda v)}{4} \\ &= \varphi(\Lambda v) + \frac{\beta}{2} - \frac{\varphi(\Lambda v)}{2} \\ &= \frac{\varphi(\Lambda v)}{2} + \frac{\beta}{2} < \frac{\beta}{2} + \frac{\beta}{2} = \beta, \end{aligned}$$

and thus

$$\varphi(\Lambda v) + \varepsilon < \beta - \varepsilon. \tag{4.1.52}$$

From (4.1.52) we deduce

$$\varphi(u) < \varphi(\Lambda v) + \varepsilon < \beta - \varepsilon < \lambda,$$

which proves that $y \in \mathrm{epi}(\varphi)$.

Therefore $\mathrm{int}(\mathrm{epi}(\varphi)) \neq \emptyset$. Since $\varphi : W \longrightarrow (-\infty, +\infty]$ is convex, it follows from Lemma 1.41 in [23] that $\mathrm{epi}(\varphi)$ is convex.

Hence, by Lemma 1.4 in [18], $\mathrm{int}(\mathrm{epi}(\varphi))$ is convex.

Observe that

$$K = S + \big(0, -u'(u) + \varphi(\Lambda u)\big),$$

where $S = \{(\Lambda v, u'(v));\ v \in V\}$, and S is a vector subspace.

Note that $K \cap \mathrm{int}(\mathrm{epi}(\varphi)) = \emptyset$, for if there existed $x \in K \cap \mathrm{int}(\mathrm{epi}(\varphi))$, then $x \in K \cap \mathrm{epi}(\varphi)$ and $x \in \mathrm{int}(\mathrm{epi}(\varphi))$. By what we have already shown, $x \in \mathrm{bdr}(\mathrm{epi}(\varphi)) \cap \mathrm{int}(\mathrm{epi}(\varphi)) = \emptyset$, since $\mathrm{int}(\mathrm{epi}(\varphi))$ is open.

Under these conditions, there exists a closed hyperplane

$$H = \{(w, t) \in W \times \mathbb{R};\ \psi(w, t) = -w^{'}(w) + t = c\}$$

which contains K and lies below $\mathrm{epi}(\varphi)$.

Thus

$$-w^{'}(\Lambda v) + u'(v - u) + \varphi(\Lambda u) = c, \quad \forall\ v \in V.$$

If $v = u$, then

$$c = -w'(\Lambda u) + \varphi(\Lambda u),$$

and hence

$$\langle w', \Lambda(v-u)\rangle = \langle u^{'}, (v-u)\rangle, \quad \forall\ v \in V.$$

Therefore we may conclude that $u^{'} = \Lambda^{'}(w^{'})$. Since $c \leq \psi(w,t)$ for all $(w,t) \in \text{epi}(\varphi)$, taking in particular $(w, \varphi(w))$ gives

$$-w^{'}(\Lambda u) + \varphi(\Lambda u) = c \leq -w^{'}(w) + \varphi(w), \quad \forall\ w \in W.$$

Consequently,

$$w^{'}(w - \Lambda u) \leq \varphi(w) - \varphi(\Lambda u), \quad \forall\ w \in W.$$

This shows that $w^{'} \in \partial\varphi(\Lambda u)$. Hence $\Lambda^{'} w^{'} \in \Lambda^{'}\partial\varphi(\Lambda u)$. Since $u^{'} = \Lambda^{'} w^{'}$, the result follows. □

Definition 4.14 (Gâteaux derivative) *Let X and Y be topological vector spaces. A mapping $\varphi : X \longrightarrow Y$ is said to be Gâteaux differentiable at a point x if there exists a linear and continuous mapping $\varphi'(x) : X \longrightarrow Y$ such that*

$$\lim_{\lambda\to 0} \frac{\varphi(x+\lambda y) - \varphi(x)}{\lambda} = \varphi'(x)y, \quad \forall y \in X.$$

The mapping $\varphi'(x)$ is called the Gâteaux derivative of φ at the point x.

Proposition 4.15 *Let K be a convex subset of a normed space V and let $\varphi : K \subset V \longrightarrow (-\infty, +\infty]$ be a function which is Gâteaux differentiable at every point $u \in K$. The following statements are equivalent:*

(a) φ is convex;

(b) $\varphi'(u)(v-u) \leq \varphi(v) - \varphi(u)$, for all $u, v \in K$;

(c) $(\varphi'(u) - \varphi'(v))(u-v) \geq 0$ for all $u, v \in K$.

Proof:

(a) ⇒ (b)

Assume that $\varphi : K \longrightarrow \mathbb{R}$ is convex, and let $u, v \in K$ and $t \in [0,1]$. By convexity of K,

$$(1-t)u + tv \in K,$$

and by convexity of φ we have

$$\varphi\big((1-t)u + tv\big) \leq (1-t)\varphi(u) + t\varphi(v),$$

or equivalently,

$$\varphi\big((1-t)u + tv\big) \leq \varphi(u) + t\big(\varphi(v) - \varphi(u)\big).$$

Thus

$$\frac{\varphi\big(u + t(v-u)\big) - \varphi(u)}{t} \leq \varphi(v) - \varphi(u).$$

Since φ is Gâteaux differentiable, by hypothesis,

$$\lim_{t\to 0} \frac{\varphi\big(u + t(v-u)\big) - \varphi(u)}{t} \leq \varphi(v) - \varphi(u),$$

that is,

$$\varphi'(u)(v-u) \leq \varphi(v) - \varphi(u),$$

which proves (b).

(b) ⇒ (c)

Assume that (b) holds and let $u, v \in K$. Then

$$\varphi'(u)(v-u) \leq \varphi(v) - \varphi(u)$$

and

$$\varphi'(v)(u-v) \leq \varphi(u) - \varphi(v).$$

Adding these two inequalities, we obtain

$$\varphi'(u)(v-u) + \varphi'(v)(u-v) \leq 0.$$

Hence

$$\varphi'(u)(v-u) - \varphi'(v)(v-u) \leq 0,$$

that is,

$$\big(\varphi'(v) - \varphi'(u)\big)(v-u) \geq 0,$$

which proves (c).

(c) $\Rightarrow$ (a)

Assume that (c) holds. Let $u, v \in K$ and consider

$$[u,v] = \{(1-t)u + tv;\ t \in [0,1]\} \subset K.$$

Define

$$\begin{array}{rccc} \psi: & [0,1] & \longrightarrow & (-\infty,+\infty] \\ & t & \longmapsto & \psi(t) = \varphi\big(u + t(v-u)\big), \end{array}$$

that is, $\psi = \varphi\big|_{[u,v]}$.

For each $t \in (0,1)$, let $\lambda > 0$ be sufficiently small so that $(t+\lambda) \in (0,1)$. Then

$$\begin{aligned} \psi'(t) &= \lim_{\lambda\to 0} \frac{\psi(t+\lambda) - \psi(t)}{\lambda} \\ &= \lim_{\lambda\to 0} \frac{\varphi(u + (t+\lambda)(v-u)) - \varphi(u + t(v-u))}{\lambda} \\ &= \lim_{\lambda\to 0} \frac{\varphi(u + t(v-u) + \lambda(v-u)) - \varphi(u + t(v-u))}{\lambda}. \end{aligned}$$

Since φ is Gâteaux differentiable on K, the above limit exists and we obtain

$$\psi'(t) = \varphi'\big(u + t(v-u)\big)(v-u), \quad t \in (0,1). \tag{4.1.53}$$

If $t = 0$ or $t = 1$, we consider respectively the right and left derivatives and obtain

$$\psi'(0) = \lim_{\lambda\to 0^+} \frac{\varphi(u + \lambda(v-u)) - \varphi(u)}{\lambda} = \varphi'(u)(v-u), \tag{4.1.54}$$

$$\psi'(1) = \lim_{\lambda\to 0^-} \frac{\varphi(v + \lambda(v-u)) - \varphi(v)}{\lambda} = \varphi'(v)(v-u). \tag{4.1.55}$$

From (4.1.53), (4.1.54) and (4.1.55) we can write

$$\psi'(t) = \varphi'\big(u + t(v-u)\big)(v-u), \quad \forall t \in [0,1]. \tag{4.1.56}$$

We claim that ψ' is increasing.

Indeed, let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. From (4.1.56) we have

$$\begin{aligned}\psi'(t_2) - \psi'(t_1) &= \varphi'\big(u + t_2(v-u)\big)(v-u) - \varphi'\big(u + t_1(v-u)\big)(v-u) \\ &= \Big(\varphi'\big(u + t_2(v-u)\big) - \varphi'\big(u + t_1(v-u)\big)\Big)(v-u). \end{aligned} \tag{4.1.57}$$

Set

$$w_1 = u + t_1(v-u) \in K, \qquad w_2 = u + t_2(v-u) \in K,$$

so that $w_2 - w_1 = (t_2 - t_1)(v-u)$.

By hypothesis,

$$\big(\varphi'(w_2) - \varphi'(w_1)\big)(w_2 - w_1) \geq 0,$$

and, using the linearity of the Gâteaux derivative and the fact that $t_2 - t_1 \geq 0$, we deduce from (4.1.57) that

$$\big[\psi'(t_2) - \psi'(t_1)\big](t_2 - t_1) \geq 0,$$

hence

$$\psi'(t_2) \geq \psi'(t_1),$$

showing that ψ' is increasing and therefore ψ is convex.

Consequently,

$$\psi\big((1-t)\cdot 0 + t \cdot 1\big) \leq (1-t)\psi(0) + t\psi(1), \quad \forall t \in [0,1],$$

that is,

$$\psi(t) \leq (1-t)\psi(0) + t\psi(1), \quad \forall t \in [0,1],$$

or equivalently,

$$\varphi\big((1-t)u + tv\big) \leq (1-t)\varphi(u) + t\varphi(v), \quad \forall t \in [0,1] \text{ and } \forall u, v \in K,$$

since u and v were chosen arbitrarily. This proves (a). □

Proposition 4.16 *Let $f : X \longrightarrow (-\infty, +\infty]$ be a proper convex function. If f is Gâteaux differentiable at a point $x \in D_e(f)$, then f is subdifferentiable at x and the Gâteaux derivative $f'(x)$ is the unique element of $\partial f(x)$.*

Proof:

Let $y \in D_e(f)$ and $\lambda \in [0,1]$. By the convexity of f and the fact that $x \in D_e(f)$, we have

$$\begin{aligned} f(y) - f(x) &= \frac{\lambda f(y) - \lambda f(x)}{\lambda} \\ &= \frac{-\lambda f(x) + f(x) - f(x) + \lambda f(y)}{\lambda} \\ &= \frac{(1-\lambda)f(x) + \lambda f(y) - f(x)}{\lambda} \\ &\geq \frac{f(x + \lambda(y-x)) - f(x)}{\lambda}. \end{aligned}$$

Taking the limit as $\lambda \longrightarrow 0$ in the inequality above, we obtain

$$f(y) - f(x) \geq \langle f'(x), y - x \rangle, \ \forall y \in D_e(f),$$

where $f'(x)$ is the Gâteaux derivative of f at x, and hence $f'(x) \in \partial f(x)$.

We now prove uniqueness. Let $x' \in \partial f(x)$. Then

$$f(y) - f(x) \geq \langle x', y - x \rangle, \ \forall y \in D_e(f).$$

In fact,

$$f(y) - f(x) \geq \langle x', y - x \rangle, \ \forall y \in X.$$

Thus

$$f(x + \lambda y) - f(x) \geq \langle x', \lambda y \rangle, \ \forall y \in X, \ \forall \lambda > 0,$$

which implies

$$\frac{f(x + \lambda y) - f(x)}{\lambda} \geq \langle x', y \rangle, \ \forall y \in X, \ \forall \lambda > 0.$$

Passing to the limit as $\lambda \to 0$, we obtain

$$\langle f'(x), y \rangle \geq \langle x', y \rangle, \ \forall y \in X.$$

Replacing y by $-y$ in the inequality above, we obtain the reverse inequality, and consequently

$$\langle f'(x), y \rangle = \langle x', y \rangle, \ \forall y \in X,$$

which implies $f'(x) = x'$, and the proof is complete. □

Example 4.17 Let X be a normed vector space. We compute the subdifferential of the norm at the point $0 \in X$, that is, we determine the elements of the set

$$\partial \| \cdot \| (0) = \Big\{ x' \in X'; \ \|y\| \geq \langle x', y \rangle, \ \forall y \in X \Big\}.$$

Note that if $x' \in \partial \| \cdot \|(0)$, then

$$\|y\| \geq \langle x', y \rangle, \ \forall y \in X.$$

If $y \in X$, then $-y \in X$ and, in particular,

$$\langle x', -y \rangle \leq \|y\| \quad \text{which implies} \quad \langle x', y \rangle \geq -\|y\|.$$

Hence $|\langle x', y \rangle| \leq \|y\|$ for all $y \in X$. Consequently $\|x'\| \leq 1$, and therefore

$$\partial \| \cdot \|(0) \subset \Big\{ x' \in X'; \ \|x'\| \leq 1 \Big\}.$$

Conversely, if $x' \in \{x' \in X'; \ \|x'\| \leq 1\}$, then $|\langle x', y \rangle| \leq \|y\|$ for all $y \in X$, and thus

$$\{x' \in X'; \ \|x'\| \leq 1\} \subset \partial \| \cdot \|(0).$$

Therefore

$$\partial \| \cdot \|(0) = \{x' \in X'; \ \|x'\| \leq 1\}.$$

Example 4.18 If $f(x) = \dfrac{1}{2}\|x\|^2$, then $\partial f(x) = F(x)$.

For $x' \in F(x)$, we have

$$\begin{aligned} f(y) - f(x) = \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 &= \frac{1}{2}(\|x\|^2 + \|y\|^2) - \|x\|^2 \\ &\geq \|x\|\|y\| - \|x\|^2 \\ &\geq \langle x', y\rangle - \langle x', x\rangle \\ &= \langle x', y - x\rangle. \end{aligned}$$

Thus $F(x) \subseteq \partial f(x)$.

Conversely, if $x' \in \partial f(x)$, then

$$\frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \geq \langle x', y - x\rangle, \quad \forall y \in D_e(f).$$

For y of the form $y = x + tx$, with $t \in \mathbb{R}$, we obtain

$$\|x + tx\|^2 - \|x\|^2 \geq 2t\langle x', x\rangle.$$

Thus, for $t = \dfrac{1}{n}$ we have

$$\left(1 + \frac{1}{2n}\right)\|x\|^2 \geq \langle x', x\rangle,$$

and for $t = -\dfrac{1}{n}$ we have

$$\left(1 - \frac{1}{2n}\right)\|x\|^2 \leq \langle x', x\rangle.$$

By the arbitrariness of n, it follows that $\langle x', x\rangle = \|x\|^2$, with $\|x\| \leq \|x'\|$. It remains to prove that $\|x'\| \leq \|x\|$. By definition, taking y of the form $y = x + tz$, with $t > 0$ and $z \in X$, we have

$$\frac{1}{2}\|x + tz\|^2 - \frac{1}{2}\|x\|^2 \geq t\langle x', z\rangle.$$

Hence

$$\begin{aligned} t\langle x', z\rangle &\leq \frac{1}{2}\left(\|x\| + t\|z\|\right)^2 - \frac{1}{2}\|x\|^2 \\ &= t\|x\|\|z\| + \frac{t^2}{2}\|z\|^2. \end{aligned}$$

Dividing both sides by t and letting $t \to 0^+$, we obtain

$$\langle x', z\rangle \leq \|x\|\|z\|, \quad \forall z \in X.$$

In particular, $\|x'\| \leq \|x\|$, and thus $\partial f(x) \subset F(x)$, so equality of the two sets holds.

4.2 Exercises

1 – Let A be a closed subset of a t.v.s. and consider the indicator function I_A defined by

$$I_A(x) = \begin{cases} 0, & \text{if } x \in A; \\ +\infty, & \text{if } x \notin A. \end{cases}$$

Prove that I_A is l.s.c.

Solution: Let E be a t.v.s. and $A \subset E$ closed. We shall show that $N(\lambda, I_A)$ is closed for every $\lambda \in \mathbb{R}$.

Indeed,

$$\text{if } \lambda < 0, \quad N(\lambda, I_A) = \{x \in E;\ I_A(x) \le \lambda\} = \emptyset,$$

$$\text{if } \lambda = 0, \quad N(\lambda, I_A) = \{x \in E;\ I_A(x) \le \lambda\} = A,$$

$$\text{if } \lambda > 0, \quad N(\lambda, I_A) = \{x \in E;\ I_A(x) \le \lambda\} = A.$$

Since both $\emptyset$ and A are closed sets, it follows that I_A is l.s.c.

2 – Let $f : X \to [-\infty, +\infty]$ be a convex function defined on a vector space X.

(a) If f takes the value $-\infty$ at some point $x_0 \in X$, prove that on any half-line Γ with origin at x_0 either $f(x) = -\infty$ for all $x \in \Gamma$, or there exists a point $x_1 \in \Gamma$ such that $f(x_1) < +\infty$, $f(x) = -\infty$ at all points $x \in \Gamma$ lying between x_0 and x_1, and $f(x) = +\infty$ at all other points of Γ.

(b) If, in addition to being convex, f is l.s.c., prove that either f never takes the value $-\infty$ or else $f(x) = -\infty$ for every $x \in X$. (Use the following result: Let f be convex, l.s.c. and proper. Then there exists a continuous affine function which minorises f (see Proposition 1.44 in the Functional Analysis book by Cavalcanti, Cavalcanti and Komornik).)

To avoid the particular cases described in (a) and (b) above, we shall consider convex functions which are not defined at $-\infty$.

Solution:

(a) Suppose that f is not identically $-\infty$. Thus there exists $x_1 \in X$ such that $f(x_1) \neq -\infty$, that is, $f(x_1) < \infty$ or $f(x_1) = +\infty$.

If $f(x_1) < \infty$, then for every $x \in \Gamma$ such that x lies between x_0 and x_1 we have $x = (1-t)x_0 + tx_1$ for some $t \in (0,1)$, and hence

$$f(x) = f((1-t)x_0 + tx_1) \le (1-t)f(x_0) + tf(x_1) = -\infty, \tag{4.2.58}$$

so f takes the value $-\infty$ between x_0 and x_1.

Now, if $x \in \Gamma$ but x lies beyond x_1, we cannot have $f(x) = -\infty$ or $f(x) < \infty$, because taking the half-line joining x to x_0 and using the same steps as in (4.2.58) we would obtain $f(y) = -\infty$ for every y between x and x_0, including x_1, which contradicts the fact that $f(x_1) < \infty$. Hence

$$f \text{ takes the value } -\infty \text{ between } x_0 \text{ and } x_1, \text{ and } f(x) = +\infty \text{ at all other points of } \Gamma.$$

Observe that, by the argument above, Γ can have at most one point $\widetilde{x}$ such that $f(\widetilde{x}) < \infty$.

Now, if $f(x_1) = +\infty$, consider $K_1 = \{x \in \Gamma;\ f(x) = +\infty\}$, $K_2 = \{x \in \Gamma;\ f(x) = -\infty\}$ and define

$$\begin{array}{rccl} \psi: & (0,1) & \longrightarrow & \Gamma_{(x_0,x_1)} \\ & t & \longmapsto & \psi(t) = (1-t)x_0 + tx_1, \end{array}$$

where $\Gamma_{(x_0,x_1)}$ is the segment joining x_0 to x_1 (excluding x_0 and x_1).

Then it is clear that ψ is a homeomorphism and moreover K_1 and K_2 are open.

Thus, since $K_1 \cap K_2 = \emptyset$, K_1 and K_2 are open in $\Gamma_{(x_0,x_1)}$, and this segment is connected (because it is homeomorphic to $(0,1)$, which is connected), it follows that $K_1 \cup K_2 \neq \Gamma_{(x_0,x_1)}$, i.e., there exists $x \in \Gamma_{(x_0,x_1)}$ such that $f(x) < \infty$. This concludes the proof of (a).

(b) Suppose that f is not identically $-\infty$ but takes the value $-\infty$ at some point, i.e., there exist $x_1, x_0 \in X$ such that $f(x_1) \neq -\infty$ and $f(x_0) = -\infty$. Consider the half-line $\Gamma_{[x_0,x_1]}$ joining x_0 to x_1. By item (a) there exists $x_2 \in \Gamma_{[x_0,x_1]}$ such that $f(x_2) < \infty$, hence f is a proper function. Therefore there exist $\varphi \in X'$ and $\beta \in \mathbb{R}$ such that

$$\varphi(x) + \beta < f(x), \quad \forall x \in X,$$

in particular

$$\varphi(x_0) + \beta < f(x_0) = -\infty,$$

which is absurd, since $\varphi \in X'$. Hence f cannot take the value $-\infty$.

3 – Let X be a uniformly convex Banach space and $x, x_n \in X$, $\quad n = 1, \cdots$. Prove that X has the following property:

$$x_n \rightharpoonup x \text{ and } \lim_{n\to\infty} \sup \|x_n\| \leq \|x\| \Rightarrow x_n \to x.$$

<u>Solution</u>: First assume that $x = 0$. Then

$$\lim_{n\to\infty} \inf \|x_n\| \leq 0. \tag{4.2.59}$$

On the other hand,

$$\lim_{n\to\infty} \sup \|x_n\| = \inf_{n\in\mathbb{N}} \sup_{k\geq n} \|x_k\| \leq 0. \tag{4.2.60}$$

Note that

$$0 \leq \sup_{k\geq n} \|x_k\|, \quad \forall n \in \mathbb{N}, \tag{4.2.61}$$

whence

$$0 \leq \inf_{n\in\mathbb{N}} \sup_{k\geq n} \|x_k\| = \lim_{n\to\infty} \sup \|x_k\|. \tag{4.2.62}$$

Comparing (4.2.59) and (4.2.62) we conclude that there exists n_0 such that

$$0 \leq \sup_{k\geq n_0} \|x_k\| < \varepsilon, \tag{4.2.63}$$

that is,

$$0 \leq \|x_k\| < \varepsilon, \quad \forall k \geq n_0. \tag{4.2.64}$$

Hence $x_n \longrightarrow 0$. Now suppose $x \neq 0$. Then

$$\|x\| \leq \lim_{n\to\infty} \inf \|x_n\|. \tag{4.2.65}$$

Indeed, from the weak convergence $x_n \rightharpoonup x$ we have

$$\begin{aligned} \forall x^* \in X', \quad |\langle x^*, x\rangle| &= \lim_{n\to\infty} |\langle x^*, x_n\rangle| \\ &= \lim_{n\to\infty} \inf |\langle x^*, x_n\rangle| \\ &\leq \lim_{n\to\infty} \inf \|x^*\|_{X'} \|x_n\|_X \\ &= \|x^*\|_{X'} \lim_{n\to\infty} \inf \|x_n\|_X. \end{aligned} \tag{4.2.66}$$

In particular,

$$\begin{aligned} &\forall x^* \in X' \text{ such that } \quad \|x^*\|_{X'} = 1, \text{ we have} \\ &|\langle x^*, x\rangle| \leq \lim_{n\to\infty} \inf \|x_n\|_X, \end{aligned} \tag{4.2.67}$$

and consequently

$$\|x\|_X = \sup_{\|x^*\|_{X'}=1} |\langle x^*, x\rangle| \leq \lim_{n\to\infty} \inf \|x_n\|_X, \tag{4.2.68}$$

which proves (4.2.65). From (4.2.65) and the hypothesis we conclude

$$\|x\|_X \leq \lim_{n\to\infty} \inf \|x_n\|_X \leq \lim_{n\to\infty} \sup \|x_n\|_X \leq \|x\|_X, \tag{4.2.69}$$

that is, $\lim_{n\to\infty} \|x_n\|$ exists and moreover

$$\lim_{n\to\infty} \|x_n\|_X = \|x\|_X > 0. \tag{4.2.70}$$

Assume, without loss of generality, that

$$\|x_n\|_X > 0, \quad \forall n \in \mathbb{N}. \tag{4.2.71}$$

Set $y_n = \dfrac{x_n}{\|x_n\|}$, $y = \dfrac{x}{\|x\|}$ and $y^* = \dfrac{x^*}{\|x^*\|}$, where $x^* \in F(x)$. Then $y_n, y \in U_X$ (the unit ball) and $y^* \in U_{X'}$. We have

$$\langle y^*, \tfrac{y_n+y}{2}\rangle \leq |\langle y^*, \tfrac{y_n+y}{2}\rangle| \leq \left\|\tfrac{y_n+y}{2}\right\|_X \leq \frac{1}{2}\Big\{\|y_n\| + \|y\|\Big\} = 1.$$

On the other hand,

$$\begin{aligned}
\lim_{n\to\infty} \langle y^*, \tfrac{y_n+y}{2}\rangle &= \lim_{n\to\infty} \left\langle \frac{x^*}{\|x^*\|_{X'}}, \frac{\frac{x_n}{\|x_n\|_X} + \frac{x}{\|x\|_X}}{2} \right\rangle \\
&= \frac{1}{2\|x^*\|_{X'}} \lim_{n\to\infty} \left\{ \langle x^*, x_n\rangle \cdot \frac{1}{\|x_n\|_X} + \langle x^*, x\rangle \frac{1}{\|x\|_X} \right\} \\
&= \frac{1}{2\|x^*\|_{X'}} \cdot \left\{ \langle x^*, x\rangle \frac{1}{\|x\|_X} + \langle x^*, x\rangle \frac{1}{\|x\|_X} \right\} \\
&= \frac{1}{2\|x^*\|_{X'}} \cdot \frac{2}{\|x\|_X} \langle x^*, x\rangle = \frac{\langle x^*, x\rangle}{\|x^*\|_{X'} \cdot \|x\|_X} = \frac{\|x^*\|\|x\|}{\|x^*\|\|x\|} = 1.
\end{aligned} \tag{4.2.72}$$

Now, since

$$\begin{aligned}
\langle y^*, \tfrac{y_n+y}{2}\rangle &\leq |\langle y^*, \tfrac{y_n+y}{2}\rangle| \\
&\leq \|y^*\|_{X'} \cdot \left\|\tfrac{y_n+y}{2}\right\|_X \\
&= \left\|\tfrac{y_n+y}{2}\right\|_X,
\end{aligned} \tag{4.2.73}$$

it follows that

$$1 = \lim_{n\to\infty} \langle y^*, \tfrac{y_n+y}{2}\rangle \leq \lim_{n\to\infty} \left\|\tfrac{y_n+y}{2}\right\|_X. \tag{4.2.74}$$

On the other hand

$$\left\|\tfrac{y_n+y}{2}\right\|_X \leq \frac{1}{2}\{\|y_n\|_X + \|y\|_X\} = 1, \tag{4.2.75}$$

and hence

$$\lim_{n\to\infty} \left\|\tfrac{y_n+y}{2}\right\|_X \leq 1. \tag{4.2.76}$$

From (4.2.74) and (4.2.76) we conclude that

$$\lim_{n\to\infty} \left\|\tfrac{y_n+y}{2}\right\|_X = 1. \tag{4.2.77}$$

By uniform convexity it follows that

$$\|y_n - y\|_X \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.2.78}$$

Indeed, we argue by contradiction. Suppose that $\|y_n - y\|_X \not\to 0$ as $n \longrightarrow \infty$. Then, without loss of generality, we may assume that there exists $\varepsilon_0 > 0$ such that

$$\|y_n - y\|_X > \varepsilon_0, \quad \forall n \in \mathbb{N}. \tag{4.2.79}$$

Hence, by the uniform convexity of X there exists $\delta_0 > 0$ such that

$$\left\|\tfrac{y_n+y}{2}\right\|_X \le 1 - \delta_0 < 1, \quad \forall n \in \mathbb{N}, \tag{4.2.80}$$

that is, $\left\|\frac{y_n+y}{2}\right\|_X \not\to 1$ as $n \to \infty$, which contradicts (4.2.77). Substituting $y_n = \dfrac{x_n}{\|x_n\|}$ and $y = \frac{x}{\|x\|_X}$ in (4.2.78) we conclude that

$$\left\|\frac{x_n}{\|x_n\|} - \frac{x}{\|x\|}\right\|_X \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{4.2.81}$$

that is,

$$\frac{x_n}{\|x_n\|} \longrightarrow \frac{x}{\|x\|} \quad \text{as } n \longrightarrow \infty. \tag{4.2.82}$$

Moreover, since $\|x_n\| \longrightarrow \|x\|$ (see (4.2.70)), the sequence $\{\|x_n\|\}_{n\in\mathbb{N}}$ is bounded, and from (4.2.82) we conclude that

$$\begin{aligned} \|x_n - x\|_X &= \left\|\frac{\|x_n\|x_n}{\|x_n\|} - \frac{\|x_n\|x}{\|x\|} + \frac{\|x_n\|x}{\|x\|} - \frac{\|x\|x}{\|x\|}\right\|_X \\ &\le \|x_n\| \left\|\frac{x_n}{\|x_n\|} - \frac{x}{\|x\|}\right\|_X + \big|\|x_n\| - \|x\|\big| \cdot 1, \quad \forall n \in \mathbb{N}, \end{aligned}$$

so $x_n \longrightarrow x$, as we wanted to prove.

Chapter 5

Monotone and Accretive Operators

5.1 Monotone Operators

In this section we study monotone operators, which generalise the notion of monotone functions.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a monotone nondecreasing function. This means that if $x, y \in D(f)$ and $x \leq y$ then $f(x) \leq f(y)$. Equivalently,

$$(x - y)(f(x) - f(y)) \geq 0, \quad \forall x, y \in D(f).$$

Our aim is to extend this concept. To that end, let us consider the following example in $\mathbb{R}^2$. For each $x = (a, b) \in \mathbb{R}^2$, consider the rotation of x by an angle θ, where $0 \leq \theta \leq \frac{\pi}{2}$.

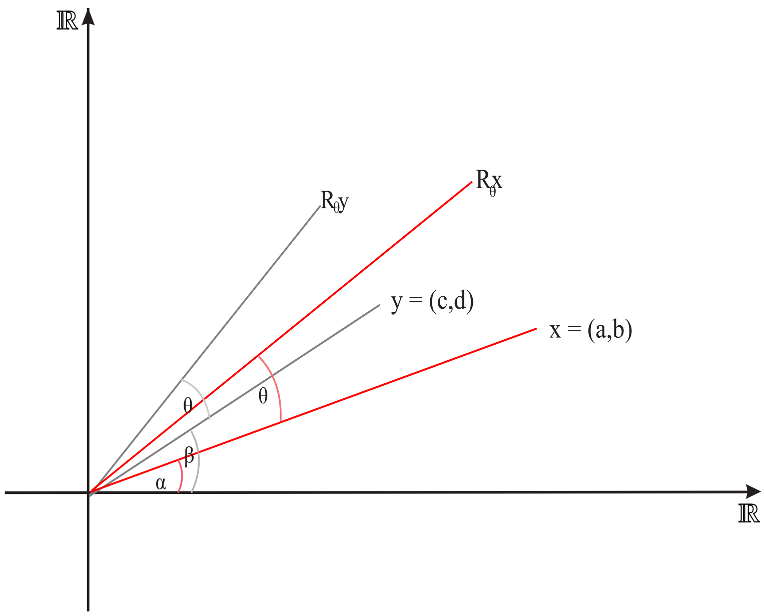

Figure 5.1: Rotation mapping.

Note that

$$\begin{aligned}
R_\theta(x) &= \big(\|x\| \cos(\alpha + \theta), \|x\| \sin(\alpha + \theta)\big) \\
&= \big(\underbrace{\|x\| \cos\alpha}_{=a} \cos\theta - \underbrace{\|x\| \sin\alpha}_{=b} \sin\theta, \underbrace{\|x\| \sin\alpha}_{=b} \cos\theta + \underbrace{\|x\| \cos\alpha}_{=a} \sin\theta\big) \\
&= \big(a\cos\theta - b\sin\theta,\ b\cos\theta + a\sin\theta\big).
\end{aligned}$$

Thus

$$\begin{array}{rccl} R_\theta: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (a,b) & \longmapsto & R_\theta(a,b) = \big(a\cos\theta - b\sin\theta,\ b\cos\theta + a\sin\theta\big). \end{array}$$

We claim that

$$\big(x-y, R_\theta(x) - R_\theta(y)\big)_{\mathbb{R}^2} \geq 0, \quad \forall x, y \in \mathbb{R}^2.$$

Indeed, let $x = (a,b)$ and $y = (c,d)$. Then

$$\begin{aligned} \big(x-y, R_\theta(x) - R_\theta(y)\big)_{\mathbb{R}^2} &= \Big((a-c, b-d), \\ &\quad \big((a-c)\cos\theta - (b-d)\sin\theta, (b-d)\cos\theta + (a-c)\sin\theta\big)\Big)_{\mathbb{R}^2} \\ &= (a-c)^2\cos\theta - (a-c)(b-d)\sin\theta \\ &\quad +(b-d)^2\cos\theta + (b-d)(a-c)\sin\theta \\ &= (a-c)^2\cos\theta + (b-d)^2\cos\theta \geq 0, \quad \text{since } 0 \leq \theta \leq \frac{\pi}{2}. \end{aligned}$$

We also note that R_θ is linear.

This example motivates the following definitions:

Definition 5.1 *Let H be a Hilbert space. A single-valued operator A in H is said to be* <u>*positive*</u> *if*

$$\big(Ax, x\big)_H \geq 0, \ \forall x \in H.$$

Definition 5.2 *Let H be a Hilbert space. A single-valued operator A in H is said to be* <u>*monotone*</u> *if*

$$\big(Ax - Ay, x-y\big)_H \geq 0, \ \forall x, y \in H.$$

We observe that if A is a single-valued linear operator on a Hilbert space, then A is monotone if and only if A is positive. The mapping R_θ considered above is an example of a single-valued, linear and monotone operator and therefore positive. However, the nondecreasing function f mentioned at the beginning of this paragraph represents a monotone operator which is not necessarily positive (unless $0 \in D(f)$ and $f(0) = 0$).

Definition 5.3 *Let H be a Hilbert space. An operator A in H is said to be* <u>*monotone*</u> *if*

$$(x_1 - x_2, y_1 - y_2)_H \geq 0, \quad \forall (x_1, y_1), (x_2, y_2) \in A.$$

We see, therefore, that the definition of a monotone operator in a Hilbert space is a natural generalisation of the concept of a monotone nondecreasing function.

Let us look at an example:

Example 5.4 Let β be a monotone operator in $\mathbb{R}$ and let Ω be a bounded open subset of $\mathbb{R}^n$. We may define an operator $\tilde{\beta}$ in the space $L^2(\Omega)$ by setting

$$\tilde{\beta} = \big\{(u,v) \in L^2\Omega \times L^2\Omega;\ v(x) \in \beta(u(x)) \text{ almost everywhere in } \Omega\big\}.$$

We shall prove that $\tilde{\beta} \neq \emptyset$. First, we claim that for each $\xi \in \mathbb{R}$ the set $\beta(\xi)$ is bounded in $\mathbb{R}$.

Indeed, suppose the contrary, i.e., that given $M > 0$ there exists $x_M \in \beta(\xi)$ such that $|x_M| > M$. Let $x_1 > \xi$. Since β is monotone we have

$$(x_1 - \xi)(y_1 - y_2) \geq 0, \ \forall y_1 \in \beta(x_1) \text{ and } \forall y_2 \in \beta(\xi).$$

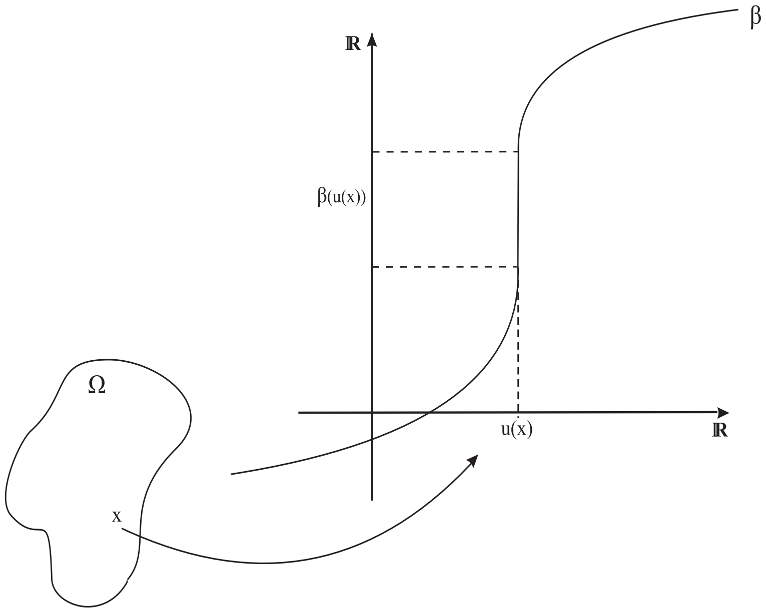

Figure 5.2: Operator $\tilde{\beta}$.

In particular, $(x_1 - \xi)(y_1 - x_M) \geq 0$. Since $(x_1 - \xi) > 0$, it follows that $(y_1 - x_M) \geq 0$ and consequently $x_M \leq y_1,\ \forall M > 0$ and $y_1 \in \beta(x_1)$. Therefore, there exists $y_1^* \in \mathbb{R}$ such that $x_M \leq y_1^*,\ \forall M > 0$.

Similarly, if $x_1 < \xi$ we have

$$(x_1 - \xi)(y_1 - y_2) \geq 0,\ \forall y_1 \in \beta(x_1),\ \forall y_2 \in \beta(\xi).$$

In particular, $(x_1 - \xi)(y_1 - x_M) \geq 0$. Since $(x_1 - \xi) < 0$, it follows that $(y_1 - x_M) \leq 0$ and consequently $x_M > y_1$, for every $M > 0$ and $y_1 \in \beta(x_1)$. Thus, there exists $y_1^0 \in \mathbb{R}$ such that $x_M \geq y_1^0,\ \forall M > 0$. Hence the sequence $(x_M)_{M>0}$ is bounded. By hypothesis, however, we have $|x_M| > M$ for all $M > 0$, i.e., $|x_M| \longrightarrow +\infty$ as $M \longrightarrow +\infty$, which is a contradiction. This shows that the set $\beta(\xi)$ is bounded.

Now take $u \in C_0^\infty(\Omega)$ and define the mapping

$$\begin{array}{rccl} v : & \Omega & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & v(x) \in \beta(u(x)). \end{array}$$

Denoting $K = \operatorname{supp}(u)$, we have

$$\int_\Omega |v(x)|^2 dx = \int_K |v(x)|^2 dx + \int_{\Omega \setminus K} |v(x)|^2 dx.$$

Note that, since u is continuous and K is compact, there exists a constant $k > 0$ such that $-k < u(x) < k$ for all $x \in K$. By the monotonicity of β and for every $x \in K$ we have

$$\big(u(x) + k\big)(y_1 - y_2) \geq 0,\ \forall y_1 \in \beta\big(u(x)\big) \text{ and } \forall y_2 \in \beta(-k),$$

$$\big(k - u(x)\big)(z_1 - z_2) \geq 0,\ \forall z_1 \in \beta(k) \text{ and } \forall z_2 \in \beta\big(u(x)\big).$$

Since $\big(u(x)+k\big)>0$ and $\big(k-u(x)\big)>0$, it follows that

$$(y_1-y_2)\geq 0,\ \forall y_1\in\beta\big(u(x)\big) \text{ and } y_2\in\beta(-k),$$

$$(z_1-z_2)\geq 0,\ \forall z_1\in\beta(k) \text{ and } \forall z_2\in\beta\big(u(x)\big).$$

In particular,

$$v(x)\geq y_2\geq c_1,$$

where c_1 is a lower bound of the set $\beta(-k)$. Moreover,

$$v(x)\leq z_1\leq c_2,$$

where c_2 is an upper bound of the set $\beta(k)$.

Hence there exists $c>0$ such that $|v(x)|\leq c$ for all $x\in K$, and therefore

$$\int_K |v(x)|^2dx\leq c^2\text{meas}(K)<\infty.$$

On the other hand, if $x\in\Omega\setminus K$, then $u(x)=0$ and consequently $v(x)\in\beta(0)$, which is a bounded subset as shown above. Hence

$$\int_{\Omega\setminus K}|v(x)|^2dx\leq k^2\text{meas}(\Omega\setminus K)\leq k^2\text{meas}(\Omega)<\infty,$$

since Ω is bounded. Thus, if $u\in C_0^\infty(\Omega)$, we have $v\in L^2(\Omega)$, and so $(u,v)\in\tilde{\beta}$. Therefore $\tilde{\beta}\neq\emptyset$ and

$$\big(u_1-u_2,v_1-v_2\big)_{L^2\Omega}=\int_\Omega \big(u_1(x)-u_2(x)\big)\big(v_1(x)-v_2(x)\big)dx.$$

Since $(u_1,v_1),(u_2,v_2)\in\tilde{\beta}$, the integral is well-defined and, moreover,

$$v_1(x)\in\beta\big(u_1(x)\big) \text{ almost everywhere in } \Omega,$$

$$v_2(x)\in\beta\big(u_2(x)\big) \text{ almost everywhere in } \Omega.$$

By the monotonicity of β it follows that

$$\big(u_1(x)-u_2(x)\big)\big(v_1(x)-v_2(x)\big)\geq 0,$$

which proves the claim.

To further generalise the notion of a monotone nondecreasing function, note that if X is a Hilbert space, then its dual X' may be identified with X and, in this way, the monotone operators on X may be regarded as operators $A:X\longrightarrow X'$. Thus the inner product can be viewed as the duality $\langle\cdot,\cdot\rangle_{X',X}$. These considerations lead us to the following definition.

Definition 5.5 *Let X be a real t.v.s., X' its dual and $A:X\longrightarrow X'$ an operator. We say that A is monotone if*

$$\big\langle x'-y',x-y\big\rangle\geq 0,\quad \textit{for all } (x',x),(y',y)\in A.$$

Example 5.6 The subdifferential operator $\partial f:X\longrightarrow X'$ is monotone. Indeed, let $(x,x'),(y,y')\in\partial f$. Then

$$x'\in\partial f(x)\quad\text{and}\quad y'\in\partial f(y).$$

Hence $x', y' \in X'$, $x, y \in D_e(f)$ and, in addition,

$$f(z) - f(x) \geq \langle x', z - x \rangle, \ \forall z \in D_e(f),$$

$$f(w) - f(y) \geq \langle y', w - y \rangle, \ \forall w \in D_e(f).$$

In particular,

$$f(y) - f(x) \geq \langle x', y - x \rangle,$$
$$f(x) - f(y) \geq \langle y', x - y \rangle,$$

and, adding these two inequalities, we obtain

$$0 \geq \langle y' - x', x - y \rangle \Rightarrow \langle x' - y', x - y \rangle \geq 0,$$

which proves the monotonicity of the operator ∂f.

□

Example 5.7 The duality map $F : X \longrightarrow X'$ is monotone. In fact, let $(x, x'), (y, y') \in F$. Then $x', y' \in X'$ and

$$\langle x', x \rangle = \|x\|^2 = \|x'\|^2,$$
$$\langle y', y \rangle = \|y\|^2 = \|y'\|^2.$$

Consequently,

$$\begin{aligned} \langle x' - y', x - y \rangle &= \|x\|^2 - \langle x', y \rangle - \langle y', x \rangle + \|y\|^2 \\ &\geq \|x\|^2 - \|x'\|\|y\| - \|y'\|\|x\| + \|y\|^2 \\ &= \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \\ &= \big(\|x\| - \|y\|\big)^2 \geq 0, \end{aligned}$$

which proves the claim.

□

In the case of Hilbert spaces, the monotonicity of an operator can be expressed by the following condition, which involves only the norm.

Proposition 5.8 *Let X be a Hilbert space. Then A is a monotone operator if and only if*

$$\big\|x_1 - x_2 + \lambda(y_1 - y_2)\big\| \geq \|x_1 - x_2\|,$$

for every $(x_1, y_1), (x_2, y_2) \in A$ and every $\lambda > 0$.

Proof: Let $(x_1, y_1), (x_2, y_2) \in A$ and $\lambda > 0$. Then

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\|^2 = \|x_1 - x_2\|^2 + \lambda^2\|y_1 - y_2\|^2 + 2\lambda(x_1 - x_2, y_1 - y_2). \tag{5.1.1}$$

If A is monotone, then

$$\big(x_1 - x_2, y_1 - y_2\big) \geq 0,$$

and therefore, from (5.1.1), it follows that

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\|^2 \geq \|x_1 - x_2\|^2, \tag{5.1.2}$$

for all $(x_1, y_1), (x_2, y_2) \in A$ and every $\lambda > 0$.

Conversely, if (5.1.2) holds, then from (5.1.1) we obtain

$$\lambda\|y_1 - y_2\|^2 + 2\big(x_1 - x_2, y_1 - y_2\big) \geq 0.$$

Letting $\lambda \longrightarrow 0$ in the inequality above, we get

$$\big(x_1 - x_2, y_1 - y_2\big) \geq 0,$$

for every $(x_1, y_1), (x_2, y_2) \in A$, which shows that A is monotone and completes the proof. □

Proposition 5.9 *Let $\mathcal{M}$ be the family of monotone operators on X. The following properties hold:*

(i) If $A, B \in \mathcal{M}$, then $A + B \in \mathcal{M}$;

(ii) If $A \in \mathcal{M}$ and $\lambda > 0$, then $\lambda A \in \mathcal{M}$;

(iii) If $A \in \mathcal{M}$, then $A^{-1} : X' \longrightarrow X''$ is monotone;

(iv) If $A \in \mathcal{M}$, then $\overline{A} \in \mathcal{M}$, where $\overline{A}$ is the closure of A in $X \times X'$, with X endowed with the strong topology and X' with the weak-$*$ topology;

(v) If $A \in \mathcal{M}$, then $\widehat{A} \in \mathcal{M}$, where $\widehat{A} = \overline{conv\, Ax}$ (the closure in X' of the convex hull of the set Ax).

Proof:

(i) If $A, B \in \mathcal{M}$, then

$$\begin{cases} \big\langle x_1' - y_1', x_1 - y_1\big\rangle \geq 0, \quad \forall (x_1, x_1'), (y_1, y_1') \in A, \\[2ex] \big\langle x_2' - y_2', x_2 - y_2\big\rangle \geq 0, \quad \forall (x_2, x_2'), (y_2, y_2') \in B. \end{cases} \tag{5.1.3}$$

Let $(z_1, z_1'), (w_1, w_1') \in A + B$. Then $z_1, w_1 \in D(A) \cap D(B)$ and

$$\begin{cases} z_1' = x_1' + x_2' \text{ where } x_1' \in Az_1 \text{ and } x_2' \in Bz_1, \\[2ex] w_1' = y_1' + y_2' \text{ where } y_1' \in Aw_1 \text{ and } y_2' \in Bw_1. \end{cases} \tag{5.1.4}$$

Thus, from (5.1.3) and (5.1.4), we obtain

$$\begin{aligned} \big\langle z_1' - w_1', z_1 - w_1\big\rangle &= \big\langle (x_1' + x_2') - (y_1' + y_2'), z_1 - w_1\big\rangle \\ &= \big\langle (x_1' - y_1') + (x_2' - y_2'), z_1 - w_1\big\rangle \\ &= \big\langle x_1' - y_1', z_1 - w_1\big\rangle + \big\langle x_2' - y_2', z_1 - w_1\big\rangle \\ &\geq 0. \end{aligned}$$

(ii) If $A \in \mathcal{M}$ and $\lambda > 0$, then

$$\big\langle x' - y', x - y\big\rangle \geq 0, \quad \forall (x, x'), (y, y') \in A. \tag{5.1.5}$$

Let $(x_1, x_1'), (y_1, y_1') \in \lambda A$. Then $x_1, y_1 \in D(A)$ and

$$x_1' = \lambda x' \text{ with } x' \in Ax_1, \qquad y_1' = \lambda y' \text{ with } y' \in Ay_1.$$

From this identity and (5.1.5), we get

$$\big\langle x_1' - y_1', x_1 - y_1\big\rangle = \lambda\big\langle x' - y', x_1 - y_1\big\rangle \geq 0.$$

(iii) Let $A \in \mathcal{M}$ and consider $A^{-1} : D(A^{-1}) \subset X' \longrightarrow X''$. Thus, for $(x, x'), (y, y') \in A$, by the monotonicity of A,

$$\langle x' - y', x - y \rangle \geq 0. \tag{5.1.6}$$

Note that $x', y' \in D(A^{-1})$. Hence, from (5.1.6),

$$\langle x - y, x' - y' \rangle_{X'' \times X} = \langle x' - y', x - y \rangle_{X' \times X} \geq 0.$$

(iv) Let (x, x') and $(y, y') \in \overline{A}$ and $\varepsilon > 0$. Consider the following neighbourhoods of (x, x') and (y, y'), respectively:

$$B_\varepsilon(x) \times V_\varepsilon(x') = \{z \in X;\ \|z - x\| < \varepsilon\} \times \{f \in X';\ |\langle f - x', x - y \rangle| < \varepsilon\},$$
$$B_\varepsilon(y) \times V_\varepsilon(y') = \{z \in X;\ \|z - y\| < \varepsilon\} \times \{f \in X';\ |\langle f - y', x - y \rangle| < \varepsilon\}.$$

Since (x, x') and $(y, y') \in \overline{A}$, there exist (x_0, x'_0) and $(y_0, y'_0) \in A$ such that $(x_0, x'_0) \in B_\varepsilon(x) \times V_\varepsilon(x')$ and $(y_0, y'_0) \in B_\varepsilon(y) \times V_\varepsilon(y')$.

Thus,

$$\|x_0 - x\| < \varepsilon, \quad \|y_0 - y\| < \varepsilon,$$
$$-\varepsilon < \langle x'_0 - x', x - y \rangle < \varepsilon, \quad -\varepsilon < \langle y'_0 - y', x - y \rangle < \varepsilon$$

and

$$\langle x_0 - y_0, x'_0 - y'_0 \rangle \geq 0.$$

Hence

$$\begin{aligned}
\langle x - y, x' - y' \rangle &= \langle x - y, x' \rangle - \langle x - y, y' \rangle + \langle x - y, x'_0 \rangle - \langle x - y, x'_0 \rangle \\
&\quad + \langle x - y, y'_0 \rangle - \langle x - y, y'_0 \rangle + \langle x_0, x'_0 - y'_0 \rangle - \langle x_0, x'_0 - y'_0 \rangle \\
&\quad + \langle y_0, x'_0 - y'_0 \rangle - \langle y_0, x'_0 - y'_0 \rangle \\
&= \langle x - y, x' - x'_0 \rangle + \langle x - y, y'_0 - y' \rangle - \langle x_0 - x, x'_0 - y'_0 \rangle \\
&\quad - \langle y - y_0, x'_0 - y'_0 \rangle + \langle x_0 - y_0, x'_0 - y'_0 \rangle \\
&> -2\varepsilon - (\|x_0 - x\| + \|y_0 - y\|)\|x'_0 - y'_0\| \\
&> -\varepsilon(2 + 2\|x'_0 - y'_0\|),
\end{aligned}$$

and therefore, letting $\varepsilon \to 0$, we obtain

$$\langle x - y, x' - y' \rangle \geq 0.$$

(v) Let $A \in \mathcal{M}$ and consider $\widehat{A} = \overline{conv Ax}$ (closure in X' of the convex hull of Ax). We first show that the operator $\breve{A} : X \longrightarrow X'$ defined by

$$\breve{A}x = conv Ax$$

is monotone. Indeed, let $(x, x'), (y, y') \in \breve{A}$. Then $x' \in conv Ax$ and $y' \in conv Ay$, hence

$$\begin{cases} x' = \displaystyle\sum_{i=1}^{n} \lambda_i x'_i \text{ where } x'_i \in Ax,\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0, \\[2ex] y' = \displaystyle\sum_{j=1}^{m} \mu_j y'_j \text{ where } y'_j \in Ay,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \geq 0. \end{cases} \tag{5.1.7}$$

Since $(x,x_i'),(y,y_j')\in A$ and A is monotone, we have

$$\big\langle x_i'-y_j',x-y\big\rangle\geq 0,\quad \forall i=1,\dots,n,\ \forall j=1,\dots,m.$$

Thus

$$\big\langle \lambda_i x_i'-\lambda_i y_j',x-y\big\rangle\geq 0,\quad \forall i=1,\dots,n,\ \forall j=1,\dots,m,$$

which implies

$$\left\langle \sum_{i=1}^{n}\lambda_i x_i'-\sum_{i=1}^{n}\lambda_i y_j',x-y\right\rangle\geq 0,\quad \forall j=1,\dots,m,$$

i.e.

$$\big\langle x'-y_j',x-y\big\rangle\geq 0,\quad \forall j=1,\dots,m.$$

From the inequality above we obtain

$$\left\langle \sum_{j=1}^{m}\mu_j x'-\sum_{j=1}^{m}\mu_j y_j',x-y\right\rangle\geq 0,$$

that is,

$$\langle x'-y',x-y\rangle\geq 0,$$

which proves that the operator $\breve{A}$ is monotone.

On the other hand, recall that

$$\begin{aligned}\breve{A}&=\big\{(x,y);\ x\in D(A),\ y\in conv Ax\big\},\\ \widehat{A}&=\big\{(x,y);\ x\in D(A),\ y\in \overline{conv Ax}\big\},\\ \overline{\breve{A}}&=\big\{(x,y);\ x\in \overline{D(A)},\ y\in \overline{conv Ax}\big\}.\end{aligned}$$

Therefore $\breve{A}\subset\widehat{A}\subset\overline{\breve{A}}$. Since $\breve{A}$ is monotone, it follows from item (iv) that $\overline{\breve{A}}$ is monotone, and, because $\overline{\breve{A}}$ extends $\widehat{A}$, we conclude that $\widehat{A}$ is monotone. □

Proposition 5.10 *Let $A\in\mathcal{M}$. Then the operator $\tilde{A}$ defined by*

$$D(\tilde{A})=D(A)-\big\{a\big\},\ \textit{where } a\in X,\qquad \tilde{A}x=A(x+a)-\big\{a'\big\},\quad a'\in X',$$

is monotone.

Proof: Let $(\tilde{x},\tilde{x}'),(\tilde{y},\tilde{y}')\in\tilde{A}$. Then

$$\begin{aligned}\tilde{x}&=x-a,\ \text{for some } x\in D(A),\\ \tilde{y}&=y-a,\ \text{for some } y\in D(A),\\ \tilde{x}'&\in\tilde{A}\tilde{x}=A(\tilde{x}+a)-\big\{a'\big\}=Ax-a',\\ \tilde{y}'&\in\tilde{A}\tilde{y}=A(\tilde{y}+a)-\big\{a'\big\}=Ay-a',\end{aligned}$$

which implies

$$\tilde{x}'+a'\in Ax,\qquad \tilde{y}'+a'\in Ay.$$

It then follows, using the monotonicity of A, that

$$\begin{aligned}\big\langle\tilde{x}'-\tilde{y}',\tilde{x}-\tilde{y}\big\rangle&=\big\langle\tilde{x}'-\tilde{y}',\tilde{x}-\tilde{y}\big\rangle+\big\langle a'-a',\tilde{x}-\tilde{y}\big\rangle\\&=\big\langle(\tilde{x}'+a')-(\tilde{y}'+a'),(\tilde{x}+a)-(\tilde{y}+a)\big\rangle\\&=\big\langle(\tilde{x}'+a')-(\tilde{y}'+a'),x-y\big\rangle\geq 0.\end{aligned}$$

□

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Definition 5.11 *Let X be a normed space. An operator A on X is said to be* <u>*locally bounded*</u> *at the point $x_0 \in X$ if there exists $\rho > 0$ such that the set*

$$\left\{ \bigcup_{x \in D(A)} Ax; \ \|x - x_0\| < \rho \right\}$$

is bounded.

Lemma 5.12 *Let (x_n) and (x'_n) be sequences of elements of X and X', respectively, such that $\|x_n\| \longrightarrow 0$ and $\|x'_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Then, given $\rho > 0$, there exist an element $z(\rho) \in X$ and subsequences (x_{n_k}) and (x'_{n_k}) of (x_n) and (x'_n), respectively, such that:*

(i) $\|z(\rho)\| < \rho$;

(ii) $\lim\limits_{k \to \infty} \left\langle x'_{n_k}, z(\rho) - x_{n_k} \right\rangle = \infty$.

Proof: Set

$$w'_n = \frac{x'_n}{1 + \left|\left\langle x'_n, x_n \right\rangle\right|},$$

then

$$\|w'_n\| = \frac{\|x'_n\|}{1 + \left|\left\langle x'_n, x_n \right\rangle\right|} \geq \frac{\|x'_n\|}{1 + \|x_n\|\|x'_n\|} = \frac{1}{\frac{1}{\|x'_n\|} + \|x_n\|},$$

which implies that $\|w'_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$.

We shall show that there exist a subsequence (w'_{n_k}) of (w'_n) and a point $z \in X$ such that $\left\langle w'_{n_k}, z \right\rangle \longrightarrow \infty$ as $k \longrightarrow \infty$. Indeed, suppose this is not the case, that is,

$$\sup_n \left|\left\langle w'_n, z \right\rangle\right| < \infty, \quad \forall z \in X.$$

Then, by the Banach–Steinhaus theorem, $\sup_n \|w'_n\| < \infty$, which contradicts the fact that $\|w'_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Hence the claim holds.

Given $\rho > 0$, define

$$z(\rho) = \frac{\rho z}{2\|z\|}.$$

Note that $z \neq 0$, for otherwise $\left\langle w'_{n_k}, z \right\rangle = 0$.

Then

$$\|z(\rho)\| = \frac{\rho}{2} < \rho. \tag{5.1.8}$$

Observe that

$$\begin{aligned} \left\langle w'_n, x_n \right\rangle &= \langle 0 \rangle \Big[\frac{x'_n}{1 + \left|\left\langle x'_n, x_n \right\rangle\right|}, x_n \Big] \\ &= \frac{\left\langle x'_n, x_n \right\rangle}{1 + \left|\left\langle x'_n, x_n \right\rangle\right|} \leq 1, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{5.1.9}$$

From what we proved above, given $M > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\left\langle w'_{n_k}, z \right\rangle > M, \quad \forall n_k \geq n_{k_0}. \tag{5.1.10}$$

Thus, from (5.1.9) and (5.1.10), for $k \geq k_0$ sufficiently large, we obtain

$$\left\langle w'_{n_k}, z(\rho)\right\rangle > \left\langle w'_{n_k}, x_{n_k}\right\rangle, \ \forall n_k \geq n_{k_0}. \tag{5.1.11}$$

Hence, from (5.1.9) and (5.1.11), for all $n_k \geq n_{k_0}$,

$$\begin{aligned}\left\langle x'_{n_k}, z(\rho) - x_{n_k}\right\rangle &= \overbrace{\left(1 + \left|\left\langle x'_{n_k}, z(\rho) - x_{n_k}\right\rangle\right|\right)}^{\geq 1} \left\langle w'_{n_k}, z(\rho) - x_{n_k}\right\rangle \\ &\geq \left\langle w'_{n_k}, z(\rho) - x_{n_k}\right\rangle \\ &\geq \left\langle w'_{n_k}, z(\rho)\right\rangle - 1 \longrightarrow \infty \quad \text{as } k \longrightarrow \infty,\end{aligned}$$

as desired. □

Theorem 5.13 *Every monotone operator $A : X \longrightarrow X'$ is locally bounded at each point of the interior of its domain.*

Proof: Let A be a monotone operator on X, let $x \in \operatorname{int} D(A)$, and let $\rho > 0$ be such that $D(A)$ contains the ball centred at x with radius ρ. We argue by contradiction. If the theorem were false, according to Definition 5.11 there would exist sequences (x_n), (x'_n) in X and X', respectively, such that $(x_n, x'_n) \in A$, $x_n \longrightarrow x$ and $\|x'_n\| \longrightarrow +\infty$. By Lemma 5.12, there would then exist subsequences $(x_{n_i}), (x'_{n_i})$ of $(x_n), (x'_n)$, respectively, and a point $z(\rho) \in X$ such that $\|z(\rho)\| < \rho$ and

$$\left\langle x'_{n_i}, z(\rho) - (x_{n_i} - x)\right\rangle \longrightarrow \infty \quad \text{as } i \longrightarrow \infty. \tag{5.1.12}$$

From $\|z(\rho)\| < \rho$ it follows that $z(\rho) + x$ belongs to the ball centred at x with radius ρ, hence $z(\rho) + x \in D(A)$. By the monotonicity of A we have

$$\left\langle y' - x'_{n_i}, z(\rho) + x - x_{n_i}\right\rangle \geq 0, \quad \forall y' \in A(z(\rho) + x),$$

and consequently

$$\left\langle y', z(\rho) + x - x_{n_i}\right\rangle \geq \left\langle x'_{n_i}, z(\rho) + x - x_{n_i}\right\rangle. \tag{5.1.13}$$

From (5.1.12) and (5.1.13) we obtain

$$\left\langle y', z(\rho) + x - x_{n_i}\right\rangle \longrightarrow \infty \quad \text{as } i \longrightarrow \infty,$$

which is a contradiction, since $x_{n_i} \longrightarrow x$ strongly in X. This concludes the proof. □

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Definition 5.14 *A function $\varphi : X \longrightarrow 2^Y$ is said to be upper semicontinuous if for every open set $W \subset Y$ the set $\left\{x;\ \varphi(x) \in W\right\}$ is open in X.*

Definition 5.15 *Let X and Y be topological vector spaces, with Y locally convex, and let $\varphi : X \longrightarrow 2^Y$. We say that φ is a Kakutani function if it is upper semicontinuous and $\varphi(x)$ is nonempty, compact and convex for every $x \in X$.* 0.5cm

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Theorem 5.16 (Kakutani) *Let S be a nonempty, compact and convex subset of a locally convex topological vector space and let $\varphi : S \longrightarrow 2^S$ be a Kakutani mapping. Then φ has a fixed point, that is, there exists $x \in S$ such that $x \in \varphi(x)$.*

Proof: See Theorem 8.6 in [49]. □

Definition 5.17 *Let A and B be arbitrary sets. A point $(x_0, y_0) \in A \times B$ is said to be a saddle point of the mapping $f : A \times B \longrightarrow \mathbb{R}$ if*

$$f(x_0, y) \leq f(x, y_0), \;\; \forall x \in A, \;\; \forall y \in B. \tag{5.1.14}$$

Remark 5.18 *If inequality* (5.1.14) holds for all $x \in A$ and all $y \in B$, then in particular it holds for $x = x_0$, which implies $f(x_0, y) \leq f(x_0, y_0)$, and for $y = y_0$, which implies $f(x_0, y_0) \leq f(x, y_0)$. Thus, (x_0, y_0) is a saddle point of f if and only if

$$f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0), \;\; \forall x \in A, \;\; \forall y \in B. \tag{5.1.15}$$

Lemma 5.19 *For every mapping $f : A \times B \longrightarrow \mathbb{R}$, one has*

$$\sup_{y\in B} \inf_{x\in A} f(x, y) \leq \inf_{x\in A} \sup_{y\in B} f(x, y).$$

Proof: We have

$$\inf_{x\in A} f(x, y) \leq f(x, y), \quad \forall x \in A, \;\; \forall y \in B,$$

hence

$$\sup_{y\in B} \inf_{x\in A} f(x, y) \leq \sup_{y\in B} f(x, y), \quad \forall x \in A,$$

and therefore

$$\sup_{y\in B} \inf_{x\in A} f(x, y) \leq \inf_{x\in A} \sup_{y\in B} f(x, y).$$

□

Example 5.20 Let A and B be arbitrary sets. A mapping $f : A \times B \to \mathbb{R}$ admits a saddle point if and only if

$$\min_{x\in A} \sup_{y\in B} f(x, y) = \max_{y\in B} \inf_{x\in A} f(x, y),$$

where we replace inf by min and sup by max to indicate that the inf and the sup are, respectively, attained.

Solution: From Observation (5.18), we know that $(x_0, y_0) \in A \times B$ is a saddle point of f if and only if

$$f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0), \quad \forall x \in A, \;\; \forall y \in B. \tag{5.1.16}$$

Suppose $(x_0, y_0) \in A \times B$ is a saddle point of f. Then we obtain

$$\left.\begin{array}{rcl} f(x_0, y_0) & \leq & \inf_{x\in A} f(x, y_0) \\ \inf_{x\in A} f(x, y_0) & \leq & f(x_0, y_0) \end{array}\right\} \Rightarrow \;\; f(x_0, y_0) = \inf_{x\in A} f(x, y_0)$$

$$\left.\begin{array}{rcl} \sup_{y\in B} f(x_0, y) & \leq & f(x_0, y_0) \\ f(x_0, y_0) & \leq & \sup_{y\in B} f(x_0, y) \end{array}\right\} \Rightarrow \;\; f(x_0, y_0) = \sup_{y\in B} f(x_0, y)$$

that is,

$$\sup_{y\in B} f(x_0, y) = f(x_0, y_0) = \inf_{x\in A} f(x, y_0). \tag{5.1.17}$$

Note that, by (5.1.17),

$$\inf_{x\in A} \sup_{y\in B} f(x, y) \leq \sup_{y\in B} f(x_0, y) = \inf_{x\in A} f(x, y_0) \leq \sup_{y\in B} \inf_{x\in A} f(x, y),$$

and consequently

$$\inf_{x\in A}\sup_{y\in B} f(x,y) \le \sup_{y\in B}\inf_{x\in A} f(x,y).$$

From this inequality and Lemma (5.19), we obtain

$$\inf_{x\in A}\sup_{y\in B} f(x,y) = \sup_{y\in B}\inf_{x\in A} f(x,y). \tag{5.1.18}$$

Using (5.1.18) and (5.1.17), we get

$$\begin{aligned}\sup_{y\in B} f(x_0,y) &\ge \inf_{x\in A}\sup_{y\in B} f(x,y)\\ &= \sup_{y\in B}\inf_{x\in A} f(x,y)\\ &\ge \inf_{x\in A} f(x,y_0) = \sup_{y\in B} f(x_0,y),\end{aligned}$$

hence

$$\inf_{x\in A}\sup_{y\in B} f(x,y) = \sup_{y\in B} f(x_0,y) = f(x_0,y_0). \tag{5.1.19}$$

Similarly, we obtain

$$\sup_{y\in B}\inf_{x\in A} f(x,y) = \inf_{x\in A} f(x,y_0) = \inf_{x\in A} f(x,y_0). \tag{5.1.20}$$

From (5.1.19) and (5.1.20), it follows that

$$\min_{x\in A}\sup_{y\in B} f(x,y) = \sup_{y\in B} f(x_0,y) = f(x_0,y_0),$$

$$f(x_0,y_0) = \inf_{x\in A} f(x,y_0) = \max_{y\in B}\inf_{x\in A} f(x,y),$$

as required.

Conversely, suppose that

$$\min_{x\in A}\sup_{y\in B} f(x,y) = \max_{y\in B}\inf_{x\in A} f(x,y), \tag{5.1.21}$$

and let x_0 and y_0 be points at which the infimum and supremum are attained, respectively. Then

$$\begin{aligned}f(x,y_0) \ge \inf_{x\in A} f(x,y_0) &= \sup_{y\in B}\inf_{x\in A} f(x,y)\\ &= \max_{y\in B}\inf_{x\in A} f(x,y)\\ &= \min_{x\in A}\sup_{y\in B} f(x,y)\\ &= \inf_{x\in A}\sup_{y\in B} f(x,y) = \sup_{y\in B} f(x_0,y) \ge f(x_0,y),\end{aligned} \tag{5.1.22}$$

for every $x \in A$ and $y \in B$, so (x_0,y_0) is a saddle point of f.

Theorem 5.21 (Minimax Theorem) *Let X and Y be reflexive Banach spaces, and let A and B be convex, bounded and closed subsets of X and Y, respectively. Suppose that the function $F : A \times B \longrightarrow \mathbb{R}$ satisfies the following conditions:*

(i) For each $y \in B$, $F(x, y)$ is a convex and l.s.c. function of x.

(ii) For each $x \in A$, $F(x, y)$ is a concave and u.s.c. function of y.

Then F has a saddle point $(x_0, y_0) \in A \times B$ and

$$\min_{x\in A} \max_{y\in B} F(x, y) = \max_{y\in B} \min_{x\in A} F(x, y) = F(x_0, y_0).$$

Proof: A proof of this theorem can be found in [47], p. 61. □ 0.5cm

Proposition 5.22 *Let X be a locally convex Hausdorff topological vector space, C a convex and compact subset of X, $A : X \longrightarrow X'$ a monotone operator such that $D(A) \subset C$, and $H : X \longrightarrow X'$ a continuous mapping such that $D(H) = C$. Then there exists an element $x \in C$ such that*

$$\langle x - y, Hx + y' \rangle \leq 0, \ \forall (y, y') \in A.$$

Proof: For each $z \in C$, define the operator $T : C \longrightarrow 2^C$ by

$$Tz = \{x \in C;\ \langle x - y, Hz + y' \rangle \leq 0, \ \forall (y, y') \in A\}.$$

To show that there exists $x \in C$ satisfying the assertion, we shall prove, by Kakutani's fixed point theorem, that T admits a fixed point, i.e., there exists $x \in C$ such that $x \in Tx$. For this, we must prove that Tz is non-empty, convex and compact. We first show that $Tz \neq \emptyset$ for every $z \in C$.

Indeed, fix $z \in C$ and, for each $(y, y') \in A$, define

$$C(y, y') = \{x \in C;\ \langle x - y, Hz + y' \rangle \leq 0\}.$$

We have $C(y, y') \neq \emptyset$ for every $(y, y') \in A$, since $y \in C(y, y')$ (note that $D(A) \subset C$, so $y \in C$). Let $(x_n) \subset C(y, y')$ be such that $x_n \longrightarrow x$. Since $x_n \in C(y, y')$, we have

$$\langle x_n - y, Hz + y' \rangle \leq 0, \ \forall n \in \mathbb{N},$$

from which it follows that

$$\langle x - y, Hz + y' \rangle \leq 0,$$

that is, $C(y, y')$ is closed.

Notice that

$$Tz = \bigcap_{(y,y')\in A} C(y, y').$$

Therefore, to show that $Tz \neq \emptyset$, it suffices to show that the family of non-empty closed subsets $\{C(y, y');\ (y, y') \in A\}$ has the finite intersection property. To this end, define

$$K = \left\{(\lambda_1, \ldots, \lambda_n);\ \sum_{i=1}^{n} \lambda_i = 1, \ \lambda_i \geq 0, \ i = 1, \ldots, n\right\}.$$

Then K is a convex and compact subset of $\mathbb{R}^n$. Let $(y_i, y_i') \in A$, $i = 1, \ldots, n$. If $x(\lambda) : K \longrightarrow X$ is defined by

$$x(\lambda) = \sum_{i=1}^{n} \lambda_i y_i,$$

then the function $f : K \times K \longrightarrow \mathbb{R}$, defined by

$$f(\lambda, \mu) = \sum_{i=1}^{n} \mu_i \langle x(\lambda) - y_i, Hz + y_i' \rangle$$

is bilinear and continuous. By Theorem 5.21 (Minimax), f admits a saddle point, and hence there exist $\lambda_0, \mu_0 \in K$ such that

$$f(\lambda_0, \mu) \le f(\lambda_0, \mu_0) \le f(\lambda, \mu_0), \quad \forall \lambda, \mu \in K,$$

and therefore

$$f(\lambda_0, \mu) \le f(\lambda_0, \mu_0) \le \sup_{\lambda \in K} f(\lambda, \lambda), \quad \forall \mu \in K. \tag{5.1.23}$$

However,

$$\begin{aligned} f(\lambda, \lambda) &= \sum_{i=1}^{n} \lambda_i \left\langle \sum_{j=1}^{n} \lambda_j y_j - y_i, Hz + y_i' \right\rangle \\ &= \sum_{i,j=1}^{n} \lambda_i \lambda_j \langle y_j - y_i, Hz + y_i' \rangle \end{aligned}$$

and from the fact that

$$\lambda_i \lambda_j \langle y_j - y_i, Hz \rangle = -\lambda_i \lambda_j \langle y_i - y_j, Hz \rangle$$

it follows that

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \langle y_j - y_i, Hz \rangle = 0,$$

which implies

$$\begin{aligned} f(\lambda, \lambda) &= \sum_{i,j=1}^{n} \lambda_i \lambda_j \langle y_j - y_i, y_i' \rangle \\ &= -\sum_{i,j=1}^{n} \lambda_i \lambda_j \langle y_i - y_j, y_j' \rangle \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \lambda_i \lambda_j \langle y_j - y_i, y_i' - y_j' \rangle. \end{aligned}$$

Since A is monotone, this last identity yields $f(\lambda, \lambda) \le 0$. Hence, from (5.1.23), we have $f(\lambda_0, \mu) \le 0$ for all $\mu \in K$ and, in particular, for $\mu^i \in K$, defined by

$$\mu^i = (\delta_{i1}, \ldots, \delta_{in}), \quad i = 1, \ldots, n,$$

where δ_{ij} is the Kronecker delta, we obtain

$$f(\lambda_0, \mu^i) = \langle x(\lambda_0) - y_i, Hz + y_i' \rangle \le 0, \quad i = 1, \ldots, n.$$

Thus we conclude that $x(\lambda_0) \in C(y_i, y_i')$ for all $i = 1, \ldots, n$. Consequently, the family $\{C(y, y'),\ (y, y') \in A\}$ has the finite intersection property, and therefore $Tz \ne \emptyset$.

Next, observe that Tz is convex for every $z \in C$. Indeed, if $x_1, x_2 \in Tz$ and $t \in [0, 1]$, then for every $(y, y') \in A$ we have

$$t\big\langle x_1 - y, Hz + y'\big\rangle \le 0, \qquad (1-t)\big\langle x_2 - y, Hz + y'\big\rangle \le 0, \tag{5.1.24}$$

and from (5.1.24) it follows immediately that

$$\big\langle tx_1 + (1-t)x_2 - y, Hz + y'\big\rangle \le 0, \ \forall (y, y') \in A,$$

or, in other words, $tx_1 + (1-t)x_2 \in Tz$, which shows that Tz is convex.

We now prove that $T : C \longrightarrow C$ is closed. To that end, let $(z_n) \subset C$ and $(x_n) \subset Tz_n$ for each $n \in \mathbb{N}$, such that $z_n \longrightarrow z$ and $x_n \longrightarrow x$. Since $x_n \in Tz_n$, we have

$$\big\langle x_n - y, Hz_n + y'\big\rangle \le 0, \ \forall (y, y') \in A, \ \forall n \in \mathbb{N}. \tag{5.1.25}$$

As C is closed, we have $z \in C$. From (5.1.25), the convergences above and the continuity of H, it follows that

$$\big\langle x - y, Hz + y'\big\rangle \le 0, \ \forall (y, y') \in A,$$

which shows that $x \in Tz$, and hence T is a closed operator.

Moreover, Tz is compact. Indeed, let $w \in \overline{Tz}$. Then there exists $(w_n)_n \subset Tz$ such that $w_n \to w$. Hence

$$\langle w_n - y, Hz + y'\rangle \le 0, \quad \forall (y, y') \in A.$$

Letting $n \to \infty$, we obtain

$$\langle w - y, Hz + y'\rangle \le 0, \quad \forall (y, y') \in A,$$

that is, $w \in Tz$, which shows that Tz is closed. Since C is compact, it follows that $Tz \subset C$ is compact for every $z \in C$.

To apply Kakutani's fixed point theorem, it remains to show that T is a Kakutani mapping. We already know that T is closed and $D(T) = C$. To ensure that T is a Kakutani mapping, it remains to show that T is upper semicontinuous. Let $W \subset C$ be open and set

$$B = \{z \in C; \ Tz \subset W\}.$$

We must show that B is open, or equivalently, that $C\backslash B = \{z \in C; \ Tz \not\subset W\}$ is closed in C.

Let $z \in \overline{C\backslash B}$. Then there exists $(z_n)_n \subset C\backslash B$ such that $z_n \to z$. Thus $Tz_n \not\subset W$, and for each n there exists $y_n \in Tz_n$ such that $y_n \notin W$. Since $C\backslash W$ is compact, there exist a subsequence $(y_{nk}) \subset C\backslash W$ and a point $y \in C\backslash W$ such that $y_{nk} \to y$. As T is closed, we obtain $y \in Tz$. Hence $Tz \not\subset W$, i.e., $z \in C\backslash B$, which proves that $C\backslash B$ is closed. Therefore T is upper semicontinuous and hence a Kakutani mapping. It follows that T admits a fixed point, and the result is proved. □

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Definition 5.23 *Let X be a normed space. An operator $A : X \longrightarrow X'$ is said to be coercive if*

$$\big\langle x, Ax\big\rangle \ge \alpha(\|x\|)\|x\|, \ \forall x \in X,$$

for some function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$\alpha(\rho) \longrightarrow \infty \quad \textit{as } \rho \longrightarrow \infty.$$

Proposition 5.24 *Let E be a finite-dimensional Banach space, $A : E \longrightarrow E'$ a monotone operator such* *that* $0 \in D(A)$, $C \subset E$ *a convex and closed subset, and* $H : E \longrightarrow E'$, *with* $D(H) = E$, *a continuous and coercive mapping, that is,*

$$\alpha(\|x\|)\|x\| \leq \langle x, Hx\rangle, \ \forall x \in E.$$

Then, for each $y_0' \in A(0)$ *there exist a constant* M *(depending on* y_0' *and on the function* α*) and an element* $x \in C$ *such that*

$$\|x\| \leq M$$

and

$$\langle x - y, Hx + y'\rangle \leq 0, \ \forall (y, y') \in A.$$

Proof: Let B denote the closed unit ball of E and $r > 0$. Denote by A_r and H_r, respectively, the restrictions of A and H to $D(A) \cap rB$ and $C_r = C \cap rB$. By Proposition 5.22 applied to A_r and H_r, there exists $x_r \in C_r$ such that

$$\langle x_r - y, Hx_r + y'\rangle \leq 0, \ \forall (y, y') \in A_r.$$

From this last inequality we obtain

$$\langle x_r, Hx_r\rangle \leq \langle y, Hx_r\rangle - \langle x_r, y'\rangle + \langle y, y'\rangle, \quad \forall (y, y') \in A_r. \tag{5.1.26}$$

Note that inequality (5.1.26) holds in particular for $y = 0$ and $y_0' \in A(0)$. Hence

$$\langle x_r, Hx_r\rangle \leq -\langle x_r, y_0'\rangle,$$

which implies, in view of the coercivity of H, that

$$\alpha(\|x_r\|)\|x_r\| \leq \|x_r\|\|y_0'\|, \tag{5.1.27}$$

for some function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\alpha(\rho) \longrightarrow +\infty$ as $\rho \longrightarrow +\infty$. If $x_r \neq 0$, from (5.1.27) we deduce that $\alpha(\|x_r\|) \leq \|y_0'\|$ and, by the property of α, there exists $M > 0$ such that $\|x_r\| \leq M$ for all $r > 0$.

Set

$$S_r = \{x \in C_r; \ \langle x - y, Hx + y'\rangle \leq 0, \ \forall (y, y') \in A_r\}.$$

Clearly $S_r \neq \emptyset$, since $x_r \in S_r$. The set S_r is closed and consequently $S_r \cap MB$ is compact and non-empty for all $r > 0$. Moreover, if $M \leq r_1 \leq r_2$, we have

$$S_{r_1} \cap MB \supset S_{r_2} \cap MB;$$

therefore

$$\bigcap_{r \geq M} (S_r \cap MB) \neq \emptyset,$$

which implies that, if x is any point of this intersection, then $x \in C$ and

$$\langle x - y, Hx + y'\rangle \leq 0, \ \forall (y, y') \in A.$$

□

5.2 Maximal Monotone and m-Monotone Operators

Definition 5.25 *We say that a monotone operator* A *is* <u>*maximal monotone*</u> *if it does not admit a proper monotone extension.*

Let X be a real topological vector space and, for each $C \subset X$, denote by $\mathcal{M}(C)$ the family of monotone operators on X whose domain is contained in C, that is,

$$\mathcal{M}(C) = \big\{A : D(A) \subset X \longrightarrow X';\ A \text{ is monotone and } D(A) \subset C\big\}.$$

We order $\mathcal{M}(C)$ by inclusion.

Remark 5.26

(i) Note that $\mathcal{M}(X) = \mathcal{M}$, and the maximal monotone operators are precisely the maximal elements of $\mathcal{M}$, that is,

$$A \in \mathcal{M} \text{ is maximal monotone if and only if } B \in \mathcal{M}(X) \text{ and } A \subset B \Rightarrow B = A. \tag{5.2.28}$$

(ii) The family $\mathcal{M}(C)$ is inductive upwards, that is, every totally ordered subset of $\mathcal{M}(C)$ admits an upper bound. Indeed, let $F \subset \mathcal{M}(C)$ be totally ordered. Defining

$$\begin{array}{rccl} B : & \bigcup_{A\in F} D(A) \subset C & \longrightarrow & X' \\ & x & \longmapsto & Bx = \bigcup_{A\in F} Ax \end{array}$$

we see that B is an upper bound for F. By Zorn's lemma, it follows that every element of $\mathcal{M}(C)$ is contained in a maximal element of $\mathcal{M}(C)$. In particular, every element of $\mathcal{M}$ is contained in a maximal element of $\mathcal{M}$. Hence, by (i), every monotone operator on X is contained in a maximal monotone operator on X.

Theorem 5.27 *Let A be a monotone operator on X. The following statements are equivalent:*

(i) A is maximal monotone;

(ii) If $x \in X$, $x' \in X'$ and

$$\big\langle x' - y', x - y\big\rangle \geq 0,\ \forall (y, y') \in A,$$

then $(x, x') \in A$.

Proof:

(i) ⇒ (ii) Suppose that A is a maximal monotone operator. Let $x \in X$, $x' \in X'$ be such that

$$\big\langle x' - y', x - y\big\rangle \geq 0,\ \forall (y, y') \in A.$$

We shall prove that $(x, x') \in A$. Indeed, define

$$B = A \cup \big\{(x, x')\big\}.$$

We claim that B is monotone and that B extends A. By the very definition of B, it is clear that B extends A. On the other hand, let $(z, z'), (w, w') \in B$. We shall prove that

$$\big\langle z' - w', z - w\big\rangle \geq 0. \tag{5.2.29}$$

If both points belong to A, there is nothing to prove. If both points do not belong to A, then

$$(z, z') = (w, w') = (x, x'),$$

and therefore

$$\langle z' - w', z - w\rangle = 0.$$

If one of these points does not belong to A, we may assume $(z, z') \in A$ and $(w, w') \notin A$; then $(w, w') = (x, x')$, and by hypothesis

$$\langle z' - w', z - w\rangle = \langle z' - x', z - x\rangle \geq 0,$$

which proves (5.2.29).

Hence B is monotone and $A \subset B$. Since A is maximal monotone, it follows from (5.2.28) that $B = A$ and, consequently, $(x, x') \in A$.

(ii) $\Rightarrow$ (i) Now suppose that condition (ii) holds and that A is not maximal monotone. Let D be a proper extension of A, that is, there exists $(z, z') \in D$ such that $(z, z') \notin A$. On the other hand, since D is monotone, given $(x, x') \in D$, we have

$$\langle x' - y', x - y\rangle \geq 0, \; \forall (y, y') \in D.$$

In particular,

$$\langle x' - y', x - y\rangle \geq 0, \; \forall (y, y') \in A. \tag{5.2.30}$$

Thus, from (5.2.30) and condition (ii), we conclude that $(x, x') \in A$ and therefore $D \subset A$, which is a contradiction, since D is a proper extension of A. Hence A does not admit a proper extension, that is, A is maximal monotone. □

Corollary 5.28 *The following statements are equivalent:*

(i) A is maximal monotone in $\mathcal{M}(C)$;

(ii) λA, $\lambda > 0$, is maximal monotone in $\mathcal{M}(C)$;

(iii) The operator $\tilde{A}$ defined in Proposition 5.10 is maximal monotone in $\mathcal{M}(C - \{a\})$.

In a reflexive space, the following statements are equivalent:

(iv) A is maximal monotone;

(v) A^{-1} is maximal monotone.

Proof: (i) $\Rightarrow$ (ii) By Proposition 5.9 it follows that $\lambda A \in \mathcal{M}(C)$, since $\lambda > 0$ and $D(\lambda A) = D(A) \subset C$. Suppose there exists $B \in \mathcal{M}(C)$ such that $\lambda A \subset B$. Then

$$A \subset \frac{1}{\lambda} B \quad \Rightarrow \quad A = \frac{1}{\lambda} B \quad \Rightarrow \quad B = \lambda A,$$

which implies that λA is maximal monotone.

(ii) $\Rightarrow$ (iii) Let $a \in X$ and $\tilde{A}x = A(x+a) - \{a'\}$, $a' \in X'$. We already know that $\tilde{A} \in \mathcal{M}\left(C - \{a\}\right)$ and $D(\tilde{A}) = D(A) - \{a\} \subset C - \{a\}$. Suppose that there exists $\tilde{B} \in \mathcal{M}\left(C - \{a\}\right)$ such that

$$\tilde{A} \subset \tilde{B}.$$

Then

$$D(\tilde{A}) \subset D(\tilde{B}) \quad \Rightarrow \quad D(A) - \{a\} \subset D(\tilde{B}) \quad \Rightarrow \quad D(A) \subset D(\tilde{B}) + \{a\}.$$

Define B by

$$D(B) = D(\tilde{B}) - \{-a\} = D(\tilde{B}) + \{a\} \supset D(A),$$

$$Bx = \tilde{B}(x-a) + \{a\}.$$

Then B is monotone and $D(A) \subset D(B)$. Moreover, if $x \in D(A)$, then

$$Bx = \tilde{B}(x-a) + \{a\} = \tilde{A}(x-a) + \{a\} = Ax - \{a\} + \{a\} = Ax.$$

Thus $A \subset B$, which implies $A = B$ since A is maximal monotone. Hence

$$D(B) = D(\tilde{B}) + \{a\} \quad \Rightarrow \quad D(\tilde{B}) = D(A) - \{a\} = D(\tilde{A}) \quad \Rightarrow \quad \tilde{A} = \tilde{B},$$

which shows that $\tilde{A}$ is maximal in $\mathcal{M}(C - \{a\})$.

(iii) $\Rightarrow$ **(i)** Let $B \in \mathcal{M}(C)$ be such that $A \subset B$. By hypothesis, $\tilde{A}$ is maximal in $\mathcal{M}(C - \{a\})$, hence $\tilde{A} \subset \tilde{B}$. In fact,

$$D(\tilde{A}) = D(A) - \{a\} \subset D(B) - \{a\} = D(\tilde{B}),$$

and if $x \in D(\tilde{A})$, then

$$\tilde{A}x = A(x+a) - \{a\} = B(x+a) - \{a\} = \tilde{B}x.$$

Therefore $\tilde{A} = \tilde{B}$, since $\tilde{A}$ is maximal in $\mathcal{M}(C - \{a\})$. It follows that $D(\tilde{A}) = D(\tilde{B})$ and consequently $D(A) = D(B)$. As $A = B$ on $D(A)$, we obtain $A = B$, as required.

Consider now a reflexive space.

(iv) $\Rightarrow$ **(v)** By definition,

$$A^{-1} : D(A^{-1}) \subset X' \to X'' = X, \quad \text{where } A^{-1} = \{(x', x);\ (x, x') \in A\}.$$

We already know that if A is monotone, then A^{-1} is monotone. It remains to show maximality. Let $x' \in X'$, $x'' \in X''$ be such that

$$\langle x' - y', x'' - y'' \rangle \geq 0,\ \forall (y', y'') \in A^{-1}.$$

We shall show that $(x', x'') \in A^{-1}$. Since X is reflexive, by the canonical isomorphism $X \equiv X''$, we may write

$$x \in X,\ x' \in X',\ \text{and } \langle x' - y', x - y \rangle \geq 0,\ \forall (y', y) \in A^{-1},$$

or equivalently,

$$x \in X,\ x' \in X',\ \text{and } \langle x' - y', x - y \rangle \geq 0,\ \forall (y, y') \in A,$$

from which it follows that $(x, x') \in A$, hence $(x', x) \in A^{-1}$. By Theorem 5.27 we conclude that A^{-1} is maximal monotone.

(v) $\Rightarrow$ **(iv)** The proof is analogous to the previous one. □

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Definition 5.29 *An operator $H : X \longrightarrow X'$ is said to be* <u>*hemicontinuous*</u> *if it is single-valued and, in addition,*

$$\forall x, y \in X, \quad H(x + ty) \overset{*}{\rightharpoonup} Hx \quad \textit{weakly-}^* \textit{ in } X' \textit{ as } t \longrightarrow 0.$$

Proposition 5.30 *Let $H : X \longrightarrow X'$ be a hemicontinuous and monotone operator such that $D(H) = X$. Then H is maximal monotone.*

Proof: Let $x \in X$ and $x' \in X'$ be such that

$$\langle x' - Hy, x - y \rangle \geq 0,\ \forall y \in D(H) = X. \tag{5.2.31}$$

According to Theorem 5.27, we have to prove that $x \in D(H) = X$ and $Hx = x'$. The first assertion is obvious. It remains to show that $Hx = x'$. Let $z \in X$ be arbitrary and, for each $t \in [0,1]$, define

$$y_t = tz + (1-t)x = x + t(z-x), \quad x \in X. \tag{5.2.32}$$

Substituting (5.2.32) into (5.2.31), we obtain

$$\big\langle x' - H(\underbrace{x + t(z-x)}_{=y_t}), \underbrace{t(x-z)}_{=x-y_t} \big\rangle \geq 0,$$

which implies, for $t \in (0,1]$,

$$\big\langle x' - H(x + t(z-x)), x - z \big\rangle \geq 0.$$

Letting $t \longrightarrow 0$ in this last inequality yields

$$\big\langle x' - Hx, x - z \big\rangle \geq 0, \ \forall z \in X.$$

In particular, for $z = x - y$, $y \in X$, we have

$$\big\langle x' - Hx, y \big\rangle \geq 0, \ \forall y \in X,$$

from which we conclude that $\big\langle x' - Hx, y \big\rangle = 0, \ \forall y \in X$, and consequently $x' = Hx$ in X'. □

0.5 cm

Definition 5.31 *We say that a monotone operator $A : X \longrightarrow X'$ is* m-monotone *if* $\mathrm{Im}(F + A) = X'$, *where F is the duality mapping according to* Definition 4.9.

Definition 5.32 *We say that a Banach space is* smooth at the point *$x \in X$ if the duality mapping $F(x)$ contains a unique element. We say that X is* smooth *if X is smooth at every point of the unit sphere*

$$U_X = \big\{x \in X;\ \|x\| = 1\big\}.$$

It follows immediately from Definition 5.11 and item (iii) of Proposition 4.4 the following result:

Proposition 5.33 *If X is smooth then X is smooth at all its points, or, in other words, the duality mapping is single-valued.*

Proof: Indeed, if X is smooth then, by definition, it is smooth on the unit sphere U_X. Hence, for each $x \in U_X$, $F(x)$ contains a unique element. Given $y \in X$ with $\|y\| > 0$, we have $x = \dfrac{y}{\|y\|} \in U_X$. Thus, by Proposition 4.4, we have $F(y) = \|y\| F\left(\dfrac{y}{\|y\|}\right)$, which contains a unique element. Therefore X is smooth at all its points. □

0.5 cm

Definition 5.34 *A normed vector space is said to be* uniformly convex *if, given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in U_X$ and $\|x - y\| > \varepsilon$ then*

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Definition 5.35 *We say that a normed space X is* <u>*strictly convex*</u> *if the unit sphere*

$$U_X = \{x \in X;\ \|x\| = 1\}$$

does not contain proper line segments, that is, U_X does not contain sets of the form

$$\{\lambda x + (1-\lambda)y;\ x, y \in U_X,\ x \neq y,\ 0 \leq \lambda \leq 1\}.$$

Lemma 5.36 *The following conditions are equivalent:*

i) *X is strictly convex;*

ii) *The equality $\|x+y\| = \|x\| + \|y\|$ implies $x = 0$ or $y = tx$ for some $t \geq 0$;*

iii) *The equality $\|\frac{x+y}{2}\| = \|x\| = \|y\|$ implies $x = y$.*

Proof:

i) $\Rightarrow$ *ii)* Let X be strictly convex and let $x, y \in X$, $x \neq 0$, be such that $\|x+y\| = \|x\| + \|y\|$. If $y = 0$, then $y = tx$ with $t = 0$. So suppose $y \neq 0$ and $0 < \lambda < 1$. Without loss of generality, assume that $\lambda\|y\| \geq (1-\lambda)\|x\|$. Then

$$\begin{aligned}
1 &\geq \left\|\lambda\frac{x}{\|x\|} + (1-\lambda)\frac{y}{\|y\|}\right\| \\
&= \left\|\lambda\frac{x}{\|x\|} + \frac{\lambda y}{\|x\|} - \frac{\lambda y}{\|x\|} + (1-\lambda)\frac{y}{\|y\|}\right\| \\
&\geq \left\|\lambda\frac{x+y}{\|x\|}\right\| - \left\|\lambda\frac{y}{\|x\|} - (1-\lambda)\frac{y}{\|y\|}\right\| \\
&= \left\|\lambda\frac{x+y}{\|x\|}\right\| - \left\|\frac{\lambda\|y\|y - (1-\lambda)\|x\|y}{\|x\|\ \|y\|}\right\| \\
&= \left\|\lambda\frac{x+y}{\|x\|}\right\| - \frac{(\lambda\|y\| - (1-\lambda)\|x\|)\|y\|}{\|x\|\ \|y\|} \\
&= \lambda\frac{\|x+y\|}{\|x\|} - \frac{\lambda(\|x\|+\|y\|) - \|x\|}{\|x\|} = 1.
\end{aligned}$$

Hence

$$\lambda\frac{x}{\|x\|} + (1-\lambda)\frac{y}{\|y\|} \in U_X,$$

and, since X is strictly convex, U_X does not contain proper line segments, so that $\frac{x}{\|x\|} = \frac{y}{\|y\|}$ and thus $y = \frac{\|y\|}{\|x\|}x$. Therefore $y = tx$ with $t = \frac{\|y\|}{\|x\|} > 0$.

ii) $\Rightarrow$ *iii)* Suppose $\|\frac{x+y}{2}\| = \|x\| = \|y\|$. Then $\|x+y\| = \|x\| + \|y\|$, and by (ii) we have $x = 0$ or $y = tx$ for some $t \geq 0$.

If $x = 0$, then $\|x\| = \|y\| = 0$ and hence $y = 0$. If $y = tx$ with $t \geq 0$, then $\|y\| = t\|x\|$ and since $\|y\| = \|x\|$, we get $t = 1$ and therefore $y = x$.

iii) $\Rightarrow$ *i)* Let $x, y \in U_X$. If $\frac{x+y}{2} \in U_X$, then

$$\left\|\frac{x+y}{2}\right\| = 1 = \|x\| = \|y\|,$$

and by (iii) we obtain $x = y$. Thus the line segment joining x and y is not proper. Hence X is strictly convex.

□

Proposition 5.37 *Every uniformly convex space is strictly convex.*

Proof: Let x and y be points of U_X with $x \neq y$. Then $\|x - y\| > \varepsilon$ for some $\varepsilon > 0$, and by hypothesis there exists $\delta > 0$ such that

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta,$$

whence $(x + y)/2 \notin U_X$, i.e., the proper segment with endpoints x and y is not contained in U_X. □

Proposition 5.38 *Let X be a Banach space. Then:*

(i) If X' is strictly convex, then X is smooth.

(ii) If X' is smooth, then X is strictly convex.

Proof:

(i) Let $x \in U_X$. By Proposition 4.4, $F(x)$ is convex. Since $x \in U_X$, we have

$$F(x) = \big\{x' \in X';\ \big\langle x', x \big\rangle = \|x'\|^2 = 1\big\} \subset \big\{x' \in X';\ \|x'\| = 1\big\} = U_{X'}.$$

By hypothesis, $U_{X'}$ does not contain proper line segments. Hence, by convexity, $F(x)$ has a unique element, that is, X is smooth at the point $x \in U_X$. Since x was arbitrary in U_X, it follows that X is smooth.

(ii) Now suppose that X' is smooth and, aiming at a contradiction, let $x, y \in U_X$, $x \neq y$, be such that the line segment $[x, y] = \big\{\lambda x + (1 - \lambda)y;\ \lambda \in [0, 1\big\}]$ is contained in the sphere U_X. It follows that $\frac{x+y}{2} \in U_X$. Let $z' \in F\big(\frac{x+y}{2}\big)$. Then

$$\big\langle \tfrac{x}{2}, z' \big\rangle + \big\langle \tfrac{y}{2}, z' \big\rangle = \big\langle \tfrac{x+y}{2}, z' \big\rangle = \|z'\|^2 = \big\| \tfrac{x+y}{2} \big\| = 1,$$

which implies

$$\big\langle x, z' \big\rangle + \big\langle y, z' \big\rangle = 2. \tag{5.2.33}$$

On the other hand, we have

$$\begin{cases} \big\langle x, z' \big\rangle \le \|x\| \|z'\| = 1, \\ \big\langle y, z' \big\rangle \le \|y\| \|z'\| = 1. \end{cases} \tag{5.2.34}$$

Hence, from (5.2.33) and (5.2.34), we conclude that

$$\big\langle x, z' \big\rangle = \big\langle y, z' \big\rangle = 1,$$

which implies, in view of the canonical embedding $X \subset X''$, that $x, y \in F(z')$. Therefore $x = y$, since X' is smooth. This contradicts our assumption that U_X contains proper line segments. Thus U_X does not contain proper line segments, i.e., X is strictly convex. □

Corollary 5.39 *Let X be a reflexive Banach space. Then:*

(i) X is strictly convex if and only if X' is smooth.

(ii) X is smooth if and only if X' is strictly convex.

Proof: This follows directly from Proposition 5.38. □

Our next goal is to show that if X is a reflexive and smooth Banach space and $f : X \longrightarrow (-\infty, +\infty]$ is a convex, proper and lower semicontinuous function, then the operator ∂f is m-monotone. Before that, however, we need to introduce an auxiliary lemma.

Since X is Banach, for $\lambda \neq 0$ and $x, y \in X$ we define

$$[x, y]_\lambda = \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

as the <u>incremental quotient</u> of the Gâteaux derivative of the norm $\|\cdot\|$.

Lemma 5.40 *The following properties hold:*

(i) The function $\lambda \longrightarrow [x, y]_\lambda$ *is non-decreasing on* $\mathbb{R} \setminus \{0\}$*;*

(ii) For all $x, y \in X$ *and* $x' \in F(x)$ *we have*

$$\langle x', y \rangle \leq \|x\| [x, y]_\lambda, \quad \textit{if } \lambda > 0;$$

$$\langle x', y \rangle \geq \|x\| [x, y]_\lambda, \quad \textit{if } \lambda < 0;$$

(iii) For all $x, y \in X$ *and* $z' \in F(x + \lambda y)$ *we have:*

$$\langle z', y \rangle \geq \|x + \lambda y\| \, [x, y]_\lambda, \quad \textit{if } \lambda > 0;$$

$$\langle z', y \rangle \leq \|x + \lambda y\| \, [x, y]_\lambda, \quad \textit{if } \lambda < 0;$$

(iv) For all $x, y \in X$ *we have:*

$$-\|y\| \leq [x, y]_\lambda, \quad \lambda > 0;$$

$$\|y\| \geq [x, y]_\lambda, \quad \lambda < 0.$$

Proof: (i) Let $x, y \in X$ and define

$$\varphi(\lambda) = \|x + \lambda y\|, \ \lambda \in \mathbb{R}.$$

We claim that φ is convex. Indeed, let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $t \in [0, 1]$. Then

$$\begin{aligned}
\varphi\big(t\lambda_1 + (1-t)\lambda_2\big) &= \big\|x + [t\lambda_1 + (1-t)\lambda_2]y\big\| \\
&= \big\|x + tx - tx + [t\lambda_1 + (1-t)\lambda_2]y\big\| \\
&= \big\|tx + (1-t)x + t\lambda_1 y + (1-t)\lambda_2 y\big\| \\
&\leq t\,\|x + \lambda_1 y\| + (1-t)\,\|x + \lambda_2 y\| \\
&= t\varphi(\lambda_1) + (1-t)\varphi(\lambda_2),
\end{aligned}$$

which proves the claim.

Now set

$$f(\lambda) = \varphi(\lambda) - \varphi(0), \ \lambda \in \mathbb{R}.$$

Note that f is convex by the convexity of φ, since, for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and $t \in [0, 1]$,

$$\begin{aligned}
f\big(t\lambda_1 + (1-t)\lambda_2\big) &= \varphi\big(t\lambda_1 + (1-t)\lambda_2\big) - \varphi(0) \\
&\leq t\varphi(\lambda_1) + (1-t)\varphi(\lambda_2) - \varphi(0) \\
&= t\varphi(\lambda_1) + (1-t)\varphi(\lambda_2) - \big(t\varphi(0) + (1-t)\varphi(0)\big) \\
&= t[\varphi(\lambda_1) - \varphi(0)] + (1-t)[\varphi(\lambda_2) - \varphi(0)] \\
&= tf(\lambda_1) + (1-t)f(\lambda_2),
\end{aligned}$$

as claimed.

Now let $0 < \lambda \le \mu$. By the convexity of f,

$$f(t\mu) = f((1-t)0 + t\mu) \le (1-t)f(0) + tf(\mu) = tf(\mu), \ \forall t \in [0,1].$$

In particular, for $t = \dfrac{\lambda}{\mu}$, we obtain

$$f(\lambda) = f\left(\frac{\lambda}{\mu}\mu\right) \le \frac{\lambda}{\mu} f(\mu),$$

which implies

$$\frac{f(\lambda)}{\lambda} \le \frac{f(\mu)}{\mu},$$

that is,

$$[x,y]_\lambda = \frac{\|x+\lambda y\| - \|x\|}{\lambda} \le \frac{\|x+\mu y\| - \|x\|}{\mu} = [x,y]_\mu,$$

proving (i).

(ii) Let $x, y \in X$ and $x' \in F(x)$. Then

$$\langle x', x \rangle = \|x\|^2 = \|x'\|^2 .$$

Thus, for any $\lambda \in \mathbb{R}$,

$$\langle x', x + \lambda y \rangle = \langle x', x \rangle + \lambda \langle x', y \rangle = \|x\|^2 + \lambda \langle x', y \rangle.$$

From this identity we get

$$\begin{aligned} \lambda \langle x', y \rangle &= \langle x', x+\lambda y \rangle - \|x\|^2 \\ &\le \|x'\| \, \|x+\lambda y\| - \|x\|^2 \\ &= \|x\| \, \|x+\lambda y\| - \|x\|^2 , \end{aligned}$$

that is,

$$\lambda \langle x', y \rangle \le \|x\| \left\{ \|x+\lambda y\| - \|x\| \right\}, \ \forall \lambda \in \mathbb{R},$$

and the desired inequalities follow by dividing by $\lambda > 0$ or $\lambda < 0$.

(iii) Let $x, y \in X$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $z' \in F(x + \lambda y)$. Then

$$\langle z', x+\lambda y \rangle = \|x+\lambda y\|^2 = \|z'\|^2,$$

and hence

$$\langle z', x \rangle \le \|z'\| \, \|x\| = \|x+\lambda y\| \, \|x\|.$$

From the identities above and for all $\lambda > 0$ we obtain

$$\begin{aligned} \|x+\lambda y\| \, [x,y]_\lambda &= \frac{\|x+\lambda y\|^2 - \|x+\lambda y\| \, \|x\|}{\lambda} \le \frac{\langle z', x+\lambda y \rangle - \langle z', x \rangle}{\lambda} \\ &= \frac{\langle z', x \rangle + \lambda \langle z', y \rangle - \langle z', x \rangle}{\lambda} = \langle z', y \rangle . \end{aligned}$$

If $\lambda < 0$, the inequalities hold with the sign reversed.

(iv) Let $x, y \in X$. From (ii) we have, for all $x' \in F(x)$,

$$\begin{aligned} -\|x'\| \, \|y\| &\le \langle x', y \rangle \le \|x\| \, [x,y]_\lambda, \quad \text{if } \lambda > 0, \\ \|x'\| \, \|y\| &\ge \langle x', y \rangle \ge \|x\| \, [x,y]_\lambda, \quad \text{if } \lambda < 0. \end{aligned}$$

Since $\|x'\| = \|x\|$, we obtain equivalently

$$\begin{aligned} -\|x\|\,\|y\| &\leq \langle x', y\rangle \leq \|x\|\,[x,y]_\lambda, \quad \text{if } \lambda > 0, \\ \|x\|\,\|y\| &\geq \langle x', y\rangle \geq \|x\|\,[x,y]_\lambda, \quad \text{if } \lambda < 0. \end{aligned}$$

If $x \neq 0$, dividing by $\|x\|$ gives

$$-\|y\| \leq [x,y]_\lambda \quad \text{for } \lambda > 0,$$

$$\|y\| \geq [x,y]_\lambda \quad \text{for } \lambda < 0.$$

If $x = 0$, then

$$[x,y]_\lambda = \frac{\|\lambda y\| - \|0\|}{\lambda} = \frac{|\lambda|\,\|y\|}{\lambda} = \begin{cases} \|y\|, & \lambda > 0, \\ -\|y\|, & \lambda < 0, \end{cases} \tag{5.2.35}$$

and, therefore,

$$[x,y]_\lambda \geq -\|y\|, \text{ if } \lambda > 0, \qquad [x,y]_\lambda \leq \|y\|, \text{ if } \lambda < 0.$$

□

It follows from (i) and (iv) of the lemma above that, for every $x, y \in X$, the mapping $\lambda \mapsto [x,y]_\lambda$ admits a *right limit* $[x,y]_+$, as $\lambda \to 0_+$, and a *left limit* $[x,y]_-$, as $\lambda \to 0_-$. Thus, by the monotone sequence theorem, we may write

$$[x,y]_+ = \lim_{\lambda \to 0_+} [x,y]_\lambda = \inf_{\lambda > 0} [x,y]_\lambda, \tag{5.2.36}$$

$$[x,y]_- = \lim_{\lambda \to 0_-} [x,y]_\lambda = \sup_{\lambda < 0} [x,y]_\lambda. \tag{5.2.37}$$

Definition 5.41 *A single-valued operator $A : X \to X'$ is said to be demicontinuous if A is continuous when X is endowed with the strong topology and X' with the weak-$*$ topology.*

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Theorem 5.42 *Let X be a Banach space and $x \in X$ with $x \neq 0$. The following are equivalent:*
(i) X is smooth at the point x;
(ii) Every duality mapping is demicontinuous at the point x;
(iii) The norm of X is Gateaux differentiable at the point x.

Proof:

$(i) \Rightarrow (ii)$ Suppose, by contradiction, that there exists a duality mapping f which is not demicontinuous at the point x. Then there exists a sequence (x_n) in X such that $x_n \to x$ strongly in X but $f(x_n)$ does not converge to $f(x)$. Passing to a subsequence, if necessary, we may find a weak-$*$ neighbourhood V of $f(x)$ such that $f(x_n) \notin V$ for $n = 1, \ldots$. Since $f : X \to X'$ is a duality mapping, by definition we have $f(x) \in F(x)$ for every $x \in X$, that is, for each $x \in X$,

$$\langle x, f(x)\rangle = \|f(x)\|^2 = \|x\|^2.$$

From the identity above, from the convergence $x_n \to x$ and by Alaoglu's theorem, $(f(x_n))$ has a weak-$*$ cluster point, say $x' \in X'$. If we prove that $x' \in F(x)$, then, since X is smooth at x by hypothesis, $F(x)$ is a singleton, hence $x' = f(x)$, contradicting the fact that $f(x_n)$ does not converge to $f(x)$. Indeed, we have

$$\begin{aligned}\left|\langle x',x\rangle - \|x\|^2\right| &\leq |\langle x',x\rangle - \langle f(x_n),x\rangle| + |\langle f(x_n),x\rangle - \langle f(x_n),x_n\rangle| + \left|\langle f(x_n),x_n\rangle - \|x\|^2\right| \\ &= |\langle x' - f(x_n),x\rangle| + |\langle f(x_n),x-x_n\rangle| + \left|\,\|x_n\|^2 - \|x\|^2\right| \\ &\leq |\langle x' - f(x_n),x\rangle| + \|f(x_n)\|\,\|x-x_n\| + \left|\,\|x_n\|^2 - \|x\|^2\right|.\end{aligned} \tag{5.2.38}$$

Since $x_n \to x$, we have $\|x_n - x\| \to 0$, $\|x_n\| \to \|x\|$ and $\{f(x_n)\}$ is a bounded set. Hence, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|f(x_n)\|\,\|x-x_n\| + \left|\,\|x_n\|^2 - \|x\|^2\right| < \varepsilon,$$

for all $n \geq n_0$. Moreover, since x' is a cluster point of $(f(x_n))$, there exists $m > n_0$ such that $f(x_m)$ belongs to the weak-$*$ neighbourhood $\{\xi \in X'; |\langle x'-\xi, x\rangle| < \varepsilon\}$ of x', that is,

$$|\langle x' - f(x_m), x\rangle| < \varepsilon.$$

Going back to (5.2.38), we obtain $\langle x',x\rangle = \|x\|^2$ and consequently $\|x\| \leq \|x'\|$. On the other hand, from the convergence $x_n \to x$ it follows that, given $\delta > 0$, there exists an index n_1 such that $\|x_n\| \leq \|x\| + \delta$ for all $n \geq n_1$ and, therefore, $\|f(x_n)\| \leq \|x\| + \delta$ for all $n \geq n_1$. Since the ball $\{\xi \in X'; \|\xi\| \leq \|x\| + \delta\}$ is weak-$*$ closed, it follows that $\|x'\| \leq \|x\| + \delta$, for all $\delta > 0$, whence $\|x'\| \leq \|x\|$. Thus $\|x'\| = \|x\|$ and hence $x' \in F(x)$.

(ii) $\Rightarrow$ *(iii)* Taking $x' = f(x)$ and $z' = f(x + \lambda y)$ in (ii) and (iii) of Lemma 5.40, we obtain, for $\lambda > 0$,

$$\langle f(x), y\rangle \leq \|x\|\,[x,y]_\lambda \quad \text{and} \quad \langle f(x+\lambda y), y\rangle \geq \|x+\lambda y\|\,[x,y]_\lambda.$$

Hence, since the duality mapping is demicontinuous at x by hypothesis, we obtain

$$\langle f(x), y\rangle = \|x\|\,[x,y]_+.$$

Arguing analogously, we deduce

$$\langle f(x), y\rangle = \|x\|\,[x,y]_-.$$

Therefore,

$$\lim_{\lambda\to 0}[x,y]_\lambda = \left\langle \frac{f(x)}{\|x\|}, y\right\rangle, \quad \forall y \in X,$$

which proves that the norm of X is Gateaux differentiable at the point x.

(iii) $\Rightarrow$ *(i)* Since the norm is Gateaux differentiable at the point x, we have $[x,y]_+ = [x,y]_-$ for every $y \in X$. But, by item (ii) of Lemma 5.40, for every $x' \in F(x)$ we have

$$\langle x', y\rangle = \|x\|\,[x,y]_+, \quad \forall y \in X.$$

It follows that $F(x)$ has only one element, that is, X is smooth at the point x. □

Proposition 5.43 *Let X be a Banach space such that X' is strictly convex. Then:*

(i) The duality mapping is single-valued and demicontinuous;

(ii) The norm of X is Gateaux differentiable at every point $x \neq 0$.

Proof: This is an immediate consequence of Proposition 5.38 and Theorem 5.42. □

Remark 5.44

a) *By Theorem 5.42, the norm of X is Gateaux differentiable at the point $x \neq 0$ if and only if $F(x)$ has a single point and, therefore, if and only if all duality mappings coincide at the point x. Hence, if the norm of X is Gateaux differentiable at the point $x \neq 0$, we have*

$$\lim_{\lambda\to 0}[x,y]_\lambda = \left\langle \frac{f(x)}{\|x\|}, y \right\rangle,$$

for every duality mapping f, that is, $\frac{f(x)}{\|x\|}$ is the Gateaux derivative of the norm of X at the point x, for any choice of the duality mapping f.

b) *In order that the norm of X be Gateaux differentiable at the point x, it is sufficient that, for every $h \in U_X$, the limit of $[x,h]_\lambda$ exist as λ tends to zero. Indeed, if this happens and $y \in X,\ y \neq 0$, then, since $\frac{y}{\|y\|} \in U_X$, the limit of $\left[x, \frac{y}{\|y\|}\right]_\lambda$ exists as λ tends to zero, and*

$$\begin{aligned}\|y\| \lim_{\lambda\to 0}\left[x, \frac{y}{\|y\|}\right]_\lambda &= \|y\| \lim_{\lambda\to 0} \frac{\left\|x + \lambda \frac{y}{\|y\|}\right\| - \|x\|}{\lambda} \\ &= \|y\| \lim_{\lambda\|y\|\to 0} \frac{\left\|x + \lambda\|y\| \frac{y}{\|y\|}\right\| - \|x\|}{\lambda\|y\|} \\ &= \lim_{\lambda\|y\|\to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda} = \lim_{\lambda\to 0}[x,y]_\lambda,\end{aligned}$$

that is, the limit of $[x,y]_\lambda$ exists when $\lambda \to 0$ and, therefore, the norm of X is Gateaux differentiable at the point x by (ii) of Lemma 5.40.

c) *If the norm of X is Gateaux differentiable at the point x, the same holds at the point kx, for every $k > 0$, and*

$$\lim_{\lambda\to 0}[kx,y]_\lambda = \left\langle \frac{f(x)}{\|x\|}, y \right\rangle, \quad \forall y \in X,$$

that is, the Gateaux derivative of the norm of X is constant and equal to $\frac{f(x)}{\|x\|}$ along the ray $\{kx;\ k > 0\}$. Indeed, we have

$$[kx,y]_\lambda = \frac{\|kx+\lambda y\| - \|kx\|}{\lambda} = \frac{\|x + \frac{\lambda}{k} y\| - \|x\|}{\frac{\lambda}{k}} = [x,y]_{\frac{\lambda}{k}}. \tag{5.2.39}$$

Hence, putting $\frac{\lambda}{k} = \mu$,

$$\lim_{\lambda\to 0}[kx,y]_\lambda = \lim_{\mu\to 0}[x,y]_\mu = \left\langle \frac{f(x)}{\|x\|}, y \right\rangle, \quad \forall y \in X.$$

By Observation 5.44 (a), if the norm of X is Gateaux differentiable on a set $C \subset X$, then all duality mappings on X coincide on C and, therefore, if f is any one of them we have, taking also Observation 5.44 (b) into account,

$$\lim_{\lambda\to 0}[x,y]_\lambda = \left\langle \frac{f(x)}{\|x\|}, y \right\rangle, \quad \forall y \in U_X,$$

at each point $x \in C$. When the convergence is uniform on C, we say that the norm of X is *uniformly Gateaux differentiable on C*. Therefore, the norm of X is uniformly Gateaux differentiable on C if and only if, given $\varepsilon > 0$, for each $y \in U_X$ one can find $\lambda_0 > 0$ such that, for any duality mapping f,

$$\left| \left\langle \frac{f(x)}{\|x\|}, y \right\rangle - [x,y]_\lambda \right| < \varepsilon, \quad \forall x \in C \quad \text{whenever} \quad 0 < |\lambda| \leq \lambda_0.$$

Proposition 5.45 *Let X be a Banach space. The following are equivalent:*

(i) The norm of X is uniformly Gateaux differentiable;

(ii) The duality mapping of X is single-valued and uniformly demicontinuous on every bounded set (that is, uniformly continuous with X endowed with the norm topology and X' with the weak-$*$ topology).

Proof: **(i) $\Rightarrow$ (ii)** Suppose that the norm of X is uniformly Gateaux differentiable. By Theorem 5.42, the duality mapping F is single-valued, and it remains to prove that F is uniformly demicontinuous on bounded sets; that is, for each $\varepsilon > 0$, $M > 0$ and $z \in X$, there exists $\delta > 0$ such that if $\|x\| \le M$ and

$$\|x - y\| < \delta, \text{ then } |\langle F(x) - F(y), z\rangle| < \varepsilon.$$

We argue by contradiction. Suppose that there exists $\varepsilon_0 > 0$ such that

$$\|x_n\| \le M, \ \|x_n - y_n\| \to 0, \text{ and } |\langle F(x_n) - F(y_n), z\rangle| \ge \varepsilon_0 > 0, \ n = 1, 2, \ldots.$$

If $x_n \to 0$, then $y_n \to 0$ and hence

$$\|F(x_n)\| = \|x_n\| \to 0 \quad \text{and} \quad \|F(y_n)\| = \|y_n\| \to 0.$$

Thus $\|F(x_n) - F(y_n)\| \to 0$, whence

$$\left|\langle F(x_n) - F(y_n), z\rangle\right| \le \|F(x_n) - F(y_n)\|\|z\| \overset{n\to\infty}{\longrightarrow} 0,$$

that is, $|\langle F(x_n) - F(y_n), z\rangle| \longrightarrow 0$, a contradiction.

If $\{x_n\}$ does not converge to zero, passing to a subsequence if necessary, we may assume that

$$\|x_n\| \ge \alpha > 0, \ n = 1, 2, \ldots.$$

Then there exists n_0 such that $\|y_n\| \ge \dfrac{\alpha}{2}$ for all $n \ge n_0$. Let $\mu > 0$ and $n \ge n_0$. By the uniform Gateaux differentiability of the norm, we may choose λ_0 such that

$$\left|\left\langle \frac{F(x_n)}{\|x_n\|}, z\right\rangle - [x_n, z]_\lambda\right| < \frac{\mu}{2} \quad \text{and} \quad \left|\left\langle \frac{F(y_n)}{\|y_n\|}, z\right\rangle - [y_n, z]_\lambda\right| < \frac{\mu}{2} \quad \text{whenever} \quad 0 < |\lambda| \le \lambda_0.$$

But

$$\begin{aligned} |[x_n, z]_{\lambda_0} - [y_n, z]_{\lambda_0}| &= \left|\frac{\|x_n + \lambda_0 z\| - \|x_n\|}{\lambda_0} - \frac{\|y_n + \lambda_0 z\| - \|y_n\|}{\lambda_0}\right| \\ &\le \left|\frac{\|x_n + \lambda_0 z\| - \|y_n + \lambda_0 z\|}{\lambda_0}\right| + \left|\frac{\|x_n\| - \|y_n\|}{\lambda_0}\right| \\ &\le \frac{2}{\lambda_0}\|x_n - y_n\| \to 0, \end{aligned}$$

as $n \to \infty$, and since

$$\begin{aligned} \left|\left\langle \frac{F(x_n)}{\|x_n\|} - \frac{F(y_n)}{\|y_n\|}, z\right\rangle\right| &\le \left|\left\langle \frac{F(x_n)}{\|x_n\|}, z\right\rangle - [x_n, z]_{\lambda_0}\right| + \left|[x_n, z]_{\lambda_0} - [y_n, z]_{\lambda_0}\right| + \left|\left\langle \frac{F(y_n)}{\|y_n\|}, z\right\rangle - [y_n, z]_{\lambda_0}\right| \\ &< \mu + \frac{2}{\lambda_0}\|x_n - y_n\|, \end{aligned}$$

we obtain

$$\lim_{n\to\infty} \sup\left|\left\langle \frac{F(x_n)}{\|x_n\|} - \frac{F(y_n)}{\|y_n\|}, z\right\rangle\right| < \mu.$$

Hence,

$$\left|\left\langle \frac{F(x_n)}{\|x_n\|} - \frac{F(y_n)}{\|y_n\|}, z\right\rangle\right| \to 0$$

as $n \to \infty$, by the arbitrariness of μ.

Moreover,

$$\begin{aligned}\left|\left\langle \frac{F(x_n)}{\|x_n\|} - \frac{F(y_n)}{\|y_n\|}, z\right\rangle\right| &= \left|\left\langle \frac{F(x_n)}{\|x_n\|} - \frac{F(y_n)}{\|x_n\|}, z\right\rangle - \left\langle \frac{F(y_n)}{\|y_n\|} - \frac{F(y_n)}{\|x_n\|}, z\right\rangle\right| \\ &\geq \left|\left\langle \frac{F(x_n)}{\|x_n\|} - \frac{F(y_n)}{\|x_n\|}, z\right\rangle\right| - \left|\left\langle \frac{F(y_n)}{\|y_n\|} - \frac{F(y_n)}{\|x_n\|}, z\right\rangle\right| \\ &= \frac{1}{\|x_n\|}\left|\left\langle F(x_n) - F(y_n), z\right\rangle\right| - \left|\frac{1}{\|y_n\|} - \frac{1}{\|x_n\|}\right| \, \left|\left\langle F(y_n), z\right\rangle\right| \\ &\geq \frac{1}{M}\left|\left\langle F(x_n) - F(y_n), z\right\rangle\right| - \frac{2}{\alpha^2}\|F(y_n)\| \, \|z\| \, \|x_n - y_n\|.\end{aligned}$$

Thus,

$$\left|\left\langle F(x_n) - F(y_n), z\right\rangle\right| \leq M\left|\left\langle \frac{F(x_n)}{\|x_n\|} - \frac{F(y_n)}{\|y_n\|}, z\right\rangle\right| + \frac{2M}{\alpha^2}\|F(y_n)\| \, \|z\| \, \|x_n - y_n\|,$$

and since $\|F(y_n)\| = \|y_n\|$ is bounded by hypothesis, we get $\left|\left\langle F(x_n) - F(y_n), z\right\rangle\right| \to 0$ as $n \to \infty$, which is a contradiction. Hence F is uniformly demicontinuous on every bounded set.

(ii) $\Rightarrow$ **(i)** Suppose that the duality mapping F is single-valued and uniformly demicontinuous on bounded sets. Denoting $x' = F(x)$ and $z' = F(x + \lambda y)$, by items **(ii)** and **(iii)** of Lemma 5.40, we have, for $x, y \in U_X$,

$$\left\langle \frac{F(x+\lambda y)}{\|x+\lambda y\|}, y\right\rangle \leq [x,y]_\lambda \leq \left\langle F(x), y\right\rangle \quad \text{if} \quad \lambda < 0$$

and

$$\left\langle F(x), y\right\rangle \leq [x,y]_\lambda \leq \left\langle \frac{F(x+\lambda y)}{\|x+\lambda y\|}, y\right\rangle \quad \text{if} \quad \lambda > 0.$$

Hence,

$$\left|[x,y]_\lambda - \left\langle F(x), y\right\rangle\right| \leq \left|\left\langle \frac{F(x+\lambda y)}{\|x+\lambda y\|}, y\right\rangle - \left\langle F(x), y\right\rangle\right|. \tag{5.2.40}$$

Let $C = B(0, 1+\rho) \setminus B(0, 1-\rho)$, $0 < \rho < 1$, be a bounded set. Then F is uniformly demicontinuous on C. Thus, given $\varepsilon > 0$ and $y \in U_X$, there exists $0 < \lambda_0 < \rho$ such that for every $x \in U_X$ we have

$$\left|\left\langle F(x+\lambda y), y\right\rangle - \left\langle F(x), y\right\rangle\right| < \varepsilon \quad \text{whenever} \quad 0 < |\lambda| \leq \lambda_0.$$

Hence $\left\langle F(x+\lambda y), y\right\rangle$ converges to $\left\langle F(x), y\right\rangle$ uniformly on U_X and, since $\dfrac{1}{\|x+\lambda y\|}$ converges to $\dfrac{1}{\|x\|}$ uniformly on U_X and these functions are bounded for $x \in U_X$ and $0 < |\lambda| \leq \rho$, the product $\left\langle \dfrac{F(x+\lambda y)}{\|x+\lambda y\|}, y\right\rangle$ converges to $\left\langle F(x), y\right\rangle$ uniformly on U_X. From (5.2.40) it follows that, for each $y \in U_X$, $[x,y]_\lambda$ converges to $\left\langle F(x), y\right\rangle$ uniformly on U_X, i.e., the norm of X is uniformly Gateaux differentiable. □

Proposition 5.46 *Let X be a reflexive and smooth Banach space, and $f : X \to (-\infty, +\infty]$ a convex, proper and lower semicontinuous function. Then ∂f is m-monotone.*

Proof: As we have seen in Example 5.6, the operator ∂f is monotone. Thus, it remains to show that $\text{Im}(F + \partial f) = X'$. Let $x_0' \in X'$. We shall prove that there exists $x_0 \in X$ such that $x_0' \in (F + \partial f)x_0$. Indeed, consider the function:

$$\varphi(x) := f(x) + \frac{\|x\|^2}{2} - \langle x_0', x\rangle, \quad \forall x \in X. \tag{5.2.41}$$

We have that φ is convex, proper and l.s.c., since f is convex, proper and l.s.c., the norm $\|\cdot\|$ is a convex, proper and continuous function, and x_0' is linear, proper and continuous.

Since f is convex, proper and l.s.c., by the first geometric form of the Hahn–Banach theorem there exist $x_1' \in X'$ and $\beta \in \mathbb{R}$ defining a continuous affine function $l(x) := \langle x_1', x\rangle - \beta$ which is a minorant of f, that is,

$$f(x) \geq \langle x_1', x\rangle - \beta, \quad \forall x \in X.$$

Hence,

$$\varphi(x) - \frac{\|x\|^2}{2} + \langle x_0', x\rangle \geq \langle x_1', x\rangle - \beta, \quad \forall x \in X,$$

which yields

$$\begin{aligned} \varphi(x) &\geq \frac{\|x\|^2}{2} + \langle x_1' - x_0', x\rangle - \beta \\ &\geq \frac{\|x\|^2}{2} - \|x_0' - x_1'\| \, \|x\| - \beta, \quad \forall x \in X. \end{aligned}$$

The inequality above implies that

$$\varphi(x) \geq \|x\| \left(\frac{\|x\|}{2} - \|x_0' - x_1'\| \right) - \beta, \quad \forall x \in X.$$

For $\|x\|$ sufficiently large, we have $\left(\frac{\|x\|}{2} - \|x_0' - x_1'\| \right) > 0$ and, therefore, when $\|x\| \to +\infty$,

$$\|x\| \left(\frac{\|x\|}{2} - \|x_0' - x_1'\| \right) \to +\infty,$$

and, consequently,

$$\lim_{\|x\| \to +\infty} \varphi(x) = +\infty. \tag{5.2.42}$$

Thus, since φ is convex and l.s.c., from (5.2.42) we deduce that there exists $x_0 \in X$ such that $\varphi(x_0) \leq \varphi(x)$ for all $x \in X$, that is,

$$f(x_0) + \frac{\|x_0\|^2}{2} - \langle x_0', x_0\rangle \leq f(x) + \frac{\|x\|^2}{2} - \langle x_0', x\rangle, \quad \forall x \in X,$$

which implies

$$f(x) - f(x_0) \geq \frac{1}{2} \left(\|x_0\|^2 - \|x\|^2 \right) + \langle x_0', x - x_0\rangle, \quad \forall x \in X. \tag{5.2.43}$$

On the other hand, since X is smooth, the duality mapping F is single-valued. Thus, for each $x \in X$, there exists a unique $x' \in X'$ such that $F(x) = x'$ and $\langle x', x\rangle = \|x'\|^2 = \|x\|^2$. It follows that

$$\begin{aligned} \frac{1}{2} \left(\|x_0\|^2 - \|x\|^2 \right) &= \frac{1}{2} \left(\|x_0\|^2 + \|x\|^2 \right) - \|x\|^2 \\ &\geq \|x_0\| \, \|x\| - \|x\|^2 \\ &= \|x_0\| \, \|F(x)\| - \|x\|^2 \\ &\geq \langle F(x), x_0\rangle - \|x\|^2 \\ &= \langle F(x), x_0\rangle - \langle F(x), x\rangle \\ &= \langle F(x), x_0 - x\rangle . \end{aligned} \tag{5.2.44}$$

Combining (5.2.43) and (5.2.44) we obtain

$$f(x) - f(x_0) \geq \langle F(x), x_0 - x\rangle + \langle x_0', x - x_0\rangle = \langle x_0' - F(x), x - x_0\rangle\,, \quad \forall x \in X.$$

From the inequality above, in particular for $x = t\,x_0 + (1-t)y$, where $y \in X$ and $t \in [0,1)$, we may write

$$f(t\,x_0 + (1-t)y) - f(x_0) \geq \langle x_0' - F(t\,x_0 + (1-t)y), t\,x_0 + (1-t)y - x_0\rangle\,, \quad \forall y \in X.$$

By the convexity of f, it follows that

$$\begin{aligned}\langle x_0' - F(t\,x_0 + (1-t)y), (1-t)(y - x_0)\rangle \;&\leq\; t\,f(x_0) + (1-t)f(y) - f(x_0)\\ &=\; (1-t)\left(f(y) - f(x_0)\right),\end{aligned}$$

and since $t \in [0,1)$, we can rewrite the above inequality as

$$\langle x_0' - F(t\,x_0 + (1-t)y), y - x_0\rangle \leq f(y) - f(x_0), \quad \forall y \in X. \tag{5.2.45}$$

Since the space X is smooth and reflexive, F is demicontinuous. Therefore, if $t \to 1$ we have $t\,x_0 + (1-t)y \to x_0$ in X, which implies $F(t\,x_0 + (1-t)y) \overset{\star}{\rightharpoonup} F(x_0)$ in X'. Taking the limit in (5.2.45) as $t \to 1$, we obtain

$$\langle x_0' - F(x_0), y - x_0\rangle \leq f(y) - f(x_0), \quad \forall y \in X.$$

Hence $x_0' - F(x_0) \in \partial f(x_0)$ and, consequently, $x_0' \in F(x_0) + \partial f(x_0) = (F + \partial f)(x_0)$. □

Remark 5.47

(i) The restrictions imposed on X in the proposition above were introduced only to simplify the proof. The general case in which X is not reflexive is treated by Rockafellar [74].

(ii) Under the same hypotheses, the operator $\lambda\partial f$, $\lambda > 0$, is m-monotone. Indeed, note that if $\lambda > 0$ and f is convex and l.s.c., then λf is convex and l.s.c. and $\lambda(\partial f) = \partial(\lambda f)$.

(iii) The duality mapping F is m-monotone. In fact, letting X be a normed space, consider $f(x) = \frac{1}{2}\|x\|^2$. We saw in Example 4.18 that $\partial f(x) = F(x)$. Since $\|\cdot\|$ is a convex, proper and l.s.c. function, the proposition above guarantees that $\partial f(x) = F(x)$ is m-monotone.

Definition 5.48 *Let X be a Banach space. We say that $D \subset X$ is almost dense in X if, for each $u \in D$, there exists a dense subset $M_u \subset X$ such that for every $v \in M_u$ we have $u + tv \in D$ for all sufficiently small $t > 0$.*

Lemma 5.49 *Let H be a monotone mapping from X into X' with $D(H)$ almost dense in X. Then H is demicontinuous if and only if it is hemicontinuous and locally bounded.*

Proof: We know that if H is demicontinuous, then H is hemicontinuous. Moreover, if $x_n \to x$ in $D(H)$, then $Hx_n \overset{\star}{\rightharpoonup} Hx$, and thus $\{Hx_n\}$ is bounded, which shows that H is locally bounded.

Conversely, suppose that H is hemicontinuous and locally bounded. Let $(x_n) \subset D(H)$ and $x \in D(H)$ with $x_n \to x$ in X. Without loss of generality, assume that $x_n \neq x$ for all n. Let M_x be the dense subset of X given by the definition of almost dense. Take $y \in M_x$ and set $t_n = \|x_n - x\|^{\frac{1}{2}}$. Then $t_n > 0$, $t_n \to 0$ and

$$w_n := x + t_n y \in D(H), \text{ for } n \text{ sufficiently large.}$$

Moreover,

$$Hw_n \overset{\star}{\rightharpoonup} Hx. \tag{5.2.46}$$

By the monotonicity of H we have

$$0 \leq \langle Hx_n - Hw_n, x_n - w_n \rangle = \langle Hx_n - Hw_n, x_n - x - t_n y \rangle. \tag{5.2.47}$$

We know that $\{Hx_n\}$ is bounded because H is locally bounded, and by (5.2.46) it follows that $\{Hw_n\}$ is also bounded. Hence

$$\frac{1}{t_n}\langle Hx_n - Hw_n, x_n - x \rangle \to 0, \tag{5.2.48}$$

since $\left\| \frac{1}{t_n}(x_n - x) \right\| = \|x_n - x\|^{\frac{1}{2}} = t_n \to 0$. Also by (5.2.46),

$$\langle Hw_n, v \rangle \to \langle Hx, v \rangle$$

for every $v \in X$. Dividing (5.2.47) by t_n we obtain

$$0 \leq \langle Hx_n - Hw_n, \frac{x_n - x}{t_n} - y \rangle. \tag{5.2.49}$$

It follows that

$$\lim_{n\to\infty} \inf \langle Hx_n - Hx, -y \rangle \geq 0. \tag{5.2.50}$$

Since M_x is dense in X and $Hx_n - Hw_n \in X'$, it follows that (5.2.49) holds for every $y \in X$, and consequently (5.2.50) also holds for every $y \in X$. Replacing y by $-y$ we obtain

$$\lim_{n\to\infty} \sup \langle Hx_n - Hx, y \rangle \leq 0, \tag{5.2.51}$$

for all $y \in X$. Therefore

$$\lim_{n\to\infty} \langle Hx_n - Hx, y \rangle = 0, \tag{5.2.52}$$

for all $y \in X$, that is, $Hx_n \overset{\star}{\rightharpoonup} Hx$, proving that H is demicontinuous. □

Lemma 5.50 *Let E be a finite-dimensional Banach space and $H : E \to E'$ a monotone and hemicontinuous operator. Then:*

(i) H is bounded on bounded sets,

(ii) H is continuous.

Proof:

(i) Suppose there exists a bounded subset $A \subset E$ such that H is not bounded on A. Then there exists a sequence $\{x_n\} \subset A$ such that $\|Hx_n\| \to \infty$. We can extract from $\{x_n\}$ a convergent subsequence (for simplicity, we keep the same notation), such that $x_n \to x_0$, where $x_0 \in \overline{A}$. Since we are assuming that H is not bounded on bounded sets, we have $\|Hx_n\| \to +\infty$ as $n \to +\infty$. As H is monotone, we have

$$\langle x_n - x, Hx_n - Hx \rangle \geq 0, \quad \forall x \in E,$$

and therefore, for n sufficiently large we can write

$$\left\langle x_n - x, \frac{Hx_n - Hx}{\|Hx_n\|} \right\rangle \geq 0, \quad \forall x \in E. \tag{5.2.53}$$

On the other hand, since $\frac{Hx_n}{\|Hx_n\|} \in U_{E'} := \{y \in E', \|y\| = 1\}$ and $U_{E'}$ is a compact (and hence sequentially compact) subset of E', it follows that $\frac{Hx_n}{\|Hx_n\|} \to y' \in E'$. We have

$$\|y'\| = \lim_{n\to+\infty} \left\| \frac{Hx_n}{\|Hx_n\|} \right\| = 1.$$

But from (5.2.53), taking the limit as $n \to +\infty$,

$$\langle x_0 - x, y' \rangle \geq 0, \quad \forall x \in E,$$

which implies $y' = 0$, a contradiction with $\|y'\| = 1$. Hence H is bounded on bounded sets.

(ii) In view of item (i), the operator H maps bounded sets into bounded sets, and thus H is locally bounded. By Lemma 5.49, H is demicontinuous. Since E is finite-dimensional, demicontinuity and continuity are equivalent notions due to the equivalence of topologies. Therefore H is continuous.

□

0.5 cm

Theorem 5.51 *Let X be a reflexive Banach space, $A : X \to X'$ a monotone operator such that $0 \in D(A) \subset C$, where C is a closed convex subset of X, and $H : X \to X'$ a monotone, hemicontinuous, coercive operator that maps bounded sets into bounded sets and satisfies $D(H) = X$. Then there exists a point $x_0 \in C$ such that*

$$\langle x_0 - y, H\, x_0 + y' \rangle \leq 0, \quad \forall (y, y') \in A. \tag{5.2.54}$$

Proof: Let Λ be the family of all finite-dimensional subspaces of X, $j_E : E \to X$ the inclusion mapping of $E \in \Lambda$ into X, and $j_{E'} : X' \to E'$ the adjoint projection of j_E. Denote by A_E the operator $j_{E'}\, A\, j_E$. Then $A_E : E \to E'$ and $D(A_E) = D(A) \cap E$. Similarly, set $H_E := j_{E'}\, H\, j_E : E \to E'$ so that $D(H_E) = X \cap E = E$. Observe that for all $x, y \in E$ we have

$$\langle x, H_E\, y \rangle = \langle x, j_{E'}\, H\, j_E\, y \rangle = \langle j_E\, x, H\, j_E\, y \rangle = \langle x, H\, y \rangle\,,$$

and similarly, if $x \in E$, $y \in D(A_E)$ and $y' \in A\, y$, then

$$\langle x, j_{E'}\, y' \rangle = \langle j_E\, x, y' \rangle = \langle x, y' \rangle\,.$$

Thus, from the above identities we can conclude that the monotonicity of H implies that of H_E, the monotonicity of A implies that of A_E and the coercivity of H implies that of H_E with the same function α. Note also that, according to Lemma 5.50, H_E is continuous, since H is hemicontinuous and $H_E = H$ on E.

Let $y'_0 \in A(0)$. Then $j_{E'}\, y'_0 \in A_E(0)$ and, therefore, by Proposition 5.24, there exist $x_E \in C_E := C \cap E$ and a constant M_E such that $\|x_E\| \leq M_E$ and, moreover,

$$\langle x_E - y, H_E\, x_E + y' \rangle \leq 0, \quad \forall (y, y') \in A_E.$$

From (5.1.27) it follows that $\alpha(\|x_E\|) \leq \|j_{E'}\, y'_0\|$ for $x_E \neq 0$, and since $\|j_{E'}\| = 1$, we have $\alpha(\|x_E\|) \leq \|y'_0\|$. Hence we may assume that M_E is a constant M independent of E. Thus, for all $E \in \Lambda$ there exists $x_E \in C_E$ such that $\|x_E\| \leq M$ and

$$\langle x_E - y, H\, x_E + y' \rangle \leq 0, \quad \forall (y, y') \in A_E. \tag{5.2.55}$$

Since, by hypothesis, H maps bounded sets into bounded sets, there exists a constant M' such that $\|H\, x_E\| \leq M'$ for all $E \in \Lambda$. The sets

$$W_{E_0} = \{(x_E, H\, x_E); E \supset E_0\}, \quad E_0 \in \Lambda$$

are therefore subsets of the bounded set $(C \cap M\, B) \times M' B' \subset X \times X'$, where B is the closed unit ball of X

and B' is the closed unit ball of X'. Moreover, the family $\mathfrak{F} = \{W_{E_0}; E_0 \in \Lambda\}$ has the finite intersection property.

Indeed, let $\{W_{E_i}; i = 1, \ldots, n\} \subset \mathfrak{F}$. Fix, for each $i = 1, \ldots, n$, a basis β_i of E_i and denote by $E = \operatorname{span}[\cup_{i=1}^n \beta_i]$. Then $\dim E < \infty$ and $E_i \subset E$ for every $i = 1, \ldots, n$. Hence $(x_E, Hx_E) \in W_{E_i}$ for all $i = 1, \ldots, n$, which implies that

$$\bigcap_{i=1}^{n} W_{E_i} \neq \emptyset.$$

Now note that $C \cap M\,B$ is convex, closed and bounded, hence, since X is reflexive, it follows that $C \cap M\,B$ is compact in the weak topology of X. On the other hand, by the Banach–Alaoglu–Bourbaki Theorem, $M'B'$ is compact in the weak-$*$ topology of X'.

Thus $C \cap M\,B \times M'B' \subset X \times X'$ is a compact topological space when endowed with the product of the weak and weak-$*$ topologies. Therefore the family $\{W_{E_0}; E_0 \in \Lambda\}$ has at least one common cluster point, that is, there exists $(x_0, x_0') \in X \times X'$ such that $(x_0, x_0') \in \bigcap_{E_0 \in \Lambda} \overline{W_{E_0}}$, where here we are taking the closure with respect to the weak topology.

Since $C \cap M\,B$ is convex and closed, it follows that $C \cap M\,B$ is weakly closed, and hence $x_0 \in C$.

It remains to prove (5.2.54). Let $(y, y') \in A$, $u \in X$ and $E_0 \in \Lambda$ such that $y \in E_0$. If $E \supset E_0$, then by (5.2.55) we have

$$\langle x_E - y, Hx_E + y' \rangle \leq 0 \tag{5.2.56}$$

and, since by hypothesis H is monotone, it follows that

$$\langle x_E - u, Hu - Hx_E \rangle \leq 0. \tag{5.2.57}$$

Combining (5.2.56) and (5.2.57) we obtain

$$\langle x_E - y, Hx_E + y' \rangle + \langle x_E - u, Hu - Hx_E \rangle \leq 0, \tag{5.2.58}$$

for each $(y, y') \in A$, $u \in X$, $y \in E_0$ and $E \supset E_0$.

Consider the function $g : X \times X' \to \mathbb{R}$ defined by

$$g(x, x') = \langle x - y, x' + y' \rangle + \langle x - u, Hu - x' \rangle.$$

By (5.2.58) we have $g(x, x') \leq 0$ for all $(x, x') \in W_{E_0}$. Moreover, from

$$g(x, x') = \langle u - y, x' \rangle + \langle x, Hu + y' \rangle - \langle y, y' \rangle - \langle u, Hu \rangle,$$

it follows that g is continuous on $X \times X'$, when $X \times X'$ are endowed with the weak and weak-$*$ topologies, respectively.

Hence $g(x, x') \leq 0$ on the weak closure of W_{E_0}, and in particular $g(x_0, x_0') \leq 0$.

Therefore

$$\langle x_0 - y, x_0' + y' \rangle + \langle x_0 - u, Hu - x_0' \rangle \leq 0,$$

for each $(y, y') \in A$, $u \in X$ and $y \in E_0$. It follows that

$$\langle x_0 - y, x_0' + y' \rangle + \langle x_0 - u, Hu - x_0' \rangle \leq 0, \tag{5.2.59}$$

for all $(y, y') \in A$ and all $u \in X$.

Setting $u = x_0$ in (5.2.59) we obtain

$$\langle x_0 - y, -x_0' - y' \rangle \geq 0, \tag{5.2.60}$$

for all $(y, y') \in A$. Hence $A' = A \cup \{(x_0, -x'_0)\}$ is a monotone extension of A, and since $x_0 \in C$, we have $A' \in \mathcal{M}(C)$.

First consider the case where A is maximal in $\mathcal{M}(C)$. Then $A' = A$, and so $(x_0, -x'_0) \in A$ and therefore $x_0 \in D(A)$. Thus we may take $y = x_0$ in (5.2.59) and hence

$$\langle x_0 - u, Hu - x'_0 \rangle \leq 0,$$

for all $u \in X = D(H)$.

By the above inequality and by Theorem 5.27 we obtain $(x_0, x'_0) \in H$, that is, $x'_0 = Hx_0$, since H is maximal monotone by Proposition 5.30.

Replacing x'_0 by Hx_0 in (5.2.60) we obtain

$$\langle x_0 - y, Hx_0 + y' \rangle \leq 0,$$

for all $(y, y') \in A$, which proves the theorem in this particular case.

If A is not maximal in $\mathcal{M}(C)$, then by Observation 5.26, item (ii), there exists an operator $\tilde{A}$, maximal in $\mathcal{M}(C)$, such that $A \subset \tilde{A}$. By what we have just proved, the theorem holds for $\tilde{A}$, that is, there exists $x_0 \in C$ such that

$$\langle x_0 - y, Hx_0 + y' \rangle \leq 0,$$

for all $(y, y') \in \tilde{A}$, and in particular for all $(y, y') \in A$. □

Next, we present a result due to Browder which characterises maximal monotone operators in Banach spaces such that both X and X' are smooth.

Theorem 5.52 *Let X be a reflexive Banach space, C a closed convex subset of X, $A : X \to X'$ a maximal operator in $\mathcal{M}(C)$ such that $0 \in D(A)$, and $H : X \to X'$ a monotone, hemicontinuous, coercive operator which maps bounded sets into bounded sets and satisfies $D(H) = X$. Then $Im(H + A) = X'$.*

Proof: Let x' be an arbitrary element of X' and let $\widetilde{A} : X \to X'$ be the operator defined by $\widetilde{A}x = Ax - x'$. Then, by Corollary 5.28, item (iii), $\widetilde{A}$ is maximal in $\mathcal{M}(C) = \mathcal{M}(C - 0)$. By Theorem 5.51, there exists $x_0 \in C$ such that

$$\left\langle x_0 - y, Hx_0 + \widetilde{y}' \right\rangle \leq 0, \;\; \forall (y, \widetilde{y}') \in \widetilde{A}$$

but

$$\begin{aligned} (y, \widetilde{y}') \in \widetilde{A} \;\; &\Leftrightarrow \;\; \widetilde{y}' \in \widetilde{A}y = Ay - x' \\ &\Leftrightarrow \;\; \widetilde{y}' + x' \in Ay \\ &\Leftrightarrow \;\; \widetilde{y}' + x' = y' \in Ay \\ &\Leftrightarrow \;\; (y, y') \in A. \end{aligned}$$

Hence,

$$\left\langle x_0 - y, Hx_0 + y' - x' \right\rangle \leq 0, \;\; \forall (y, y') \in A.$$

From this last inequality and by Theorem 5.27, we obtain that $(x_0, x' - Hx_0) \in A$, i.e., $x' - Hx_0 \in Ax_0$ and therefore $x' \in Hx_0 + Ax_0 = (H + A)x_0$, which yields the desired conclusion. □

Corollary 5.53 *Let X be a reflexive and smooth Banach space and let C be a closed convex subset of X. Then every operator A maximal in $\mathcal{M}(C)$ is m-monotone.*

Proof: First, observe that the duality mapping F satisfies the assumptions on the operator H in Theorem 5.51. Indeed, according to Example 5.7, F is monotone. Since X is smooth, by Corollary 5.39 we have that X' is strictly convex and thus, by Proposition 5.43, we obtain that the duality mapping is single-valued and demicontinuous and hence hemicontinuous. Moreover, by definition, $D(F) = X$, and F is coercive and maps bounded sets into bounded sets.

Now note that, given x_0 an arbitrary element of $D(A)$, the operator $\widetilde{F}$ defined by

$$\widetilde{F}(x) = F(x + x_0)$$

still satisfies the same hypotheses. Indeed,

i) $D(\widetilde{F}) = X$, since $D(\widetilde{F}) = D(F) - x_0 = X - x_0 = X$;

ii) $\widetilde{F}$ is monotone, by Proposition 5.10;

iii) $\widetilde{F}$ maps bounded sets into bounded sets. In fact, let $A \subset X$ be such that $\|x\| \leq M$ for all $x \in A$. Then
$$\|\widetilde{F}(x)\|_{X'} = \|F(x + x_0)\|_{X'} = \|x + x_0\|_X \leq \|x\|_X + \|x_0\|_X \leq M + \|x_0\|.$$

iv) $\widetilde{F}$ is coercive. Indeed, we know that, by definition,
$$\big\langle F(x), x\big\rangle = \|x\|^2 = \|F(x)\|^2.$$

Thus,

$$\begin{aligned}
\big\langle \widetilde{F}(x), x\big\rangle &= \big\langle F(x + x_0), x\big\rangle = \big\langle F(x + x_0), x + x_0\big\rangle - \big\langle F(x + x_0), x_0\big\rangle \\
&= \|x + x_0\|^2 - \big\langle F(x + x_0), x_0\big\rangle \\
&\geq \|x - (-x_0)\|^2 - \|F(x + x_0)\|\|x_0\| \\
&= \|x\|^2 - 2\|x\|\|x_0\| + \|x_0\|^2 - \|x + x_0\|\|x_0\| \\
&\geq \|x\|^2 - 2\|x\|\|x_0\| + \|x_0\|^2 - \|x\|\|x_0\| - \|x_0\|^2 \\
&= \|x\|^2 - 3\|x\|\|x_0\| \\
&= (\|x\| - 3\|x_0\|)\|x\|.
\end{aligned}$$

Setting $\alpha(t) = t - 3\|x_0\|$ we have $\lim\limits_{t\to+\infty} \alpha(t) = +\infty$ and $\big\langle \widetilde{F}(x), x\big\rangle \geq \alpha(\|x\|)\|x\|$, which proves that $\widetilde{F}$ is coercive.

v) $\widetilde{F}$ is hemicontinuous. Indeed, since F is hemicontinuous, we have

$$\big\langle \widetilde{F}(x + ty), z\big\rangle = \big\langle F(x + ty + x_0), z\big\rangle \overset{t\to 0}{\longrightarrow} \big\langle F(x + x_0), z\big\rangle = \big\langle \widetilde{F}(x), z\big\rangle$$

for all $x, y, z \in X$.

Now, since A is maximal in $\mathcal{M}(C)$, the operator $\tilde{A}$ defined by $\tilde{A}(x) = A(x + x_0)$ satisfies $0 \in D(\widetilde{A})$ and, by item (iii) of Corollary 5.28, is maximal in $\mathcal{M}(C - x_0)$. By Theorem 5.52, we have

$$Im(\widetilde{F} + \widetilde{A}) = X'.$$

Finally, we show that $Im(F + A) = Im(\widetilde{F} + \widetilde{A})$. Indeed, if $y' \in Im(\widetilde{F} + \widetilde{A})$, then there exists $x \in D(\widetilde{F}) \cap D(\widetilde{A}) = X \cap [D(A) - x_0] = D(A) - x_0$ such that

$$\begin{aligned}
y' \in \widetilde{F}(x) + \widetilde{A}(x) &= F(x + x_0) + A(x + x_0) \\
&= F(z) + A(z); \quad z \in D(A) \\
&\Rightarrow y' \in Im(F + A).
\end{aligned}$$

Conversely, if $y' \in Im(F + A)$, then there exists $z \in D(F + A) = D(F) \cap D(A) = D(A)$, that is,

$z = \underbrace{(z - x_0)}_{x} + x_0$, such that

$$y' \in F(z) + A(z) \Rightarrow y' \in F(x + x_0) + A(x + x_0) = \widetilde{F}(x) + \widetilde{A}(x).$$

Hence $Im(F + A) = Im(\widetilde{F} + \widetilde{A}) = X'$, and therefore A is m-monotone. □

Theorem 5.54 *Let X be a reflexive and smooth Banach space with X' smooth. Then a monotone operator $A : X \to X'$ is maximal if and only if A is m-monotone.*

Proof: By Corollary 5.53, if A is maximal monotone, then A is m-monotone.

Conversely, suppose that $Im(F + A) = X'$ and consider $(x, x') \in X \times X'$ such that

$$\langle x' - y', x - y \rangle \geq 0, \ \forall (y, y') \in A. \tag{5.2.61}$$

Thus, in view of Theorem 5.27, we must prove that $(x, x') \in A$. Indeed, since $Fx + x'$ belongs to $X' = Im(F + A)$, there exists $x_1 \in D(A)$ such that $Fx + x' \in Fx_1 + Ax_1$. Therefore there exists $x_1' \in Ax_1$, i.e., $(x_1, x_1') \in A$ such that

$$Fx + x' = Fx_1 + x_1'. \tag{5.2.62}$$

We show that $(x_1, x_1') = (x, x')$. Taking $(y, y') = (x_1, x_1')$ in (5.2.61) and using (5.2.62), we obtain

$$\langle x - x_1, x' - x_1' \rangle = \langle x - x_1, Fx_1 - Fx \rangle \geq 0,$$

which implies

$$\langle x - x_1, Fx - Fx_1 \rangle \leq 0.$$

From this last inequality it follows that

$$\langle x, Fx \rangle + \langle x_1, Fx_1 \rangle - \langle x, Fx_1 \rangle - \langle x_1, Fx \rangle \leq 0. \tag{5.2.63}$$

By the definition of F we have

$$\|x\|^2 + \|x_1\|^2 - \|x\| \underbrace{\|Fx_1\|}_{=\|x_1\|} - \|x_1\| \underbrace{\|Fx\|}_{=\|x\|} \leq 0,$$

which implies

$$(\|x\| - \|x_1\|)^2 = \|x\|^2 + \|x_1\|^2 - 2\|x\|\|x_1\| \leq 0,$$

and hence $\|x\| = \|x_1\|$. Thus, from (5.2.63) we have

$$2\|x\|^2 = \langle x, Fx \rangle + \langle x_1, Fx_1 \rangle \leq \langle x, Fx_1 \rangle + \langle x_1, Fx \rangle \leq 2\|x\|^2,$$

and from this last inequality we conclude that

$$\langle x_1, Fx \rangle = \|x\|^2, \tag{5.2.64}$$

for otherwise, if we assumed that $\langle x_1, Fx \rangle < \|x\|^2$ or $\langle x, Fx \rangle > \|x\|^2$, we would arrive at a contradiction. Hence, from (5.2.64) and since

$$\|x_1\|^2 = \|x\|^2 = \|Fx\|^2 = \langle x, Fx \rangle,$$

it follows that $x, x_1 \in F'(Fx)$, where $F' : X' \to X$ is the duality mapping of X' (here we use the fact that X is reflexive). Therefore $x_1 = x$, since X' is smooth by hypothesis, and since by (5.2.62) we have $Fx - Fx_1 = x_1' - x'$, it follows that $x_1' = x'$. Hence $(x, x') = (x_1, x_1') \in A$, as desired. □

Corollary 5.55 *Let X be a reflexive Banach space and assume that both X and X' are smooth. Then an operator A is m-monotone if and only if λA is m-monotone for every $\lambda > 0$.*

Proof: This is an immediate consequence of Theorem 5.54 and of the equivalence of items (i) and (ii) of Corollary 5.28. □

Example 5.56 Let X be a smooth and reflexive Banach space. Let $f : X \longrightarrow (-\infty, +\infty]$ be a convex, proper, lower semicontinuous function. According to Proposition 5.46, the subdifferential ∂f is an m-monotone operator. When X' is smooth, Theorem 5.54 guarantees that the subdifferential is maximal monotone.

Example 5.57 Let Ω be a bounded open subset of $\mathbb{R}^n$ with regular boundary $\partial\Omega$ and let Δ be the Laplacian. The operator A on $L^2(\Omega)$ defined by

$$D(A) = \{v \in H^2(\Omega);\ \partial_\nu v = 0 \text{ on } \partial\Omega\}, \quad Au = -\Delta u,\ \forall u \in D(A)$$

is maximal monotone. Indeed,

i) A is monotone since

$$\begin{aligned} \langle Au - Av, u - v\rangle &= \int_\Omega -\Delta(u - v)(u - v)\, dx \\ &= \int_\Omega |\nabla(u - v)|^2\, dx \geq 0,\ \forall u, v \in D(A). \end{aligned}$$

ii) A is m-monotone, since from elliptic partial differential equation theory, for each $v \in L^2(\Omega)$, there exists $u \in D(A)$ such that $-\Delta u + u = v$, that is, $\mathrm{Im}(I + A) = L^2(\Omega)$. Taking $Fu = u$ as the duality mapping of $L^2(\Omega)$, so that $F = I$, we conclude that A is m-monotone.

By Theorem 5.54, A is maximal monotone.

Under suitable hypotheses, we can verify that the sum of two operators is a maximal monotone operator. For this purpose, consider the following lemma, whose proof can be found in [19].

Lemma 5.58 *Let X be a reflexive Banach space with norm $\|\cdot\|$. Then, for each $a > 1$, there exists a norm $\|\cdot\|_a$ on X such that X and X' are strictly convex when endowed with the norm $\|\cdot\|_a$ and the corresponding dual norm $\|\cdot\|_a'$.*

Example 5.59 Let X be a reflexive Banach space and A a maximal monotone operator from X into X'. If B is a monotone, hemicontinuous and bounded operator from X into X', then $A + B$ is maximal monotone.

Proposition 5.9 guarantees that $A+B$ is monotone. Thus, in order to ensure that $A+B$ is maximal monotone it suffices to show that $A + B$ is m-monotone, that is, $\mathrm{Im}((A + B) + F) = X'$, since we are in the setting of Theorem 5.54 – indeed, as X and X' are strictly convex spaces, they are smooth.

We define the operator

$$\begin{aligned} H : X &\longrightarrow X' \\ x &\longmapsto H(x) = F_0(x) + B(x), \end{aligned}$$

where F_0 is the duality mapping on $(X, \|\cdot\|_0)$, a strictly convex space.

We have:

- H is hemicontinuous.

  Indeed, since $(X, \|\cdot\|_0)$ and $(X', \|\cdot\|_0^{'})$ are strictly convex, by Proposition 5.43 the operator F_0 is single-valued and demicontinuous. Hence H is hemicontinuous, since F_0 and B are hemicontinuous.

- H is monotone.

  As F_0 is monotone (Example 5.7) and B is monotone by hypothesis, we have that $H = B + F_0$ is monotone.

- H is coercive.

  Indeed,

$$\begin{aligned}\langle (B+F_0)x, x\rangle &= \langle Bx, x\rangle + \langle F_0x, x\rangle \\ &= \langle Bx, x\rangle + \|x\|_0^2 \\ &\geq -\|Bx\|_0^{'}\|x\|_0 + \|x\|_0^2 \\ &\geq (\|x\|_0 - \|Bx\|_0^{'})\|x\|_0.\end{aligned}$$

  Since B is hemicontinuous, if $t \to 0$ then $B(tx) \stackrel{*}{\rightharpoonup} B(0)$ and, therefore,

$$\|B(0)\|_0 \leq \liminf_{t>0} \|B(tx)\|_0 \leq \|Bx\|_0.$$

  Consequently,

$$\langle (B+F_0)x, x\rangle \geq (\|x\|_0 - \|B0\|_0^{'})\|x\|_0.$$

  Defining

$$\begin{aligned}\alpha : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \alpha(t) = t - \|B0\|_0,\end{aligned}$$

  we obtain

$$\langle (B+F_0)x, x\rangle \geq \alpha(\|x\|_0)\|x\|_0,$$

  and hence H is coercive.

Assuming, without loss of generality, that $0 \in D(A)$, we see that H satisfies the hypotheses of Theorem 5.52 and therefore

$$\operatorname{Im}(A+H) = X',$$

that is,

$$\operatorname{Im}(A+B+F_0) = X'.$$

Thus, $A+B$ is m-monotone on $(X, \|\cdot\|_0)$ and therefore maximal monotone on $(X, \|\cdot\|_0)$. Since monotonicity and maximality are independent of the chosen norm, we conclude that $A+B$ is a maximal monotone operator on X.

Proposition 5.60 *Let X be a reflexive and smooth Banach space with X' smooth, let $C \subset X$ be a closed convex subset, and let $A : X \to X'$ be a monotone operator such that $D(A) \subset C$. Then A admits a maximal monotone extension whose domain is contained in C. In particular, every monotone operator on a Banach space under these conditions admits a maximal monotone extension whose domain is contained in $\overline{\text{conv}\ D(A)}$.*

Proof: By item (ii) of Observation 5.26, the monotone operator A admits a maximal extension in C. By Corollary 5.53, this extension is m-monotone and therefore maximal monotone in view of Theorem 5.54. The second assertion is immediate, since $\overline{\text{conv}\ D(A)}$ is a closed convex set containing $D(A)$. □

Theorem 5.61 *Let X be a reflexive Banach space such that X and X' are smooth. Let $A : X \to X'$ be a maximal monotone, coercive operator such that $0 \in D(A)$. Then Im $A = X'$.*

Proof: For each $\lambda > 0$, the operator λF is monotone, hemicontinuous, coercive, maps bounded sets into bounded sets, and $D(\lambda F) = X$. Moreover, since X is smooth, the duality mapping $F : X \longrightarrow X'$ is single-valued, and we have a unique $x' \in X'$ satisfying $x' = F(x)$ and

$$\langle x', x\rangle = \langle F(x), x\rangle = \|x\|^2 = \|x'\|^2.$$

Hence the element $y' := \lambda x' = \lambda F(x)$ satisfies

$$\langle y', x\rangle = \langle \lambda x', x\rangle = \lambda\langle x', x\rangle = \lambda\|x\|^2 = \lambda\|x'\|^2 = \frac{1}{\lambda}\|\lambda x'\|^2 = \frac{1}{\lambda}\|y'\|^2.$$

By Proposition 5.60, A is maximal in $\mathcal{M}(\overline{\text{conv } D(A)})$, and since $0 \in D(A)$ we can apply Theorem 5.52 to obtain $\text{Im}(\lambda F + A) = X'$.

We shall prove that $\text{Im}(A) = X'$. Indeed, let $y' \in X'$. For each $\lambda > 0$ there exists $x_\lambda \in D(\lambda F + A) = D(\lambda F) \cap D(A) = X \cap D(A) = D(A)$ such that $y' \in (\lambda F + A)x_\lambda$, that is, there exists $x'_\lambda \in Ax_\lambda$ such that

$$\lambda F x_\lambda + x'_\lambda = y'. \tag{5.2.65}$$

Since A is coercive, there exists a function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\alpha(\rho) \to \infty$ as $\rho \to \infty$ and

$$\alpha(\|x\|)\|x\| \le \langle x, x'\rangle\,,\ \forall (x, x') \in A.$$

The above relation holds for $(x_\lambda, x'_\lambda) \in A$, $\lambda > 0$, and therefore

$$\begin{aligned}
\alpha\left(\|x_\lambda\|\right)\|x_\lambda\| &\le \langle x_\lambda, x'_\lambda\rangle \\
&\le \langle x_\lambda, x'_\lambda\rangle + \lambda\|x_\lambda\|^2 \\
&= \langle x_\lambda, x'_\lambda\rangle + \langle x_\lambda, \lambda F(x_\lambda)\rangle \\
&= \langle x_\lambda, x'_\lambda + \lambda F(x_\lambda)\rangle \\
&= \langle x_\lambda, y'\rangle \\
&\le \|x_\lambda\|\ \|y'\| \quad \forall\ \lambda > 0.
\end{aligned}$$

Hence, if $x_\lambda \neq 0$,

$$\alpha\left(\|x_\lambda\|\right) \le \|y'\|,\ \forall\ \lambda > 0,$$

and thus the set $\{x_\lambda;\ \lambda > 0\}$ is bounded. Consequently, $\{F(x_\lambda);\ \lambda > 0\}$ is bounded (since $\|F(x_\lambda)\| = \|x_\lambda\|$). From (5.2.65) we obtain

$$x'_\lambda = y' - \lambda F(x_\lambda) \to y' \quad \text{as } \lambda \to 0 \tag{5.2.66}$$

in the norm topology of X'.

Moreover, since $\{x_\lambda;\ \lambda > 0\}$ is bounded and X is reflexive, we can extract a sequence (λ_n), with $\lambda_n \to 0$, such that (x_{λ_n}) converges weakly to some $y \in X$, that is,

$$x_{\lambda_n} \rightharpoonup y. \tag{5.2.67}$$

Thus, for each $(x, x') \in A$, by the monotonicity of A we obtain

$$\left\langle x - x_{\lambda_n}, x' - x'_{\lambda_n}\right\rangle \ge 0.$$

By the convergences (5.2.66) and (5.2.67), we get

$$\langle x-y, x'-y'\rangle \geq 0, \ \forall (x,x') \in A.$$

Since A is maximal monotone, Theorem 5.27 yields $(y,y') \in A$, that is, $y' \in Ay$, or equivalently, $y' \in \mathrm{Im}(A)$. Therefore $\mathrm{Im}(A) = X'$. □

Corollary 5.62 *Let X be a reflexive Banach space such that X and X' are smooth, and let $H : X \longrightarrow X'$ be a monotone, hemicontinuous, coercive operator with $D(H) = X$. Then $Im(H) = X'$.*

Proof: Since H is monotone, hemicontinuous and $D(H) = X$, it follows from Proposition 5.30 that H is maximal monotone. As $D(H) = X$, we have $0 \in D(H)$ and Theorem 5.61 implies that $\mathrm{Im}(H) = X'$. □

5.3 Accretive Operators

According to Proposition 5.8, in a Hilbert space monotonicity is equivalent to the condition

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\| \geq \|x_1 - x_2\| \quad \forall (x_1,y_1),(x_2,y_2) \in A, \ \forall \lambda > 0.$$

Since this condition involves only the norm, it makes sense in any normed space, which allows us to generalise the notion of a monotone operator in Hilbert spaces.

Definition 5.63 *Let X be a Banach space. We say that the operator $A : X \longrightarrow X$ is accretive if*

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\| \geq \|x_1 - x_2\| \tag{5.3.68}$$

for all $(x_1,y_1),(x_2,y_2) \in A$ and for all $\lambda > 0$.

Definition 5.64 *Let X be a Banach space and let $A : X \longrightarrow X$ be an operator. We say that A is dissipative if $-A$ is accretive.*

We wish to present other characterisations of accretive operators. For this, given a Banach space X, we recall the definition of the incremental quotient of the Gâteaux derivative of the norm,

$$[x,y]_\lambda = \frac{\|x+\lambda y\| - \|x\|}{\lambda}, \ \lambda \neq 0, \ x,y \in X.$$

Lemma 5.65 *The following properties hold:*

(i) $[\alpha x, \beta y]_+ = |\beta|[x,y]_+$ *if* $\alpha\beta > 0$;

(ii) $[x, \alpha x + y]_+ = \alpha\|x\| + [x,y]_+$;

(iii) $-[x,-y]_+ \leq [x,y]_+$;

(iv) $|[x,y]_+| \leq \|y\|$;

(v) $[x,\beta y]_+ \geq \beta[x,y]_+$.

Proof: It is enough to prove that these properties hold for the incremental quotient $[x,y]_\lambda$, since, by taking the infimum, we obtain the corresponding results for $[x,y]_+$.

(i) If $\alpha\beta > 0$,

$$\begin{aligned}[\alpha x, \beta y]_\lambda &= \frac{\|\alpha x + \lambda \beta y\| - \|\alpha x\|}{\lambda} \\ &= |\beta| \frac{\|\frac{\alpha}{\beta} x + \lambda y\| - \|\frac{\alpha}{\beta} x\|}{\lambda} \\ &= |\beta| \frac{\|x + \lambda \frac{\beta}{\alpha} y\| - \|x\|}{\lambda \frac{\beta}{\alpha}} \\ &= |\beta| \, [x, y]_{\lambda \frac{\beta}{\alpha}} .\end{aligned}$$

Taking the limit as $\lambda \longrightarrow 0^+$ we obtain the desired conclusion.

(ii) If $\alpha = 0$ there is nothing to prove. For $\alpha \neq 0$, take $\lambda_0 > 0$ such that $1 + \lambda\alpha > 0$ whenever $0 < \lambda < \lambda_0$. Then

$$\begin{aligned}&\Big|[x, \alpha x + y]_\lambda - (\alpha \|x\| + [x, y]_\lambda)\Big| \\ =& \left| \frac{\|x + \lambda \alpha x + \lambda y\| - \|x\| - \|x + \lambda y\| + \|x\|}{\lambda} - \alpha \|x\| \right| \\ =& \left| \frac{\|(1+\lambda\alpha) x + \lambda y\| - \|x + \lambda y\| - \lambda\alpha\|x\|}{\lambda} \right| \\ =& \left| \frac{1}{\lambda} \left((1+\lambda\alpha) \left\| x + \frac{\lambda}{1+\lambda\alpha} y \right\| - \|x + \lambda y\| - \lambda\alpha \|x\| \right) \right| \\ =& \left| \frac{1}{\lambda} \left(\left\| x + \frac{\lambda}{1+\lambda\alpha} y \right\| - \|x + \lambda y\| \right) + \alpha \left(\left\| x + \frac{\lambda}{1+\lambda\alpha} y \right\| - \|x\| \right) \right| \\ \leq& \left| \frac{1}{\lambda} \left(\left\| x + \frac{\lambda}{1+\lambda\alpha} y \right\| - \|x + \lambda y\| \right) \right| + \left| \alpha \left(\left\| x + \frac{\lambda}{1+\lambda\alpha} y \right\| - \|x\| \right) \right| \\ \leq& \frac{1}{\lambda} \left\| x + \frac{\lambda}{1+\lambda\alpha} y - x - \lambda y \right\| + \left| \alpha \left(\left\| x + \frac{\lambda}{1+\lambda\alpha} y - x \right\| \right) \right| \\ \leq& \left| \frac{1}{1+\lambda\alpha} - 1 \right| \|y\| + |\alpha| \frac{\lambda}{1+\lambda\alpha} \|y\|.\end{aligned}$$

Taking the limit as $\lambda \longrightarrow 0^+$ we obtain the desired conclusion.

(iii) Note that

$$\begin{aligned}[x, y]_\lambda + [x, -y]_\lambda &= \frac{\|x + \lambda y\| - \|x\| + \|x - \lambda y\| - \|x\|}{\lambda} \\ &= \frac{1}{\lambda} (\|x + \lambda y\| + \|x - \lambda y\| - 2\|x\|) \\ &\geq \frac{1}{\lambda} (\|x + \lambda y + x - \lambda y\| - 2\|x\|) = 0.\end{aligned}$$

Hence $-[x, y]_\lambda \leq [x, y]_\lambda$ for all $\lambda > 0$.

(iv) If $\lambda > 0$, by item (iv) of Lemma 5.40 we have

$$-\|y\| \leq [x, y]_\lambda = \frac{\|x + \lambda y\| - \|x\|}{\lambda} \leq \frac{\|x\| + \|\lambda y\| - \|x\|}{\lambda} = \|y\|.$$

(v) If $\beta = 0$, there is nothing to prove. The case $\beta > 0$ is covered by item (i). Let now $\beta < 0$. Then

$$\beta[x, y]_+ = -|\beta|[x, y]_+ = -[x, |\beta| y]_+ = -[x, -\beta y]_+ \leq [x, \beta y]_+.$$

Lemma 5.66 *For each $x, y \in X$, there exists $x' \in F(x)$ such that*

$$\langle x', y \rangle = \|x\| [x, y]_+.$$

Proof: Given $x \in X$, we have two cases to consider.

Case 1: $y = \rho x$, for some $\rho \in \mathbb{R}$. Let $\lambda > 0$ be such that $1 + \lambda\rho > 0$. Then

$$[x, y]_\lambda = [x, \rho x]_\lambda = \frac{\|x + \lambda\rho x\| - \|x\|}{\lambda} = \frac{\|(1 + \lambda\rho)x\| - \|x\|}{\lambda} = \rho\|x\|.$$

But for every $x' \in F(x)$ we have

$$\langle x', y\rangle = \rho\,\langle x', x\rangle = \rho\|x\|^2 = \|x\|[x, y]_\lambda.$$

Taking the infimum we obtain the result.

Case 2: x and y are linearly independent. Let $V = \text{span}\{x, y\} \subset X$ and define $\xi' : V \longrightarrow \mathbb{R}$ by

$$\langle \xi', \alpha x + \beta y\rangle = \alpha\|x\| + \beta[x, y]_+.$$

By Lemma 5.65 we have

$$\begin{aligned}
\langle \xi', \alpha x + \beta y\rangle &= \alpha\|x\| + \beta[x, y]_+ \\
&\leq \alpha\|x\| + [x, \beta y]_+ \\
&= [x, \alpha x + \beta y]_+ \\
&\leq \|\alpha x + \beta y\|.
\end{aligned}$$

By the Hahn–Banach Theorem, there exists $\xi_1' \in X'$ extending ξ' such that $\|\xi_1'\| \leq 1$. Since ξ_1' extends ξ', we have

$$\langle \xi_1', x\rangle = \|x\| \quad \text{and} \quad \langle \xi_1', y\rangle = [x, y]_+.$$

Set $x' = \|x\|\xi_1'$. Then

$$\|x\|^2 = \langle x', x\rangle \leq \|x'\|\|x\|.$$

On the other hand,

$$\|x'\| = \|\|x\|\xi_1'\| = \|x\|\|\xi_1'\| \leq \|x\|.$$

Thus $x' \in F(x)$ and $\langle x', y\rangle = \|x\|[x, y]_+$.

Proposition 5.67 *Define*

$$\langle y, x\rangle_s = \sup\{\langle x', y\rangle\,;\ x' \in F(x)\}.$$

Then

$$\langle y, x\rangle_s = \|x\|[x, y]_+.$$

Proof: Let $x, y \in X$. By item (ii) of Lemma 5.40, for every $\lambda > 0$ we have

$$\langle x', y\rangle \leq \|x\|[x, y]_\lambda,\ \forall x' \in F(x).$$

Taking the infimum over $\lambda > 0$ and the supremum over x' in $F(x)$, we obtain

$$\langle y, x\rangle_s \leq \|x\|[x, y]_+. \tag{5.3.69}$$

On the other hand, by Lemma 5.66, there exists $x' \in F(x)$ such that

$$\|x\|[x, y]_+ = \langle x', y\rangle \leq \langle y, x\rangle_s. \tag{5.3.70}$$

From (5.3.69) and (5.3.70) the desired identity follows. □

Proposition 5.68 *The following statements are equivalent:*

(i) $\|x + \lambda y\| \geq \|x\|,\ \forall x, y \in X,\ \forall\lambda > 0$;

(ii) $[x, y]_+ \geq 0, \ \forall x, y \in X$;

(iii) $\langle y, x \rangle_s \geq 0, \ \forall x, y \in X$;

(iv) $\forall x, y \in X$, $\exists x' \in F(x)$ *such that* $\langle x', y \rangle \geq 0$.

Proof:

$(i) \Rightarrow (ii)$ If $\|x + \lambda y\| \geq \|x\|$, then $[x, y]_\lambda = \dfrac{\|x + \lambda y\| - \|x\|}{\lambda} \geq 0$. Hence $[x, y]_+ \geq 0$.

$(ii) \Rightarrow (i)$

$$\frac{\|x + \lambda y\| - \|x\|}{\lambda} = [x, y]_\lambda \geq [x, y]_+ \geq 0 \Longrightarrow \|x + \lambda y\| \geq \|x\|.$$

$(ii) \Leftrightarrow (iii)$ This follows immediately from the previous proposition, since $\langle y, x \rangle_s = \|x\| [x, y]_+$.

$(ii) \Leftrightarrow (iv)$ By Lemma 5.66, there exists $x' \in F(x)$ such that $\langle x', y \rangle = \|x\| [x, y]_+ \geq 0$.

Finally, we obtain a new characterisation of accretive operators.

Corollary 5.69 *The following statements are equivalent:*

(i) A *is an accretive operator;*

(ii) $[x_1 - x_2, y_1 - y_2]_+ \geq 0$, $\forall (x_1, y_1), (x_2, y_2) \in A$;

(iii) $\langle x_1 - x_2, y_1 - y_2 \rangle_s \geq 0$, $\forall (x_1, y_1), (x_2, y_2) \in A$;

(iv) $\forall (x_1, y_1), (x_2, y_2) \in A$, $\exists x' \in F(x_1 - x_2)$ *such that* $\langle x', y_1 - y_2 \rangle \geq 0$.

Proof: The operator A is accretive if

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\| \geq \|x_1 - x_2\| \quad \forall (x_1, y_1), (x_2, y_2) \in A, \ \forall \lambda > 0.$$

Thus it is enough to take $x = x_1 - x_2$ and $y = y_1 - y_2$ in Proposition 5.68. □

Remark 5.70

(a) If X is a Hilbert space, then the accretivity condition

$$\langle x', y_1 - y_2 \rangle \geq 0, \ x' \in F(x_1 - x_2), \tag{5.3.71}$$

coincides with the notion of monotonicity.

(b) If X is a complex vector space, condition (5.3.71) is replaced by

$$\Re \langle x', y_1 - y_2 \rangle \geq 0.$$

Example 5.71

(a) If $T : X \longrightarrow X$ is a contraction, that is,

$$\|Tx_1 - Tx_2\| \leq \|x_1 - x_2\| \quad \forall x_1, x_2 \in X,$$

then $A := I - T$ is accretive. Indeed, let $\lambda > 0$ and $x_1, x_2 \in X$. Then

$$\begin{aligned} \|x_1 - x_2 + \lambda(Ax_1 - Ax_2)\| &= \|x_1 - x_2 + \lambda(x_1 - Tx_1 - x_2 + Tx_2)\| \\ &= \|(1 + \lambda)(x_1 - x_2) - \lambda(Tx_1 - Tx_2)\| \\ &\geq (1 + \lambda)\|x_1 - x_2\| - \lambda\|Tx_1 - Tx_2\| \\ &\geq \|x_1 - x_2\|. \end{aligned}$$

(b) Since $L^p(\Omega)$ is strictly convex for $1 < p < \infty$, the duality mapping is single-valued. We claim that

$$F(u) = \|u\|_p^{2-p}\, u|u|^{p-2}, \quad \forall u \in L^p(\Omega). \tag{5.3.72}$$

The element $F(u)$ is the unique element of $L^{p'}(\Omega)$ satisfying

$$\langle F(u), u\rangle = \|u\|_p^2 = \|F(u)\|_{p'}^2. \tag{5.3.73}$$

It suffices to prove that the right-hand side of (5.3.72) belongs to $L^{p'}(\Omega)$ and satisfies (5.3.73). Indeed,

$$\begin{aligned} \big|\|u\|_p^{2-p} u|u|^{p-2}\big|^{p'} &= \|u\|_p^{(2-p)p'}|u|^{(p-1)p'} \\ &= \|u\|_p^{(2-p)\frac{p}{p-1}}|u|^p. \end{aligned}$$

Hence,

$$\begin{aligned} \big\|\|u\|_p^{2-p} u|u|^{p-2}\big\|_{p'}^{p'} &= \|u\|_p^{(2-p)\frac{p}{p-1}}\|u\|_p^p \\ &= \|u\|_p^{p'}. \end{aligned}$$

This shows that $\|u\|_p^{2-p}u|u|^{p-2} \in L^{p'}(\Omega)$ and $\big\|\|u\|_p^{2-p}u|u|^{p-2}\big\|_{p'} = \|u\|_p$.
Moreover,

$$\big\langle \|u\|_p^{2-p}u|u|^{p-2}, u\big\rangle = \|u\|_p^{2-p}\int_\Omega u^2|u|^{p-2} = \|u\|_p^2.$$

Therefore, $\|u\|_p^{2-p}u|u|^{p-2} = F(u)$.

Definition 5.72 *Let X be a Banach space, $A : X \longrightarrow X$ an operator and $\lambda \in \mathbb{R}$. Denote $J_\lambda := (I + \lambda A)^{-1}$. For $\lambda \neq 0$, we define the Yosida approximation of A by*

$$A_\lambda := \frac{1}{\lambda}(I - J_\lambda).$$

Proposition 5.73 *The following statements hold:*

(i) $D(A_\lambda) = D(J_\lambda) = Im(I + \lambda A)$ and $Im(J_\lambda) = D(A)$;

(ii) $J_\lambda = (I + \lambda A)^{-1} = \{(x + \lambda y, x), (x, y) \in A\}$;

(iii) $A_\lambda = \frac{1}{\lambda}(I - J_\lambda) = \{(x + \lambda y, y); (x, y) \in A\}$, $\lambda \neq 0$;

(iv) If $x \in J_\lambda z$, then there exists $y \in X$ such that $(x, y) \in A$ and $z = x + \lambda y$;

(v) If $\lambda \neq 0$ and $y \in A_\lambda z$, then there exists $x \in X$ such that $(x, y) \in A$ and $z = x + \lambda y$.

Proof: (i) $D(A_\lambda) = X \cap D(J_\lambda) = D(J_\lambda) = \mathrm{Im}(I + \lambda A)$, and $\mathrm{Im}(J_\lambda) = \mathrm{Im}[(I + \lambda A)^{-1}] = D(I + \lambda A) = D(A)$.

(ii) Define $B = \{(x + \lambda y, x); (x, y) \in A\}$, $\lambda \in \mathbb{R}$. Let $z = (\overline{y}, x) \in J_\lambda = (I + \lambda A)^{-1}$. Then $(x, \overline{y}) \in (I + \lambda A)$ with $x \in D(I + \lambda A) = D(A)$ and $\overline{y} \in (I + \lambda A)x$. Hence $\overline{y} = x + \lambda y$ for some $y \in Ax$, that is, $z = (x + \lambda y, x)$ with $x \in D(A)$ and $y \in Ax$. Therefore $z \in B$.

Conversely, let $z \in B$. Then $z = (x + \lambda y, x)$ for some $(x, y) \in A$. It follows that $x \in D(A)$ and $y \in Ax$. Therefore $x + \lambda y \in (I + \lambda A)x \Rightarrow (x, x + \lambda y) \in (I + \lambda A) \Rightarrow z = (x + \lambda y, x) \in (I + \lambda A)^{-1} = J_\lambda$.

(iii) Define $B = \{(x+\lambda y, y); (x, y) \in A\}$, $\lambda \neq 0$. Let $z = (\overline{y}, \overline{x}) \in A_\lambda$, then $\overline{y} \in D(A_\lambda)$ and $\overline{x} \in A_\lambda\overline{y}$. Since $D(A_\lambda) = \mathrm{Im}(I+\lambda A)$, we have $\overline{y} = x+\lambda y$ for some $x \in D(A)$ and $y \in Ax$. On the other hand, since

$A_\lambda \overline{y} = \frac{1}{\lambda}(\overline{y} - J_\lambda \overline{y})$ and $\overline{x} \in A_\lambda \overline{y}$, it follows that $\overline{x} = \frac{1}{\lambda}(\overline{y} - \xi)$ for some $\xi \in J_\lambda \overline{y}$. Hence $(x + \lambda y, \xi) \in J_\lambda$. But by (ii), $\xi = x$ and therefore $\overline{x} = \frac{1}{\lambda}(\overline{y} - \xi) = \frac{1}{\lambda}(x + \lambda y - x) = y$. Thus $z = (\overline{y}, \overline{x}) = (x + \lambda y, y)$ for some $(x, y) \in A \Rightarrow z \in B$.

Conversely, let $z \in B$. Then $z = (x + \lambda y, y)$ for some $(x, y) \in A$. We must show that $x + \lambda y \in D(A_\lambda)$ and $y \in A_\lambda(x + \lambda y)$. Indeed, by (i) we have $D(A_\lambda) = \text{Im}(I + \lambda A)$ and since $x + \lambda y \in (I + \lambda A)x$, it follows that $x + \lambda y \in D(A_\lambda)$.

Moreover, $A_\lambda(x + \lambda y) = \frac{1}{\lambda}(I - J_\lambda)(x + \lambda y)$. By item (ii), $(x + \lambda y, x) \in J_\lambda$, so $x \in J_\lambda(x + \lambda y)$ and therefore

$$y = \frac{1}{\lambda}(x + \lambda y - x) \in \frac{1}{\lambda}(I - J_\lambda)(x + \lambda y) = A_\lambda(x + \lambda y).$$

We conclude that $z = (x + \lambda y, y) \in A_\lambda$.

(iv) Let $x \in J_\lambda z$. Then $J_\lambda z \neq \emptyset$ and therefore $z \in D(J_\lambda)$. Hence $(z, x) \in J_\lambda = \{(x + \lambda y, x); (x, y) \in A\}$. Consequently, $(z, x) = (\overline{x} + \lambda y, \overline{x})$ for some $(\overline{x}, y) \in A$. Thus $z = \overline{x} + \lambda y$, $x = \overline{x}$ and therefore $z = x + \lambda y$ for some $y \in Ax$.

(v) Let $\lambda \neq 0$ and $y \in A_\lambda z$. Then $z \in D(A_\lambda)$ and $(z, y) \in A_\lambda = \{(x + \lambda \overline{y}, \overline{y}); (x, \overline{y}) \in A\}$. Hence there exists $(x, \overline{y}) \in A$ such that $(z, y) = (x + \lambda \overline{y}, \overline{y})$, that is, $z = x + \lambda \overline{y}$ and $y = \overline{y}$ and, therefore, $z = x + \lambda y$ with $(x, y) \in A$.

Notation: For simplicity, we denote by D_λ the set $\text{Im}(I + \lambda A) = D(J_\lambda) = D(A_\lambda)$.

Remark 5.74 *Let $J_\lambda : D_\lambda \to D(A) \subset X$ and consider $z = x + \lambda y$, with $(x, y) \in A$. Then $z \in D_\lambda$ and, by item (ii) of Proposition 5.73, we have $(z, x) = (x + \lambda y, x) \in J_\lambda$, or equivalently, $x \in J_\lambda z$. If J_λ is single-valued, then $x = J_\lambda z = J_\lambda(x + \lambda y)$. Similarly, if A_λ is single-valued, then $y = A_\lambda z = A_\lambda(x + \lambda y)$.*

Proposition 5.75 *Let $A : X \to X$ be an accretive operator. Then:*

(i) J_λ is a single-valued operator;

(ii) If $\lambda > 0$, A_λ is single-valued;

(iii) If $z \in D_\lambda$ then $(J_\lambda z, A_\lambda z) \in A$, for all $\lambda > 0$.

Proof:

(i) Let $\lambda \geq 0$, $z \in D_\lambda$ and $x_1, x_2 \in J_\lambda z$. We want to show that $x_1 = x_2$. In fact, by item (iv) of Proposition 5.73, there exist $y_1 \in Ax_1$ and $y_2 \in Ax_2$ such that

$$z = x_1 + \lambda y_1 = x_2 + \lambda y_2 \Rightarrow 0 = z - z = (x_1 - x_2) + \lambda(y_1 - y_2).$$

If $\lambda > 0$, the accretivity of A yields

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\| = \|z - z\| = 0.$$

Therefore $x_1 = x_2$. The case $\lambda = 0$ is immediate.

(ii) Let $\lambda > 0$. Then $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ is single-valued since both I and J_λ are.

(iii) Let $\lambda > 0$ and $z \in D_\lambda$. Then there exists $(x, y) \in A$ such that $z = x + \lambda y$. By Observation 5.74 and since J_λ and A_λ are single-valued, we have $J_\lambda z = x$ and $A_\lambda z = y$. Thus $(J_\lambda z, A_\lambda z) \in A$. □

Proposition 5.76 *The operator $A : X \to X$ is accretive if and only if J_λ is a contraction for every $\lambda \geq 0$.*

Proof: Let $A : X \to X$ be an accretive operator and take $z_1, z_2 \in D_\lambda$, $\lambda \geq 0$.

We have $z_1 = x_1 + \lambda y_1$ for some $(x_1, y_1) \in A$ and $z_2 = x_2 + \lambda y_2$ for some $(x_2, y_2) \in A$. Since A is accretive, it follows from Proposition 5.75(i) that J_λ is single-valued, so $J_\lambda z_1 = x_1$, $J_\lambda z_2 = x_2$ and moreover

$$\|J_\lambda z_1 - J_\lambda z_2\| = \|x_1 - x_2\| \leq \|(x_1 - x_2) + \lambda(y_1 - y_2)\| = \|z_1 - z_2\|,$$

which proves that J_λ is a contraction for $\lambda \geq 0$.

Conversely, suppose that J_λ is a contraction for $\lambda \geq 0$.

Let $(x_1, y_1),\ (x_2, y_2) \in A$. Then $z_1 = x_1 + \lambda y_1,\ z_2 = x_2 + \lambda y_2 \in D_\lambda$. Since J_λ is a contraction, it is single-valued, and thus, by Observation 5.74, $J_\lambda z_1 = x_1$ and $J_\lambda z_2 = x_2$. Consequently,

$$\|x_1 - x_2\| = \|J_\lambda z_1 - J_\lambda z_2\| \leq \|z_1 - z_2\| = \|(x_1 - x_2) + \lambda(y_1 - y_2)\|,\ \forall \lambda \geq 0,$$

in particular for $\lambda > 0$, which shows that A is accretive. □

Notation: Let $\omega \in \mathbb{R}$. We denote by $\mathcal{A}(\omega)$ the class of operators $A : X \to X$ such that $A + \omega I$ is accretive. Therefore $\mathcal{A}(0)$ is the class of accretive operators.

Proposition 5.77 *$A \in \mathcal{A}(\omega)$ if and only if for every $(x_1, y_1),\ (x_2, y_2) \in A$ there exists $x' \in F(x_1 - x_2)$ such that*

$$\langle x', y_1 - y_2 \rangle + \omega \|x_1 - x_2\|^2 \geq 0. \tag{5.3.74}$$

Proof: $(\Rightarrow)$ Let $A \in \mathcal{A}(\omega)$ and $(x_1, y_1),\ (x_2, y_2) \in A$. Since $A + \omega I$ is accretive by hypothesis, it follows from Corollary 5.69(iv) that there exists $x' \in F(x_1 - x_2)$ such that

$$\langle x', y_1 + \omega x_1 - (y_2 + \omega x_2) \rangle \geq 0.$$

Hence

$$\langle x', y_1 - y_2 \rangle + \omega \langle x', x_1 - x_2 \rangle \geq 0. \tag{5.3.75}$$

Since $x' \in F(x_1 - x_2)$ we obtain

$$\langle x', y_1 - y_2 \rangle + \omega \|x_1 - x_2\|^2 \geq 0. \tag{5.3.76}$$

$(\Leftarrow)$ Conversely, suppose that for every $(x_1, y_1),\ (x_2, y_2) \in A$ there exists $x' \in F(x_1 - x_2)$ such that (5.3.74) holds. We prove that $A \in \mathcal{A}(\omega)$. Indeed, let $(x_1, z_1),\ (x_2, z_2) \in A + \omega I$. Then $z_1 = y_1 + \omega x_1$ and $z_2 = y_2 + \omega x_2$, where $y_1 \in Ax_1$ and $y_2 \in Ax_2$.

By hypothesis, there exists $x' \in F(x_1 - x_2)$ such that

$$\langle x', y_1 - y_2 \rangle + \omega \|x_1 - x_2\|^2 \geq 0,$$

that is,

$$\langle x', (y_1 + \omega x_1) - (y_2 + \omega x_2) \rangle \geq 0 \Rightarrow \langle x', z_1 - z_2 \rangle \geq 0.$$

By Corollary 5.69(iv), it follows that $A + \omega I$ is accretive. □

Remark 5.78 *(i) Let $\omega \leq 0$ and $A \in \mathcal{A}(\omega)$. Take $(x_1, y_1),\ (x_2, y_2) \in A$. Then, by Proposition 5.77, there exists $x' \in F(x_1 - x_2)$ such that*

$$\langle x', y_1 - y_2 \rangle + \omega \|x_1 - x_2\|^2 \geq 0,$$

that is,

$$\langle x', y_1 - y_2 \rangle \geq -\omega \|x_1 - x_2\|^2.$$

Consequently, by Corollary 5.69(iv), A is accretive.

(ii) Let $\omega \geq 0$ and $A \in \mathcal{A}(\omega)$. Take $(x_1, y_1),\ (x_2, y_2) \in A$. By Corollary 5.69(iv), there exists $x' \in F(x_1 - x_2)$ such that

$$\langle x', y_1 - y_2 \rangle \geq 0 \Rightarrow \langle x', y_1 - y_2 \rangle + \omega \|x_1 - x_2\|^2 \geq 0.$$

Therefore, by Proposition 5.77, we have $A \in \mathcal{A}(\omega)$.

In summary:

- $\omega \leq 0 \Rightarrow \mathcal{A}(\omega) \subset \mathcal{A}(0)$
- $\omega \geq 0 \Rightarrow \mathfrak{A}(0) \subset \mathfrak{A}(\omega)$

Moreover, if

$$0 < \omega_1 < \omega_2 \Rightarrow \mathcal{A}(\omega_1) \subset \mathcal{A}(\omega_2). \tag{5.3.77}$$

Indeed, let $A \in \mathcal{A}(\omega_1)$. By Proposition 5.77 we obtain that for all $(x_1, y_1),\ (x_2, y_2) \in A$ there exists $x' \in F(x_1 - x_2)$ such that

$$\langle x', y_1 - y_2 \rangle + \omega_1 \|x_1 - x_2\|^2 \geq 0 \Rightarrow \langle x', y_1 - y_2 \rangle + \omega_2 \|x_1 - x_2\|^2 \geq 0,$$

and by Proposition 5.77 it follows that $A \in \mathcal{A}(\omega_2)$.

Theorem 5.79 *Let $\omega \in \mathbb{R}$ and $\lambda \geq 0$ with $\lambda\omega < 1$ and $A \in \mathcal{A}(\omega)$. Then:*

i) J_λ (and consequently A_λ) is single-valued and Lipschitz with constant $(1 - \lambda\omega)^{-1}$;

ii) $\|J_\lambda x - x\| \leq \lambda(1 - \lambda\omega)^{-1}|Ax|$ for every $x \in D(A) \cap D_\lambda$, where

$$|Ax| := \inf\{\|y\|;\ y \in Ax\};$$

iii) If $n \in \mathbb{N}$, $x \in D(J_\lambda^n)$ and $\lambda|\omega| < 1$, then

$$\|J_\lambda^n x - x\| \leq n(1 - \lambda|\omega|)^{-n+1}\|J_\lambda x - x\|;$$

iv) If $x \in D_\lambda$, with $\lambda \neq 0$ and $\mu \in \mathbb{R}$, then $\left(\frac{\mu}{\lambda}\right)x + \frac{(\lambda - \mu)}{\lambda}J_\lambda x \in D_\mu$ and, moreover,

$$J_\mu\left(\frac{\mu}{\lambda}x + \frac{(\lambda - \mu)}{\lambda}J_\lambda x\right) = J_\lambda x;$$

v) $A_\lambda \in \mathcal{A}\left(\frac{\omega}{1 - \lambda\omega}\right)$;

vi) $\|A_\lambda x - A_\lambda y\| \leq \lambda^{-1}\big[1 + (1 - \lambda|\omega|)^{-1}\big]\|x - y\|$ for all $x, y \in D_\lambda$;

vii) If $x \in D_\lambda \cap D_\mu$ and $0 < \mu \leq \lambda$, then $(1 - \lambda\omega)\|A_\lambda x\| \leq (1 - \mu\omega)\|A_\mu x\|$;

viii) $\lim\limits_{\lambda \to 0^+} J_\lambda x = x$, for all $x \in \overline{D(A) \cap \bigcap\limits_{\lambda > 0} D_\lambda}$.

Proof:

i) Let $z \in D_\lambda = D(J_\lambda)$ and $x_1, x_2 \in J_\lambda z$. We first show that $x_1 = x_2$. Indeed, by Proposition 5.73(iv), there exist $y_1 \in Ax_1$ and $y_2 \in Ax_2$ such that

$$z = x_1 + \lambda y_1 = x_2 + \lambda y_2.$$

Let $x' \in F(x_1 - x_2)$. Then

$$\begin{aligned} 0 &= \langle x', 0\rangle = \langle x', z - z\rangle = \langle x', (x_1 - x_2) + \lambda(y_1 - y_2)\rangle \\ &= \langle x', x_1 - x_2\rangle + \lambda\langle x', y_1 - y_2 + \omega x_1 - \omega x_1 + \omega x_2 - \omega x_2\rangle \\ &= \langle x', x_1 - x_2\rangle + \lambda\langle x', (y_1 + \omega x_1) - (y_2 + \omega x_2)\rangle - \lambda\omega\langle x', x_1 - x_2\rangle \\ &= \|x_1 - x_2\|^2 + \lambda\langle x', (y_1 + \omega x_1) - (y_2 + \omega x_2)\rangle - \lambda\omega\|x_1 - x_2\|^2 \\ &= \underbrace{(1 - \lambda\omega)}_{>0}\|x_1 - x_2\|^2 + \lambda\langle x', (y_1 + \omega x_1) - (y_2 + \omega x_2)\rangle. \end{aligned} \tag{5.3.78}$$

On the other hand, since $A \in \mathcal{A}(\omega)$, we have that $A + \omega I$ is accretive and therefore, by Corollary 5.69(iv), there exists $\xi' \in F(x_1 - x_2)$ such that

$$\langle \xi', (y_1 + \omega x_1) - (y_2 + \omega x_2)\rangle \geq 0.$$

Taking $x' = \xi'$ in (5.3.78) we obtain

$$(1 - \lambda\omega)\|x_1 - x_2\|^2 = -\lambda\langle \xi', (y_1 + \omega x_1) - (y_2 + \omega x_2)\rangle \leq 0,$$

which implies $\|x_1 - x_2\| \leq 0$, hence $x_1 = x_2$, proving that J_λ is single-valued.

It follows that A_λ is also single-valued, since $A_\lambda = \dfrac{I - J_\lambda}{\lambda}$.

It remains to prove that J_λ is Lipschitz. Since $A + \omega I$ is accretive, by Proposition 5.76 its resolvent

$$J_t^{A+\omega I} = [I + t(A + \omega I)]^{-1},$$

is a contraction for all $t \geq 0$. Let $t > 0$ so that $1 + \omega t \neq 0$. Then

$$\begin{aligned} J_t^{A+\omega I} &= [I + t(A + \omega I)]^{-1} = [(1 + \omega t)I + tA]^{-1} \\ &= \left[(1 + \omega t)\left(I + \frac{t}{1 + \omega t}A\right)\right]^{-1} = \left(I + \frac{t}{1 + \omega t}A\right)^{-1}(1 + \omega t)^{-1}. \end{aligned}$$

Hence

$$J_t^{A+\omega I} = \left(I + \frac{t}{1 + \omega t}A\right)^{-1}(1 + \omega t)^{-1},$$

or equivalently,

$$(1 + \omega t)J_t^{A+\omega I} = \left(I + \frac{t}{1 + \omega t}A\right)^{-1}. \tag{5.3.79}$$

Since $J_t^{A+\omega I}$ is a contraction, it follows from (5.3.79), for $x, y \in D_t^{A+\omega I} = D(J_t^{A+\omega I}) = \mathrm{Im}[I + t(A + \omega I)]$, that

$$\begin{aligned} \left\|\left(I + \frac{t}{1 + \omega t}A\right)^{-1}x - \left(I + \frac{t}{1 + \omega t}A\right)^{-1}y\right\| &= \|(1 + \omega t)J_t^{A+\omega I}x - (1 + \omega t)J_t^{A+\omega I}y\| \\ &\leq |1 + \omega t|\|x - y\|, \end{aligned} \tag{5.3.80}$$

so $\left(I + \dfrac{t}{1 + \omega t}A\right)^{-1}$ is Lipschitz with constant $|1 + \omega t|$.

Let $\lambda > 0$ and set

$$t = \frac{\lambda}{\underbrace{1 - \lambda\omega}_{>0}} > 0,$$

which implies $\dfrac{t}{1+\omega t} = \lambda$.

Moreover,

$$1+\omega t = 1+\omega\left(\frac{\lambda}{1-\lambda\omega}\right) = 1+\frac{\omega\lambda}{1-\lambda\omega} = \frac{1-\lambda\omega+\omega\lambda}{1-\lambda\omega} = (1-\lambda\omega)^{-1},$$

so $|1+\omega t| = (1-\lambda\omega)^{-1}$. Thus $(1+\lambda A)^{-1} = J_\lambda$ is Lipschitz with constant $(1-\lambda\omega)^{-1}$.

If $\lambda = 0$, then $J_\lambda = I$ and the result is immediate.

ii) Consider the set $V_\lambda = D(A)\cap D_\lambda$. If $V_\lambda = \emptyset$, the result holds trivially, so there is nothing to prove. Suppose then $V_\lambda \neq \emptyset$. Let $y \in Ax$; then $x+\lambda y \in D_\lambda = \operatorname{Im}(I+\lambda A)$ and, moreover, by item (i), J_λ is single-valued. Hence, by Observation 5.74, $J_\lambda(x+\lambda y) = x$.

Thus

$$\begin{aligned}\|J_\lambda x - x\| &= \|J_\lambda x - J_\lambda(x+\lambda y)\|\\ &\le (1-\lambda\omega)^{-1}\|x-(x+\lambda y)\|\\ &= (1-\lambda\omega)^{-1}\lambda\|y\|.\end{aligned}$$

By the arbitrariness of $y\in Ax$ we obtain

$$\|J_\lambda x - x\| \le (1-\lambda\omega)^{-1}\lambda\|y\|, \quad \forall\, y\in Ax.$$

Setting $|Ax| = \inf\limits_{y\in Ax}\{\|y\|\}$, we clearly have $0\le |Ax| < +\infty$. Therefore

$$\|J_\lambda x - x\| \le \lambda(1-\lambda\omega)^{-1}|Ax|.$$

iii) Let $n\in\mathbb{N}$, $\lambda\ge 0$ with $\lambda|\omega|<1$ and $x\in D(J_\lambda^n)$. Then

$$\begin{aligned}\|J_\lambda^n x - x\| &= \|J_\lambda^n x - J_\lambda^{n-1}x + J_\lambda^{n-1}x - J_\lambda^{n-2}x + J_\lambda^{n-2}x - \cdots - J_\lambda x + J_\lambda x - x\|\\ &= \left\|\sum_{i=1}^{n}\left(J_\lambda^{n-i+1}x - J_\lambda^{n-i}x\right)\right\|. \end{aligned}\tag{5.3.81}$$

But

$$\begin{aligned}\|J_\lambda^{n-i+1}x - J_\lambda^{n-i}x\| &= \|J_\lambda(J_\lambda^{n-i}x) - J_\lambda(J_\lambda^{n-i-1}x)\|\\ &\overset{(i)}{\le} (1-\lambda\omega)^{-1}\|J_\lambda^{n-i}x - J_\lambda^{n-i-1}x\|, \quad \forall i = 1,\dots,n-1.\end{aligned}$$

By induction, after another $(n-i-1)$ steps we obtain

$$\begin{aligned}\|J_\lambda^{n-i+1}x - J_\lambda^{n-i}x\| &\le (1-\lambda\omega)^{-1}[(1-\lambda\omega)^{-1}]^{n-i-1}\|J_\lambda x - x\|\\ &= [(1-\lambda\omega)^{-1}]^{n-i}\|J_\lambda x - x\|\\ &= (1-\lambda\omega)^{-n+i}\|J_\lambda x - x\|.\end{aligned}\tag{5.3.82}$$

On the other hand, $\lambda\omega \le \lambda|\omega|$, which implies

$$1-\lambda\omega \ge 1-\lambda|\omega| \Longrightarrow (1-\lambda\omega)^{-1} \le (1-\lambda|\omega|)^{-1},$$

and consequently

$$(1-\lambda\omega)^{-n+i} \le (1-\lambda|\omega|)^{-n+i}.\tag{5.3.83}$$

We also have $i\ge 1$ and thus $n-i\le n-1$. Observing that

$$0\le \lambda|\omega| < 1 \Longrightarrow -1 < -\lambda|\omega| \le 0 \Longrightarrow 0 < 1-\lambda|\omega| \le 1,$$

we obtain

$$\frac{1}{1-\lambda|\omega|} \geq 1.$$

Hence

$$[(1-\lambda|\omega|)^{-1}]^{n-i} \leq [(1-\lambda|\omega|)^{-1}]^{n-1},$$

so

$$(1-\lambda|\omega|)^{-n+i} \leq (1-\lambda|\omega|)^{-n+1}. \tag{5.3.84}$$

From (5.3.82), (5.3.83) and (5.3.84) we have

$$\|J_\lambda^{n-i+1}x - J_\lambda^{n-i}x\| \leq (1-\lambda|\omega|)^{-n+1}\|J_\lambda x - x\|,$$

and therefore, from (5.3.81),

$$\begin{aligned}\|J_\lambda^n x - x\| &\leq \sum_{i=1}^{n} \|J_\lambda^{n-i+1}x - J_\lambda^{n-i}x\| \\ &\leq \sum_{i=1}^{n}(1-\lambda|\omega|)^{-n+1}\|J_\lambda x - x\| \\ &= n(1-\lambda|\omega|)^{-n+1}\|J_\lambda x - x\|.\end{aligned}$$

iv) Let $\lambda > 0$, $x \in D_\lambda$ and $\mu \in \mathbb{R}$. Since $x \in D_\lambda = \operatorname{Im}(I+\lambda A)$, there exists $(x_1, y_1) \in A$ such that $x = x_1 + \lambda y_1$. Also, $x_1 + \mu y_1 \in D_\mu = \operatorname{Im}(I+\mu A)$. Hence $J_\lambda x = x_1$. Thus

$$\begin{aligned}\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x &= \frac{\mu}{\lambda}(x_1+\lambda y_1) + \frac{\lambda-\mu}{\lambda}x_1 \\ &= \frac{\mu}{\lambda}x_1 + \mu y_1 + x_1 - \frac{\mu}{\lambda}x_1 \\ &= x_1 + \mu y_1 \in D_\mu.\end{aligned}$$

Moreover,

$$J_\lambda x = x_1 = J_\mu(x_1+\mu y_1) = J_\mu\left(\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x\right),$$

as claimed.

v) We prove that $A_\lambda + \dfrac{\omega}{1-\lambda\omega}I$ is accretive; thus we must show that for all $x_1, x_2 \in D_\lambda$ and all $t \geq 0$,

$$\left\|(x_1-x_2) + t\left[\left(A_\lambda + \frac{\omega}{1-\lambda\omega}I\right)x_1 - \left(A_\lambda + \frac{\omega}{1-\lambda\omega}I\right)x_2\right]\right\| \geq \|x_1-x_2\|. \tag{5.3.85}$$

Indeed,

$$\begin{aligned}&\left\|(x_1-x_2) + tA_\lambda x_1 + \frac{t\omega}{1-\lambda\omega}x_1 - tA_\lambda x_2 + \frac{t\omega}{1-\lambda\omega}x_2\right\| \\ =\ &\left\|(x_1-x_2) + \frac{t\omega}{1-\lambda\omega}(x_1-x_2) + \frac{t}{\lambda}(I-J_\lambda)x_1 - \frac{t}{\lambda}(I-J_\lambda)x_2\right\| \\ =\ &\left\|\left[1+\frac{t\omega}{1-\lambda\omega}+\frac{t}{\lambda}\right](x_1-x_2) - \frac{t}{\lambda}[J_\lambda x_1 - J_\lambda x_2]\right\|.\end{aligned}$$

Moreover,

$$1+\frac{t\omega}{1-\lambda\omega}+\frac{t}{\lambda} = 1+\frac{\lambda t\omega + t(1-\lambda\omega)}{\lambda(1-\lambda\omega)} = 1+\frac{\lambda t\omega + t - t\lambda\omega}{\lambda(1-\lambda\omega)} = 1+\frac{t}{\lambda(1-\lambda\omega)} > 0.$$

From this and (5.3.86) we can write

$$\begin{aligned}
&\left\|(x_1-x_2)+tA_\lambda x_1+\frac{t\omega}{1-\lambda\omega}x_1-tA_\lambda x_2+\frac{t\omega}{1-\lambda\omega}x_2\right\| \\
\geq\ &\left(1+\frac{t}{\lambda(1-\lambda\omega)}\right)\|x_1-x_2\|-\frac{t}{\lambda}\|J_\lambda x_1-J_\lambda x_2\|\,. \qquad (5.3.86)
\end{aligned}$$

Now, by item (i),

$$\|J_\lambda x_1-J_\lambda x_2\|\leq(1-\lambda\omega)^{-1}\|x_1-x_2\|\,,$$

and consequently

$$-\|J_\lambda x_1-J_\lambda x_2\|\geq-(1-\lambda\omega)^{-1}\|x_1-x_2\|\,. \qquad (5.3.87)$$

From (5.3.86) and (5.3.87) we obtain

$$\begin{aligned}
&\left\|(x_1-x_2)+tA_\lambda x_1+\frac{t\omega}{1-\lambda\omega}x_1-tA_\lambda x_2+\frac{t\omega}{1-\lambda\omega}x_2\right\| \\
\geq\ &\left(1+\frac{t}{\lambda(1-\lambda\omega)}\right)\|x_1-x_2\|-\frac{t}{\lambda(1-\lambda\omega)}\|x_1-x_2\| \\
=\ &\|x_1-x_2\|\,,
\end{aligned}$$

which proves (5.3.85).

vi) Let $\lambda>0$ and $x,\,y\in D_\lambda=\mathrm{Im}(I+\lambda A)$. Then

$$\begin{aligned}
\|A_\lambda x-A_\lambda y\| &= \left\|\frac{1}{\lambda}(I-J_\lambda)\,x-\frac{1}{\lambda}(I-J_\lambda)\,y\right\|=\frac{1}{\lambda}\|(x-y)-(J_\lambda x-J_\lambda y)\| \\
&\leq \frac{1}{\lambda}\left[\|x-y\|+\|J_\lambda x-J_\lambda y\|\right].
\end{aligned}$$

By item (i), since J_λ is Lipschitz, we have

$$\begin{aligned}
\|A_\lambda x-A_\lambda y\| &\leq \frac{1}{\lambda}\|x-y\|+\frac{1}{\lambda(1-\lambda\omega)}\|x-y\| \\
&= \lambda^{-1}\left[1+(1-\lambda\omega)^{-1}\right]\|x-y\|.
\end{aligned}$$

vii) Let $\mu,\lambda\in\mathbb{R}$ with $0<\mu\leq\lambda$. If $D_\mu\cap D_\lambda=\emptyset$, there is nothing to prove. Suppose $D_\mu\cap D_\lambda\neq\emptyset$ and let $x\in D_\mu\cap D_\lambda$. Then

$$\begin{aligned}
\|A_\lambda x\| &= \frac{1}{\lambda}\|(I-J_\lambda)x\|=\frac{1}{\lambda}\|x-J_\lambda x+J_\mu x-J_\mu x\| \\
&\leq \frac{1}{\lambda}\left[\|x-J_\mu x\|+\|J_\lambda x-J_\mu x\|\right]. \qquad (5.3.88)
\end{aligned}$$

Since $A_\mu=\frac{1}{\mu}(I-J_\mu)$, we have $\mu A_\mu=(I-J_\mu)$, and from (5.3.88) we obtain

$$\|A_\lambda x\|\leq\frac{1}{\lambda}\left[\mu\|A_\mu x\|+\|J_\lambda x-J_\mu x\|\right]. \qquad (5.3.89)$$

On the other hand, by item (iv),

$$J_\lambda x=J_\mu\left(\frac{\mu}{\lambda}x+\frac{\lambda-\mu}{\lambda}J_\lambda x\right),$$

and by item (i) J_μ is Lipschitz; hence

$$\begin{aligned}
\|J_\lambda x - J_\mu x\| &= \left\|J_\mu\left(\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x\right) - J_\mu x\right\| \\
&\le (1-\mu\omega)^{-1}\left\|\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x - x\right\| \\
&= (1-\mu\omega)^{-1}\left\|\frac{\lambda-\mu}{\lambda}(J_\lambda x - x)\right\| \\
&= (1-\mu\omega)^{-1}(\lambda-\mu)\left\|\frac{1}{\lambda}(J_\lambda - I)x\right\| \\
&= (1-\mu\omega)^{-1}(\lambda-\mu)\|A_\lambda x\|.
\end{aligned}$$

Substituting into (5.3.89), we get

$$\|A_\lambda x\| \le \frac{1}{\lambda}\left[\mu\|A_\mu x\| + (1-\mu\omega)^{-1}(\lambda-\mu)\|A_\lambda x\|\right].$$

Multiplying by $\lambda(1-\mu\omega) > 0$ yields

$$\lambda(1-\mu\omega)\|A_\lambda x\| \le \mu(1-\mu\omega)\|A_\mu x\| + (\lambda-\mu)\|A_\lambda x\|,$$

and since $\lambda(1-\mu\omega) - (\lambda-\mu) = \lambda - \lambda\mu\omega - \lambda + \mu = \mu(1-\lambda\omega)$, we obtain

$$\mu(1-\lambda\omega)\|A_\lambda x\| \le \mu(1-\mu\omega)\|A_\mu x\|,$$

which implies

$$(1-\lambda\omega)\|A_\lambda x\| \le (1-\mu\omega)\|A_\mu x\|.$$

viii) Consider the set $D(A) \cap \bigcap_{\lambda>0} D_\lambda$. If this set is empty, there is nothing to prove. Suppose it is non-empty and take $x \in D(A) \cap \bigcap_{\lambda>0} D_\lambda$. By item (ii) we have

$$\|J_\lambda x - x\| \le \lambda(1-\lambda\omega)^{-1}|Ax|, \qquad \lambda > 0.$$

Taking the limit as $\lambda \to 0^+$ we obtain

$$\lim_{\lambda\to 0^+}\|J_\lambda x - x\| = 0,$$

that is,

$$\lim_{\lambda\to 0^+} J_\lambda x = x,\ \forall x \in D(A) \cap \bigcap_{\lambda>0} D_\lambda. \tag{5.3.90}$$

Now let $x \in \overline{D(A) \cap \bigcap_{\lambda>0} D_\lambda}$ and $\varepsilon > 0$. Choose $y \in D(A) \cap \bigcap_{\lambda>0} D_\lambda$ such that

$$\|x - y\| < \frac{\varepsilon}{2}. \tag{5.3.91}$$

Then, for this $x \in \overline{D(A) \cap \bigcap_{\lambda>0} D_\lambda}$ we have

$$^{1}\ \|J_\lambda x - x\| \le \|J_\lambda x - J_\lambda y\| + \|J_\lambda y - y\| + \|y - x\|. \tag{5.3.92}$$

By (i),

$$\|J_\lambda x - J_\lambda y\| \le (1-\lambda\omega)^{-1}\|x - y\|. \tag{5.3.93}$$

[1] Since x may not belong to the domain of J_λ, we are considering its extension defined as $\lim_{y\to x} J_\lambda y$, $y \in D_\lambda$, which is still Lipschitz.

Combining (5.3.92) and (5.3.93) we obtain

$$\|J_\lambda x - x\| \le \left[(1-\lambda\omega)^{-1} + 1\right]\|x-y\| + \|J_\lambda y - y\|\,. \tag{5.3.94}$$

Since $y \in D(A) \cap \bigcap_{\lambda>0} D_\lambda$, as $\lambda \to 0^+$ we have

$$0 \le \lim_{\lambda\to 0^+} \|J_\lambda x - x\| \le 2\|x-y\|,$$

and, by (5.3.91),

$$0 \le \lim_{\lambda\to 0^+} \|J_\lambda x - x\| \le \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, the result follows.

Example 5.80 Let $T : X \longrightarrow X$ be Lipschitz with constant α, that is,

$$\|Tx_1 - Tx_2\| \le \alpha\|x_1 - x_2\|, \ \forall x_1,\ x_2 \in D(T).$$

Then, for $t > 0$, we have $\dfrac{I-T}{t} \in \mathcal{A}\left(\dfrac{\alpha-1}{t}\right)$, that is,

$$\left\|(x_1-x_2) + \lambda\left[\left(\frac{I-T}{t} + \frac{\alpha-1}{t}I\right)x_1 - \left(\frac{I-T}{t} + \frac{\alpha-1}{t}I\right)x_2\right]\right\| \ge \|x_1-x_2\|. \tag{5.3.95}$$

Indeed,

$$\begin{aligned}
&\left\|(x_1-x_2) + \lambda\left[\left(\frac{I-T}{t} + \frac{\alpha-1}{t}I\right)x_1 - \left(\frac{I-T}{t} + \frac{\alpha-1}{t}I\right)x_2\right]\right\| \\
= &\left\|(x_1-x_2) + \lambda\left[\frac{x_1 - Tx_1 + \alpha x_1 - x_1}{t} - \frac{x_2 - Tx_2 + \alpha x_2 - x_2}{t}\right]\right\| \\
= &\left\|(x_1-x_2) + \frac{\lambda}{t}\left[\alpha(x_1-x_2) - (Tx_1 - Tx_2)\right]\right\| \\
\ge &\left\|\left(1 + \frac{\lambda\alpha}{t}\right)(x_1-x_2)\right\| - \frac{\lambda}{t}\|Tx_1 - Tx_2\| \\
\ge &\left(1 + \frac{\lambda\alpha}{t}\right)\|x_1-x_2\| - \frac{\lambda}{t}\alpha\|x_1-x_2\| \\
= &\|x_1-x_2\|
\end{aligned}$$

which proves the claim.

Note that if $T : X \longrightarrow X$ is non-expansive, then $I - T$ is an accretive operator (Example 5.71). In fact, since T is non-expansive, T is Lipschitz with constant $\alpha = 1$. From the argument above, taking $t = 1$, we obtain that $I - T$ is accretive.

Remark 5.81 *Let $\omega \in \mathbb{R}$ and $A \in \mathcal{A}(\omega)$. Consider*

$$\mathcal{D} = \bigcup_{\mu>0}\left(\bigcap_{0<\lambda<\mu} D_\lambda\right).$$

Take $x \in \mathcal{D}$, $\lambda_0 \in \mathbb{R}$ such that $\lambda_0\omega < 1$. The map

$$\begin{aligned} g_x :\ (0,\lambda_0) &\longrightarrow \mathbb{R} \\ \lambda &\longmapsto g_x(\lambda) = (1-\lambda\omega)\|A_\lambda x\| \end{aligned}$$

is decreasing on the interval $(0,\lambda_0)$, since by item (vii) of Theorem (5.79) we have that, if $0 < \mu \le \lambda < \lambda_0$, *then*

$$(1-\lambda\omega)\|A_\lambda x\| \le (1-\mu\omega)\|A_\mu x\|.$$

This function admits a limit as $\lambda \to 0^+$. Define

$$|||Ax||| = \lim_{\lambda\to 0_+} g_x(\lambda) = \sup_{\lambda>0}(1-\lambda\omega)\|A_\lambda x\|. \tag{5.3.96}$$

(Observe that nothing prevents us from having $|||Ax||| = +\infty$).

Proposition 5.82 *Let $A \in \mathcal{A}(\omega)$ and $\lambda > 0$ such that $\lambda\omega < 1$. Then*

(i) $\|A_\lambda x\| \le (1-\lambda\omega)^{-1}|||Ax|||,\ \forall x \in \mathcal{D}\cap D_\lambda$;

(ii) $\lim_{\lambda\to 0^+}\|A_\lambda x\| = |||Ax|||$;

(iii) *If $D(A) \subset \mathcal{D}$ and $x \in D(A)$, then $|||Ax||| \le |Ax|$.*

Proof:

(i) We have that, if $\lambda \in (0,\lambda_0)$ and $0<\xi<\lambda$ in such a way that $x \in \bigcap_{0<\xi<\lambda} D_\xi$, then

$$|||Ax||| = \sup_{\xi>0}(1-\xi\omega)\|A_\xi x\| \ge (1-\lambda\omega)\|A_\lambda x\|.$$

Thus, if $\lambda \in (0,\lambda_0)$, there is nothing else to prove. If $\lambda \ge \lambda_0$, by item (vii) of Theorem 5.79 it follows that

$$(1-\lambda\omega)\|A_\lambda x\| \le (1-\lambda_0\omega)\|A_{\lambda_0}x\| \le |||Ax|||.$$

(ii) Let $x \in \mathcal{D}$.

Case 1: $|||Ax||| < \infty$.

$$\|A_\lambda x\| = (1-\lambda\omega)^{-1}(1-\lambda\omega)\|A_\lambda x\| \quad \Rightarrow \quad \lim_{\lambda\to 0^+}\|A_\lambda x\| = |||Ax|||.$$

Case 2: $|||Ax||| = \infty$.

Let $M>0$. There exists $\lambda_0>0$ such that $(1-\lambda_0\omega)\|A_{\lambda_0}x\| \ge M$. If $0<\lambda<\lambda_0$, then

$$\|A_\lambda x\| \ge \frac{(1-\lambda_0\omega)}{1-\lambda\omega}\|A_{\lambda_0}x\| \ge M(1-\lambda\omega)^{-1}.$$

Hence, $\lim_{\lambda\to 0^+}\|A_\lambda x\| \ge M$. Since $M>0$ is arbitrary, the result follows.

(iii) By item (ii) of Theorem 5.79, we have

$$\|J_\mu x - x\| \le \mu(1-\mu\omega)^{-1}|Ax|, \quad \forall x \in D(A)\cap D_\mu. \tag{5.3.97}$$

Hence,

$$(1-\mu\omega)\|A_\mu x\| \le |Ax|, \quad \forall x \in D(A)\cap D_\mu. \tag{5.3.98}$$

By hypothesis, $x \in D(A)\cap D_\mu$ for every $\mu\omega<1$, $0<\mu<\mu_0$. Passing to the limit in (5.3.98), the result follows.

□

Proposition 5.83 *Let $\omega \in \mathbb{R}$, $A \in \mathcal{A}(\omega)$ and $\lambda>0$ such that $\lambda\omega<1$. Suppose that $D(A)$ is dense in X and $J_\lambda : \overline{D(A)} \to X$ is one–to–one*[2]*. Then A is one–to–one.*

[2] Equivalently, $A_\lambda = \frac{1}{\lambda}(I-J_\lambda)$ is one–to–one on $\overline{D(A)}$.

Proof: Let $x_1, x_2 \in D(A)$ be such that $Ax_1 = Ax_2$. Then, for every $\lambda > 0$, it follows that

$$x_1 + \lambda Ax_1 = x_2 + \lambda Ax_2.$$

Since $x_1 + \lambda Ax_1 \in D_\lambda = \mathrm{Im}(I + \lambda A)$ and J_λ is single–valued, we have

$$J_\lambda(x_1 + \lambda Ax_1) = J_\lambda(x_2 + \lambda Ax_2).$$

By the resolvent identity, $J_\lambda(x_i + \lambda Ax_i) = x_i$. Thus, $x_1 = x_2$, proving that A is injective. □

Proposition 5.84 *Let $\omega \in \mathbb{R}$ and $A \in \mathcal{A}(\omega)$. Suppose that $D(A)$ is dense in X and that for some $\lambda_0 > 0$ the resolvent $J_{\lambda_0} : \overline{D(A)} \to X$ is one–to–one. Then:*

(i) J_λ is one–to–one for every $\lambda > 0$ and $\lambda\omega < 1$;

(ii) A_λ is one–to–one on $\overline{D(A)}$ for every $\lambda > 0$ and $\lambda\omega < 1$;

(iii) A is one–to–one.

Proof:

(i) Fix any $\lambda > 0$ with $\lambda\omega < 1$. Let $x_1, x_2 \in \overline{D(A)}$ satisfy

$$J_\lambda x_1 = J_\lambda x_2.$$

We want to show that $x_1 = x_2$.

By the resolvent identity (cf. Theorem 5.79, item (iv)), for any $\mu > 0$ such that $\mu\omega < 1$ we may write

$$J_\mu\left(\frac{\mu}{\lambda}x + \frac{\lambda - \mu}{\lambda}J_\lambda x\right) = J_\lambda x.$$

Applying the identity to x_1 and x_2, and using the hypothesis $J_\lambda x_1 = J_\lambda x_2$, we obtain

$$J_\mu\left(\frac{\mu}{\lambda}x_1 + \frac{\lambda - \mu}{\lambda}J_\lambda x_1\right) = J_\mu\left(\frac{\mu}{\lambda}x_2 + \frac{\lambda - \mu}{\lambda}J_\lambda x_2\right).$$

Choose $\mu = \lambda_0$. Since J_{λ_0} is one–to–one, it follows that

$$\frac{\mu}{\lambda}x_1 + \frac{\lambda - \mu}{\lambda}J_\lambda x_1 = \frac{\mu}{\lambda}x_2 + \frac{\lambda - \mu}{\lambda}J_\lambda x_2.$$

But $J_\lambda x_1 = J_\lambda x_2$, so the last equality reduces to

$$\frac{\mu}{\lambda}x_1 = \frac{\mu}{\lambda}x_2,$$

hence $x_1 = x_2$.

Thus, J_λ is one–to–one.

(ii) Immediate, since

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda),$$

and $I - J_\lambda$ is one–to–one iff J_λ is one–to–one.

(iii) Follows immediately from Proposition 5.83.

□

5.4 Maximal Accretive and m-Accretive Operators

Definition 5.85 *Let $A : X \to X$ be an accretive operator with $D(A) \subset C \subset X$. We say that A is* ***maximal accretive in*** *C if A does not admit a proper accretive extension with domain contained in C.*
We say that A is ***maximal accretive*** *if A is maximal accretive in X.*

Definition 5.86 *We say that an operator $A : X \to X$ is* ***m-accretive*** *if*

$$\mathrm{Im}(I + A) = X.$$

From the definitions above, every **m-accretive operator is maximal accretive**. Indeed, let $A : X \to X$ be m-accretive and suppose $A \subset B$, where B is accretive. We prove that $A = B$.

Let $(x, y) \in B$ and set $z = x + y$. Then

$$(z, x) = (x + y, x) \in J_1^B,$$

since $(x, y) \in B$.

As B is accretive, Proposition 5.75 implies that J_1^B is single–valued; therefore

$$J_1^B z = x. \tag{5.4.99}$$

Since A is m-accretive, there exists $x_1 \in D(A)$ such that $z \in (I + A)x_1$. Thus, $z = x_1 + y_1$ with $(x_1, y_1) \in A \subset B$. Hence, by Observation 5.74,

$$J_1^B z = x_1. \tag{5.4.100}$$

From (5.4.99)–(5.4.100) we deduce $x = x_1$, hence $y = y_1$, and so $(x, y) \in A$. Thus $B \subset A$.

The converse is false: a maximal accretive operator is not necessarily m-accretive (see [21]), even when X and X' are uniformly convex.

On the other hand, if X is a Hilbert space, accretivity coincides with monotonicity, and since X is reflexive and both X and X' are smooth, Theorem 5.54 gives that A is m-accretive iff it is maximal accretive.

Proposition 5.87 *Let A be an accretive operator such that $\mathrm{Im}(I + \mu A) = X$ for some $\mu > 0$. Then $\mathrm{Im}(I + \lambda A) = X$ for every $\lambda > 0$.*

Proof: Given $\lambda > 0$, set $k = \lambda/\mu$. We show that $\mathrm{Im}(I + \lambda A) = X$.

Let $x \in X$ and consider

$$\frac{x}{k} + \left(1 - \frac{1}{k}\right) y \in D_\mu = \mathrm{Im}(I + \mu A) = X.$$

Define

$$z_y = J_\mu \left(\frac{x}{k} + \left(1 - \frac{1}{k}\right) y\right), \qquad y \in X.$$

Since $J_\mu : X \to D(A)$ and J_μ is single–valued (because A is accretive), we may define the map

$$\begin{array}{rccl} B: & X & \longrightarrow & D(A) \\ & y & \longmapsto & By = J_\mu\left(\frac{x}{k} + \left(1-\frac{1}{k}\right)y\right) \end{array}.$$

As J_μ is a contraction (Proposition 5.76), we have for all $y_1, y_2 \in X$:

$$\|By_1 - By_2\| \le \left|1 - \frac{1}{k}\right| \|y_1 - y_2\|.$$

Now $\left|1 - \frac{1}{k}\right| < 1$ iff $k > \frac{1}{2}$. Thus, if $\lambda > \mu/2$, then B is a contraction and hence has a unique fixed point $y_0 \in D(A)$ such that

$$y_0 = By_0 = J_\mu\left(\frac{x}{k} + \left(1 - \frac{1}{k}\right) y_0\right).$$

Thus,

$$\frac{x}{k} \in \frac{y_0}{k} + \mu A y_0,$$

and multiplying by k gives

$$x \in y_0 + k\mu A y_0 = y_0 + \lambda A y_0 = (I + \lambda A) y_0.$$

Hence $\mathrm{Im}(I + \lambda A) = X$ for all $\lambda > \mu/2$.

Iterating the argument yields

$$\mathrm{Im}(I + \lambda A) = X \quad \text{for all } \lambda > \frac{\mu}{2^n},\ n \in \mathbb{N}.$$

Thus, $\mathrm{Im}(I + \lambda A) = X$ for every $\lambda > 0$. □

Corollary 5.88

(i) An accretive operator A is m-accretive iff $\mathrm{Im}(I + \lambda A) = X$ for all $\lambda > 0$;

(ii) If A is m-accretive, then $D(A_\lambda) = D(J_\lambda) = D_\lambda = X$ for every $\lambda > 0$.

Proof:

(i) If A is m-accretive, then $\mathrm{Im}(I + A) = X$, and by Proposition 5.87, $\mathrm{Im}(I + \lambda A) = X$ for all $\lambda > 0$. Conversely, if $\mathrm{Im}(I + \lambda A) = X$ for all $\lambda > 0$, choosing $\lambda = 1$ shows A is m-accretive.

(ii) If A is m-accretive, then $\mathrm{Im}(I + \lambda A) = X$ for all $\lambda > 0$, hence

$$D(A_\lambda) = D(J_\lambda) = D_\lambda = X.$$

□

Proposition 5.89 *Every m-accretive operator is closed.*

Proof: Let $A : X \to X$ be m-accretive, and let $(x_n) \subset D(A)$ satisfy $x_n \to x$ and $y_n \in Ax_n$ with $y_n \to y$. We prove that

$$(x, y) \in A. \tag{5.4.101}$$

Since $x_n + y_n \to x + y$ and $D(J_1) = X$, both $x_n + y_n$ and $x + y$ lie in $D(J_1)$.

By Proposition 5.76, J_1 is a contraction and single–valued. Thus,

$$J_1(x_n + y_n) = x_n. \tag{5.4.102}$$

Passing to the limit yields

$$J_1(x + y) = x,$$

which means $(x, y) \in A$, proving (5.4.101). □

Example 5.91 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary Γ, and consider the operator

$$A : L^p(\Omega) \to L^p(\Omega), \qquad 1 < p < \infty,$$

defined by

$$D(A) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), \qquad Au = -\Delta u.$$

We show that A is m-accretive. Since A is linear and single–valued, for $u_1, u_2 \in D(A)$,

$$\langle Au_1 - Au_2,\ F(u_1 - u_2)\rangle = \langle -\Delta u,\ Fu\rangle,$$

where $u = u_1 - u_2$ and $F(u) = u|u|^{p-2}\|u\|_p^{2-p}$.

Integrating by parts,

$$\langle -\Delta u,\ Fu\rangle = (p-1)\|u\|_p^{2-p} \int_\Omega |\nabla u|^2 |u|^{p-2}\, dx \geq 0.$$

Thus A is accretive. By elliptic regularity [14], for each $v \in L^p(\Omega)$ there exists $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ solving

$$u - \Delta u = v,$$

so $\mathrm{Im}(I - \Delta) = L^p(\Omega)$, and hence A is m-accretive.

In particular, when $p = 2$,

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad Au = -\Delta u,$$

is m-accretive and therefore maximal monotone (Theorem 5.54).

Theorem 5.92 *The following statements are equivalent:*

(i) A is maximal accretive in $C \supset D(A)$;

(ii) If $x \in C$, $y \in X$ and

$$\|x - u + \lambda(y - v)\| \geq \|x - u\| \quad \textit{for all } (u, v) \in A,$$

then $(x, y) \in A$;

(iii) If $x \in C$, $y \in X$ and there exists $\xi' \in F(x - u)$ such that

$$\langle \xi', y - v\rangle \geq 0 \quad \forall (u, v) \in A,$$

then $(x, y) \in A$.

Proof: Same as the proof of Theorem 5.27 for $(i) \Leftrightarrow (ii)$. The equivalence $(ii) \Leftrightarrow (iii)$ follows from Proposition 5.68. □

The class of accretive, maximal accretive, and m-accretive operators satisfies an analogous translation invariance as in Proposition 5.24 for monotone operators.

Proposition 5.93 *If A is accretive, maximal accretive, or m-accretive, then any translate of A in $D(A)$ and $\operatorname{Im}(A)$ remains, respectively, accretive, maximal accretive, or m-accretive.*

Proposition 5.94 *Let X be smooth, $A \in \mathcal{A}(\omega)$, $D(A) \subset C \subset X$, and assume that $A + \omega I$ is maximal in C. Then Ax is convex and closed.*

Proof: Let $x \in D(A)$ and let $y_1, y_2 \in Ax$, $t \in [0,1]$. We show that $ty_1 + (1-t)y_2 \in Ax$.

Since $A \in \mathcal{A}(\omega)$, $A + \omega I$ is accretive. Hence, for every $(u,v) \in A$ and for $i = 1, 2$:

$$(x,\, y_i + \omega x),\ (u,\, v + \omega u) \in A + \omega I.$$

Thus, by Corollary 5.69, there exists $x_i' \in F(x-u)$ such that

$$\langle x_i',\, y_i + \omega x - v - \omega u\rangle \geq 0.$$

Since X is smooth, F is single–valued, hence $x_1' = x_2' = F(x-u)$, and so

$$\langle F(x-u), y_i + \omega x - v - \omega u\rangle \geq 0.$$

Multiplying by t and $1-t$ and adding yields

$$\langle F(x-u),\, ty_1 + (1-t)y_2 + \omega x - v - \omega u\rangle \geq 0. \tag{5.4.103}$$

Since $x \in C$ and $A + \omega I$ is maximal accretive in C, Theorem 5.92(iii) applied to (5.4.103) gives

$$(x,\, ty_1 + (1-t)y_2 + \omega x) \in A + \omega I,$$

and therefore $ty_1 + (1-t)y_2 \in Ax$.

Closedness follows analogously by taking limits. □

Definition 5.95 *An operator $A : X \to X$, where X is a Banach space, is called* ***demiclosed*** *if*

$$(x_n, y_n) \subset A,\ x_n \to x,\ y_n \rightharpoonup y \quad \Longrightarrow \quad (x,y) \in A.$$

Definition 5.96 *A map $\varphi : X \to Y$ between Banach spaces is* ***Fréchet differentiable*** *at $x \in X$ if there exists a bounded linear map $L(x) : X \to Y$ such that*

$$\varphi(x+y) - \varphi(x) = L(x)y + \omega(x,y),$$

with

$$\lim_{y \to 0} \frac{\omega(x,y)}{\|y\|} = 0.$$

(The long Lemmas 2.4.11–2.4.12 are kept in the Portuguese source; only the introduction is repeated here.)

Proposition 5.97 *Let X be a Banach space whose norm is Fréchet differentiable, and let $A \in \mathcal{A}(\omega)$ such that $A + \omega I$ is maximal in $\overline{D(A)}$. Then A is demiclosed.*

Proof: As the norm is Fréchet differentiable, see [47] Lemma 6.11 p53, and [47] Theorem 6.12 p54 together imply that the duality map F is single–valued and continuous. (for the reader's convenience we proved these results below)

Let $(x_n, y_n) \subset A$ with $x_n \to x$ and $y_n \rightharpoonup y$. Since $A + \omega I$ is accretive,

$$\langle F(x_n - u), y_n + \omega x_n - v - \omega u\rangle \geq 0 \quad \forall (u,v) \in A.$$

Passing to the limit yields

$$\langle F(x-u),\, y + \omega x - v - \omega u\rangle \geq 0.$$

Since $x \in \overline{D(A)}$ and $A + \omega I$ is maximal in $\overline{D(A)}$, Theorem 5.92(iii) gives

$$y + \omega x \in (A + \omega I)x,$$

hence $(x, y) \in A$. □

Proposition 5.98 *Every operator $A \in \mathcal{A}(\omega)$ such that $A + \omega I$ is maximal in $\overline{D(A)}$ is closed as a subset of $X \times X$.*

Proof: Let $(x_n, y_n) \subset A$ with $x_n \to x$ and $y_n \to y$. For each $(u, v) \in A$ and $\lambda \geq 0$,

$$\|x_n - u\| \leq \|x_n - u + \lambda(y_n + \omega x_n - v - \omega u)\|.$$

Passing to the limit gives

$$\|x - u\| \leq \|x - u + \lambda(y + \omega x - v - \omega u)\|.$$

As $x \in \overline{D(A)}$ and $A + \omega I$ is maximal accretive in $\overline{D(A)}$, Theorem 5.92 yields $(x, y) \in A$. □

Proposition 5.99 *Let X' be uniformly convex, $A \in \mathcal{A}(\omega)$, $A + \omega I$ maximal in $\overline{D(A)}$ and*

$$D(A) \subset \operatorname{Im}(I + \lambda A), \quad 0 < \lambda < \lambda_0, \quad \lambda_0 \omega < 1.$$

Then

$$\lim_{\lambda \to 0} \|A_\lambda x\| = |Ax|, \qquad \forall x \in D(A),$$

where $|Ax| = \inf\{\|y\|;\ y \in Ax\}$.

Proof: Since $D(A) \subset D_\lambda$ for all $0 < \lambda < \lambda_0$, we have $D(A) \subset D = \bigcap_{0<\lambda\leq\lambda_0} D_\lambda$.

Proposition 5.82(ii) gives

$$\lim_{\lambda \to 0} \|A_\lambda x\| = |||Ax|| = \lim_{\lambda \to 0} (1 - \lambda\omega)\|A_\lambda x\|.$$

Moreover, Proposition 5.82(iii) gives

$$|||Ax|| \leq |Ax|.$$

Since X' is uniformly convex, X is reflexive; thus $(A_\lambda x)$ is bounded and admits a weakly convergent subsequence

$$A_{\lambda_n} x \rightharpoonup y.$$

By lower semicontinuity,

$$\|y\| \leq |||Ax||.$$

Also, $A_{\lambda_n}x \in A(J_{\lambda_n}x)$ and $J_{\lambda_n}x \to x$, and since A is demiclosed (Proposition 5.97), it follows that $y \in Ax$. Therefore

$$|Ax| \leq \|y\| \leq |||Ax||.$$

Thus $|Ax| = |||Ax||$, and hence

$$\lim_{\lambda \to 0} \|A_\lambda x\| = |Ax|.$$

□

Lemma 5.100 ***(Kato)*** *Let $C \subset X$ be a nonempty closed convex subset of a smooth Banach space X. Then $x \in \overset{\circ}{C}$ iff*

$$x \in C \quad \textit{and} \quad \|x\|^2 \leq \langle F(x), y\rangle \ \ \forall\, y \in C.$$

Proof: If x satisfies the inequality, then

$$\|x\|^2 = \langle F(x), x\rangle \leq \langle F(x), y\rangle \leq \|F(x)\|\|y\| = \|x\|\|y\|,$$

so $\|x\| \leq \|y\|$ for all $y \in C$, hence x is a minimum–norm point in C.

Conversely, suppose $x \in \overset{\circ}{C}$ and $y \in C$. For $t \in (0,1)$,

$$\|(1-t)x + ty\|^2 \leq \|(1-t)x + ty\|\|x\| + t\langle F((1-t)x + ty), y - x\rangle.$$

Since $\|x\| \leq \|(1-t)x + ty\|$, the difference yields

$$\langle F((1-t)x + ty), y - x\rangle \geq 0.$$

As $t \to 0$, using demicontinuity of F, we obtain

$$\langle F(x), y - x\rangle \geq 0.$$

Hence

$$\|x\|^2 = \langle F(x), x\rangle \leq \langle F(x), y\rangle,$$

which completes the proof. □

Lemma 5.101 *Let X be a Banach space and let $C \neq \emptyset$ be a convex, closed subset of X. If $(x_n) \subset C$ is a sequence such that $\|x_n\| \to |C|$ and $x_n \rightharpoonup x$, then $x \in \overset{\circ}{C}$.*

Proof: Since C is convex and closed, it is weakly closed. As $x_n \rightharpoonup x$, we have $x \in C$ and hence $\|x\| \geq |C|$. On the other hand, by the lower semicontinuity of the norm, we have

$$\|x\| \leq \liminf_n \|x_n\| = \lim_n \|x_n\| = |C|.$$

It follows that $\|x\| = |C|$ and therefore $x \in \overset{\circ}{C}$. □

Theorem 5.102 *Let X be a Banach space.*

i) X is reflexive if and only if for each nonempty, convex, closed subset C of X, one has $\overset{\circ}{C} \neq \emptyset$;

ii) X is strictly convex if and only if for each nonempty, convex, closed subset C of X, the set $\overset{\circ}{C}$ contains at most one element;

iii) X is reflexive and strictly convex if and only if for each nonempty, convex, closed subset C of X, the set $\overset{\circ}{C}$ contains a unique point.

Proof: **(i)** Suppose that X is reflexive and let $(x_n) \subset C$ be such that $\|x_n\| \longrightarrow |C|$. Then $(x_n)_{n\in\mathbb{N}} \subset C$ is bounded, so there exist a subsequence $(x_{n_k}) \subset (x_n)$ and an $x \in X$ such that

$$x_{n_k} \rightharpoonup x,$$

by Theorem III.27, p. 50, in Brézis.

Now, by Lemma 5.36, it follows that $x \in \overset{\circ}{C}$, that is, $\overset{\circ}{C} \neq \emptyset$.

We now prove the converse by using James' Theorem (see [36], p. 16, Theorem 3), which states that X is reflexive if every $x' \in X'$ attains its norm, i.e., there exists $z \in \{x \in X;\ \|x\| \leq 1\}$ such that

$$\langle x', z\rangle = \sup_{\substack{y\in X\\ \|y\|\leq 1}} \big|\langle x', y\rangle\big| = \|x'\|.$$

Let $x' \in X'$ and define

$$C = \big\{x \in X;\ \langle x', x\rangle \geq \|x'\|\big\}.$$

We claim that C is convex and closed. Indeed, if $x, y \in C$ and $t \in [0,1]$, then

$$\langle x', tx + (1-t)y\rangle = t\langle x', x\rangle + (1-t)\langle x', y\rangle \geq t\|x'\| + (1-t)\|x'\| = \|x'\|,$$

so $tx + (1-t)y \in C$ for all $t \in [0,1]$, proving that C is convex.

To see that C is closed, let $(x_n) \subset C$ be such that $x_n \longrightarrow x$. We must show that $x \in C$. Since $x_n \to x$, by the continuity of x' we have

$$\langle x', x_n\rangle \longrightarrow \langle x', x\rangle \quad \text{as } n \longrightarrow \infty,$$

that is, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \quad \big|\langle x', x_n\rangle - \langle x', x\rangle\big| < \varepsilon,$$

which is equivalent to

$$-\varepsilon < \langle x', x\rangle - \langle x', x_n\rangle < \varepsilon.$$

Hence,

$$\langle x', x\rangle > -\varepsilon + \langle x', x_n\rangle \geq -\varepsilon + \|x'\|.$$

By the arbitrariness of $\varepsilon > 0$, it follows that

$$\langle x', x\rangle \geq \|x'\|.$$

Thus $x \in C$ and therefore C is closed.

Now let $(y_n)_{n\in\mathbb{N}} \subset C$ with $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and such that

$$\lim_{n\to\infty} |\langle x', y_n\rangle| = \|x'\| = \sup_{\substack{y\in X\\ \|y\|=1}} |\langle x', y\rangle|.$$

Define

$$z_n = \frac{\|x'\|\, y_n}{\langle x', y_n\rangle}.$$

Then

$$\langle x', z_n\rangle = \left\langle x', \frac{\|x'\|\, y_n}{\langle x', y_n\rangle}\right\rangle = \frac{\|x'\|}{\langle x', y_n\rangle}\langle x', y_n\rangle = \|x'\|,$$

so $z_n \in C$ for all $n \in \mathbb{N}$, and

$$\begin{aligned}\lim_{n\to+\infty} \|z_n\| &= \lim_{n\to+\infty} \frac{\|x'\|\cdot\|y_n\|}{|\langle x', y_n\rangle|} = \lim_{n\to+\infty} \frac{\|x'\|}{|\langle x', y_n\rangle|}\\ &= \|x'\|\cdot \frac{1}{\lim\limits_{n\to+\infty} |\langle x', y_n\rangle|} = \|x'\|\cdot\frac{1}{\|x'\|} = 1.\end{aligned}$$

By the definition of $|C|$ we have

$$|C| \leq \|x\|, \quad \forall x \in C.$$

In particular,

$$|C| \leq \|z_n\|, \quad \forall n \in \mathbb{N}.$$

Passing to the limit, we obtain

$$|C| \leq 1.$$

By hypothesis, $\overset{\circ}{C} = \{x \in C;\ \|x\| = |C|\} \neq \emptyset$, hence there exists $x_0 \in C$ such that $\|x_0\| = |C|$.

Since $x_0 \in C$, we have

$$\langle x', x_0\rangle \geq \|x'\|. \tag{5.4.104}$$

On the other hand, from $\|x_0\| = |C| \leq 1$, we get

$$\langle x', x_0\rangle \leq |\langle x', x_0\rangle| \leq \|x'\|\|x_0\| \leq \|x'\|. \tag{5.4.105}$$

From (5.4.104) and (5.4.105), we deduce

$$\langle x', x_0\rangle = \|x'\|,$$

with x_0 belonging to the closed unit ball. Since $x' \in X'$ was arbitrary, James' Theorem yields that X is reflexive.

(ii) Suppose that X is strictly convex and let $C \subset X$ be convex, closed and nonempty. If $\overset{\circ}{C} = \emptyset$, there is nothing to prove. Assume $\overset{\circ}{C} \neq \emptyset$ and let $x, y \in \overset{\circ}{C}$. Then $\frac{x+y}{2} \in C$, thus

$$\left\|\frac{x+y}{2}\right\| \geq |C|. \tag{5.4.106}$$

However,

$$\left\|\frac{x+y}{2}\right\| \leq \frac{1}{2}\|x\| + \frac{1}{2}\|y\| = |C|. \tag{5.4.107}$$

From (5.4.106) and (5.4.107) it follows that

$$\left\|\frac{x+y}{2}\right\| = |C| = \|x\| = \|y\|.$$

Since X is strictly convex, item (iii) of Lemma 5.36 gives $x = y$. Hence $\overset{\circ}{C}$ contains at most one element.

Conversely, assume that for every convex, closed subset $C \subset X$, the set $\overset{\circ}{C}$ contains at most one element. Let $x, y \in U_X$, with $x \neq y$, where $U_X = \{x \in X;\ \|x\| = 1\}$. Consider the subset

$$K = \{tx + (1-t)y;\ 0 \leq t \leq 1\} \subset U_X.$$

We claim that K is convex, closed and that $\overset{\circ}{K} = K$.

- **K is convex.**

Let $z_1, z_2 \in K$ and $\lambda \in [0,1]$. We must show that

$$\lambda z_1 + (1-\lambda)z_2 \in K.$$

Since $z_1 = t_1 x + (1-t_1)y$ and $z_2 = t_2 x + (1-t_2)y$, we obtain

$$\begin{aligned}\lambda z_1 + (1-\lambda)z_2 &= \lambda(t_1 x + (1-t_1)y) + (1-\lambda)(t_2 x + (1-t_2)y)\\ &= (\lambda t_1 + (1-\lambda)t_2)x + (\lambda(1-t_1) + (1-\lambda)(1-t_2))y.\end{aligned} \tag{5.4.108}$$

Observe that

$$\begin{aligned}\lambda t_1 + (1-\lambda)t_2 + \lambda(1-t_1) + (1-\lambda)(1-t_2) &= \lambda t_1 + (1-\lambda)t_2 + \lambda - \lambda t_1 + (1-\lambda) - (1-\lambda)t_2\\ &= \lambda + (1-\lambda) = 1, \quad \forall \lambda \in [0,1].\end{aligned}$$

Setting $t_3 = \lambda t_1 + (1-\lambda)t_2$, we have $\lambda(1-t_1) + (1-\lambda)(1-t_2) = 1 - t_3$, and from (5.4.108) it follows that

$$\lambda z_1 + (1-\lambda)z_2 = t_3 x + (1-t_3)y, \quad t_3 \in [0,1], \quad \forall \lambda \in [0,1],$$

that is, $\lambda z_1 + (1-\lambda)z_2 \in K$, proving that K is convex.

- **K is closed.**

Let $(z_n)_{n\in\mathbb{N}} \subset K$ be such that $z_n \longrightarrow z$ in X. Then there exists $(t_n)_{n\in\mathbb{N}} \subset [0,1]$ with

$$z_n = t_n x + (1-t_n)y.$$

The sequence $(t_n)_{n\in\mathbb{N}} \subset [0,1]$ is bounded, so there exists a subsequence, again denoted (t_n), converging in $[0,1]$:

$$t_n \longrightarrow t \in [0,1].$$

We claim that

$$z_n \longrightarrow tx + (1-t)y \quad \text{in } X.$$

Indeed,

$$\begin{aligned}\|z_n - (tx + (1-t)y)\| &= \|(t_n - t)x + (t_n - t)y\| \\ &\le |t_n - t|\|x\| + |t_n - t|\|y\| \longrightarrow 0, \quad n \longrightarrow \infty.\end{aligned}$$

By the uniqueness of the limit, $z = tx + (1-t)y \in K$, with $t \in [0,1]$, proving that K is closed.

- **We show that $\overset{\circ}{K} = K$.**

By definition, $\overset{\circ}{K} \subset K$. On the other hand, since $K \subset U_X$, we have

$$\|z\| = 1, \quad \forall z \in K.$$

Thus

$$|K| = \inf\{\|z\|;\ z \in K\} = 1,$$

and if $z \in K$, then $z \in \overset{\circ}{K}$, because $\|z\| = |K| = 1$. Hence $\overset{\circ}{K} = K$.

By hypothesis, $\overset{\circ}{K}$ contains at most one element and $K \neq \emptyset$, so K contains exactly one element, namely $x = y$. This is a contradiction. Therefore,

$$\{tx + (1-t)y;\ x \neq y,\ 0 \le t \le 1\} \not\subset U_X,$$

which proves that X is strictly convex.

(iii) ($\Rightarrow$) Suppose X is strictly convex and reflexive.

Let $C \subset X$ be convex, closed and nonempty. By item (i), $\overset{\circ}{C} \neq \emptyset$, and by item (ii), $\overset{\circ}{C}$ contains at most one element. Hence $\overset{\circ}{C}$ contains a unique element.

($\Leftarrow$) Conversely, assume that for every convex, closed, nonempty subset $C \subset X$, the set $\overset{\circ}{C}$ contains a unique element. Then $\overset{\circ}{C} \neq \emptyset$, so by item (i) X is reflexive and, by item (ii), X is strictly convex. □

Theorem 5.103 *Let X be a Banach space. The following assertions are equivalent:*
(i) X is reflexive, strictly convex and satisfies the property

$$x_n \rightharpoonup x \text{ and } \limsup_{n\to\infty} \|x_n\| \le \|x\| \ \Rightarrow\ x_n \to x. \tag{5.4.109}$$

(ii) For each convex, closed subset $C \subset X$ and each sequence $(x_n) \subset C$ such that $\|x_n\| \to |C|$, there exists $x \in X$ such that $x_n \to x$.

Proof: (i)⇒(ii) Let $C \subset X$ be convex and closed (in the strong topology) and let $(x_n)_{n\in\mathbb{N}} \subset C$ be such that

$$\|x_n\| \longrightarrow |C| = \inf\{\|x\|;\ x \in C\} \quad \text{as } n \longrightarrow \infty. \tag{5.4.110}$$

From (5.4.110), the sequence (x_n) is bounded. Since X is reflexive,

$$\exists (x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}} \text{ and } x \in X \text{ such that } x_{n_k} \rightharpoonup x, \text{ as } k \longrightarrow \infty. \tag{5.4.111}$$

By Lemma 5.101, we have

$$x \in \overset{\circ}{C}, \tag{5.4.112}$$

so $\overset{\circ}{C} \neq \emptyset$. Thus, by the strict convexity of X (and its reflexivity), Theorem 5.102 (iii) implies that x is the unique element of $\overset{\circ}{C}$.

Claim: $x_n \rightharpoonup x$ as $n \longrightarrow \infty$.

We prove this by contradiction. Suppose $x_n \not\rightharpoonup x$ as $n \longrightarrow \infty$. Then there exist $\mathbb{N}' \subset \mathbb{N}$ and a weak neighbourhood V_x of x such that $x_k \notin V_x$ for all $k \in \mathbb{N}'$. However, $\{x_k\}_{k\in\mathbb{N}'} \subset \{x_n\}_{n\in\mathbb{N}}$ is bounded, hence $\{x_k\}_{k\in\mathbb{N}'}$ is bounded and there exists a subsequence $\{x_{k_j}\} \subset \{x_k\}$ such that

$$x_{k_j} \rightharpoonup x_0. \tag{5.4.113}$$

It follows that $x_0 \in \overset{\circ}{C}$. Indeed, by lower semicontinuity of the norm with respect to the weak topology,

$$\|x_0\| \leq \lim_{n\to\infty} \inf \|x_n\| \leq |C|.$$

On the other hand, since C is convex and closed, its weak closure coincides with its strong closure, and $\{x_{k_j}\} \subset C$ converges weakly to an element of C, that is, $x_0 \in C$. Hence

$$\|x_0\| \geq |C|.$$

Therefore, $\|x_0\| = |C|$ and we conclude that $x_0 \in \overset{\circ}{C} = \{x\}$. Thus,

$$x_0 = x. \tag{5.4.114}$$

Consequently, there exists $j_0 \in \mathbb{N}$ such that

$$x_{k_j} \in V_x, \ \forall j \geq j_0, \tag{5.4.115}$$

which contradicts the fact that $x_k \notin V_x$ for all $k \in \mathbb{N}'$.

Hence $x_n \rightharpoonup x$ as $n \longrightarrow \infty$. From this and from

$$\|x\| = |C| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \sup \|x_n\|, \tag{5.4.116}$$

it follows, by hypothesis (5.4.109), that

$$x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \tag{5.4.117}$$

as desired.

(ii)⇒(i)

Let $C \subset X$ be convex, closed and nonempty.

Claim: $\overset{\circ}{C} \neq \emptyset$.

If C is finite, then $C = \{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$, and

$$\begin{aligned} |C| &= \inf\{\|x\|;\ x \in C\} = \inf\{\|x_1\|, \ldots, \|x_n\|\} \\ &= \|x_{n_0}\| \quad \text{for some } n_0 \in \{1, \ldots, n\}, \end{aligned} \tag{5.4.118}$$

so $x_{n_0} \in \overset{\circ}{C}$, proving the claim.

Now suppose that C is infinite. By the definition of $|C|$, there exists $(x_n)_{n\in\mathbb{N}} \subset C$ such that

$$\|x_n\| \longrightarrow |C| \quad \text{as } n \longrightarrow \infty. \tag{5.4.119}$$

By hypothesis (ii), there exists $x \in X$ such that

$$x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty. \tag{5.4.120}$$

It follows that

$$\|x_n\| \longrightarrow \|x\|, \quad \text{as } n \longrightarrow \infty, \tag{5.4.121}$$

and by uniqueness of the limit,

$$\|x\| = |C|. \tag{5.4.122}$$

Since C is closed, $x \in C$, and from (5.4.122) we conclude that $x \in \overset{\circ}{C}$, i.e., $\overset{\circ}{C} \neq \emptyset$, as claimed.

We now show that $\overset{\circ}{C}$ has exactly one element.

If $y \in \overset{\circ}{C}$ then $y \in C$ and $\|y\| = |C|$. Define a sequence $(z_n)_{n\in\mathbb{N}} \subset X$ by

$$z_{2n} = x_n \quad \text{and} \quad z_{2n+1} = y, \quad \forall n \in \mathbb{N}. \tag{5.4.123}$$

Then $(z_n)_{n\in\mathbb{N}} \subset C$ and

$$\begin{aligned} \|z_{2n}\| = \|x_n\| \longrightarrow |C|, &\quad \text{as } n \longrightarrow \infty, \\ \|z_{2n+1}\| = \|y\| = |C|, &\quad \forall n \in \mathbb{N}. \end{aligned} \tag{5.4.124}$$

From (5.4.124),

$$\|z_n\| \longrightarrow |C|, \quad \text{as } n \longrightarrow \infty. \tag{5.4.125}$$

By hypothesis (ii), there exists $z \in X$ such that

$$z_n \longrightarrow z, \quad \text{as } n \longrightarrow \infty, \tag{5.4.126}$$

so $z \in \overline{C} = C$ since C is closed.

Moreover, the subsequences $(z_{2n}) = (x_n) \subset (z_n)$ and $(z_{2n+1}) = (y) \subset (z_n)$ are convergent and

$$\begin{aligned} z_{2n} = x_n \longrightarrow x, &\quad \text{as } n \longrightarrow \infty, \\ z_{2n+1} = y \longrightarrow y, &\quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{5.4.127}$$

From (5.4.126) and (5.4.127), we conclude that

$$x = z = y,$$

i.e., $y = x$. Thus $\overset{\circ}{C}$ has a unique element. Since C was arbitrary (convex, closed and nonempty), Theorem 5.102 (iii) shows that X is reflexive and strictly convex.

It remains to prove that X satisfies property (5.4.109). Let $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$ be such that

$$x_n \rightharpoonup x \quad \text{and} \quad \limsup_{n\to\infty} \|x_n\| \leq \|x\|. \tag{5.4.128}$$

By lower semicontinuity of the norm in the weak topology (see Brézis, Proposition III.5 (iii)), we have from (5.4.128)

$$\|x\| \leq \lim_{n\to\infty} \inf \|x_n\|. \tag{5.4.129}$$

From (5.4.128) and (5.4.129) it follows that the limit $\lim_{n\to\infty} \|x_n\|$ exists and

$$\lim_{n\to\infty} \|x_n\| = \|x\|. \tag{5.4.130}$$

Clearly, from (5.4.128) we have

$$\frac{x_n}{\|x\|} \rightharpoonup \frac{x}{\|x\|} \quad \text{as } n \to \infty, \tag{5.4.131}$$

and from (5.4.130),

$$\left\| \frac{x_n}{\|x\|} \right\| \longrightarrow \left\| \frac{x}{\|x\|} \right\| = 1, \quad \text{as } n \to \infty. \tag{5.4.132}$$

Let $y_n = \frac{x_n}{\|x\|}$ and $y = \frac{x}{\|x\|}$ for each $n \in \mathbb{N}$. Then (5.4.131) and (5.4.132) become

$$y_n \rightharpoonup y \quad \text{as } n \to \infty, \tag{5.4.133}$$

and

$$\|y_n\| \longrightarrow \|y\|, \quad \text{as } n \to \infty. \tag{5.4.134}$$

Let $y^* \in F(y)$. Then

$$\langle y^*, y\rangle = \|y^*\| = \|y\| = 1. \tag{5.4.135}$$

Consider the set

$$C = \big\{\omega \in X \text{ such that } \langle y^*, \omega\rangle \geq 1\big\}. \tag{5.4.136}$$

Claim: C is convex and closed.

Indeed, if $\omega_1, \omega_2 \in C$ and $\lambda \in [0,1]$, then

$$\langle y^*, \lambda\omega_1 + (1-\lambda)\omega_2\rangle = \lambda\langle y^*, \omega_1\rangle + (1-\lambda)\langle y^*, \omega_2\rangle \geq \lambda + (1-\lambda) = 1, \tag{5.4.137}$$

so $\lambda\omega_1 + (1-\lambda)\omega_2 \in C$ and C is convex.

Now let $\omega_0 \in \overline{C}$. Then there exists $(\omega_n)_{n\in\mathbb{N}} \subset C$ such that

$$\omega_n \longrightarrow \omega_0 \quad \text{as } n \longrightarrow \infty. \tag{5.4.138}$$

Thus $\omega_n \rightharpoonup \omega_0$ as $n \to \infty$, and hence

$$\langle y^*, \omega_0\rangle = \lim_{n\to\infty} \langle y^*, \omega_n\rangle \geq 1, \tag{5.4.139}$$

so $\omega_0 \in C$. Therefore, $\overline{C} \subset C$, which shows that C is closed.

Moreover, $y \in C$ (see (5.4.135)) and $\|y\| = 1$. Hence

$$|C| = \inf\{\|\omega\|;\ \omega \in C\} \leq \|y\| = 1. \tag{5.4.140}$$

Assume, for a contradiction, that $|C| < 1$. Then there exists $\omega_0 \in C$ such that

$$|C| \leq \|\omega_0\| < \|y\| = 1. \tag{5.4.141}$$

Notice that $0 \notin C$, so $\omega_0 \neq 0$. From (5.4.141),

$$\left\langle y^*, \frac{\omega_0}{\|\omega_0\|} \right\rangle = \frac{1}{\|\omega_0\|}\langle y^*, \omega_0\rangle \geq \frac{1}{\|\omega_0\|} > 1. \tag{5.4.142}$$

Hence, from (5.4.135) and (5.4.142),

$$1 = \|y^*\| = \sup_{\substack{\omega \in X \\ \|\omega\|=1}} \langle y^*, \omega\rangle \geq \left\langle y^*, \frac{\omega_0}{\|\omega_0\|} \right\rangle > 1, \tag{5.4.143}$$

a contradiction. Thus we must have

$$|C| \geq 1. \tag{5.4.144}$$

From (5.4.140) and (5.4.144), we conclude that

$$|C| = 1 = \|y\|, \tag{5.4.145}$$

so $y \in \overset{\circ}{C}$. Since X has already been shown to be reflexive and strictly convex, Theorem 5.102 (iii) implies that y is the unique member of $\overset{\circ}{C}$.

From (5.4.133) and (5.4.135) we have

$$\lim_{n\to\infty} \langle y^*, y_n\rangle = \langle y^*, y\rangle = 1. \tag{5.4.146}$$

We may assume, without loss of generality, that

$$\langle y^*, y_n\rangle > 0, \quad \forall n \in \mathbb{N}. \tag{5.4.147}$$

Define $z_n = \dfrac{y_n}{\langle y^*, y_n\rangle}$ for each $n \in \mathbb{N}$. Then

$$\langle y^*, z_n\rangle = \frac{\langle y^*, y_n\rangle}{\langle y^*, y_n\rangle} = 1, \quad \forall n \in \mathbb{N}, \tag{5.4.148}$$

so $(z_n)_{n\in\mathbb{N}} \subset C$. Furthermore, from (5.4.134), (5.4.135) and (5.4.146),

$$\|z_n\| = \frac{\|y_n\|}{\langle y^*, y_n\rangle} \longrightarrow \frac{1}{1} = 1 = |C| \quad \text{as } n \longrightarrow \infty. \tag{5.4.149}$$

Hence, by hypothesis (ii), there exists $z \in X$ such that

$$z_n \longrightarrow z \quad \text{as } n \longrightarrow \infty, \tag{5.4.150}$$

and thus $z_n \rightharpoonup z$ as $n \to \infty$. By Lemma 5.101, we conclude that $z \in \overset{\circ}{C} = \{y\}$, that is, $z = y$. Therefore,

$$z_n \longrightarrow y \quad \text{as } n \longrightarrow \infty. \tag{5.4.151}$$

From this and (5.4.146) it follows that

$$y_n = \langle y^*, y_n\rangle z_n \longrightarrow 1 \cdot y = y \quad \text{as } n \longrightarrow \infty, \tag{5.4.152}$$

and consequently

$$x_n = \|x\| y_n \longrightarrow \|x\| \frac{x}{\|x\|} = x \quad \text{as } n \longrightarrow \infty, \tag{5.4.153}$$

that is,

$$x_n \longrightarrow x \quad \text{as } n \longrightarrow \infty, \tag{5.4.154}$$

which completes the proof. □

5.5 Sections

Definition 5.104 *Let X be a normed space and $A : X \longrightarrow X$ an operator. We define the operator $\overset{\circ}{A}: X \longrightarrow X$ by $\overset{\circ}{A} x = (Ax)^\circ$; that is,*

$$\overset{\circ}{A} x = \{y \in Ax;\ \|y\| = |Ax|\},$$

*where $|Ax| = \inf\{\|y\|;\ y \in Ax\}$. The operator $\overset{\circ}{A}$ is called the **minimal section** of A.*

Theorem 5.105 *Let X be a reflexive, strictly convex and smooth Banach space and let $A \in \mathcal{A}(\omega)$ be a demiclosed operator such that*

$$D(A) \subset \mathrm{Im}(I + \lambda A),\ 0 < \lambda < \lambda_0 \ \textit{with}\ \lambda_0\omega < 1.$$

Then $\overset{\circ}{A}$ is a single-valued operator and $D(\overset{\circ}{A}) = D(A)$.

Proof: Let B be the operator defined by $D(B) = D(A)$ and $Bx = \overline{\mathrm{conv}}\, Ax$, where $\mathrm{conv}\, Ax$ is the convex hull of Ax.

By definition of B, if $x \in D(A)$ then $Bx \neq \emptyset$. Note that Bx is convex and closed. By Theorem 5.14 p35 [47] (iii), the set $\overset{\circ}{B} x = (Bx)^\circ$ has a unique element, that is, there exists a unique element $\overset{\circ}{B} x \in Bx$ such that $\| \overset{\circ}{B} x\| = |Bx|$.

We assume, for the moment, that

$$\overset{\circ}{B} x \in Ax,\ \forall x \in D(A). \tag{5.5.155}$$

Assuming (5.5.155), we claim that $\overset{\circ}{B} x$ is the unique element of $\overset{\circ}{A} x$. Indeed, since $A \subset B$, from (5.5.155) we obtain

$$\inf\{\|y\|;\ y \in Bx\} \leq \inf\{\|y\|;\ y \in Ax\},$$

i.e., $|Bx| \leq |Ax|$, and since $\| \overset{\circ}{B} x\| = |Bx|$,

$$\| \overset{\circ}{B} x\| \leq |Ax|.$$

On the other hand, as we are assuming $\overset{\circ}{B} x \in Ax$, we get

$$|Ax| = \inf\{\|y\|;\ y \in Ax\} \leq \| \overset{\circ}{B} x\|.$$

From these two inequalities it follows that $|Ax| = \| \overset{\circ}{B} x\|$, and therefore $\overset{\circ}{B} x \in \overset{\circ}{A} x$.

Furthermore,

$$\begin{aligned}\overset{\circ}{A}x &= \{y \in Ax;\ \|y\| = |Ax|\}\\ &= \{y \in Ax;\ \|y\| = \|\overset{\circ}{B}x\|\}\\ &\subset \{y \in Bx;\ \|y\| = \|\overset{\circ}{B}x\|\}\\ &= \{y \in Bx;\ \|y\| = |Bx|\} = \overset{\circ}{B}x,\end{aligned}$$

since $A \subset B$. Thus $\overset{\circ}{A}x \subset \overset{\circ}{B}x$, and because $\overset{\circ}{B}x$ is a singleton, $\overset{\circ}{A}x$ is at most a singleton. Since $\overset{\circ}{B}x \in \overset{\circ}{A}x$, it follows that the unique element of $\overset{\circ}{A}x$ is $\overset{\circ}{B}x$.

Hence, if $x \in D(A)$, then $\overset{\circ}{A}x$ is a singleton, so $\overset{\circ}{A}x \neq \emptyset$, and therefore $x \in D(\overset{\circ}{A})$. This proves that $D(A) \subset D(\overset{\circ}{A})$.

Conversely, if $x \in D(\overset{\circ}{A})$, then $\overset{\circ}{A}x \neq \emptyset$, i.e., the set $\{y \in Ax;\ \|y\| = |Ax|\} \neq \emptyset$. Thus there exists $y \in Ax$ such that $\|y\| = |Ax|$, which implies $Ax \neq \emptyset$, and hence $x \in D(A)$. Therefore $D(\overset{\circ}{A}) \subset D(A)$, and we conclude that $D(A) = D(\overset{\circ}{A})$.

It remains to prove (5.5.155).

Let $x \in D(A)$ and $y \in Ax$. By accretivity of $A + \omega I$ and the fact that F is single-valued (since X is smooth), we have

$$\langle F(x-u), y + \omega x - (v + \omega u)\rangle \geq 0, \quad \forall (u,v) \in A. \tag{5.5.156}$$

Let $z \in \operatorname{conv} Ax$. Then there exist $y_i \in Ax$ and $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, such that $z = \sum_{i=1}^n \lambda_i y_i$. Thus, from (5.5.156),

$$\begin{aligned}\langle F(x-u), z + \omega x - (v+\omega u)\rangle &= \left\langle F(x-u), \sum_{i=1}^n \lambda_i y_i + \omega x - (v+\omega u)\right\rangle\\ &= \sum_{i=1}^n \lambda_i \langle F(x-u), y_i + \omega x - (v+\omega u)\rangle.\end{aligned}$$

From (5.5.156),

$$\langle F(x-u), y_i + \omega x - (v+\omega u)\rangle \geq 0, \quad i = 1, \ldots, n,$$

hence

$$\langle F(x-u), z + \omega x - (v+\omega u)\rangle \geq 0, \quad \forall z \in \operatorname{conv} Ax,\ \forall (u,v) \in A. \tag{5.5.157}$$

Since $F(x-u) \in X'$, the inequality (5.5.157) remains valid for all $z \in \overline{\operatorname{conv} Ax} := Bx$. Therefore,

$$\langle F(x-u), z + \omega x - (v+\omega u)\rangle \geq 0, \quad \forall (x,z) \in B,\ \forall (u,v) \in A.$$

Proceeding analogously, we obtain

$$\langle F(x-u), z + \omega x - (v+\omega u)\rangle \geq 0, \quad \forall (x,z), (u,v) \in B,$$

that is, $B \in \mathcal{A}(\omega)$.

By hypothesis, $D(A) \subset \operatorname{Im}(I + \lambda A)$ for $0 < \lambda < \lambda_0$. Hence, by item (ii) of Theorem 5.79, for each $x \in D(A)$,

$$\|A_\lambda x\| \leq (1 - \omega\lambda)^{-1}|Ax|, \quad 0 < \lambda < \lambda_0,\ \lambda\omega < 1. \tag{5.5.158}$$

Observe that if $\omega < 0$, then $1 - \lambda\omega > 1$ and thus $\frac{1}{1-\lambda\omega} < 1$. If $\omega \geq 0$, then $1 - \lambda\omega \geq 1 - \lambda_0\omega$, hence

$\frac{1}{1-\lambda\omega} \leq \frac{1}{1-\lambda_0\omega}$.

From (5.5.158), for each $x \in D(A)$ there exists $L > 0$ such that

$$\|A_\lambda x\| \leq L, \ \forall \lambda \in (0, \mu_0), \quad \mu_0 = \min\{\lambda_0, 1/\omega\}.$$

Thus, for every sequence $(\lambda_k) \subset (0, \mu_0)$ with $\lambda_k \to 0^+$, we have $\|A_{\lambda_k} x\| \leq L$. By reflexivity of X, there exists a subsequence $(\lambda_n) \subset (\lambda_k)$, $\lambda_n \to 0^+$, and $y \in X$ such that $A_{\lambda_n} x \rightharpoonup y$.

Now take a sequence $(\lambda_n) \subset (0, \mu_0)$ such that $\lambda_n \to 0^+$ and $A_{\lambda_n} x \rightharpoonup y \in X$. Note that $\lambda_n \omega < 1$ for all $n \in \mathbb{N}$, and by item (viii) of Theorem 5.79, $J^A_{\lambda_n} x \to x$ since $x \in D(A) \cap \left(\bigcap_{0<\lambda<\mu_0} D_\lambda\right)$. Moreover, by Proposition 5.99, we have $(J^A_{\lambda_n} x, A_{\lambda_n} x) \in A$. Thus

$$J^A_{\lambda_n} x \longrightarrow x, \quad A_{\lambda_n} x \rightharpoonup y, \quad \text{and } (J^A_{\lambda_n} x, A_{\lambda_n} x) \in A. \tag{5.5.159}$$

Since A is demiclosed by hypothesis, we deduce that $(x, y) \in A$, i.e., $y \in Ax$.

We now claim that

$$J^B_\lambda x = J^A_\lambda x, \quad \forall x \in D(A), \ \forall \lambda \in (0, \mu_0). \tag{5.5.160}$$

Indeed, let $x \in D(A)$. Since $D(A) \subset \mathrm{Im}(I + \lambda A) \subset \mathrm{Im}(I + \lambda B)$, we can write $x = x_1 + \lambda y_1 = x_2 + \lambda y_2$, with $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$. Thus $J^A_\lambda x = x_1$ and $J^B_\lambda x = x_2$, because in this case both J^A_λ and J^B_λ are single-valued. On the other hand, since $A \subset B$, $(x_1, y_1) \in B$ and thus $J^B_\lambda x = x_1 = J^A_\lambda x$, which proves (5.5.160). Consequently $J^B_{\lambda_n} x = J^A_{\lambda_n} x$, and so $A_{\lambda_n} x = B_{\lambda_n} x$. By lower semicontinuity of the norm in the weak topology of X,

$$\|y\| \leq \lim_{n\to+\infty} \inf \|A_{\lambda_n} x\| = \lim_{n\to+\infty} \inf \|B_{\lambda_n} x\|. \tag{5.5.161}$$

Furthermore, by item (ii) of Theorem 5.79,

$$\|B_{\lambda_n} x\| \leq (1 - \lambda_n \omega)^{-1} |Bx|, \tag{5.5.162}$$

so combining (5.5.161) and (5.5.162), we obtain

$$\begin{aligned} \|y\| &\leq \lim_{n\to+\infty} \inf (1 - \lambda_n \omega)^{-1} |Bx| \\ &= \lim_{n\to+\infty} (1 - \lambda_n \omega)^{-1} |Bx| = |Bx|. \end{aligned}$$

Since $y \in Ax \subset Bx$ and $\|y\| \leq |Bx|$, it follows that $\|y\| = |Bx|$ and therefore $y = \overset{\circ}{B} x$, because $\overset{\circ}{B} x$ is the unique element of Bx with this property. Thus $\overset{\circ}{B} x \in Ax$, proving (5.5.155) and completing the proof. □

Theorem 5.106 *Let X and X' be uniformly convex and let $A \in \mathcal{A}(\omega)$ be a closed operator such that*

$$D(A) \subset Im(I + \lambda A), \quad 0 < \lambda < \lambda_0 \quad (\lambda_0 \omega < 1).$$

Then there exists a demiclosed extension $\widetilde{A}$ of A such that

0.5cm

(i) $\widetilde{A} \in \mathcal{A}(\omega)$ and $D(\widetilde{A}) \subset Im(I + \lambda A) \subset Im(I + \lambda \widetilde{A})$, for $0 < \lambda < \lambda_0$ with $\lambda\omega < 1$;

(ii) $D(\widetilde{A}) = D(\overset{\circ}{\widetilde{A}}) = D(\overset{\circ}{A}) = D(A)$ and $\overset{\circ}{\widetilde{A}} x = \overset{\circ}{A} x, \quad \forall x \in D(A)$.

Proof:

(i) Let

$$\mathcal{F} = \left\{B : X \longrightarrow 2^X;\ B \in \mathcal{A}(\omega) \text{ and } D(B) \subset \overline{D(A)}\right\},$$

partially ordered by inclusion, and let $\mathcal{G}$ be a totally ordered subset of $\mathcal{F}$. Define the operator $T : X \longrightarrow 2^X$ by $D(T) = \bigcup\limits_{B\in\mathcal{G}} D(B)$ and

$$Tx = \bigcup \{Bx;\ x \in D(B) \text{ and } B \in \mathcal{G}\}.$$

From the definition of $\mathcal{G}$ and $D(T)$, we have $D(T) \subset \overline{D(A)}$ and, if $B \in \mathcal{G}$, then $T \supset B$. We now prove that $T \in \mathcal{A}(\omega)$. Indeed, take $(x, y), (u, v) \in T$, so that $(x, y) \in B_1$ and $(u, v) \in B_2$ for some $B_1, B_2 \in \mathcal{G}$. Since $\mathcal{G}$ is totally ordered, we may assume, without loss of generality, that $B_2 \supset B_1$. Hence $(x, y) \in B_2$. As $B_2 \in \mathcal{A}(\omega)$, Corollary 5.69 yields $f \in F(x - u)$ such that

$$\langle y + \omega x - v - \omega u, f\rangle \geq 0.$$

This shows that $T \in \mathcal{A}(\omega)$ and hence T is an upper bound for $\mathcal{G}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element $\widetilde{A} \supset A$. Since A is closed and $A \in \mathcal{A}(\omega)$, Proposition 5.84 implies that, if $\lambda \in (0, \lambda_0)$, then $D_\lambda = Im(I + \lambda A)$ is a closed subset of X. Thus, for $0 < \lambda < \lambda_0$ we have $\overline{D(A)} \subset Im(I + \lambda A)$. Therefore, from $A \subset \widetilde{A}$ it follows that $D(\widetilde{A}) \subset Im(I + \lambda A) \subset Im(I + \lambda\widetilde{A})$.

(ii) We prove that $\widetilde{A} + \omega I$ is maximal in $\overline{D(A)}$. Let B be an accretive extension of $\widetilde{A} + \omega I$ with $D(B) \subset \overline{D(A)}$. Then the operator $\widetilde{B} := B - \omega I$ is m-accretive and satisfies $D(\widetilde{B}) = D(B) \subset \overline{D(A)}$. If $x \in D(\widetilde{A})$, then $x \in D(\widetilde{B})$, since $D(\widetilde{A}) = D(\widetilde{A} + \omega I) \subset D(B) = D(\widetilde{B})$. Moreover, if $y \in \widetilde{A}x$, then $y + \omega x \in (\widetilde{A} + \omega I)x \subset Bx$, that is, $y \in (B - \omega I)x = \widetilde{B}x$. Thus, $\widetilde{B} \in \mathcal{F}$ and extends $\widetilde{A}$. Hence, $\widetilde{B} = \widetilde{A}$. Returning to the definition of $\widetilde{B}$, we obtain $B = \widetilde{A} + \omega I$.

Since X' is uniformly convex, Theorem 6.15 p57 in [47] implies that the norm on X is uniformly Fréchet differentiable and, by Proposition 5.97, $\widetilde{A}$ is demiclosed. Consequently, $D(\widetilde{A}) = D(\overset{\circ}{\widetilde{A}})$. It is clear that $D(A) \subset D(\widetilde{A})$. We now prove the reverse inclusion. Indeed, let $x \in D(\widetilde{A})$. By item (i), we already know that $D(\widetilde{A}) \subset Im(I + \lambda A)$ for $0 < \lambda < \lambda_0$. This implies that the set $\left\{\|\widetilde{A}_\lambda x\|;\ \lambda \in (0, \lambda_0)\right\}$ is bounded. Again, since X' is uniformly convex, Milman's theorem implies that X' is reflexive and therefore X is reflexive as well. Thus, there exist $\{\lambda_n\} \subset (0, \lambda_0)$ and $y \in X$ such that $\lambda_n \longrightarrow 0^+$ and $\widetilde{A}_{\lambda_n} x \rightharpoonup y$ as $n \longrightarrow \infty$.

Since $x \in D(\widetilde{A})$, we have

$$J^{\widetilde{A}}_{\lambda_n} x \longrightarrow x \text{ and } \left(J^{\widetilde{A}}_{\lambda_n} x, \widetilde{A}_{\lambda_n} x\right) \in \widetilde{A}. \tag{5.5.163}$$

Because $\widetilde{A}$ is demiclosed, it follows that $(x, y) \in \widetilde{A}$. Furthermore, from the convergence $\widetilde{A}_{\lambda_n} x \rightharpoonup y$ and Theorem 5.79, item (ii), we obtain

$$\begin{aligned} \|y\| &\leq \liminf_{n\to+\infty} \|\widetilde{A}_{\lambda_n} x\| \\ &\leq \liminf_{n\to+\infty} (1 - \lambda_n\omega)^{-1}|\widetilde{A}x| \\ &= |\widetilde{A}x| = \|\overset{\circ}{\widetilde{A}}\, x\|, \end{aligned}$$

since $\overset{\circ}{\widetilde{A}}$ is single-valued by Theorem 5.105.

But $y \in \widetilde{A}x$, hence $\| \overset{\circ}{\widetilde{A}} x\| = |\widetilde{A}x| \le \|y\|$. This, together with (5.5.164), implies that $\lim_{n\to\infty} \|\widetilde{A}_{\lambda_n} x\| = \|y\|$. Since X is uniformly convex, we conclude that $\widetilde{A}_{\lambda_n} x \to y$.

For $x \in D(\widetilde{A})$, it makes sense to consider both $J_\lambda^{\widetilde{A}} x$ and $J_\lambda^A x$, since we already know that

$$D(\widetilde{A}) \subset Im(I + \lambda A) \subset Im(I + \lambda \widetilde{A}).$$

If $x = x_1 + \lambda y_1 = x_2 + \lambda y_2$, with $(x_1, y_1) \in A$ and $(x_2, y_2) \in \widetilde{A}$, then, since $A \subset \widetilde{A}$, we also have $(x_1, y_1) \in \widetilde{A}$, and therefore

$$J_\lambda^{\widetilde{A}} x = x_1 = J_\lambda^A x.$$

It follows that $\widetilde{A}_\lambda x = A_\lambda x$ and hence

$$J_{\lambda_n}^{\widetilde{A}} x \to x \text{ and } A_{\lambda_n} x \to y.$$

Since A is a closed operator, we conclude that $(x, y) \in A$. This means that $x \in D(A)$, $y \in Ax$ and $\|y\| = |\widetilde{A}x|$. Consequently, $D(\widetilde{A}) = D(A)$.

We next show that $D(\widetilde{A}) \subset D(\overset{\circ}{A})$. Indeed, let $x \in D(\widetilde{A})$. From the previous arguments there exists $y \in X$ such that $y \in Ax$ and $\|y\| = |\widetilde{A}x|$. Since $Ax \subset \widetilde{A}x$, we obtain

$$\|y\| = |\widetilde{A}x| \le |Ax| \le \|y\|,$$

because $y \in Ax$. Hence $\|y\| = |Ax|$, so $y \in \overset{\circ}{A} x$, and therefore $x \in D(\overset{\circ}{A})$.

By the definition of $\overset{\circ}{A}$ we have $D(\overset{\circ}{A}) \subset D(A)$. Thus we have shown that

$$D(\widetilde{A}) \subset D(\overset{\circ}{A}) \subset D(A) = D(\widetilde{A}).$$

To conclude the proof, we show that for every $x \in D(A)$, $\overset{\circ}{\widetilde{A}} x = \overset{\circ}{A} x$. Since $\overset{\circ}{\widetilde{A}}$ is single-valued, it suffices to prove that $\overset{\circ}{A} x \subset \overset{\circ}{\widetilde{A}} x$.

Let $y_1 \in \overset{\circ}{A} x = \{y \in Ax;\ \|y\| = |Ax|\}$. Since $x \in D(\widetilde{A})$, there exists $y_2 \in Ax$ such that $\|y_2\| = |Ax|$. But $Ax \subset \widetilde{A}x$, so $|\widetilde{A}x| \le |Ax|$. Thus

$$\|y_2\| = |\widetilde{A}x| \le |Ax| \le \|y_2\|.$$

Now $y_1 \in Ax$ and $\|y_1\| = |Ax| = |\widetilde{A}x|$. Therefore $y_1 \in \overset{\circ}{\widetilde{A}} x$. □

Lemma 5.107 *Let X' be a uniformly convex space and $A \in \mathcal{A}(\omega)$ such that $A + \omega I$ is maximal in $C \supseteq D(A)$. Then $F(\overset{\circ}{A} x)$ has a single element for every $x \in D(A)$.*

Proof: By Proposition 5.11 p35 in [47], X is strictly convex, and by Proposition 5.38, X is smooth. Hence, by Proposition 5.94, Ax is convex and closed. Since X is reflexive, Milman's theorem and Theorem teo 5.14 p35 in [47] yield $\overset{\circ}{A} x \ne \emptyset$. Let $y_1, y_2 \in \overset{\circ}{A} x$. Then $y_1, y_2 \in Ax$ and $\|y_1\| = \|y_2\| = |Ax|$. By Lemma

5.100 (Kato's lemma),

$$\begin{aligned}\|y_1\|^2 &\le \langle y_2, F(y_1)\rangle \\ &\le \|y_2\|\|F(y_1)\| \\ &= \|y_2\|\|y_1\| \\ &= \|y_2\|^2 = \|y_1\|^2.\end{aligned}$$

Thus

$$\|y_2\|^2 = \langle y_2, F(y_1)\rangle = \|y_2\|^2.$$

By the definition of F, this implies $F(y_1) \in F(y_2)$. Since X is smooth, it follows that $F(y_1) = F(y_2)$. □

Proposition 5.108 *Let X' be a uniformly convex space, let $A \in \mathcal{A}(\omega)$ with $A + \omega I$ maximal in $\overline{D(A)}$, and assume*

$$D(A) \subset Im(I + \lambda A),\ 0 < \lambda < \lambda_0;\ \lambda_0\omega < 1.$$

Then:

(i) There exists a sequence $(\lambda_n) \subset (0, \lambda_0)$ such that

$$\lim_{n\to\infty} F(A_{\lambda_n} x) = F(\overset{\circ}{A}x), \quad \forall x \in D(A);$$

(ii) If X is uniformly convex, there exists a sequence $(\lambda_n) \subset (0, \lambda_0)$ such that

$$\lim_{n\to\infty} A_{\lambda_n} x = \overset{\circ}{A}x, \quad \forall x \in D(A).$$

Proof:

(i) Fix $x \in D(A)$. Since $A \in \mathcal{A}(\omega)$ and $\big(J_\lambda x, A_\lambda x\big) \in A$, by the definition of an accretive operator we have

$$\langle y + \omega x - A_\lambda x - \omega J_\lambda x, F(x - J_\lambda x)\rangle \ge 0, \quad \forall y \in Ax.$$

Using that $\lambda F(z) = F(\lambda z)$ and $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$, we obtain

$$\lambda\langle y, F(A_\lambda x)\rangle - \lambda\langle A_\lambda x, F(A_\lambda x)\rangle + \lambda^2\omega\langle A_\lambda x, F(A_\lambda x)\rangle \ge 0,$$

for every $y \in Ax$. Hence

$$(1 - \lambda\omega)\|A_\lambda x\|^2 \le \langle y, F(A_\lambda x)\rangle, \quad \forall y \in Ax. \tag{5.5.164}$$

By Proposition 5.99, the family $\{A_\lambda x\}$ is bounded and therefore $\{F(A_\lambda x)\}$ is also bounded. By Milman's theorem, X' is reflexive, and thus there exist $\lambda_n \longrightarrow 0^+$ and $u' \in X'$ such that

$$F\big(A_{\lambda_n} x\big) \rightharpoonup u'. \tag{5.5.165}$$

Passing to the limit in (5.5.164), and using (5.5.165) together with Proposition **??**, we get

$$|Ax|^2 \le \langle y, u'\rangle \le \|y\|\|u'\|, \quad \forall y \in Ax. \tag{5.5.166}$$

In particular, this inequality holds for $y \in \overset{\circ}{A}x$, i.e., for $\|y\| = |Ax|$. Hence

$$|Ax| \le \|u'\|. \tag{5.5.167}$$

By weak lower semicontinuity of the norm, from (5.5.165) and Theorem 5.79, item (ii), we obtain

$$\|u'\| \le \lim_{n\to+\infty}\inf\|F(A_{\lambda_n} x)\| = \lim_{n\to+\infty}\inf\|A_{\lambda_n} x\| \le \lim_{n\to+\infty}\inf(1 - \lambda_n\omega)^{-1}|Ax| = |Ax|,$$

which implies

$$\|u'\| \le |Ax|. \tag{5.5.168}$$

Thus,

$$|Ax| = \|u'\|. \tag{5.5.169}$$

Again, for $y \in \overset{\circ}{A} x$ we have

$$|Ax|^2 \le \langle y, u' \rangle \le \|y\|\|u'\| = |Ax|\|u'\| = |Ax|^2.$$

Therefore,

$$\langle y, u' \rangle = \|u'\|^2 = \|y\|^2,$$

that is, $u' \in F(y)$ for every $y \in \overset{\circ}{A} x$. By Lemma 5.107, $F(\overset{\circ}{A} x)$ has a single element, i.e., $u' = F(\overset{\circ}{A} x)$. Since X' is uniformly convex and

$$F(A_{\lambda_n} x) \rightharpoonup u' \quad \text{and} \quad \limsup_{n\to\infty} \|A_{\lambda_n} x\| \le \|u'\|,$$

it follows that

$$F(A_{\lambda_n} x) \longrightarrow u' = F(\overset{\circ}{A} x), \tag{5.5.170}$$

which proves (i).

(ii) Now assume that X is uniformly convex. Let G be the duality mapping of X'. Then

$$G(x') = \{x \in X;\ \langle x', x \rangle = \|x\| = \|x'\|\},$$

since X is reflexive (because X' is reflexive). Hence

$$x' \in F(x) \Longleftrightarrow x \in G(x'),$$

that is, $FG = GF = I$, which implies $G = F^{-1}$, since G is single-valued.

By Theorem 6.15 p57 in [47], G is uniformly continuous on bounded sets. As $\{A_\lambda x\}$ $(0 < \lambda < \lambda_0)$ is bounded in X and $\overset{\circ}{A} x$ is single-valued, by (5.5.170) we have

$$\overset{\circ}{A} x = F^{-1}F(\overset{\circ}{A} x) = F^{-1}\left(\lim_{\lambda\to 0^+} F(A_\lambda x)\right) = \lim_{\lambda\to 0^+} F^{-1}F(A_\lambda x) = \lim_{\lambda\to 0^+} A_\lambda x.$$

□

Definition 5.109 *Let $A+\omega I$ be an m-accretive operator. We say that a single-valued operator $A' \subset A$ is a* <u>*principal section*</u> *of A if $D(A') = D(A)$ and, whenever $(x, y) \in \overline{D(A)} \times X$ satisfies*

$$\langle y + \omega x - A'u - \omega u, \xi' \rangle \ge 0,\ \forall u \in D(A) \text{ and } \forall \xi' \in F(x - u),$$

then $(x, y) \in A$. In other words, A' is a principal section if every extension of A' with domain contained in $\overline{D(A)}$, belonging to $\mathcal{A}(\omega)$, is contained in A.

If X is a Hilbert space and A is m-accretive, then the minimal section $\overset{\circ}{A}$ is a principal section (see [17]). Later we shall prove a more general result in this direction. For the moment, we prove the following auxiliary result:

Proposition 5.110 *Let X be a separable Banach space, X' uniformly convex, $A + \omega I$ an m-accretive operator and $A' \subset A$ an operator such that*

(i) $D(A') = D(A)$;

(ii) For every $u \in D(A)$ there exists $v \in A'u$ such that

$$\|v\| \leq \theta\big(|Au|\big), \tag{5.5.171}$$

where $\theta : [0, +\infty) \longrightarrow \mathbb{R}$ is bounded on bounded intervals.

If $x \in D(A)$ and $y \in X$ are such that

$$\langle y + \omega x - v - \omega u, F(x-u)\rangle \geq 0, \quad \forall (u,v) \in A',$$

then $(x,y) \in A$.

Proof: Let $y \in X$ and define $\widetilde{A} = A - y$. Then, by Proposition 5.93, $\widetilde{A} + \omega I$ is m-accretive. Set

$$\widetilde{\theta}(|\widetilde{A}x|) := \theta(|Ax|) + \|y\|.$$

The operators $\widetilde{A}$ and $\widetilde{A}'$, as well as $\widetilde{\theta}$, satisfy the assumptions imposed on A, A' and θ. Indeed, we have $D(\widetilde{A}') = D(\widetilde{A})$, the function $\widetilde{\theta}$ is bounded on bounded intervals, and, from (5.5.171), for every $u \in D(\widetilde{A}) = D(A)$ there exists $v - y \in \widetilde{A}'u$ such that

$$\|v - y\| \leq \|v\| + \|y\| \leq \theta(|Au|) + \|y\| = \widetilde{\theta}(|\widetilde{A}u|).$$

Note that $(x,y) \in A$ if and only if $(x,0) \in \widetilde{A}$. Therefore, without loss of generality, we may assume $y = 0$. In this case, the hypothesis reads: if $x \in D(A)$, then

$$\langle \omega x - v - \omega u, F(x-u)\rangle \geq 0, \quad \forall (u,v) \in A'. \tag{5.5.172}$$

We must prove that $(x,0) \in A$.

Let $\lambda > 0$ be such that $\lambda\omega < 1$. Since $A + \omega I$ is m-accretive, it is maximal accretive in $\overline{D(A)}$. Moreover, $D(A) \subset \operatorname{Im}(I + \lambda A)$ for every $\lambda > 0$ with $\lambda\omega < 1$. Indeed,

$$\operatorname{Im}(A + \omega I + \mu I) = X,\ \forall \mu > 0 \ \Rightarrow\ \operatorname{Im}\big((\omega+\mu)\big(\tfrac{1}{\omega+\mu}A + I\big)\big) = X \ \Rightarrow\ \operatorname{Im}\big(\tfrac{1}{\omega+\mu}A + I\big) = X.$$

Taking $\mu = \frac{1-\lambda\omega}{\lambda} > 0$, we obtain $\operatorname{Im}(I + \lambda A) = X$, and hence $D(A) \subset \operatorname{Im}(I + \lambda A)$, as desired.

Let $u = J_\lambda x$ (with $\lambda > 0$ such that $\lambda\omega < 1$). From (5.5.172) we have

$$\langle \omega x - v - \omega J_\lambda x, F(x - J_\lambda x)\rangle \geq 0, \quad \forall v \in A'(J_\lambda x).$$

Multiplying and dividing by λ^2, we get

$$\lambda^2 \langle \omega A_\lambda x - \tfrac{v}{\lambda}, F(A_\lambda x)\rangle \geq 0, \quad \forall v \in A'(J_\lambda x),$$

which implies

$$\lambda^2 \langle \omega A_\lambda x, F(A_\lambda x)\rangle \geq \lambda \langle v, F(A_\lambda x)\rangle, \quad \forall v \in A'(J_\lambda x),$$

and hence

$$\lambda\omega \langle A_\lambda x, F(A_\lambda x)\rangle \geq \langle v, F(A_\lambda x)\rangle, \quad \forall v \in A'(J_\lambda x),\ \forall \lambda > 0,$$

so that

$$\langle v, F(A_\lambda x)\rangle \leq \lambda\omega \|A_\lambda x\|^2, \quad \forall v \in A'(J_\lambda x). \tag{5.5.173}$$

On the other hand, by Proposition 5.99, $A_\lambda x \in AJ_\lambda x$. Thus, by Theorem 5.79, item (ii), we obtain

$$|AJ_\lambda x| \leq \|A_\lambda x\| \leq (1 - \lambda\omega)^{-1} |Ax|. \tag{5.5.174}$$

Since $J_\lambda x \in D(A')$, hypothesis (ii) gives $v_\lambda \in A'(J_\lambda x)$ such that

$$\|v_\lambda\| \le \theta(|A(J_\lambda x)|).$$

By (5.5.174), the set $\{|A(J_\lambda x)|\}_{0<\lambda<\lambda_0}$, $\lambda_0\omega < 1$, is bounded. As θ is bounded on bounded sets, it follows that $\{\|v_\lambda\|\}_{0<\lambda<\lambda_0}$, $\lambda_0\omega < 1$, is bounded.

Since X' is uniformly convex, it is reflexive, and consequently X is reflexive as well. Therefore, passing to a subsequence if necessary, there exist $(\lambda_n) \to 0$ and $z \in X$ such that $v_{\lambda_n} \rightharpoonup z$. From (5.5.173) and Proposition 5.108, item (i), in the limit we have

$$\langle z, F(\overset{\circ}{A}\, x)\rangle \le 0. \tag{5.5.175}$$

On the other hand, since X' is uniformly convex, Theorem teo 6.15 p57 in [47] implies that the norm is Fréchet differentiable. By Proposition 5.97, the operator A is demiclosed and, moreover, by Theorem 5.79, item (viii), we have $J_{\lambda_n} x \to x$. Since $v_{\lambda_n} \rightharpoonup z$ and $(J_{\lambda_n} x, v_{\lambda_n}) \in A' \subset A$, it follows that $(x, z) \in A$, i.e., $z \in Ax$.

By Lemma 5.107, the set $F(\overset{\circ}{A}\, x)$ has a single element. Furthermore, since A is demiclosed, Theorem 5.105 implies that $\overset{\circ}{A}\, x$ is also single-valued. Hence, by Lemma 5.100,

$$\overset{\circ}{A}\, x \in Ax \text{ and } \|\overset{\circ}{A}\, x\|^2 \le \langle F(\overset{\circ}{A}\, x), y\rangle, \ \forall y \in Ax.$$

In particular, for $z \in Ax$, (5.5.175) yields

$$\|\overset{\circ}{A}\, x\|^2 \le \langle F(\overset{\circ}{A}\, x), z\rangle \le 0 \quad \Rightarrow \quad \overset{\circ}{A}\, x = 0 \quad \Rightarrow \quad 0 \in Ax.$$

Thus $(x, 0) \in A$, as required. □

Corollary 5.111 *Let X be a Banach space with X' uniformly convex, and let A and B be such that $D(A) = D(B)$, $A + \omega I$ and $B + \omega I$ are m-accretive, and*

$$\overset{\circ}{A}\, x \cap \overset{\circ}{B}\, x \neq \emptyset, \quad \textit{for every } x \in D(A).$$

Then $A = B$. In particular, if $\overset{\circ}{A} = \overset{\circ}{B}$, then $A = B$.

Proof: If $A + \omega I$ and $B + \omega I$ are m-accretive, then these operators are maximal accretive for every $C \supset D(A) = D(B)$. Since X' is uniformly convex, X' is strictly convex and, consequently, X is smooth. Moreover, X' is reflexive and therefore X is also reflexive. For each $x \in D(A) = D(B)$, the sets Ax and Bx are convex, closed and nonempty. Consequently, $\overset{\circ}{A}\, x \neq \emptyset$ and $\overset{\circ}{B}\, x \neq \emptyset$ for all $x \in D(A)$.

Fix $x \in D(A) = D(B)$ and $y \in \overset{\circ}{A}\, x \cap \overset{\circ}{B}\, x$. Define the operator S by $Sx = y$. Then $S \subset A$, $S \subset B$ and, in addition,

$$\|Sx\| = \|y\| = |Ax| \quad \text{and} \quad \|Sx\| = \|y\| = |Bx|, \ \forall x \in D(A) = D(B).$$

Thus S satisfies the assumptions of Proposition 5.110. Hence, if $(x, y) \in A$, then

$$\langle y + \omega x - Su - \omega u, F(x - u)\rangle \ge 0, \ \forall u \in D(A) = D(B),$$

and Proposition 5.110 implies $(x, y) \in B$, that is, $A \subset B$. Similarly, we obtain $B \subset A$. □

5.6 Perturbation of Accretive Operators

The sum of two accretive operators in a smooth Banach space is an accretive operator and, more generally, if $A \in \mathcal{A}(\omega_1)$ and $B \in \mathcal{A}(\omega_2)$, then $(A+B) \in \mathcal{A}(\omega_1+\omega_2)$; this follows immediately from item (iv) of Corollary 5.69. Indeed, let

$$(x_1, y_1+z_1+(\omega_1+\omega_2)x_1),\ (x_2, y_2+z_2+(\omega_1+\omega_2)x_2) \in A+B+(\omega_1+\omega_2)I,$$

where $(x_1,y_1),(x_2,y_2) \in A$ and $(x_1,z_1),(x_2,z_2) \in B$. We shall prove that $A+B+(\omega_1+\omega_2)I$ is accretive. In fact, since $A+\omega_1 I$ and $B+\omega_2 I$ are accretive, it follows from item (iv) of Corollary 5.69 that there exists $x' \in F(x_1-x_2)$ such that

$$\big\langle F(x_1-x_2), y_1+\omega_1 x_1-(y_2+\omega_1 x_2)\big\rangle \geq 0$$

and

$$\big\langle F(x_1-x_2), z_1+\omega_2 x_1-(z_2+\omega_2 x_2)\big\rangle \geq 0,$$

because X' being smooth implies $F(x_1-x_2)=x'$.

It then follows that

$$\big\langle F(x_1-x_2), y_1+\omega_1 x_1-(y_2+\omega_1 x_2)+z_1+\omega_2 x_1-(z_2+\omega_2 x_2)\big\rangle \geq 0,$$

that is,

$$\big\langle F(x_1-x_2), y_1+z_1+(\omega_1+\omega_2)x_1-[y_2+z_2+(\omega_1+\omega_2)x_2\big\rangle] \geq 0.$$

Therefore, by Corollary 5.69, $A+B+(\omega_1+\omega_2)I$ is accretive.

However, even if $A+\omega_1 I$ and $B+\omega_2 I$ are m-accretive, the operator $A+B+(\omega_1+\omega_2)I$ is not necessarily m-accretive. In what follows, we establish sufficient conditions for the sum of an m-accretive operator $A+\omega I$ with an m-accretive operator B to be m-accretive.

Remark 5.112 *In practice, and more specifically in PDE applications, what really matters to us is the condition*

$$D(A) \subset Im(I+\lambda A), \quad \lambda \in]0,\lambda_0[,\ \lambda\omega<1,\ A \in \mathcal{A}(\omega),$$

imposed in the previous sections, which is strictly weaker than requiring A to be m-accretive.

Lemma 5.113 *Let X be a smooth Banach space, $A+\omega I$ an m-accretive operator on X and B a single-valued, Lipschitz and accretive operator such that $D(B)=X$. Then $A+B+\omega I$ is m-accretive.*

Proof: Since $A+\omega I$ and B are accretive and X is smooth, we have that $A+B+\omega I$ is accretive. It remains to show that

$$Im\big[I+\lambda(A+B+\omega I)\big]=X, \tag{5.6.176}$$

for some $\lambda>0$, in view of Proposition 5.87. In other words, we must prove that for every $y \in X$ there exists $x \in D(A)$ such that

$$y \in \big[I+\lambda(A+B+\omega I)\big]x, \quad \text{for some } \lambda>0.$$

Showing this is equivalent to showing that for every $y \in X$ there exists $x \in D(A)$ such that

$$(y-\lambda Bx) \in \big[I+\lambda(A+\omega I)\big]x, \quad \text{for some } \lambda>0,$$

or, using the fact that $J_\lambda^{A+\omega I}$ is single-valued and defined on the whole of X (since $A+\omega I$ is m-accretive), that we must show

$$\begin{cases} \text{For each } y \in X, \text{ there exists } x \in D(A) \text{ such that} \\ J_\lambda^{A+\omega I}(y-\lambda Bx)=x, \text{ for some } \lambda>0. \end{cases} \tag{5.6.177}$$

Indeed, assuming for a moment that (5.6.177) holds, we have

$$(y - \lambda Bx, x) \in J_\lambda^{A+\omega I} = \{(v + \lambda z, v);\ (v, z) \in A + \omega I\},$$

and thus $y - \lambda Bx = v + \lambda z$ and $v = x$ for some $(v, z) \in A + \omega I$. Hence

$$y - \lambda Bx \in [I + \lambda(A + \omega I)]v = [I + \lambda(A + \omega I)]x,$$

which proves (5.6.176). Let us show that (5.6.177) indeed holds. Fix $y \in X$ and define the map

$$\begin{array}{rccc} G: & X & \longrightarrow & X \\ & x & \longmapsto & G(x) = J_\lambda^{A+\omega I}(y - \lambda Bx) \end{array}.$$

We claim that G is a contraction for $\lambda = \frac{1}{2L}$, where $L > 0$ is the Lipschitz constant of B. In fact, if $x_1, x_2 \in X$, then

$$\|G(x_1) - G(x_2)\| = \left\| J_\lambda^{A+\omega I}(y - \lambda Bx_1) - J_\lambda^{A+\omega I}(y - \lambda Bx_2) \right\|. \tag{5.6.178}$$

Note that $A + \omega I \in \mathcal{A}(0)$ and $\lambda \cdot 0 = 0 < 1$ for all $\lambda > 0$. By item (i) of Theorem 5.79, $J_\lambda^{A+\omega I}$ is Lipschitz with constant $(1 - \lambda \cdot 0)^{-1} = 1$.

Therefore, from (5.6.178) we obtain

$$\begin{aligned} \|G(x_1) - G(x_2)\| &\leq \|y - \lambda Bx_1 - y + \lambda Bx_2\| \\ &= \lambda \|Bx_1 - Bx_2\| \\ &\leq \frac{1}{2L} L \|x_1 - x_2\| = \frac{1}{2} \|x_1 - x_2\|. \end{aligned}$$

Hence G is a contraction and thus has a unique fixed point, that is, there exists a unique $x \in X$ such that $G(x) = x$, or equivalently,

$$J_\lambda^{A+\omega I}(y - \lambda Bx) = x,$$

for $\lambda = \frac{1}{2L} > 0$. Since $J_\lambda^{A+\omega I} : X \longrightarrow D(A)$, we deduce that $x \in D(A)$, which proves (5.6.177) and hence the lemma. □

Lemma 5.114 *If B is an accretive, single-valued, Lipschitz operator with $D(B) = X$, then B is m-accretive.*

Proof: For each $y \in X$ we must find $x \in D(B) = X$ such that $y = (I + \lambda B)x$ for some $\lambda > 0$, that is, for each $y \in X$ there must exist $x \in X$ such that $x = y - \lambda Bx$ for some $\lambda > 0$. To this end, it suffices to show that the map

$$\begin{array}{rccc} G: & X & \longrightarrow & X \\ & x & \longmapsto & G(x) = y - \lambda Bx \end{array}$$

has a fixed point. Indeed, for $x_1, x_2 \in X$ we have

$$\begin{aligned} \|G(x_1) - G(x_2)\| &= \|-\lambda Bx_1 + \lambda Bx_2\| \\ &= \lambda \|Bx_1 - Bx_2\| \\ &\leq \lambda L \|x_1 - x_2\|, \end{aligned}$$

where $L > 0$ is the Lipschitz constant of B. Taking $\lambda = \frac{1}{2L}$ we obtain the desired contraction. □

Proposition 5.115 *Let X be a smooth Banach space and let $A + \omega I$ and B be m-accretive operators on*

X. Then, for each $y \in X$ and each $\lambda > 0$, there exist $x_\lambda \in D(A)$ and $u_\lambda \in Ax_\lambda$ such that

$$y = (1+\omega)x_\lambda + u_\lambda + B_\lambda x_\lambda,$$

where $B_\lambda := \dfrac{1}{\lambda}(I - J_\lambda^B)$ is the Yoshida approximation of B. Moreover, if $D(A) \cap D(B) \neq \emptyset$, then $(x_\lambda)_\lambda$ is bounded.

Proof: Since B is m-accretive, B is accretive, i.e., $B \in \mathcal{A}(0)$ and $Im(I + \lambda B) = X$ for every $\lambda > 0$. Thus $\lambda \cdot 0 = 0 < 1$ for all $\lambda > 0$. Therefore, by item (i) of Theorem 5.79, J_λ^B is single-valued for all $\lambda > 0$, and so B_λ is single-valued for all $\lambda > 0$. By item (v) of the same theorem, $B_\lambda \in \mathcal{A}(0)$, that is, B_λ is accretive for every $\lambda > 0$. In addition, by item (vi) of that theorem, B_λ is Lipschitz with constant $\lambda^{-1}[1 + (1 - \lambda|0|)]^{-1} = \frac{2}{\lambda}$, for every $\lambda > 0$. Moreover,

$$D(B_\lambda) = Im(I + \lambda B) = X, \quad \forall \lambda > 0,$$

and from the remarks above we conclude that B_λ is single-valued, Lipschitz, accretive and $D(B_\lambda) = X$ for all $\lambda > 0$. By Lemma 5.113, it follows that $A + B_\lambda + \omega I$ is m-accretive for all $\lambda > 0$. Hence

$$Im\big[I + (A + B_\lambda + \omega I)\big] = X, \quad \forall \lambda > 0. \tag{5.6.179}$$

Fix $y \in X$ and $\lambda > 0$. Then, for each $\lambda > 0$, in view of (5.6.179) there exists $x_\lambda \in D(A)$ such that

$$y \in x_\lambda + Ax_\lambda + B_\lambda x_\lambda + \omega x_\lambda,$$

that is,

$$y = (1+\omega)x_\lambda + u_\lambda + B_\lambda x_\lambda, \tag{5.6.180}$$

for some $u_\lambda \in Ax_\lambda$. It remains to show that if $D(A) \cap D(B) \neq \emptyset$ then (x_λ) is bounded. Take $x_0 \in D(A) \cap D(B)$ and consider $y \in X$ and $\lambda > 0$. Then, by (5.6.180) there exists $x_\lambda \in D(A)$ such that

$$y \in (1+\omega)x_\lambda + Ax_\lambda + B_\lambda x_\lambda.$$

Let also

$$y_\lambda \in (1+\omega)x_0 + Ax_0 + B_\lambda x_0.$$

Then

$$y_\lambda - x_0 \in (A + B_\lambda + \omega I)x_0$$

and

$$y - x_\lambda \in (A + B_\lambda + \omega I)x_\lambda.$$

By the accretivity of $(A + B_\lambda + \omega I)$ and since X is smooth (and thus F is single-valued), Corollary 5.69 yields

$$\big\langle F(x_\lambda - x_0), y - x_\lambda - (y_\lambda - x_0)\big\rangle \geq 0,$$

that is,

$$\big\langle F(x_\lambda - x_0), y - y_\lambda\big\rangle - \big\langle F(x_\lambda - x_0), x_\lambda - x_0\big\rangle \geq 0.$$

But

$$\big\langle F(x_\lambda - x_0), x_\lambda - x_0\big\rangle = \|x_\lambda - x_0\|^2,$$

hence

$$\begin{aligned}\|x_\lambda - x_0\|^2 &\leq \big\langle F(x_\lambda - x_0), y - y_\lambda\big\rangle \\ &\leq \|F(x_\lambda - x_0)\| \, \|y - y_\lambda\| = \|x_\lambda - x_0\| \, \|y - y_\lambda\| .\end{aligned} \tag{5.6.181}$$

Note that if $\|x_\lambda - x_0\| = 0$, i.e. $x_\lambda = x_0$, then $\|x_\lambda\| = \|x_0\|$.

If instead $\|x_\lambda - x_0\| \neq 0$, then from (5.6.181) we have

$$\|x_\lambda - x_0\| \leq \|y - y_\lambda\|,$$

or yet

$$\|x_\lambda\| \leq \|x_0\| + \|y\| + \|y_\lambda\|. \tag{5.6.182}$$

Since $y_\lambda \in (1+\omega)x_0 + Ax_0 + B_\lambda x_0$, we may write

$$y_\lambda = (1-\omega)x_0 + u_0 + B_\lambda x_0, \tag{5.6.183}$$

for some $u_0 \in Ax_0$. On the other hand, since $B \in \mathcal{A}(0)$, Theorem 5.79, item (ii), gives

$$\|B_\lambda x\| \leq |Bx|, \quad \forall x \in D(B) \cap D_\lambda^B, \ \forall \lambda > 0,$$

and $D_\lambda^B = X$. Thus

$$\|B_\lambda x_0\| \leq |Bx_0|, \quad \forall \lambda > 0, \tag{5.6.184}$$

and combining (5.6.183) and (5.6.184) we obtain

$$\|y_\lambda\| \leq \|(1-\omega)x_0\| + \|u_0\| + |Bx_0|, \quad \forall \lambda > 0. \tag{5.6.185}$$

Combining (5.6.182) and (5.6.185) we conclude that

$$\|x_\lambda\| \leq k, \quad \forall \lambda > 0 \text{ whenever } x_\lambda \neq x_0.$$

Therefore

$$\|x_\lambda\| \leq M, \quad \forall \lambda > 0,$$

where $M = \max\{\|x_0\|, k\}$, which completes the proof. □

Proposition 5.116 *Let X' be uniformly convex, and let $A + \omega I$ and B be m-accretive operators such that $D(A) \cap D(B) \neq \emptyset$. Suppose that, for each $y \in X$ and $\lambda > 0$, there exists $x_\lambda \in D(A)$ such that*

$$y = (1+\omega)x_\lambda + u_\lambda + B_\lambda x_\lambda, \quad \textit{for some } u_\lambda \in Ax_\lambda.$$

Moreover, assume that $B_\lambda x_\lambda$ is bounded on some interval $(0, \lambda_0)$. Then, for each $y \in X$, there exists a unique $x \in D(A) \cap D(B)$ such that $x_\lambda \to x$ as $\lambda \to 0^+$ and

$$y \in (1+\omega)x + Ax + Bx.$$

Hence $A + B + \omega I$ is m-accretive.

Proof: Since X' is uniformly convex, X' is strictly convex and consequently X is smooth. Furthermore, X' is reflexive and therefore X is also reflexive. Let $y \in X$ and $\lambda, \mu > 0$. By Proposition 5.115 there exist (x_λ, u_λ), $(x_\mu, u_\mu) \in A$ such that

$$y = (1+\omega)x_\lambda + u_\lambda + B_\lambda x_\lambda$$

and

$$y = (1+\omega)x_\mu + u_\mu + B_\mu x_\mu.$$

It follows that

$$0 = (x_\lambda - x_\mu) + \omega(x_\lambda - x_\mu) + (u_\lambda - u_\mu) + (B_\lambda x_\lambda - B_\mu x_\mu). \tag{5.6.186}$$

Since X is smooth, the duality map F is single-valued and therefore, from (5.6.186),

$$\langle F(x_\lambda - x_\mu), (x_\lambda - x_\mu) + \omega(x_\lambda - x_\mu) + (u_\lambda - u_\mu) + (B_\lambda x_\lambda - B_\mu x_\mu)\rangle = 0,$$

that is,

$$\begin{aligned} \|x_\lambda - x_\mu\|^2 + \langle F(x_\lambda - x_\mu), (u_\lambda + \omega x_\lambda) - (u_\mu + \omega x_\mu)\rangle \\ + \langle F(x_\lambda - x_\mu), B_\lambda x_\lambda - B_\mu x_\mu\rangle = 0. \end{aligned} \tag{5.6.187}$$

However, by the accretivity of $A + \omega I$, Corollary 5.69 gives

$$\langle F(x_\lambda - x_\mu), (u_\lambda + \omega x_\lambda) - (u_\mu + \omega x_\mu)\rangle \geq 0.$$

From this and (5.6.187) we obtain

$$\|x_\lambda - x_\mu\|^2 + \langle F(x_\lambda - x_\mu), B_\lambda x_\lambda - B_\mu x_\mu\rangle \leq 0. \tag{5.6.188}$$

Moreover, by the accretivity of B, Proposition 5.75 yields

$$(J_\lambda^B x_\lambda, B_\lambda x_\lambda),\ (J_\mu^B x_\mu, B_\mu x_\mu) \in B.$$

It then follows from Corollary 5.69 that

$$\langle F(J_\lambda^B x_\lambda - J_\mu^B x_\mu), B_\lambda x_\lambda - B_\mu x_\mu\rangle \geq 0,$$

or equivalently,

$$\langle -F(J_\lambda^B x_\lambda - J_\mu^B x_\mu), B_\lambda x_\lambda - B_\mu x_\mu\rangle \leq 0. \tag{5.6.189}$$

Adding (5.6.188) and (5.6.189), we get

$$\|x_\lambda - x_\mu\|^2 + \langle F(x_\lambda - x_\mu) - F(J_\lambda^B x_\lambda - J_\mu^B x_\mu), B_\lambda x_\lambda - B_\mu x_\mu\rangle \leq 0. \tag{5.6.190}$$

On the other hand,

$$\begin{aligned} F(J_\lambda^B x_\lambda - J_\mu^B x_\mu) &= F(x_\lambda - x_\mu - x_\lambda + x_\mu + J_\lambda^B x_\lambda - J_\mu^B x_\mu) \\ &= F(x_\lambda - x_\mu - (x_\lambda - J_\lambda^B x_\lambda) + (x_\mu - J_\mu^B x_\mu)) \\ &= F(x_\lambda - x_\mu - \lambda B_\lambda x_\lambda + \mu B_\mu x_\mu). \end{aligned} \tag{5.6.191}$$

Substituting (5.6.191) into (5.6.190) gives

$$\|x_\lambda - x_\mu\|^2 + \langle F(x_\lambda - x_\mu) - F(x_\lambda - x_\mu - \lambda B_\lambda x_\lambda + \mu B_\mu x_\mu), B_\lambda x_\lambda - B_\mu x_\mu\rangle \leq 0,$$

or equivalently,

$$\|x_\lambda - x_\mu\|^2 \leq \|F(x_\lambda - x_\mu) - F(x_\lambda - x_\mu - \lambda B_\lambda x_\lambda + \mu B_\mu x_\mu)\|\|B_\lambda x_\lambda - B_\mu x_\mu\|. \tag{5.6.192}$$

By hypothesis, $B_\lambda x_\lambda$ is bounded on some interval $(0, \lambda_0)$, and hence there exists $k > 0$ such that

$$\|B_\lambda x_\lambda\| \leq k, \quad 0 < \lambda < \lambda_0.$$

Thus, if $0 < \lambda, \mu < \lambda_0$, we have

$$\|B_\lambda x_\lambda - B_\mu x_\mu\| \leq 2k,$$

and from (5.6.192) we deduce

$$\|x_\lambda - x_\mu\|^2 \leq 2k\|F(x_\lambda - x_\mu) - F(x_\lambda - x_\mu - \lambda B_\lambda x_\lambda + \mu B_\mu x_\mu)\|. \tag{5.6.193}$$

Now

$$\begin{aligned}&\|(x_\lambda - x_\mu) - (x_\lambda - x_\mu - \lambda B_\lambda x_\lambda + \mu B_\mu x_\mu)\| \\ &\quad = \|\lambda B_\lambda x_\lambda - \mu B_\mu x_\mu\| \le \lambda\|B_\lambda x_\lambda\| + \mu\|B_\mu x_\mu\| \le (\lambda+\mu)k < (\lambda+\mu)2k, \end{aligned} \tag{5.6.194}$$

with $0 < \lambda, \mu < \lambda_0$.

Since X' is uniformly convex, by Theorem 6.15 p57 in [47], F is uniformly continuous on bounded sets. Thus, given $\varepsilon > 0$ and $M > 0$, there exists $\delta = \delta(\varepsilon)$ such that if $\|x_1\| < M$ and $\|x_1 - x_2\| < \delta$, then $\|F(x_1) - F(x_2)\| < \frac{\varepsilon}{2k}$. Let $\xi_0 = \min\{\frac{\delta}{2k},\ \lambda_0\} > 0$. Then, if $\lambda, \mu \le \xi_0$ we have $(\mu+\lambda)2k < \delta$. Hence, under this condition, by (5.6.194),

$$\|(x_\lambda - x_\mu) - (x_\lambda - x_\mu - \lambda B_\lambda x_\lambda + \mu B_\mu x_\mu)\| < (\mu+\lambda)2k < \delta.$$

Moreover, since $D(A) \cap D(B) \neq \emptyset$, Proposition 5.115 implies that x_λ is bounded. That is, there exists $M > 0$ such that $\|x_\lambda\| < \frac{M}{2}$ for all $\lambda > 0$. Thus

$$\|x_\lambda - x_\mu\| \le \|x_\lambda\| + \|x_\mu\| < M, \quad \forall \lambda, \mu > 0.$$

Therefore, from (5.6.193) we obtain

$$\begin{aligned}\|x_\lambda - x_\mu\|^2 &\le 2k\|F(x_\lambda - x_\mu) - F(x_\lambda - x_\mu - \lambda B_\lambda x_\lambda + \mu B_\mu x_\mu)\| \\ &< 2k\frac{\varepsilon}{2k} = \varepsilon, \quad 0 < \lambda, \mu < \xi_0,\end{aligned}$$

that is,

$$\|x_\lambda - x_\mu\| \longrightarrow 0, \quad \text{as } \lambda,\ \mu \longrightarrow 0^+.$$

Hence (x_λ) is a Cauchy net and, since X is Banach, there exists $x \in X$ such that

$$x_\lambda \longrightarrow x \quad \text{in } X. \tag{5.6.195}$$

It remains to show that $x \in D(A) \cap D(B)$ and, moreover, that $y \in (1+\omega)x + Ax + Bx$. Indeed, going back to the beginning of the proof, recall that

$$y \in (1+\omega)x_\lambda + u_\lambda + B_\lambda x_\lambda, \quad (x_\lambda, u_\lambda) \in A. \tag{5.6.196}$$

From the boundedness of $B_\lambda x_\lambda$ and x_λ, $0 < \lambda < \lambda_0$, we deduce from (5.6.196) that u_λ is bounded on $(0, \lambda_0)$. Since X is reflexive, there exist $(\lambda_n) \subset (0, \lambda_0)$, $\lambda_n \to 0^+$, such that $u_{\lambda_n} \rightharpoonup u$ for some $u \in X$.

At the beginning of Section 5.4 we proved that if an operator A is m-accretive, then it is maximal accretive in every $C \supset D(A)$. In particular, A is maximal accretive in $\overline{D(A)}$. Therefore, since by hypothesis $A + \omega I$ and B are m-accretive, it follows that $A + \omega I$ is maximal accretive in $\overline{D(A)}$ and B is maximal accretive in $\overline{D(B)}$. Then, by Theorem teo 6.15 p57 [47], the norm of X is Fréchet differentiable. From this and Proposition 5.97 we infer that A and B are demiclosed. Since $(x_{\lambda_n}, u_{\lambda_n}) \in A$, $x_{\lambda_n} \to x$ and $u_{\lambda_n} \rightharpoonup u$, it follows that $(x, u) \in A$, that is, $x \in D(A)$ and $u \in Ax$.

On the other hand, from (5.6.196), we have

$$B_{\lambda_n} x_{\lambda_n} = y - (1+\omega)x_{\lambda_n} - u_{\lambda_n}.$$

Therefore,

$$B_{\lambda_n} x_{\lambda_n} \rightharpoonup v, \quad v = y - (1+\omega)x - u. \tag{5.6.197}$$

We now show that $x \in D(B)$ and $v \in Bx$. Indeed,

$$\begin{aligned}\|J_\lambda^B x_\lambda - x\| &\le \|J_\lambda^B x_\lambda - x_\lambda\| + \|x_\lambda - x\| \\ &= \lambda\|B_\lambda x_\lambda\| + \|x_\lambda - x\| \longrightarrow 0 \quad \text{as } \lambda \to 0^+,\end{aligned}$$

hence $J_\lambda^B x_\lambda \longrightarrow x$ as $\lambda \to 0^+$. In particular,

$$J_{\lambda_n}^B x_{\lambda_n} \longrightarrow x \quad \text{as } n \to \infty. \tag{5.6.198}$$

By Proposition 5.75, item (iii),

$$\left(J_{\lambda_n}^B x_{\lambda_n}, B_{\lambda_n} x_{\lambda_n}\right) \in B. \tag{5.6.199}$$

Since B is demiclosed, (5.6.197), (5.6.198) and (5.6.199) imply that $(x, v) \in B$, that is, $x \in D(B)$ and $v \in Bx$. Consequently, $x \in D(A) \cap D(B)$ and

$$y = (1 + \omega)x + u + v,$$

where $u \in Ax$ and $v \in Bx$. Thus there exists $x \in D(A) \cap D(B)$ such that

$$y \in (1 + \omega)x + Ax + Bx.$$

It remains to show that the solution is unique. Indeed, suppose there exist $x_1,\ x_2 \in D(A) \cap D(B)$ such that

$$y \in [(1 + \omega)I + A + B]x_1 = [I + (A + B + \omega I)]x_1$$

and

$$y \in [(1 + \omega)I + A + B]x_2 = [I + (A + B + \omega I)]x_2,$$

that is, $x_1 \in J_1^{\omega I + A + B} y$ and $x_2 \in J_1^{\omega I + A + B} y$. Since X is smooth, $A + \omega I$ and B are accretive, and hence $A + B + \omega I$ is accretive. By Proposition 5.75, item (i), $J_\lambda^{A+B+\omega I}$ is single-valued for every $\lambda > 0$, in particular for $\lambda = 1$. Thus $x_1 = x_2$. □

Theorem 5.117 *Let X' be uniformly convex, and let $A + \omega I$ and B be m-accretive operators such that*

i) $D(A) \subset D(B)$;

ii) For every $r > 0$, there exist constants $K(r)$ and $C(r)$ with $K(r) < 1$ such that

$$|Bx| \le K(r)|Ax| + C(r), \quad \forall x \in D(A) \text{ with } \|x\| \le r. \tag{5.6.200}$$

Then $A + B + \omega I$ is m-accretive.

Proof: Since X' is uniformly convex, X is smooth, and hence by Proposition 5.115, for every $y \in X$ and every $\lambda > 0$ there exist $x_\lambda \in D(A)$ and $u_\lambda \in Ax_\lambda$ such that

$$y = (1 + \omega)x_\lambda + u_\lambda + B_\lambda x_\lambda. \tag{5.6.201}$$

Moreover, since $D(A) \cap D(B) = D(A) \neq \emptyset$, Proposition 5.115 also gives that $(x_\lambda)_{\lambda > 0}$ is bounded and therefore there exists $r > 0$ such that

$$\|x_\lambda\| \le r, \quad \forall \lambda > 0. \tag{5.6.202}$$

On the other hand, recall that

$$|Ax_\lambda| = \inf\{\|z\|;\ z \in Ax_\lambda\} \le \|u_\lambda\|.$$

From this and (5.6.201), we obtain

$$|Ax_\lambda| \le \|u_\lambda\| \le \|y\| + |1 + \omega|\, \|x_\lambda\| + \|B_\lambda x_\lambda\|.$$

Since $B \in \mathcal{A}(0)$ and $\lambda \cdot 0 < 1$ for all $\lambda > 0$, and since $x_\lambda \in D(A) \subset D(B)$ and $x_\lambda \in D(B_\lambda) = D_\lambda^B$, i.e.

$x_\lambda \in D(B) \cap D_\lambda^B$, Theorem 5.79, item (ii), yields

$$\|B_\lambda x_\lambda\| = \left\| \frac{1}{\lambda}\left(I - J_\lambda\right) x_\lambda \right\| \le \frac{1}{\lambda}\lambda(1-\lambda\cdot 0)^{-1}\left|Bx_\lambda\right| = \left|Bx_\lambda\right|,$$

and therefore

$$|Ax_\lambda| \le \|u_\lambda\| \le \|y\| + |1+\omega|\,\|x_\lambda\| + |Bx_\lambda|\,. \tag{5.6.203}$$

From (5.6.200), (5.6.202) and (5.6.203), we deduce

$$|Ax_\lambda| \le \|y\| + r|1+\omega| + K(r)\,|Ax_\lambda| + C(r),$$

that is,

$$(1-K(r))\,|Ax_\lambda| \le \|y\| + r|1+\omega| + C(r), \quad \forall \lambda > 0,$$

which implies that $|Ax_\lambda|$ is bounded for all $\lambda > 0$, since $(1-K(r)) > 0$. By hypothesis (ii) and the boundedness of $|Ax_\lambda|$, it follows that $|Bx_\lambda|$ is also bounded for all $\lambda > 0$. As

$$\|B_\lambda x_\lambda\| \le |Bx_\lambda|\,, \quad \forall \lambda > 0,$$

we conclude that $\|B_\lambda x_\lambda\|$ is bounded for all $\lambda > 0$. Therefore, by Proposition 5.116, for each $y \in X$ there exists a unique $x \in D(A)$ such that

$$y \in (1+\omega)x + Ax + Bx = [I + (A+B+\omega I)]\,x,$$

that is,

$$Im\,[I + (A+B+\omega I)] = X,$$

and thus $A+B+\omega I$ is m-accretive. □

Theorem 5.118 *Let X' be uniformly convex, and let $A+\omega I$ and B be m-accretive operators such that*

(i) $D(A) \cap D(B) \neq \emptyset$;

(ii) There exist $b \in [0,1)$ and a function $\psi : [0,\infty) \longrightarrow \mathbb{R}$ non-negative and non-decreasing such that

$$\langle u + \omega x, F(B_\lambda x)\rangle \ge -\psi(\|x\|) - b\|B_\lambda x\|^2.$$

Then $A+B+\omega I$ is m-accretive.

Proof: As before, by Proposition 5.115, for every $y \in X$ and $\lambda > 0$ there exist $x_\lambda \in D(A)$ and $u_\lambda \in Ax_\lambda$ such that

$$y = (1+\omega)x_\lambda + u_\lambda + B_\lambda x_\lambda, \quad (x_\lambda, u_\lambda) \in A. \tag{5.6.204}$$

Moreover, since $D(A)\cap D(B) \neq \emptyset$, there exists $C > 0$ such that $\|x_\lambda\| \le C$ for all $\lambda > 0$, and as $B \in \mathcal{A}(0)$, Theorem 5.79, item (i), implies that B_λ is single-valued. By Proposition 5.116, it suffices to show that $\{B_\lambda x_\lambda\}$ is bounded. We have

$$\begin{aligned} \langle F(B_\lambda x_\lambda), y - x_\lambda\rangle &= \langle F(B_\lambda x_\lambda), \omega x_\lambda + u_\lambda + B_\lambda x_\lambda\rangle \\ &= \langle F(B_\lambda x_\lambda), \omega x_\lambda + u_\lambda\rangle + \|B_\lambda x_\lambda\|^2 \\ &\ge -\psi(\|x_\lambda\|) - b\|B_\lambda x_\lambda\|^2 + \|B_\lambda x_\lambda\|^2. \end{aligned} \tag{5.6.205}$$

Hence

$$\begin{aligned} \|F(B_\lambda x_\lambda)\|\|y - x_\lambda\| &\ge \langle F(B_\lambda x_\lambda), y - x_\lambda\rangle \\ &\ge (1-b)\|B_\lambda x_\lambda\|^2 - \psi(\|x_\lambda\|), \end{aligned}$$

or

$$\alpha t^2 - \beta t - \gamma \le 0, \tag{5.6.206}$$

where

$$\begin{cases} \alpha := & (1-b) > 0, \\ \beta := & \|y - x_\lambda\|, \\ \gamma := & \psi(\|x_\lambda\|), \\ t := & \|B_\lambda x_\lambda\|. \end{cases}$$

Therefore

$$\frac{\beta - \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha} \le t \le \frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha},$$

and so

$$t \le \frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha} \le \frac{\beta + \beta + 2\sqrt{\alpha\gamma}}{2\alpha} = \frac{\beta + \sqrt{\alpha\gamma}}{\alpha}.$$

Thus

$$(1-b)\|B_\lambda x_\lambda\| \le \|y - x_\lambda\| + [(1-b)\psi(\|x_\lambda\|)]^{\frac{1}{2}}, \ \forall \lambda > 0. \tag{5.6.207}$$

Since ψ is non-decreasing and $\|x_\lambda\| \le C$ for every $\lambda > 0$, we obtain

$$\|B_\lambda x_\lambda\| \le \frac{1}{1-b}\left[\|y\| + C + \sqrt{(1-b)\psi(C)}\right],$$

so that $B_\lambda x_\lambda$ is bounded and Proposition 5.116 yields that $A + B + \omega I$ is m-accretive. □

Corollary 5.119 *Under the assumptions of Theorem 5.118, if*

$$\langle u + \omega x, F(B_\lambda x)\rangle \ge 0, \ \forall (x,u) \in A,$$

then $A + B + \omega I$ is m-accretive.

Proof: It suffices to take $b = 0$ and $\psi \equiv 0$ in Theorem 5.118. □

Theorem 5.120 *Let X' be uniformly convex, and let $A + \omega I$ and B be m-accretive operators such that B is linear. Assume that*

(i) $D(B) \subseteq D(A)$;

(ii) $\langle F(Bx), u + \omega x\rangle \ge -\psi(\|x\|) - b\|Bx\|^2, \ \forall (x,u) \in A,$

with b and ψ as in the assumptions of Theorem 5.118. Then $A + B + \omega I$ is m-accretive.

Proof: The idea is to recover condition (ii) of Theorem 5.118. Indeed, since B is accretive, J_λ^B is single-valued and Lipschitz with constant 1, for all $\lambda > 0$. Consequently, B_λ is single-valued. Thus, for all $\lambda > 0$,

$$\begin{aligned} B_\lambda = \frac{1}{\lambda}\left[I - (I + \lambda B)^{-1}\right] &= \frac{1}{\lambda}\left[(I + \lambda B)(I + \lambda B)^{-1} - (I + \lambda B)^{-1}\right] \\ &= \frac{1}{\lambda}\left[(I + \lambda B) - I\right](I + \lambda B)^{-1} = BJ_\lambda^B. \end{aligned} \tag{5.6.208}$$

If $x \in X$, then $J_\lambda^B x \in D(B) \subset D(A)$. In particular, if $x \in D(A)$ and $v \in A(J_\lambda^B x)$, then for every $u \in Ax$ we have

$$\begin{aligned} \lambda\langle F(B_\lambda x), u + \omega x\rangle &= \langle F(\lambda B_\lambda x), u + \omega x\rangle \\ &= \langle F(x - J_\lambda^B x), u + \omega x\rangle \\ &= \langle F(x - J_\lambda^B x), u + \omega x - v - \omega J_\lambda^B x\rangle + \langle F(x - J_\lambda^B x), v + \omega J_\lambda^B x\rangle. \end{aligned} \tag{5.6.209}$$

Since $A+\omega I$ is accretive and X is smooth, the first term on the right-hand side of (5.6.209) is non-negative, hence

$$\begin{aligned}\langle F(B_\lambda x), u+\omega x\rangle &\geq \langle F(B_\lambda x), v+\omega J_\lambda^B x\rangle \\ &= \langle F(BJ_\lambda^B x), v+\omega J_\lambda^B x\rangle.\end{aligned} \tag{5.6.210}$$

But $(J_\lambda^B x, v) \in A$. By the hypothesis, we obtain

$$\begin{aligned}\langle F(B_\lambda x), u+\omega x\rangle &\geq \langle F(BJ_\lambda^B x), v+\omega J_\lambda^B x\rangle \\ &\geq -\psi(\|J_\lambda^B x\|) - b\|BJ_\lambda^B x\|^2 \\ &= -\psi(\|J_\lambda^B x\|) - b\|B_\lambda x\|^2.\end{aligned} \tag{5.6.211}$$

Since B is linear, J_λ^B is linear as well. From the accretivity of B we have

$$\|J_\lambda^B x\| \leq \|x\|, \tag{5.6.212}$$

and from this and the fact that ψ is non-decreasing, we deduce

$$\langle F(B_\lambda x), u+\omega x\rangle \geq -\psi(\|x\|) - b\|B_\lambda x\|^2,$$

which is precisely condition (ii) in Theorem 5.118. □

5.7 Linear contraction semigroups: Hille–Yosida theory and some applications

5.7.1 m-accretive operators

In this section, X is a Banach space endowed with the norm $\|\cdot\|$.

5.7.1.1 Unbounded operators in Banach spaces

Definition 5.121 *An unbounded linear operator in X is a pair (D, A), where D is a vector subspace of X and A is a linear mapping $D \to X$. If*

$$\sup\{\|Ax\|;\ x \in D,\ \|x\| \leq 1\} < \infty,$$

then A is bounded. If

$$\sup\{\|Ax\|;\ x \in D,\ \|x\| \leq 1\} = \infty,$$

then A is unbounded.

Remark 5.122 *It follows from the Hahn–Banach Theorem that A is bounded if and only if there exists a closed vector subspace Y of X such that $D \subset Y$ and an operator $\bar{A} \in \mathcal{L}(Y, X)$ such that $Ax = \bar{A}x$ for all $x \in D$.*

Definition 5.123 *Let (D, A) be an unbounded linear operator in X. The domain $D(A)$ of A is the set*

$$D(A) = D,$$

the range $Im(A)$ of A is the set

$$Im(A) = A(D),$$

and the graph $G(A)$ of A is the set

$$G(A) = \{(x, f) \in X \times X;\ x \in D \text{ and } f = Ax\}.$$

Both $D(A)$ and $Im(A)$ are vector subspaces of X, and $G(A)$ is a vector subspace of $X \times X$.

Remark 5.124 *The pair (A, D) is often called "the operator A with domain $D(A) = D$" or simply "the operator A". However, we must notice that an operator is not determined only by the value Ax, but also by its domain. In other words, when we define an operator it is absolutely necessary to specify its domain. In particular, the same formula may define several operators, depending on what the domain is.*

For example, let $X = L^2(\mathbb{R}^n)$. Let A_1 be defined by $D(A_1) = X$ and $A_1 u = u$ for all $u \in X$ (A_1 is the identity on X), and let

$$D(A_2) = \{u \in H^1(\mathbb{R}^n);\ u(x) = 0 \text{ for almost all } x \text{ with } |x| \geq 1\}$$

and $A_2 u = u$ for all $u \in D(A_2)$. Both A_1 and A_2 are defined by the same formula, but A_1 and A_2 have different properties. For instance, the domain of A_1 is dense in X, while the domain of A_2 is not.

Remark 5.125 *When there is no risk of confusion, an unbounded linear operator in X is simply called a linear operator in X or an operator in X.*

Definition 5.126 *An operator A in X is m-accretive if the following hold:*
i) A is accretive;
ii) For every $\lambda > 0$ and every $f \in X$, there exists $x \in D(A)$ such that $x + \lambda Ax = f$.

Lemma 5.127 *If A is an m-accretive operator in X, then for each $\lambda > 0$ and each $f \in X$ there exists a unique solution $x \in D(A)$ of the equation*

$$x + \lambda Ax = f.$$

Moreover, $\|x\| \leq \|f\|$. In particular, given $\lambda > 0$, the mapping $f \mapsto x$ is a contraction $X \to X$, and is one-to-one $X \to D(A)$.

Proof: The result follows immediately from Definition 5.126. □

Proposition 5.128 *If A is an m-accretive operator in X, then the graph $G(A)$ of A is closed in $X \times X$.*

Proof: This follows from Proposition 5.89. □

Corollary 5.129 *Let A be an m-accretive operator in X. For each $x \in D(A)$ set $\|x\|_{D(A)} = \|x\| + \|Ax\|$ and $|||x|||_{D(A)} = \|x + Ax\|$. Then*
i) $\|\cdot\|_{D(A)}$ is a norm on $D(A)$, and $(D(A), \|\cdot\|_{D(A)})$ is a Banach space; $\|\cdot\|_{D(A)}$ is called the graph norm;
ii) $D(A) \hookrightarrow X$;
iii) The restriction of A to $D(A)$ is continuous $D(A) \to X$ and $\|A\|_{\mathcal{L}(D(A),X)} < 1$;
iv) $|||\cdot|||_{D(A)}$ is an equivalent norm on $D(A)$;
v) J_1 is an isomorphism from X onto $D(A)$.

Proof: It is clear that $\|\cdot\|_{D(A)}$ is a norm on $D(A)$. Moreover, the mapping

$$D(A) \to X \times X, \qquad g : x \mapsto (x, Ax)$$

satisfies $\|g(x)\|_{X\times X} = \|x\|_{D(A)}$. Since $g(D(A)) = G(A)$, which is closed by Proposition 5.89, it follows that $(D(A), \|\cdot\|_{D(A)})$ is a Banach space. Indeed, let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(D(A), \|\cdot\|_{D(A)})$, where $\|x\|_{D(A)} = \|x\| + \|Ax\|$. Then

$$\|x_n - x_m\| \to 0 \quad \text{and} \quad \|Ax_n - Ax_m\| \to 0 \text{ as } m, n \to \infty,$$

which implies that there exist $x \in X$ and $y \in X$ such that $x_n \to x$ in X and $Ax_n \to y$ in X.

However, since $(x_n, Ax_n) \in G(A)$ and $G(A)$ is closed, we obtain $(x, y) \in G(A)$, that is, $y = Ax$. Thus $x_n \to x$ in $(D(A), \|\cdot\|_{D(A)})$. This proves (i).

Item (ii) follows from the inequality $\|x\| \leq \|x\|_{D(A)}$, whereas (iii) follows from the inequality $\|Ax\| \leq \|x\|_{D(A)}$. Moreover,

$$\begin{aligned} \|A\|_{\mathcal{L}(D(A),X)} &= \sup\{\|Ax\|;\ x \in D(A) \text{ and } \|x\|_{D(A)} \leq 1\} \\ &\leq \sup\{\|x\|;\ x \in D(A) \text{ and } \|x\|_{D(A)} \leq 1\} \leq 1. \end{aligned}$$

To prove (iv), note that $\|x\| \leq \|x\|_{D(A)}$, and also

$$\|x\|_{D(A)} \leq 2\|x\| + |\!|\!|x|\!|\!|_{D(A)} \leq 3\, |\!|\!|x|\!|\!|_{D(A)}.$$

Indeed,

$$\begin{aligned} \|x\|_{D(A)} = \|x\| + \|Ax\| &\leq \|x + 2Ax\| + \|Ax\| \\ &\leq \|x + Ax\| + \|Ax\| + \|Ax\| \\ &= |\!|\!|x|\!|\!|_{D(A)} + 2\|Ax\| \\ &\leq |\!|\!|x|\!|\!|_{D(A)} + 2\|x + Ax\| \\ &= |\!|\!|x|\!|\!|_{D(A)} + 2|\!|\!|x|\!|\!|_{D(A)} \\ &= 3|\!|\!|x|\!|\!|_{D(A)}. \end{aligned}$$

Since A is accretive, (iv) follows. Finally, we have $Im(J_1) = D(A)$ by Lemma 5.127, and it is immediate that $|\!|\!|J_1 x|\!|\!|_{D(A)} = \|x\|$ for all $x \in X$, because

$$|\!|\!|J_1 x|\!|\!|_{D(A)} = \|J_1 x + Ax\| = \|(I + A)J_1 x\| = \|x\|,$$

and hence J_1 is an isometry from X onto $D(A)$ endowed with the equivalent norm $|\!|\!|\cdot|\!|\!|_{D(A)}$. This completes the proof. □

Remark 5.130 *From now on, we shall regard $D(A)$ as a Banach space $(D(A), \|\cdot\|_{D(A)})$.*

Corollary 5.131 *If A is an m-accretive operator in X, then*

(i) $\|J_1 x\|_{D(A)}$ defines a norm on X, equivalent to the original norm $\|\cdot\|$;

(ii) $J_\lambda \in \mathcal{L}(X, D(A))$ for every $\lambda > 0$.

Proof: It follows from Corollary 5.129, item (iv), that $|\!|\!|J_1 x|\!|\!|_{D(A)} = \|x\|$. Hence (i) holds. Given $\lambda > 0$ and $x \in X$, we have $\lambda A J_\lambda x = x - J_\lambda x$, and thus

$$\|J_\lambda x\|_{D(A)} = \|J_\lambda x\| + \frac{1}{\lambda}\|x - J_\lambda x\| \leq \left(1 + \frac{2}{\lambda}\right)\|x\|.$$

Therefore, (ii) follows. □

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Definition 5.132 *Let A be an operator in X, and let J_λ be as defined above. For each $x \in X$ and $\lambda > 0$, we define $A_\lambda x \in X$ by $A_\lambda x = AJ_\lambda x$. The operator A_λ is called the* ***Yosida approximation of*** *A.*

Lemma 5.133 *Let A be an operator in X and let A_λ be as above. The following properties hold:*

(i) $A_\lambda x = \dfrac{x - J_\lambda x}{\lambda}$ *for each* $x \in X$;

(ii) $A_\lambda \in \mathcal{L}(X)$ *and* $\|A_\lambda\|_{\mathcal{L}(X)} \leq \dfrac{2}{\lambda}$ *for every* $\lambda > 0$;

(iii) $A_\lambda x = J_\lambda A x$ *for each* $x \in D(A)$;

(iv) $(J_\lambda)_{|D(A)} \in \mathcal{L}(D(A))$ *and* $\|(J_\lambda)_{|D(A)}\|_{\mathcal{L}(D(A))} \leq 1$ *for each* $\lambda > 0$;

(v) A_λ *is m-accretive.*

Proof: (i) Let $x \in X$ and set $z = J_\lambda x$. We have $z + \lambda A z = x$, and hence $\lambda A_\lambda x = \lambda A z = x - z$, which proves (i). Item (ii) then follows immediately. Indeed,

$$\begin{aligned} \|A_\lambda\| &= \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|A_\lambda x\| \\ &= \sup_{\substack{x \in X \\ \|x\| \leq 1}} \left\| \frac{x - J_\lambda x}{\lambda} \right\| \\ &\leq \sup_{\substack{x \in X \\ \|x\| \leq 1}} \left\| \frac{x}{\lambda} \right\| + \sup_{\substack{x \in X \\ \|x\| \leq 1}} \left\| \frac{J_\lambda x}{\lambda} \right\| \\ &\leq \frac{1}{\lambda} + \frac{1}{\lambda} \|J_\lambda\| \\ &\leq \frac{2}{\lambda}. \end{aligned}$$

(iii) Finally, let $x \in D(A)$ and set $z = J_\lambda x$. Then

$$z + \lambda A z = x.$$

Since both x and z belong to $D(A)$, it follows that $Az \in D(A)$ and

$$Az + \lambda A(Az) = Ax.$$

Now set $w = J_\lambda A x$. Then

$$w + \lambda A w = Ax,$$

and hence $(w - Az) + \lambda A(w - Az) = 0$. Since A is accretive, we conclude that $w = Az$, which proves (iii).

(iv) We have

$$\begin{aligned} \|J_\lambda x\|_{D(A)} &= \|J_\lambda x\| + \|A(J_\lambda x)\| = \|J_\lambda x\| + \|A_\lambda x\| \\ &= \|J_\lambda x\| + \|J_\lambda A x\| \leq \|x\| + \|Ax\| = \|x\|_{D(A)}. \end{aligned}$$

(v) This follows from Theorem 5.79. □

Remark 5.134 *If A is an m-accretive operator in X and if X is reflexive, then $D(A)$ is dense in X.*
Proof: *See [83], Theorem (4.6), p. 16.* □

Remark 5.135 *If X is a Hilbert space, then we cannot prove the estimate in (ii). In this case, we have $\|A_\lambda\|_{\mathcal{L}(X)} \le \frac{1}{\lambda}$. Indeed, given $x \in X$, let $f = J_\lambda x$, so that $f + \lambda Af = x$. Taking the inner product with Af, we obtain*

$$(f, Af) + \lambda(Af, Af) = (x, Af) \le \|x\|\,\|Af\|.$$

Thus, by Lemma 5.156 we obtain

$$\lambda\|Af\|^2 \le \|x\|\,\|Af\|.$$

If $\|Af\| \neq 0$, then

$$\begin{aligned} \|Af\| &\le \frac{1}{\lambda}\|x\|, \\ \|AJ_\lambda x\| &\le \frac{1}{\lambda}\|x\|, \\ \|A_\lambda x\| &\le \frac{1}{\lambda}\|x\|, \\ \|A_\lambda\|_{\mathcal{L}(X)} &\le \frac{1}{\lambda}. \end{aligned}$$

The purpose of the next proposition is to show that J_λ is a good approximation of the identity, and that the (bounded) operator A_λ is an approximation of the unbounded operator A as $\lambda \to 0^+$.

We now state the following result, which will be used in the next proposition.

Proposition 5.136 *Let X and Y be Banach spaces, let E be a subset of X, and let $(A_\lambda)_{\lambda\in(-1,1)}$ be a bounded family in $\mathcal{L}(X,Y)$. If $\lim_{\lambda\to 0} A_\lambda x = 0$ for every $x \in E$, then $\lim_{\lambda\to 0} A_\lambda x = 0$ for every $x \in \overline{E}$.*

Proof: Let $x \in \overline{E}$ and let $(x_n)_{n\in\mathbb{N}} \subset E$ be a sequence converging to x as $n \to \infty$. Then there exists $C < \infty$ such that, for all $n \in \mathbb{N}$,

$$\|A_\lambda x\| \le \|A_\lambda x_n\| + C\|x - x_n\|.$$

Given $\varepsilon > 0$, we can choose n_0 sufficiently large such that $C\|x - x_{n_0}\| \le \frac{\varepsilon}{2}$. Then, for λ sufficiently small, we have $\|A_\lambda x_{n_0}\| \le \frac{\varepsilon}{2}$. The result follows. □

Proposition 5.137 *Let A be an m-accretive operator in X. If $D(A)$ is dense in X, then*

(i) $\|J_\lambda x - x\| \le \lambda\|Ax\|$ *for every* $\lambda > 0$ *and every* $x \in D(A)$;

(ii) $\|J_\lambda x - x\| \to 0^+$ *as* $\lambda \to 0^+$ *for every* $x \in X$;

(iii) $\|A_\lambda x - Ax\| \to 0^+$ *as* $\lambda \to 0^+$ *for every* $x \in D(A)$;

(iv) $\|J_\lambda x - x\|_{D(A)} \to 0$ *as* $\lambda \to 0^+$ *for every* $x \in D(A)$.

Proof: (i) Let $x \in D(A)$. We have $J_\lambda x - x = -\lambda A_\lambda x$, and thus (i) follows from Lemma 5.133, item (iii).

(ii) We have $\|J_\lambda - I\|_{\mathcal{L}(X)} \le \|J_\lambda\| + \|I\| \le 2$. Moreover, for $x \in D(A)$, item (i) gives $\|J_\lambda x - x\| \le \lambda\|Ax\|$. Letting $\lambda \to 0$, we obtain $\|J_\lambda x - x\| \to 0$. Since $D(A)$ is dense in X, (ii) follows from Proposition 5.136.

(iii) Given $x \in D(A)$, it follows from (ii) that $J_\lambda Ax - Ax \to 0$ as $\lambda \to 0^+$ in X. Hence (iii) follows, since $J_\lambda Ax = A_\lambda x$ by Lemma 5.133.

(iv) Finally, (iv) follows from (ii) and (iii). □

Remark 5.138 *Property (i) also holds if $D(A)$ is not dense. Therefore, if A is an m-accretive operator, then $J_\lambda x \to x$ as $\lambda \to 0$ for each $x \in D(A)$ and, consequently, for each $x \in \overline{D(A)}$.*

Finally, the next proposition provides a short and usual characterization of m-accretive operators.

Proposition 5.139 *Let A be an accretive operator in X. Then the following properties are equivalent:*

(i) A is m-accretive;

(ii) There exists $\lambda_0 > 0$ such that, for every $f \in X$, there exists a solution $x \in D(A)$ of the equation $x + \lambda_0 Ax = f$.

Proof: $(i) \Rightarrow (ii)$ follows from Corollary 5.88.
$(ii) \Rightarrow (i)$ follows from Proposition 5.87. □

Remark 5.140 *Let A be an accretive operator in X. In order to verify that A is m-accretive, the natural approach is to solve the equation $x + \lambda Ax = f$ for every $f \in X$ and every $\lambda > 0$. Proposition 5.139 means that, in fact, it suffices to solve this equation for every $f \in X$ and some fixed $\lambda > 0$.*

Corollary 5.141 *Let A and B be two operators in X. If $Im(I + A) = X$, if B is accretive and if $G(A) \subset G(B)$, then $A = B$ and A is m-accretive.*

Proof: Let $(x, f) \in G(B)$ and set $g = f + x$. In particular, $x \in D(B)$ and $x + Bx = g$. Since $Im(I + A) = X$, there exists $y \in D(A)$ such that $y + Ay = g$. As $G(A) \subset G(B)$, it follows that $y \in D(B)$ and $y + By = g$. In particular,

$$(x - y) + B(x - y) = 0.$$

Therefore $y = x$, since B is accretive. It follows that $(x, f) \in G(A)$, and hence $A = B$.

Finally, A is accretive (because B is accretive) and $Im(I + A) = X$, so A is m-accretive by Proposition 5.139. □

Corollary 5.142 *Let A and B be two m-accretive operators in X. If $G(A) \subset G(B)$, then $A = B$.*

5.7.2 Accretive operators and duality applications: sum of accretive operators

Recall the definition of the duality mapping F. For each $x \in X$, we define the duality set $F(x) \subset X'$ by

$$F(x) = \{\xi \in X';\ \|\xi\|_{X'} = \|x\| \text{ and } \langle \xi, x \rangle = \|x\|^2\}.$$

It follows from the Hahn–Banach theorem that $F(x) \neq \emptyset$.

Lemma 5.143 *Let A be a linear operator in X. The following properties are equivalent:*

(i) A is accretive;

(ii) for every $x \in D(A)$ there exists $x' \in F(x)$ such that $\langle x', Ax \rangle \geq 0$.

Proof: This follows from Proposition 5.68. □

Lemma 5.144 *Let A be an m-accretive operator in X. Then*

$$\langle x', Ax\rangle \geq 0, \quad \text{for every } x \in D(A) \text{ and every } x' \in F(x).$$

Proof: Let $x \in D(A)$ and $x' \in F(x)$. For every $\lambda > 0$ we have

$$\langle x', (I+\lambda A)^{-1}x\rangle \leq \|x'\| \, \|(I+\lambda A)^{-1}x\| \leq \|x\|^2 = \langle x', x\rangle.$$

Hence,

$$\langle x', x - (I+\lambda A)^{-1}x\rangle \geq 0.$$

Dividing both sides of the inequality by λ we obtain

$$\left\langle x', \frac{x-(I+\lambda A)^{-1}x}{\lambda}\right\rangle \geq 0.$$

By Lemma 5.133 we have $\langle x', A_\lambda x\rangle \geq 0$. Then, by item (iii) of Proposition 5.137, letting $\lambda \to 0^+$ we obtain $\langle x', Ax\rangle \geq 0$ for every $x \in D(A)$. □

Corollary 5.145 *Let A and B be operators in X. Define the operator $A+B$ by*

$$D(A+B) = D(A) \cap D(B), \qquad (A+B)x = Ax + Bx.$$

If A is m-accretive and B is accretive, then $A+B$ is accretive.

Proof: Since A is m-accretive, it follows from Lemma 5.144 that

$$\langle x', Ax\rangle \geq 0, \quad \forall x \in D(A),\ \forall x' \in F(x).$$

As B is accretive, Lemma 5.143 yields: for every $x \in D(B)$ there exists $x' \in F(x)$ such that $\langle x', Bx\rangle \geq 0$. Let $x \in D(A+B) = D(A) \cap D(B)$. Then $x \in D(A)$ and $x \in D(B)$. Thus, there exists $x' \in F(x)$ such that

$$\langle x', (A+B)x\rangle = \langle x', Ax\rangle + \langle x', Bx\rangle \geq 0.$$

By Lemma 5.143, $A+B$ is accretive. □

5.7.3 Restriction and extrapolation

In this section we show that, given an m-accretive operator with dense domain, we can either restrict the domain to a smaller space or extend it to a larger space in such a way that the restricted or extended operator is again m-accretive.

Theorem 5.146 *Let A be an m-accretive operator in X with dense domain and let X_1 be the Banach space $(D(A), \|\cdot\|_{D(A)})$. The operator $A_{(1)}$ in X_1 defined by*

$$\begin{cases} D(A_{(1)}) = \{x \in X_1;\ Ax \in X_1\}, \\ A_{(1)}x = Ax, \quad \forall x \in D(A_{(1)}) \end{cases}$$

is m-accretive in X_1 and $D(A_{(1)})$ is dense in X_1.

Proof: Let $x \in D(A_{(1)})$, $f \in X_1$ and $\lambda > 0$ be such that

$$x + \lambda A_{(1)}x = f.$$

In particular,

$$x + \lambda Ax = f. \tag{5.7.213}$$

Since $x \in D(A_{(1)})$ we have $Ax \in X_1$, hence $Ax \in D(A)$. From (5.7.213) we obtain

$$Ax + \lambda A(Ax) = Af. \tag{5.7.214}$$

As A is accretive, it follows from (5.7.213) and (5.7.214) that $\|x\| \leq \|f\|$ and $\|Ax\| \leq \|Af\|$. Thus,

$$\|x\|_{X_1} = \|x\| + \|Ax\| \leq \|f\| + \|Af\| = \|f\|_{X_1} = \|x + \lambda A_{(1)}x\|_{X_1}.$$

Therefore $A_{(1)}$ is accretive.

Now let $\lambda > 0$, $f \in X_1$ and set $x = J_\lambda f$. Then $x = (I + \lambda A)^{-1} f$, that is, $x + \lambda Ax = f$. In particular, $Ax \in D(A)$ (since $f, x \in D(A)$), which means $x \in D(A_{(1)})$ and $x + \lambda A_{(1)}x = f$. Hence $A_{(1)}$ is m-accretive.

Let $x \in X_1$ and set $x_\lambda = J_\lambda x$. As above we can check that $x_\lambda \in D(A_{(1)})$. Moreover, by item (iv) of Proposition 5.137,

$$x_\lambda \longrightarrow x \quad \text{as } \lambda \to 0^+ \text{ in } X_1.$$

Therefore $D(A_{(1)})$ is dense in X_1. □

Remark 5.147 *Some remarks concerning Theorem 5.146:*

(i) We have seen in Theorem 5.146 that the operator

$$A_{(1)} : X_1 \to X_1, \quad X_1 = (D(A), \|\cdot\|_{D(A)}),$$

defined by

$$\begin{cases} D(A_{(1)}) = \{x \in X_1;\ Ax \in X_1\}, \\ A_{(1)}x = Ax, \quad \forall x \in D(A_{(1)}) \end{cases}$$

is m-accretive in X_1 and $\overline{D(A_{(1)})} = X_1$. Hence we may apply Theorem 5.146 to the operator $A_{(1)}$. Thus, setting $X_2 = (D(A_{(1)}), \|\cdot\|_{D(A_{(1)})})$, which is a Banach space, the operator $A_{(2)}$ defined by

$$\begin{cases} D(A_{(2)}) = \{x \in X_2;\ A_{(1)}x \in X_2\}, \\ A_{(2)}x = A_{(1)}x, \quad \forall x \in D(A_{(2)}) \end{cases}$$

is m-accretive in X_2 and $\overline{D(A_{(2)})} = X_2$.

Applying Theorem 5.146 successively, and setting $X_{n+1} = (D(A_{(n)}), \|\cdot\|_{D(A_{(n)})})$, which is again a Banach space, the operator $A_{(n+1)}$ defined by

$$\begin{cases} D(A_{(n+1)}) = \{x \in X_{n+1};\ A_{(n)}x \in X_{n+1}\}, \\ A_{(n+1)}x = A_{(n)}x, \quad \forall x \in D(A_{(n+1)}) \end{cases}$$

is m-accretive in X_{n+1} and $\overline{D(A_{(n+1)})} = X_{n+1}$.

Concerning the norm $\|\cdot\|_{D(A_{(n)})}$, recall that $\|\cdot\|_{D(A)}$ is defined by $\|x\|_{D(A)} = \|x\| + \|Ax\|$. Thus,

$$\|x\|_{D(A_{(1)})} = \|x\|_{D(A)} + \|Ax\|_{D(A)} = \|x\| + \|Ax\| + \|Ax\| + \|A^2x\|.$$

We have $\|x\|_{D(A_{(1)})} \approx \|x\| + \|Ax\| + \|A^2x\|$. Indeed,

$$\|x\|_{D(A_{(1)})} = \|x\| + 2\|Ax\| + \|A^2x\| \leq 2\big(\|x\| + \|Ax\| + \|A^2x\|\big),$$

and

$$\|x\| + \|Ax\| + \|A^2x\| \leq \|x\| + 2\|Ax\| + \|A^2x\| = \|x\|_{D(A_{(1)})}.$$

Similarly, we obtain

$$\|x\|_{D(A_{(n)})} \approx \|x\| + \|Ax\| + \|A^2x\| + \cdots + \|A^nx\| = \sum_{j=0}^{n} \|A^jx\|.$$

Note that

$$X_1 = (D(A), \|\cdot\|_{D(A)}) \subset X, \quad X_2 = (D(A_{(1)}), \|\cdot\|_{D(A_{(1)})}) \subset X_1 \subset X, \ \ldots,$$

$$X_{n+1} = (D(A_{(n)}), \|\cdot\|_{D(A_{(n)})}) \subset X_n \subset \cdots \subset X_2 \subset X_1 \subset X = X_0.$$

Moreover,

$$\overline{D(A)} = X \Rightarrow \overline{X_1} = X, \quad \overline{D(A_{(1)})} = X_1 \Rightarrow \overline{X_2} = X_1, \ \ldots, \ \overline{D(A_{(n+1)})} = X_{n+1} \Rightarrow \overline{X_{n+1}} = X_n.$$

Since $A_{(n)}$ is m-accretive for every $n \in \mathbb{N}$, Corollary 5.129 yields a family $(X_n)_{n\in\mathbb{N}}$ of Banach spaces such that

$$\cdots \hookrightarrow X_{n+1} \hookrightarrow X_n \hookrightarrow \cdots \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow X_0 = X,$$

all continuously and densely embedded.

Also note that the family $(A_{(n)})_{n\in\mathbb{N}}$ of operators is such that $A_{(n)}$ is m-accretive in X_n with domain X_{n+1} and $A_{(n)}x = Ax$ for every $x \in X_{n+1}$.

When A is bounded we have $X_n = X$ for all $n \in \mathbb{N}$. Indeed, since $A : D(A) \subset X \to X$ is bounded and closed (because A is m-accretive), Proposition 2.39 of [23] implies that $D(A)$ is closed. Under the assumptions of Theorem 5.146 we have $\overline{D(A)} = X$. Therefore $D(A) = X$, so $X_1 = X$. From the definition of $D(A_{(1)})$ it follows that $D(A_{(1)}) = X$. Considering the definition of $D(A_{(n)})$, $n \in \mathbb{N}$, successively, we obtain $D(A_{(n)}) = X$ for all $n \in \mathbb{N}$, or equivalently $X_n = X$ for all $n \in \mathbb{N}$.

If A is not bounded, the family $(X_n)_{n\in\mathbb{N}}$ is strictly decreasing. In fact, if A is not bounded, then by Theorem 2.39 of [23], $D(A)$ is not closed and consequently $X_1 \neq X$. Also, if A is not bounded then $A_{(1)}$ is not bounded. Indeed, suppose by contradiction that $A_{(1)}$ is bounded. Then

$$\begin{aligned} \infty > \alpha &= \sup_{\|x\|\leq 1, \ x\in D(A_{(1)})} \|A_{(1)}x\|_{D(A)} \\ &= \sup_{\|x\|\leq 1, \ x\in D(A_{(1)})} \left(\|A_{(1)}x\|_X + \|A(A_{(1)}x)\|_X\right) \\ &\geq \sup_{\|x\|\leq 1, \ x\in D(A_{(1)})} \|Ax\|_X. \end{aligned}$$

Hence A would be bounded, a contradiction.

(ii) It follows from Corollary 5.131 that $X_1 = J_1(X)$ and that $\|J_1x\|_{X_1} \approx \|x\|$. By iteration we obtain $X_n = J_1^n(X)$, for every non-negative integer n, and $\|J_1^nx\|_{X_n} \approx \|x\|$.

Remark 5.148 *Given an operator A in X, we can define "powers of A" as follows:*
We define A^2 by:

$$\begin{cases} D(A^2) = \{x \in D(A); \ Ax \in D(A)\}, \\ A^2x = A(Ax) \quad \text{for every } x \in D(A^2). \end{cases}$$

More generally, by induction we define the operator A^n, for $n \geq 2$, by

$$\begin{cases} D(A^n) = \{x \in D(A^{n-1}); \ A^{n-1}x \in D(A)\}, \\ A^nx = A(A^{n-1}x) \quad \text{for every } x \in D(A^n). \end{cases}$$

The spaces X_n defined in Observation 5.147 coincide with $D(A^n)$ with equivalent norms if $D(A^n)$

is endowed with the norm

$$\|x\|_{D(A^n)} = \sum_{j=0}^{n} \|A^j x\|.$$

Indeed, we have $X_n = (D(A_{(n-1)}), \|\cdot\|_{D(A_{(n-1)})})$ *and*

$$\|x\|_{X_n} = \|x\|_{D(A_{(n-1)})} \simeq \|x\| + \|Ax\| + \cdots + \|A^n x\| = \|x\|_{D(A^n)}.$$

We now show by induction that $D(A_{(n-1)}) = D(A^n)$ *for every* $n \geq 2$.
For $n = 2$, *we have* $D(A_1) = \{x \in X_1;\ Ax \in X_1\}$, *where* $X_1 = (D(A), \|\cdot\|_{D(A)})$. *Thus,*

$$x \in D(A_1) \iff x, Ax \in X_1 \iff x, Ax \in D(A) \iff x, Ax \in D(A^2).$$

Hence $D(A_1) = D(A^2)$.

Assume that for some $r \in \mathbb{N}$, $r \geq 2$, *we have* $D(A_{(r-1)}) = D(A^r)$. *We prove that the same holds for* $r+1$, *i.e.,* $D(A_{(r)}) = D(A^{r+1})$. *First observe that*

$$Ax \in D(A^r) \iff A^r x \in D(A). \tag{5.7.215}$$

Indeed, let $x \in D(A_r)$. *Then* $x \in D(A_{(r-1)}) = D(A^r)$ *by the induction hypothesis. Hence* $x \in D(A^r)$ *and therefore* $A^{r-1}x \in D(A)$; *from* (5.7.215) *we infer* $Ax \in D(A^{r-1})$. *Thus* $A^{r-1}(Ax) = A^r x \in D(A)$, *and so* $x \in D(A^{r+1})$.

Conversely, let $x \in D(A^{r+1})$ *(we want to show that* $x \in D(A_r)$, *that is,* $x \in D(A_{(r-1)})$ *and* $A_{(r-1)}x \in D(A_{(r-1)})$*). Since* $x \in D(A^{r+1})$, *we have* $x \in D(A^r) = D(A_{(r-1)})$ *by the induction hypothesis, hence* $x \in D(A_{(r-1)})$. *On the other hand, from* $x \in D(A^{r+1})$ *we also obtain* $A^r x \in D(A)$, *and thus by* (5.7.215) *we get* $Ax \in D(A^r) = D(A_{(r-1)})$ *by the induction hypothesis. Therefore* $Ax \in D(A_{(r-1)}) = X_r$, *and by Observation 5.147 (namely,* $A_{(r)}x = Ax$ *for every* $x \in X_{r+1}$*) it follows that* $A_{(r-1)}x = Ax$ *for every* $x \in X_r = D(A_{(r-1)})$. *Since* $x \in D(A_{(r-1)})$, *we have* $A_{(r-1)}x = Ax \in D(A_{(r-1)})$. *Hence* $x \in D(A_r)$.

Therefore $D(A_{(n-1)}) = D(A^n)$ *for every* $n \in \mathbb{N}$, $n \geq 2$, *as desired.*

Theorem 5.149 *If* A *is an m-accretive operator in* X *with dense domain, then there exist a Banach space* X_{-1} *and an operator* $A_{(-1)}$ *in* X_{-1} *such that:*

- *(i)* $X \hookrightarrow X_{-1}$ *with dense embedding;*
- *(ii) For every* $x \in X$, *the norm of* x *in* X_{-1} *is given by* $\|x\|_{X_{-1}} = \|J_1 x\|$;
- *(iii)* $A_{(-1)}$ *is m-accretive in* X_{-1};
- *(iv)* $D(A_{(-1)}) = X$ *and the norms* $\|\cdot\|$ *and* $\|\cdot\|_{X_{-1}}$ *are equivalent on* X;
- *(v) For every* $x \in D(A)$ *we have* $A_{(-1)}x = Ax$. *Moreover,* X_{-1} *and* $A_{(-1)}$ *satisfying (i)–(iv) are unique.*

Proof: Define $|||x||| = \|J_1 x\|$ for every $x \in X$. Then $|||\cdot|||$ is a norm on X. Thus, considering $(X, |||\cdot|||)$ as a normed space, there exists a unique Banach space $(X_{-1}, \|\cdot\|_{X_{-1}})$ such that the embedding $X \hookrightarrow X_{-1}$ is dense, which proves (i); see [60], p. 69.

Moreover, since $|||x||| = \|x\|_{X_{-1}}$ by construction, we have $\|x\|_{X_{-1}} = \|J_1 x\|$ for all $x \in X$, proving (ii).

Note also that

$$AJ_1 x = x - J_1 x, \quad \text{for every } x \in X.$$

Thus, by Lemma 5.133,

$$J_1 Ax = x - J_1 x, \quad \text{for every } x \in D(A),$$

and hence

$$|||Ax||| = \|J_1 Ax\| = \|x - J_1 x\| \le \|x\| + \|J_1 x\| \le 2\|x\|, \quad \text{for every } x \in D(A).$$

Therefore $|||Ax||| \le 2\|x\|$ for all $x \in D(A)$, so A is bounded as an operator from $(D(A), \|\cdot\|)$ into $(X, |||\cdot|||)$. Since A is linear, there exists a unique operator $\overline{A} \in \mathcal{L}(X, X_{-1})$ such that $\overline{A}x = Ax$ for all $x \in D(A)$ and

$$|||\overline{A}x||| \le 2\|x\| \quad \text{for every } x \in X. \tag{5.7.216}$$

We now define the operator $A_{(-1)}$ in X_{-1} by

$$\begin{cases} D(A_{(-1)}) = X, \\ A_{(-1)}x = \overline{A}x, \quad \text{for every } x \in X. \end{cases}$$

Let $\lambda > 0$, $x \in D(A)$ and set $v = J_1 x$. Then

$$v + \lambda A v = J_1(x + \lambda A x).$$

Since A is m-accretive, we have

$$|||x + \lambda Ax||| = \|J_1(x + \lambda Ax)\| = \|v + \lambda Av\| \ge \|v\| = \|J_1 x\| = |||x|||.$$

Because $\overline{D(A)} = X$ and $\overline{A}$ is continuous, it follows that

$$|||x + \lambda \overline{A}x||| \ge |||x||| \quad \text{for every } x \in X.$$

Therefore $A_{(-1)}$ is accretive.

Now let $f \in X_{-1}$ and choose a sequence $(f_n) \subset X$ such that $f_n \to f$ as $n \to \infty$. Set $x_n = J_1 f_n$. Then (x_n) is a Cauchy sequence in X. Indeed, using linearity of J_1 we have

$$\|x_n - x_m\| = \|J_1 f_n - J_1 f_m\| = \|J_1(f_n - f_m)\| \to 0,$$

since (f_n) is Cauchy in X_{-1} (being convergent there). As X is a Banach space, (x_n) converges in X; let x denote its limit. We have

$$f_n = x_n + Ax_n = x_n + \overline{A}x_n.$$

Passing to the limit as $n \to \infty$ yields

$$f = x + \overline{A}x = x + A_{(-1)}x.$$

Hence, by Proposition 5.139, $A_{(-1)}$ is m-accretive in X_{-1}, proving (iii).

By the definition of $A_{(-1)}$ we have $D(A_{(-1)}) = X$. Furthermore, the norms $\|\cdot\|_{X_{-1}}$ and $\|\cdot\|$ are equivalent. Indeed, we have already observed that

$$\|x\|_{X_{-1}} = \|J_1 x\| \le \|x\| \quad \text{for all } x \in X.$$

On the other hand, for every $x \in D(A)$ we have

$$\begin{aligned} \|x\| &= \|AJ_1 x + J_1 x\| \\ &\le \|\overline{A}\|_{\mathcal{L}(X, X_{-1})} \|J_1 x\| + \|J_1 x\| \\ &\le c\|J_1 x\| + \|J_1 x\| \\ &= c\|x\|_{X_{-1}} + \|x\|_{X_{-1}} \\ &= d\|x\|_{X_{-1}}, \end{aligned}$$

where $c, d > 0$ are suitable constants. Since $\overline{D(A)} = X$, it follows that $\|x\| \leq d\|x\|_{X_{-1}}$ for every $x \in X$. Thus

$$\|x\|_{X_{-1}} \leq \|x\| \leq d\|x\|_{X_{-1}},$$

which proves (iv).

From the definition of $A_{(-1)}$ we have $A_{(-1)}x = \overline{A}x$ for every $x \in X$. Since $\overline{A}x = Ax$ for all $x \in D(A)$, we obtain (v). Finally, the uniqueness of $\overline{A}$ implies the uniqueness of $A_{(-1)}$. □

Remark 5.150 *Observe that, under the assumptions of Theorem 5.149, the operator $A_{(-1)}$ is m-accretive in X_{-1}, where X_{-1} is a Banach space and $\overline{D(A_{(-1)})} = X_{-1}$, since $D(A_{(-1)}) = X$ and $\overline{X} = X_{-1}$. Hence, applying this theorem to the operator $A_{(-1)}$, we obtain a Banach space $(X_{-2}, \|\cdot\|_{X_{-2}})$ and an operator $A_{(-2)}$ which is m-accretive in X_{-2} and satisfies $\overline{D(A_{(-2)})} = X_{-2}$. Proceeding in this way and applying the theorem successively, we construct a family of operators $(A_{(-n)})_{n\in\mathbb{N}}$ such that $A_{(-n)}$ is m-accretive in X_{-n} with domain X_{-n+1} and $A_{(-n)}x = Ax$ for every $x \in D(A)$.*

Moreover, we can construct a family $(X_{-n})_{n\in\mathbb{N}}$ of Banach spaces such that

$$X_0 \hookrightarrow X_{-1} \hookrightarrow \cdots \hookrightarrow X_{-n+1} \hookrightarrow X_{-n} \hookrightarrow \cdots,$$

all embeddings being dense. Arguing as in Observation 5.147, we prove that if A is bounded then $X_{-n} = X$ for every $n \in \mathbb{N}$, whereas if A is not bounded the family $(X_{-n})_{n\in\mathbb{N}}$ is strictly decreasing.

Combining this with Observation 5.147, we obtain the bi-infinite scale

$$\cdots \hookrightarrow X_{n+1} \hookrightarrow X_n \hookrightarrow \cdots \hookrightarrow X_0 = X \hookrightarrow \cdots \hookrightarrow X_{-1} \hookrightarrow \cdots \hookrightarrow X_{-n+1} \hookrightarrow X_{-n} \hookrightarrow \cdots$$

with all embeddings dense, and we obtain a family of operators $(A_{(n)})_{n\in\mathbb{Z}}$ such that $A_{(n)}$ is m-accretive in X_n with domain X_{n+1} and

$$A_{(n)}x = A_{(j)}x, \quad \text{for all } x \in X_n \cap X_j.$$

Remark 5.151 *Some remarks about Theorem 5.149 and Observation 5.150:*

(i) Note that restriction and extrapolation commute, i.e.,

$$A_{(-n)}(A_{(n)}x) = A_{(n)}(A_{(-n)}x).$$

In particular, $(X_1)_{-1} = (X_{-1})_1 = X$ and $(A_{(1)})_{(-1)} = A_{(-1)}(A_{(1)}) = A$.

(ii) Note also that X_{-n} is the completion of X with respect to the norm $\|J_\lambda^n x\|$. In particular, J_λ^n can be extended by continuity to an isomorphism from X_{-n} onto X. For every $x \in D(A_{(-n)}) = X_{-n+1}$, the element $A_{(-n)}x$ is the limit in X_{-n} of $A(J_\lambda^n x)$, where $J_\lambda^n x \in D(A)$.

Corollary 5.152 *With the notation of Theorem 5.149, if $x \in X$ is such that $A_{(-1)}x \in X$, then $x \in D(A)$.*

Proof: Let A be an m-accretive operator in X with $\overline{D(A)} = X$. Let $f = x + A_{(-1)}x \in X$. Since A is m-accretive, there exists $y \in D(A)$ such that $y + Ay = f$ and then $y + A_{(-1)}y = f$. As $A_{(-1)}$ is m-accretive, it follows that $x = y \in D(A)$. □

Corollary 5.153 *If A is an m-accretive operator in X with dense domain, then:*

(i) $\|J_\lambda x - x\|_{X_{-1}} \leq 2\lambda\|x\|$ for every $x \in X$;

(ii) If $(x_\lambda)_{\lambda>0}$ is a bounded family in X and X is reflexive, then $J_\lambda x_\lambda - x_\lambda \to 0$ as $\lambda \to 0^+$.

To prove Corollary 5.153 we shall use the following results.

Lemma 5.154 *Let $X \hookrightarrow Y$ and $(x_n)_{n\in\mathbb{N}} \subset X$. If $x_n \rightharpoonup x$ in X as $n \to \infty$, then $x_n \rightharpoonup x$ in Y as $n \to \infty$.*

Proof: The embedding $X \to Y$ is continuous; hence it is also continuous with respect to the weak topologies. The result follows. □

Lemma 5.155 *Let $X \hookrightarrow Y$ be Banach spaces and let $(x_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence in X such that $x_n \rightharpoonup y$ in Y as $n \to \infty$ for some $y \in Y$. If X is reflexive, then $y \in X$ and $x_n \rightharpoonup y$ in X as $n \to \infty$.*

Proof: First we show that $y \in X$. Since X is reflexive and (x_n) is bounded in X, there exist $x \in X$ and a subsequence (x_{n_k}) such that $x_{n_k} \rightharpoonup x$ in X as $k \to \infty$. By Lemma 5.154,

$$x_{n_k} \rightharpoonup x \quad \text{in } Y \text{ as } k \to \infty.$$

By uniqueness of the weak limit in Y, we must have $x = y \in X$.

We now prove that $x_n \rightharpoonup y$ in X. Suppose, by contradiction, that this is not the case. Then there exist $x' \in X'$, $\varepsilon > 0$ and a subsequence (x_{n_k}) such that

$$|\langle x', x_{n_k} - y\rangle| \geq \varepsilon, \quad \text{for every } k \in \mathbb{N}.$$

On the other hand, since (x_{n_k}) is bounded in X, there exist a further subsequence $(x_{n_{k_j}})$ and $z \in X$ such that $x_{n_{k_j}} \rightharpoonup z$ in X as $j \to \infty$. By the first part we must have $z = y$, which contradicts the inequality above. Hence $x_n \rightharpoonup y$ in X. □

Proof of Corollary 5.153.
Proof: By item (i) of Proposition 5.137 applied to $A_{(-1)}$, we have

$$\|J_\lambda x - x\|_{X_{-1}} \leq \lambda \|A_{(-1)}x\|_{X_{-1}} \quad \text{for all } \lambda > 0,\ x \in D(A_{(-1)}) = X.$$

From (5.7.216) we know that

$$\|A_{(-1)}x\|_{X_{-1}} = |||A_{(-1)}x||| \leq 2\|x\| \quad \text{for all } x \in X,$$

which proves (i).

For (ii), we have $X \hookrightarrow X_{-1}$, and X and X_{-1} are Banach spaces. Let $(x_\lambda)_{\lambda>0} \subset X$ be bounded and suppose that $J_\lambda x_\lambda - x_\lambda \to 0$ in X_{-1} as $\lambda \to 0^+$. By Lemma 5.155, since X is reflexive, it follows that $J_\lambda x_\lambda - x_\lambda \to 0$ in X as $\lambda \to 0^+$. □

5.7.4 Hilbert spaces and self-adjoint and skew-adjoint operators

In this section we assume that H is a Hilbert space and denote its inner product by $(\cdot,\cdot)$.

Lemma 5.156 *If A is a linear operator in H, then the following properties are equivalent:*

(i) *A is accretive;*

(ii) *$(Ax, x) \geq 0$ for every $x \in D(A)$.*

Proof: Since H is a Hilbert space, it follows from Observation 5.70 that A is accretive if and only if A is monotone. On the other hand, from the remark following Definition 5.2 we know that A is monotone if and only if A is positive (here we are assuming that A is linear and single-valued). □

Corollary 5.157 *If A is m-accretive in H, then $D(A)$ is dense in H.*

Proof: Let $z \in \overline{D(A)}^{\perp}$ and write $J_1 z = x \in D(A)$. Then

$$0 = (z, x) = ((A+I)x, x) = (Ax, x) + \|x\|^2 \geq \|x\|^2,$$

since $(Ax, x) \geq 0$ by the previous lemma. Hence $x = 0$.

As J_1 is a bijection, there exists a unique $y \in H$ such that $J_1 y = 0$. But J_1 is linear, so $J_1 0 = 0$, and thus $y = 0$. Therefore $z = 0$ and we conclude that $\overline{D(A)} = H$. □

Remark 5.158 *The spaces H_n defined above are Hilbert spaces with inner product*

$$(x, y)_{H_n} = (x, y)_H + (Ax, Ay)_{H_1} + \cdots + (A^{n-1}x, A^{n-1}y)_{H_{n-1}}.$$

In the case of the spaces H_{-n}, the inner product is given by

$$(x, y)_{H_{-n}} = (J_1^n x, J_1^n y)_H.$$

Before proceeding, recall that given a linear operator $A : D(A) \subset X \to X$, with X a Banach space, we define

$$D(A^*) = \{u^* \in X';\ \exists v^* \in X' \text{ such that } \langle u^*, Au\rangle = \langle v^*, u\rangle,\ \forall u \in D(A)\}.$$

It is well known that if $D(A)$ is dense in X, then, for each u^*, the corresponding v^* is unique, which allows us to define the *adjoint operator* A^* by

$$\begin{aligned} A^* &: D(A^*) \subset X' \to X', \\ &u^* \mapsto A^* u^* = v^*. \end{aligned}$$

Clearly A^* is linear. If X is a reflexive Banach space and $A : D(A) \subset X \to X$ is a linear, closed, bounded operator with dense domain $D(A)$, then $D(A^*)$ is also dense in X' and the following hold:

(i) If $B \in \mathcal{L}(X)$, then $(A+B)^* = A^* + B^*$; in particular,

$$(A+I)^* = A^* + I;$$

(ii) $(\operatorname{Im} A)^{\perp} = \ker(A^*)$.

Finally, if $A : D(A) \subset X \to X$ is a closed, densely defined linear operator, then the following properties are equivalent:

$$\begin{aligned} &(i) \quad D(A) = X; \\ &(ii) \quad A \text{ is continuous}; \\ &(iii) \quad D(A^*) = X'; \\ &(iv) \quad A^* \text{ is continuous}. \end{aligned} \tag{5.7.217}$$

Remark 5.159 *If A is m-accretive in H, then, by Corollary 5.157, we have $D(A)$ dense in H, and therefore A^* is well defined.*

Lemma 5.160 *Let A be a densely defined operator in H and A^* its adjoint. Then:*

(i) $G(A^*) = \{(x, f) \in H \times H;\ (f, y) = (x, g),\ \forall (y, g) \in G(A)\}$*, that is,*

$$(x, f) \in G(A^*) \iff (-f, x) \in G(A)^{\perp};$$

(ii) $G(A^*)$ *is closed in* $H \times H$.

Proof: See [23]. □

Proposition 5.161 *If A is m-accretive, then:*

(i) *A^* is m-accretive;*

(ii) *$(I+\lambda A^*)^{-1} = ((I+\lambda A)^{-1})^*$ for every $\lambda > 0$;*

(iii) *$(A^*)_\lambda = (A_\lambda)^*$ for every $\lambda > 0$;*

(iv) *$e^{-t(A^*)_\lambda} = (e^{-tA_\lambda})^*$ for every $\lambda > 0$ and $t \in \mathbb{R}$.*

Proof: (i) Let $x \in D(A^*)$ and $\lambda > 0$. Then

$$(A^*x, J_\lambda x) = (x, AJ_\lambda x) = (x, A_\lambda x) = \frac{1}{\lambda}\big(\|x\|^2 - (x, J_\lambda x)\big) \geq 0.$$

For λ small enough we have $\frac{1}{\lambda} \geq 1$, and thus

$$\frac{1}{\lambda}\big(\|x\|^2 - (x, J_\lambda x)\big) \geq \|x\|^2 - (x, J_\lambda x),$$

that is,

$$(A^*x, J_\lambda x) \geq \|x\|^2 - (x, J_\lambda x).$$

As $\lambda \to 0^+$ we have $J_\lambda x \to x$, and therefore

$$(A^*x, J_\lambda x) \longrightarrow (A^*x, x),$$

while

$$\|x\|^2 - (x, J_\lambda x) \longrightarrow \|x\|^2 - \|x\|^2 = 0.$$

Hence

$$(A^*x, x) \geq 0,$$

and by Lemma 5.156 we conclude that A^* is accretive.

To show that A^* is m-accretive, let $\lambda > 0$ and set $L_\lambda = ((I+\lambda A)^{-1})^* \in \mathcal{L}(H)$, which is bounded by (5.7.217). Let $z \in H$ and $x = L_\lambda z$. If $y \in D(A)$, then

$$\begin{aligned}
(x, Ay) &= \frac{1}{\lambda}\big[(x, y+\lambda Ay) - (x,y)\big] \\
&= \frac{1}{\lambda}\big[(L_\lambda z, (I+\lambda A)y) - (x,y)\big] \\
&= \frac{1}{\lambda}\big[(z, (I+\lambda A)^{-1}(I+\lambda A)y) - (x,y)\big] \\
&= \frac{1}{\lambda}\big[(z,y) - (x,y)\big] \\
&= \frac{1}{\lambda}(z-x, y).
\end{aligned}$$

Thus $\big(x, \frac{z-x}{\lambda}\big) \in G(A^*)$, hence $x \in D(A^*)$ and

$$A^*x = \frac{z-x}{\lambda} \quad \Rightarrow \quad z = (I+\lambda A^*)x.$$

Therefore A^* is m-accretive, proving (i). Moreover,

$$(I+\lambda A^*)^{-1}z = x = ((I+\lambda A)^{-1})^* z,$$

and by the arbitrariness of $z \in H$ we obtain (ii).

To prove (iii), note that

$$\begin{aligned}(A^*)_\lambda &= \frac{I-(I+\lambda A^*)^{-1}}{\lambda} = \frac{I-((I+\lambda A)^{-1})^*}{\lambda} \\ &= \left(\frac{I-(I+\lambda A)^{-1}}{\lambda}\right)^* = (A_\lambda)^*.\end{aligned}$$

Finally, for (iv) we have

$$\begin{aligned}e^{-t(A^*)_\lambda} &= \sum_{n=0}^{\infty}\frac{(-t(A^*)_\lambda)^n}{n!} \\ &= \lim_{k\to\infty}\sum_{n=0}^{k}\frac{(-t(A^*)_\lambda)^n}{n!} \\ &= \lim_{k\to\infty}\sum_{n=0}^{k}\frac{(-t(A_\lambda)^*)^n}{n!} \\ &= \lim_{k\to\infty}\sum_{n=0}^{k}\frac{\left((-tA_\lambda)^n\right)^*}{n!} \\ &= \lim_{k\to\infty}\left(\sum_{n=0}^{k}\frac{(-tA_\lambda)^n}{n!}\right)^* \\ &= \left(e^{-tA_\lambda}\right)^*.\end{aligned}$$

□

Proposition 5.162 *Let A be accretive with dense domain $D(A)$ in H. If $G(A)$ is closed and A^* is accretive, then A is m-accretive.*

Proof: From the discussion at the beginning of this section we have

$$(\mathrm{Im}(I+A))^\perp = \ker(I+A^*) = \{x\in D(A^*);\ x+A^*x=0\}.$$

Thus, if $x\in(\mathrm{Im}(I+A))^\perp$, then

$$x+A^*x=0,$$

and since A^* is accretive we obtain

$$\|x\|\le\|x+A^*x\|=0,$$

hence $x=0$ and consequently

$$(\mathrm{Im}(I+A))^\perp=\{0\},$$

which implies

$$\overline{\mathrm{Im}(I+A)}=H.$$

Now let $f\in H$ and choose $(f_n)\subset\mathrm{Im}(I+A)$ such that $f_n\to f$ as $n\to\infty$. Set

$$x_n=(I+A)^{-1}f_n.$$

Since A is accretive, we have

$$\begin{aligned}\|x_n-x_m\| &\le \|(I+A)(x_n-x_m)\| \\ &= \|(I+A)(I+A)^{-1}(f_n-f_m)\| \\ &= \|f_n-f_m\| \longrightarrow 0 \quad \text{as } n,m\to\infty.\end{aligned}$$

Thus (x_n) is a Cauchy sequence in H, and hence there exists $x\in H$ such that $x_n\to x$.

The graph $G(A)$ is closed, and consequently $G(I+A)$ is also closed. Since

$$x_n \in D(A) = D(I+A), \quad x_n \to x, \quad (I+A)x_n = f_n \to f,$$

we conclude that $x \in D(A)$ and $f = (I+A)x$. Therefore $\text{Im}(I+A)$ is closed and hence

$$\text{Im}(I+A) = H,$$

so A is m-accretive. □

0.5cm

Definition 5.163 *An operator A with dense domain in H is said to be* symmetric *(respectively,* skew-symmetric*) if*

$$G(A) \subset G(A^*) \quad (\textit{respectively } G(A) \subset G(-A^*)).$$

We say that A is self-adjoint *(respectively,* skew-adjoint*) if*

$$A = A^* \quad (\textit{respectively } A = -A^*).$$

Remark 5.164 *From Definition 5.163 we deduce that*

$$\begin{aligned} A \text{ is symmetric} &\iff (Ax, y) = (x, Ay), \ \forall x, y \in D(A), \\ A \text{ is skew-symmetric} &\iff (Ax, y) = (x, -Ay), \ \forall x, y \in D(A), \\ A \text{ self-adjoint} &\implies A \text{ symmetric}, \\ A \text{ skew-adjoint} &\implies A \text{ skew-symmetric}. \end{aligned}$$

Corollary 5.165 *Let A be a densely defined operator in H. Then:*

(i) *If A is skew-adjoint, then A and $-A$ are m-accretive and $(Ax, x) = 0$ for every $x \in D(A)$;*

(ii) *If A is self-adjoint and accretive, then A is m-accretive.*

Proof: (i) Let $x \in D(A)$. Then

$$(Ax, x) = (x, A^*x) = (x, -Ax) = -(Ax, x),$$

hence

$$(Ax, x) = 0 = (-Ax, x), \quad \forall x \in D(A),$$

and by Lemma 5.156 both A and $-A$ are accretive.

Next observe that

- $-A$ is accretive;
- $(-A)^* = -A^* = -(-A) = A$ is accretive;
- $G(-A) = G(A^*)$ is closed by Lemma 5.160;

thus, by Proposition 5.162, $-A$ is m-accretive.

Similarly,

- A is accretive;
- $A^* = -A$ is accretive;

- $G(A)$ is closed, since $G(-A)$ is;

hence, again by Proposition 5.162, A is m-accretive.

(ii) We have $A^* = A$ and A is accretive, while $G(A) = G(A^*)$ is closed by Lemma 5.160. The result then follows from Proposition 5.162. □

Corollary 5.166 *If A is m-accretive in H, then the following are equivalent:*

(i) *A is self-adjoint;*

(ii) *$(Ax, y) = (x, Ay)$ for all $x, y \in D(A)$.*

Proof: $(i) \Rightarrow (ii)$ is immediate.

$(ii) \Rightarrow (i)$ If A satisfies (ii), then $G(A) \subset G(A^*)$. We show the reverse inclusion. Let $(x, f) \in G(A^*)$ and set

$$g = x + A^*x = x + f.$$

Since A is m-accretive, there exists $y \in D(A)$ such that

$$g = y + Ay,$$

and as $G(A) \subset G(A^*)$, we have $y \in D(A^*)$ and

$$g = y + A^*y.$$

Thus

$$y + A^*y = x + A^*x.$$

Since A^* is accretive,

$$\|x - y\| \leq \|(I + A^*)(x - y)\| = 0,$$

so $x = y$. Therefore $(x, Ax) = (x, f) \in G(A)$, and hence $G(A^*) \subset G(A)$ and $A = A^*$. □

Corollary 5.167 *If A is m-accretive, then the following are equivalent:*

(i) *A is skew-adjoint;*

(ii) *$(Ax, x) = 0$ for every $x \in D(A)$;*

(iii) *$-A$ is m-accretive.*

Proof: Observe that $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ follow from Corollary 5.165.

$(iii) \Rightarrow (ii)$ Since A and $-A$ are m-accretive, we have

$$(Ax, x) \geq 0 \quad \text{and} \quad (-Ax, x) \geq 0, \quad \forall x \in D(A),$$

whence

$$(Ax, x) = 0, \quad \forall x \in D(A).$$

$(ii) \Rightarrow (i)$ Let $x, y \in D(A)$. Then

$$(Ax, y) + (x, Ay) = (A(x + y), x + y) - (Ax, x) - (Ay, y) = 0,$$

so

$$(Ax, y) = (x, -Ay), \quad \forall x, y \in D(A),$$

which implies $G(A) \subset G(-A^*)$.

We claim that $(A^*x, x) = 0$ for every $x \in D(A^*)$.

Indeed, if $x \in D(A) \subset D(-A^*) = D(A^*)$, then

$$(A^*x, x) = (x, Ax) = 0.$$

Now let $x \in D(A^*)$. For each $\lambda > 0$ we have $J_\lambda x \in D(A)$, hence

$$(A^*J_\lambda x, J_\lambda x) = 0.$$

But

$$(A^*J_\lambda x, J_\lambda x) = ((J_\lambda)^* A^* J_\lambda x, x) = ((A_\lambda)^*(J_\lambda x), x), \quad \forall \lambda > 0,$$

that is,

$$((A_\lambda)^*(J_\lambda x), x) = 0, \quad \forall \lambda > 0. \tag{5.7.218}$$

Since $J_\lambda, (A_\lambda)^* \in \mathcal{L}(H)$ and $J_\lambda x \to x$, $(A_\lambda)^* x \to A^*x$ as $\lambda \to 0^+$, we have

$$(A_\lambda)^*(J_\lambda x) \longrightarrow A^*x$$

as $\lambda \to 0^+$. Passing to the limit in (5.7.218) we obtain $(A^*x, x) = 0$.

From this we also get $(-A^*x, x) = 0$ for every $x \in D(-A^*)$, and thus $-A^*$ is accretive.

Since $D(A) \subset D(-A^*)$ and A is m-accretive (hence maximal), we conclude that $A = -A^*$, proving (i). □

Corollary 5.168 *Let A be m-accretive and $A_{(n)}$ the operator defined in Observation 5.150, for $n \in \mathbb{Z}$. If A is self-adjoint, then $A_{(n)}$ is self-adjoint (the same holds if A is skew-adjoint).*

Proof: We argue by induction on $n \in \mathbb{N}$. We first prove the result for $A_{(n)}$ with $n \geq 0$ and then for $A_{(-n)}$, $n \geq 1$.

Assume that A is self-adjoint and let $x, y \in D(A_1)$. Then

$$\begin{aligned}
(A_1 x, y)_{H_1} &= (A_1 x, y)_H + (A(A_1)x, y)_H \\
&= (Ax, y)_H + (A^2 x, y)_H \\
&= (x, Ay)_H + (x, A^2 y)_H \\
&= (x, A_1 y)_{H_1}.
\end{aligned}$$

Since A_1 is m-accretive, it follows from Corollary 5.166 that A_1 is self-adjoint.

Assume now that A_k is self-adjoint and let us show that A_{k+1} is also self-adjoint. For $x, y \in D(A_{k+1})$ we have

$$\begin{aligned}
(A_{k+1} x, y)_{H_{k+1}} &= (A_{k+1} x, y)_{H_k} + (A_k(A_{k+1})x, y)_{H_k} \\
&= (A_k x, y)_{H_k} + (A_k^2 x, y)_{H_k} \\
&= (x, A_k y)_{H_k} + (x, A_k^2 y)_{H_k} \\
&= (x, A_{k+1} y)_{H_{k+1}},
\end{aligned}$$

and again, since A_{k+1} is m-accretive, Corollary 5.166 gives that A_{k+1} is self-adjoint.

Now let $x, y \in D(A_{(-1)}) = H$. Then there exist sequences (x_n), $(y_n) \subset D(A)$ such that $x_n \to x$ and $y_n \to y$ in H. Moreover, by Theorem 5.149(v),

$$A_{(-1)} x_n = Ax_n \in H, \quad A_{(-1)} y_n = Ay_n \in H,$$

and by Theorem 5.149(iv) the norms in $D(A_{(-1)})$ and H are equivalent. Since, by Corollary 5.129, $A_{(-1)}$ is continuous on $D(A_{(-1)})$, we obtain

$$A_{(-1)}x_n \to A_{(-1)}x, \quad A_{(-1)}y_n \to A_{(-1)}y \quad \text{in } H,$$

and, as $H \hookrightarrow H_{(-1)}$, also

$$A_{(-1)}x_n \to A_{(-1)}x, \quad A_{(-1)}y_n \to A_{(-1)}y \quad \text{in } H_{(-1)}.$$

Thus,

$$\begin{aligned}(A_{(-1)}x_n, y_n)_{H_{(-1)}} &= (J_1(A_{(-1)}x_n), J_1 y_n)_H \\ &= (J_1 A x_n, J_1 y_n)_H \\ &= (A J_1 x_n, J_1 y_n)_H \\ &= (J_1 x_n, A J_1 y_n)_H \\ &= (J_1 x_n, J_1 A y_n)_H \\ &= (J_1 x_n, J_1 A_{(-1)} y_n)_H \\ &= (x_n, A_{(-1)} y_n)_{H_{(-1)}},\end{aligned}$$

that is,

$$(A_{(-1)}x_n, y_n)_{H_{(-1)}} = (x_n, A_{(-1)}y_n)_{H_{(-1)}}.$$

Letting $n \to \infty$ we obtain

$$(A_{(-1)}x, y)_{H_{(-1)}} = (x, A_{(-1)}y)_{H_{(-1)}}. \tag{5.7.219}$$

From Observation 5.150 we know that $A_{(-1)}$ is m-accretive, and thus, by (5.7.219) and Corollary 5.166, $A_{(-1)}$ is self-adjoint.

Assume now that $A_{(-k)}$ is self-adjoint and let us show that $A_{(-(k+1))}$ is also self-adjoint. All the properties of $A_{(-1)}$ with respect to A and of $H_{(-1)}$ with respect to H hold, in an analogous way, for $A_{(-(k+1))}$ with respect to $A_{(-k)}$ and for $H_{(-(k+1))}$ with respect to $H_{(-k)}$. Hence the argument is the same as in the case $n = 1$. Therefore $A_{(-(k+1))}$ is self-adjoint and the proof is complete. □

Lemma 5.169 *Let A be an m-accretive operator in H. Consider a family $(x_\varepsilon)_{\varepsilon>0} \subset D(A)$. If $x_\varepsilon \rightharpoonup x$ in H as $\varepsilon \to 0$, and if (Ax_ε) is bounded in H, then $x \in D(A)$ and $Ax_\varepsilon \rightharpoonup Ax$ in H as $\varepsilon \to 0$.*

Proof: Since H is a Hilbert space, it is reflexive. This ensures the existence of a sequence $\varepsilon_n \to 0$ and of some $y \in H$ such that $Ax_{\varepsilon_n} \rightharpoonup y$ in H as $n \to +\infty$. In particular, $(x_{\varepsilon_n}, Ax_{\varepsilon_n}) \rightharpoonup (x, y)$ in $H \times H$ as $n \to +\infty$. On the other hand, Proposition 5.89 shows that $G(A)$ is closed, and in particular closed in the weak topology of $H \times H$, hence $x \in D(A)$ and $y = Ax$.

Assume now that $Ax_\varepsilon \not\rightharpoonup Ax$ as $\varepsilon \to 0$. Then there exist $\mathbb{N}' \subset \mathbb{N}$ and a subsequence $(\varepsilon_n)_{n\in\mathbb{N}'}$ with $\varepsilon_n \to 0$ such that $Ax_{\varepsilon_n} \not\rightharpoonup Ax$ as $n \to \infty$. By definition of weak convergence, there exist $\varphi_0 \in H'$ and $\eta_0 > 0$ with the property that, for every $\delta_n = \frac{1}{n} > 0$, we can find $\varepsilon_n \in \mathbb{R}$ such that $0 < |\varepsilon_n| < \delta_n$ and

$$\big|\langle \varphi_0, Ax_{\varepsilon_n}\rangle - \langle \varphi_0, Ax\rangle\big| \geq \eta_0,$$

as required.

Since (Ax_{ε_n}) is bounded, there exist $\mathbb{N}^* \subset \mathbb{N}'$ and $h \in H$ such that $(Ax_{\varepsilon_{n_k}})$ satisfies $Ax_{\varepsilon_{n_k}} \rightharpoonup h$. As $G(A)$ is closed and $x_{\varepsilon_{n_k}} \rightharpoonup x$, we obtain $Ax = h = Ax$.

On the other hand, from the existence of $\mathbb{N}' \subset \mathbb{N}$ and $(\varepsilon_n)_{n\in\mathbb{N}'}$ with $\varepsilon_n \to 0$ such that $Ax_{\varepsilon_n} \not\rightharpoonup Ax$ as $n \to \infty$, we infer the existence of a neighbourhood V of Ax such that, for each $n \in \mathbb{N}'$, there exists $n_0 \in \mathbb{N}'$, $n_0 > n$, with $Ax_{\varepsilon_{n_0}} \notin V$.

Taking this same neighbourhood V and using the fact that $Ax_{\varepsilon_{n_k}} \rightharpoonup Ax$, there exists $m_0 \in \mathbb{N}^* \subset \mathbb{N}'$

such that $Ax_{\varepsilon_{n_k}} \in V$ for all $k \geq m_0$, which contradicts the property above. □

Remark 5.170 *The following properties hold:*

(i) If A is self-adjoint and n is a non-negative integer, then A^{2n} is self-adjoint and accretive and therefore m-accretive.

(ii) If A is self-adjoint and accretive and if n is a non-negative integer, then A^{2n+1} is self-adjoint and hence m-accretive.

(iii) If A is skew-adjoint and n is a non-negative integer, then A^{2n} is self-adjoint and accretive and therefore m-accretive.

(iv) If A is skew-adjoint and n is a non-negative integer, then A^{2n+1} is skew-adjoint and hence m-accretive.

Remark 5.171 *If A is an m-accretive operator, then A^n is not necessarily accretive.*

Indeed, consider $H = \mathbb{R}^2$ and for each $x = (a, b) \in \mathbb{R}^2$ let A be the operator that rotates x by an angle $\theta = \frac{\pi}{2}$. Then A^2 is the rotation by an angle $\theta = \pi$.

In particular, for $x = (\|x\| \cos\theta, \|x\| \sin\theta)$ we have

$$\begin{aligned}
(A^2x, x) &= \big(\|x\| \cos(\theta + \pi), \|x\| \sin(\theta + \pi)\big) \cdot \big(\|x\| \cos\theta, \|x\| \sin\theta\big) \\
&= \|x\|^2 \cos(\theta + \pi) \cos\theta + \|x\|^2 \sin(\theta + \pi) \sin\theta \\
&= \|x\|^2 \cos\big((\theta + \pi) - \theta\big) \\
&= \|x\|^2 \cos(\pi) \\
&= -\|x\|^2 \leq 0.
\end{aligned}$$

Consequently, A^2 is not m-accretive. We already know that A is monotone from the first example in the chapter on Monotone and Accretive Operators. Since H is a Hilbert space, A is accretive. Now, given $f = (y, z) \in \mathbb{R}^2$, it suffices to take $x = (a, b) = \big(\frac{y+z}{2}, \frac{y-z}{2}\big)$ to see that $x + Ax = f$, and consequently A is m-accretive.

Let A be an m-accretive operator in H and let A^* be its adjoint. It follows from Proposition 5.161 that A^* is also m-accretive. In particular, $D((A^*)^n)$ is dense in H for each non-negative integer n. Hence, if $D((A^*)^n)$ is endowed with the norm $\|x\|_{D((A^*)^n)} = \sum_{j=1}^{n} \|(A^*)^j x\|$, then

$$D((A^*)^n) \hookrightarrow H \hookrightarrow D((A^*)^n)',$$

where both embeddings are dense. Thus we obtain the following result.

Proposition 5.172 *Let A be as above and $(H_{-n})_{n \geq 0}$ the spaces defined in Observation 5.150. Then $H_{-n} = D((A^*)^n)'$ with equivalent norms.*

Proof: It is enough to prove that $\|x\|_{H_{-n}} \approx \|x\|_{D((A^*)^n)'}$. By density, we may assume that $x \in H$. It follows from Observation 5.151(ii), Proposition 5.162, Observation 5.148 and Observation 5.147(ii) that

$$\begin{aligned}
\|x\|_{H_{-n}} &= \|J_1(A)^n x\| \\
&= \sup_{\|y\|=1} (J_1(A)^n x, y)_{H,H} \\
&= \sup_{\|y\|=1} (x, J_1(A^*)^n y)_{H,H} \\
&= \sup_{\|y\|=1} (x, J_1(A^*)^n y)_{D((A^*)^n)', D((A^*)^n)} \\
&= \sup_{\|z\|_{D((A^*)^n)}=1} (x, z)_{D((A^*)^n)', D((A^*)^n)} \\
&= \|x\|_{D((A^*)^n)'}.
\end{aligned}$$

This proves the claim. □

Corollary 5.173 *Let A be a self-adjoint and accretive operator, or a skew-adjoint operator, in H, and let $(H_n)_{n\in\mathbb{Z}}$ be the spaces introduced in Observation 5.150. Then $H_{-n} = H_n'$ with equivalent norms for each $n \in \mathbb{Z}$.*

Proof: Consider $n \geq 0$. By Observation 5.148 we have $H_n = D(A^n) = D((A^*)^n)$, and thus $H_{-n} = H_n'$ by Proposition 5.172. By Observation 5.158, the spaces H_n are Hilbert spaces and therefore reflexive. Hence $H_{-n}' = H_n'' = H_n$. □

Recall that H is said to be a complex Hilbert space if there exists a mapping $b : H \times H \to \mathbb{C}$ satisfying:

$$\begin{cases}
b(\lambda x + \mu y, z) = \lambda b(x,z) + \mu b(y,z), & \text{for all } x, y, z \in H \text{ and all } \lambda, \mu \in \mathbb{R}; \\
b(y,x) = \overline{b(x,y)}, & \text{for all } x, y \in H; \\
b(ix, y) = ib(x,y), & \text{for all } x, y \in H; \\
b(x,x) = \|x\|^2, & \text{for all } x \in H.
\end{cases}$$

Moreover, H is a Banach space with respect to the norm induced by $b : H \times H \to \mathbb{C}$.

It is easy to see that H, endowed with the inner product

$$(x, y) = \mathrm{Re}(b(x,y)),$$

is a real Hilbert space.

Lemma 5.174 *Let H be a complex Hilbert space and let A be an operator in H. Assume that A is $\mathbb{C}$-linear and define iA by*

$$\begin{cases}
D(iA) = D(A), \\
(iA)x = iAx, \quad \text{for every } x \in D(A).
\end{cases}$$

If $D(A)$ is dense in H, then A^* is $\mathbb{C}$-linear and $(iA)^* = -iA^*$.

Proof: Recall that $G(A^*) = \{(x,f) \in H \times H;\ (f,y) = (x,g),\ \forall (y,g) \in G(A)\}$. Let $(x,f) \in G(A^*)$. We want to show that for each $\lambda \in \mathbb{C}$ we have $(\lambda x, \lambda f) \in G(A^*)$, or equivalently, that for each $(y,g) \in G(A)$,

$$(\lambda f, y) = (\lambda x, g).$$

Indeed,

$$\begin{aligned}(\lambda f, y) &= (f, \overline{\lambda} y) \\ &= (A^* x, \overline{\lambda} y) \\ &= (x, A(\overline{\lambda} y)) \\ &= (x, \overline{\lambda} A y) \\ &= (\lambda x, A y) \\ &= (A^*(\lambda x), y), \quad \text{for every } y \in D(A).\end{aligned}$$

By density we obtain $A^*(\lambda x) = \lambda A^*(x)$ for every $x \in D(A^*)$. Similarly,

$$\begin{aligned}(A^*(x+y), z) &= (x+y, Az) \\ &= (x, Az) + (y, Az) \\ &= (A^* x, z) + (A^* y, z) \\ &= (A^*(x+y), z), \quad \text{for every } z \in D(A),\end{aligned}$$

so A^* is $\mathbb{C}$-linear.

Now, if $(x, f) \in G(A^*)$, then $x \in D(A^*)$ and $f = A^* x$, hence $-if = A^*(-ix) = -iA^* x$, that is, $(x, -if) \in G(-iA^*)$. We now show that $(x, -if) \in G((iA)^*)$. Indeed, for $(y, g) \in G(A)$ we have

$$\begin{aligned}(-if, y) &= (f, iy) \\ &= (A^* x, iy) \\ &= (x, A(iy)) \\ &= (x, iAy) \\ &= (x, ig).\end{aligned}$$

Therefore $(x, -if) \in G((iA)^*)$, and so $G(-iA^*) \subset G((iA)^*)$. Applying this to the operator iA, we get $G(-i(iA)^*) \subset G(-A^*)$. By $\mathbb{C}$-linearity it follows that $G((iA)^*) \subset G(-iA^*)$. Consequently,

$$G((iA)^*) = G(-iA^*).$$

□

Corollary 5.175 *Let H be a complex Hilbert space and let A be an operator in H. If A is $\mathbb{C}$-linear, then the following properties are equivalent:*

(i) A is self-adjoint;

(ii) iA is skew-adjoint.

Proof: Assume that A is self-adjoint. It follows from Lemma 5.174 that

$$(iA)^* = -iA^* = -iA = -(iA),$$

so iA is skew-adjoint.

Conversely, if iA is skew-adjoint, then

$$A^* = (-i(iA))^* = i(iA)^* = -i(iA) = A,$$

that is, A is self-adjoint. □

5.7.5 Examples of m-accretive operators and partial differential operators

In this section we describe some examples of partial differential operators associated with classical evolution equations.

5.7.5.1 First-order operators

We now present a series of examples related to transport equations.

Example 5.176 A first-order operator in $\mathbb{R}$. *Let $X = C_b(\mathbb{R})$ and define the operator A in X by*

$$\begin{cases} D(A) = \{u \in C^1(\mathbb{R}) \cap X;\ u' = \dfrac{du}{dx} \in X\}, \\ Au = u', \quad \text{for } u \in D(A). \end{cases} \tag{5.7.220}$$

Proposition 5.177 *If A is defined by (5.7.220), then A and $-A$ are m-accretive.*

Proof: We first show that A is accretive. To this end, take $\lambda > 0$ and $(u, f) \in D(A) \times X$ satisfying $u + \lambda Au = f$. This implies that

$$u + \lambda u' = f, \quad \text{for all } x \in \mathbb{R}. \tag{5.7.221}$$

Let

$$Lf(x) = \frac{1}{\lambda}\int_{-\infty}^{x} e^{\frac{s-x}{\lambda}} f(s)\, ds. \tag{5.7.222}$$

Then

$$\begin{aligned} |Lf(x)| &\le \frac{1}{\lambda}\int_{-\infty}^{x} e^{\frac{s-x}{\lambda}} |f(s)|\, ds \\ &\le \frac{1}{\lambda}\|f\|_\infty \int_{-\infty}^{x} e^{\frac{s-x}{\lambda}}\, ds \\ &= \|f\|_\infty. \end{aligned}$$

Indeed,

$$\begin{aligned} \int_{-\infty}^{x} e^{\frac{s-x}{\lambda}}\, ds &= \lim_{b\to+\infty}\int_{-b}^{x} e^{\frac{s-x}{\lambda}}\, ds \\ &= \lim_{b\to+\infty}\int_{\frac{-b-x}{\lambda}}^{0} \lambda e^{u}\, du, \quad \text{via the change of variables } u = u(s) = (s-x)\lambda^{-1} \\ &= \lambda \lim_{b\to+\infty} e^{u}\Big|_{\frac{-b-x}{\lambda}}^{0} \\ &= \lambda\big(1 - \lim_{b\to+\infty} e^{\frac{-b-x}{\lambda}}\big) \\ &= \lambda. \end{aligned}$$

Therefore,

$$\|Lf\|_\infty \le \|f\|_\infty, \tag{5.7.223}$$

since

$$\|f\|_\infty = \inf\{c;\ |f(x)| \le c,\ \forall x \in \mathbb{R}\} = \sup_{x\in\mathbb{R}} |f(x)|.$$

In general, the solution of (5.7.221) is given by

$$u(x) = Lf(x) + ae^{-\frac{x}{\lambda}}.$$

Indeed, we know that the general solution of an equation of the form $y' + p(x)y = q(x)$ is

$$y(x) = e^{-\int p(x)\,dx}\left(\int q(x)e^{\int p(x)\,dx}dx + a\right).$$

In our case the equation has the form

$$y' + \frac{1}{\lambda}y = \frac{1}{\lambda}f,$$

and hence

$$\begin{aligned} y(x) &= e^{-\int \frac{1}{\lambda}\,dx}\left(\frac{1}{\lambda}\int f(x)e^{\int \frac{1}{\lambda}\,dx}dx + a\right) \\ &= e^{-\frac{x}{\lambda}}\left(\frac{1}{\lambda}\int e^{\frac{x}{\lambda}}f(x)\,dx + a\right) \\ &= e^{-\frac{x}{\lambda}}\left(\frac{1}{\lambda}\int e^{\frac{s}{\lambda}}f(s)\,ds + a\right) \\ &= \frac{1}{\lambda}\int e^{\frac{s-x}{\lambda}}f(s)\,ds + ae^{-\frac{x}{\lambda}}. \end{aligned}$$

Since u and Lf are bounded, we must have $a = 0$. If we assume $a \neq 0$, then there exists $K \in \mathbb{N}$ such that

$$|ae^{-\frac{x}{\lambda}}| \leq |u(x) - Lf(x)| \leq K, \quad \forall x \in \mathbb{R}.$$

As λ is fixed, this leads to a contradiction as $x \to -\infty$.

Therefore $u = Lf$, and from (5.7.223) we obtain

$$\|u\|_\infty = \|Lf\|_\infty \leq \|f\|_\infty = \|u + \lambda u'\|_\infty = \|u + \lambda Au\|_\infty.$$

Hence A is accretive.

Lemma 5.178 *Let $f : [a,b] \to \mathbb{R}$ be continuous and $\alpha, \beta : I \to [a,b]$ differentiable. Let $\varphi : I \to \mathbb{R}$ be defined by*

$$\varphi(x) = \int_{\alpha(x)}^{\beta(x)} f(t)\,dt, \quad x \in I.$$

Then φ is differentiable and

$$\varphi'(x) = f(\beta(x))\beta'(x) - f(\alpha(x))\alpha'(x).$$

Let $\lambda > 0$ and $f \in X$. By the lemma above and by (5.7.223), we have $Lf \in X \cap C^1(\mathbb{R})$. Indeed, defining $\alpha, \beta : \mathbb{R} \to [-b, x]$ by $\alpha(x) = b$ and $\beta(x) = x$, it is clear from the lemma that $(Lf)' = f$. Thus $Lf \in C^1(\mathbb{R})$ and satisfies (5.7.221). Consequently $Lf \in D(A)$, since $(Lf)' = f \in X$. In summary, $Lf \in D(A)$ and $Lf + \lambda(Lf)' = f$. Hence A is m-accretive. The same argument shows that $-A$ is m-accretive. □

Remark 5.179 *Note that in the previous example $D(A)$ is* not *dense in X. For instance, $u(x) = \sin(x^2)$ belongs to X. However, if $z \in C^1(\mathbb{R})$ satisfies $\|z - u\|_\infty \leq \frac{1}{4}$, then $\sup_{x\in\mathbb{R}} |z'(x)| = \infty$, so $z \notin D(A)$. Therefore u cannot be approximated by elements of $D(A)$.*

Remark 5.180 *We can modify the examples above as follows:*

(i) *Let $X = L^\infty(\mathbb{R})$ and define A by*

$$\begin{cases} D(A) = W^{1,\infty}(\mathbb{R}), \\ Au = u', \quad \text{for } u \in D(A). \end{cases}$$

Then A and $-A$ are m-accretive. The proof is essentially the same as that of Proposition 5.177. Note that in this example it is easy to see that $D(A)$ is not dense in X.

Indeed, let $u(x) = \sin(x^2)$. Then $u(x) \in X = L^\infty(\mathbb{R})$. However, given $\varepsilon = \frac{1}{4}$, if there existed $z \in W^{1,\infty}(\mathbb{R})$ such that $\|z-u\|_\infty \leq \frac{1}{4}$, we would have $\sup_{x\in\mathbb{R}} |z'(x)| = \infty$, contradicting the fact that $z' \in L^\infty(\mathbb{R})$. Thus $W^{1,\infty}(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$.

(ii) *Let now $X = C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ is the closure of $\mathcal{D}(\mathbb{R})$ in L^∞, and $\mathcal{D}(\mathbb{R})$ is the Fréchet space of C^∞ functions from $\mathbb{R}$ to $\mathbb{R}$ with compact support in $\mathbb{R}$, endowed with the topology of uniform convergence of all derivatives on compact subsets of $\mathbb{R}$. Define A in X by*

$$\begin{cases} D(A) = \{u \in C^1(\mathbb{R}) \cap X;\ u' \in X\}, \\ Au = u', \quad \text{for } u \in D(A). \end{cases}$$

Then A and $-A$ are m-accretive with dense domain. The proof that A and $-A$ are m-accretive is the same as in Proposition 5.177. To show that $D(A)$ is dense in X, note that

$$X = \overline{C_c^\infty(\mathbb{R})}^{L^\infty} \subset \overline{D(A)}^{L^\infty} \subset \overline{X}^{L^\infty} = X.$$

(iii) *Now let $1 \leq p < \infty$, take $X = L^p(\mathbb{R})$ and define A by*

$$\begin{cases} D(A) = W^{1,\infty}(\mathbb{R}), \\ Au = u', \quad \text{for } u \in D(A). \end{cases}$$

Then A and $-A$ are m-accretive with dense domain. If $p = 2$, then A is skew-adjoint. Since $\mathcal{D}(\mathbb{R}) \subset D(A)$, we have

$$X = L^p(\mathbb{R}) = \overline{C_c^\infty(\mathbb{R})} \subset \overline{W^{1,p}(\mathbb{R})} = \overline{D(A)}.$$

Hence $X \subset \overline{D(A)}$ and therefore $\overline{D(A)} = X$.

We show that A is m-accretive; following the proof of Proposition 5.177, it suffices to show that $L \in \mathcal{L}(L^p)$ and $\|L\|_{\mathcal{L}(L^p(\mathbb{R}))} \leq 1$, i.e., $\|Lf\|_p \leq \|f\|_p$.

For $p = 1$ we have

$$\begin{aligned} |Lf(x)| &= \left| \frac{1}{\lambda} \int_{-\infty}^{x} e^{\frac{s-x}{\lambda}} f(s)\, ds \right| \\ &= \left| \frac{1}{\lambda} \int_{-\infty}^{0} e^{\frac{s}{\lambda}} f(s+x)\, ds \right| \\ &\leq \frac{1}{\lambda} \int_{-\infty}^{0} e^{\frac{s}{\lambda}} |f(s+x)|\, ds. \end{aligned}$$

Thus

$$\int_{\mathbb{R}} |Lf(x)|\, dx \leq \frac{1}{\lambda} \int_{\mathbb{R}} \int_{-\infty}^{0} e^{\frac{s}{\lambda}} |f(s+x)|\, ds\, dx.$$

By Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}} |Lf(x)|\, dx &\leq \frac{1}{\lambda} \left(\int_{-\infty}^{0} e^{\frac{s}{\lambda}}\, ds \right) \left(\int_{\mathbb{R}} |f(s+x)|\, dx \right) \\ &= \frac{1}{\lambda} \left(\lambda \int_{-\infty}^{0} e^{s}\, ds \right) \left(\int_{\mathbb{R}} |f(x)|\, dx \right) \\ &= \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1. \end{aligned}$$

Therefore $\|Lf\|_1 \leq \|f\|_1$.

Now let $1 < p < \infty$ and let p' be the conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\begin{aligned}
|Lf(x)| &= \left| \frac{1}{\lambda} \int_{-\infty}^{x} e^{\frac{s-x}{\lambda}} f(s)\, ds \right| \\
&= \left| \frac{1}{\lambda} \int_{-\infty}^{0} e^{\frac{s}{\lambda}} f(s+x)\, ds \right| \\
&\leq \frac{1}{\lambda} \int_{-\infty}^{0} e^{\frac{s}{\lambda}} |f(s+x)|\, ds \\
&= \frac{1}{\lambda} \int_{-\infty}^{0} e^{\frac{s}{\lambda}(\frac{1}{p'}+\frac{1}{p})} \left(|f(s+x)|^p\right)^{1/p} ds \\
&= \frac{1}{\lambda} \int_{-\infty}^{0} e^{\frac{s}{\lambda p'}} e^{\frac{s}{\lambda p}} \left(|f(s+x)|^p\right)^{1/p} ds.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
|Lf(x)| &\leq \frac{1}{\lambda} \left(\int_{-\infty}^{0} \left| e^{\frac{s}{\lambda p'}} \right|^{p'} ds \right)^{1/p'} \left(\int_{-\infty}^{0} \left(e^{\frac{s}{\lambda}} |f(s+x)|^p \right) ds \right)^{1/p} \\
&= \frac{1}{\lambda} \left(\int_{-\infty}^{0} e^{\frac{s}{\lambda}}\, ds \right)^{1/p'} \left(\int_{-\infty}^{0} e^{\frac{s}{\lambda}} |f(s+x)|^p\, ds \right)^{1/p} \\
&= \frac{1}{\lambda} \left(\lambda \int_{-\infty}^{0} e^{s}\, ds \right)^{1/p'} \left(\int_{-\infty}^{0} e^{\frac{s}{\lambda}} |f(s+x)|^p\, ds \right)^{1/p} \\
&= \lambda^{-1/p} \left(\int_{-\infty}^{0} e^{\frac{s}{\lambda}} |f(s+x)|^p\, ds \right)^{1/p}.
\end{aligned}$$

Therefore

$$|Lf(x)|^p \leq \frac{1}{\lambda} \int_{-\infty}^{0} e^{\frac{s}{\lambda}} |f(s+x)|^p\, ds.$$

Hence

$$\int_{\mathbb{R}} |Lf(x)|^p\, dx \leq \frac{1}{\lambda} \int_{\mathbb{R}} \int_{-\infty}^{0} e^{\frac{s}{\lambda}} |f(s+x)|^p\, ds\, dx.$$

By Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}} |Lf(x)|^p\, dx &\leq \frac{1}{\lambda} \left(\int_{-\infty}^{0} e^{\frac{s}{\lambda}}\, ds \right) \left(\int_{\mathbb{R}} |f(s+x)|^p\, dx \right) \\
&= \frac{1}{\lambda} \left(\lambda \int_{-\infty}^{0} e^{s}\, ds \right) \left(\int_{\mathbb{R}} |f(x)|^p\, dx \right) \\
&= \int_{\mathbb{R}} |f(x)|^p\, dx = \|f\|_p^p.
\end{aligned}$$

Thus $\|Lf\|_p \leq \|f\|_p$.

As in the proof of Proposition 5.177, it follows that A is m-accretive, and by the same argument $-A$ is m-accretive.

If $p = 2$, then $W^{1,2}(\mathbb{R}) = H^1(\mathbb{R})$ is a Hilbert space and, by Corollary 5.167, A is skew-adjoint.

Example 5.181 A first-order operator on a bounded interval. *Let $X = \{u \in C([0,1]);\ u(0) = u'(0) = 0\}$ endowed with the supremum norm. Define the operator A in X by*

$$\begin{cases} D(A) = \{u \in C^1([0,1]);\ u(0) = u'(0) = 0\}, \\ Au = u', \quad \text{for } u \in D(A). \end{cases} \tag{5.7.224}$$

Before stating the first result we recall a proposition that will be used in its proof.

Proposition 5.182 *Assume that $f \in C(\mathbb{R}^n)$. Then $(\rho_n * f) \to f$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^n$.*

Proof: See [14], Proposition IV.2.1, p. 70. □

Proposition 5.183 *The operator A defined above is m-accretive with dense domain.*

Proof: Following the proof of Proposition 5.177, given $f \in X$ and $\lambda > 0$, the unique solution of

$$u + \lambda u' = f$$

is

$$u(x) = \frac{1}{\lambda} \int_0^x e^{\frac{s-x}{\lambda}} f(s)\, ds,$$

from which it follows that A is m-accretive. We now show that $D(A)$ is dense in X.

Let $u \in X$ and $\delta > 0$, and define $u_\delta \in X$ by

$$u_\delta(x) = \begin{cases} 0, & x \in [0, \delta], \\ u(x-\delta), & x \geq \delta. \end{cases}$$

Then $\|u_\delta - u\| \to 0$ in X as $\delta \to 0^+$.

Given $\varepsilon > 0$, choose δ sufficiently small so that

$$\|u_\delta - u\| \leq \frac{\varepsilon}{2}.$$

Define $v_\delta \in C_c(\mathbb{R})$ by

$$v_\delta(x) = \begin{cases} 0, & x \leq 0, \\ u_\delta(x), & 0 \leq x \leq 1, \\ (2-x)u_\delta(1), & 1 \leq x \leq 2, \\ 0, & x \geq 2. \end{cases}$$

Note that $\operatorname{supp} v_\delta \subset [0, 2]$ and the intervals on which it is defined are closed; moreover, if $A = [0, 1]$ and $B = [1, 2]$, then $u_\delta(x) = (2-x)u_\delta(1)$ for all $x \in A \cap B$. By the gluing lemma, v_δ is continuous.

Let $(\rho_n) \subset \mathbb{R}$ be a mollifier sequence. By Proposition 5.182 we have $\rho_n * v_\delta \to v_\delta = u_\delta$ uniformly on $[0, 1]$. Thus, for n sufficiently large,

$$\|u - (\rho_n * v_\delta)_{|[0,1]}\| \leq \|u - u_\delta\| + \|u_\delta - (\rho_n * v_\delta)\| \leq \varepsilon.$$

Clearly $(\rho_n * v_\delta)_{|[0,1]} \in D(A)$ for n large enough, since $v_\delta \in L^1_{\text{loc}}(\mathbb{R})$ and $\rho_n \in C_c^\infty(\mathbb{R})$, and as the supports shrink we have

$$(\rho_n * v_\delta)'(0) = (\rho_n * v_\delta')(0) = 0.$$

□

Remark 5.184 *We can modify the examples above as follows:*

(i) *Let $X = L^\infty(0, 1)$ and define A by*

$$\begin{cases} D(A) = \{u \in W^{1,\infty}(0,1);\ u(0) = 0\}, \\ Au = u', \quad \text{for } u \in D(A). \end{cases}$$

Then A is m-accretive. The proof is an adaptation of the proof of Proposition 5.183. Note that $D(A)$ is not dense in X.

(ii) *Let $1 \leq p < \infty$, $X = L^p(0,1)$, and define A by*

$$\begin{cases} D(A) = \{u \in W^{1,p}(0,1);\ u(0) = 0\}, \\ Au = u', \quad \text{for } u \in D(A). \end{cases}$$

Then A is m-accretive with dense domain. Since $\mathcal{D}(0,1) \subset D(A)$, it follows that $D(A)$ is dense in X. The rest of the proof is an adaptation of the proof of Proposition 5.183.

(iii) *Let $X = \{u \in C([0,1]);\ u(0) = u(1)\}$ and define A by*

$$\begin{cases} D(A) = \{u \in C^1([0,1]);\ u(0) = u(1) \text{ and } u'(0) = u'(1)\}, \\ Au = u', \quad \text{for } u \in D(A). \end{cases}$$

Then A is m-accretive.

Example 5.185 First-order operators in $\mathbb{R}^+$. *We may modify the examples above by considering operators on the half-line. The proof of the corresponding result is almost the same as in the case of the whole line. For instance, let*

$$X = C_0(\mathbb{R}^+) = \{u \in C^1([0,\infty));\ u(0) = 0 \text{ and } \lim_{x\to\infty} u(x) = 0\}$$

and define

$$\begin{cases} D(A) = \{u \in C^1([0,\infty)) \cap X;\ u' \in X\}, \\ Au = u', \quad \text{for } u \in D(A). \end{cases} \tag{5.7.225}$$

We have the following result.

Proposition 5.186 *If A is as above, then A is m-accretive with dense domain.*

Proof: We first show that A is accretive. Let $\lambda > 0$ and $(u,f) \in D(A) \times X$ satisfy $u + \lambda Au = f$. Then

$$u + \lambda u' = f, \quad \forall x \in [0,\infty). \tag{5.7.226}$$

Set

$$Lf(x) = \frac{1}{\lambda}\int_{-\infty}^{x} e^{\frac{s-x}{\lambda}} f(s)\,ds.$$

Then

$$|Lf(x)| \leq \frac{1}{\lambda}\|f\|_\infty \int_{-\infty}^{x} e^{\frac{s-x}{\lambda}}\,ds = \frac{1}{\lambda}\|f\|_\infty \lambda = \|f\|_\infty,$$

so

$$\|Lf\|_\infty \leq \|f\|_\infty. \tag{5.7.227}$$

The general solution of (5.7.226) is

$$u(x) = Lf(x) + ce^{\frac{x}{\lambda}}.$$

Since u and Lf are bounded, we must have $c = 0$. Therefore $u = Lf$, and from (5.7.227) it follows that A is accretive.

Now let $\lambda > 0$ and $f \in X$. We show that $Lf \in X$. In fact,

$$\begin{aligned} \lim_{x\to+\infty} Lf(x) &= \lim_{x\to+\infty} \frac{1}{\lambda}\int_0^{\infty} e^{\frac{s-x}{\lambda}} f(s)\,ds \\ &= \lim_{x\to+\infty}\lim_{b\to+\infty} \frac{1}{\lambda}\int_0^{b} e^{\frac{s-x}{\lambda}} f(s)\,ds. \end{aligned}$$

Using integration by parts we obtain

$$\begin{aligned}\left|\frac{1}{\lambda}\int_0^x e^{\frac{s-x}{\lambda}} f(s)\,ds\right| &= \left|[f(s)e^{\frac{s-x}{\lambda}}]_0^x - \int_0^x e^{\frac{s-x}{\lambda}} f'(s)\,ds\right| \\ &\leq |f(x)-f(0)| + \left|\int_0^x f'(s)\,ds\right| \\ &\leq |f(x)-f(0)| + |f(x)-f(0)| \longrightarrow 0\end{aligned}$$

as $x \to +\infty$.

Hence $\lim_{x\to\infty} Lf(x) = 0$. Moreover, $Lf(0) = 0$, and since Lf is the integral of continuous functions, Lf is continuous, and so $Lf \in X$. Furthermore, $(Lf)'(x) = f(x) \in C[0,\infty)$, hence $Lf \in C^1[0,\infty)$ and $(Lf)' \in X$, because $f \in X$.

Thus $Lf \in D(A)$. □

Remark 5.187 *We can modify the example above as follows.*

(i) Let $p = \infty$, $X = L^\infty(\mathbb{R}_+)$ and A be defined by

$$\begin{cases} D(A) = \{u \in W^{1,\infty}(\mathbb{R}_+);\ u(0) = 0\}, \\ Au = u',\ \text{for } u \in D(A). \end{cases}$$

Then A is m-accretive and $D(A)$ is not dense in X.

Arguing as in the proof of Proposition 5.177, one checks that A is accretive. Let $f \in X$ and $\lambda > 0$. Consider

$$Lf(x) = \frac{1}{\lambda}\int_0^x e^{\frac{s-x}{\lambda}} f(s)\,ds.$$

We have

$$|Lf(x)| \leq \frac{1}{\lambda}\int_0^x e^{\frac{s-x}{\lambda}}|f(s)|\,ds \leq \frac{1}{\lambda}\|f\|_\infty \int_0^x e^{\frac{s-x}{\lambda}}\,ds = \|f\|_\infty\big(1 - e^{-\frac{x}{\lambda}}\big) \leq \|f\|_\infty.$$

Hence

$$\sup_{x\in\mathbb{R}_+} |Lf(x)| \leq \|f\|_\infty,$$

that is, $\|Lf\|_\infty \leq \|f\|_\infty$. Consequently, $Lf \in X$. Moreover, $Lf \in D(A)$ since

$$(Lf)' = f \in L^\infty(\mathbb{R}_+) \quad \text{and} \quad Lf(0) = \frac{1}{\lambda}\int_0^0 e^{\frac{s}{\lambda}} f(s)\,ds = 0.$$

Therefore A is m-accretive.

Arguing as in Observation 5.179, one sees that $D(A)$ is not dense in X.

(ii) Let $1 \leq p < \infty$, $X = L^p(\mathbb{R}_+)$ and A be defined by

$$\begin{cases} D(A) = \{u \in W^{1,p}(\mathbb{R}_+);\ u(0) = 0\}, \\ Au = u',\ \text{for } u \in D(A). \end{cases}$$

Then A is m-accretive with dense domain.

As in item (iii) of Observation 5.180, one verifies that A is accretive. Let $f \in X$ and $\lambda > 0$ and define

$$Lf(x) = \frac{1}{\lambda}\int_0^x e^{\frac{s-x}{\lambda}} f(s)\,ds.$$

Again, as in item (iii) of Observation 5.180, we obtain

$$\|Lf\|_p \leq \|f\|_p.$$

Hence $Lf \in X$. Since $(Lf)' = f \in X$ and $Lf(0) = 0$, we have $Lf \in D(A)$, and therefore A is m-accretive.

Moreover,

$$L^p(\mathbb{R}_+) = X = \overline{C_0^\infty(\mathbb{R}_+)} \subset \overline{D(A)} \subset X,$$

so that $\overline{D(A)} = X$.

We can also modify the examples above by considering the operator $-u'$ instead of u'. For instance, let

$$X = \{u \in C([0,\infty));\ \lim_{x\to\infty} u(x) = 0\},$$

and let A be the operator defined by

$$\begin{cases} D(A) = \left\{u \in C^1([0,\infty));\ \lim_{x\to\infty} u(x) = \lim_{x\to\infty} u'(x) = 0\right\}, \\ Au = -u',\ \text{for } u \in D(A). \end{cases}$$

We have the following result.

Proposition 5.188 *If A is as above, then A is m-accretive with dense domain.*

Proof: Let $\lambda > 0$ and $(u,f) \in D(A) \times X$ satisfy

$$u + \lambda A u = f,$$

or, equivalently,

$$u - \lambda u' = f, \quad \forall x \in [0,\infty). \tag{5.7.228}$$

Define

$$Lf(x) = -\frac{1}{\lambda}\int_x^\infty e^{\frac{x-s}{\lambda}} f(s)\, ds.$$

Then

$$|Lf(x)| \leq \frac{1}{\lambda}\|f\|_\infty \int_x^\infty e^{\frac{x-s}{\lambda}}\, ds = \frac{1}{\lambda}\|f\|_\infty\, \lambda \int_{-\infty}^0 e^u\, du = \|f\|_\infty.$$

Hence $\|Lf\|_\infty \leq \|f\|_\infty$.

We may rewrite (5.7.228) as

$$u' - \frac{1}{\lambda}u = -\frac{1}{\lambda}f,$$

whose general solution is

$$u(x) = -\frac{1}{\lambda}\int_x^\infty e^{\frac{x-s}{\lambda}} f(s)\, ds + c e^{\frac{x}{\lambda}} = Lf(x) + c e^{\frac{x}{\lambda}}.$$

Since $\lim_{x\to\infty} u(x) = 0$, given, say, $\varepsilon = 1$, there exists x_0 such that $|u(x)| < 1$ for all $x > x_0$. By continuity of u on the compact interval $[0, x_0]$, there is $M > 0$ such that $|u(x)| \leq M$ for all $x \in [0, x_0]$. Let $K = \max\{M, 1\}$. Then $|u(x)| \leq K$ for all $x \in \mathbb{R}_+$. Thus $u \in L^\infty(\mathbb{R}_+)$.

Since $f \in X$, we have $f \in C([0,\infty))$ and $\lim_{x\to\infty} f(x) = 0$. As $(Lf)'(x) = f(x)$, it follows that $\lim_{x\to\infty}(Lf)'(x) = 0$ and $(Lf)' \in C([0,\infty))$, so $Lf \in C^1([0,\infty))$. Moreover,

$$\lim_{x\to\infty} Lf(x) = -\frac{1}{\lambda}\lim_{b\to\infty}\int_b^b e^{\frac{b-s}{\lambda}} f(s)\, ds = 0.$$

Therefore $Lf \in D(A)$ and A is m-accretive. In addition, $\overline{D(A)} = X$. □

Remark 5.189 *We may modify the preceding example as follows.*

(i) Let $X = C_b([0,\infty))$ and A be defined by

$$\begin{cases} D(A) = \{u \in C^1([0,\infty)) \cap X;\ u' \in X\}, \\ Au = -u',\ \textit{for } u \in D(A). \end{cases}$$

Then A is m-accretive and $D(A)$ is not dense in X.

(ii) Let $X = L^p(0,\infty)$ and A be defined by

$$\begin{cases} D(A) = W^{1,\infty}(0,\infty), \\ Au = -u',\ \textit{for } u \in D(A). \end{cases}$$

Then A is m-accretive and $D(A)$ is not dense in X.

(iii) Let $1 \le p < \infty$, $X = L^p(0,\infty)$ and A be defined by

$$\begin{cases} D(A) = W^{1,p}(0,\infty), \\ Au = -u',\ \textit{for } u \in D(A). \end{cases}$$

Then A is m-accretive with dense domain.

Remark 5.190 *Let $X = \{u \in C([0,1]);\ u(0) = 0\}$, endowed with the supremum norm, and consider the operator A in X defined by*

$$\begin{cases} D(A) = \{u \in C([0,1]);\ u(0) = u'(0) = 0\}, \\ Au = u',\ \textit{for } u \in D(A). \end{cases}$$

In Observations 5.189, 5.184 and 5.187, the operator $-A$ is not m-accretive.

Example 5.191 (**A first-order operator in** $\mathbb{R}^n$) Let $X = C_b(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$. Define the operator A in X by

$$\begin{cases} D(A) = \{u \in X;\ a \cdot \nabla u \in X\}, \\ Au = a \cdot \nabla u = \displaystyle\sum_{j=1}^n a_j \frac{\partial u}{\partial x_j},\ \text{for } u \in D(A). \end{cases} \tag{5.7.229}$$

The condition $a \cdot \nabla u \in X$ is understood in the sense of distributions.

We have the following result.

Proposition 5.192 *If A is defined by (5.7.229), then A and $-A$ are m-accretive.*

The proof relies on two lemmas.

Lemma 5.193 *Let $\lambda > 0$ and $1 \le p \le \infty$. If $u \in L^p(\mathbb{R}^n)$ satisfies*

$$u + \lambda a \cdot \nabla u = 0 \quad \textit{in } \mathbb{R}^n,$$

then $u = 0$ almost everywhere.

Proof: Let $(\rho_n)_{n\in\mathbb{N}}$ be a mollifying sequence and set $u_n = \rho_n * u$. Then $u_n \in C^\infty(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Indeed, by Proposition 4.20 in [18], $u_n \in C^\infty(\mathbb{R}^n)$. Moreover,

$$|u_n(x)| = |(\rho_n * u)(x)| \le \int_{\mathbb{R}^n} |\rho_n(x-y)u(y)|\,dy \le \|\rho_n(x)\|_{L^q}\|u\|_{L^p}.$$

Thus

$$\sup_{x\in\mathbb{R}^n} |u_n(x)| \le \sup_{x\in\mathbb{R}^n} \big(\|\rho_n(x)\|_{L^q}\|u\|_{L^p}\big) = c \sup_{x\in\mathbb{R}^n} \|\rho_n(x)\|_{L^q}.$$

Since $\rho_n \in C_c^\infty(\mathbb{R}^n)$, there exists $k(n) > 0$ such that

$$\sup_{x\in\mathbb{R}^n} |u_n(x)| \le c\,k(n),$$

and therefore $u_n \in L^\infty(\mathbb{R}^n)$ for each n.

Furthermore,

$$0 \le \|u_n + \lambda a\cdot\nabla u_n\| \longrightarrow \|u + \lambda a\cdot\nabla u\| = 0, \quad \text{as } n\to\infty,$$

so that

$$u_n + \lambda a\cdot\nabla u_n = 0. \tag{5.7.230}$$

For a fixed $x\in\mathbb{R}^n$, define

$$h(t) = e^t u_n(x+\lambda a t), \quad t\in\mathbb{R}.$$

From (5.7.230) we have

$$h'(t) = e^t u_n(x+\lambda a t) + e^t\nabla u_n(x+\lambda a t)\cdot(\lambda a) = e^t\big(u_n(x+\lambda a t) + \lambda a\cdot\nabla u_n(x+\lambda a t)\big) = 0.$$

Hence h is constant. Since u_n is bounded, there exists $C>0$ such that

$$0 \le |h(t)| = |e^t u_n(x+\lambda a t)| \le C e^t,$$

and letting $t\to-\infty$ yields $|h(t)|\to 0$ and therefore $h(t)\equiv 0$. In particular, $h(0)=0$ implies $u_n(x)=0$. As $x\in\mathbb{R}^n$ was arbitrary, we deduce that $u_n\equiv 0$.

Since $u_n = \rho_n * u \to u$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, it follows that $u=0$ almost everywhere. □

Lemma 5.194 *Let $\lambda>0$ and $f\in C_b(\mathbb{R}^n)$, and define*

$$Lf(x) = \frac{1}{\lambda}\int_0^\infty e^{-\frac{s}{\lambda}} f(x-as)\,ds.$$

Then

$$Lf + \lambda a\cdot\nabla(Lf) = f \tag{5.7.231}$$

in $\mathcal{D}'(\mathbb{R}^n)$. Moreover,

$$\|Lf\|_{L^p} \le \|f\|_{L^p} \quad \textit{for every } 1\le p\le\infty \textit{ such that } f\in L^p(\mathbb{R}^n). \tag{5.7.232}$$

Proof: Define

$$Mf(x) = \frac{1}{\lambda}\int_0^\infty e^{-\frac{s}{\lambda}} f(x+as)\,ds, \quad f\in C_b(\mathbb{R}^n).$$

Observe that $Lf\in L^1_{\text{loc}}(\mathbb{R}^n)$, since $Lf(x)$ is continuous (being an integral of continuous functions) and thus integrable on any compact set.

By Fubini's theorem, for every $f \in C_b(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\langle Lf, \varphi\rangle = \int_{\mathbb{R}^n} f\, M\varphi\, dx.$$

Indeed,

$$\begin{aligned}
\langle Lf, \varphi\rangle &= \int_{\mathbb{R}^n} Lf(x)\,\varphi(x)\,dx \\
&= \int_{\mathbb{R}^n} \frac{1}{\lambda}\int_0^\infty e^{-\frac{s}{\lambda}} f(x-as)\,ds\,\varphi(x)\,dx \\
&= \frac{1}{\lambda}\int_0^\infty \int_{\mathbb{R}^n} e^{-\frac{s}{\lambda}} f(x-as)\varphi(x)\,dx\,ds \\
&= \frac{1}{\lambda}\int_0^\infty \int_{\mathbb{R}^n} e^{-\frac{s}{\lambda}} f(u)\varphi(u+as)\,du\,ds \quad (u = x-as) \\
&= \int_{\mathbb{R}^n} f(u)\left[\frac{1}{\lambda}\int_0^\infty e^{-\frac{s}{\lambda}}\varphi(u+as)\,ds\right] du \\
&= \int_{\mathbb{R}^n} f(x)\,M\varphi(x)\,dx.
\end{aligned}$$

In addition,

$$\begin{aligned}
M(\lambda a\cdot\nabla\varphi)(x) &= \frac{1}{\lambda}\int_0^\infty e^{-\frac{s}{\lambda}}\lambda a\cdot\nabla\varphi(x+as)\,ds \\
&= \int_0^\infty e^{-\frac{s}{\lambda}}\frac{d}{ds}\big(\varphi(x+as)\big)\,ds.
\end{aligned}$$

Using integration by parts with $u = e^{-\frac{s}{\lambda}}$ and $dv = \frac{d}{ds}(\varphi(x+as))\,ds$, we obtain

$$\begin{aligned}
\int_0^\infty e^{-\frac{s}{\lambda}}\frac{d}{ds}(\varphi(x+as))\,ds &= e^{-\frac{s}{\lambda}}\varphi(x+as)\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\frac{s}{\lambda}}\varphi(x+as)\,ds \\
&= -\varphi(x) + M\varphi(x).
\end{aligned}$$

That is,

$$M(\lambda a\cdot\nabla\varphi)(x) = -\varphi(x) + M\varphi(x).$$

Therefore,

$$\begin{aligned}
\langle Lf, \varphi\rangle &= \int_{\mathbb{R}^n} f\,M\varphi\,dx \\
&= \int_{\mathbb{R}^n} f\big(M(\lambda a\cdot\nabla\varphi)(x) + \varphi(x)\big)\,dx \\
&= \int_{\mathbb{R}^n} f\,M(\lambda a\cdot\nabla\varphi)\,dx + \int_{\mathbb{R}^n} f\varphi(x)\,dx \\
&= \langle Lf, \lambda a\cdot\nabla\varphi\rangle + \langle f, \varphi\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle Lf, \lambda a\cdot\nabla\varphi\rangle &= \left\langle Lf, \sum_{i=1}^{n}\lambda a_i\frac{\partial\varphi}{\partial x_i}\right\rangle\\ &= \sum_{i=1}^{n}\lambda a_i\left\langle Lf, \frac{\partial\varphi}{\partial x_i}\right\rangle\\ &= \sum_{i=1}^{n}\lambda a_i\left\langle -\frac{\partial Lf}{\partial x_i}, \varphi\right\rangle\\ &= \left\langle -\sum_{i=1}^{n}\lambda a_i\frac{\partial Lf}{\partial x_i}, \varphi\right\rangle\\ &= \langle -\lambda a\cdot\nabla(Lf), \varphi\rangle.\end{aligned}$$

Hence

$$\langle Lf, \varphi\rangle = \langle \lambda a\cdot\nabla(Lf) + f, \varphi\rangle.$$

Since $\varphi\in\mathcal{D}(\mathbb{R}^n)$ is arbitrary, we obtain

$$Lf + \lambda a\cdot\nabla(Lf) = f,$$

which proves (5.7.231).

We now prove (5.7.232).

For $p=\infty$, we have

$$\begin{aligned}|Lf(x)| &= \left|\frac{1}{\lambda}\int_0^\infty e^{-\frac{s}{\lambda}} f(x-as)\,ds\right|\\ &\le \frac{1}{\lambda}\int_0^\infty e^{-\frac{s}{\lambda}}|f(x-as)|\,ds\\ &\le \frac{1}{\lambda}\|f\|_\infty\int_0^\infty e^{-\frac{s}{\lambda}}\,ds\\ &= \|f\|_\infty,\end{aligned}$$

and therefore $\|Lf\|_\infty \le \|f\|_\infty$.

For $1\le p<\infty$, one argues as in item (iii) of Observation 5.180 to obtain $\|Lf\|_{L^p}\le\|f\|_{L^p}$. □

Returning to the proof of Proposition 5.192.
Proof: We first show that A is m-accretive. Let $\lambda>0$, $f\in X$ and $u\in D(A)$ satisfy

$$u+\lambda Au = f.$$

Set $w=Lf$, where L is defined in Lemma 5.194. Then $Lf+\lambda a\cdot\nabla(Lf)=f$. Hence

$$Lf+\lambda a\cdot\nabla(Lf) = u+\lambda Au = u+\lambda a\cdot\nabla u,$$

so that

$$(u-w)+\lambda a\cdot\nabla(u-w)=0$$

in $\mathcal{D}'(\mathbb{R}^n)$. Applying Lemma 5.193, we obtain $u=w=Lf$.

Accretivity follows from Lemma 5.194, since for every $1\le p\le\infty$,

$$\|u\|_{L^p} = \|Lf\|_{L^p}\le\|f\|_{L^p} = \|u+\lambda Au\|_{L^p}.$$

To prove m-accretivity, let $\lambda > 0$ and $f \in X$. Then $u = Lf \in D(A)$. Indeed, by Lemma 5.194, Lf is bounded and continuous (as an integral of continuous functions), hence $Lf \in X$. Moreover, Lemma 5.194 yields

$$a \cdot \nabla(Lf) = \frac{1}{\lambda}(f - Lf) \in X.$$

Thus $u = Lf \in D(A)$ and

$$Lf + \lambda a \cdot \nabla(Lf) = f,$$

that is, $f = u + \lambda Au$. Therefore A is m-accretive. The same argument, with a replaced by $-a$, shows that $-A$ is m-accretive as well. □

Remark 5.195 *In Proposition 5.192, the domain $D(A)$ is not dense in X.*

Remark 5.196 *We can slightly modify the example above as follows:*

(i) *Let $X = C_0(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$. Define the operator A in X by*

$$\begin{cases} D(A) = \{u \in X;\ a \cdot \nabla u \in X\}, \\ Au = a \cdot \nabla u = \displaystyle\sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j}, \ \text{for } u \in D(A). \end{cases} \tag{5.7.233}$$

Then A and $-A$ are m-accretive with dense domain.
Proof: *The proof that A and $-A$ are m-accretive follows exactly as in Proposition 5.192. To see that $\overline{D(A)} = X$, note that*

$$X = \overline{C_0^\infty(\mathbb{R}^n)}^{L^\infty} \subset \overline{D(A)}^{L^\infty} \subset \overline{X}^{L^\infty} = X.$$

□

(ii) *Let $X = L^\infty(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$. Define the operator A in X as in (5.7.233). Then A and $-A$ are m-accretive and $D(A)$ is not dense in X.*

(iii) *Let $X = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and $a \in \mathbb{R}^n$. Define the operator A in X as in (5.7.233). Then A and $-A$ are m-accretive with dense domain in X. Moreover, if $X = L^2(\mathbb{R}^n)$, then A is anti-adjoint.*
Proof: *The proof that A and $-A$ are m-accretive is essentially the same as in Proposition 5.192. Density follows from the fact that*

$$X = L^p(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)} \subset \overline{D(A)}.$$

If $p = 2$, then $L^2(\mathbb{R}^n)$ is a Hilbert space and, by Corollary 5.167, A is anti-adjoint. □

5.7.5.2 The Laplacian with Dirichlet boundary condition

Example 5.197 *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $X = H^{-1}(\Omega)$ and define the operator A in X by*

$$\begin{cases} D(A) = H_0^1(\Omega), \\ Au = -\Delta u \quad \text{for every } u \in D(A). \end{cases} \tag{5.7.234}$$

We equip $H_0^1(\Omega)$ with the usual norm

$$\|u\|_{H_0^1} = \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

We have the following result.

Proposition 5.198 *The operator A defined by (5.7.234) is self-adjoint, accretive, and $\|\cdot\|_{D(A)}$ is equivalent to $\|\cdot\|_{H^1}$. In particular, A is m-accretive with dense domain.*

To prove Proposition 5.198, we first recall the following facts.

Remark 5.199 **(i)** *It is well known that*

$$H_0^1(\Omega) \subset L^2(\Omega) \equiv L^2(\Omega)' \subset H^{-1}(\Omega),$$

see [23], p. 445. In particular, if $u \in H_0^1(\Omega)$ and $v \in L^2(\Omega)$, then

$$\langle u, v\rangle_{H_0^1, H^{-1}} = \int_\Omega u(x)v(x)\, dx. \tag{5.7.235}$$

(ii) *The Laplace operator Δ is linear and continuous from $H^1(\Omega)$ into $H^{-1}(\Omega)$. Note that, for $u \in H^1(\Omega)$, the linear functional*

$$\Delta u \in H^{-1}(\Omega)$$

on $H_0^1(\Omega)$ is defined by

$$\langle \Delta u, v\rangle = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx, \quad \text{for } v \in H_0^1(\Omega), \tag{5.7.236}$$

see [23], p. 452.

Lemma 5.200 *For each $f \in H^{-1}(\Omega)$ there exists a unique solution $u \in H_0^1(\Omega)$ of the equation*

$$-\Delta u + u = f \quad \text{in } H^{-1}(\Omega).$$

Moreover,

$$\|f\|_{H^{-1}} = \|u\|_{H^1} \tag{5.7.237}$$

and

$$\|u\|_{H^1} \le \|f\|_{L^2} \tag{5.7.238}$$

whenever $f \in L^2(\Omega)$.

Proof: By the Lax–Milgram Theorem (see [23], p. 181), for each $f \in H^{-1}(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u, v)_{H^1} = \langle f, v\rangle_{H^{-1}, H_0^1} \quad \text{for every } v \in H_0^1(\Omega). \tag{5.7.239}$$

On the other hand, (5.7.239) is equivalent, by density, to

$$\int_\Omega \nabla u \cdot \nabla v + uv = \langle f, v\rangle_{H^{-1}, H_0^1}, \quad \text{for every } v \in \mathcal{D}(\Omega),$$

which is equivalent to $-\Delta u + u = f$ in $H^{-1}(\Omega)$.

Moreover, taking $v = u$ in (5.7.239) we obtain

$$\|u\|_{H^1}^2 \le \|f\|_{H^{-1}} \|u\|_{H^1},$$

and hence $\|u\|_{H^1} \le \|f\|_{H^{-1}}$.

Furthermore, it follows again from (5.7.239) that

$$|\langle f, v\rangle_{H^{-1}, H_0^1}| \le \|u\|_{H^1} \|v\|_{H^1}, \quad \text{for every } v \in H^1(\Omega).$$

Therefore $\|f\|_{H^{-1}} \le \|u\|_{H^1}$, and (5.7.237) follows. Finally, from (5.7.239) we also have

$$\|u\|_{H^1}^2 = \langle f, u\rangle_{H^{-1}, H_0^1} \le \|f\|_{L^2} \|u\|_{L^2} \le \|f\|_{L^2} \|u\|_{H^1},$$

which proves (5.7.238). □

Remark 5.201 *Some applications of Lemma 5.200.*

(i) *It follows from Lemma 5.200 that the differential operator $-\Delta + I$ defines an isometry from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$.*

(ii) *In particular, from (5.7.237) and (5.7.238) we obtain $\|f\|_{H^{-1}} \leq \|f\|_{L^2}$ for every $f \in L^2(\Omega)$.*

We now prove Proposition 5.198.

Proof: By Lemma 5.200, for each $f \in X = H^{-1}(\Omega)$ there exists a unique solution $u \in H_0^1(\Omega)$ of

$$-\Delta u + u = f \quad \text{in } X = H^{-1}.$$

We denote by J the operator $f \mapsto u$. It follows from Observation 5.201(i) that J is an isometry from $H^{-1}(\Omega)$ onto $H_0^1(\Omega)$. In particular,

$$(u, v)_{H^{-1}} = (Ju, Jv)_{H_0^1}. \tag{5.7.240}$$

Let $u, v \in H_0^1(\Omega)$. From (5.7.236) and (5.7.235) we obtain

$$\begin{aligned}
(u, Jv)_{H_0^1} &= \int_\Omega \nabla u \cdot \nabla (Jv)\, dx + (u, Jv)_{L^2} \\
&= \langle u, -\Delta(Jv)\rangle_{H_0^1, H^{-1}} + \langle u, Jv\rangle_{H_0^1, H^{-1}} \\
&= \langle u, v\rangle_{H_0^1, H^{-1}} = (u, v)_{L^2}.
\end{aligned}$$

Hence

$$(u, Jv)_{H_0^1} = (u, v)_{L^2}. \tag{5.7.241}$$

Moreover, from (5.7.240) we have

$$\begin{aligned}
(-\Delta u, v)_{H^{-1}} &= (-\Delta u + u, v)_{H^{-1}} - (u, v)_{H^{-1}} \\
&= (J(-\Delta u + u), Jv)_{H_0^1} - (u, v)_{H^{-1}} \\
&= (-\Delta(Ju) + Ju, Jv)_{H_0^1} - (u, v)_{H^{-1}} \\
&= (u, Jv)_{H_0^1} - (u, v)_{H^{-1}}.
\end{aligned}$$

Applying (5.7.241) yields

$$(-\Delta u, v)_{H^{-1}} = (u, v)_{L^2} - (u, v)_{H^{-1}}. \tag{5.7.242}$$

In particular, for each $u \in H_0^1(\Omega)$, combining (5.7.242) with Observation 5.201(ii), we obtain

$$(Au, u)_{H^{-1}} = (-\Delta u, u)_{H^{-1}} = (u, u)_{L^2} - (u, u)_{H^{-1}} = \|u\|_{L^2}^2 - \|u\|_{H^{-1}}^2 \geq 0.$$

Therefore, by Lemma 5.156, the operator A is accretive.

We now prove that A is m-accretive. Given $f \in X = H^{-1}$, from the above observations we have $u = Jf \in D(A)$ and $u + Au = f$. Thus A is m-accretive.

Furthermore, it follows from (5.7.242) that

$$(Au, v)_{H^{-1}} = (u, Av)_{H^{-1}} \quad \text{for all } u, v \in D(A).$$

Indeed,

$$(Au, v)_{H^{-1}} = (u, v)_{L^2} - (u, v)_{H^{-1}} = (v, u)_{L^2} - (v, u)_{H^{-1}} = (Av, u)_{H^{-1}} = (u, Av)_{H^{-1}}.$$

Hence, by Corollary 5.166, A is self-adjoint.

Finally, by Corollary 5.129,

$$\|u\|_{D(A)} = \|u\|_X + \|Au\|_X = \|u\|_{H^{-1}} + \|-\Delta u\|_{H^{-1}} \approx \|u - \Delta u\|_{H^{-1}} \quad \text{on } D(A).$$

On the other hand, by Lemma 5.200 we have

$$\|u - \Delta u\|_{H^{-1}} = \|f\|_{H^{-1}} = \|u\|_{H^1} = \|u\|_{H_0^1},$$

which completes the proof. □

Proposition 5.202 *Let $\Omega \subset \mathbb{R}^n$ be open and let A be as defined in the previous proposition. Then the following properties hold:*

(i) $J_\lambda \in \mathcal{L}(H^{-1}(\Omega))$ *and*

$$\|J_\lambda\|_{\mathcal{L}(H^{-1}(\Omega))} \le 1 \quad \textit{for every } \lambda > 0;$$

(ii) $J_\lambda \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ *for every* $\lambda > 0$*;*

(iii) $J_\lambda \,|_{H_0^1(\Omega)} \in \mathcal{L}(H_0^1(\Omega))$ *and*

$$\left\|J_\lambda \,|_{H_0^1(\Omega)}\right\|_{\mathcal{L}(H_0^1(\Omega))} \le 1 \quad \textit{for every } \lambda > 0;$$

(iv) $J_\lambda u \to u$ *in* $H^{-1}(\Omega)$ *as* $\lambda \to 0^+$*, for every* $u \in H^{-1}(\Omega)$*;*

(v) $J_\lambda u \to u$ *in* $H_0^1(\Omega)$ *as* $\lambda \to 0^+$*, for every* $u \in H_0^1(\Omega)$*.*

Proof: This follows from Definition 5.128, Corollary 5.131, Lemma 5.133 and Proposition 5.137. □

5.8 The Hille–Yosida–Phillips Theorem

In this section we study the evolution equation

$$\frac{du}{dt} + Au = 0,$$

where A is an m-accretive operator with dense domain.

5.8.1 The semigroup generated by $-A$, where A is an m-accretive operator

In this section, X is a Banach space endowed with the norm $\|\cdot\|$.

Lemma 5.203 *Let A be an m-accretive operator in X with dense domain. Then, for every $\lambda > 0$, the operator A_λ belongs to $\mathcal{L}(X)$ and the following hold:*

(i) $\|e^{-tA_\lambda}\|_{\mathcal{L}(X)} \le 1$ *for all* $t \ge 0$*;*

(ii) $\|e^{-tA_\lambda}x - e^{-tA_\mu}x\| \le t\|A_\lambda x - A_\mu x\|$ *for every* $x \in X$*, every* $t \ge 0$ *and every* $\lambda, \mu > 0$*.*

Proof: From Lemma 5.133 we know that $A_\lambda \in \mathcal{L}(X)$.

(i) Let $x \in X$. We have

$$e^{-tA_\lambda}x = e^{-\frac{t}{\lambda}I + \frac{t}{\lambda}J_\lambda}x = e^{-\frac{t}{\lambda}}e^{\frac{t}{\lambda}J_\lambda}x,$$

and thus

$$\begin{aligned}\|e^{-tA_\lambda}x\| &= \left\|e^{-\frac{t}{\lambda}}e^{\frac{t}{\lambda}J_\lambda}x\right\| \\ &= e^{-\frac{t}{\lambda}}\left\|e^{\frac{t}{\lambda}J_\lambda}x\right\| \\ &\le e^{-\frac{t}{\lambda}}e^{\frac{t}{\lambda}\|J_\lambda\|_{\mathcal{L}(X)}}\|x\| \\ &= e^{-\frac{t}{\lambda}}e^{\frac{t}{\lambda}}\|x\| \\ &= \|x\|,\end{aligned}$$

which proves (i).

(ii) Let $\lambda, \mu > 0$. We already know that A_λ and A_μ commute. For every $x \in X$, $t \ge 0$ and $s \in [0,1]$ we have

$$e^{-stA_\lambda}e^{-(1-s)tA_\mu}x = e^{-tA_\mu}e^{-st(A_\lambda - A_\mu)}x.$$

Now,

$$\begin{aligned}\frac{d}{ds}\Big(e^{-stA_\lambda}e^{-(1-s)tA_\mu}x\Big) &= \frac{d}{ds}\Big(e^{-tA_\mu}e^{-st(A_\lambda-A_\mu)}x\Big) \\ &= e^{-tA_\mu}e^{-st(A_\lambda-A_\mu)}t(A_\mu - A_\lambda)x \\ &= t\,e^{-stA_\lambda}e^{-(1-s)tA_\mu}(A_\mu - A_\lambda)x,\end{aligned}$$

and hence

$$\begin{aligned}\left\|\frac{d}{ds}\Big(e^{-stA_\lambda}e^{-(1-s)tA_\mu}x\Big)\right\| &= \left\|t\,e^{-stA_\lambda}e^{-(1-s)tA_\mu}(A_\mu - A_\lambda)x\right\| \\ &\le t\,\|(A_\mu - A_\lambda)x\|.\end{aligned}$$

On the other hand,

$$e^{-tA_\lambda}x - e^{-tA_\mu}x = \int_0^1 \frac{d}{ds}\Big(e^{-stA_\lambda}e^{-(1-s)tA_\mu}x\Big)\,ds,$$

and therefore

$$\begin{aligned}\|e^{-tA_\lambda}x - e^{-tA_\mu}x\| &\le \int_0^1 \left\|\frac{d}{ds}\Big(e^{-stA_\lambda}e^{-(1-s)tA_\mu}x\Big)\right\|ds \\ &\le \int_0^1 t\,\|(A_\mu - A_\lambda)x\|\,ds \\ &= t\,\|(A_\mu - A_\lambda)x\|,\end{aligned}$$

which proves (ii). □

Corollary 5.204 *Let A be an m-accretive operator in X with dense domain. Then there exists a family $\{T(t)\}_{t\ge 0} \subset \mathcal{L}(X)$ such that:*

(i) $\|T(t)\|_{\mathcal{L}(X)} \le 1$ *for all* $t \ge 0$*;*

(ii) $e^{-tA_\lambda}x \to T(t)x$ *as* $\lambda \to 0^+$*, for every* $x \in X$*, uniformly on bounded subsets of* $[0, +\infty)$.

Proof: (i) Let $T_\lambda(t) = e^{-tA_\lambda}$. From the previous lemma we have

$$\|T_\lambda(t)\|_{\mathcal{L}(X)} \le 1, \quad \forall\,\lambda > 0,\ \forall\,t \ge 0. \tag{5.8.243}$$

Now let $x \in D(A)$. For $\lambda, \mu > 0$ and fixed $T > 0$ we obtain

$$\|T_\lambda(t)x - T_\mu(t)x\| = \|e^{-tA_\lambda}x - e^{-tA_\mu}x\| \le t\|A_\lambda x - A_\mu x\| \le T\|A_\lambda x - A_\mu x\|,$$

and therefore

$$\sup_{t\in[0,T]} \|T_\lambda(t)x - T_\mu(t)x\| \leq T\|A_\lambda x - A_\mu x\| \longrightarrow 0, \quad \text{as } \lambda,\mu \to 0^+.$$

Hence $\{T_\lambda(\cdot)x\}_\lambda$ is a Cauchy sequence in $C([0,T];X)$.

Define $T(t)x = \lim_{\lambda\to 0^+} T_\lambda(t)x$. It is clear that $T(t)$ is a linear map from $D(A)$ into X. Moreover, by (5.8.243),

$$\|T(t)x\| \leq \|x\|, \quad \forall\, t \geq 0.$$

Since $D(A)$ is dense in X, we may extend $T(t)$ by continuity to a unique operator $T(t) \in \mathcal{L}(X)$, and

$$\|T(t)\|_{\mathcal{L}(X)} \leq 1.$$

(ii) Item (ii) follows directly from the construction of $T(t)$ in (i). □

Remark 5.205 *The family $\{T(t)\}$ constructed in the previous corollary is sometimes denoted by e^{-tA}. Note that, when A is bounded, this coincides with the usual definition of the exponential of an operator.*

Proposition 5.206 *Let A be an m-accretive operator in X with dense domain, and consider the family $\{T(t)\}_{t\geq 0}$ constructed in Corollary 5.204. For each $x \in D(A)$ and every $t > 0$, the following properties hold:*

(i) $\left\|\dfrac{T(t)x - x}{t}\right\| \leq \|Ax\|$ *for all $t \geq 0$;*

(ii) *the map $t \longmapsto T(t)x$ belongs to*

$$C\big([0,\infty); D(A)\big) \cap C^1\big([0,\infty); X\big);$$

(iii) $AT(t)x = T(t)Ax$ *for all $t \geq 0$.*

In addition, the function $u(t) = T(t)x$ is the unique solution of the problem

$$\begin{cases} \dfrac{du}{dt} + Au &= 0, \quad t > 0, \\ u(0) &= x, \end{cases} \tag{5.8.244}$$

in $C\big([0,\infty); D(A)\big) \cap C^1\big([0,\infty); X\big)$.

Proof: (i) Let $x \in D(A)$ and set

$$u(t) = T(t)x, \quad u_\lambda(t) = T_\lambda(t)x, \quad v_\lambda(t) = -u_\lambda'(t) = A_\lambda u_\lambda(t) = T_\lambda(t)A_\lambda x.$$

From Lemma 5.133 we have

$$J_\lambda|_{D(A)} \in \mathcal{L}(D(A)),$$

and consequently

$$A_\lambda|_{D(A)} \in \mathcal{L}(D(A)) \quad \text{and} \quad T_\lambda(t)|_{D(A)} = e^{-tA_\lambda}|_{D(A)} \in \mathcal{L}(D(A)).$$

Thus $u_\lambda(t) \in D(A)$ for every $t \geq 0$. Moreover,

$$v_\lambda(t) - T(t)Ax = T_\lambda(t)(A_\lambda x - Ax) + (T_\lambda(t) - T(t))Ax,$$

and hence

$$\begin{aligned} \|v_\lambda(t) - T(t)Ax\| &\leq \|T_\lambda(t)(A_\lambda x - Ax)\| + \|(T_\lambda(t) - T(t))Ax\| \\ &\leq \|A_\lambda x - Ax\| + \|(T_\lambda(t) - T(t))Ax\| \longrightarrow 0 \end{aligned}$$

as $\lambda \to 0^+$, uniformly on bounded intervals of $[0, \infty)$.

On the other hand,

$$u_\lambda(t) = x - \int_0^t v_\lambda(s)\, ds,$$

and therefore, as $\lambda \to 0^+$,

$$u(t) = x - \int_0^t T(s)Ax\, ds. \tag{5.8.245}$$

Thus

$$T(t)x = x - \int_0^t T(s)Ax\, ds,$$

or equivalently,

$$\frac{T(t)x - x}{t} = -\frac{1}{t}\int_0^t T(s)Ax\, ds,$$

and consequently

$$\left\| \frac{T(t)x - x}{t} \right\| \le \frac{1}{t}\int_0^t \|T(s)Ax\|\, ds \le \frac{\|Ax\|}{t}\int_0^t ds = \|Ax\|.$$

(ii) From (5.8.245) we obtain

$$\frac{du}{dt} = -T(t)Ax \in C([0, \infty); X),$$

so $u \in C^1([0, \infty); X)$.

Now let $w_\lambda(t) = J_\lambda u_\lambda(t)$. Then $w_\lambda(t) \in D(A)$ and $w_\lambda(t) \to u(t)$ in X as $\lambda \to 0^+$, for each fixed $t \ge 0$. Moreover, $v_\lambda(t) = Aw_\lambda(t)$, hence

$$\big(w_\lambda(t), Aw_\lambda(t)\big) \longrightarrow \big(u(t), T(t)Ax\big) \quad \text{in } X \times Y.$$

Since the graph $G(A)$ is closed, it follows that $u(t) \in D(A)$ and

$$Au(t) = T(t)Ax, \tag{5.8.246}$$

so that $u \in C([0, \infty); D(A))$.

(iii) This is precisely (5.8.246).

Finally,

$$\frac{du}{dt} + Au(t) = -T(t)Ax + T(t)Ax = 0$$

and

$$u(0) = T(0)x = x,$$

that is, u is a solution of (5.8.244).

It remains to prove uniqueness. Let

$$w \in C([0, \infty); D(A)) \cap C^1([0, \infty); X)$$

be another solution of (5.8.244). Given $t > 0$, define

$$z(s) = T(t - s)w(s), \quad s \in [0, t].$$

Then

$$z \in C([0, t]; D(A)) \cap C^1([0, t]; X)$$

and

$$\begin{aligned} \frac{dz}{ds} &= \frac{d}{ds}\big(T(t-s)w(s)\big) \\ &= T(t-s)Aw(s) + T(t-s)\frac{d}{ds}w(s) \\ &= T(t-s)\Big(Aw(s) + \frac{d}{ds}w(s)\Big) \\ &= T(t-s)\cdot 0 = 0, \end{aligned}$$

so z is constant on $[0,t]$. Now,

$$z(t) = w(t) \quad \text{and} \quad z(0) = T(t)x,$$

whence

$$w(t) = T(t)x.$$

By the arbitrariness of $t > 0$ we obtain uniqueness of the solution. □

5.8.2 Semigroups and their generators

In this subsection we deal with contraction semigroups and their generators.

Definition 5.207 *A family* $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ *is called a* contraction semigroup *if it has the following properties:*

(i) $T(0) = I$*;*

(ii) $T(t+s) = T(t)T(s)$ *for all* $s,t \geq 0$*;*

(iii) for each $x \in X$ *the map* $t \mapsto T(t)x$ *is continuous from* $[0,+\infty)$ *into* X*;*

(iv) $\|T(t)\|_{\mathcal{L}(X)} \leq 1$ *for all* $t \geq 0$.

Remark 5.208 *In this definition we explicitly require the continuity of the map* $t \mapsto T(t)x$*. Many authors do not include this condition and use instead the terminology "(contraction) semigroup of class* C^0*".*

Definition 5.209 *Let* $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ *be a contraction semigroup. The* ***generator*** L *of* $(T(t))_{t\geq 0}$ *is the linear operator in* X *defined by:*

(i)

$$D(L) = \left\{ x \in X;\ \frac{T(t)x - x}{t} \text{ has a limit in } X \text{ as } t \to 0^+ \right\};$$

(ii)

$$Lx = \lim_{t\to 0^+} \frac{T(t)x - x}{t}, \quad \text{for all } x \in D(L).$$

Remark 5.210 *Note that if* $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ *is a contraction semigroup, then for each* $x \in X$ *the function* $t \mapsto \|T(t)x\|$ *is non-increasing on* $[0,+\infty)$*. Indeed,*

$$\|T(t+s)x\| = \|T(s)T(t)x\| \leq \|T(t)x\|.$$

Proposition 5.211 *If* $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ *is a contraction semigroup in* X *and* L *is its generator, then* $-L$ *is m-accretive with dense domain.*

The proof of Proposition 5.211 is based on the following lemma.

Lemma 5.212 *If $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is a contraction semigroup in X and L is its generator, then the following properties hold:*

(i) Given $x \in X$ and $t > 0$, define

$$I(t,x) = \int_0^t T(s)x\,ds.$$

Then $I(t,x) \in D(L)$ and $LI(t,x) = T(t)x - x$;

(ii) Given $x \in X$, define

$$Jx = \int_0^{+\infty} e^{-t}T(t)x\,dt.$$

Then $Jx \in D(L)$ and $Jx - LJx = x$.

Proof: Fix $t > h > 0$. We have

$$\begin{aligned}
\left(\frac{T(h)-I}{h}\right) I(t,x) &= \left(\frac{T(h)-I}{h}\right)\int_0^t T(s)x\,ds \\
&= \frac{1}{h}\int_0^t T(h)(T(s)x)\,ds - \frac{1}{h}\int_0^t T(s)x\,ds \\
&= \frac{1}{h}\int_0^t T(s+h)x\,ds - \frac{1}{h}\int_0^t T(s)x\,ds \\
&= \frac{1}{h}\int_h^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^t T(s)x\,ds \\
&= \frac{1}{h}\int_h^t T(s)x\,ds + \frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^h T(s)x\,ds - \frac{1}{h}\int_h^t T(s)x\,ds \\
&= \frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^h T(s)x\,ds \longrightarrow T(t)x - x,
\end{aligned}$$

as $h \to 0^+$, since for each $t \geq h \geq 0$,

$$\begin{aligned}
\left\|\frac{1}{h}\int_t^{t+h} T(s)x\,ds - T(t)x\right\| &= \left\|\frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_t^{t+h} T(t)x\,ds\right\| \\
&\leq \frac{1}{h}\int_t^{t+h} \|T(s)x - T(t)x\|\,ds \\
&\leq \sup_{s\in[t,t+h]} \|T(s)x - T(t)x\| \longrightarrow 0,
\end{aligned}$$

as $h \to 0^+$. This implies that $I(t,x) \in D(L)$ and $LI(t,x) = T(t)x - x$.

On the other hand,

$$\begin{aligned}\left(\frac{T(h)-I}{h}\right)Jx &= \left(\frac{T(h)-I}{h}\right)\int_0^{+\infty} e^{-t}T(t)x\,dt\\ &= \frac{1}{h}\int_0^{+\infty} e^{-t}(T(t+h)x-T(t)x)\,dt\\ &= \frac{1}{h}\int_h^{+\infty} e^{-(u-h)}T(u)x\,du-\frac{1}{h}\int_0^{+\infty} e^{-t}T(t)x\,dt\\ &= \frac{1}{h}\int_h^{+\infty} e^{-(t-h)}T(t)x\,dt-\frac{1}{h}\int_0^{+\infty} e^{-t}T(t)x\,dt\\ &= \frac{e^h}{h}\int_h^{+\infty} e^{-t}T(t)x\,dt-\frac{1}{h}\int_0^{+\infty} e^{-t}T(t)x\,dt\\ &= \frac{e^h-1}{h}\int_h^{+\infty} e^{-t}T(t)x\,dt-\frac{1}{h}\int_0^{h} e^{-t}T(t)x\,dt.\end{aligned}$$

Using that $(e^h-1)/h\to 1$ and the continuity of $t\mapsto T(t)x$, together with the dominated convergence theorem, we conclude that

$$\left(\frac{T(h)-I}{h}\right)Jx\longrightarrow Jx-x \quad \text{in } X \text{ as } h\to 0^+.$$

Thus $\lim_{h\to 0^+}\frac{T(h)-I}{h}Jx=Jx-x$ in X, which implies that $Jx\in D(L)$ and $Jx-LJx=x$. □

Proof:[Proposition 5.211] Let $x\in D(L)$ and $\lambda,h>0$. We have

$$x-\lambda\frac{T(h)x-x}{h}=\left(1+\frac{\lambda}{h}\right)x-\frac{\lambda}{h}T(h)x.$$

Hence

$$\left\|x-\lambda\frac{T(h)x-x}{h}\right\|\geq\left(1+\frac{\lambda}{h}\right)\|x\|-\frac{\lambda}{h}\|x\|=\|x\|.$$

In view of the inequality above, we obtain in the limit as $h\to 0^+$ that $-L$ is accretive.

Moreover, given $f\in X$, define $x=Jf$, where J is defined in Lemma 5.212. Then $x=Jf\in D(L)$ and $x-Lx=f$. Therefore, $-L$ is m-accretive.

It remains to show that $D(-L)=D(L)$ is dense in X. Indeed, given $x\in X$ and $\varepsilon>0$, consider

$$x_\varepsilon=\frac{1}{\varepsilon}I(\varepsilon,x),\qquad I(\varepsilon,x)=\int_0^\varepsilon T(s)x\,ds.$$

Clearly $x_\varepsilon\to x$ in X. Since $x_\varepsilon\in D(L)$, it follows that $D(L)$ is dense in X. □

Proposition 5.213 *Let A be an m-accretive operator in X with dense domain. The family $(T(t))_{t\geq 0}\subset\mathcal{L}(X)$ introduced in Corollary 5.204 has the following properties:*

(i) $(T(t))_{t\geq 0}$ is a contraction semigroup in X;

(ii) the generator of $(T(t))_{t\geq 0}$ is $-A$;

(iii) if a contraction semigroup $(S(t))_{t\geq 0}$ has generator $-A$, then $S(t)=T(t)$ for each $t\geq 0$.

Proof: From Corollary 5.204 we know that $\|T(t)\|_{\mathcal{L}(X)}\leq 1$ for each $t\geq 0$. Moreover,

$$\begin{aligned}T(t+s)x &= \lim_{\lambda\to 0^+} e^{-(t+s)A_\lambda}x\\ &= \lim_{\lambda\to 0^+} e^{-tA_\lambda}e^{-sA_\lambda}x\\ &= T(t)T(s)x, \qquad \text{for each } x\in D(A).\end{aligned}$$

By density, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$ in X. From Proposition 5.206(ii) it follows that, for each $x \in X$, the map $t \mapsto T(t)x$ is continuous from $[0, +\infty)$ into X. Also,

$$T(0)x = \lim_{\lambda \to 0^+} e^{-0A_\lambda} x = x$$

for each $x \in D(A)$, and again by density we have $T(0) = I$.

From Proposition 5.206 we have

$$T(t)x = x - \int_0^t T(s)Ax\,ds \quad \Longleftrightarrow \quad \frac{T(t)x - x}{t} = -\frac{1}{t}\int_0^t T(s)Ax\,ds.$$

Letting $t \to 0^+$ we obtain that $x \in D(L)$ and $Lx = -Ax$. In other words, $G(A) \subset G(-L)$. Since both A and $-L$ are m-accretive, it follows from Corollary **??** that $A = -L$.

Assume now that another contraction semigroup $(S(t))_{t\geq 0}$ has generator $-A$. We shall prove that $T(t) = S(t)$ for each $t \geq 0$. Take $x \in D(A)$ and set $u(t) = S(t)x$. Given $t \geq 0$ and $h > 0$, we have

$$\begin{aligned} \frac{u(t+h) - u(t)}{h} &= \frac{S(h) - I}{h} u(t) \\ &= \frac{S(t)S(h)x - S(t)x}{h} \\ &= S(t)\frac{S(h)x - x}{h} \longrightarrow -S(t)Ax, \end{aligned}$$

as $h \to 0^+$. Hence $u(t) \in D(A)$ and the right derivative $\frac{d^+u}{dt}$ exists for each $t \geq 0$, with

$$Au(t) = S(t)Ax = \frac{d^+u}{dt}.$$

By Dini's lemma, together with the definition of semigroup, we obtain $u \in C^1([0, +\infty); X)$. We now show that $u \in C^1([0, +\infty); D(A))$.

Given a sequence $(t_n) \subset [0, +\infty)$ such that $t_n \to t$, we have

$$\begin{aligned} \|u(t_n) - u(t)\|_{D(A)} &= \|u(t_n) - u(t)\|_X + \|A(u(t_n)) - A(u(t))\|_X \\ &= \|u(t_n) - u(t)\|_X + \|S(t_n)Ax - S(t)Ax\|_X \longrightarrow 0, \end{aligned}$$

which follows from property (iii) in the definition of semigroup, combined with the fact that u is continuous in X and $x \in D(A)$. Dini's lemma also yields $\frac{d^+u}{dt} = \frac{du}{dt}$, that is, u is the solution of the problem

$$\begin{cases} \dfrac{du}{dt} + Au = 0, & t > 0, \\ u(0) = x. \end{cases}$$

In view of Proposition 5.206 it follows that $S(t)x = T(t)x$ for each $t \geq 0$ and $x \in D(A)$. By density, we conclude that $T(t) = S(t)$ for all $t \geq 0$. This completes the proof. □

Remark 5.214 *Property (iii) in Proposition 5.213 ensures that if A is an m-accretive operator, then the contraction semigroup generated by $-A$ is unique. In particular, there exists a bijection between the sets*

$$\mathfrak{X} = \{\textit{contraction semigroups}\} \quad \textit{and} \quad \mathfrak{U} = \{\textit{m-accretive operators with dense domain}\},$$

given by $(T(t))_{t\geq 0} \mapsto -L$.

Applying Propositions 5.211 and 5.213 we obtain the following result, known as the Hille–Yosida–Phillips theorem.

Theorem 5.215 *A linear operator A in X is the generator of a contraction semigroup in X if, and only if, $-A$ is m-accretive with dense domain.*

Theorem 5.216 *Let A be an m-accretive operator in X with dense domain, and let $(T(t))_{t\geq 0}$ be the contraction semigroup generated by $-A$. Let $L \in \mathcal{L}(X)$ be such that $L|_{D(A)} \in \mathcal{L}(D(A))$. If $ALx = LAx$ for every $x \in D(A)$, then $T(t)L = LT(t)$ for all $t \geq 0$. In particular, if $Lx = 0$, then $LT(t)x = 0$ for all $t \geq 0$.*

Proof: Let $x \in D(A)$ and consider $u(t) = T(t)x$. Then u is the solution of problem (5.8.244). Setting $v(t) = Lu(t)$, we have, for each $h > 0$ and $t \geq 0$,

$$\begin{aligned} \frac{v(t+h)-v(t)}{h} &= \frac{Lu(t+h)-Lu(t)}{h} \\ &= L\left(\frac{u(t+h)-u(t)}{h}\right). \end{aligned}$$

Thus

$$\frac{d^+v}{dt} = L\frac{d^+u}{dt} = L\frac{du}{dt} \in C^0([0,+\infty);X).$$

If $(t_n) \subset [0,+\infty)$ is such that $t_n \to t$, then

$$\begin{aligned} \|v(t_n)-v(t)\|_{D(A)} &= \|v(t_n)-v(t)\|_X + \|A(v(t_n))-A(v(t))\|_X \\ &= \|Lu(t_n)-Lu(t)\| + \|A(Lu(t_n))-A(Lu(t))\| \\ &= \|Lu(t_n)-Lu(t)\| + \|L(A(u(t_n)))-L(A(u(t)))\| \\ &= \|Lu(t_n)-Lu(t)\| + \|L(A(T(t_n)x))-L(A(T(t)x))\| \\ &= \|Lu(t_n)-Lu(t)\| + \|L(T(t_n)Ax)-L(T(t)Ax)\| \longrightarrow 0, \end{aligned}$$

because $L \in \mathcal{L}(X)$ and $L \circ u$ and $t \mapsto T(t)x$ are continuous. Hence

$$v \in C([0,+\infty);D(A)) \cap C^1([0,+\infty);X),$$

and moreover,

$$\begin{aligned} \frac{dv}{dt} + Av &= L\frac{du}{dt} + ALu(t) \\ &= L\frac{du}{dt} + LAu(t) \\ &= L\left(\frac{du}{dt} + Au(t)\right) \\ &= L(0) = 0. \end{aligned}$$

Note that $v(0) = Lu(0) = LT(0)x = LI(x) = Lx$. By uniqueness of the solution, we obtain $v(t) = T(t)Lx$. Therefore, $T(t)Lx = LT(t)x$ for each $x \in D(A)$. The result follows by density. □

Corollary 5.217 *Let A be an m-accretive operator in X with dense domain and let $(T(t))_{t\geq 0}$ be the contraction semigroup generated by $-A$. If J_λ is the operator introduced in Definition 5.128, then*

$$T(t)J_\lambda = J_\lambda T(t), \qquad \textit{for each } \lambda > 0 \textit{ and } t \geq 0.$$

Proof: This follows from Lemma 5.133 combined with the previous proposition with $L = J_\lambda$. □

We conclude this subsection by characterising the domain of an m-accretive operator in reflexive Banach spaces. For the next proposition we shall need the following result:

Corollary 5.218 *Assume that X is reflexive. If $f : I \longrightarrow X$ is Lipschitz and bounded, then $f \in$*

$W^{1,\infty}(I,X)$ and

$$\|f'\|_{L^\infty(I,X)} \leq L,$$

where L is the Lipschitz constant of f.

Proposition 5.219 *Let A be an m-accretive operator in X and let $(T(t))_{t\geq 0}$ be the contraction semigroup generated by $-A$. If X is reflexive, then any $x \in X$ such that*

$$\sup_{h>0} \frac{1}{h}\|T(h)x - x\| < +\infty$$

belongs to $D(A)$. In particular,

$$D(A) = \{x \in X;\ \exists C > 0 \text{ such that } \|T(h)x - x\| \leq Ch,\ \forall h > 0\}.$$

Proof: Let x be as in the statement and define $u(t) = T(t)x$. Given $0 \leq s < t$, we have

$$\begin{aligned} \|u(t) - u(s)\| &= \|T(t)x - T(s)x\| \\ &= \|T(s + t - s)x - T(s)x\| \\ &= \|T(s)(T(t - s)x - x)\| \\ &\leq \|T(t - s)x - x\| \\ &\leq C(t - s). \end{aligned}$$

It follows that u is Lipschitz and hence continuous from $[0, +\infty)$ into X. Note that

$$\|u(t)\|_X = \|T(t)x\|_X \leq \|x\|,$$

so u is bounded and, by Corollary 5.218, we have $u \in W^{1,\infty}((0, +\infty), X)$. Thus there exists a sequence $t_n \to 0$ such that u is differentiable at each t_n and $\|u'(t_n)\| \leq C$. In particular,

$$\frac{u(t_n + h) - u(t_n)}{h} = \frac{T(h) - I}{h} T(t_n)x$$

has a limit as $h \to 0$, for each $n \in \mathbb{N}$. This implies that $T(t_n)x \in D(A)$ and $\|AT(t_n)x\| \leq C$ for each $n \in \mathbb{N}$. Since X is reflexive, there exist a subsequence (still denoted $(t_n)_{n\in\mathbb{N}}$) and $y \in X$ such that $AT(t_n)x \rightharpoonup y$ in X as $n \to +\infty$. As $T(t_n)x \to x$ when $n \to +\infty$, it follows that

$$(T(t_n)x, AT(t_n)x) \rightharpoonup (x, y) \quad \text{in } X \times X.$$

Since $G(A)$ is closed, we conclude that $x \in D(A)$. □

5.8.3 Regularity properties

In this subsection we show that certain subspaces of X are invariant under the action of contraction semigroups.

Proposition 5.220 *Let A be an m-accretive operator in X and let $(T(t))_{t\geq 0}$ be the contraction semigroup generated by $-A$. If $T_{(1)}(t) = T(t)|_{D(A)}$ and $A_{(1)}$ is the operator defined in Theorem 5.146, then $(T_{(1)}(t))_{t\geq 0}$ is a contraction semigroup in $D(A)$ and its generator is $-A_{(1)}$.*

Proof: From Proposition 5.206 we know that $T(t)(D(A)) \subset D(A)$. Moreover, if $t \geq 0$ and $x \in D(A)$, then

$$\begin{aligned} \|T(t)x\|_{D(A)} &= \|T(t)x\|_X + \|AT(t)x\|_X \\ &= \|T(t)x\|_X + \|T(t)Ax\|_X \\ &\leq \|x\|_X + \|Ax\|_X = \|x\|_{D(A)}. \end{aligned}$$

Therefore $T(t)|_{D(A)} \in \mathcal{L}(D(A))$ and $\|T(t)|_{D(A)}\| \leq 1$. It follows from Proposition 5.206(ii) that $(T_{(1)}(t))_{t\geq 0}$ is a contraction semigroup in $D(A)$, since the other properties are immediate.

Let L be its generator and consider $x \in D(A_{(1)}) = D(A^2)$. Then

$$\frac{T_{(1)}(h)x - x}{h} = \frac{T(h)x - x}{h} \longrightarrow -Ax \quad \text{in } X,$$

as $h \to 0^+$. Moreover, $Ax \in D(A)$ and by Proposition 5.206 we obtain

$$A\frac{T_{(1)}(h)x - x}{h} = \frac{T_{(1)}(h)Ax - Ax}{h} \longrightarrow -A(Ax) \quad \text{in } X,$$

as $h \to 0^+$. Consequently,

$$\left\|\frac{T_{(1)}(h)x - x}{h} - Ax\right\|_{D(A)} = \left\|\frac{T_{(1)}(h)x - x}{h} - Ax\right\|_X + \left\|A\left(\frac{T_{(1)}(h)x - x}{h}\right) - A(Ax)\right\|_X \longrightarrow 0.$$

Therefore, $x \in D(L)$ and $Lx = -Ax$. In other words, $G(A_{(1)}) \subset G(-L)$. Since $-L$ and $A_{(1)}$ are m-accretive in $D(A)$, it follows from Corollary 5.142 that $A_{(1)} = -L$. □

Corollary 5.221 *Let A be an m-accretive operator in X, and let $(T(t))_{t\geq 0}$ be the contraction semigroup generated by $-A$. Given a positive integer n, consider the space X_n and the operator $A_{(n)}$ defined in Observation 5.147. If $T_{(n)}(t) = T(t)|_{X_n}$ for each $t \geq 0$, then $(T_{(n)}(t))_{t\geq 0}$ is a contraction semigroup in X_n and its generator is $-A_{(n)}$.*

Proof: It suffices to iterate Proposition 5.220 and Observation 5.147, noting that

$$\|x\|_n = \sum_{j=0}^{n} \|A^j x\|_X.$$

□

Corollary 5.222 *Let A be an m-accretive operator in X and let $(X_n)_{n\geq 0}$ be the spaces defined in Observation 5.147. Given $x \in D(A)$, let*

$$u \in C([0, +\infty); D(A)) \cap C^1([0, +\infty); X)$$

be the solution of problem (5.8.244). If $x \in X_n$ for some $n \geq 1$, then

$$u(\cdot) = T(\cdot)x \in \bigcap_{j=0}^{n} C_j^b([0, +\infty); X_{n-j}). \tag{5.8.247}$$

Moreover,

$$\frac{d^j u}{dt^j} = (-1)^j T(t)A^j x = (-1)^j A^j u(t), \tag{5.8.248}$$

for each $t \geq 0$ and $0 \leq j \leq n$, and

$$\frac{d}{dt}\left(\frac{d^j u}{dt^j}\right) + A\left(\frac{d^j u}{dt^j}\right) = 0, \tag{5.8.249}$$

for each $t \geq 0$ and every $0 \leq j \leq n-1$. In particular, if $x \in \bigcap_{n\geq 0} D(A^n)$, then

$$u \in C^\infty([0, +\infty); X_n) \quad \textit{for each } n \geq 0.$$

Proof: The case $n = 1$ follows from Proposition 5.206. Assume that the result holds for some $n > 1$, and let $x \in X_{n+1}$. In particular, $A^j x \in X_{n-j+1}$ for each $0 \leq j \leq n+1$, and for $j = 1$ we have $Ax \in X_n$.

Thus

$$v(\cdot) = T(\cdot)Ax \in \bigcap_{j=0}^{n} C_j^b([0,+\infty);X_{n-j}). \tag{5.8.250}$$

Equivalently,

$$v(\cdot) = T(\cdot)Ax \in \bigcap_{j=1}^{n+1} C_j^b([0,+\infty);X_{n+1-j}). \tag{5.8.251}$$

Taking $j=0$ in (5.8.250), we see that $v \in C_b([0,+\infty);X_n)$. Since $v = Au \in C_b([0,+\infty);X_n)$ and

$$\|u\|_{n+1} = \|u\|_n + \|Au\|_n,$$

it follows that $u \in C_b([0,+\infty);X_{n+1})$. Hence

$$u \in \bigcap_{j=0}^{n+1} C_j^b([0,+\infty);X_{n+1-j}).$$

Now observe that

$$\begin{aligned}\frac{d}{dt}\left(\frac{d^j u}{dt^j}\right) &= (-1)^j \frac{d}{dt}\big(T(t)A^j x\big)\\ &= (-1)^j(-A)T(t)A^j x\\ &= (-1)^{j+1}T(t)A^{j+1}x\\ &= (-1)^{j+1}A^{j+1}u(t).\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{d}{dt}\left(\frac{d^{j+1}u}{dt^{j+1}}\right) + A\left(\frac{d^{j+1}u}{dt^{j+1}}\right) &= (-1)^{j+2}A^{j+2}u(t) + A\big((-1)^{j+1}A^{j+1}u(t)\big)\\ &= (-1)^{j+2}A^{j+2}u(t) + (-1)^{j+1}A^{j+2}u(t)\\ &= 0,\end{aligned}$$

which proves (5.8.249) and completes the induction. □

5.8.4 Weak Solutions and Extrapolation

If $x \in D(A)$ then $u(t) = T(t)x$ is the solution of problem 5.204, according to Proposition 5.206. On the other hand, if $x \in X \backslash D(A)$ then $u \notin C([0,\infty), D(A))$ and, in particular, u is not a solution of 5.204 on $[0,\infty)$.

In this section, we will show that u is a solution in a "weak" form of problem 5.204.

Lemma 5.223 *Let A be an m-accretive operator in X and $(T(t))_{t\geq 0}$ be the contraction semigroup generated by $-A$. Consider the space X_{-1} and the operator $A_{(-1)}$ defined by Theorem 5.149. If $(T_{-1}(t))_{t\geq 0}$ is the contraction semigroup in X_{-1} generated by A_{-1}, then $T_{-1}(t)|_X = T(t)$ for all $t \geq 0$.*

Proof: Let $x \in D(A)$. We have

$$\begin{aligned}\left\|\frac{T_{(-1)}x - x}{t} + Ax\right\|_X &\approx \left\|\frac{T_{(-1)}x - x}{t} + Ax\right\|_{D(A_{(-1)})}\\ &= \left\|\frac{T_{(-1)}x - x}{t} + A_{(-1)}x\right\|_{X_{-1}} + \left\|A_{(-1)}\left(\frac{T_{(-1)}x - x}{t} + A_{(-1)}x\right)\right\|_{X_{-1}} \longrightarrow 0,\end{aligned}$$

when $\lambda \to 0^+$, since $(T_{-1}(t))_{t\geq 0}$ is the contraction semigroup in X_{-1} generated by A_{-1} and A_{-1} is continuous. Let L be the generator of $(T_{-1}(t)|_X)_{t\geq 0}$. From the limit above, we have that $Lx = -Ax$ and $G(A) \subset G(-L)$. Since A and $-L$ are m-accretive, it follows from Corollary 5.142 that $L = -A$. Thus, from item (iii) of Proposition 5.213 we have the result. □

Corollary 5.224 *Let A be an m-accretive operator in X and $(T(t))_{t\geq 0}$ be the contraction semigroup in X generated by $-A$. Consider the space X_{-1} and the operator $A_{(-1)}$ defined in Theorem 5.149. Let $x \in X$ and $u(t) = T(t)x$, for all $t \geq 0$. Then, u is the unique solution of the problem*

$$\begin{cases} \dfrac{du}{dt} + A_{(-1)}u = 0; \\ u(0) = x; \end{cases}$$

in the space $C([0,\infty), X) \cap C^1([0,\infty), X_{-1})$.

Proof: We know that $A_{(-1)}$ is m-accretive in X_{-1}, $D(A_{(-1)}) = X$ and $\overline{X} = X_{-1}$. By Proposition 5.206, for all $x \in D(A_{(-1)}) = X$ and all $t \geq 0$, $u(t) = T_{(-1)}(t)x$ is the unique solution of the problem

$$\begin{cases} \dfrac{du}{dt} + A_{(-1)}u = 0; \\ u(0) = x; \end{cases}$$

in the space $C([0,\infty), X) \cap C^1([0,\infty), X_{-1})$. But, by Lemma 5.223, $T_{-1}(t)|_X = T(t)$. Thus, we have the desired result. □

Corollary 5.225 *Let A be an m-accretive operator in X and $(T(t))_{t\geq 0}$ be the contraction semigroup in X generated by $-A$. Given $n \geq 0$, consider the space X_{-n} and the operator A_{-n} defined in Remark 5.234. If $(T_{(-n)})_{t\geq 0}$ is the contraction semigroup in X_{-n} generated by $A_{(-n)}$, then $T_{(-n)}(t)|_{X_{-j}} = T_{(-j)}(t)$ for all $0 \leq j \leq n$ and all $t \geq 0$.*

Proof: The result follows by applying Lemma 5.223 iteratively and Remark 5.151. □

Corollary 5.226 *Let A be an m-accretive operator in X and $(T(t))_{t\geq 0}$ be the contraction semigroup in X generated by $-A$. Given $n \geq 0$, consider the space X_{-n} and the operator $A_{(-n)}$ defined in Remark 5.150 and let $(T_{(-n)})_{t\geq 0}$ be the contraction semigroup in X_{-n} generated by $A_{(-n)}$. Let $x \in X$ and consider $u(t) = T(t)x$, for $t \geq 0$. Then, $u \in C_b^m([0,\infty), X_{-n})$ for all $n \geq 0$. In addition,*

$$\frac{d^n u}{dt^n} = (-1)^n T_{(-n)}(t) A^n_{(-n)} X = (-1)^n A^n_{(-n)} u(t),$$

and,

$$\frac{d}{dt}\left(\frac{d^{n-1}u}{dt^{n-1}}\right) + (-1)^{n+1} A_{(-n)} \left(\frac{d^{n-1}u}{dt^{n-1}}\right) = 0$$

for all $t \geq 0$ and all $n \geq 1$.

Proof: The result follows by applying Corollary 5.222 to the operator $A_{(-n)}$, for all $n \geq 0$. □

5.8.5 Group of Isometries

We will show that, under some appropriate hypotheses, some contraction semigroups can be embedded into larger families of operators.

A family $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is called a group of isometries if it satisfies the following properties:

(i) $T(0) = I$;

(ii) $T(t+s) = T(t)T(s)$, for all $s, t \in \mathbb{R}$;

(iii) The mapping $t \mapsto T(t)x$ is continuous from $\mathbb{R}$ into X, for all $x \in X$;

(iv) $\|T(t)x\| = \|x\|$, for all $t \in \mathbb{R}$ and all $x \in X$.

Remark 5.227 *(i) If $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is a group of isometrias, then $(T(t))_{t\geq 0}$ is a contraction semigroup. In addition, if we set $S(t) = T(-t)$, for all $t \in \mathbb{R}$, then $(S(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is also a group of isometries, and thus, $(S(t))_{t\geq 0}$ is a contraction semigroup.*

(ii) Recall that in a Banach space, an isometry, that is, a linear map $T : X \to X$ such that $\|Tx\| = \|x\|$ for all $x \in X$, need not be surjective. For example, $T\varphi(t) = \varphi(t + h)$ in $X = L^p(0, \infty)$ with $h > 0$.

Note, also, that if $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is a group of isometries, then $T(t) : X \to X$ is surjective for all $t \in \mathbb{R}$, that is, $T(t)X = X$ for all $t \in \mathbb{R}$. Indeed, we have $T(t)X \subset X$. On the other hand, given $t \in \mathbb{R}$ and $x \in X$, we have $x = T(t)y$ with $y = T(-t)x$. Thus, $x \in T(t)X$. Hence, $X \subset T(t)X$.

Conversely, if $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is a contraction semigroup such that $T(t)$ is a surjective isometry for all $t \geq 0$, then $(T(t))_{t\in\mathbb{R}}$ can be embedded into a group of isometries $(S(t))_{t\in\mathbb{R}}$. For this, it suffices to consider the map i given by

$$\begin{array}{rccl} i: & \mathcal{L}(X) & \longrightarrow & \mathcal{L}(X) \\ & T(t) & \longmapsto & \begin{cases} S(t) = T(t), & \text{if } t \geq 0 \\ S(t) = (T(-t)^{-1}, & \text{if } t < 0. \end{cases} \end{array}$$

Note that $S(t)$ is a group of isometries. Indeed,

(a) $S(0) = T(0) = I$.

(b) <u>Case 1</u>: if $t \geq 0$ and $s \geq 0$ then $t + s \geq 0$ and

$$S(t + s) = T(t + s) = T(t)T(s) = S(t)S(s);$$

<u>Case 2</u>: if $t < 0$ and $s < 0$ then $t + s < 0$ and

$$\begin{aligned} S(t+s) &= (T(-(t+s)))^{-1} \\ &= (T(-t-s))^{-1} \\ &= (T(-t)T(-s))^{-1} \\ &= (T(-s))^{-1}(T(-t))^{-1} \\ &= S(s)S(t) \\ &= S(t)S(s) \end{aligned}$$

<u>Case 3</u>: if $t < 0$, $s \geq 0$ and $t + s < 0$ then

$$\begin{aligned} S(t+s) &= (T(-(t+s)))^{-1} \\ &= (T(s))^{-1}(T(-(t+s)))^{-1}T(s) \\ &= (T(-(t+s))T(s))^{-1}T(s) \\ &= (T(-t))^{-1}T(s) \\ &= S(t)S(s) \end{aligned}$$

<u>Case 4</u>: if $t \geq 0$, $s < 0$ and $t + s < 0$ is similar to the previous item;

<u>Case 5</u>: if $t < 0$, $s \geq 0$ and $t + s \geq 0$ then

$$S(t+s) = T(t+s) = (T(-t))^{-1}T(t+s)T(-t) = (T(-t))^{-1}T(t+s-t) = (T(-t))^{-1}T(s) = S(t)S(s);$$

Case 6: if $t \geq 0$, $s < 0$ and $t + s \geq 0$ is similar to the previous item.

In all cases, it holds that $S(t+s) = S(t)S(s)$.

(c) The mapping $t \mapsto S(t)x$ is continuous from $\mathbb{R}$ into X, for all $x \in \mathbb{R}$. Indeed, for $t \geq 0$ it follows from the continuity of $t \mapsto T(t)x$. For $t < 0$, it follows from the fact that the inverse of a linear, bijective and continuous mapping is a linear and continuous mapping.

(d) $\|S(t)x\| = \|T(t)x\| = \|x\|$, if $t \geq 0$. And, $\|S(t)x\| = \|(T(-t))^{-1}x\| = \|x\|$, since the inverse of an isometry is an isometry.

Also, i is an embedding. Indeed, if $t \geq 0$, $\|S(t)\| = \|T(t)\|$. If $t < 0$,

$$\|S(t)\| = \|(T(t))^{-1}\| = 1 = \|T(t)\|.$$

Lemma 5.228 *Let $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ be a group of isometrias. If L is the generator of the contraction semigroup $(T(t))_{t\geq 0}$ and $\tilde{L}$ is the generator of the contraction semigroup $(S(t))_{t\geq 0}$, where $S(t) = T(-t)$, then $L = \tilde{L}$. In particular, L and $-L$ are m-accretive with dense domain.*

Proof: Let $x \in D(L)$. Given $h > 0$, we have

$$\frac{S(h)x - x}{h} = \frac{T(-h)x - x}{h} = -T(-h)\frac{T(h)x - x}{h} \longrightarrow -Lx,$$

when $h \to 0^+$. Thus, $x \in D(\tilde{L})$ and $\tilde{L}x = -Lx$. Hence, $G(L) \subset G(-\tilde{L})$.

Now, given $x \in D(\tilde{L})$ and $h > 0$, we have

$$\frac{T(h)x - x}{h} = -T(h)\frac{T(-h)x - x}{h} = -T(h)\frac{S(h)x - x}{h} \longrightarrow -\tilde{L}x,$$

when $h \to 0^+$. Thus, $x \in D(L)$ and $Lx = -\tilde{L}x$. Hence, $G(\tilde{L}) \subset G(-L)$. Therefore, $\tilde{L} = -L$. Moreover, by Proposition 5.211, L and $-L$ are m-acretivos with dense domain. □

Lemma 5.229 *Let A be an m-accretive operator with dense domain such that $-A$ is m-accretive. Let $(T(t))_{t\geq 0}$ be the contraction semigroup in X generated by A and $(S(t))_{t\geq 0}$ be the contraction semigroup in X generated by $-A$. Define $(U(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ by*

$$U(t) = \begin{cases} T(t), & \text{if } t \geq 0; \\ S(-t), & \text{if } t < 0. \end{cases}$$

Then, $(U(t))_{t\in\mathbb{R}}$ is a group of isometries.

Proof: Given $\lambda > 0$, consider the operator $A_\lambda \in \mathcal{L}(X)$ introduced in Definition 5.132. Let $x \in X$ and $t \in \mathbb{R}$. We have that $(e^{-tA_\lambda})_{t\in\mathbb{R}}$ is a group of isometries, indeed

(i) $e^{0A_\lambda} = I$;

(ii) $e^{-(t+s)A_\lambda} = e^{-tA_\lambda}e^{-sA_\lambda}$ for all $s, t \in \mathbb{R}$;

(iii) The mapping $t \mapsto e^{-tA_\lambda}x$ is continuous from $\mathbb{R}$ into X, for all $x \in X$, since it is the composition of continuous maps;

(iv) We have $\|e^{-tA_\lambda}x\| \leq \|x\| = \|e^{tA_\lambda}e^{-tA_\lambda}x\| \leq \|e^{-tA_\lambda}x\|$. Thus, $\|e^{-tA_\lambda}x\| = \|x\|$.

From Corollary 5.204, for all $x \in X$,

$$e^{-tA_\lambda}x \longrightarrow U(t)x,$$

when $t \to 0^+$, uniformly on bounded intervals. Therefore, $(U(t))_{t\in\mathbb{R}}$ is a group of isometries. □

Proposition 5.230 *If $(T(t))_{t\geq 0}$ is a contraction semigroup in X with generator A, then the following properties are equivalent:*

(i) $-A$ *is m-accretive;*

(ii) *There exists a group of isometries $(U(t))_{t\in R}$ such that $T(t) = U(t)$, for all $t \geq 0$.*

Proof: From Lemma 5.229 we have that $(i) \Rightarrow (ii)$. And, from Lemma 5.228 we have $(ii) \Rightarrow (i)$. □

Corollary 5.231 *Let $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ be a group of isometries and $-A$ be the generator of the contraction semigroup $(T(t))_{t\geq 0}$. Then, for all $x \in D(A)$, the function $u(t) = T(t)x$, $t \in \mathbb{R}$, is the unique solution of the problem*

$$\begin{cases} \dfrac{du}{dt} + Au = 0; \\ u(0) = x; \end{cases}$$

in the space $C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, X)$.

Proof: For $t > 0$, it follows from Proposition 5.206. For $t < 0$, we have $(S(-t))_{t<0}$ is a contraction semigroup with generator A. Applying Proposition 5.206 for $S(-t)$ we have that $u(t) = S(t)x$ is the unique solution of the problem

$$\begin{cases} \dfrac{du}{dt} + Au = 0; \\ u(0) = x; \end{cases}$$

in the space $C((-\infty, 0), D(A)) \cap C^1((-\infty, 0), X)$.

For $t = 0$ we have

$$\frac{d^+u}{dt}(0) = \lim_{h\to 0^+} \frac{u(h) - u(0)}{h} = \lim_{h\to 0^+} \frac{T(h)x - x}{h} = -Ax.$$

And,

$$\frac{d^-u}{dt}(0) = \lim_{h\to 0^-} \frac{u(h) - u(0)}{h} = \lim_{h\to 0^-} \frac{S(-h)x - x}{h} = -\lim_{t\to 0^+} \frac{S(t)x - x}{t} = -Ax.$$

Thus, u is differentiable at the origin, therefore, continuous and

$$\frac{du}{dt}(0) = -Ax = -Au(0).$$

□

Remark 5.232 *Consider the group of isometries $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ and let $x \in X$. If $T(t_0)x \in D(A)$ for some $t_0 \in \mathbb{R}$, then $T(t)x \in D(A)$ for all $t \in \mathbb{R}$. Indeed, given $t \in \mathbb{R}$, there exists $s \in \mathbb{R}$ such that $t = s + t_0$. Thus, $T(t)x = T(s + t_0)x = T(s)T(t_0)x$ which belongs to $D(A)$, since $T(t_0)x \in D(A)$.*

Therefore, if $x \notin D(A)$ then $T(t)x \notin D(A)$ for all $t \in \mathbb{R}$. Indeed, if $T(t)x \in D(A)$ for some $t \in \mathbb{R}$ then, by what was done above, $T(t)x \in D(A)$ for all $t \in \mathbb{R}$. In particular, for $t = 0$ we have $T(0)x = x \in D(A)$. Contradiction!

5.8.6 The case of Hilbert Spaces

In this section, X is a Hilbert space endowed with scalar product $(\cdot, \cdot)$.

Lemma 5.233 *If $(T(t))_{t\geq 0}$ is a contraction semigroup with generator $-A$, then:*

(i) *$(T(t)^*)_{t\geq 0}$ is a contraction semigroup;*

(ii) The generator of $(T(t)^*)_{t\geq 0}$ is $-A^*$.

Proof: Since $(T(t))_{t\geq 0}$ is a contraction semigroup generated by $-A$, then by Proposition (5.211), A is m-accretive in X with dense domain. Thus by Proposition (5.161) and Corollary (5.157), it follows that A^* is also m-accretive with dense domain. Thus, by Proposition (5.215) $-A^*$ is the generator of a contraction semigroup in X. Let then $(S(t))_{t\geq 0}$ be the contraction semigroup generated by $-A^*$. By Corollary (5.204) and by Proposition (5.161), we have

$$S(t)x = \lim_{\lambda\to 0^+} e^{-t(A^*)_\lambda} = \lim_{\lambda\to 0^+} (e^{-tA_\lambda})^* x = T(t)^* x \quad \text{for all} \quad t\geq 0 \quad \text{and all} \quad x\in X.$$

Thus, by Proposition (5.213) $S(t) = T(t)^*$ whose generator is $-A^*$. □

Remark 5.234 *Some comments on Lemma (5.233). Let $(T(t))_{t\geq 0}$ be a contraction semigroup in a Banach space X and let A be its generator.*

(i) One can always consider $T(t)^*$.
The family $(T(t)^*)_{t\geq 0}$ satisfies properties $(i),(ii)$ and (iv) of Definition (5.207). However, property (iii) may not hold. For example, let $X = L^1(\mathbb{R})$ and let $(T(t))_{t\geq 0}$ be defined by $T(t)\varphi(x) = \varphi(x-t)$. We have that $(T(t)^*)_{t\geq 0}$ is defined in $X' = L^\infty(\mathbb{R})$ by $T(t)^*\varphi(x) = \varphi(x+t)$ and it can be verified that $(T(t)^*)$ is not continuous in X'.

(ii) Since $D(A)$ is dense in X, we can consider the operator A^* in X'. If A^* is m-accretive with dense domain, then the proof of Lemma (5.233) shows that $(T(t)^*)_{t\geq 0}$ is a contraction semigroup in X' and that its generator is $-A^*$.

(iii) In particular, if X is reflexive, then $-A^*$ is m-accretive with dense domain. see [83] Thm.4.6, p. 16. And then $(T(t)^*)_{t\geq 0}$ is a contraction semigroup and its generator is $-A^*$.

Corollary 5.235 *If A is a self-adjoint and accretive operator in X and if $(T(t))_{t\geq 0}$ is a contraction semigroup generated by $-A$, then $(T(t)) = (T(t))^*$ for all $t\geq 0$.*

Proof: By Corollary (5.165), we have that A is m-accretive with dense domain and, since A is self-adjoint, then $A = A^*$ whence $-A = -A^*$. But from Lemma (5.233), $(T(t)^*)_{t\geq 0}$ is the contraction semigroup generated by $-A^* = -A$. Thus, since $(T(t))_{t\geq 0}$ and $(T(t)^*)_{t\geq 0}$ have the same generators then $(T(t)) = (T(t)^*)$ for all $t\geq 0$. □

Corollary 5.236 *If A is a skew-adjoint operator in X, then there exists a group of isometries $(T(t)_{t\in\mathbb{R}}$ such that $-A$ is the generator of the contraction semigroup $(T(t))_{t\geq 0}$. Moreover, $(T(t))^* = T(-t)$ for all $t\in\mathbb{R}$.*

Proof: By Corollary (5.165), we have that $-A$ and A are m-accretive with dense domain. Thus, by Proposition (5.215), there exists a contraction semigroup $(T(t))_{t\geq 0}$ generated by $-A$. On the other hand, from Proposition (5.230), since $-A$ is m-accretive, then there exists a group of isometries $(T(t))_{t\in\mathbb{R}}$ such that for $t\geq 0$ it coincides with the semigroup $(T(t))_{t\geq 0}$ generated by $-A$.
Let us show that $(T(t))^* = T(-t)$ for all $t\in\mathbb{R}$. Given $x,y\in D(A)$, note that

$$\begin{aligned}
\frac{d}{dt}(T(t)x, T(t)y) &= (\frac{d}{dt}(T(t)x), T(t)y) + (T(t)x, \frac{d}{dt}(T(t)y)) \\
&= (-AT(t)x, T(t)y) + (T(t)x, -AT(t)y) \\
&= (T(t)x, -A^*T(t)y) - (T(t)x, AT(t)y) \\
&= (T(t)x, AT(t)y) - (T(t)x, AT(t)y) = 0
\end{aligned}$$

Therefore, $(x,y) = (T(t)x, T(t)y)$ for all $x,y\in D(A)$. In particular, taking $y = T(-t)z$, we have

$$(x, T(-t)z) = (T(t)x, T(t)T(-t)z) = (T(t)x, T(0)z) = (T(t)x, z).$$

Thus, $(T(t))^* = T(-t)$ for all $t\in\mathbb{R}$. Then, the result follows by density. □

Theorem 5.237 *Let A be an accretive and self-adjoint operator in X and let $(T(t))_{t\geq 0}$ be the contraction semigroup generated by $-A$. For each $x \in X$ the function $u(t) = T(t)x$ for $t \geq 0$ has the following properties:*

(i) $u \in C([0,\infty), X) \cap C((0,\infty), D(A)) \cap C^1((0,\infty), X)$ and u is the unique solution of the problem

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad \text{for all} \quad t > 0 \\ u(0) = x \end{cases} \tag{5.8.252}$$

in this class;

(ii) $\|Au(t)\| \leq \dfrac{1}{t\sqrt{2}}\|x\|$ for all $t > 0$. Moreover, the function $t \mapsto \sqrt{t}\|Au(t)\|$ belongs to $L^2(0,\infty)$ and $\displaystyle\int_0^\infty s\|Au(s)\|^2 ds \leq \frac{1}{4}\|x\|^2$;

(iii) $(Au(t), u(t)) \leq \dfrac{1}{2t}\|x\|^2$ for all $t > 0$. Moreover, the function $t \mapsto (Au(t), u(t))$ belongs to $L^2(0,\infty)$ and $\displaystyle\int_0^\infty (Au(t), u(t))ds \leq \frac{1}{2}\|x\|^2$;

(iv) If $x \in D(A)$ then $\|Au(t)\|^2 \leq \dfrac{1}{2t}(Ax, x)$ for all $t > 0$. Moreover, $Au \in L^2((0,\infty), X)$ and $\|Au\|^2_{L^2((0,\infty),X)} \leq \dfrac{1}{2}(Ax, x)$.

Proof: Let $x \in X$ and let $u(t) = T(t)x$. Given $\lambda > 0$, consider the operator A_λ and $u_\lambda(t) = e^{-tA_\lambda}x$. By Lemma (5.133), it follows that A_λ is m-accretive and it follows from Proposition (5.161) and from the fact that A is self-adjoint, that A_λ is self-adjoint since $(A_\lambda)^* = (A^*) = A_\lambda$. Therefore, $(e^{-tA_\lambda})_{t\geq 0}$ is a contraction semigroup. Applying Remark (5.210) we obtain that the map

$$t \mapsto \|u'_\lambda(t)\| = \|e^{-tA_\lambda}A_\lambda x\| \quad \text{is non-increasing.} \tag{5.8.253}$$

Indeed, for $0 \leq t \leq t+s$, we have

$$\|u'_\lambda(t+s)\| = \|e^{-(t+s)A_\lambda}A_\lambda x\| = \|e^{-sA_\lambda}e^{-tA_\lambda}A_\lambda x\| \leq \|e^{-tA_\lambda}A_\lambda\|x = \|u'_{\lambda(t)}\|.$$

Moreover, we have the following identities:

$$\frac{d}{dt}\|u_\lambda(t)\|^2 = -2(A_\lambda u_\lambda(t), u_\lambda(t)) \quad \text{for all} \quad t \geq 0. \tag{5.8.254}$$

$$\frac{d}{dt}(A_\lambda u_\lambda(t), u_\lambda(t)) = 2(A_\lambda u_\lambda(t), u'_\lambda(t)) = -2\|u'_\lambda(t)\|^2 \quad \text{for all} \quad t \geq 0. \tag{5.8.255}$$

Indeed, since $u'_\lambda(t) = -A_\lambda e^{-tA_\lambda} = -A_\lambda u_\lambda(t)$, we have

$$\begin{aligned} \frac{d}{dt}\|u_\lambda(t)\|^2 &= \frac{d}{dt}(u_\lambda(t), u_\lambda(t)) \\ &= (u'_\lambda(t), u_\lambda(t)) + (u_\lambda(t), u'_\lambda(t)) \\ &= 2(u'_\lambda(t), u_\lambda(t)) \\ &= -2(A_\lambda u_\lambda(t), u_\lambda(t)) \end{aligned}$$

and thus, proving (5.8.254). Similarly, using the fact that A_λ is self-adjoint, we have

$$\begin{aligned} \frac{d}{dt}(A_\lambda u_\lambda(t), u_\lambda(t)) &= (A_\lambda u'_\lambda(t), u_\lambda(t)) + (A_\lambda u_\lambda(t), u'_\lambda(t)) \\ &= (u'_\lambda(t), A_\lambda u_\lambda(t)) + (A_\lambda u_\lambda(t), u'_\lambda(t)) \\ &= 2(A_\lambda u_\lambda(t), u'_\lambda(t)). \end{aligned}$$

proving (5.8.255). From (5.8.255), we have that $(A_\lambda u_\lambda(t), u_\lambda(t))$ is a non-increasing function in t. Then integrating (5.8.254) from 0 to t, we have:

$$t(A_\lambda u_\lambda(t), u_\lambda(t)) \leq \int_0^t (A_\lambda u_\lambda(s), u_\lambda(s))ds \leq \frac{1}{2}\|x\|^2. \tag{5.8.256}$$

Indeed, since $(A_\lambda u_\lambda(t), u_\lambda(t))$ is non-increasing, then for $0 \leq s \leq t$, we have

$$(A_\lambda u_\lambda(t), u_\lambda(t)) \leq (A_\lambda u_\lambda(s), u_\lambda(s))$$

whence

$$t(A_\lambda u_\lambda(t), u_\lambda(t)) = \int_0^t (A_\lambda u_\lambda(t), u_\lambda(t))ds \leq \int_0^t (A_\lambda u_\lambda(s), u_\lambda(s))ds.$$

On the other hand, from (5.8.254) and since $s \geq 0$, it follows that $\frac{d}{ds}\|u_\lambda(s)\|^2 = -2(A_\lambda u_\lambda(s), u_\lambda(s))$ whence

$$\int_0^t (A_\lambda u_\lambda(s), u_\lambda(s))ds = -\frac{1}{2}\int_0^t \frac{d}{ds}\|u_\lambda(s)\|^2 ds = -\frac{\|u_\lambda(s)\|^2}{2}\Bigg|_0^t = -\frac{\|u_\lambda(t)\|^2}{2} + \frac{\|x\|^2}{2} \leq \frac{\|x\|^2}{2}.$$

Applying (5.8.253) and integrating (5.8.255) between 0 and $t > 0$ we obtain:

$$2t\|u'_\lambda(t)\|^2 \leq \int_0^t \|u'_\lambda(s)\|^2 ds = (A_\lambda x, x) - (A_\lambda u_\lambda(t), u_\lambda(t)) \leq (A_\lambda x, x). \tag{5.8.257}$$

Indeed, using (5.8.253) and integrating (5.8.255) between 0 and $t > 0$ we have

$$2t\|u'_\lambda(t)\|^2 \leq 2\int_0^t \|u'_\lambda(s)\|^2 ds.$$

On the other hand, from (5.8.255), we have

$$\begin{aligned}
2\int_0^t \|u'_\lambda(s)\|^2 ds &= -\int_0^t \frac{d}{ds}(A_\lambda u_\lambda(s), u_\lambda(s))ds \\
&= -(A_\lambda u_\lambda(s), u_\lambda(s))\Bigg|_0^t \\
&= -(A_\lambda u_\lambda(t), u_\lambda(t)) + (A_\lambda u_\lambda(0), u_\lambda(0)) \\
&= (A_\lambda x, x) - (A_\lambda u_\lambda(t), u_\lambda(t)) \\
&\leq (A_\lambda x, x)
\end{aligned}$$

where the last inequality follows from Lemma (5.156). Thus proving (5.8.257). Multiplying (5.8.255) by $t \geq 0$, using the fact that $u'_\lambda(t)$ is non-increasing and integrating from 0 to t, we have for $0 \leq s \leq t$ that

$$\begin{aligned}
t^2\|u'_\lambda(t)\|^2 &= \int_0^t s\|u'_\lambda(s)\|^2 ds \\
&\leq 2\int_0^t s\|u'_\lambda(s)\|^2 ds \\
&= -\int_0^t s\frac{d}{ds}(A_\lambda u_\lambda(s), u_\lambda(s))ds
\end{aligned}$$

Using integration by parts with $u = s$ and $dv = \frac{d}{ds}(A_\lambda u_\lambda(s), u_\lambda(s))ds$, we obtain

$$\begin{aligned} -\int_0^t s\frac{d}{ds}(A_\lambda u_\lambda(s), u_\lambda(s))ds &= -t(A_\lambda u_\lambda(s), u_\lambda(s)) + \int_0^t (A_\lambda u_\lambda(s), u_\lambda(s))ds \\ &\leq \int_0^t (A_\lambda u_\lambda(s), u_\lambda(s))ds \\ &\leq \frac{1}{2}\|x\|^2. \end{aligned}$$

Applying (5.8.256), it follows that

$$2t^2\|u'_\lambda(t)\|^2 \leq \|x\|^2. \tag{5.8.258}$$

Consider now $t > 0$. By Corollary (5.204) and Proposition (5.137), it follows that $J_\lambda u_\lambda(t) \to u(t)$ in X when $\lambda \to 0^+$. On the other hand, $A(J_\lambda u_\lambda(t)) = A_\lambda u_\lambda(t) = u'_\lambda(t)$ is bounded in X. Applying Lemma (5.169), since A_λ is m-accretive, we have that $u_\lambda \in D(A)$ and $A_\lambda u_\lambda(t) \rightharpoonup Au(t)$ when $\lambda \to 0^+$.
Thus, property (i) follows now by applying Proposition (5.206) with the initial value $u(\varepsilon)$ for any $\varepsilon > 0$ and letting $\varepsilon \to 0$ and using the fact that $u'_\lambda(t)$ converges to $u'(t)$. Furthermore, passing the limit as $\lambda \to 0^+$ in (5.8.256) we obtain (iii), passing the limit as $\lambda \to 0^+$ in (5.8.257) since $x \in D(A)$ we obtain (iv) and finally, passing the limit as $\lambda \to 0^+$ in (5.8.258) we obtain (ii). □

5.9 Exponential Formula

Just as in the case of linear semigroups, one can define the exponential of an operator under certain hypotheses:

Theorem 5.238 (Crandall-Liggett) *Let $A \in \mathcal{A}(\omega)$ such that $\overline{D(A)} \subset Im(I + \lambda A)$, for $0 < \lambda < \lambda_0$ with $\lambda_0|\omega| < 1$. Then, for any $x \in \overline{D(A)}$ and $t > 0$, the limit exists*

$$\lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} x, \tag{5.9.259}$$

and the convergence is uniform on bounded intervals. Setting

$$S(t)x := \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} x,$$

we have that $S \in Q_\omega(\overline{D(A)})$ and for all $x \in D(A)$, $S(t)x$ is Lipschitz continuous on bounded intervals.

To prove this theorem, we will make use of two technical lemmas, which will be used to obtain estimates that enable us to prove the existence of the limit given by (5.9.259).

Lemma 5.239 *Let $A \in \mathcal{A}(\omega)$, $0 < \mu \leq \lambda < \lambda_0$, such that $\omega\lambda_0 < 1$ and $x \in D(J_\lambda^m) \cap D(J_\mu^n)$ with m and n positive integers such that $m \leq n$. Then*

$$\begin{aligned} \left\|J_\mu^n x - J_\lambda^m x\right\| &\leq (1-\mu\omega)^{-n} \sum_{j=0}^{m-1} \alpha^j \beta^{n-j} \binom{n}{j} \left\|J_\lambda^{m-j} x - x\right\| \\ &\quad + \sum_{j=m}^{n} (1-\mu\omega)^{-j} \alpha^m \beta^{j-m} \binom{j-1}{m-1} \left\|J_\mu^{n-j} x - x\right\|, \end{aligned} \tag{5.9.260}$$

where $\alpha = \dfrac{\mu}{\lambda}$ and $\beta = \dfrac{\lambda - \mu}{\lambda}$.

Proof: To avoid overburdening the notation, for $0 \leq j \leq n$ and $0 \leq k \leq m$, let us set

$$a(k;j) := \left\| J_\mu^j x - J_\lambda^k x \right\|.$$

By Theorem 5.79, item (iv), we can write

$$\begin{aligned} a(k;j) &= \left\| J_\mu^j x - J_\mu \left(\alpha J_\lambda^{k-1} x + \beta J_\lambda^k x \right) \right\| \\ &\leq (1-\mu\omega)^{-1} \left\| J_\mu^{j-1} x - \left(\alpha J_\lambda^{k-1} x + \beta J_\lambda^k x \right) \right\| \\ &\leq (1-\mu\omega)^{-1} \left(\alpha \left\| J_\mu^{j-1} x - J_\lambda^{k-1} \right\| + \beta \left\| J_\mu^{j-1} x - J_\lambda^k x \right\| \right) \\ &= \alpha_1 a(k-1;j-1) + \beta_1 a(k;j-1), \end{aligned}$$

where $\alpha_1 = \alpha(1-\mu\omega)^{-1}$ and $\beta_1 = \beta(1-\mu\omega)^{-1}$. We will use the formula

$$a(k;j) \leq \alpha_1 a(k-1;j-1) + \beta_1 a(k;j-1), \tag{5.9.261}$$

to demonstrate (5.9.260) by induction.

Let us say that the validity of (5.9.260) is property $P_{m,n}$. Let us prove $P_{m,n}$ for all $n \geq m \geq 1$.

Firstly, let us prove that $P_{1,n}$ holds, for all $n \geq 1$, by induction on n. Indeed, we have that

$$\begin{aligned} a(1;1) &\leq \alpha_1\ a(0;0) + \beta_1\ a(1;0) \\ &= (1-\mu\omega)^{-1} \beta \| J_\lambda x - x \|, \end{aligned}$$

whence it follows that $P_{1,1}$ is verified. Now, suppose $P_{1,n-1}$ is valid and let us prove that $P_{1,n}$ is verified. We have

$$\begin{aligned} a(1;n) &\leq \alpha_1\ a(0;n-1) + \beta_1\ a(1;n-1) \\ &\leq \alpha_1\ a(0;n-1) + \beta_1 \left\{ \beta_1^{n-1} \binom{n-1}{0} a(1;0) + \sum_{j=1}^{n-1} \alpha_1 \beta_1^{j-1} \binom{j-1}{0} a(0;n-1-j) \right\} \\ &= \alpha_1\ a(0;n-1) + \beta_1^n\ a(1;0) + \sum_{j=1}^{n-1} \alpha_1 \beta_1^j\ a(0;n-1-j) \\ &= \beta_1^n\ a(1;0) + \sum_{j=1}^{n} \alpha_1 \beta_1^{j-1}\ a(0;n-j), \end{aligned}$$

and thus, $P_{1,n}$ holds.

Let us assume that $P_{m-1,n}$ is true for $n \geq m-1$. We want to prove $P_{m,n}$ for $n \geq m$. Again, we will use induction on n. The first case is $n = m$, which must be verified. Note that

$$\begin{aligned} a(m;m) &\leq \alpha_1\ a(m-1;m-1) + \beta_1\ a(m;m-1) \\ &\leq \alpha_1^2\ a(m-2;n-2) + 2\alpha_1\beta_1\ a(m-1;n-2) + \beta_1^2\ a(m;m-2) \\ &\vdots \\ &\leq \sum_{j=0}^{m-1} \alpha_1^j \beta_1^{m-j} \binom{m}{j} a(m-j;0), \end{aligned}$$

which proves that $P_{m,m}$,.

Now, suppose that $P_{m,n-1}$ is true for $n-1 \geq m$. Then

$$
\begin{aligned}
a(m;n) &\leq \alpha_1\ a(m-1;n-1) + \beta_1\ a(m;n-1) \\
&\leq \alpha_1 \left\{ (1-\mu\omega)^{-(n-1)} \sum_{j=0}^{m-2} \alpha^j \beta^{n-1-j} \binom{n-1}{j} a(m-1-j;0) \right. \\
&\quad \left. + \sum_{j=m-1}^{n-1} (1-\mu\omega)^{-j} \alpha^{m-1} \beta^{j-(m-1)} \binom{j-1}{m-2} a(0;n-1-j) \right\} \\
&\quad + \beta_1 \left\{ (1-\mu\omega)^{-(n-1)} \sum_{j=0}^{m-1} \alpha^j \beta^{n-1-j} \binom{n-1}{j} a(m-j;0) \right. \\
&\quad \left. + \sum_{j=m}^{n-1} (1-\mu\omega)^{-j} \alpha^m \beta^{j-m} \binom{j-1}{m-1} a(0;n-1-j) \right\} \\
&= (1-\mu\omega)^{-n} \sum_{j=0}^{m-2} \alpha^{j+1} \beta^{n-(j+1)} \binom{n-1}{j} a(m-(j+1);0) \qquad (5.9.262) \\
&\quad + \sum_{j=m-1}^{n-1} (1-\mu\omega)^{-(j+1)} \alpha^m \beta^{(j+1)-m} \binom{j-1}{m-2} a(0;n-(j+1)) \qquad (5.9.263) \\
&\quad + (1-\mu\omega)^{-n} \sum_{j=0}^{m-1} \alpha^j \beta^{n-j} \binom{n-1}{j} a(m-j;0) \qquad (5.9.264) \\
&\quad + \sum_{j=m}^{n-1} (1-\mu\omega)^{-(j+1)} \alpha^m \beta^{(j+1)-m} \binom{j-1}{m-1} a(0;n-(j+1)). \qquad (5.9.265)
\end{aligned}
$$

Let us set $j' = j+1$ in (5.9.262), rewriting j' by j (just a change in indices so that it does not alter the sum). In (5.9.264), let us separate from the sum the term corresponding to $j = 0$ and group the remaining terms with the terms of (5.9.262) to obtain

$$
(1-\mu\omega)^{-n} \left\{ \beta^n a(m;0) + \sum_{j=1}^{m-1} \alpha^j \beta^{n-j} \left[\binom{n-1}{j-1} + \binom{n-1}{j} \right] a(m-j;0) \right\}. \qquad (5.9.266)
$$

Now decoupling the term corresponding to $j = m-1$ in (5.9.263) and adding with the expression in (5.9.265) we obtain,

$$
(1-\mu\omega)^{-m} \alpha^m a(0;n-m) + \sum_{j=m}^{n-1} (1-\mu\omega)^{-(j+1)} \alpha^m \beta^{(j+1)-m} \left[\binom{j-1}{m-2} + \binom{j-1}{m-1} \right] a(0;n-(j+1)). \qquad (5.9.267)
$$

From Stiefel's formula,

$$
\binom{n-1}{j-1} + \binom{n-1}{j} = \binom{n}{j},
$$

making the change $j' = j+1$, and adding the resulting expression with (5.9.266) we obtain,

$$
\begin{aligned}
a(m;n) &\leq (1-\mu\omega)^{-n} \sum_{j=0}^{m-1} \alpha^j \beta^{n-j} \binom{n}{j} a(m-j;0) \\
&\quad + \sum_{j=m}^{n} (1-\mu\omega)^{-j} \alpha^m \beta^{j-m} \binom{j-1}{m-1} a(0;n-j).
\end{aligned}
$$

□

Lemma 5.240 *Let m and n be positive integers with $m \leq n$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Then* *(i)* $\displaystyle\sum_{j=0}^{m}\binom{n}{j}\alpha^j\beta^{n-j}(m-j) \le \sqrt{(n\alpha-m)^2+n\alpha\beta}$;

(ii) $\displaystyle\sum_{j=m}^{n}\binom{j-1}{m-1}\alpha^m\beta^{j-m}(n-j) \le \sqrt{\frac{m\beta}{\alpha^2}+\left(\frac{m\beta}{\alpha}+m-n\right)^2}$.

Proof:

(i) Initially, considering $n=1$, we have $m=1$ and the inequality becomes

$$\sum_{j=0}^{1}\binom{1}{j}\alpha^j\beta^{1-j}(1-j)=\beta\le\sqrt{(\alpha-1)^2+\alpha\beta}=\sqrt{\beta},$$

which is true, since $0<\beta<1$.

For $n\ge 2$, from the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{j=0}^{m}\binom{n}{j}\alpha^j\beta^{n-j}(m-j) &\le \sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}|m-j|\\ &\le \left[\sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}\right]^{\frac{1}{2}}\left[\sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}(m-j)^2\right]^{\frac{1}{2}}\\ &= \left[\sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}(m-j)^2\right]^{\frac{1}{2}},\end{aligned}$$

since

$$(\alpha+\beta)^n=\sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}\quad\text{and}\quad\alpha+\beta=1.$$

Now note that if $\alpha,\beta>0$, we have

$$\begin{aligned}\alpha n(\alpha+\beta)^{n-1} &= \alpha n\sum_{j=0}^{n-1}\frac{(n-1)!}{j!(n-1-j)!}\alpha^j\beta^{n-1-j}\\ &= \sum_{j=0}^{n-1}(j+1)\frac{n(n-1)!}{(j+1)j!(n-(j+1))!}\alpha^{j+1}\beta^{n-(j+1)}\\ &= \sum_{j=0}^{n}j\binom{n}{j}\alpha^j\beta^{n-j}\end{aligned}\tag{5.9.268}$$

and

$$\begin{aligned}&\alpha^2 n(n-1)(\alpha+\beta)^{n-2}+\alpha n(\alpha+\beta)^{n-1}\\ =\;&\alpha n\left(\alpha(n-1)(\alpha+\beta)^{n-2}+(\alpha+\beta)^{n-1}\right)\\ =\;&\alpha n\left(\sum_{j=0}^{n-1}j\binom{n-1}{j}\alpha^j\beta^{n-1-j}+\sum_{j=0}^{n-1}\binom{n-1}{j}\alpha^j\beta^{n-1-j}\right)\\ =\;&\sum_{j=0}^{n-1}n(j+1)\frac{(n-1)!}{j!(n-(j+1))!}\alpha^{j+1}\beta^{n-(j+1)}\\ =\;&\sum_{j=0}^{n}j^2\binom{n}{j}\alpha^j\beta^{n-j}.\end{aligned}\tag{5.9.269}$$

Therefore, from (5.9.268) and (5.9.269), besides the fact that $\beta = 1-\alpha$, it follows that

$$\begin{aligned}
\sum_{j=0}^{m}\binom{n}{j}\alpha^j\beta^{n-j}(m-j) &\le \left(\sum_{j=0}^{n}\binom{n}{j}\alpha^j\beta^{n-j}\left[m^2-2mj+j^2\right]\right)^{\frac{1}{2}}\\
&= \left(m^2-2\alpha mn+\alpha^2 n(n-1)+\alpha n\right)^{\frac{1}{2}}\\
&= \left(m^2-2\alpha mn+\alpha^2n^2-\underbrace{\alpha^2 n+\alpha n}_{\alpha n(1-\alpha)}\right)^{\frac{1}{2}}\\
&= \left(m^2-2\alpha mn+\alpha^2n^2+n\alpha\beta\right)^{\frac{1}{2}}\\
&= \left((\alpha n-m)^2+n\alpha\beta\right)^{\frac{1}{2}}.
\end{aligned}$$

(ii) For $0<\beta<1$, consider

$$F_m(\beta)=\sum_{j=m}^{\infty}\binom{j-1}{m-1}\beta^{j-m}.$$

We claim that $F_m(\beta)$ is convergent and $F_m(\beta)=(1-\beta)^{-m}$.

Indeed, let

$$a_j := \binom{j-1}{m-1} = \frac{(j-1)(j-2)\dots(j-m+1)}{(m-1)!},$$

for $j\ge m$. Then

$$\begin{aligned}
1\le \sqrt[j]{a_j} &= \sqrt[j]{\frac{j-1}{m-1}}\sqrt[j]{\frac{j-2}{m-2}}\dots\sqrt[j]{\frac{j-(m-1)}{1}}\\
&= \sqrt[j]{j-1}\sqrt[j]{\frac{j}{2}-1}\dots\sqrt[j]{\frac{j}{m-1}-1}\le(\sqrt[j]{j})^{m-1},
\end{aligned}$$

whence,

$$1\le \lim_{j\to\infty}\sqrt[j]{a_j}\le(\lim_{j\to\infty}\sqrt[j]{j})^{m-1}=1^{m-1}=1,$$

which shows us that $\lim_{j\to\infty}\sqrt[j]{a_j}=1$, and thus, the radius of convergence of the series given by $F_m(\beta)$ is 1. Hence, $F_m(\beta)$ converges absolutely for all $\beta\in(-1,1)$. We shall prove that $F_m(\beta)=(1-\beta)^{-m}$ by induction. We have

$$F_1(\beta)=\sum_{j=1}^{\infty}\binom{j-1}{0}\beta^{j-1}=\sum_{j=0}^{\infty}\beta^j=(1-\beta)^{-1}.$$

Assume

$$F_{m-1}(\beta)=\sum_{j=m-1}^{\infty}\binom{j-1}{m-2}\beta^{j-(m-1)}=(1-\beta)^{-(m-1)}.$$

Thus,

$$\begin{aligned}
(1-\beta)^{-m} &= F_{m-1}(\beta)F_1(\beta)\\
&= \left[\sum_{j=m-1}^{\infty}\binom{j-1}{m-2}\beta^{j-(m-1)}\right]\left[\sum_{j=1}^{\infty}\binom{j-1}{0}\beta^{j-1}\right]\\
&= \left[\sum_{j=m-1}^{\infty}\binom{j-1}{m-2}\beta^{j-(m-1)}\right]\left[\sum_{j=m-1}^{\infty}\beta^{j-(m-1)}\right]=\sum_{j=m-1}^{\infty}c_j,
\end{aligned}$$

where $c_j = \sum_{k=m-1}^{j} a_k\ b_{j+(m-1)-k}$, with $a_k = \binom{k-1}{m-2}\beta^{k-(m-1)}$ and $b_k = \beta^{k-(m-1)}$.

Thus,

$$\begin{aligned}
(1-\beta)^{-m} &= \sum_{j=m-1}^{\infty}\left[\sum_{k=m-1}^{j}\binom{k-1}{m-2}\beta^{k-(m-1)}\beta^{j+(m-1)-k-(m-1)}\right]\\
&= \sum_{j=m-1}^{\infty}\left[\sum_{k=m-1}^{j}\binom{k-1}{m-2}\beta^{j-(m-1)}\right]\\
&= \sum_{j=m-1}^{\infty}\left[\sum_{k-1=m-2}^{j-1}\binom{k-1}{m-2}\beta^{j-(m-1)}\right]\\
&= \sum_{j'=m}^{\infty}\binom{j'-1}{m-1}\beta^{j'-m} = F_m(\beta),
\end{aligned}$$

where the penultimate identity is justified by the Pascal's column theorem (the sum of the first j elements of a column of Pascal's triangle is equal to the element that is advanced one row and one column over the last term of the sum).

Since the power series $F_m(\beta)$ converges absolutely in $(0,1)$, the same happens with its derivatives $F_m'(\beta)$ and $F_m''(\beta)$. Moreover,

$$\begin{aligned}
F_m(\beta) &= \sum_{j=m}^{\infty}\binom{j-1}{m-1}\beta^{j-m} = (1-\beta)^{-m};\\
F_m'(\beta) &= \sum_{j=m+1}^{\infty}\binom{j-1}{m-1}(j-m)\beta^{j-m-1} = m(1-\beta)^{-m-1};\\
F_m''(\beta) &= \sum_{j=m+2}^{\infty}\binom{j-1}{m-1}(j-m)(j-m-1)\beta^{j-m-2} = m(m+1)(1-\beta)^{-m-2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
m(1-\beta)^{-m-1} &= \sum_{j=m+1}^{\infty}\binom{j-1}{m-1}(j-m)\beta^{j-m-1}\\
&= \frac{1}{\beta}\left[\sum_{j=m+1}^{\infty}\binom{j-1}{m-1}j\beta^{j-m} - m(1-\beta)^{-m} + m\right],
\end{aligned}$$

whence

$$\sum_{j=m}^{\infty}\binom{j-1}{m-1}j\beta^{j-m} = \beta m(1-\beta)^{-m-1} + m(1-\beta)^{-m}.$$

Also

$$m(m+1)(1-\beta)^{-m-2} = \frac{1}{\beta^2}\left[\sum_{j=m+2}^{\infty}\binom{j-1}{m-1}(j-m)^2\beta^{j-m} - \sum_{j=m+2}^{\infty}\binom{j-1}{m-1}(j-m)\beta^{j-m}\right],$$

or even,

$$\sum_{j=m+2}^{\infty}\binom{j-1}{m-1}(j-m)^2\beta^{j-m} = \beta^2 m(m+1)(1-\beta)^{-m-2} + \sum_{j=m+2}^{\infty}\binom{j-1}{m-1}(j-m)\beta^{j-m},$$

whence

$$\begin{aligned}\sum_{j=m}^{\infty}\binom{j-1}{m-1}j^2\beta^{j-m} &= \beta^2 m(m+1)(1-\beta)^{-m-2}+\beta m(1-\beta)^{-m-1}\\ &+ 2m\left[\beta m(1-\beta)^{-m-1}+m(1-\beta)^{-m}\right]-m^2(1-\beta)^{-m}.\end{aligned}$$

If $\alpha+\beta=1$,

$$\begin{aligned}\sum_{j=m}^{\infty}\binom{j-1}{m-1}\beta^{j-m} &= \alpha^{-m};\\ \sum_{j=m}^{\infty}\binom{j-1}{m-1}j\beta^{j-m} &= m\alpha^{-m}\left(1+\frac{\beta}{\alpha}\right)=\frac{m}{\alpha}\alpha^{-m};\\ \sum_{j=m}^{\infty}\binom{j-1}{m-1}j^2\beta^{j-m} &= \left(\frac{m^2+\beta m}{\alpha^2}\right)\alpha^{-m}.\end{aligned}$$

Thus, from the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{j=m}^{n}\binom{j-1}{m-1}\alpha^m\beta^{j-m}(n-j) &\le \sum_{j=m}^{\infty}\binom{j-1}{m-1}\alpha^m\beta^{j-m}|n-j|\\ &\le \left[\sum_{j=m}^{\infty}\binom{j-1}{m-1}\alpha^m\beta^{j-m}\right]^{\frac{1}{2}}\left[\sum_{j=m}^{\infty}\binom{j-1}{m-1}\alpha^m\beta^{j-m}(j-n)^2\right]^{\frac{1}{2}}\\ &= \left[\frac{m^2+\beta m}{\alpha^2}-2n\frac{m}{\alpha}+n^2\right]^{\frac{1}{2}}\\ &= \left[\frac{m\beta}{\alpha^2}+\left(\frac{m}{\alpha}-n\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Note that $\dfrac{m}{\alpha}=\dfrac{m\beta}{\alpha}+m$, and we obtain the desired inequality.

□

Proof of Theorem 5.238. Let $x\in D(A)$, $0<\mu\le\lambda<\lambda_0$, $n\ge m$ and $\lambda_0|\omega|<1$. Since, by hypothesis, $\overline{D(A)}\subset Im(I+\lambda A)$, for all $\lambda\in(0,\lambda_0)$, it follows that

$$D(A)\subset\overline{D(A)}\subset\bigcap_{\lambda\in(0,\lambda_0)}D_\lambda.$$

But by Proposition 5.73, item (i), we have that $J_\lambda:D_\lambda\to D(A)$, thus $x\in D(J_\mu^n)\cap D(J_\lambda^m)$.

By Theorem 5.79, item (iv), if $x\in D_\lambda$, $\lambda\neq 0$ and $\mu\in\mathbb{R}$ then

$$\frac{\mu}{\lambda}x+\frac{\lambda-\mu}{\lambda}J_\lambda x\in D_\mu$$

and

$$J_\mu\left(\frac{\mu}{\lambda}x+\frac{\lambda-\mu}{\lambda}J_\lambda x\right)=J_\lambda x.$$

Let us denote $\alpha = \frac{\mu}{\lambda}$ and $\beta = \frac{\lambda-\mu}{\lambda}$. By Lemma 5.239 we have

$$\begin{aligned} \|J_\mu^n x - J_\lambda^m x\| &\leq (1-\mu|\omega|)^{-n}\sum_{j=0}^{m-1}\alpha^j\beta^{n-j}\binom{n}{j}\|J_\lambda^{m-j}x - x\| \\ &+ \sum_{j=m}^{n}(1-\mu|\omega|)^{-j}\alpha^m\beta^{j-m}\binom{j-1}{m-1}\|J_\mu^{n-j}x - x\|. \end{aligned} \tag{5.9.270}$$

By Theorem 5.79, items (ii) and (iii), it follows that

$$\begin{aligned} \|J_\lambda^{m-j}x - x\| &\leq (m-j)(1-\lambda|\omega|)^{-m+j+1}\|J_\lambda x - x\| \\ &\leq (m-j)(1-\lambda|\omega|)^{-m+j+1}\lambda(1-\lambda\omega)^{-1}|Ax| \\ &\leq (m-j)(1-\lambda|\omega|)^{-m+j}\lambda|Ax|. \end{aligned} \tag{5.9.271}$$

Analogously

$$\|J_\mu^{n-j}x - x\| \leq (n-j)(1-\mu|\omega|)^{-n+j}\mu|Ax|. \tag{5.9.272}$$

Substituting (5.9.271) and (5.9.272) into (5.9.270) and recalling that $(1-\lambda|\omega|)^j < 1$, we obtain

$$\begin{aligned} \|J_\mu^n x - J_\lambda^m x\| &\leq (1-\mu|\omega|)^{-n}(1-\lambda|\omega|)^{-m}\lambda\sum_{j=0}^{m-1}\alpha^j\beta^{n-j}\binom{n}{j}(m-j)|Ax| \\ &+ (1-\mu|\omega|)^{-n}\mu\sum_{j=m}^{n}\alpha^m\beta^{j-m}\binom{j-1}{m-1}(n-j)|Ax|. \end{aligned} \tag{5.9.273}$$

Let $f(\xi) = (1-\xi)^n e^{2n\xi}$, so $f'(\xi) \geq 0$ for $\xi \in [0, \frac{1}{2}]$, that is, f is increasing on $[0, \frac{1}{2}]$, whence it follows that

$$1 = f(0) \leq f(\xi), \quad \forall \xi \in \left[0, \frac{1}{2}\right],$$

that is,

$$(1-\xi)^{-n} \leq e^{2n\xi}, \quad \forall \xi \in \left[0, \frac{1}{2}\right]. \tag{5.9.274}$$

If $\lambda|\omega| \leq \frac{1}{2}$, from (5.9.273), (5.9.274) and Lemma 5.240, we obtain

$$\begin{aligned} \|J_\mu^n x - J_\lambda^m x\| &\leq e^{2n|\omega|\mu}e^{2m|\omega|\lambda}\lambda\left[(n\alpha - m)^2 + n\alpha\beta\right]^{\frac{1}{2}}|Ax| \\ &+ e^{2n|\omega|\mu}\mu\left[\frac{m\beta}{\alpha^2} + \left(\frac{m\beta}{\alpha} + m - n\right)^2\right]^{\frac{1}{2}}|Ax|. \end{aligned} \tag{5.9.275}$$

But recalling that $\alpha = \frac{\mu}{\lambda}$ and $\beta = \frac{\lambda-\mu}{\lambda}$, it follows that

$$\begin{aligned} \lambda\left[(n\alpha - m)^2 + n\alpha\beta\right]^{\frac{1}{2}} &= \left[(n\mu - m\lambda)^2 + n\mu(\lambda-\mu)\right]^{\frac{1}{2}} \quad \text{and} \\ \mu\left[\frac{m\beta}{\alpha^2} + \left(\frac{m\beta}{\alpha} + m - n\right)^2\right]^{\frac{1}{2}} &= \left[m\lambda(\lambda-\mu) + (m\lambda - n\mu)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{5.9.276}$$

Substituting the identities above into (5.9.275) we obtain

$$\begin{aligned} \|J_\mu^n x - J_\lambda^m x\| &\leq e^{2|\omega|(n\mu+m\lambda)}\left[(n\mu-m\lambda)^2+n\mu(\lambda-\mu)\right]^{\frac{1}{2}}|Ax| \\ &+ e^{2|\omega|n\mu}\left[m\lambda(\lambda-\mu)+(m\lambda-n\mu)^2\right]^{\frac{1}{2}}|Ax|, \end{aligned} \tag{5.9.277}$$

for $0<\mu\leq\lambda<\lambda_0$ and $\lambda|\omega|\leq\frac{1}{2}$.

Setting $\mu=\frac{t}{n}$ and $\lambda=\frac{t}{m}$, $t>0$, we obtain that $0<\mu\leq\lambda$, since $m\leq n$. Furthermore

$$\lambda=\frac{t}{m}<\lambda_0 \Leftrightarrow t<m\lambda_0$$

and

$$\lambda|\omega|\leq\frac{1}{2}\Leftrightarrow t\leq\frac{m}{2|\omega|},$$

$\omega\neq 0$, that is, if $m\to\infty$, we can take values for t arbitrarily large.

From (5.9.277), it follows that

$$\|J_{\frac{t}{n}}^n x - J_{\frac{t}{m}}^m x\| \leq [2te^{4|\omega|t}+te^{2|\omega|t}]\left(\frac{1}{m}-\frac{1}{n}\right)^{\frac{1}{2}}|Ax|. \tag{5.9.278}$$

Therefore the sequence $\left\{J_{\frac{t}{n}}^n x\right\}_n$ is Cauchy, $\forall\ t>0$ and $\forall\ x\in D(A)$. Since X is a Banach space, it follows that there exists

$$S(t)x := \lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x, \tag{5.9.279}$$

for all $x\in D(A)$, and such convergence is uniform on bounded intervals by virtue of (5.9.278).

Since for each $x,y\in\overline{D(A)}$ we have

$$\|J_{\frac{t}{n}}^n x - J_{\frac{t}{n}}^n y\| \leq (1-\frac{t}{n}\omega)^{-n}\|x-y\|, \tag{5.9.280}$$

it follows that the limit given in (5.9.279) exists for all $x\in\overline{D(A)}$. Indeed, let $x\in\overline{D(A)}$ then given $\epsilon>0$ there exists $y\in D(A)$ such that $\|x-y\|<\epsilon$.

Note that

$$\begin{aligned} \|J_{\frac{t}{n}}^n x - J_{\frac{t}{m}}^m x\| &\leq \|J_{\frac{t}{n}}^n x - J_{\frac{t}{n}}^n y\| + \|J_{\frac{t}{n}}^n y - J_{\frac{t}{m}}^m y\| + \|J_{\frac{t}{m}}^m y - J_{\frac{t}{m}}^m x\| \\ &\leq (1-\frac{t}{n}\omega)^{-n}\|x-y\| + \|J_{\frac{t}{n}}^n y - J_{\frac{t}{m}}^m y\| + (1-\frac{t}{m}\omega)^{-m}\|x-y\| \\ &\leq \epsilon\left[(1-\frac{t}{n}\omega)^{-n}+(1-\frac{t}{m}\omega)^{-m}\right] + \|J_{\frac{t}{n}}^n y - J_{\frac{t}{m}}^m y\|. \end{aligned}$$

Since

$$\left(\lim_{n\to\infty} 1-\frac{\omega t}{n}\right)^{-n} = e^{\omega t},$$

the sequence $\{(1-\frac{t}{n}\omega)^{-n}\}_n$ is convergent, hence, bounded, it follows that

$$\|J_{\frac{t}{n}}^n x - J_{\frac{t}{m}}^m x\| \leq C\epsilon + \|J_{\frac{t}{n}}^n y - J_{\frac{t}{m}}^m y\|,$$

and due to the fact that $\left\{J_{\frac{t}{n}}^n y\right\}_n$ is Cauchy, it follows that for the given ϵ there exists $n_0\in\mathbb{N}$ such that if $m,n\geq n_0$, then $\|J_{\frac{t}{n}}^n y - J_{\frac{t}{m}}^m y\|<\epsilon$.

Therefore,

$$\|J^n_{\frac{t}{n}}x - J^m_{\frac{t}{m}}x\| \le (C+1)\epsilon,$$

for all $m, n \ge n_0$, proving that the sequence $\left\{J^n_{\frac{t}{n}}x\right\}_n$ is Cauchy, for $x \in \overline{D(A)}$, thus, convergent.

Moreover, the map

$$\begin{aligned} S(t): \ \overline{D(A)} &\longrightarrow \overline{D(A)} \\ x &\mapsto S(t)x = \lim_{n\to\infty} J^n_{\frac{t}{n}}x \end{aligned}$$

is Lipschitzian with constant $e^{\omega t}$.

Let now $x \in D(A)$ and $0 \le t \le \tau$. Taking $m = n$, $\mu = \frac{t}{n}$ and $\lambda = \frac{\tau}{n}$ in (5.9.277) we obtain

$$\begin{aligned} \|J^n_{\frac{t}{n}}x - J^n_{\frac{\tau}{n}}x\| &\le e^{2|\omega|(t+\tau)}\left[(t-\tau)^2 + t(\frac{\tau}{n} - \frac{t}{n})\right]^{\frac{1}{2}}|Ax| \\ &+ e^{2|\omega|t}\left[\tau(\frac{\tau}{n} - \frac{t}{n}) + (\tau - t)^2\right]^{\frac{1}{2}}|Ax|. \end{aligned}$$

Letting $n \to \infty$ we obtain

$$\|S(t)x - S(\tau)x\| \le \left(e^{2|\omega|(t+\tau)} + e^{2|\omega|t}\right)|Ax||t-\tau|. \tag{5.9.281}$$

Showing that the map $t \mapsto S(t)x$ is Lipschitz continuous on bounded intervals, $\forall\, x \in D(A)$.

From inequality (5.9.281), the strong continuity of S follows, that is, $\lim_{t\to 0^+} S(t)x = x$, for all $x \in \overline{D(A)}$. To show that $S \in Q_\omega(\overline{D(A)})$, it remains to prove that $S(t+s) = S(t)S(s)$ in $\overline{D(A)}$, since $S(0) = I$ is trivially satisfied.

Indeed, let $m \in \mathbb{N}$ and $x \in \overline{D(A)}$, then

$$S(mt)x = \lim_{n\to\infty} J^n_{\frac{mt}{n}}x = \lim_{k\to\infty} J^{mk}_{\frac{t}{k}}x = \lim_{k\to\infty}\left(J^k_{\frac{t}{k}}\right)^m x = [S(t)]^m x. \tag{5.9.282}$$

Now, if $l, k, r, s \in \mathbb{N}$, from (5.9.282) we have

$$\begin{aligned} S\left(\frac{l}{k} + \frac{r}{s}\right)x &= S\left(\frac{ls+rk}{ks}\right)x = \left[S\left(\frac{1}{ks}\right)\right]^{ls+rk} x \\ &= \left[S\left(\frac{1}{ks}\right)\right]^{ls}\left[S\left(\frac{1}{ks}\right)\right]^{rk} x = S\left(\frac{l}{k}\right)S\left(\frac{r}{s}\right)x, \end{aligned} \tag{5.9.283}$$

proving that $S(t+s) = S(t)S(s)$ for all $s, t \in \mathbb{Q}$, $s, t \ge 0$. From the strong continuity of S and the density of $\mathbb{Q}$ in $\mathbb{R}$, the desired result follows.

Theorem 5.241 () *Let $(\epsilon_n)_n$ be a sequence of positive real numbers such that $\epsilon_n \to 0$ when $n \to \infty$, $\left[\frac{t}{\epsilon_n}\right]$ the integer part of $\frac{t}{\epsilon_n}$ and A an operator satisfying the hypotheses of Theorem 5.238. Then $\forall\, x \in \overline{D(A)}$ one has that*

$$S(t)x = \lim_{n\to\infty}(I + \epsilon_n A)^{-\left[\frac{t}{\epsilon_n}\right]}x = \lim_{n\to\infty}(I + \epsilon_n A)^{-\left[\frac{t}{\epsilon_n}\right]-1}x,$$

uniformly on bounded intervals.

Proof: Let $x \in D(A)$. We define $k_n = \left[\frac{t}{\epsilon_n}\right]$. Observe that we can take n sufficiently large so that $0 < \frac{t}{k_n} < \lambda_0$, since

$$\epsilon_n \to 0 \Rightarrow \frac{t}{\epsilon_n} \to \infty \Rightarrow k_n \to \infty \Rightarrow \frac{t}{k_n} \to 0.$$

Therefore, it makes sense to speak of $J^{k_n}_{\frac{t}{k_n}} x$. Note that

$$\|J^{k_n}_{\epsilon_n} x - S(t)x\| \leq \|J^{k_n}_{\epsilon_n} x - J^{k_n}_{\frac{t}{k_n}} x\| + \|J^{k_n}_{\frac{t}{k_n}} x - S(t)x\|. \tag{5.9.284}$$

Since $0 \leq \frac{t}{\epsilon_n} - k_n < 1$, it follows that $0 \leq t - k_n\epsilon_n < \epsilon_n$, so $k_n\epsilon_n \to t$ when $n \to \infty$. Moreover, $t - k_n\epsilon_n \geq 0 \Rightarrow \frac{t}{k_n} \geq \epsilon_n$.

Thus, applying inequality (5.9.277) for $n = m = k_n$, $\mu = \epsilon_n$ and $\lambda = \frac{t}{k_n}$ we have

$$\begin{aligned} \|J^{k_n}_{\epsilon_n} x - J^{k_n}_{\frac{t}{k_n}} x\| &\leq e^{2|\omega|(k_n\epsilon_n + t)} \left[(k_n\epsilon_n - t)^2 + k_n\epsilon_n(\frac{t}{k_n} - \epsilon_n)\right]^{\frac{1}{2}} |Ax| \\ &+ e^{2|\omega|k_n\epsilon_n} \left[t(\frac{t}{k_n} - \epsilon_n) + (t - k_n\epsilon_n)^2\right]^{\frac{1}{2}} |Ax| \longrightarrow 0, \end{aligned} \tag{5.9.285}$$

since $k_n\epsilon_n \to t$, $\epsilon_n \to 0$ and $\frac{t}{k_n} \to 0$.

Moreover, applying Theorem 5.238, for $n = k_n$, we have

$$\|J^{k_n}_{\frac{t}{k_n}} x - S(t)x\| \longrightarrow 0. \tag{5.9.286}$$

Therefore, from (5.9.284), (5.9.285) and (5.9.286), we obtain

$$\lim_{n\to\infty} (I + \epsilon_n A)^{-\left[\frac{t}{\epsilon_n}\right]} x = S(t)x, \quad \forall\, x \in D(A).$$

Now if $x \in \overline{D(A)}$, then for each $\epsilon > 0$ there exists $y \in D(A)$, such that $\|x - y\| < \epsilon$.

Note that

$$\begin{aligned} \|J^{k_n}_{\epsilon_n} x - J^{k_n}_{\frac{t}{k_n}} x\| &\leq \|J^{k_n}_{\epsilon_n} x - J^{k_n}_{\epsilon_n} y\| + \|J^{k_n}_{\epsilon_n} y - J^{k_n}_{\frac{t}{k_n}} y\| + \|J^{k_n}_{\frac{t}{k_n}} y - J^{k_n}_{\frac{t}{k_n}} x\| \\ &\leq (1 - |\omega|\epsilon_n)^{-k_n}\|x - y\| + \|J^{k_n}_{\epsilon_n} y - J^{k_n}_{\frac{t}{k_n}} y\| + (1 - |\omega|\frac{t}{k_n})^{-k_n}\|x - y\| \\ &\leq C\epsilon + \|J^{k_n}_{\epsilon_n} y - J^{k_n}_{\frac{t}{k_n}} y\|. \end{aligned}$$

Therefore,

$$\|J^{k_n}_{\epsilon_n} x - J^{k_n}_{\frac{t}{k_n}} x\| \longrightarrow 0, \tag{5.9.287}$$

for all $x \in \overline{D(A)}$.

Since for $x \in \overline{D(A)}$ we have

$$\|J^{k_n}_{\epsilon_n} x - S(t)x\| \leq \|J^{k_n}_{\epsilon_n} x - J^{k_n}_{\frac{t}{k_n}} x\| + \|J^{k_n}_{\frac{t}{k_n}} x - S(t)x\|,$$

we obtain, by (5.9.287) and by Theorem 5.238, that

$$S(t)x = \lim_{n\to\infty} (I+\epsilon_n A)^{-\left[\frac{t}{\epsilon_n}\right]} x, \ \forall \ x \in \overline{D(A)}$$

proving the first equality.

The second equality is proved analogously by replacing k_n with $k_n + 1$. □

Remark 5.242 *More generally, if (ϵ_n) is a sequence of non-negative numbers and such that $\epsilon_n \to 0$, when $n \to \infty$, (k_n) a sequence of non-negative integers such that $k_n\epsilon_n \to t$ and A is an operator satisfying the conditions of Theorem 5.238, then*

$$S(t)x = \lim_{n\to\infty} (I+\epsilon_n A)^{-k_n} x, \ \forall \ x \in \overline{D(A)}.$$

Proposition 5.243 *Let $x \in D(A)$ and $0 \le \tau \le t$. Assuming the hypotheses of Theorem 5.238 are satisfied, we have*

$$\|S(t)x - S(\tau)x\| \le e^{\omega^+(t-\tau)} e^{\omega\tau} (t-\tau)|Ax| \tag{5.9.288}$$

where $\omega^+ = \max\{\omega, 0\}$.

Proof: Take $x \in D(A)$ and $0 \le \tau \le t$. Since $S \in Q_\omega(\overline{D(A)})$ it follows that

$$\|S(t)x - S(\tau)x\| = \|S(\tau)S(t-\tau)x - S(\tau)x\| \le e^{\omega\tau}\|S(t-\tau)x - x\| \tag{5.9.289}$$

We have two cases to consider:

a) $\omega \le 0$

In this case, we have, by the proof of item (iii) of Theorem 5.79, for n sufficiently large and $s \ge 0$

$$\begin{aligned} \|J^n_{\frac{s}{n}}x - x\| &\le \sum_{i=1}^{n} \|J^{n-i+1}_{\frac{s}{n}} - J^{n-i}_{\frac{s}{n}}\| \le \sum_{i=1}^{n} \left(1 - \frac{s}{n}\omega\right)^{-(n-i)} \|J_{\frac{s}{n}}x - x\| \\ &\le \sum_{i=1}^{n} \left(1 - \frac{s}{n}\omega\right)^{-(n-i)} \frac{s}{n} \left(1 - \frac{s}{n}\omega\right)^{-1} |Ax| \\ &= \sum_{i=1}^{n} \left(1 - \frac{s}{n}\omega\right)^{-n+i-1} \frac{s}{n}|Ax| \end{aligned} \tag{5.9.290}$$

Now, bearing in mind that $1-(s/n)\omega > 1$, since $\omega \le 0$ and n is sufficiently large, and since $-n+i-1 \le -i \le 0$, $\forall i$ such that $0 \le i \le n$, it follows from (5.9.290) that

$$\|J^n_{\frac{s}{n}}x - x\| \le \sum_{i=1}^{n} \left(1 - \frac{s}{n}\omega\right)^{-i} \frac{s}{n}|Ax| \le \sum_{i=1}^{n} \frac{s}{n}|Ax| = n\frac{s}{n}|Ax| = s|Ax|$$

From this last inequality it follows, when $n \to \infty$, by Theorem 5.238

$$\|S(s)x - x\| \le s|Ax|, \quad \forall s \ge 0$$

whence by (5.9.289) follows (5.9.288), since, in this case, $\omega^+ = 0$ and $s = t - \tau$.

b) $\omega > 0$

From item (iii) of Theorem 5.79 for n sufficiently large and $s \ge 0$, noting that $(s/n)\omega < 1$, we have

$$\|J^n_{\frac{s}{n}}x - x\| \le n\left(1 - \frac{s}{n}\omega\right)^{-n+1} \|J_{\frac{s}{n}}x - x\| \tag{5.9.291}$$

and from item (ii) of the same theorem, we obtain

$$\|J_{\frac{s}{n}}x - x\| \le \frac{s}{n}\left(1-\frac{s}{n}\omega\right)^{-1}|Ax| \tag{5.9.292}$$

From (5.9.291) and (5.9.292) it follows that

$$\begin{aligned}\|J^n_{\frac{s}{n}}x - x\| &\le n\left(1-\frac{s}{n}\omega\right)^{-n+1}\frac{s}{n}\left(1-\frac{s}{n}\omega\right)^{-1}|Ax| \\ &= s\left(1-\frac{s}{n}\omega\right)^{-n}|Ax|\end{aligned}$$

whence,

$$\|S(s)x - x\| \le se^{\omega s}|Ax|.$$

From there and from (5.9.289) follows (5.9.288), since $\omega^+ = \omega$ and writing $s = t - \tau$. □

Definition 5.244 *The semigroup associated to $A \in \mathcal{A}(\omega)$ by Theorem 5.238 will be called the semigroup generated by $-A$ and $-A$ the exponential generator of S.*

5.10 Abstract Cauchy Problem

Let X be a Banach space, $A : X \to X$ an operator and consider the following Abstract Cauchy Problem

$$\frac{du}{dt} + Au \ni 0 \tag{5.10.293}$$
$$u(0) = x \tag{5.10.294}$$

Definition 5.245 *A function $u : [0,\infty) \to X$ is called a strong solution of (5.10.293) if*

i) u is continuous on $[0,\infty)$ and Lipschitz continuous on every compact subset of $(0,\infty)$;

ii) u is differentiable at almost every point of $(0,\infty)$;

iii) $u(t) \in D(A)$ for almost every $t \in (0,\infty)$;

iv) $-\frac{du}{dt}(t) \in Au(t)$ for almost every $t \in (0,\infty)$.

If u is a strong solution of (5.10.293), then by i) and iii), $u(t) \in \overline{D(A)}$, for all $t \in [0,T]$.

Indeed, suppose by contradiction that, for some $t_0 \in (0,T)$, we have $u(t_0) \notin \overline{D(A)}$. Then, there exists $r > 0$ such that $B(u(t_0), r) \cap D(A) = \emptyset$. Since u is continuous, there exists $\delta > 0$ such that $u(t_0-\delta, t_0+\delta) \subset B(u(t_0), r)$, that is, $u(t_0-\delta, t_0+\delta) \cap D(A) = \emptyset$. Which contradicts iii). If $t_0 \in \{0,T\}$ then $t_0 = \lim t_n$ with $t_n \in (0,T)$, for each $n \in \mathbb{N}$. Hence, $u(t_n) \in \overline{D(A)}$. Since $u(t_n) \to u(t_0)$, we have that $u(t_0) \in \overline{D(A)}$.

In particular, if u satisfies (5.10.294), then $x \in \overline{D(A)}$.

Lemma 5.246 *Let u be a function defined on an interval and taking values in a Banach space X. Suppose that u has a weak derivative, $u'(t)$ at the point t (i.e., that the derivative $\frac{d}{dt}\langle u(t), x^*\rangle$ exists at the point t and $\frac{d}{dt}\langle u(t), x^*\rangle = \langle u'(t), x^*\rangle$ for each $x^* \in X'$) and that the function $\|u(t)\|$ is differentiable at the point t. Then,*

$$\|u(t)\|\frac{d}{dt}\|u(t)\| = \langle u'(t), u^*\rangle \quad \forall u^* \in F(u(t)).$$

Proof: Note that $F(u(t)) = \{u^* \in X', \langle u^*, u(t)\rangle = \|u(t)\|^2 = \|u^*\|^2\}$. For all $u^* \in F(u(t))$, we have

$$\begin{aligned}\big\langle u(t+h), u^*\big\rangle - \big\langle u(t), u^*\big\rangle &\leq \|u(t+h)\|\|u^*\| - \|u(t)\|^2 = \\ &= \|u(t+h)\|\|u(t)\| - \|u(t)\|^2 = (\|u(t+h)\| - \|u(t)\|)\|u(t)\|.\end{aligned}$$

Thus, if $h > 0$ we have, dividing by h and passing to the limit as $h \to 0$,

$$\big\langle u'(t), x^*\big\rangle \leq \|u(t)\|\frac{d}{dt}\|u(t)\|$$

and, if $h < 0$,

$$\big\langle u'(t), x^*\big\rangle \geq \|u(t)\|\frac{d}{dt}\|u(t)\|$$

whence we conclude the desired result. □

Proposition 5.247 *Let $A : X \to X$, $A \in \mathcal{A}(\omega)$ and u and v be strong solutions of (5.10.293) with $u(0) = x$ and $v(0) = y$. Then*

$$\|u(t) - v(t)\| \leq e^{\omega t}\|x - y\|; \quad \forall t \in [0, \infty). \tag{5.10.295}$$

Proof: By definition, we have

$$-\frac{du}{dt} \in Au \quad \text{and} \quad -\frac{dv}{dt} \in Av \quad \text{almost everywhere in } (0, \infty),$$

whence, by the accretivity of $A + \omega I$, and from what was observed above,

$$\left(u(t), -\frac{du}{dt}(t) + \omega u(t)\right) \in A + \omega I \quad \text{and} \quad \left(v(t), -\frac{dv}{dt}(t) + \omega v(t)\right) \in A + \omega I,$$

by corollary 5.69, there exists $u^* \in F(u(t) - v(t))$ such that

$$\left\langle -\frac{du}{dt}(t) + \omega u(t) + \frac{dv}{dt}(t) - \omega v(t), u^*\right\rangle \geq 0 \quad \text{a.e. in } (0, \infty).$$

Thus, it follows

$$\begin{aligned}\left\langle \frac{du}{dt}(t) - \frac{dv}{dt}(t), u^*\right\rangle &\leq \omega\big\langle u(t) - v(t), u^*\big\rangle \\ &\leq \omega\|u(t) - v(t)\|\|u^*\| \\ &= \omega\|u(t) - v(t)\|^2 \quad \text{a.e. in } (0, \infty),\end{aligned}$$

that is,

$$\left\langle \frac{d(u - v)}{dt}(t), u^*\right\rangle \leq \omega\|u(t) - v(t)\|^2 \quad \text{a.e. in } (0, \infty) \tag{5.10.296}$$

and since by hypothesis u and v are Lipschitz continuous on each compact subset of $(0, \infty)$, it follows that $t \mapsto u(t) - v(t)$ is Lipschitz continuous on each compact subset of $(0, \infty)$ and consequently is absolutely continuous. Applying Lemma 5.246 on the left side of inequality (5.10.296) we obtain

$$\|u(t) - v(t)\|\frac{d}{dt}\|u(t) - v(t)\| \leq \omega\|u(t) - v(t)\|^2.$$

If $\|u(t) - v(t)\| \neq 0$, it follows that

$$\frac{d}{dt}\|u(t) - v(t)\| \leq \omega\|u(t) - v(t)\| \quad \text{a.e. in } (0, \infty). \tag{5.10.297}$$

Multiplying (5.10.297) by $e^{-\omega t}$, we obtain

$$\begin{aligned} & e^{-\omega t}\frac{d}{dt}\|u(t)-v(t)\| - \omega e^{-\omega t}\|u(t)-v(t)\| \leq 0 \quad \text{a.e. in } (0,\infty) \\ \Rightarrow \quad & \frac{d}{dt}(e^{-\omega t}\|u(t)-v(t)\|) \leq 0 \quad \text{a.e. in } (0,\infty). \end{aligned}$$

Integrating from 0 to t, since $e^{-\omega t}\|u(t)-v(t)\|$ is absolutely continuous, we have

$$e^{-\omega t}\|u(t)-v(t)\| - e^{-\omega 0}\|u(0)-v(0)\| \leq 0, \quad \forall t \in [0,\infty).$$

Therefore

$$\|u(t)-v(t)\| \leq e^{\omega t}\|x-y\|, \quad \forall t \in [0,\infty).$$

□

Corollary 5.248 *Let $A \in \mathcal{A}(\omega)$. Then, (5.10.293) has at most one strong solution satisfying (5.10.294).*

Proof: Immediate consequence of Proposition 5.247 □

Lemma 5.249 *Let $A \in \mathcal{A}(\omega)$, u be a strong solution of (5.10.293) and $h > 0$. Then the function*

$$\varphi(t) = e^{-\omega t}\|u(t+h)-u(t)\|$$

is monotone.

Proof: Since $u(t)$ is a strong solution of (5.10.293) then $v(t) := u(t+h)$ also satisfies each of the items of Definition 5.245 and therefore $v(t)$ is a strong solution of (5.10.293) with initial value $v(0) = u(h)$. Proceeding analogously to proposition 5.247, using lemma 5.246, we obtain

$$\frac{d}{dt}\|v(t)-u(t)\| \leq \omega\|v(t)-u(t)\|.$$

Multiplying by $e^{-\omega t}$ it follows that

$$\frac{d}{dt}\left[e^{-\omega t}\|v(t)-u(t)\|\right] \leq 0 \quad \text{a.e. in} \quad [0,\infty),$$

and from the definition of v and φ we have

$$\frac{d}{dt}\left[\varphi(t)\right] = \frac{d}{dt}\left[e^{-\omega t}\|u(t+h)-u(t)\|\right] \leq 0 \quad \text{a.e. in} \quad [0,\infty).$$

Since $\varphi(t) = e^{-\omega t}\|u(t+h)-u(t)\|$ is absolutely continuous, we can integrate from t_1 to t_2, $t_1 \leq t_2$ and obtain

$$\varphi(t_2) - \varphi(t_1) = \int_{t_1}^{t_2} \frac{d}{dt}\varphi(t) \quad dt \leq 0,$$

that is,

$$\varphi(t_2) \leq \varphi(t_1),$$

i.e., φ is monotonically decreasing. □

Theorem 5.250 *Let $A \in \mathcal{A}(\omega)$, u be a strong solution of the abstract Cauchy problem (5.10.293)-(5.10.294). Then:*

(i) $\|u(t)-u(s)\| \leq e^{\omega^+(t-s)}(t-s)\,|Au(s)|$, for almost every $t \in (0,\infty)$ and every s such that $u(s) \in D(A)$, $0 \leq s \leq t$.

(ii) $\left\| \dfrac{d}{dt}u(t) \right\| = |Au(t)|$ *almost everywhere in* $(0,\infty)$.

(iii) The function $e^{-\omega t}|Au(t)|$ *is monotonically decreasing.*

Proof: Let Ω be the set of points $t \in (0,\infty)$ such that $u(t) \in D(A)$, u is differentiable at the point t and $\dfrac{d}{dt}u(t) + Au(t) \ni 0$. Since u is a strong solution of (5.10.293)-(5.10.294), it follows that $(0,\infty) \setminus \Omega$ has measure zero.

(i) Let $t \in \Omega$. Then $-\dfrac{d}{dt}u(t) \in Au(t)$. Let us fix s such that $u(s) \in D(A)$, $0 \le s \le t$ and take $y \in Au(s)$. Since $A + \omega I$ is accretive, there exists $u^* \in F(u(t) - u(s))$ such that

$$\left\langle u^*, -\frac{d}{dt}u(t) + \omega u(t) - y - \omega u(s) \right\rangle \ge 0.$$

From this it follows that

$$\begin{aligned} \left\langle u^*, \frac{d}{dt}u(t) \right\rangle &\le \omega \langle u^*, u(t) - u(s)\rangle - \langle u^*, y\rangle \\ &\le \omega \|u(t) - u(s)\|^2 + \|u(t) - u(s)\| \, \|y\|. \end{aligned}$$

Bearing in mind that $u(s)$ does not depend on t, from Lemma 5.246 we have

$$\begin{aligned} \|u(t) - u(s)\| \frac{d}{dt}\|u(t) - u(s)\| &= \left\langle u^*, \frac{d}{dt}(u(t) - u(s)) \right\rangle \\ &= \left\langle u^*, \frac{d}{dt}u(t) \right\rangle \\ &\le \omega \|u(t) - u(s)\|^2 + \|u(t) - u(s)\| \, \|y\|, \end{aligned}$$

that is,

$$\frac{d}{dt}\|u(t) - u(s)\| \le \|y\| + \omega\|u(t) - u(s)\| \le \|y\| + \omega^+\|u(t) - u(s)\|, \tag{5.10.298}$$

where $\omega^+ = \max\{0, \omega\}$. Therefore

$$\frac{d}{dt}\|u(t) - u(s)\| \le \|y\| + \omega^+\|u(t) - u(s)\|.$$

Multiplying by $e^{-\omega^+(t-s)}$, we have

$$\frac{d}{dt}\left[e^{-\omega^+(t-s)}\|u(t) - u(s)\|\right] \le e^{-\omega^+(t-s)}\|y\|. \tag{5.10.299}$$

Let us consider two cases:

Case I ($\omega^+ = 0$): From (5.10.298) we have that

$$\frac{d}{dt}\|u(t) - u(s)\| \le \|y\|, \quad \text{for all } y \in A(u(s)),$$

and integrating from s to t,

$$\|u(t) - u(s)\| \le (t - s)\|y\|.$$

Taking the infimum over $y \in Au(s)$, we have

$$\|u(t) - u(s)\| \le (t - s)\,|Au(s)|\,.$$

Case II ($\omega^+ > 0$): Integrating (5.10.299) from s to t, we have

$$\int_s^t \frac{d}{d\tau}\left[e^{-\omega^+(\tau-s)}\|u(\tau)-u(s)\|\right]\, d\tau \leq \int_s^t e^{-\omega^+(\tau-s)}\|y\|\, d\tau.$$

And since $e^{-\omega^+(\tau-s)}\|u(\tau)-u(s)\|$ is absolutely continuous in τ, it follows that

$$\begin{aligned} e^{-\omega^+(t-s)}\|u(t)-u(s)\| &\leq e^{\omega^+ s}\|y\| \int_s^t e^{-\omega^+\tau}\, d\tau \\ &= \frac{e^{\omega^+ s}}{-\omega^+}\|y\| \left[e^{-\omega^+ t} - e^{-\omega^+ s}\right] \\ &= \frac{1-e^{-\omega^+(t-s)}}{\omega^+}\|y\|. \end{aligned}$$

Whence,

$$\|u(t)-u(s)\| \leq \frac{e^{\omega^+(t-s)}-1}{\omega^+}\|y\|.$$

Considering that $e^x - 1 \leq xe^x$ for all $x \geq 0$, we obtain

$$\|u(t)-u(s)\| \leq (t-s)e^{\omega^+(t-s)}\|y\|.$$

Since this relation holds for all $y \in Au(s)$, we can take the infimum over $y \in Au(s)$ and conclude that

$$\|u(t)-u(s)\| \leq (t-s)e^{\omega^+(t-s)}\left|Au(s)\right|,$$

for all $t \in \Omega$ and s such that $u(s) \in D(A)$ and $0 \leq s \leq t$.

(ii) Let $s, t \in \Omega$, $0 \leq t \leq s$. By i) we have:

$$\begin{aligned} \left\|\frac{d}{dt}u(t)\right\| &= \left\|\lim_{s\to t}\frac{u(s)-u(t)}{s-t}\right\| \\ &= \lim_{s\to t}\frac{\|u(s)-u(t)\|}{s-t} \\ &\overset{i)}{\leq} \lim_{s\to t}\frac{e^{\omega^+(s-t)}(s-t)\left|Au(s)\right|}{s-t} \\ &= \left|Au(t)\right|. \end{aligned}$$

On the other hand, since $-\dfrac{d}{dt}u(t) \in Au(t)$, it follows that $|Au(t)| \leq \left\|\dfrac{d}{dt}u(t)\right\|$ and therefore,

$$|Au(t)| = \left\|\frac{d}{dt}u(t)\right\|.$$

(iii) By Lemma 5.249, for $h > 0$ and $0 \leq s \leq t$, we have

$$e^{-\omega t}\|u(t+h)-u(t)\| = \varphi(t) \leq \varphi(s) = e^{-\omega s}\|u(s+h)-u(s)\|. \tag{5.10.300}$$

Assuming $s, t \in \Omega$, then

$$u(s), u(t) \in D(A),\ -\frac{d}{dt}u(t) \in Au(t), -\frac{d}{ds}u(s) \in Au(s).$$

Dividing (5.10.300) by h and taking the limit as $h \longrightarrow 0$, we obtain

$$e^{-\omega t}\left\|\frac{d}{dt}u(t)\right\| \leq e^{-\omega s}\left\|\frac{d}{ds}u(s)\right\|.$$

From item ii) it follows that

$$e^{-\omega t}\left|Au(t)\right| \leq e^{-\omega s}\left|Au(s)\right|, \quad 0 \leq s \leq t,$$

that is, the function $e^{-\omega t}\left|Au(t)\right|$ is monotonically decreasing.

□

Corollary 5.251 *Let X be a reflexive, strictly convex and smooth space and $A+\omega I$ be an accretive and maximal operator in C, $D(A) \subset C$. If u is a strong solution of (5.10.293), then $\overset{\circ}{A}\, u(t)$ has a unique element and*

$$\frac{d}{dt}u(t) + \overset{\circ}{A}\, u(t) = 0 \quad a.e.\ in \quad (0,\infty).$$

Proof: Since X is smooth, $A \in \mathcal{A}(\omega)$, $D(A) \subset C$ and $A+\omega I$ is maximal in C then, by Proposition 5.94, $Au(t)$ is convex and closed for all t such that $u(t) \in D(A)$.
Since X is reflexive and strictly convex, $Au(t) \subset X$, $Au(t) \neq \emptyset$ for each t such that $u(t) \in D(A)$, and $Au(t)$ is convex and closed, by Theorem 5.102, we have that $(Au(t))^{\circ}$ has a unique element.
By item ii) of Theorem 5.250 we know that $|Au(t)| = \left\|\frac{d}{dt}u(t)\right\|$. It follows that $-\frac{d}{dt}u(t)$ is the unique element of $(Au(t))^{\circ}$, and thus, from definition 5.104 we have

$$-\frac{d}{dt}u(t) = (Au(t))^{\circ} = \overset{\circ}{A}\, u(t),$$

or even,

$$\frac{d}{dt}u(t) + \overset{\circ}{A}\, u(t) = 0 \quad \text{a.e. in} \quad (0,\infty).$$

□

Definition 5.252 *Let π be the partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0,T]$. The scheme*

$$\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + Ax_i \ni 0, \quad i = 1,\cdots,N, \tag{5.10.301}$$

is called discretization of the equation (5.10.293).

If $\max\limits_{1\leq i\leq N}\{t_i - t_{i-1}\} \leq \varepsilon$, (5.10.301) will be called ε-discretization of (5.10.293) on $[0,T]$.

If the sequence $x_0, x_1, \cdots, x_N$ satisfies (5.10.301), the function u_π, defined by

$$u_\pi(0) = x_0 \quad and \quad u_\pi(t) = x_i \ if\ t \in (t_{i-1}, t_i],$$

is called a solution of (5.10.293) on $[0,T]$ with initial value x_0.

If (5.10.301) is an ε-discretization, u_π will be called ε-approximate solution of (5.10.293) with initial value x_0.

If u_π is an ε-approximate solution of (5.10.293) on $[0,T]$ with initial value x_0 and $\|x - x_0\| \leq \varepsilon$, u_π will be called ε-approximate solution of the problem (5.10.293)-(5.10.294) on $[0,T]$.

Proposition 5.253 *Let A be an operator under the conditions of Theorem 5.238, that is,*

$$A \in \mathcal{A}(\omega) \ such\ that\ \overline{D(A)} \subset Im(I+\lambda A),\ 0 < \lambda < \lambda_0 \ with\ \lambda_0|\omega| < 1.$$

Then, for each partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0,T]$ such that $t_i - t_{i-1} < \lambda_0$ and for each $x_0 \in \overline{D(A)}$, the discretization (5.10.301) admits a unique solution with initial value x_0.

Proof: For $i = 1$, the discretization (5.10.301) with initial value x_0 is equivalent to

$$0 \in \frac{x_1 - x_0}{t_1} + Ax_1$$

and therefore,

$$0 \in x_1 - x_0 + t_1 A x_1,$$

or equivalently

$$x_0 \in x_1 + t_1 A x_1.$$

That is,

$$x_0 \in (I + t_1 A)x_1. \tag{5.10.302}$$

Since $A \in \mathcal{A}(\omega)$ and $0 < t_1 < \lambda_0$, we have that $0 < t_1|\omega| < \lambda_0|\omega| < 1$, and by Theorem 5.79, J_{t_1} is single-valued. Moreover, from (5.10.302) it follows that

$$J_{t_1} x_0 = x_1, \quad x_0 \in \overline{D(A)} \subset \mathrm{Im}(I + t_1 A) = D_{t_1}, \tag{5.10.303}$$

that is, there exists a unique $x_1 \in \mathrm{Im}(J_{t_1}) = D(A)$ such that

$$x_0 \in (I + t_1 A)x_1 \quad \text{and} \quad (I + t_1 A)^{-1} x_0 = J_{t_1} x_0 = x_1.$$

By recurrence, it is seen that, for each $i = 1, 2, \ldots, N$ there exists a unique $x_i \in D(A)$ that satisfies (5.10.301), or equivalently

$$x_i = (I + (t_i - t_{i-1})A)^{-1} x_{i-1}, \quad 0 < t_i - t_{i-1} < \lambda_0, \tag{5.10.304}$$

or even,

$$x_i = (I + (t_i - t_{i-1})A)^{-1} (I + (t_{i-1} - t_{i-2})A)^{-1} \cdots (I + t_1 A)^{-1} x_0 \tag{5.10.305}$$

and thus the sequence $\{x_i\}$, $i = 1, 2, \ldots, N$ defines a solution for (5.10.301) with initial value x_0.

For uniqueness, suppose there exists another solution of (5.10.301) with initial value x_0 given by a sequence $\{y_i\}$. Then y_1 satisfies

$$0 \in \frac{y_1 - x_0}{t_1} + Ay_1$$

and, analogously to what was done for x_1, we have $J_{t_1} x_0 = y_1$. Considering (5.10.303) and the fact that J_{t_1} is single-valued, it follows that $x_1 = y_1$. Recursively we have that $x_i = y_i$ for all $i = 1, 2, \ldots, N$, thus guaranteeing the uniqueness of the solution. □

Proposition 5.254 *Let $A \in \mathcal{A}(\omega)$ such that $\overline{D(A)} \subseteq Im(I + \lambda A)$, for $0 < \lambda < \lambda_0$, $\lambda_0|\omega| < 1$. If u_{ε_N} is the solution of the ε_N-discretization (5.10.301) in the form*

$$0 = t_0 < t_1 = \frac{T}{N} < \cdots < t_{N-1} = \frac{(N-1)}{N} T < t_N = T,$$

with initial value $x_0 \in \overline{D(A)}$, then

$$\lim_{N \to \infty} u_{\varepsilon_N}(t) = S(t)x_0, \tag{5.10.306}$$

uniformly on $[0, T]$, where $S(t)$ is the semigroup generated by $-A$.

Proof: Let $N \in \mathbb{N}$ such that $\varepsilon_N = \dfrac{T}{N} < \lambda_0$. According to Proposition 5.253, the sequence

$$\begin{cases} x_0 \\ x_i = (I + \varepsilon_N A)^{-i} x_0 \end{cases}$$

defines the unique solution of the discretization

$$\frac{x_i - x_{i-1}}{\varepsilon_N} + Ax_i \ni 0,\ i = 1, \ldots, N.$$

If $t_{i-1} < t \leq t_i$, then $\frac{T(i-1)}{N} < t \leq \frac{Ti}{N}$ and consequently $i - 1 < \frac{t}{\varepsilon_N} \leq i$. Thus by (5.10.305),

$$u_{\varepsilon_N}(t) = x_i = (I + \varepsilon_N A)^{-i} x_0 = \begin{cases} (I + \varepsilon_N A)^{-\left[\frac{t}{\varepsilon_N}\right]-1}, \text{ if } \frac{(i-1)T}{N} < t < \frac{iT}{N} \\ (I + \varepsilon_N A)^{-\left[\frac{t}{\varepsilon_N}\right]}, \text{ if } t = \frac{iT}{N} \end{cases}$$

In both cases, by Theorem 5.241, we have

$$\lim_{N\to\infty} u_{\varepsilon_N}(t) = S(t)x_0.$$

□

Theorem 5.255 *Let $A \in \mathcal{A}(\omega)$ such that $\overline{D(A)} \subseteq Im(I + \lambda A)$, for $0 < \lambda < \lambda_0$, $\lambda_0|\omega| < 1$. Take $x \in \overline{D(A)}$ and let u be a strong solution of (5.10.293) with $u(0) = x$. Then $u(t) = S(t)x$, $t \in [0, \infty)$, where S is the semigroup generated by $-A$.*

Proof: Suppose initially that $x \in D(A)$. Consider $s > 0$ and $N \in \mathbb{N}$ such that $\varepsilon = \frac{s}{N} \leq \lambda_0$. Let us extend the strong solution u to negative values so that it remains continuous: $u(t) = x$, for $t < 0$. We can then define the function

$$g_\varepsilon(t) = \frac{u(t) - u(t-\varepsilon)}{\varepsilon} - \frac{du}{dt}(t),$$

so that g_ε is defined almost everywhere in $(0, s)$.

Since $-\frac{du}{dt}(t) \in Au(t)$ almost everywhere in $(0, s)$, or even, $\left(u(t), -\frac{du}{dt}(t)\right) \in A$, it follows that

$$\left(u(t), -\varepsilon\frac{du}{dt}(t)\right) \in \varepsilon A.$$

From the definition of g_ε, we have $(u(t), \varepsilon g_\varepsilon(t) + u(t-\varepsilon) - u(t)) \in \varepsilon A$ whence

$$(u(t), \varepsilon g_\varepsilon(t) + u(t-\varepsilon)) \in (I + \varepsilon A).$$

Since $\varepsilon \leq \lambda_0$, we have that $u(t) = (I + \varepsilon A)^{-1}(\varepsilon g_\varepsilon(t) + u(t-\varepsilon))$ almost everywhere in $(0, s)$. Now let us set $u_\varepsilon(t) = x$ for $t \leq 0$.

By (5.10.304),

$$u_\varepsilon(t) = (I + \varepsilon A)^{-1} u_\varepsilon(t-\varepsilon),\ t \geq 0.$$

Thus,

$$\begin{aligned} \|u_\varepsilon(t) - u(t)\| &= \|J_\varepsilon u_\varepsilon(t-\varepsilon) - J_\varepsilon(u(t-\varepsilon) + \varepsilon g_\varepsilon(t))\| \\ &\leq (1 - \varepsilon\omega)^{-1}\|u_\varepsilon(t-\varepsilon) - u(t-\varepsilon) - \varepsilon g_\varepsilon(t)\| \\ &\leq (1 - \varepsilon|\omega|)^{-1}(\|u_\varepsilon(t-\varepsilon) - u(t-\varepsilon)\| + \varepsilon\|g_\varepsilon(t)\|), \end{aligned}$$

almost everywhere in $(0, s)$.

Therefore,

$$\begin{aligned}\int_0^s \|u_\varepsilon(t)-u(t)\|dt &\leq (1-\varepsilon|\omega|)^{-1}\int_0^s \|u_\varepsilon(t-\varepsilon)-u(t-\varepsilon)\|dt \\ &+ \varepsilon(1-\varepsilon|\omega|)^{-1}\int_0^s \|g_\varepsilon(t)\|dt,\end{aligned}$$

whence

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{s-\varepsilon}^s \|u_\varepsilon(t)-u(t)\|dt \leq \frac{(1-\varepsilon|\omega|)^{-1}}{\varepsilon}\int_0^s \|u_\varepsilon(t-\varepsilon)-u(t-\varepsilon)\|dt \\ &-\frac{1}{\varepsilon}\int_0^{s-\varepsilon} \|u_\varepsilon(t)-u(t)\|dt + (1-\varepsilon|\omega|)^{-1}\int_0^s \|g_\varepsilon(t)\|dt \\ &= \frac{(1-\varepsilon|\omega|)^{-1}-1}{\varepsilon}\int_0^{s-\varepsilon} \|u_\varepsilon(t)-u(t)\|dt + (1-\varepsilon|\omega|)^{-1}\int_0^s \|g_\varepsilon(t)\|dt.\end{aligned} \tag{5.10.307}$$

If $\varepsilon < t < s$, from Theorem 5.250, items (i) and (iii),

$$\begin{aligned}\|u(t)-u(t-\varepsilon)\| &\leq e^{\omega^+\varepsilon}\varepsilon|Au(t-\varepsilon)| \\ &\leq e^{\omega^+\varepsilon}\varepsilon e^{\omega(t-\varepsilon)}|Ax| \\ &\leq e^{\omega^+\varepsilon+\omega(t-\varepsilon)}\varepsilon|Ax| \\ &\leq e^{\omega^+ s}\varepsilon|Ax|,\end{aligned}$$

almost everywhere in (ε, s).

If $0 < t < \varepsilon$,

$$\|u(t)-u(t-\varepsilon)\| = \|u(t)-x\| \leq e^{\omega^+ t}t|Ax| \leq e^{\omega^+ s}\varepsilon|Ax|,$$

almost everywhere in $(0, \varepsilon)$.

Thus,

$$\|u(t)-u(t-\varepsilon)\| \leq e^{\omega^+ s}\varepsilon|Ax| \text{ almost everywhere in } (0,s).$$

Moreover,

$$\left\|\frac{du}{dt}(t)\right\| = |Au(t)| \leq e^{\omega t}|Ax| \leq e^{\omega^+ s}|Ax|$$

almost everywhere in $(0, T)$.

Consequently,

$$\begin{aligned}\|g_\varepsilon(t)\| &\leq \varepsilon^{-1}\|u(t)-u(t-\varepsilon)\| + \left\|\frac{du}{dt}(t)\right\| \\ &\leq 2e^{\omega^+ s}|Ax|.\end{aligned}$$

Since $g_\varepsilon(t) \to 0$ almost everywhere in $(0, s)$, as u is a strong solution of (5.10.293), by the dominated convergence theorem

$$\lim_{\varepsilon\to 0}\int_0^s \|g_\varepsilon(t)\|dt = 0. \tag{5.10.308}$$

We have that

$$\lim_{\varepsilon\to 0}\frac{(1-\varepsilon|\omega|)^{-1}-1}{\varepsilon} = |\omega| \tag{5.10.309}$$

and, from Proposition 5.254,

$$\lim_{\varepsilon\to 0}\int_0^{s-\varepsilon}\|u_\varepsilon(t)-u(t)\|dt=\int_0^s\|S(t)x-u(t)\|dt. \tag{5.10.310}$$

Combining (5.10.308), (5.10.309) and (5.10.310), we obtain

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u_\varepsilon(t)-u(t)\|dt\le|\omega|\int_0^s\|S(t)x-u(t)\|dt. \tag{5.10.311}$$

Let us calculate the limit on the left side of (5.10.311):

$$\begin{aligned}&\left|\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u_\varepsilon(t)-u(t)\|dt-\|S(s)x-u(s)\|\right|\\ &=\left|\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u_\varepsilon(t)-u(t)\|dt-\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|S(s)x-u(s)\|dt\right|\\ &\le\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u_\varepsilon(t)-S(s)x\|dt+\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u(t)-u(s)\|dt.\end{aligned}$$

Let $s_1\in(s-\varepsilon,s]$ be a Lebesgue point of both $u(t)$ and $S(t)x$. Then

$$\begin{aligned}&\left|\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u_\varepsilon(t)-u(t)\|dt-\|S(s)x-u(s)\|\right|\\ &\le\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u_\varepsilon(t)-S(s_1)x\|dt+\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|S(s_1)x-S(s)x\|dt\\ &+\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u(t)-u(s_1)\|dt+\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u(s_1)-u(s)\|dt\\ &\le\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|u_\varepsilon(t)-S(t)x\|dt+\frac{1}{\varepsilon}\int_{s-\varepsilon}^s\|S(t)x-S(s)x\|dt\\ &+\varepsilon\|x\|+\int_{s-\varepsilon}^s\|u(t)-u(s_1)\|dt+\varepsilon\longrightarrow 0,\ \text{when } \varepsilon\to 0,\end{aligned}$$

since $u_\varepsilon(t)\to S(t)x$ uniformly on bounded intervals. Thus, from (5.10.311), it follows that

$$\|S(s)x-u(s)\|\le|\omega|\int_0^s\|S(t)x-u(t)\|dt.$$

Consider $\varphi(t)=\|S(t)x-u(t)\|$. Since $s>0$ is arbitrary, we have

$$\varphi(t)\le|\omega|\int_0^t\varphi(\tau)d\tau.$$

By Gronwall's lemma, $\varphi(t)\equiv 0$.

Let now $x\in\overline{D(A)}$. Since u is a strong solution, we have that $u(t)\in D(A)$ almost everywhere. Choose a sequence $\{\varepsilon_n\}$ converging to zero, such that $u(\varepsilon_n)\in D(A)$ for all n. The function

$$v_n(t):=u_n(t+\varepsilon_n)$$

is a strong solution of the equation

$$\begin{cases} \dfrac{dv_n}{dt}(t) + Av_n(t) \ni 0 \\ v_n(0) = u(\varepsilon_n). \end{cases}$$

By what we have already demonstrated, $S(t)u(\varepsilon_n) = v_n(t)$. Thus, in the limit $S(t)x = u(t)$. □

Our interest now is to prove the converse of the previous theorem. For this, we will need the following lemma.

Lemma 5.256 *Let $A \in \mathcal{A}(\omega)$ such that $\overline{D(A)} \subset Im(I + \lambda A)$, for $0 < \lambda < \lambda_0$, $\lambda_0|\omega| < 1$ and S the semigroup generated by $-A$. If $x \in \overline{D(A)}$ and $(x_0, y_0) \in A$ then*

$$\sup_{\xi' \in F(x-x_0)} \limsup_{t\to 0} \left\langle \frac{S(t)x - x}{t} + \omega(x_0 - x), \xi' \right\rangle \leq \langle y_0, x_0 - x\rangle_s$$

Proof: Let us denote by $\left[\frac{t}{\lambda}\right]$ the integer part of $\frac{t}{\lambda}$, $t \geq 0$ and $\lambda > 0$. By (ii) and (iii) of Theorem 5.79, for λ sufficiently small we have that

$$\begin{aligned} \|J_\lambda^{\left[\frac{t}{\lambda}\right]} x_0 - x_0\| &\leq \left[\frac{t}{\lambda}\right](1 - \lambda|\omega|)^{-\left[\frac{t}{\lambda}\right]+1}\|J_\lambda x_0 - x_0\| \\ &\leq \left[\frac{t}{\lambda}\right](1 - \lambda|\omega|)^{-\left[\frac{t}{\lambda}\right]+1}\lambda(1-\lambda\omega)^{-1}|Ax_0| \\ &\leq t(1 - \lambda|\omega|)^{-\left[\frac{t}{\lambda}\right]}|Ax_0| \end{aligned}$$

By hypothesis, $x \in \overline{D(A)} \subset D_\lambda$, $0 < \lambda < \lambda_0$. According to item (i) of Theorem 5.79, using the previous inequality we have, for λ sufficiently small, that

$$\begin{aligned} \|J_\lambda^{\left[\frac{t}{\lambda}\right]} x - x_0\| &\leq \|J_\lambda^{\left[\frac{t}{\lambda}\right]} x - J_\lambda^{\left[\frac{t}{\lambda}\right]} x_0\| + \|J_\lambda^{\left[\frac{t}{\lambda}\right]} x_0 - x_0\| \\ &\leq (1-\lambda\omega)^{-\left[\frac{t}{\lambda}\right]}\|x - x_0\| + t(1-\lambda|\omega|)^{-\left[\frac{t}{\lambda}\right]}|Ax_0| \end{aligned} \tag{5.10.312}$$

For each $\lambda > 0$ and $k \in \mathbb{N}$, define

$$y_{\lambda,k} = \frac{1}{\lambda}(J_\lambda^{k-1}x - J_\lambda^k x).$$

We have,

$$y_{\lambda,k} = \frac{1}{\lambda}(I - J_\lambda)(J_\lambda^{k-1}x) = A_\lambda J_\lambda^{k-1}x \in AJ_\lambda^k x,$$

that is, $(J_\lambda^k x, A_\lambda J_\lambda^{k-1}x) \in A$.

Since $A \in \mathcal{A}(\omega)$, it follows by Proposition 5.77 that there exists $\eta' \in F(x_0 - J_\lambda^k x)$ such that

$$\langle \eta', y_0 - y_{\lambda,k}\rangle + \omega\langle \eta', x_0 - J_\lambda^k x\rangle \geq 0,$$

$$\langle \eta', y_0 - y_{\lambda,k}\rangle + \omega\|x_0 - J_\lambda^k x\|^2 \geq 0,$$

thus

$$\langle \eta', y_{\lambda,k}\rangle \leq \langle \eta', y_0\rangle + \omega\|x_0 - J_\lambda^k x\|^2. \tag{5.10.313}$$

But,

$$\begin{aligned}\langle \eta', y_{\lambda,k}\rangle &= \lambda^{-1}\langle \eta', J_\lambda^{k-1}x - J_\lambda^k x\rangle \\ &= \lambda^{-1}\langle \eta', (x_0 - J_\lambda^k x) - (x_0 - J_\lambda^{k-1}x\rangle \\ &\geq \lambda^{-1}\left(\|x_0 - J_\lambda^k x\|^2 - \|x_0 - J_\lambda^{k-1}x\|\|x_0 - J_\lambda^k x\|\right) \\ &\geq (2\lambda)^{-1}(\|x_0 - J_\lambda^k x\|^2 - \|x_0 - J_\lambda^{k-1}x\|^2)\end{aligned}$$

Therefore, by (5.10.313) it follows that

$$\begin{aligned}\|x_0 - J_\lambda^k x\|^2 - \|x_0 - J_\lambda^{k-1}x\|^2 &\leq 2\lambda\langle y_0, \eta'\rangle + 2\lambda\omega\|x_0 - J_\lambda^k x\|^2 \\ &\leq 2\lambda\langle y_0, x_0 - J_\lambda^k x\rangle_s + 2\lambda\omega\|x_0 - J_\lambda^k .x\|^2.\end{aligned}$$

For $\tau \in [k\lambda, (k+1)\lambda)$ we have that $k \leq \frac{\tau}{\lambda} < k+1$, so

$$\left[\frac{\tau}{\lambda}\right] = k \Rightarrow J_\lambda^{\left[\frac{\tau}{\lambda}\right]}x = J_\lambda^k x.$$

Thus by the previous inequality we obtain

$$\begin{aligned}\|x_0 - J_\lambda^k x\|^2 - \|x_0 - J_\lambda^{k-1}x\|^2 &\leq 2\int_{\lambda k}^{(k+1)\lambda}\langle y_0, x_0 - J_\lambda^k x\rangle_s d\tau + 2\lambda\omega\|x_0 - J_\lambda^k x\|^2 \\ &= 2\int_{\lambda k}^{(k+1)\lambda}\langle y_0, x_0 - J_\lambda^{\left[\frac{\tau}{\lambda}\right]}x\rangle_s d\tau + 2\lambda\omega\|x_0 - J_\lambda^k x\|^2\end{aligned} \tag{5.10.314}$$

Let $t \geq \lambda$. Summing (5.10.314) from $k=1$ to $k = \left[\frac{t}{\lambda}\right]$, it follows that

$$\|x_0 - J_\lambda^{\left[\frac{t}{\lambda}\right]}x\|^2 - \|x_0 - x\|^2 \leq 2\int_\lambda^{(\left[\frac{t}{\lambda}\right]+1)\lambda}\langle y_0, x_0 - J_\lambda^{\left[\frac{\tau}{\lambda}\right]}x\rangle_s d\tau + 2\lambda\omega\sum_{k=1}^{\left[\frac{t}{\lambda}\right]}\|x_0 - J_\lambda^k x\|^2 \tag{5.10.315}$$

We have that $\langle \cdot, \cdot\rangle_s$ is upper semicontinuous (u.s.c.). So defining

$$f(\lambda) = \begin{cases} \langle y_0, x_0 - J_\lambda^{\left[\frac{\tau}{\lambda}\right]}x\rangle_s, & \text{if } \lambda > 0 \\ \langle y_0, x_0 - S(\tau)x\rangle_s, & \text{if } \lambda = 0\end{cases}$$

and,

$$g(\lambda) = \begin{cases} \langle y_0, -x_0 + J_\lambda^{\left[\frac{\tau}{\lambda}\right]}x\rangle_s, & \text{if } \lambda > 0 \\ \langle y_0, -x_0 + S(\tau)x\rangle_s, & \text{if } \lambda = 0\end{cases}$$

it follows that f and g are u.s.c. in $[0, \infty)$.

Observe that $-f(\lambda) \leq g(\lambda)$, for $\lambda > 0$ and due to the u.s.c. of f and g at $\lambda = 0$, we have that for all $\epsilon > 0$, there exists $V_\epsilon(0)$ such that

$$f(\lambda) < f(0) + \epsilon \;\text{ and }\; g(\lambda) < g(0) + \epsilon,$$

for all $\lambda \in V_\epsilon(0)$.

Thus setting $\epsilon = 1$, it follows for λ sufficiently small that

$$f(\lambda) < |f(0)| + 1 \;\text{ and }\; -f(\lambda) \leq g(\lambda) < |g(0)| + 1.$$

Note also that
$$|f(0)| \leq \|y_0\|\|x_0 - S(\tau)x\| \ \text{ and } \ |g(0)| \leq \|y_0\|\|x_0 - S(\tau)x\|.$$

Therefore
$$|f(\lambda)| \leq \|y_0\|\|x_0 - S(\tau)x\| + 1 \tag{5.10.316}$$

Now letting $\lambda \to 0$ in (5.10.312) we obtain
$$\|S(t)x - x_0\| \leq \|x - x_0\|e^{\omega t} + t|Ax_0|e^{|\omega|t}, \tag{5.10.317}$$
since
$$\lim_{\lambda\to 0}(1-\lambda\omega)^{-[\frac{t}{\lambda}]} = \lim_{\lambda\to 0}(1-\frac{\lambda}{t}(t\omega))^{-[\frac{t}{\lambda}]} = e^{t\omega},$$
and
$$\lim_{\lambda\to 0}(1-\lambda|\omega|)^{-[\frac{t}{\lambda}]} = e^{t|\omega|},$$
We conclude then by (5.10.316) and (5.10.317) (we used (5.10.317) with $t = \tau$), that
$$|\langle y_0, x_0 - J_\lambda^{[\frac{\tau}{\lambda}]}x\rangle_s| \leq \|y_0\|\|x - x_0\|e^{\omega\tau} + \|y_0\|\tau e^{|\omega|\tau}|Ax_0| + 1 := h(\tau) \in L^1(0, 2t).$$

Note that the integral appearing in (5.10.315) can be written as
$$\int_\lambda^{([\frac{t}{\lambda}]+1)\lambda} \langle y_0, x_0 - J_\lambda^{[\frac{\tau}{\lambda}]}x\rangle_s d\tau = \int_0^{2t} \langle y_0, x_0 - J_\lambda^{[\frac{\tau}{\lambda}]}x\rangle_s \chi_\lambda(\tau)d\tau$$
where
$$\chi_\lambda(\tau) = \begin{cases} 1, & \text{if } \tau \in [\lambda, ([\frac{t}{\lambda}]+1)\lambda] \\ 0, & \text{if } \tau \in [0, 2t]\backslash[\lambda, ([\frac{t}{\lambda}]+1)\lambda] \end{cases}$$
Thus applying the Dominated Convergence Theorem (replacing the hypothesis of convergence almost everywhere by lim sup), we obtain

$$\limsup_{\lambda\to 0} \int_\lambda^{([\frac{t}{\lambda}]+1)\lambda} \langle y_0, x_0 - J_\lambda^{[\frac{\tau}{\lambda}]}x\rangle_s d\tau = \limsup_{\lambda\to 0} \int_0^{2t} \langle y_0, x_0 - J_\lambda^{[\frac{\tau}{\lambda}]}x\rangle_s \chi_\lambda(\tau)d\tau$$
$$\leq \int_0^{2t} \langle y_0, x_0 - S(\tau)x\rangle_s \chi_{[0,t]}(\tau)d\tau = \int_0^t \langle y_0, x_0 - S(\tau)x\rangle_s d\tau$$

Therefore by (5.10.315) we have, taking the lim sup on both sides, that
$$\|x_0 - S(t)x\|^2 - \|x - x_0\|^2 \leq 2\int_0^t \langle y_0, x_0 - S(\tau)x\rangle_s d\tau + I, \tag{5.10.318}$$
where
$$I = \limsup_{\lambda\to 0} 2\lambda\omega \sum_{k=1}^{[\frac{t}{\lambda}]} \|x_0 - J_\lambda^k x\|^2$$

Since our interest is to let $\lambda \to 0$ then we can consider $n \in \mathbb{N}$ such that
$$\frac{t}{n+1} < \lambda \leq \frac{t}{n} \Rightarrow \left[\frac{t}{\lambda}\right] = n.$$

Thus using (5.10.312) it follows that

$$\begin{aligned}
&2\lambda\omega \sum_{k=1}^{[\frac{t}{\lambda}]} \|x_0 - J_\lambda^k x\|^2 \leq 2\omega \frac{t}{n} \sum_{k=1}^{n} \|x_0 - J_\lambda^k x\|^2 \\
&\leq 2\omega \frac{t}{n} \sum_{k=1}^{n} \left[\left(1 - \frac{t}{n}\omega\right)^{-k} \|x - x_0\| + \frac{tk}{n} \left(1 - \frac{t}{n}|\omega|\right)^{-k} |Ax_0| \right]^2
\end{aligned} \tag{5.10.319}$$

Observe now that setting

$$\varphi_n(\tau) = \left[\left(1 - \frac{t}{n}\omega\right)^{-[\frac{\tau n}{t}]} \|x - x_0\| + \tau \left(1 - \frac{t}{n}|\omega|\right)^{-[\frac{\tau n}{t}]} |Ax_0| \right]^2$$

we have

$$\lim_{n\to\infty} \varphi_n(\tau) = \left[e^{\omega\tau} \|x - x_0\| + \tau e^{|\omega|\tau} |Ax_0| \right]^2 := \varphi(\tau)$$

uniformly on $[0, t]$.

Next observe that the Riemann sum of φ_n relative to the decomposition of $(0, t)$ into n equal parts is given by

$$S_{\varphi_n} = \sum_{k=1}^{n} \varphi(\tau_k) \frac{t}{n},$$

for some $\tau_k \in [\frac{tk}{n,}, \frac{t(k+1)}{n}]$.

Thus for each $\tau_k = \frac{tk}{n}$, it follows by (5.10.319) and from the definition of φ_n that

$$2\lambda\omega \sum_{k=1}^{n} \|x_0 - J_\lambda^k x\|^2 \leq 2\omega S_{\varphi_n} \tag{5.10.320}$$

Since $\varphi_n \to \varphi$ uniformly on $[0, t]$ we have that

$$\begin{aligned}
\|S_{\varphi_n} - \int_0^t \varphi d\tau\| &\leq \|S_{\varphi_n} - S_\varphi\| + \|S_\varphi - \int_0^t \varphi d\tau\| \\
&\leq \frac{t}{n} \sum_{k=1}^{n} \|\varphi_n(\tau_k) - \varphi(\tau_k)\| + \|S_\varphi - \int_0^t \varphi d\tau\| \\
&\leq \frac{t}{n} \sup_{[0,t]} \|\varphi_n(\tau) - \varphi(\tau)\| n + \|S_\varphi - \int_0^t \varphi d\tau\| \to 0
\end{aligned} \tag{5.10.321}$$

when $n \to \infty$.

Thus, by (5.10.320) we obtain

$$\begin{aligned}
I &\leq \lim_{n\to\infty} \sup 2\lambda\omega \sum_{k=1}^{n} \|x_0 - J_\lambda^k x\|^2 \leq \lim_{n\to\infty} 2\omega S_{\varphi_n} \\
&= 2\omega \int_0^t \varphi(\tau) d\tau = 2\omega \int_0^t \left[e^{\omega\tau} \|x - x_0\| + \tau e^{|\omega|\tau} |Ax_0| \right]^2 d\tau
\end{aligned}$$

Consider now, $\xi' \in F(x - x_0)$. We have then

$$\begin{aligned}2\langle S(t)x-x,\xi'\rangle &= 2\langle S(t)x-x_0,\xi'\rangle+2\langle x_0-x,\xi'\rangle \le 2\|x_0-S(t)x\|\|x-x_0\|-2\|x-x_0\|^2\\ &\le \|x_0-S(t)x\|^2-\|x-x_0\|^2 \le 2\int_0^t \langle y_0,x_0-S(\tau)x\rangle_s d\tau + I \end{aligned} \quad (5.10.322)$$

But, $\langle y_0, x_0-S(\cdot)x\rangle_s : [0,\infty)\to\mathbb{R}$ is u.s.c. Then for all $\epsilon>0$, there exists $\delta>0$ such that

$$\langle y_0,x_0-S(\tau)x\rangle_s \le \langle y_0,x_0-x\rangle_s+\epsilon, \quad (5.10.323)$$

for all $\tau\in[0,\delta)$, and therefore if $t\in(0,\delta)$, we have according to (5.10.322) and (5.10.323) that

$$\begin{aligned}\left\langle \frac{S(t)x-x}{t},\xi'\right\rangle &\le \frac{1}{t}\int_0^t \langle y_0,x_0-S(\tau)x\rangle_s d\tau+\frac{I}{2t}\\ &\le \langle y_0,x_0-x\rangle_s+\epsilon+\frac{I}{2t}.\end{aligned} \quad (5.10.324)$$

Since $\frac{I}{2t}\le\frac{\omega}{t}\int_0^t\left[e^{\omega\tau}\|x-x_0\|+\tau e^{|\omega|\tau}|Ax_0|\right]^2 d\tau$, we obtain according to the Mean Value Theorem that

$$\limsup_{t\to0}\frac{I}{2t}\le\omega\|x-x_0\|^2=\omega\langle x-x_0,\xi'\rangle.$$

Therefore taking $\limsup_{t\to0}$ on both sides of (5.10.324) we obtain

$$\limsup_{t\to0}\left\langle\frac{S(t)x-x}{t},\xi'\right\rangle\le\langle y_0,x_0-x\rangle_s+\omega\langle x-x_0,\xi'\rangle, \quad (5.10.325)$$

or even,

$$\limsup_{t\to0}\left\langle\frac{S(t)x-x}{t}+\omega(x_0-x),\xi'\right\rangle\le\langle y_0,x_0-x\rangle_s,\quad\forall\,\xi'\in F(x-x_0). \quad (5.10.326)$$

Proving the desired result. □

Theorem 5.257 *Let $A\in A(\omega)$, $\overline{D(A)}\subset Im(I+\lambda A)$, for $0<\lambda<\lambda_0$, $\lambda_0|\omega|<1$, A be a closed operator, $S\in Q_\omega(\overline{D(A)})$ the semigroup generated by $-A$, $z\in\overline{D(A)}$ and $S(t)z$ differentiable almost everywhere in $(0,\infty)$. Then $S(t)z$ is a strong solution of (5.10.293)-(5.10.294).*

Proof: Let $z\in\overline{D(A)}$ and $t_0>0$ such that $\frac{d}{dt}S(t)z$ exists at the point $t=t_0$. We can write

$$S(t_0)z-S(t_0-h)z=\left(\frac{d}{dt}S(t_0)z\right)h+\alpha(h),\quad 0<h<t_0, \quad (5.10.327)$$

where $\lim_{h\to0}\frac{\|\alpha(h)\|}{h}=0$.

Since $S(t_0-h)z\in\overline{D(A)}$ and by hypothesis $\overline{D(A)}\subset Im(I+\lambda A)$, $0<\lambda<\lambda_0$, it follows that if $0<h<\lambda_0$, then

$$S(t_0-h)\in Im(I+hA),$$

that is, there exists $(x_h,y_h)\in A$ such that

$$S(t_0-h)z=x_h+hy_h.$$

Thus, from (5.10.327) it follows that

$$S(t_0)z - x_h = \left(\frac{d}{dt}S(t_0)z + y_h\right)h + \alpha(h), \quad 0 < h < \lambda_0. \tag{5.10.328}$$

Setting $(x_0, y_0) = (x_h, y_h) \in A$ and $S(t_0)z \in \overline{D(A)}$ in Lemma 5.256 it follows that

$$\sup_{\xi' \in F(S(t_0)z - x_h)} \limsup_{t \to 0} \left\langle \frac{S(t+t_0)z - S(t_0)z}{t} + \omega(x_h - S(t_0)z), \xi' \right\rangle \leq \langle y_h, x_h - S(t_0)z\rangle_s \tag{5.10.329}$$

Since by Proposition 4.4, $F(x_h - S(t_0)z)$ is compact in the weak-$*$ topology, it follows that there exists $\eta' \in F(x_h - S(t_0)z)$ such that

$$\sup_{\xi' \in F(S(t_0)z - x_h)} \limsup_{t \to 0} \left\langle \frac{S(t+t_0)z - S(t_0)z}{t} + \omega(x_h - S(t_0)z), \xi' \right\rangle \leq \langle y_h, \eta'\rangle$$

Since $S(\cdot)z$ is differentiable, we obtain that

$$\left\langle \frac{d}{dt}S(t_0)z + \omega(x_h - S(t_0)z), \xi' \right\rangle \leq \langle y_h, \eta'\rangle,$$

for all $\xi' \in F(S(t_0)z - x_h)$.

In particular for $\xi' = \eta'$, we have

$$\left\langle \frac{d}{dt}S(t_0)z + \omega(x_h - S(t_0)z) + y_h, \eta' \right\rangle \geq 0.$$

By (5.10.328), it follows that

$$\langle S(t_0)z - x_h - \alpha(h) + \omega h x_h - \omega h S(t_0)z, \eta'\rangle \geq 0,$$

that is,

$$(1 - \omega h)\langle x_h - S(t_0)z, \eta'\rangle + \langle \alpha(h), \eta'\rangle \leq 0,$$

with $\eta' \in F(x_h - S(t_0)z)$.

Whence,

$$(1 - \omega h)\|x_h - S(t_0)z\|^2 \leq \|\alpha(h)\|x_h - S(t_0)z\|. \tag{5.10.330}$$

It then follows that

$$\lim_{h \to 0} \frac{S(t_0)z - x_h}{h} = 0,$$

therefore $x_h \to S(t_0)z$ when $h \to 0$. And considering (5.10.328), letting $h \to 0$, we conclude that

$$\lim_{h \to 0} y_h = -\frac{d}{dt}S(t_0)z$$

But $(x_h, y_h) \in A$ and A is closed. Thus

$$\left(S(t_0)z, -\frac{d}{dt}S(t_0)z\right) \in A.$$

Therefore setting $u(t) = S(t)z$, we have that

$$-\frac{d}{dt}u(t) \in Au(t), \text{ for almost every } t \in (0,\infty).$$

The other properties of the definition of strong solution follow immediately from the semigroup properties. □

Remark 5.258 *In the previous Theorem it is sufficient to assume that $D(A) \subset Im(I+\lambda A)$, $0 < \lambda < \lambda_0$, since by hypothesis A is closed, consequently, by Proposition 5.84, it follows that $D_\lambda = Im(I+\lambda A)$ is closed.*

Corollary 5.259 *Let X be a reflexive Banach space. If A satisfies the conditions of Theorem 5.257 and S is the semigroup generated by $-A$, then $S(t)x$ is the strong solution of (5.10.293)-(5.10.294), for all $x \in D(A)$.*

Proof: By Theorem **??**, we have that, for all $x \in D(A)$, $S(\cdot)x$ is Lipschitz continuous on bounded intervals, hence absolutely continuous on $[0,T]$, $\forall\, T > 0$, and therefore differentiable almost everywhere in $(0,\infty)$, since X is reflexive. Then by Theorem 5.257, $u(t) = S(t)x$ is the strong solution of 5.10.293)-(5.10.294). □

Remark 5.260 *If A satisfies the hypotheses of Theorem 5.257, Theorems 5.255 and 5.257 show that problem (5.10.293)-(5.10.294) has a strong solution if and only if the function $S(\cdot)x$ is differentiable almost everywhere, and in case differentiability occurs, the strong solution is $S(t)x$, for all $x \in \overline{D(A)}$. This fact, combined with what was established in Proposition 5.254, suggests considering $S(t)x$ as a solution of problem (5.10.293)-(5.10.294) even if $S(t)x$ is not differentiable, and therefore, does not satisfy conditions (ii) and (iv) of Definition 5.245.*

Thus when A is under the conditions of Theorem 5.238, the function $S(\cdot)x$ will be called ***generalized solution*** *of (5.10.293)-(5.10.294).*

Theorem 5.261 *Let X' be uniformly convex, $A+\omega I$ and $B+\omega I$ be m-accretive and S_A and S_B be the semigroups generated by $-A$ and $-B$ on $\overline{D(A)}$ and $\overline{D(B)}$ respectively. If $S_A(t) = S_B(t)$, $\forall\, t \geq 0$, then $A = B$.*

Proof: Initially observe that since $A+\omega I$ and $B+\omega I$ are m-accretive, hence maximal accretive on $\overline{D(A)}$ and $\overline{D(B)}$ respectively, it follows by Proposition 5.97 that A and B are demiclosed and therefore closed.

Moreover, X' is uniformly convex, thus X is smooth, and also reflexive.

We will now prove that $D(A) = D(B)$. Let $x \in D(A)$. By hypothesis $S_A(t) = S_B(t)$, then $\overline{D(A)} = \overline{D(B)}$. Let us set $S(t) = S_A(t) = S_B(t)$, thus by Lemma 5.256 we have

$$\limsup_{t\to 0} \left\langle \frac{S(t)x - x}{t} + \omega(x_0 - x), F(x - x_0) \right\rangle \leq \langle y_0, F(x_0 - x)\rangle, \tag{5.10.331}$$

for all $(x_0, y_0) \in B$.

Since by Corollary 5.259, $S(t)x$ is a strong solution of (5.10.293)-(5.10.294) it follows by Theorem 5.250 (i) for $s = 0$ that

$$\|\frac{S(t)x - x}{t}\| \leq e^{\omega^+ t}|Ax| \leq e^{\omega^+ T}|Ax|, \quad t \in (0,T).$$

Without loss of generality we can then assume that

$$\frac{S(t)x - x}{t} \rightharpoonup y \text{ in } X,$$

when $t \to 0$. Thus by (5.10.331) we obtain that

$$\langle y + \omega(x_0 - x), F(x - x_0)\rangle \le \langle y_0, F(x_0 - x)\rangle,\ \forall\ (x_0, y_0) \in B.$$

Therefore,

$$\langle -y + \omega x - \omega x_0 - y_0, F(x - x_0)\rangle \ge 0,\ \forall\ (x_0, y_0) \in B,$$

and since $B + \omega I$ is m-accretive it follows by Theorem 5.92(iii) that $(x, -y) \in B$, that is, $x \in D(B)$, proving that $D(A) \subset D(B)$. Analogously it is proved that $D(B) \subset D(A)$, which shows the desired equality.

We now denote $D = D(A) = D(B)$. If $x \in D$, then by Corollary 5.259, $u(t) = S(t)x$ is a strong solution of (5.10.293)-(5.10.294). Thus setting $\varphi(t) = -\frac{d}{dt}S(t)x$, we have that $\varphi(t) \in AS(t)x$ and $\varphi(t) \in BS(t)x$ almost everywhere in $(0, \infty)$.

By (iii), of Theorem 5.250 we have that $e^{-\omega t}|Au(t)|$ is decreasing. Thus, if $t > 0$,

$$|Au(t)| \le e^{\omega t}|Ax| \le e^{\omega^+ T}|Ax|,$$

where $\omega^+ = \max\{\omega, 0\}$ and $t \in [0, T]$.

Therefore

$$\|\varphi(t)\| = \|\frac{d}{dt}u(t)\| = |Au(t)| \le e^{\omega^+ T}|Ax|.$$

Since X is reflexive, we obtain the existence of a sequence (t_n) such that $t_n \to 0$ and $\varphi(t_n) \rightharpoonup y$, when $n \to \infty$.

Due to the strong continuity of $S(t)x$ we have that

$$S(t_n)x \to x,$$

and since

$$\varphi(t_n) \in AS(t_n)x \cap BS(t_n)x,$$

it follows that $y \in Ax \cap Bx$, since A and B are demiclosed. Thus, we obtain that $\|y\| \ge |Ax|$ and $\|y\| \ge |Bx|$.

On the other hand, since $\varphi(t_n) \rightharpoonup y$ and $\|\varphi(t_n)\| = |Au(t_n)| \le e^{\omega t_n}|Ax|$, we have

$$\|y\| \le \liminf \|\varphi(t_n)\| \le \lim e^{\omega t_n}|Ax| = |Ax|.$$

Analogously, $\|y\| \le |Bx|$. Therefore $\|y\| = |Ax| = |Bx|$, that is, $y \in A^0x \cap B^0x$ and by Corollary 5.111 it follows that $A = B$. □

Theorem 5.262 *Let X' be uniformly convex. If $A+\omega I$ is an m-accretive operator, then A^0 is a principal section of A.*

Proof: Let $x_0 \in \overline{D(A)}$ and $y_0 \in X$ be such that

$$\langle y_0 + \omega x_0 - v - \omega u, F(x_0 - u)\rangle \ge 0\ , \quad \forall (u, v) \in A^0 \tag{5.10.332}$$

and S be the semigroup generated by $-A$ on $\overline{D(A)}$.

We will show that $(x_0, y_0) \in A$. In order to use proposition 5.110 we will show that $x_0 \in D(A)$. Indeed, since X is reflexive, $S(t)x$ is, by corollary 5.259, a solution of (5.10.293)-(5.10.294), for all $x \in$

$D(A)$. Thus, by item ii) of theorem 5.250, it follows that

$$-\frac{d}{dt}S(t)x \in A^0 S(t)x \text{ a.e. in } (0,\infty)$$

From (5.10.332) it follows, then

$$\left\langle y_0 + \omega x_0 + \frac{d}{dt}S(t)x - \omega S(t)x, F(x_0 - S(t)x) \right\rangle \geq 0 \text{ a.e. in } (0,\infty)$$

Thus,

$$\begin{aligned} -\left\langle \frac{d}{dt}S(t)x, F(x_0 - S(t)x) \right\rangle &\leq \omega\langle x_0 - S(t)x, F(x_0 - S(t)x)\rangle + \langle y_0, F(x_0 - S(t)x)\rangle \\ &\leq \omega\|S(t)x - x_0\|^2 + \|y_0\|\|S(t)x - x_0\| \end{aligned}$$

a.e. in $(0,\infty)$, but, by lemma 5.246, it follows that

$$\begin{aligned} \|S(t)x - x_0\|\frac{d}{dt}\|S(t)x - x_0\| &= \left\langle \frac{d}{dt}S(t)x, F(S(t)x - x_0) \right\rangle \\ &\leq \omega\|S(t)x - x_0\|^2 + \|y\|\|S(t)x - x_0\| \end{aligned}$$

a.e. in $(0,\infty)$, thus

$$\frac{d}{dt}\|S(t)x - x_0\| \leq \omega^+\|S(t)x - x_0\| + \|y_0\| \text{ a.e. in } (0,\infty) \tag{5.10.333}$$

integrating from 0 to t, if $\omega^+ = 0$, we have

$$\|S(t)x - x_0\| \leq \|x - x_0\| + t\|y_0\|$$

and multiplying (5.10.333) by $e^{-\omega^+ t}$ and integrating from 0 to t, we obtain

$$\int_0^t \frac{d}{d\tau}(e^{-\omega^+\tau}\|S(\tau)x - x_0\|)d\tau \leq \int_0^t e^{-\omega^+\tau}\|y_0\|d\tau$$

Thus,

$$\begin{aligned} e^{-\omega^+ t}\|S(t)x - x_0\| &\leq \|x - x_0\| + \|y_0\|\left[\frac{-e^{-\omega^+\tau}}{\omega^+}\right]_{\tau=0}^{\tau=t} \\ &= \|x - x_0\| + \frac{1 - e^{-\omega^+ t}}{\omega^+}\|y_0\| \end{aligned}$$

whence, if $\omega^+ > 0$

$$\begin{aligned} \|S(t)x - x_0\| &\leq e^{\omega^+ t}\|x - x_0\| + \frac{e^{\omega^+ t} - 1}{\omega^+}\|y_0\| \\ &\leq e^{\omega^+ t}\|x - x_0\| + te^{\omega^+ t}\|y_0\| \end{aligned}$$

for all $x \in D(A)$. Since $x_0 \in \overline{D(A)}$, it follows, in both cases

$$\|S(t)x - x_0\| \leq te^{\omega^+ t}\|y_0\|$$

thus, $\left\|\frac{S(t)x_0 - x_0}{t}\right\|$ is bounded in every bounded interval. Therefore, there exists a sequence (t_n) with $t_n \to 0$ when $n \to \infty$, and a $z \in X$ such that

$$\frac{S(t_n)x_0 - x_0}{t_n} \rightharpoonup z \text{ when } n \to \infty$$

By lemma 5.256 and since $F(u-x_0)$ is single-valued, we have, then

$$\langle z+\omega(u-x_0), F(x_0-u)\rangle \leq \langle v, F(u-x_0)\rangle,\ \forall (u,v)\in A$$

that is,

$$\langle -z+\omega x_0 - v - \omega u, F(x_0-u)\rangle \geq 0\ ,\ \forall (u,v)\in A$$

hence, by the maximality of $A+\omega I$, $(x_0,-z)\in A$ and, therefore, $x_0\in D(A)$. From (5.10.332) it follows, then, by proposition 5.110 that $(x_0,y_0)\in A$. Thus A^0 is a principal section of A. □

For the next result we will state two lemmas, whose proofs can be found in [62] and [47] respectively.

Lemma 5.263 *Let M and N be metric spaces. A function $f : M \to N$ is continuous at a point a, if $x_n \to a$ implies that $\{f(x_n)\}$ has a subsequence converging to $f(a)$.*

Lemma 5.264 *If X is reflexive, every absolutely continuous function, $u : [0,T] \to X$ is differentiable a.e. in $(0,T)$ and*

$$u(t)-u(0)=\int_0^t \frac{d}{d\tau}u(\tau)d\tau,\quad \forall t\in[0,T].$$

Theorem 5.265 *Let X and X' be uniformly convex Banach spaces, $A\in\mathcal{A}(\omega)$ a closed operator such that*

$$D(A)\subset Im(I+\lambda A),\quad 0<\lambda\leq\lambda_0,$$

with $\lambda_0|\omega|<1$. Then, for all $x\in D(A)$,

i) The set Ax has a unique element of minimal norm, $\overset{\circ}{A}x$;

ii) If S is the semigroup generated by $-A$, then $S(t)x$ is the unique strong solution of (5.10.293)-(5.10.294);

iii) The function $\varphi(t)=e^{-\omega t}\|\overset{\circ}{A}S(t)x\|$ is monotonically decreasing;

iv) $\overset{\circ}{A}S(t)x$ is right continuous at every $t\geq 0$;

v) $S(t)x$ is differentiable from the right at every $t\geq 0$ and

$$\frac{d^+}{dt}S(t)x+\overset{\circ}{A}S(t)x=0,\ \forall t\geq 0;$$

vi) The function $S(t)x$ is continuously differentiable, except on an at most countable set and

$$-\frac{d}{dt}S(t)x=\overset{\circ}{A}S(t)x$$

at the points where differentiability occurs.

Proof:

i) By Theorem 5.106 there exists a demiclosed extension $\tilde{A}$ of A, which satisfies $\overset{\circ}{\tilde{A}}x=\overset{\circ}{A}x$, for all $x\in D(A)$. By Theorem 5.105, $\overset{\circ}{\tilde{A}}$ is single-valued, and thus $\overset{\circ}{A}$ is single-valued, since

$$D(\tilde{A})=D(\overset{\circ}{\tilde{A}})=D(\overset{\circ}{A})=D(A).$$

Let $y\in Ax$ with $\|y\|=|Ax|$. Then $y\in\overset{\circ}{\tilde{A}}x=\overset{\circ}{A}x$, for some $x\in D(A)$. Since $\overset{\circ}{\tilde{A}}$ is single-valued, it follows that $y=\overset{\circ}{A}x$.

ii) Since X is reflexive (Milman's Theorem), Corollary 5.259 guarantees that $S(t)x$ is a strong solution and by Corollary 5.248, the solution is unique.

iii) It is an immediate consequence of item (iii) of Theorem 5.250.

iv) Let $t \geq 0$, $\{t_n\}$ be a sequence such that $t_n \geq t$, for all $n = 1, 2, \ldots$, with $t_n \to t$ when $n \to \infty$. We have that $S(t_n)x \in D(A)$, for all $n \geq 1$. By iii) we have

$$e^{-\omega t_n} \| \overset{\circ}{A} S(t_n)x \| \leq \| \overset{\circ}{A} x \| \quad \forall n.$$

Since X is reflexive, there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and $y \in X$ such that

$$\overset{\circ}{A} S(t_{n_k})x \rightharpoonup y \text{ when } k \to \infty. \tag{5.10.334}$$

Since $t_{n_k} \to t$ then $S(t_{n_k})x \to S(t)x$.

Moreover, let us note that

$$(S(t_{n_k})x, \overset{\circ}{A} S(t_{n_k})x) \in \overset{\circ}{A} \subset A \subset \tilde{A}.$$

Since $\tilde{A}$ is demiclosed, it results that

$$S(t)x \in D(\tilde{A}) = D(A) \quad \text{and} \quad y \in \tilde{A}S(t)x.$$

Now, since the norm is lower semicontinuous in the weak topology of X,

$$\begin{aligned} e^{-\omega t}\|y\| &\leq \liminf_{k\to\infty} e^{-\omega t_{n_k}} \| \overset{\circ}{A} S(t_{n_k})x\| \\ &\leq \limsup_{k\to\infty} e^{-\omega t_{n_k}} \| \overset{\circ}{A} S(t_{n_k})x\| \\ &\leq e^{-\omega t} \| \overset{\circ}{A} S(t)x\| \\ &= e^{-\omega t} \| \overset{\circ}{\tilde{A}} S(t)x\| \end{aligned} \tag{5.10.335}$$

$$\Rightarrow \|y\| \leq \| \overset{\circ}{\tilde{A}} S(t)x\|.$$

Since $y \in \tilde{A}S(t)x$, it follows that $\|y\| = \| \overset{\circ}{\tilde{A}} S(t)x\|$, and hence, $y \in \overset{\circ}{\tilde{A}} S(t)x = \overset{\circ}{A} S(t)x$. Therefore, from (5.10.334) it follows that

$$\overset{\circ}{A} S(t_{n_k})x \rightharpoonup \overset{\circ}{A} S(t)x. \tag{5.10.336}$$

We also have

$$\begin{aligned} \limsup_{k\to\infty} \| \overset{\circ}{A} S(t_{n_k})x\| &= \limsup_{k\to\infty} e^{-\omega t_{n_k}} \| \overset{\circ}{A} S(t_{n_k})x\| e^{\omega t_{n_k}} \\ &\leq \limsup_{k\to\infty} e^{-\omega t_{n_k}} \| \overset{\circ}{A} S(t_{n_k})x\| \limsup_{k\to\infty} e^{\omega t_{n_k}} \\ &\leq e^{-\omega t} \| \overset{\circ}{A} S(t)x\| e^{\omega t} \\ &= \| \overset{\circ}{A} S(t)x\|. \end{aligned} \tag{5.10.337}$$

Thus, from (5.10.336) and (5.10.337), considering that X is uniformly convex, we have that

$$\overset{\circ}{A} S(t_{n_k})x \to \overset{\circ}{A} S(t)x.$$

Thus, considering the function

$$\begin{aligned} f : [0,\infty) &\longrightarrow X \\ t &\longmapsto f(t) = \mathring{A}\, S(t)x \end{aligned}$$

it follows by Lemma 5.263 that $\mathring{A}\, S(t)x$ is continuous.

v) By Corollary 5.259, $S(t)x$ is a strong solution of (5.10.293)-(5.10.294) and, by item ii) of Theorem 5.250,

$$\left\| -\frac{d}{dt}S(t)x \right\| = |AS(t)x|, \text{ for almost every } t \in (0,\infty), \tag{5.10.338}$$

which implies

$$-\frac{d}{dt}S(t)x = \mathring{A}\, S(t)x, \quad \text{for almost every } t \in (0,\infty).$$

Since $S(t)x$ is Lipschitz continuous on bounded intervals, by Lemma 5.264 we have

$$\begin{aligned} S(t+h)x - S(t)x &= \int_t^{t+h} \frac{d}{d\tau}S(\tau)x d\tau \\ &= -\int_t^{t+h} \mathring{A}\, S(\tau)x d\tau \end{aligned}$$

for all $t \geq 0$ and $h > 0$. Then

$$\frac{S(t+h)x - S(t)x}{h} + \mathring{A}\, S(t)x = -\frac{1}{h}\int_t^{t+h} \mathring{A}\, S(\tau)x d\tau + \mathring{A}\, S(t)x.$$

Knowing that

$$\lim_{h\to 0}\frac{1}{h}\int_h^{t+h} \mathring{A}\, S(\tau)x d\tau = \mathring{A}\, S(t)x,$$

it follows

$$\frac{d^+}{dt}S(t)x + \mathring{A}\, S(t)x = 0, \quad \forall t \geq 0.$$

vi) By iv) the function $\mathring{A}\,(t)x$ is defined for all $t \geq 0$. Since $\varphi(t) = e^{-\omega t}\|\mathring{A}\, S(t)x\|$ is monotonically decreasing, it follows that the set of discontinuity points of φ is, at most, countable.

If $\|\mathring{A}\, S(\cdot)x\|$ is continuous at a point t, then $\mathring{A}\, S(\cdot)x$ also is. Indeed, if $\{t_n\}$ is a sequence with $t_n \to t$, we have that

$$e^{-\omega t_n}\|\mathring{A}\, S(t_n)x\| \leq M,$$

for some constant $M > 0$ and for all $n \in \mathbb{N}$. With this, following the same reasoning used in iv), we conclude that $\lim\limits_{k\to\infty} \mathring{A}\, S(t_{n_k})x = \mathring{A}\, S(t)x$, for some subsequence $\{t_{n_k}\} \subset \{t_n\}$. With this, we conclude that the set of discontinuity points of $\mathring{A}\, S(\cdot)x$ is, at most, countable.

As we have already seen,

$$-\frac{d}{dt}S(t)x = \mathring{A}\, S(t)x, \text{ for almost every } t > 0.$$

Hence, it follows that $S(\cdot)x$ is continuously differentiable at every continuity point of $\mathring{A}\, S(\cdot)x$, and with this, we conclude the desired result.

□

We now wish to prove Theorem 5.265, which guarantees us that if

$$Im(I + \lambda A) \supset \overline{\text{conv } D(A)}, \; 0 < \lambda \leq \lambda_0, \tag{5.10.339}$$

then the strong solution of the Cauchy problem (5.10.293)-(5.10.294) can be obtained as the limit of the solution of the approximate problem

$$\begin{cases} \dfrac{du_\lambda}{dt} + A_\lambda u_\lambda = 0, \\ \qquad\quad u_\lambda(0) = 0, \end{cases} \tag{5.10.340}$$

where $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ is the Yosida approximation of A.

To prove Theorem 5.265, we need some auxiliary results which we present below.

Theorem 5.266 *Let X be a Banach space, $C \subset X$ a closed convex cone with vertex 0 and J a Lipschitzian map from C into C, i.e.*

$$\|Jx - Jy\| \leq \alpha\|x - y\|, \;\forall x, y \in C, \;\alpha > 0. \tag{5.10.341}$$

If $f \in L^1(0, T; X)$, $T > 0$, is such that $f(t) \in C$ for almost every $t \in (0, T)$, then for each $x_0 \in C$ there exists a unique function $u : [0, T] \to C$ satisfying

i) *u is absolutely continuous on $[0, T]$, differentiable almost everywhere in $(0, T)$ and*
$$u(t) \in C, \;\forall t \in [0, T]; \tag{5.10.342}$$

ii) $$\frac{d}{dt}u(t) + (I - J)u(t) = f(t) \text{ a.e. in } (0, T); \tag{5.10.343}$$

iii) $$u(0) = x_0. \tag{5.10.344}$$

Proof: Initially, observe that since C is a convex cone with vertex 0, then $a + b \in C$ and $\lambda a \in C$ for any $\lambda > 0$ and $a, b \in C$. Therefore, defining

$$j(t)x = Jx + f(t), \quad \forall x \in C, \text{ for almost every } t \in [0, T],$$

we have that $j(t)$ is a map from C into C such that

$$\|j(t)x - j(t)y\| \leq \alpha\|x - y\|, \quad \forall x, y \in C, \text{ for almost every } t \in [0, T],$$

whence it follows that $j(t)x$ is integrable on $[0, T]$ for each $x \in C$. With the definition of the map j, expression (5.10.343) becomes

$$\frac{d}{dt}u(t) + (I - j(t))u(t) = 0 \text{ a.e. in } (0, T).$$

Consider

$$\widetilde{C} = \{u \in C(0, T; X);\; u(t) \in C, \;\forall t \in [0, T]\},$$

which is a closed convex subset of $C(0, T; X)$. Let us consider a contraction with fixed point in $\widetilde{C}$. Given $x_0 \in C$, let us define $\phi : C(0, T; X) \to C(0, T; X)$ by

$$\phi u(t) = e^{-t}x_0 + \int_0^t e^{s-t} j(s)(u(s))ds.$$

First, let us verify that $\phi(\widetilde{C}) \subset \widetilde{C}$. Let $u \in \widetilde{C}$. It is clear that $e^{-t}x_0 \in C$, for all $t \in [0, T]$, and also $j(s)(u(s)) \in C$, whence $e^{s-t}j(s)(u(s)) \in C$, for any $s, t \in [0, T]$. Furthermore, since

$$\|Ju(s) - Ju(t)\| \leq \alpha\|u(s) - u(t)\|, \quad \forall s, t \in [0, T],$$

it follows that $Ju : [0, T] \to C$ is a continuous map, whence $\int_0^t e^{s-t} Ju(s)ds \in C$.

Now, observe that, for each fixed $t \in [0, T]$, the function $g(s) = e^{s-t}f(s)$ belongs to $L^1(0, T; X)$, and, in particular, $L^1(0, T; C)$, which is a complete subset of $L^1(0, T; X)$. Since $C^0(0, T; C)$ is dense in

$L^1(0,T;C)$, we can obtain a sequence $\{g_n\} \subset C^0(0,T;C)$ that converges to g in $L^1(0,T;C)$. In particular, we have

$$\int_0^t g_n(s)ds \to \int_0^t g(s)ds,$$

with $\int_0^t g_n(s)ds \in C$, for all $t \in [0,T]$. Since C is closed, we conclude $\int_0^t g(s)ds \in C$, for all $t \in [0,T]$. Therefore,

$$\int_0^t e^{s-t} j(s)(u(s))ds \in C, \quad \forall t \in [0,T],$$

and thus, $\phi u(t) \in C$, for all $t \in [0,T]$. Hence, $\phi u \in \widetilde{C}$.

We will prove that for $n \in \mathbb{N}$ sufficiently large, ϕ^n is a strict contraction. For this, we will prove by induction, that

$$\|\phi^n u(t) - \phi^n v(t)\| \leq \frac{\alpha^n t^n}{n!} \|u - v\|_{C(0,T;X)},$$

for any $u, v \in C(0,T;X)$, $t \in [0,T]$ and for all $n \in \mathbb{N}$.

Indeed, we have

$$\begin{aligned} \|\phi u(t) - \phi v(t)\| &\leq \int_0^t e^{s-t} \|j(s)(u(s)) - j(s)(v(s))\| ds \\ &\leq \alpha \int_0^t e^{s-t} \|u(s) - v(s)\| ds \\ &\leq \alpha \int_0^t ds \|u - v\|_{C(0,T;X)} \\ &= \alpha t \|u - v\|_{C(0,T;X)}, \quad \forall t \in [0,T]. \end{aligned}$$

Suppose, now, that

$$\|\phi^{n-1} u(t) - \phi^{n-1} v(t)\| \leq \frac{(\alpha t)^{n-1}}{(n-1)!} \|u - v\|_{C(0,T;X)}, \quad \forall t \in [0,T].$$

Therefore,

$$\begin{aligned} \|\phi^n u(t) - \phi^n v(t)\| &\leq \int_0^t e^{s-t} \|j(s)(\phi^{n-1} u(t)) - j(s)(\phi^{n-1} v(t))\| ds \\ &\leq \alpha \int_0^t e^{s-t} \|\phi^{n-1} u(t) - \phi^{n-1} v(t)\| ds \\ &\leq \alpha^n \int_0^t e^{s-t} \frac{s^{n-1}}{(n-1)!} \|u - v\|_{C(0,T;X)} ds \\ &= \frac{\alpha^n}{(n-1)!} \|u - v\|_{C(0,T;X)} \int_0^t s^{n-1} ds \\ &= \frac{(\alpha t)^n}{n!} \|u - v\|_{C(0,T;X)}, \quad \forall t \in [0,T], \end{aligned}$$

whence we conclude the desired result.

Since

$$\|\phi^n u(t) - \phi^n v(t)\| \leq \frac{\alpha^n T^n}{n!} \|u - v\|_{C(0,T;X)}, \quad \forall u, v \in C(0,T;X),$$

it follows that for n sufficiently large, ϕ^n is a strict contraction (see [60]). Therefore, ϕ has a unique fixed point in $\widetilde{C}$. Let $\widetilde{u}$ be such fixed point. Then, since,

$$\widetilde{u}(t) = e^{-t} x_0 + \int_0^t e^{s-t} j(s)(\widetilde{u}(s)) ds \tag{5.10.345}$$

it results that $\widetilde{u}$ is absolutely continuous on $[0,T]$, differentiable almost everywhere in $(0,T)$ and $\widetilde{u}(t) \in C$, for all $t \in [0, T]$. Moreover, $\widetilde{u}(0) = x_0$ and, differentiating both sides of (5.10.345) we see that

$$\frac{d}{dt}\widetilde{u}(t) + (I - J)\widetilde{u}(t) = f(t) \text{ a.e. in } (0, T).$$

Now, let us prove the uniqueness of the function satisfying i), ii) and iii). Let, then, u and v be functions satisfying such conditions. In particular, we have

$$u_t(\tau) + (I - J)u(\tau) = f(\tau) \text{ a.e. in } (0, T). \tag{5.10.346}$$

and

$$v_t(\tau) + (I - J)v(\tau) = f(\tau) \text{ a.e. in } (0, T). \tag{5.10.347}$$

Subtracting (5.10.347) from (5.10.346) we obtain

$$(u(\tau) - v(\tau)) + (u_t(\tau) - v_t(\tau)) = Ju(\tau) - Jv(\tau) \text{ a.e. in } (0, T).$$

Multiplying both sides of the equality above by $(u(\tau) - v(\tau))_t$, integrating in X and using Hölder's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(\tau) - v(\tau)\|^2 + \|u_t(\tau) - v_t(\tau)\|^2 \leq \|Ju(\tau) - Jv(\tau)\|\|u_t(\tau) - v_t(\tau)\| \text{ a.e. in } (0, T).$$

Integrating the inequality above on $(0, t)$, for $t \in [0, T]$ given, we have

$$\frac{1}{2}\|u(t) - v(t)\|^2 + \int_0^t \|u_t(\tau) - v_t(\tau)\|^2 d\tau \leq \int_0^t \|Ju(\tau) - Jv(\tau)\|^2 d\tau + \int_0^t \|u_t(\tau) - v_t(\tau)\|^2 d\tau.$$

In particular,

$$\|u(t) - v(t)\|^2 \leq \alpha \int_0^t \|u(\tau) - v(\tau)\|^2 d\tau,$$

and by Grönwall's Inequality it follows $u(t) = v(t)$. Thus, we obtain the desired uniqueness. □

Corollary 5.267 *Under the same hypotheses of Theorem 5.266, for each $x_0 \in C$ and each $\lambda > 0$ there exists a unique function $u : [0, T] \to C$ satisfying (5.10.342), (5.10.344) and*

$$\frac{d}{dt}u(t) + \frac{1}{\lambda}(I - J)u(t) = f(t) \textit{ a.e. in } (0, T). \tag{5.10.348}$$

Proof: Let v be the unique function satisfying (5.10.342), (5.10.344) and

$$\frac{d}{dt}v(t) + (I - J)v(t) = \lambda f(\lambda t), \text{ a.e. in } (0, \lambda T).$$

Considering $u(t) = v\left(\frac{t}{\lambda}\right)$, we have $u(\lambda t) = v(t)$ and

$$\frac{d}{dt}[u(\lambda t)] + (I - J)u(\lambda t) = \lambda f(\lambda t), \text{ a.e. in } (0, \lambda T),$$

whence it follows

$$\frac{d}{dt}u(t) + \frac{1}{\lambda}(I - J)u(t) = f(t), \text{ a.e. in } (0, T),$$

with u satisfying (5.10.342) and (5.10.344). □

Corollary 5.268 *Let C be a closed convex subset of X, $J : C \to C$ Lipschitzian with constant $\alpha > 0$, and $\lambda > 0$ and $T > 0$. Then, for each $x_0 \in C$ there exists a unique function $u : [0, T] \to C$ satisfying (5.10.342), (5.10.344) and*

$$\frac{d}{dt}u(t) + \frac{1}{\lambda}(I - J)u(t) = 0 \textit{ a.e. in } (0, T). \tag{5.10.349}$$

Proof: It follows from the previous Corollary, with $f \equiv 0$. In this case, the proof of Theorem 5.266 is applicable without assuming that C is a cone, since the map $j(t) : C \to C$ given by $j(t) = J$, for all $t \in [0, T]$, is well defined. □

Corollary 5.269 *Under the conditions of Corollary 5.268, if $x_0, y_0 \in C$ and u and v satisfy (5.10.342) and $u(0) = x_0$, $v(0) = y_0$, then*

i) $\|u(t) - v(t)\| \le e^{\frac{(\alpha-1)t}{\lambda}} \|x_0 - y_0\|$;

ii) $\left\|\frac{d}{dt}u(t+t_0)\right\| \le e^{\frac{(\alpha-1)}{\lambda}(t+t_0)} \left\|\frac{d}{dt}u(t_0)\right\|$ *for all t, t_0 such that u is differentiable at $t + t_0$ and t_0.*

Proof: i) We have

$$u(t) = e^{-\frac{t}{\lambda}}x_0 + \frac{1}{\lambda}\int_0^t e^{\frac{s-t}{\lambda}} Ju(s)ds,$$

$$v(t) = e^{-\frac{t}{\lambda}}y_0 + \frac{1}{\lambda}\int_0^t e^{\frac{s-t}{\lambda}} Jv(s)ds,$$

whence

$$e^{\frac{t}{\lambda}}\|u(t) - v(t)\| \le \|x_0 - y_0\| + \frac{\alpha}{\lambda}\int_0^t e^{\frac{s}{\lambda}}\|u(s) - v(s)\|ds$$

and, therefore, by Gronwall's Inequality

$$\|u(t) - v(t)\| \le e^{\frac{(\alpha-1)}{\lambda}t}\|x_0 - y_0\|.$$

ii) Let $t \in [0, T]$ and $t_0 \in [0, T-h)$ such that u is differentiable at t_0 and at $t+t_0$. Let also $h \in (0, T-t_0-t)$. Considering $x_0 = u(t_0)$ and $y_0 = u_0(t_0+h)$, we observe that the solutions of (5.10.349) associated to the respective initial data x_0 and y_0 are given by $u(t + t_0)$ and $u(t + t_0 + h)$, and by *i)* we have

$$\left\|\frac{u(t+t_0+h) - u(t+t_0)}{h}\right\| \le e^{\frac{(\alpha-1)}{\lambda}(t+t_0)} \left\|\frac{u(t_0+h) - u(t_0)}{h}\right\|,$$

and letting $h \to 0$ we obtain

$$\left\|\frac{d}{dt}u(t+t_0)\right\| \le e^{\frac{(\alpha-1)}{\lambda}(t+t_0)} \left\|\frac{d}{dt}u(t_0)\right\|.$$

□

The estimate that will be established next is due to Chernoff [25] in the linear case and to Miyadera and Oharu [70] in the general case. The proof we will give is found in Brezis [17] and Pazy [83].

Lemma 5.270 *Let $\{\varphi_n\}$ be a sequence of locally integrable functions on $[0, \infty)$ such that*

$$\varphi_n(t) \le n\alpha^n e^{\frac{-t}{\lambda}} + \frac{\alpha}{\lambda}\int_0^t e^{\frac{(s-t)}{\lambda}}\varphi_{n-1}(s)ds, \;\; n = 1, ..., \;\; \alpha \ge 1 \;\; \text{and } \lambda > 0 \tag{5.10.350}$$

and $\varphi_0(t) \le \frac{t}{\lambda}e^{\frac{(\alpha-1)t}{\lambda}}$. Then, for every non-negative integer, n, and $t \ge 0$ we have

$$\varphi_n(t) \le \alpha^n e^{\frac{(\alpha-1)t}{\lambda}} \left[\left(n - \alpha\frac{t}{\lambda}\right)^2 + \alpha\frac{t}{\lambda}\right]^{\frac{1}{2}}. \tag{5.10.351}$$

Proof: We have that

$$\varphi_0(t) \le \frac{t}{\lambda}e^{\frac{(\alpha-1)t}{\lambda}} \le e^{\frac{(\alpha-1)t}{\lambda}}(\alpha^2\frac{t^2}{\lambda^2} + \alpha\frac{t}{\lambda})^{\frac{1}{2}}$$

since $\alpha \geq 1$ and $t \geq 0$. Thus, (5.10.351) is valid for $n = 0$. Assuming valid for n, that is,

$$\varphi_n(t) \leq \alpha^n e^{\frac{(\alpha-1)t}{\lambda}} \left[\left(n - \alpha \frac{t}{\lambda} \right)^2 + \alpha \frac{t}{\lambda} \right]^{\frac{1}{2}}.$$

Then, (5.10.351) will be demonstrated if we prove that

$$\varphi_{n+1}(t) \leq \alpha^{n+1} e^{\frac{(\alpha-1)t}{\lambda}} \left[\left(n + 1 - \alpha \frac{t}{\lambda} \right)^2 + \alpha \frac{t}{\lambda} \right]^{\frac{1}{2}},$$

but, thanks to (5.10.350) it suffices to show that

$$(n+1)\alpha^{n+1} e^{\frac{-t}{\lambda}} + \frac{\alpha^{n+1}}{\lambda} \int_0^t e^{\frac{\alpha s - t}{\lambda}} \left[\left(n - 1 - \frac{\alpha s}{\lambda} \right)^2 + \frac{\alpha s}{\lambda} \right]^{\frac{1}{2}} ds$$
$$\leq \alpha^{n+1} e^{\frac{(\alpha-1)t}{\lambda}} \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{\frac{1}{2}}$$

or, equivalently,

$$n + 1 + \frac{1}{\lambda} \int_0^t e^{\frac{\alpha s}{\lambda}} \left[\left(n - \frac{\alpha s}{\lambda} \right)^2 + \frac{\alpha s}{\lambda} \right]^{\frac{1}{2}} ds \leq e^{\frac{\alpha t}{\lambda}} \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{\frac{1}{2}}$$

and since both members are equal to $n + 1$ at the point $t = 0$ it is sufficient to demonstrate that the derivative of the first is less than or equal to the derivative of the second, that is, that

$$\frac{1}{\lambda} e^{\frac{\alpha t}{\lambda}} \left[\left(n - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{\frac{1}{2}} \leq \frac{\alpha}{\lambda} e^{\frac{\alpha t}{\lambda}} \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{\frac{1}{2}}$$
$$+ \frac{\alpha}{\lambda} e^{\frac{\alpha t}{\lambda}} \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{-\frac{1}{2}} \left(-\frac{1}{2} - n + \frac{\alpha t}{\lambda} \right).$$

But this inequality is true because the second member is positive, since

$$\left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{\frac{1}{2}} + \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{-\frac{1}{2}} \left(-\frac{1}{2} - n + \frac{\alpha t}{\lambda} \right)$$
$$= \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{-\frac{1}{2}} \left[\left(n - \frac{\alpha t}{\lambda} \right)^2 + \frac{1}{2} + n \right] \geq 0,$$

and since $n + 1 \geq n$ and $(a + b)^2 \geq a^2$, with $a, b \geq 0$ we have

$$\left\{ \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{\frac{1}{2}} + \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{-\frac{1}{2}} \left(-\frac{1}{2} - n + \frac{\alpha t}{\lambda} \right) \right\}^2$$
$$\geq \left(n - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda}$$

it follows that

$$\left[\left(n - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{\frac{1}{2}} \leq \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{\frac{1}{2}}$$
$$+ \left[\left(n + 1 - \frac{\alpha t}{\lambda} \right)^2 + \frac{\alpha t}{\lambda} \right]^{-\frac{1}{2}} \left(-\frac{1}{2} - n + \frac{\alpha t}{\lambda} \right).$$

which multiplying by $\frac{1}{\lambda}e^{\frac{\alpha t}{\lambda}}$ and using that $\alpha \geq 1$ gives us the desired result. $\square$

Theorem 5.271 *Let C be a closed convex subset of X, $J : C \to C$ a Lipschitzian map with constant $\alpha \geq 1$, $x_0 \in C$, $T > 0$ and $u : [0, T] \to C$ satisfying (5.10.342), (5.10.344) and (5.10.349). Then, for each positive integer n and each $t \geq 0$ we have*

$$\|u(t) - J^n x_0\| \leq \alpha^n e^{\frac{(\alpha-1)t}{\lambda}} \|x_0 - Jx_0\| \left[\left(n - \alpha\frac{t}{\lambda}\right)^2 + \alpha\frac{t}{\lambda}\right]^{\frac{1}{2}}. \tag{5.10.352}$$

Proof: By ii) of Corollary 5.269, we have for all t, t_0 such that u is differentiable at $t + t_0$ and t_0,

$$\left\|\frac{d}{dt}u(t+t_0)\right\| \leq e^{\frac{(\alpha-1)(t+t_0)}{\lambda}} \left\|\frac{d}{dt}u(t_0)\right\| \leq e^{\frac{(\alpha-1)(t+t_0)}{\lambda}} \left\|\frac{1}{\lambda}(J-I)u(t_0)\right\|.$$

Taking the limit as t_0 tends to 0 we have

$$\left\|\frac{d}{dt}u(t)\right\| \leq e^{\frac{(\alpha-1)(t)}{\lambda}} \left\|\frac{1}{\lambda}(J-I)x_0\right\|. \tag{5.10.353}$$

Therefore, if $Jx_0 = x_0$, then $u(t) = x_0$, for all $t \geq 0$ and, in this case, the estimate (5.10.352) is valid. Let, then, $Jx_0 \neq x_0$ and let us set for $n = 0, 1, ...$

$$\varphi_n(t) = \|u(t) - J^n x_0\| \|x_0 - Jx_0\|^{-1}. \tag{5.10.354}$$

Since,

$$u(t) = e^{-\frac{t}{\lambda}} x_0 + \frac{1}{\lambda}\int_0^t e^{\frac{s-t}{\lambda}} Ju(s) ds$$

for $n = 1, 2, \ldots$ we have

$$\|u(t) - J^n x_0\| \leq e^{-\frac{t}{\lambda}} \|x_0 - J^n x_0\| + \frac{\alpha}{\lambda}\int_0^t e^{\frac{s-t}{\lambda}} \|u(s) - J^n x_0\| ds$$

Thus,

$$\varphi_n(t) \leq e^{-\frac{t}{\lambda}} \|x_0 - J^n x_0\| \|x_0 - Jx_0\|^{-1} + \frac{\alpha}{\lambda}\int_0^t e^{\frac{s-t}{\lambda}} \varphi_{n-1}(s) ds. \tag{5.10.355}$$

But,

$$\begin{aligned}
\|x_0 - J^n x_0\| &= \|J^0 x_0 - Jx_0 + Jx_0 - ... - J^n x_0\| \\
&\leq \sum_{i=1}^n \|J^{i-1}x_0 - J^i x_0\| \\
&\leq \sum_{i=1}^n \alpha^{i-1} \|x_0 - Jx_0\|
\end{aligned}$$

and since, by hypothesis, $\alpha \geq 1$, $\|x_0 - J^n x_0\| \leq n\alpha^n \|x_0 - Jx_0\|$. From there and from (5.10.355) it follows, for $n = 1, ...,$

$$\varphi_n(t) \leq n\alpha^n e^{-\frac{t}{\lambda}} + \alpha\int_0^t \frac{s-t}{\lambda}\varphi_{n-1}(s) ds. \tag{5.10.356}$$

Moreover,

$$\begin{aligned}
u(t) - x_0 &= \frac{1}{\lambda}\int_0^t e^{-\frac{s-t}{\lambda}} (Ju(s) - x_0) ds \\
&= \frac{1}{\lambda}\int_0^t e^{\frac{s-t}{\lambda}} (J-I)u(s) ds + \frac{1}{\lambda}\int_0^t e^{\frac{s-t}{\lambda}} (u(s) - x_0) ds.
\end{aligned}$$

But from (5.10.353), we have

$$\|\frac{1}{\lambda}(J-I)u(s)\| = \left\|\frac{d}{dt}u(s)\right\| \le e^{\frac{(\alpha-1)s}{\lambda}} \left\|\frac{1}{\lambda}(x_0 - Jx_0)\right\|.$$

Thus,

$$\|u(t) - x_0\| \le \frac{e^{-\frac{t}{\lambda}}}{\lambda}\left[\|x_0 - Jx_0\| \int_0^t e^{\frac{\alpha s}{\lambda}}ds + \int_0^t e^{\frac{s}{\lambda}}\|u(s) - x_0\|ds\right], \tag{5.10.357}$$

that is, the formula

$$\|u(t) - x_0\| \le \frac{e^{-\frac{t}{\lambda}}}{\lambda}\|x_0 - Jx_0\| \int_0^t \sum_{k=0}^{n} \frac{(t-s)^k}{\lambda^k k!} e^{\frac{\alpha s}{\lambda}} ds + \frac{e^{-\frac{t}{\lambda}}}{\lambda^{n+1} n!} \int_0^t (t-s)^n e^{\frac{s}{\lambda}} \|u(s) - x_0\| ds$$

is valid for $n = 0$. Let us show that it is valid for all n. Assume that it is valid for n and let us prove that it remains valid for $n+1$.

Indeed, observe that

$$\begin{aligned}\frac{1}{\lambda^n n!}\int_0^t (t-s)^n e^{\frac{s}{\lambda}}\|u(s)-x_0\|ds &\le \frac{1}{\lambda^n n!}\int_0^t (t-s)^n e^{\frac{s}{\lambda}}\left\{\frac{e^{-\frac{s}{\lambda}}}{\lambda}\left[\|x_0 - Jx_0\| \int_0^s e^{\frac{\alpha\xi}{\lambda}}d\xi + \int_0^s e^{\frac{\xi}{\lambda}}\|u(\xi) - x_0\|ds\right]\right\} ds \\ &= \frac{1}{\lambda^n n!}\int_0^t (t-s)^n \frac{1}{\lambda}\left[\|x_0 - Jx_0\| \int_0^s e^{\frac{\alpha\xi}{\lambda}}d\xi + \int_0^t e^{\frac{\xi}{\lambda}}\|u(s) - x_0\|d\xi\right] ds\end{aligned}$$

and, since,

$$\int_0^t (t-s)^n \int_0^s e^{\frac{\alpha\xi}{\lambda}} d\xi ds = \int_0^t e^{\frac{\alpha\xi}{\lambda}} \frac{(t-\xi)^{n+1}}{n+1} d\xi$$

and

$$\int_0^t (t-s)^n \int_0^s e^{\frac{\xi}{\lambda}} \|u(\xi) - x_0\| d\xi ds = \int_0^t e^{\frac{\xi}{\lambda}} \frac{(t-\xi)^{n+1}}{n+1} \|u(\xi) - x_0\| d\xi$$

we have

$$\frac{1}{\lambda^n n!}\int_0^t (t-s)^n e^{\frac{s}{\lambda}}\|u(s)-x_0\|ds \le \frac{1}{\lambda^{n+1}(n+1)!}\left[\|x_0 - Jx_0\| \int_0^t (t-s)^{n+1} e^{\frac{\alpha s}{\lambda}} ds + \int_0^t e^{\frac{s}{\lambda}} (t-s)^{n+1} \|u(s) - x_0\| ds\right]$$

which, comparing with the induction hypothesis gives us the desired result and the formula is valid for all $n = 0, 1, \ldots$.

Thus, letting $n \to \infty$, we obtain

$$\|u(t) - x_0\| \le \frac{1}{\lambda}\|x_0 - Jx_0\| \int_0^t e^{\frac{(\alpha-1)s}{\lambda}} ds \le \frac{t}{\lambda} e^{\frac{(\alpha-1)t}{\lambda}} \|x_0 - Jx_0\|.$$

So

$$\varphi_0(t) = \|u(t) - x_0\| \|x_0 - Jx_0\|^{-1} \le \frac{1}{\lambda} t e^{\frac{(\alpha-1)t}{\lambda}}. \tag{5.10.358}$$

From (5.10.354), (5.10.356) and (5.10.358) follows (5.10.352) from the previous lemma follows the result. □

Now we are in conditions to state and prove the following result:

Theorem 5.272 *Let $A \in \mathcal{A}(\omega)$, such that $\overline{D(A)} \subset Im(I + \lambda A)$ for $0 < \lambda < \lambda_0$ with $\lambda_0|\omega| < 1$ and satisfies condition (5.10.339). Then, for each $\lambda > 0$ such that $\lambda|\omega| < \frac{1}{2}$, $\forall x \in C$ and for all $T > 0$, there exists a unique function $u_\lambda : [0, T] \longrightarrow C$, absolutely continuous in $(0, T)$, differentiable a.e. in $(0, T)$, $\frac{du_\lambda}{dt} + A_\lambda u_\lambda = 0$ a.e. in $(0, T)$ and $u_\lambda(0) = x$. Moreover,*

$$\lim_{\lambda \to 0} u_\lambda(t) = S(t)x, \ \forall x \in \overline{D(A)} \text{ and } \forall T \ge 0, \tag{5.10.359}$$

Proof: Since $C \subset Im(I + \lambda A) = D_\lambda$ and $J_\lambda : D_\lambda \longrightarrow Im(D_\lambda) = D(A) \subset \overline{\text{conv } D(A)}$ and J_λ is Lipschitzian with constant $(1-\lambda\omega)^{-1}$ (see Theorem 5.79), then the restriction of J_λ to C is a Lipschitzian map from C into C with Lipschitz constant $\alpha = (1 - \lambda|\omega|)^{-1} \geq 1$, since $(1 - \lambda\omega)^{-1} \leq (1 - \lambda|\omega|)^{-1}$.

By Corollary 5.268 there exists a unique function u_λ satisfying the stated conditions. It remains to demonstrate (5.10.359).

First let us show (5.10.359) for $x \in D(A)$. Let, then, $x \in D(A)$, $t \in [0, T]$ and $n \in \mathbb{N}$ such that $n = \left[\frac{t}{\lambda}\right]$. Setting $S_\lambda(t)x = u_\lambda(t)$ we have:

$$\begin{aligned} & \|u_\lambda(t) - S(t)x\| = \|S_\lambda(t)x - S(t)x\| \\ \leq \; & \underbrace{\|S_\lambda(t)x - S_\lambda(n\lambda)x\|}_{I} + \underbrace{\|S_\lambda(n\lambda)x - J_\lambda^n x\|}_{II} + \underbrace{\|J_\lambda^n x - S(n\lambda)x\|}_{III} + \underbrace{\|S(n\lambda)x - S(t)x\|}_{IV} \end{aligned} \tag{5.10.360}$$

Now let us estimate each of the terms above.
<u>Estimate I:</u> From Corollary 5.269, item ii), we have that, for almost every $t_0 \in [0, T - t]$ it holds that

$$\begin{aligned} & \|u_\lambda(t + t_0)x - u_\lambda(n\lambda + t_0)x\| \leq \left\| \int_{n\lambda + t_0}^{t + t_0} \frac{du_\lambda}{ds}(\xi) d\xi \right\| = \left\| \int_{n\lambda}^{t} \frac{du_\lambda}{ds}(s + t_0) ds \right\| \\ \leq \; & \int_{n\lambda}^{t} e^{(\alpha-1)\frac{s+t_0}{\lambda}} \|A_\lambda u_\lambda(t_0)\| ds \leq e^{(\alpha-1)\frac{t+t_0}{\lambda}} (t - n\lambda) \|A_\lambda u_\lambda(t_0)\|. \end{aligned}$$

Taking the limit as $t_0 \to 0$ it follows that

$$\|u_\lambda(t)x - u_\lambda(n\lambda)x\| \leq e^{(\alpha-1)\frac{t}{\lambda}} (t - n\lambda) \|A_\lambda x\|.$$

Note that

$$\alpha = \frac{1}{1 - \lambda|\omega|} \Rightarrow \alpha - 1 = \frac{\lambda|\omega|}{1 - \lambda|\omega|}.$$

And,

$$\frac{t}{\lambda} - \left[\frac{t}{\lambda}\right] < 1 \Leftrightarrow t - n\lambda < \lambda.$$

Whence it follows that

$$\begin{aligned} & \|S_\lambda(t)x - S_\lambda(n\lambda)(x)\| = \|u_\lambda(t)x - u_\lambda(n\lambda)x\| \leq e^{(\alpha-1)\frac{t}{\lambda}} (t - n\lambda) \|A_\lambda x\| \\ = \; & e^{\frac{|\omega| t}{1 - \lambda|\omega|}} (t - n\lambda) \|A_\lambda x\| \leq e^{\frac{|\omega| t}{1 - \lambda|\omega|}} (t - n\lambda) \frac{1}{1 - \lambda|\omega|} |Ax| \\ < \; & e^{\frac{|\omega| t}{1 - \lambda|\omega|}} \lambda (1 - \lambda|\omega|)^{-1} |Ax|. \end{aligned}$$

Note that the term on the right converges to zero when $\lambda \to 0$ independently of $t \in [0, T]$.
<u>Estimate II:</u> Bearing in mind that $S_\lambda(t)x = u_\lambda(t)$ one has, by Theorem 5.271, that:

$$\begin{aligned} & \|S_\lambda(n\lambda)x - J_\lambda^n x\| = \|u_\lambda(n\lambda) - J_\lambda^n x\| \\ \leq \; & \alpha^n e^{\frac{(\alpha-1)n\lambda}{\lambda}} \|x - J_\lambda x\| \left[\left(n - \alpha \frac{n\lambda}{\lambda} \right)^2 + \alpha \frac{n\lambda}{\lambda} \right]^{\frac{1}{2}} \\ \leq \; & \alpha^n e^{(\alpha-1)n} \|x - J_\lambda x\| [(n - \alpha n)^2 - \alpha n]^{\frac{1}{2}} \\ \leq \; & (1 - \lambda|\omega|)^{-n} e^{\frac{n\lambda|\omega|}{1 - \lambda|\omega|}} \lambda (1 - \lambda|\omega|)^{-1} |Ax| \left[(n - \alpha n)^2 + \alpha n \right]^{\frac{1}{2}} \\ \leq \; & (1 - \lambda|\omega|)^{-\frac{t}{\lambda}} e^{\frac{t|\omega|}{1 - \lambda|\omega|}} \frac{\sqrt{\lambda}}{1 - \lambda|\omega|} |Ax| \left[\lambda \left(n^2 (1 - \alpha)^2 + \alpha n \right) \right]^{\frac{1}{2}} \\ \leq \; & (1 - \lambda|\omega|)^{-\frac{t}{\lambda}} e^{\frac{t|\omega|}{1 - \lambda|\omega|}} \frac{\sqrt{\lambda}}{1 - \lambda|\omega|} \left[t^2 \frac{\lambda|\omega|^2}{(1 - \lambda|\omega|)^2} + \frac{t}{1 - \lambda|\omega|} \right]^{\frac{1}{2}} |Ax|, \end{aligned} \tag{5.10.361}$$

(5.10.362)

since

$$\begin{aligned} n = \left[\frac{t}{\lambda}\right] \leq \frac{t}{\lambda} \quad &\Rightarrow \quad -n \geq -\frac{t}{\lambda} \;\text{ and} \\ \lambda(n^2(1-\alpha)^2 + \alpha n) \quad &= \quad \lambda\left(\left[\frac{t}{\lambda}\right]^2\left(1-\frac{1}{1-\lambda|\omega|}\right)^2 + \frac{\left[\frac{t}{\lambda}\right]}{1-\lambda|\omega|}\right) \\ &\leq \quad \lambda\left(\frac{t}{\lambda}\right)^2\left(\frac{\lambda|\omega|}{1-\lambda|\omega|}\right)^2 + \frac{\frac{t}{\lambda}}{1-\lambda|\omega|} \\ &\leq \quad t^2\frac{\lambda|\omega|^2}{(1-\lambda|\omega|)^2} + \frac{t}{1-\lambda|\omega|}. \end{aligned}$$

The last term of (5.10.361) also converges to zero when $\lambda \to 0$ independently of $t \in [0, T]$.
Estimate III: Since $\lambda|\omega| < \frac{1}{2}$ one has, by (5.9.279) that

$$\begin{aligned} &\|J_\lambda^n x - S(n\lambda)x\| = \lim_{m\to\infty} \|J_\lambda^n x - J_{\frac{n\lambda}{m}}^m x\| = \lim_{m\to\infty} \|J_{\frac{n\lambda}{n}}^n x - J_{\frac{n\lambda}{m}}^m x\| \\ \leq\;& \lim_{m\to\infty} 2n\lambda e^{4|\omega|n\lambda}\left(\frac{1}{n}-\frac{1}{m}\right)^{\frac{1}{2}} |Ax| \\ =\;& 2n\lambda e^{4|\omega|n\lambda}\frac{1}{\sqrt{n}}|Ax| \\ =\;& 2\sqrt{n\lambda}\sqrt{\lambda}e^{4|\omega|n\lambda}|Ax| \\ =\;& 2\sqrt{n}\lambda e^{4|\omega|n\lambda}|Ax| \\ \leq\;& 2\sqrt{t}\sqrt{\lambda}e^{4|\omega|t}|Ax| \end{aligned}$$

where the term on the right converges to zero when $\lambda \to 0$ uniformly with respect to $t \in [0, T]$.
Estimate IV: By Proposition 5.243 it follows that

$$\begin{aligned} \|S(n\lambda)x - S(t)x\| \quad &\leq \quad e^{\omega^+(t-n\lambda)}e^{\omega^+n\lambda}(t-n\lambda)|Ax| \\ &\leq \quad e^{\omega^+\lambda}e^{\omega^+t}\lambda|Ax| \end{aligned}$$

which converges to zero independently of the value of $t \in [0, T]$.

From estimates I, II, III and IV we have that (5.10.360) tends to zero when $\lambda \to 0$, uniformly on $[0, T]$, that is,

$$\lim_{\lambda\to 0} u_\lambda(t) = S(t)x, \;\forall x \in D(A) \text{ and } \forall t \geq 0,$$

or even, given $x \in D(A)$ and $\varepsilon > 0$, there exists $\delta(\varepsilon, x)$ such that

$$\|u_\lambda(t) - S(t)x\| < \varepsilon \text{ whenever } \lambda < \delta(\varepsilon, x). \tag{5.10.363}$$

Thus, given $x \in \overline{D(A)}$ there exists $y \in D(A)$ such that $\|y - x\| < \varepsilon$ and

$$\begin{aligned} &\|u_\lambda(t) - S(t)x\| = \|S_\lambda(t)x - S_\lambda(t)y + S_\lambda(t)y - S(t)y + S(t)y - S(t)x\| \\ \leq\;& \underbrace{\|S_\lambda(t)x - S_\lambda(t)y\|}_{V} + \underbrace{\|S_\lambda(t)y - S(t)y\|}_{<\varepsilon \text{ by } (5.10.363)} + \underbrace{\|S(t)y - S(t)x\|}_{VI} \end{aligned}$$

Estimate V: Note that, setting $S_\lambda(t)y = v_\lambda(t)$, by Corollary 5.269

$$\begin{aligned} &\|S_\lambda(t)x - S_\lambda(t)y\| = \|u_\lambda(t) - v_\lambda(t)\| \\ \leq\;& e^{\frac{(\alpha-1)t}{\lambda}}\|x-y\| = e^{\frac{|\omega|t}{1-\lambda|\omega|}}\|x-y\| \\ <\;& \underbrace{e^{\frac{|\omega|T}{1-\lambda|\omega|}}}_{\text{bounded}}\varepsilon \end{aligned} \tag{5.10.364}$$

Estimate VI: Since we are under the hypotheses of Theorem **??**, $S(t)$ is Lipschitz continuous on bounded intervals, and therefore

$$\|S(t)y - S(t)x\| \leq L\|x - y\| < L\varepsilon. \tag{5.10.365}$$

In this way, from (5.10.363), (5.10.364) and (5.10.365) it follows that

$$\lim_{\lambda \to 0} u_\lambda(t) = S(t)x, \ \forall x \in \overline{D(A)} \text{ and } \forall t \geq 0$$

which concludes the proof. □

Before stating the next theorem, we will prove some auxiliary results.

Lemma 5.273 *Let X be a Hilbert space and $\varphi : X \to (-\infty, \infty]$ be a convex, proper and lower semi-continuous function. Considering $A = \partial\varphi$ and S the semigroup generated by $-A$, define, for $\lambda > 0$, the function $\varphi_\lambda : X \longrightarrow (-\infty, \infty]$ given by*

$$\varphi_\lambda(x) = \min_{y \in X} \left\{ \frac{1}{\lambda}\|y - x\|^2 + \varphi(y) \right\}.$$

Then:

(i) $\varphi_\lambda(x) = \dfrac{\lambda}{2}\|A_\lambda x\|^2 + \varphi(J_\lambda x)$

(ii) φ_λ *is convex, Gateaux differentiable and* $\partial\varphi_\lambda = A_\lambda$.

Proof:

(i) Set

$$\psi(y) = \frac{1}{2\lambda}\|y - x\|^2 + \varphi(y).$$

Then, $\partial\psi(y) \supset \dfrac{1}{\lambda}(y - x) + \partial\varphi(y)$. By definition, $J_\lambda x$ is the unique solution of

$$0 \in \frac{1}{\lambda}(y - x) + \partial\varphi(y).$$

Thus, $J_\lambda x$ is a minimum of the function ψ, that is,

$$\begin{aligned} \varphi_\lambda(x) &= \min_{y \in X} \left\{ \frac{1}{\lambda}\|y - x\|^2 + \varphi(y) \right\} \\ &= \frac{1}{2\lambda}\|J_\lambda x - x\|^2 + \varphi(J_\lambda x) = \frac{\lambda}{2}\|A_\lambda x\|^2 + \varphi(J_\lambda x). \end{aligned}$$

(ii) Since $A_\lambda x \in A(J_\lambda x) = \partial\varphi(J_\lambda x)$, we have

$$\varphi(J_\lambda y) - \varphi(J_\lambda x) \geq (A_\lambda x, J_\lambda y - J_\lambda x).$$

Thus,

$$\begin{aligned} \varphi_\lambda(y) - \varphi_\lambda(x) &= \frac{\lambda}{2}\|A_\lambda y\|^2 - \frac{\lambda}{2}\|A_\lambda x\|^2 + (\varphi(J_\lambda y) - \varphi(J_\lambda x)) \\ &\geq \frac{\lambda}{2}\left(\|A_\lambda y\|^2 - \|A_\lambda x\|^2\right) + (A_\lambda x, J_\lambda y - y + y - x + x - J_\lambda x) \\ &= \frac{\lambda}{2}\left(\|A_\lambda y\|^2 - \|A_\lambda x\|^2\right) + (A_\lambda x, J_\lambda y - y) + (A_\lambda x, x - J_\lambda x) + (A_\lambda x, y - x) \\ &\geq \frac{\lambda}{2}\left(\|A_\lambda y\|^2 + \|A_\lambda x\|^2\right) - \lambda\|A_\lambda x\|\|A_\lambda y\| + (A_\lambda x, y - x) \\ &= \frac{\lambda}{2}\left(\|A_\lambda y\| - \|A_\lambda x\|\right)^2 + (A_\lambda x, y - x). \end{aligned} \tag{5.10.366}$$

Therefore,

$$\varphi_\lambda(y)-\varphi_\lambda(x)-(A_\lambda x, y-x)\geq \frac{\lambda}{2}\left(\|A_\lambda y\|-\|A_\lambda x\|\right)^2\geq 0.$$

Multiplying (5.10.366) by -1 we have

$$\begin{aligned}\varphi_\lambda(x)-\varphi_\lambda(y) &\leq -\frac{\lambda}{2}\left(\|A_\lambda y\|-\|A_\lambda x\|\right)^2-(A_\lambda x+A_\lambda y-A_\lambda y, y-x)\\ &= -\frac{\lambda}{2}\left(\|A_\lambda y\|-\|A_\lambda x\|\right)^2+(A_\lambda y-A_\lambda x, y-x)+(A_\lambda y, x-y),\end{aligned}$$

whence

$$\varphi_\lambda(x)-\varphi_\lambda(y)-(A_\lambda y, x-y)\leq (A_\lambda y-A_\lambda x, y-x). \tag{5.10.367}$$

Thus, exchanging x with y in (5.10.367) and combining with (5.10.366) it follows that

$$0\leq \varphi_\lambda(y)-\varphi_\lambda(x)-(A_\lambda x, y-x)\leq (A_\lambda y-A_\lambda x, y-x), \tag{5.10.368}$$

for all $x, y\in X$ and $\lambda>0$.

If y is of the form $y=x+tz$, $t>0$, we have

$$0\leq \varphi_\lambda(x+tz)-\varphi_\lambda(x)-(A_\lambda x, tz)\leq (A_\lambda(x+tz)-A_\lambda x, tz),$$

whence

$$\begin{aligned}0\leq \frac{\varphi_\lambda(x+tz)-\varphi_\lambda(x)}{t}-(A_\lambda x, z) &\leq (A_\lambda(x+tz)-A_\lambda x, z)\\ &\leq \|z\|\|A_\lambda(x+tz)-A_\lambda x\|\\ &\leq \frac{2\|z\|}{\lambda}\|x+tz-x\|.\end{aligned}$$

Therefore,

$$\lim_{t\to 0}\frac{\varphi_\lambda(x+tz)-\varphi_\lambda(x)}{t}=(A_\lambda x, z).$$

Let us prove that φ_λ is convex. From (5.10.366),

$$\varphi_\lambda(y)-\varphi_\lambda(x)\geq (A_\lambda x, y-x), \tag{5.10.369}$$

for all $x, y\in X$. Setting $y:=x$ and $x:=(1-t)x+ty$, we have

$$\varphi_\lambda(x)-\varphi_\lambda((1-t)x+ty)\geq t(A_\lambda((1-t)x+ty), x-y). \tag{5.10.370}$$

Substituting now in (5.10.369) $x:=(1-t)x+ty$ it follows

$$\begin{aligned}\varphi_\lambda(y)-\varphi_\lambda((1-t)x+ty) &\geq (A_\lambda((1-t)x+ty), y-(1-t)x+ty)\\ &= -(1-t)(A_\lambda((1-t)x+ty), x-y)\end{aligned} \tag{5.10.371}$$

Do $(1-t)$(5.10.370) $+\,t$(5.10.371) to obtain

$$(1-t)\varphi_\lambda(x)+t\varphi_\lambda(y)-\varphi_\lambda((1-t)x+ty)\geq 0.$$

It only remains to calculate $\partial\varphi_\lambda$. Now, φ_λ is convex and proper. Since it is also Gateaux differentiable, by proposition 4.16 we have that φ_λ is subdifferentiable at every point and $A_\lambda x$ is the unique element of $\partial\varphi_\lambda(x)$.

□

Corollary 5.274 *φ_λ is Frechet differentiable.*

Proof: From (5.10.368),

$$0 \leq \varphi_\lambda(y) - \varphi_\lambda(x) - (A_\lambda x, y - x) \leq (A_\lambda y - A_\lambda x, y - x),$$

whence

$$0 \leq |\varphi_\lambda(y) - \varphi_\lambda(x) - (A_\lambda x, y - x)| \leq \frac{2}{\lambda}\|y - x\|^2.$$

□

Lemma 5.275 *If A is an m-monotone operator on a Hilbert space, then $\overline{D(A)}$ is convex.*

We will also use the following properties:

(i) If $x \in De(f) \cap De(g)$, then $\partial(f+g)(x) \supset \partial f(x) + \partial g(x)$.

(ii) x is a minimum point of f if, and only if, $0 \in \partial f(x)$.

Now we can state the following result:

Theorem 5.276 *Let X be a Hilbert space and $\varphi : X \longrightarrow (-\infty, \infty]$ be a convex, proper and lower semicontinuous function. Considering $A = \partial\varphi$ and S the semigroup generated by $-A$, if $x \in \overline{D(A)}$ and $t > 0$, then $S(t)x \in D(A)$ and the following hold:*

(i) $\| \overset{\circ}{A} S(t)x\| \leq \frac{1}{t}\|S(t)x - x\|$;

(ii) $\| \overset{\circ}{A} S(t)x\| \leq \| \overset{\circ}{A} v\| + \frac{1}{t}\|v - x\|, \ \forall v \in D(A)$.

Proof: Let $x \in \overline{D(A)}$. We have that $A = \partial\varphi$ is m-accretive, and thus, $\overline{D(A)}$ is convex. Thus, according to Theorem 5.272, the problem

$$\begin{cases} \dfrac{du_\lambda}{dt} + A_\lambda u_\lambda = 0, \ \lambda > 0, \\ u_\lambda(0) = x, \end{cases} \tag{5.10.372}$$

possesses a strong solution u_λ, for all $x \in \overline{D(A)}$.

According to Lemma 5.273, for all $v \in X$

$$\begin{aligned} \varphi_\lambda(v) - \varphi_\lambda(u_\lambda(t)) &\geq (\partial\varphi_\lambda(u_\lambda(t)), v - u_\lambda(t)) \\ &= (A_\lambda u_\lambda(t), v - u_\lambda(t)) \\ &= \left(-\frac{du_\lambda}{dt}(t), v - u_\lambda(t)\right) \\ &= \frac{1}{2}\frac{d}{dt}\|v - u_\lambda(t)\|^2. \end{aligned}$$

Integrating from 0 to T

$$\frac{1}{2}\|v - u_\lambda(T)\|^2 - \frac{1}{2}\|v - x\|^2 \leq T\varphi_\lambda(v) - \int_0^T \varphi_\lambda(u_\lambda(t))dt. \tag{5.10.373}$$

From (5.10.372), composing with $t\dfrac{du_\lambda}{dt}(t)$ we have

$$t\left\|\frac{du_\lambda}{dt}(t)\right\|^2 + t\left\langle A_\lambda u_\lambda(t), \frac{du_\lambda}{dt}(t)\right\rangle = 0. \tag{5.10.374}$$

By Corollary (5.274), φ_λ is Frechet differentiable and its derivative is A_λ. Thus,

$$\frac{d\varphi_\lambda}{dt}(u_\lambda(t)) = \left\langle A_\lambda u_\lambda(t), \frac{du_\lambda}{dt}(t) \right\rangle$$

by a generalization of the chain rule. Thus, from (5.10.374)

$$t\left\|\frac{du_\lambda}{dt}(t)\right\|^2 + t\frac{d}{dt}\varphi_\lambda(u_\lambda(t)) = 0.$$

Therefore, we have

$$\begin{aligned}
\int_0^T t\left\|\frac{du_\lambda}{dt}(t)\right\|^2 dt &= -\int_0^T t\frac{d}{dt}(\varphi_\lambda u_\lambda)(t)dt \\
&= -T\varphi_\lambda(u_\lambda(T)) + \int_0^T \varphi_\lambda(u_\lambda(t))dt \\
&\leq -T\varphi_\lambda(u_\lambda(T)) + T\varphi_\lambda(v) - \frac{1}{2}\|v-u_\lambda(T)\|^2 + \frac{1}{2}\|v-x\|^2.
\end{aligned}$$

By Theorem 5.250, $\left\|\frac{du_\lambda}{dt}\right\|$ is non-increasing. Thus,

$$\frac{T^2}{2}\left\|\frac{du_\lambda}{dt}(T)\right\|^2 \leq T\varphi_\lambda(v) - T\varphi_\lambda(u_\lambda(T)) - \frac{1}{2}\|v-u_\lambda(T)\|^2 + \frac{1}{2}\|v-x\|^2. \tag{5.10.375}$$

Setting $v = u_\lambda(T)$,

$$\left\|\frac{du_\lambda}{dt}(T)\right\| \leq \frac{1}{T}\|u_\lambda(T) - x\|.$$

Since T is arbitrary,

$$\|A_\lambda u_\lambda(t)\| \leq \frac{1}{t}\|u_\lambda(t) - x\|, \ t > 0. \tag{5.10.376}$$

We also have, by Theorem 5.272,

$$\lim_{\lambda\to 0} u_\lambda(t) = S(t)x \quad \forall x \in \overline{D(A)}.$$

Then, for $t > 0$,

$$\limsup_{\lambda\to 0}\|A_\lambda u_\lambda(t)\| \leq \frac{1}{t}\|S(t)x - x\|.$$

Therefore, for each $t > 0$, there exists a sequence λ_n converging to zero and $y(t) \in X$ such that $A_{\lambda_n}u_{\lambda_n}(t) \rightharpoonup y(t)$. Since

$$\|u_{\lambda_n}(t) - J_{\lambda_n}(u_{\lambda_n})(t)\| = \lambda_n\|A_{\lambda_n}u_{\lambda_n}(t)\|,$$

we have that $J_{\lambda_n}u_{\lambda_n}(t) \longrightarrow S(t)x$. By Proposition 5.97, A is demiclosed and by Proposition 5.75,

$$(J_{\lambda_n}u_{\lambda_n}(t), A_{\lambda_n}u_{\lambda_n}(t)) \in A,$$

hence it follows that $S(t)x \in D(A)$. It remains to prove estimates (i) and (ii).

Since A is demiclosed, we have that $y(t) \in AS(t)x$. From Theorem 5.265, item (i), $AS(t)x$ has a unique element of minimal norm,

$$\begin{aligned} \| \overset{\circ}{A} S(t)x\| &\leq \|y(t)\| \\ &\leq \liminf \|A_{\lambda_n} u_{\lambda_n}(t)\| \\ &\leq \limsup \|A_\lambda u_\lambda(t)\| \\ &\leq \frac{1}{t}\|S(t)x - x\|. \end{aligned}$$

Now, if $v \in D(A)$, note that

$$\begin{aligned} \varphi_\lambda(v) - \varphi_\lambda(u_\lambda(T)) &\leq |(A_\lambda v, u_\lambda(T) - v)| \\ &\leq \|A_\lambda v\|\|u_\lambda(T) - v\|. \end{aligned}$$

From (5.10.375),

$$\frac{T^2}{2}\left\|\frac{du_\lambda}{dt}(T)\right\|^2 \leq T\|A_\lambda v\|\|u_\lambda(T) - v\| + \frac{1}{2}\|v - x\|^2 - \frac{1}{2}\|v - u_\lambda(T)\|^2.$$

By Young's inequality we have

$$T\|A_\lambda v\|\|u_{\lambda(T)} - v\| \leq \frac{1}{2}T^2\|A_\lambda v\|^2 + \frac{1}{2}\|v - u_{\lambda(T)}\|^2.$$

Thus,

$$\left\|\frac{du_\lambda}{dt}(T)\right\|^2 \leq \|A_\lambda v\|^2 + \frac{1}{T^2}\|v - x\|^2 \leq \left(\|A_\lambda v\| + \frac{1}{T}\|v - x\|\right)^2.$$

Since T is arbitrary

$$\left\|\frac{du_\lambda}{dt}(T)\right\| = \|A_\lambda u_\lambda(t)\| \leq \|A_\lambda v\| + \frac{1}{t}\|v - x\|, \ \forall t > 0, \ \forall v \in D(A).$$

By Proposition 5.108, item (ii), it follows that

$$\| \overset{\circ}{A} S(t)x\| \leq \| \overset{\circ}{A} v\| + \frac{1}{t}\|v - x\|,$$

completing the proof. □

Corollary 5.277 *Under the conditions of Theorem 5.276, $\forall x \in \overline{D(A)}$:*

(i) $S(t)x$ is a strong solution of (5.10.293)-(5.10.294);

(ii) $\overset{\circ}{A} S(t)x$ is right continuous at every $t \geq 0$;

(iii) $S(t)x$ is differentiable from the right at every $t > 0$ and

$$\frac{d^+}{dt}S(t)x + \overset{\circ}{A} S(t)x = 0, \ \forall t > 0.$$

Proof: Immediate consequence of Theorems 5.265 and 5.276. □

5.11 Examples

Example 5.278 *Let X be a Hilbert space and $f : X \to (-\infty, +\infty]$ be a convex, proper and l.s.c. function. By Proposition 5.46, ∂f is an m-monotone operator and, therefore, m-accretive. Since $\overline{D(\partial f)} = \overline{D_e(f)}$,* *it follows by Corollary 5.277, that the problem*

$$\begin{cases} \dfrac{d}{dt}u + \partial f(u) \ni 0 \\ u(0) = x \quad \forall x \in \overline{D_e(f)} \end{cases} \tag{5.11.377}$$

has a strong solution, $S(t)x$, where S is the semigroup generated by $-\partial f$ on $\overline{D_e(f)}$.

(1) Consider in particular $X = \mathbb{R}$,

$$\begin{aligned} f : \mathbb{R} &\longrightarrow (-\infty, +\infty] \\ x &\longmapsto f(x) = \|x\| \end{aligned}$$

In this case, the operator $A = \partial f$ is defined by

$$D(A) = \mathbb{R}, \quad Ax = \begin{cases} -1, & \text{if } x < 0 \\ [-1, 1], & \text{if } x = 0 \\ 1, & \text{if } x > 0. \end{cases} \tag{5.11.378}$$

Indeed, we have that

$$D_e(f) = \{x \in \mathbb{R};\ f(x) < +\infty\} = \{x \in \mathbb{R};\ \|x\| < +\infty\} = \mathbb{R}.$$

Thus, $D(\partial f) = \mathbb{R}$.

We saw in Example 4.17 that $\partial f(0) = [-1, 1]$. Moreover, since $\mathbb{R}$ is smooth (since it is Hilbert), it follows from Theorem 5.42 that the norm in $\mathbb{R}$ is Gateaux differentiable. Thus, by Proposition 4.16, the norm in $\mathbb{R}$ is subdifferentiable on $\mathbb{R} \setminus \{0\}$ and the Gateaux derivative $f'(x)$ is the unique element of $\partial f(x)$ for all $x \in \mathbb{R} \setminus \{0\}$. Now, by Remark 5.44, we have

$$f'(x) = \frac{I(x)}{\|x\|} = \frac{x}{\|x\|} = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0. \end{cases}$$

Therefore,

$$\partial f(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0. \end{cases}$$

Thus, we obtain the characterization of the operator $A = \partial f$ given in (5.11.378). Since the norm is a convex, proper and l.s.c. function, it follows that problem (5.11.377) has a strong solution for all $x \in \mathbb{R}$ when $\partial f = A$.

(2) Let us consider another particular case of (5.11.377). Take $X = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open, bounded and with smooth boundary, and $f : L^2(\Omega) \longrightarrow (-\infty, +\infty]$ the function given by

$$f(u) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\nabla u|^2 dx, & \text{if } u \in H^1(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that the function f is proper, since

$$D_e(f) = \{u \in L^2;\ f(u) < +\infty\} = H^1(\Omega) \neq \emptyset.$$

We saw in Example 4.11 that f is l.s.c., and moreover, f is convex. Indeed,

$$\begin{aligned}
f(tu+(1-t)v) &= \frac{1}{2}\int_\Omega |\nabla(tu+(1-t)v)|^2 dx \\
&\leq \frac{1}{2}\int_\Omega (t|\nabla u|+(1-t)|\nabla v|)^2 dx \\
&= \frac{1}{2}\int_\Omega t^2|\nabla u|^2+2t(1-t)|\nabla u||\nabla v|+(1-t)^2|\nabla v|^2 dx \\
&= \frac{1}{2}\int_\Omega t|\nabla u|^2+(1-t)|\nabla v|^2-t(1-t)(|\nabla u|-|\nabla v|)^2 dx \\
&\leq \frac{1}{2}\int_\Omega t|\nabla u|^2+(1-t)|\nabla v|^2 dx \\
&= tf(u)+(1-t)f(v).
\end{aligned}$$

We conclude that f is convex, proper and l.s.c.. Therefore, by Proposition 5.46, the operator ∂f is m-monotone. Hence, by Theorem 5.54, ∂f is maximal monotone.

Let A be the operator on $L^2(\Omega)$ defined by

$$\begin{aligned}
D(A) &= \{u\in H^2(\Omega);\ \partial_\nu v=0 \text{ on } \partial\Omega\} \\
Au &= -\Delta u,\ \forall u\in D(A).
\end{aligned}$$

We saw in Example 5.57 that A is maximal monotone.

We claim that $A=\partial f$, i.e., $-\Delta=\partial f$. Indeed, let $u\in D(A)$ and $v\in D_e(f)$. We have,

$$\begin{aligned}
\langle Au, v-u\rangle &= \int_\Omega(-\Delta u)(v-u)\ dx \\
&= \int_\Omega(-\Delta u)v\ dx+\int_\Omega(\Delta u)u\ dx \\
&= \int_\Omega \nabla u\nabla v\ dx-\int_\Omega|\nabla u|^2\ dx \\
&\leq \frac{1}{2}\int_\Omega(|\nabla u|^2+|\nabla v|^2)\ dx-\int_\Omega|\nabla u|^2\ dx \\
&= \frac{1}{2}\int_\Omega|\nabla v|^2\ dx-\frac{1}{2}\int_\Omega|\nabla u|^2\ dx \\
&= f(v)-f(u).
\end{aligned}$$

It follows that $Au\in\partial f(u)$, $\forall u\in D(A)$. Thus, $-\Delta\subset\partial f$ and, since $-\Delta$ is maximal monotone, $-\Delta=\partial f$.

Therefore, problem (5.11.377) with $x=u_0$ has a strong solution $S(t)u_0$ for all $u_0\in L^2(\Omega)$, where S is the semigroup generated by $-\partial f=\Delta$ on $L^2(\Omega)$.

Example 5.279 *If $\Omega\subset\mathbb{R}^n$ is open, bounded and has smooth boundary, the operator A of $L^p\Omega$, $1<p<\infty$, defined by*

$$\begin{aligned}
&D(A)=W^{2,p}\Omega\cap W_0^{1,p}\Omega \\
&Au=-\Delta u
\end{aligned}$$

is m-accretive, as was seen in Example 5.91. Since A is closed (because A is m-accretive) and $L^p\Omega$ is reflexive, if S is the semigroup generated by $-A$, the function $S(t)u_0$ is, by Corollary 5.259, a strong solution of the problem

$$\begin{cases} \dfrac{d}{dt}u+Au=0 \\ u(0)=u_0 \end{cases}$$

for all $u_0 \in W^{2,p}\Omega \cap W_0^{1,p}\Omega$.

Example 5.280 *Let* $\beta : \mathbb{R} \to \mathbb{R}$ *be a monotone operator and* Ω *an open set of* $\mathbb{R}^n$. *Let us define the operator* $\tilde{\beta} : L^p\Omega \to L^p\Omega$, $1 < p < \infty$, *by*

$$D(\tilde{\beta}) = \{u \in L^p\Omega;\ \exists v \in L^p\Omega \text{ such that } v(x) \in \beta(u(x)) \text{ a.e. in } \Omega\}$$
$$\tilde{\beta}(u) = \{v \in L^p\Omega;\ v(x) \in \beta(u(x)) \text{ a.e. in } \Omega\},\ \forall u \in D(\tilde{\beta})$$

we will show that $\tilde{\beta}$ *is m-accretive.*

Claim 1: $\tilde{\beta}$ *is accretive.*

Indeed, by item b) of Example 5.71, we have $F(u) = u|u|^{p-2}\|u\|_p^{2-p}$, $\forall u \in L^p\Omega$. *Thus, if* $(u_1, v_1), (u_2, v_2) \in \tilde{\beta}$, *we have*

$$\langle v_1 - v_2, F(u_1 - u_2)\rangle = \|u_1 - u_2\|_p^{2-p} \int_\Omega (u_1 - u_2)|u_1 - u_2|^{p-2}(v_1 - v_2)dx \geq 0$$

since $(u_1(x) - u_2(x))(\underbrace{v_1(x)}_{\in\beta(u_1(x))} - \underbrace{v_2(x)}_{\in\beta(u_2(x))}) \geq 0$, *since* β *is monotone. It follows from Corollary 5.69 that* $\tilde{\beta}$ *is accretive.*

Claim 2: $\tilde{\beta}$ *is m-accretive if* Ω *is bounded or if* $0 \in \beta(0)$. *Indeed, it suffices to show that* $Im(I+\tilde{\beta}) = L^p\Omega$.

Case I: Ω *bounded*

Let $v \in L^p\Omega$, *since* β *is, by hypothesis, accretive (since, in Hilbert spaces, monotonicity is equivalent to the accretivity condition),* $(I + \beta)^{-1}$ *is, by Proposition 5.75, a single-valued operator. Thus, setting, for each* $x \in \Omega$,

$$u(x) = (I + \beta)^{-1}v(x)$$

it suffices to show that $u \in L^p\Omega$, *since from there it follows* $v(x) \in (I + \beta)u(x)$, *that is,* $v(x) - u(x) \in \beta(u(x))$ *with* $v - u \in L^p\Omega$, *hence* $u \in D(\tilde{\beta})$ *and* $v - u \in \tilde{\beta}(u)$, *or even,* $v \in (I + \tilde{\beta})u$ *as we wanted.*

Let us show that $u \in L^p\Omega$. *Indeed, by Proposition 5.76, since the operator* $\beta : \mathbb{R} \to \mathbb{R}$ *is accretive, then,* $J_1 = (I + \beta)^{-1}$ *is a contraction, and since* u *is measurable, set* $c = (I + \beta)^{-1}(0)$, *then*

$$\begin{aligned} |u(x)| = |u(x) + c - c| &\leq |u(x) - c| + |c| \\ &= |(I + \beta)^{-1}v(x) - (I + \beta)^{-1}(0)| + |c| \\ &\leq |v(x)| + |c| \end{aligned}$$

in this way, if Ω *is bounded,* $c \in L^p\Omega$ *and, therefore,* $u \in L^p\Omega$.

Case II: $0 \in \beta(0)$

If $0 \in \beta(0)$ *then* $c = (I + \beta)^{-1}(0) = 0$, *whence* $|u(x)| \leq |v(x)|$ *which implies* $u \in L^p\Omega$. *Now, since* $\tilde{\beta}$ *is closed and* $L^p\Omega$ *is reflexive for* $1 < p < \infty$, *if* S *is the semigroup generated by* $-\tilde{\beta}$, *then the function* $S(t)u_0$, $u_0 \in D(\tilde{\beta})$, *is, by Corollary 5.259, in both cases, a strong solution of the problem*

$$\begin{cases} \dfrac{d}{dt}u + \tilde{\beta}(u) \ni 0 \\ \quad u(0) = u_0 \end{cases}$$

Example 5.281 *The operator* $A + \tilde{\beta}$ *of* $L^p\Omega$, $1 < p < \infty$, *where* $A = -\Delta$ *and* $\tilde{\beta}$ *are the operators described in the examples above, is m-accretive. Indeed, first let us make considerations in order to use Corollary 5.119.*

If $u \in L^p\Omega$, then $F(u)(x) = u(x)|u(x)|^{p-2}\|u\|_p^{2-p}$ and, therefore, if $(u, v) \in A$, we have

$$\begin{aligned}\langle v, F(\tilde{\beta}_\lambda u)\rangle &= \|\beta_\lambda u\|_p^{2-p} \int_\Omega -\Delta u(x)(\beta_\lambda(u(x))|\beta_\lambda(u(x))|^{p-2}dx \\ &= (p-1)\|\beta_\lambda u\|_p^{2-p} \int_\Omega |\beta_\lambda(u(x))|^{p-2}|\nabla u(x)|^2\beta'_\lambda(u(x))dx \geq 0\end{aligned}$$

since the derivative β'_λ of β_λ is non-negative since, by Theorem 5.79, β_λ is accretive. By Corollary 5.119, $A + \tilde{\beta} = -\Delta + \tilde{\beta}$ is m-accretive. Thus, by Proposition 5.89, it is closed and since $L^p\Omega$ is reflexive, the problem

$$\begin{cases} \dfrac{d}{dt}u + (A + \tilde{\beta})u \ni 0 \\ \qquad\qquad u(0) = u_0 \end{cases}$$

has, by Corollary 5.259, for all $u_0 \in D(A + \tilde{\beta})$, a strong solution $S(t)u_0$, where S is the semigroup generated by $-(A + \tilde{\beta})$.

Example 5.282 *Let C be a closed convex subset of a reflexive Banach space, $T : C \to C$ a Lipschitzian map with constant α and $t \geq 0$. By Example 5.80, item a) $\dfrac{I-T}{t} \in \mathcal{A}\left(\dfrac{\alpha-1}{t}\right)$.*

Note that

$$D\left(\frac{I-T}{t}\right) = D(T) = C. \tag{5.11.379}$$

Let us show that there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$

$$C \subset Im\left(I + \lambda\frac{I-T}{t}\right). \tag{5.11.380}$$

Indeed, let $x \in C$, $\lambda > 0$ and define

$$G(y) = \frac{t}{t+\lambda}x + \frac{\lambda}{t+\lambda}Ty$$

Note that $G : C \to C$. Indeed, let $y \in C$. We have to show that $G(y) \in C$. Since $T : C \to C$, $Ty \in C$ and from the fact that $G(y) = \frac{t}{t+\lambda}x + \frac{\lambda}{t+\lambda}Ty$ and $\frac{t}{t+\lambda} + \frac{\lambda}{t+\lambda} = 1$ it follows that $G(y)$ is a convex combination of x and Ty. Thus $G(y) \in C$. Moreover,

$$\|G(y) - G(z)\| = \frac{\lambda}{t+\lambda}\|Ty - Tz\| \leq \frac{\lambda\alpha}{t+\lambda}\|y - z\|.$$

- *If $\alpha \leq 1$ then G is a strict contraction for all $\lambda > 0$.*
- *If $\alpha < 1$ then G is a strict contraction whenever $\lambda < \dfrac{t}{\alpha-1}$.*

In any case, taking $\lambda_0 = \dfrac{t}{\alpha-1}$ we have that G is a strict contraction. Thus it has a unique fixed point $y \in C$, that is,

$$y = G(y) = \frac{t}{t+\lambda}x + \frac{\lambda}{t+\lambda}Ty$$

or even,

$$\frac{t+\lambda}{t}y - \frac{\lambda}{t}Ty = x$$

which implies

$$\left[I - \lambda\frac{I+T}{t}\right]y = x$$

and therefore,

$$x \in Im\left(I - \lambda\frac{I+T}{t}\right)$$

which shows (5.11.380) *and from* (5.11.379) *it follows that*

$$D\left(\frac{I-T}{t}\right) = C \subset Im\left(I - \lambda\frac{I+T}{t}\right).$$

Observe that $\dfrac{I-T}{t}$ *is a closed operator. Setting* $A = \frac{I-T}{t}$ *and* $x \in C$*, by Corollary* 5.259 $S(t)x$ *is a strong solution of* (5.10.293)-(5.10.294) *where* S *is the semigroup generated by* $-A = \dfrac{T-I}{t}$.

Example 5.283 *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a continuous, strictly increasing function such that* $\varphi(0) = 0$, $\varphi(\mathbb{R}) = \mathbb{R}$*. Consider the operator*

$$A : L^1(0,1) \to L^1(0,1),$$

defined by

$$D(A) = \left\{u \in C\left([0,1]; L^1(0,1)\right); u(0) = 0 \text{ and } \varphi(u) \text{ is absolutely continuous } \right\}$$

and

$$Au = \varphi(u)' = \frac{d}{dx}\left[\varphi(u(x))\right].$$

Note that $Au \in L^1(0,1)$ *since* $\varphi \circ u$ *is continuous on* $[0,1]$*. Then the problem*

$$\begin{cases} u_t + (\varphi(u))_x = 0, \ t > 0, \ 0 < x < 1 \\ u(0,x) = u_0(x), \ 0 < x < 1 \\ u(t,0) = 0, \ t > 0 \end{cases}$$

Can be rewritten as

$$\begin{cases} \frac{d}{dt}u + Au = 0, \ t > 0, \ 0 < x < 1 \\ u(0,x) = u_0(x), \ 0 < x < 1 \end{cases}$$

since $u(t) \in D(A)$ *implies that* $u(t,0) = 0$ *for all* $t > 0$.

Let us show that A *is m-accretive.*
A is accretive: Indeed, let $p : \mathbb{R} \to \mathbb{R}$ *be a Lipschitzian, non-decreasing function, such that* $|p| \leq 1$ *and* $p(0) = 0$*, and* $j : \mathbb{R} \to \mathbb{R}$ *defined by*

$$j(s) = \int_0^s p(\tau)d\tau.$$

Let $u, v \in D(A)$*. Then*

$$\begin{aligned}
\|u - v + \lambda(Au - Av)\|_{L^1(0,1)} &= \int_0^1 |u - v + \lambda(Au - Av)|dx \\
= & \int_0^1 |u - v + \lambda(\varphi(u)' - \varphi(v)')|dx \\
\geq & \int_0^1 |u - v + \lambda(\varphi(u)' - \varphi(v)')||p(\varphi(u) - \varphi(v))|dx \\
\geq & \int_0^1 [u - v + \lambda(\varphi(u)' - \varphi(v)')]\, p(\varphi(u) - \varphi(v))dx \\
= & \int_0^1 (u-v)p(\varphi(u) - \varphi(v))dx + \lambda\int_0^1 (\varphi(u) - \varphi(v))'p(\varphi(u) - \varphi(v))dx
\end{aligned} \quad (5.11.381)$$

(5.11.382)

From the definition of j *we have that*

$$\frac{d}{dx}j(s(x)) = \frac{d}{dx}\int_0^{s(x)} p(\tau)d\tau = \frac{d}{dx}p(s(x)) = p(s(x))\frac{d}{dx}s(x).$$

Setting $s = \varphi(u) - \varphi(v)$ *it follows that*

$$(j(\varphi(u) - \varphi(v)))' = p(\varphi(u) - \varphi(v))(\varphi(u) - \varphi(v))',$$

that is,

$$\int_0^1 (\varphi(u) - \varphi(v))' p(\varphi(u) - \varphi(v))dx = \int_0^1 (j(\varphi(u) - \varphi(v)))' dx \tag{5.11.383}$$

$$= \underbrace{j(\varphi(u(1)) - \varphi(v(1))}_{\geq 0} - \underbrace{j(\varphi(u(0)) - \varphi(v(0)))}_{=0} \geq 0, \tag{5.11.384}$$

since $p(0) = 0$ *and* p *does not decrease, i.e.,* $j(s) = \int_0^s \underbrace{p(\tau)}_{\geq 0} d\tau \geq 0$. *Thus,* (5.11.381) *and* (5.11.383) *we obtain*

$$\|u - v + \lambda(Au - Av)\|_{L^1(0,1)} \geq \int_0^1 (u - v)p(\varphi(u) - \varphi(v))dx \tag{5.11.385}$$

In particular, (5.11.385) *holds for* $p = p_n$ *where, for each* $n \in \mathbb{N}$, $p_n : \mathbb{R} \to \mathbb{R}$ *is defined by*

$$p_n(s) = \begin{cases} ns, & \text{if } |s| < \frac{1}{n} \\ sign(s), & \text{if } |s| \geq \frac{1}{n} \end{cases}$$

where

$$sign(s) = \begin{cases} 1, & \text{if } s > 0 \\ 0, & \text{if } s = 0 \\ -1, & \text{if } s < 0 \end{cases}$$

But p_n *converges at every point* $s \in \mathbb{R}$ *to* $sign(s)$. *Thus,* $p_n(\varphi(u) - \varphi(v))$ *converges, at each point of* $[0,1]$, *to* $sign(\varphi(u) - \varphi(v))$. *Moreover, since* φ *is strictly increasing,* $sign(\varphi(u) - \varphi(v)) = sign(u - v)$. *Hence,* $p_n(\varphi(u) - \varphi(v))$ *converges at each point of* $[0,1]$ *to* $sign(u - v)$. *Since*

$$|(u - v)p_n(\varphi(u) - \varphi(v))| = |u - v| \underbrace{|p_n(\varphi(u) - \varphi(v))|}_{\leq 1} \leq |u - v|$$

and by hypothesis $u - v$ *is integrable, then, by the Lebesgue Dominated Convergence Theorem, it follows that*

$$\|u - v + \lambda(Au - Av)\|_{L^1(0,1)} \geq \int_0^1 (u - v)sign(u - v)dx = \int_0^1 |u - v|dx,$$

that is,

$$\|u - v + \lambda(Au - Av)\|_{L^1(0,1)} \geq \|u - v\|_{L^1(0,1)}dx,$$

that is, A *is accretive.*
<u>*A is m-accretive*</u>: *For this, we must show that for all* $h \in L^1(0,1)$ *there exists* $u \in D(A)$ *such that*

$$u + Au = u + \varphi(u)' = h$$

or, setting $\beta = \varphi^{-1}$ *and* $v = \varphi(u)$, *show that there exists* v *absolutely continuous satisfying*

$$\begin{cases} v' + \beta(v) = h \\ v(0) = 0 \end{cases} \tag{5.11.386}$$

Consider first $h \in C([0,1])$. *By Peano's Theorem, equation* (5.11.386) *has a local solution* v *on an interval* $[0,a)$, $0 < a \leq 1$, *such that* $v(0) = 0$. *Let us show that* v *is unique. Indeed, suppose there exists another solution* ω, *then*

$$\begin{cases} \omega' + \beta(\omega) = h \\ \omega(0) = 0 \end{cases}$$

Hence,

$$(v - \omega)' + \beta(v) - \beta(\omega) = 0$$

which implies that

$$\frac{1}{2}\frac{d}{dx}|v-\omega|^2 + [\beta(v)-\beta(\omega)]\,(v-\omega) = 0.$$

Since β *is increasing we have that* $[\beta(v)-\beta(\omega)]\,(v-\omega) \geq 0$ *and therefore*

$$\frac{d}{dx}|v-\omega|^2 \leq 0.$$

Integrating from 0 *to* x*,* $0 \leq x < a$ *we have:*

$$0 \geq |v(x)-\omega(x)|^2 - \underbrace{|v(0)-\omega(0)|^2}_{=0}$$

which implies $v = \omega$ *in* $[0,a)$ *proving the uniqueness of solution.*

To extend v *to the interval* $[0,1]$*, observe that, exactly as it was shown that* A *is an accretive operator of* $L^1(0,1)$*, it is shown that* A *is accretive in* $L^1(0,a)$*. Observe also that* $0 \in D(A)$ *and* $A(0) = \dfrac{d}{dx}[\varphi(0)] = 0$*. Hence, setting* $u = \beta(v)$ *we have:*

$$\begin{aligned}\int_0^a |u|dx &= \int_0^a |u-0|dx \leq \int_0^a |u-0+Au-A0|dx\\ &= \int_0^a |u+Au|dx \leq \int_0^1 |h|dx = \|h\|_{L^1(0,1)}\end{aligned}$$

and therefore, if $0 \leq x < a$*,*

$$\begin{aligned}|v(x)| &= \left|\int_0^x v'(s)ds\right| \leq \int_0^a |v'|ds\\ &= \int_0^a |h-\beta(v)|ds \leq \int_0^a |h|ds + \int_0^a |u|ds\\ &\leq 2\|h\|_{L^1(0,1)}\end{aligned}$$

that is,

$$|v(x)| \leq 2\|h\|_{L^1(0,1)}$$

and therefore the solution v *can be extended to the interval* $[0,1]$*.*

Let now $h \in L^1(0,1)$ *and* $(h_n) \subset C([0,1])$ *such that* $h_n \to h$ *in* $L^1(0,1)$*. From what has already been demonstrated, for each* $n \in \mathbb{N}$ *there exists* u_n *such that* $u_n + Au_n = h_n$*. From this and the accretivity of* A *it follows*

$$\|u_n-u_m\|_{L^1(0,1)} \leq \|u_n-u_m+Au_n-Au_m\|_{L^1(0,1)} = \|h_n-h_m\| \longrightarrow 0$$

when $m,n \to \infty$*. Thus there exists* $u \in L^1(0,1)$ *such that* $u_n \to u$ *in* $L^1(0,1)$*. But then,* $Au_n = h_n - u_n \to h-u$ *in* $L^1(0,1)$*.*

To complete the proof it suffices to demonstrate that A *is a closed operator. Let, for this,* $(u_n) \subset D(A)$*,* $u_n \to u$ *and* $Au_n \to \omega$ *in* $L^1(0,1)$*. We must show that* $u \in D(A)$ *and* $Au = \omega$*. By hypothesis* $\varphi(u_n)$ *is absolutely continuous, thus*

$$\begin{aligned}&\left|\varphi(u_n)(x) - \int_0^x \omega(\tau)d\tau\right| = \left|\int_0^x (\varphi(u_n)'-\omega)(\tau)d\tau\right|\\ \leq\ &\int_0^1 |(\varphi(u_n)'-\omega)(\tau)|d\tau = \underbrace{\|Au_n-\omega\|}_{\to 0},\ \forall x \in [0,1].\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \varphi(u_n)(x) = \int_0^x \omega(\tau)d\tau.$$

By the continuity of β

$$\lim_{n\to\infty} u_n(x) = \beta\left(\int_0^x \omega(\tau)d\tau\right).$$

On the other hand, since $u_n \to u$ *in* $L^1(0,1)$*, there exists a sequence* n_k *such that*

$$\lim_{k\to\infty} u_{n_k}(x) = u(x) \ \text{a.e. in } [0,1].$$

Hence,

$$u(x) = \beta\left(\int_0^x \omega(\tau)d\tau\right) \ \text{a.e. in } [0,1].$$

Now, redefine u *so that the equality holds at every point of* $[0,1]$*, from where it follows that* u *is continuous,* $u(0) = 0$ *and*

$$\varphi(u)(x) = \varphi(\beta(\int_0^x \omega(\tau)d\tau)) = \int_0^x \omega(\tau)d\tau.$$

Note that $\varphi(u)$ *is absolutely continuous. Thus* $u \in D(A)$ *and* $Au = \varphi(u)' = \omega$*. Therefore* A *is m-accretive.*

From this it follows that A *is under the conditions of Theorem* **??** *(Crandall-Liggett) and by Remark 5.260 the function* $S(t)u_0$*, where* S *is the semigroup generated by* $-A$*, is a generalized solution for all* $u_0 \in \overline{D(A)}$*.*

Example 5.284 *Let* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$*, be strictly increasing and such that* $\varphi(0) = 0$ *and* $\varphi(\mathbb{R}) = \mathbb{R}$*. Consider* $A : L^1(0,1) \longrightarrow L^1(0,1)$ *defined by*

$$D(A) = \{u \in C([0,1]); u(0) = u(1) = 0, \varphi(u) \text{ and } [\varphi(u)]' \text{ are absolutely continuous}\}$$

and

$$Au = -[\varphi(u)]'', u \in D(A).$$

Let us prove that A *is m-accretive.*

Let p *and* j *be as in the previous example. From the fact that* $|p| \le 1$*,* $\||a| - |b|\| \le |a - b|$ *and* $a \le |a|$*, it follows that*

$$\|u - v + \lambda(Au - Av)\|_{L^1(0,1)} = \int_0^1 |u - v - \lambda(\varphi(u) - \varphi(v))''|dx$$
$$\ge \int_0^1 |u - v - \lambda(\varphi(u) - \varphi(v))''| \cdot |p(\varphi(u) - \varphi(v))|dx$$
$$\ge \int_0^1 (u - v)p(\varphi(u) - \varphi(v))dx - \lambda \int_0^1 (\varphi(u) - \varphi(v))'' \cdot p(\varphi(u) - \varphi(v))dx.$$

The idea now is to prove that the second term of the sum above is positive, for if this is the case,

$$\|u - v - \lambda(Au - Av)\|_{L^1(0,1)} \ge \int_0^1 (u - v)p(\varphi(u) - \varphi(v))dx \longrightarrow \|u - v\|_{L^1(0,1)},$$

as already done previously.

If $s = \varphi(u) - \varphi(v)$*, then* $j'(s) = s'p(s)$ *is absolutely continuous. In this way*

$$\int_0^1 (j(s))''dx = (j(s))'(1) - (j(s))'(0) = 0.$$

But

$$(j(s))'' = [s'\ p(s)]' = (s')^2\ p'(s) + s''\ p(s),$$

whence

$$\int_0^1 (\varphi(u) - \varphi(v))''\ p(\varphi(u) - \varphi(v))dx = -\int_0^1 (\varphi(u)' - \varphi(v)')^2\ p'(\varphi(u) - \varphi(v)) \leq 0,$$

since p *is increasing* ($p' \geq 0$).

This proves that A *is accretive.*

To prove that A *is m-accretive, set* $\beta = \varphi^{-1}$ *and* $v = \varphi(u)$*. As in the previous example, we must prove that given* $h \in L^1(0,1)$*, there exists* v *such that* v *and* v' *are absolutely continuous,* $v(0) = v(1) = 0$ *and* $\beta(v) - v'' = h$.

Observe initially that if v *satisfies the statement above, then* v *is bounded and* $\|v\|_\infty \leq 2\|h\|_{L^1(0,1)}$. *Indeed, since* A *is accretive and* $A(0) = 0$,

$$\|\beta(v)\|_{L^1(0,1)} = \|u\|_{L^1(0,1)} = \|u - 0\|_{L^1(0,1)} \leq \|u - 0 + (Au - A0)\|_{L^1(0,1)} = \|h\|_{L^1(0,1)}.$$

Since $v(0) = v(1) = 0$*, there exists* $\xi \in (0,1)$ *such that* $v'(\xi) = 0$*. Thus,*

$$|v'(x)| \leq \int_\xi^x |v''(\tau)|d\tau \leq \int_0^1 |\beta(v)(\tau) - h(\tau)|d\tau \leq 2\|h\|_{L^1(0,1)}$$

whence

$$|v(x)| \leq \int_0^x |v'(\tau)|d\tau \leq 2\|h\|_1,\ \forall x \in [0,1]. \tag{5.11.387}$$

Let us keep, for a moment, this information and consider an auxiliary map $\tilde{\beta} : \mathbb{R} \longrightarrow \mathbb{R}$*, bounded, non-decreasing and such that* $\tilde{\beta}(0) = 0$

And let us define,

$$Tv(x) = \int_0^1 g(x,y)(\tilde{\beta}(v(y)) - h(y))dy,\ v \in L^1(0,1), \tag{5.11.388}$$

where

$$g(x,y) = \begin{cases} y(x-1) & if \quad 0 \leq y < x \leq 1; \\ x(y-1) & if \quad 0 \leq x \leq y \leq 1. \end{cases}$$

Then it is obvious that $Tv(0) = Tv(1) = 0$*,* Tv *and* $(Tv)'$ *are absolutely continuous. To conclude the proof, it suffices to show that* $T : S \longrightarrow S$*,* $S \subset C([0,1])$*), has a fixed point.*

Let $K > 0$*, such that* $|\tilde{\beta}(s)| \leq K, \forall s \in \mathbb{R}$*. Then*

$$\begin{aligned} |Tv(x)| &\leq \int_0^1 |g(x,y)| \cdot |\tilde{\beta}(v)(y) - h(y)|dy \\ &\leq \int_0^1 |\tilde{\beta}(v)(y) - h(v)|dy \leq K + \|h\|_1, \end{aligned}$$

and also

$$|(Tv)'(x)| \leq \int_0^1 |\tilde{\beta}(v)(y) - h(y)|dy \leq K + \|h\|_1.$$

Then, $T(L^1(0,1)) \subseteq S$ *where*

$$S = \{w \in C([0,1]); w(0) = w(1) = 0, \|w\|_\infty, \|w'\|_\infty \leq K + \|h\|_1\}.$$

By the Arzelà-Ascoli theorem, S *is relatively compact in* $C([0,1])$*. By Schauder's fixed point theorem, it suffices to prove that* T *is continuous.*

Let $\{v_n\} \subset C([0,1])$ *such that* $v_n \to v$ *in* $C([0,1])$*. Then* $\tilde{\beta}(v_n)(y) \to \tilde{\beta}(v)(y)$ *and* $\tilde{\beta}$ *is bounded. We have*

$$Tv_n(x) - Tv(x) = \int_0^1 g(x,y)(\tilde{\beta}(v_n)(y) - \tilde{\beta}(v)(y))dy,$$

whence

$$|Tv_n(x) - Tv(x)| \leq \int_0^1 |\tilde{\beta}(v_n)(y) - \tilde{\beta}(v)(y)|dy \longrightarrow 0,$$

by the Lebesgue dominated convergence theorem. Therefore, T *has a fixed point which is a solution of* $\tilde{\beta}(v) - v'' = h$*.*

Let us return then to the proof, let $h \in C([0,1])$*, and consider*

$$\tilde{\beta}(s) = \begin{cases} 2\|h\|_\infty & if \quad \beta(s) > 2\|h\|_\infty; \\ \beta(s) & if \quad |\beta(s)| \leq 2\|h\|_\infty; \\ -2\|h\|_\infty & if \quad \beta(s) < -2\|h\|_\infty. \end{cases}$$

Thus, $\tilde{\beta} : \mathbb{R} \longrightarrow \mathbb{R}$ *is non-decreasing and* $\tilde{\beta}(0) = 0$*, so, by what was done above, there exists* $v \in S$*, such that* $\tilde{\beta}(v) - v'' = h$*. Now, let* $y_0 \in [0,1]$ *such that* $v(y_0) = \max_{x\in[0,1]}\{v(x)\}$*, so,* $v''(y_0) < 0$*, hence*

$$\tilde{\beta}(v(x)) \leq \tilde{\beta}(v(y_0)) \leq \tilde{\beta}(v(y_0)) - v''(y_0) = h(y_0) \leq \max_{x\in[0,1]}\{h(x)\}$$

for all $x \in [0,1]$*. Analogously, let* y_1 *such that* $v(y_1) = \min_{x\in[0,1]}\{v(x)\}$*, in this case, we have*

$$\tilde{\beta}(v(x)) \geq \min_{x\in[0,1]}\{h(x)\}$$

for all $x \in [0,1]$*. That is, for all* $x \in [0,1]$

$$|\tilde{\beta}(v(x))| \leq \|h\|_\infty$$

and, therefore, $\tilde{\beta}(v) = \beta(v)$ *and* v *satisfies* $\beta(v) - v'' = h$*.*

If $h \in L^1(0,1)$*, there exists* $\{h_n\} \subset C([0,1])$ *such that* $h_n \to h$ *in* $L^1(0,1)$*. We can consider* $\|h_n\|_1 < \|h\|_1$*. With this, define for each* $n \in \mathbb{N}$

$$\tilde{\beta_n}(s) = \begin{cases} 2\|h_n\|_1 & if \quad \beta(s) > 2\|h_n\|_1; \\ \beta(s) & if \quad |\beta(s)| \leq 2\|h_n\|_1; \\ -2\|h_n\|_1 & if \quad \beta(s) < -2\|h_n\|_1. \end{cases}$$

and

$$\tilde{\beta}(s) = \begin{cases} 2\|h\|_\infty & if \quad \beta(s) > 2\|h\|_\infty; \\ \beta(s) & if \quad |\beta(s)| \leq 2\|h\|_\infty; \\ -2\|h\|_\infty & if \quad \beta(s) < -2\|h\|_\infty. \end{cases}$$

It is left as an exercise to the reader to verify that $\tilde{\beta_n} \to \tilde{\beta}$ *uniformly on* $\mathbb{R}$*. Thus, if* T_n *is the operator, as in (5.11.388), but associated to* $\tilde{\beta}_n$ *and* h_n *and* T *the operator associated to* $\tilde{\beta}$ *and* h*, we have that* $T_n \to T$ *uniformly on* $[0,1]$*.*

On the other hand, if v_n is a fixed point of T_n, we have, by the argument above, that

$$\beta(v_n) - v_n'' = h_n$$

and from (5.11.387) $\|v\|_\infty \leq 2\|h_n\|_1$, *but,* $\|h_n\|_1 \leq \|h\|_1$, *so,*

$$\|v_n\|_\infty \leq \|h\|_1,$$

since $\{v_n\} \subset S = S(\tilde{\beta}, h)$ *which is relatively compact, we have that there exist* $v \in \bar{S}$ *and* $\{v_{n_k}\} \subset \{v_{n_k}\}$ *(which we will continue to denote by* $\{v_n\}$*) such that* $v_n \to v$ *uniformly on* $[0,1]$*, consequently,*

$$T_n(v_n) \longrightarrow T(v) \text{ uniformly on } [0,1]$$

but,

$$T_n(v_n) = v_n \forall n \in \mathbb{N}$$

therefore,

$$v_n \longrightarrow T(v) \text{ uniformly on } [0,1]$$

and, by uniqueness of the limit, we have

$$Tv = v.$$

Thus, v *is a fixed point of* T *and, therefore, satisfies,* $\tilde{\beta}(v) - v'' = h$*. Still, from* $\tilde{\beta}_n \to \tilde{\beta}$ *uniformly on* $\mathbb{R}$ *and* $v_n \to v$ *uniformly on* $[0,1]$*, we have*

$$\tilde{\beta}_n(v_n) \to \tilde{\beta}(v) \text{ uniformly on } [0,1]$$

but, $\tilde{\beta}_n(v_n) = \beta(v_n)$*, for all* $n \in \mathbb{N}$ *and,* β *is continuous, so*

$$\beta(v_n) \to \beta(v) \text{ uniformly on } [0,1]$$

thus, by uniqueness of the limit, we have

$$\beta(v) = \tilde{\beta}(v)$$

as we wanted.

Therefore, A *is m-accretive. And thus, for all* $u_0 \in \overline{D(A)}$, $S(t)u_0$ *is a generalized solution of the problem*

$$\begin{cases} u_t - (\varphi(u))_{xx} = 0, & t > 0,\ 0 < x < 1; \\ u(0,x) = u_0(x), & 0 < x < 1; \\ u(t,0) = u(t,1) = 0, & t > 0. \end{cases} \tag{5.11.389}$$

Example 5.285 *Let* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *be continuous, non-decreasing and such that* $\varphi(0) = 0$*. Define* $A : C([0,1]) \longrightarrow C([0,1])$ *by*

$$D(A) = \{u \in C([0,1]); u(0) = u(1) = 0,\ u, u', u'' \in C([0,1])\},$$

$$Au = \{w \in C([0,1]); \varphi(-w) = u''\}.$$

Let us prove that A *is m-accretive.*

Let $u_1, u_2 \in D(A)$, $w_1 \in Au_1$ *and* $w_2 \in Au_2$ *and suppose* $u_1 \neq u_2$*. Then*

$$\|u_1 - u_2\| = |u_1(x_0) - u_2(x_0)| > 0,\ x_0 \in (0,1).$$

Since $u_1 - u_2$ *is continuous, the set of points where this function attains its maximum is closed, so it has a smallest element* x_0*. To fix ideas, suppose* $u_1(x_0) > u_2(x_0)$*. We must have* $w_1(x_0) \geq w_2(x_0)$*.*

Indeed, if $w_1(x_0) < w_2(x_0)$, then $w_1(x) < w_2(x)$ for all x in some open interval J containing x_0. Since φ is non-decreasing,

$$\begin{aligned}(u_1 - u_2)''(x) &= \varphi(-w_1(x)) - \varphi(-w_2(x)) \\ &\geq -w_1(x) + w_2(x) > 0, \ \forall x \in J.\end{aligned}$$

Thus, $(u_1 - u_2)$ is a convex function on J. But x_0 is a maximum of $u_1 - u_2$, whence $u_1 - u_2$ must be constant on J, which contradicts the minimality of x_0.

Thus, for $\lambda > 0$

$$\begin{aligned}\|u_1 - u_2 + \lambda(w_1 - w_2)\| &\geq |u_1(x_0) - u_2(x_0) + \lambda(w_1(x_0) - w_2(x_0))| \\ &\geq |u_1(x_0) - u_2(x_0)| = \|u_1 - u_2\|,\end{aligned}$$

proving that A is accretive.

It remains to prove that given $h \in C([0,1])$, there exists $u \in D(A)$ such that $h \in (I + A)u$, that is, $u'' = \varphi(u - h)$.

Let us assume initially that $|\varphi(s)| \leq K$, $\forall s \in \mathbb{R}$. Consider, as in the previous example, $T : C([0,1]) \longrightarrow C([0,1])$

$$Tu(x) = \int_0^1 g(x,y) \cdot \varphi(u(y) - h(y))dy.$$

Then $(Tu)'' = \varphi(u - h)$. It suffices to prove that T has a fixed point. We have

$$|Tu(x)| \leq \int_0^1 |\varphi(u(y)) - h(y)|dy \leq K$$

and $|(Tu)'(x)| \leq K$.

Thus $T(C([0,1])) \subset S = \{w;\ \|w\|, \|w'\| \leq K\}$. It is sufficient then to prove that T is continuous.

Let $u_n \longrightarrow u$ in $C([0,1])$. Then $\varphi(u_n(y) - h(y)) \longrightarrow \varphi(u(y) - h(y))$ and $|\varphi(u_n(y) - h(y))| \leq K$. By the Lebesgue dominated convergence theorem,

$$|Tu_n(x) - Tu(x)| \leq \int_0^1 |\varphi(u_n(y) - h(y)) - \varphi(u(y) - h(y))|dy \longrightarrow 0.$$

Since $Tu = u$, we have that $u \in D(A)$.

Observe now that if v is a solution of our problem, then it is bounded, whether φ is bounded or not:

$$\|v\| \leq \|v - 0 + 1((h - v) - 0)\| = \|h\|.$$

Hence, if φ is unbounded, define

$$\tilde{\varphi} = \begin{cases} \varphi(2\|h\|), & if \quad s > 2\|h\|; \\ \varphi(s), & if \quad |s| \leq 2\|h\|; \\ \varphi(-2\|h\|) & if \quad s < -\|h\|. \end{cases}$$

Then $\tilde{\varphi}$ is bounded and satisfies the conditions imposed on φ. Therefore, if u is a solution of $u'' = \tilde{\varphi}(u - h)$, since $\|u\| \leq \|h\|$, it follows that $\tilde{\varphi}(u - h) = \varphi(u - h)$. In any case, $h \in (I + A)u$.

Therefore, for all $u_0 \in \overline{D(A)}$, $S(t)u_0$ is a generalized solution of the problem

$$\begin{cases} \varphi(u_t) - u_{xx} = 0, & t > 0,\ 0 < x < 1; \\ u(0,x) = u_0(x), & 0 < x < 1; \\ u(t,0) = u(t,1) = 0, & t > 0. \end{cases} \tag{5.11.390}$$

Indeed

$$0 \in u_t + Au \Longleftrightarrow -u_t \in Au \Longleftrightarrow \varphi(u_t) = u_{xx}.$$

Example 5.286 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with sufficiently smooth boundary. Consider the Cauchy problem:*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + g(u_t) = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{5.11.391}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a monotone increasing, continuous function satisfying the conditions:

$$g(s)s > 0 \text{ for } s \neq 0 \text{ and } ks \leq g(s) \leq Ks \text{ for } |s| > 1,\ k, K > 0.$$

Also, assume that $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$ satisfies $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. We will show the existence of strong and generalized solutions for problem (5.11.391).

Initially, consider the set

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega);\ \ \gamma_0 u = 0 \text{ on } \Gamma_0\}.$$

This set is a closed subspace of $H^1(\Omega)$, when both are endowed with the inner product given by

$$(u,v)_1 = (\nabla u, \nabla v).$$

Moreover, there exists a constant $c > 0$ such that

$$\|u\| \leq c\|\nabla u\|, \quad \forall u \in H^1_{\Gamma_0}(\Omega).$$

Let us denote $\mathcal{V} = H^1_{\Gamma_0}(\Omega)$. Consider also the Laplacian operator $-\Delta : D(-\Delta) \subset L^2(\Omega) \to L^2(\Omega)$ with domain

$$D(-\Delta) = \left\{ v \in \mathcal{V} \cap H^2(\Omega);\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_1 \right\}.$$

Such operator is defined by the triple $\{\mathcal{V}, L^2(\Omega), a\}$, where $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is given by $a(u,v) = (\nabla u, \nabla v)$, $\forall u, v \in \mathcal{V}$. From this it follows that

$$D(-\Delta) = \{u \in \mathcal{V};\, \exists f_u \in L^2(\Omega) \text{ such that } a(u,v) = (f,v), \quad \forall v \in \mathcal{V}\}$$

and that $D(-\Delta)$ is dense in $\mathcal{V}$. Also from the fact that $-\Delta$ is defined by a triple, with a coercive, it follows that $-\Delta$ admits an extension, which we will still denote by $-\Delta$, that is, we have $-\Delta : \mathcal{V} \to \mathcal{V}'$, which satisfies $\|-\Delta u\|_{\mathcal{V}'} = \|u\|_{\mathcal{V}}$, for all $u \in \mathcal{V}$, and also

$$\langle -\Delta u, w \rangle_{\mathcal{V}',\mathcal{V}} = a(u,w), \quad \forall u, w \in \mathcal{V}.$$

Regarding the operator $-\Delta$, we can state:

Claim 1: *If $v \in \mathcal{V}$ then $\gamma_0 v \in H^{1/2}(\Gamma_1)$.*

Claim 2: *If $u \in \mathcal{V} \cap H^2(\Omega)$ with $\dfrac{\partial u}{\partial \nu} = 0$ on Γ_1 and $v \in \mathcal{V}$ then $\langle \gamma_1 u, \gamma_0 v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 0$.*

Claim 3: *The map $\gamma_0 : \mathcal{V} \to H^{1/2}(\Gamma_1)$ is surjective.*

For the proofs of the claims made about $-\Delta$, consult the appendix.

The operator N

Consider the Neumann operator $N : H^{-1/2}(\Gamma_1) \to \mathcal{V}$ given by

$$Nq = p \Longleftrightarrow \begin{cases} -\Delta p & = & 0 & \text{in} & \Omega; \\ p & = & 0 & \text{on} & \Gamma_0; \\ \dfrac{\partial p}{\partial \nu} & = & q & \text{on} & \Gamma_1. \end{cases}$$

Note that, in principle, the conditions $p = 0$ on Γ_0 and $\dfrac{\partial p}{\partial \nu} = q$ on Γ_1 are in the sense of the trace of order zero and one, respectively.

Let us show that this operator is well defined and continuous.

Let $q \in H^{-1/2}(\Gamma_1)$. Define $\varphi : \mathcal{V} \to \mathbb{R}$ by

$$\varphi(v) = \langle q, \gamma_0 v \rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)}.$$

We have $\gamma_0 v \in H^{1/2}(\Gamma_1)$, for all $v \in \mathcal{V}$, by Claim 1.

From the continuity of q and the continuity of the trace map of order 0, we have that $\varphi \in \mathcal{V}'$.

Since $a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)}$ is a bilinear, continuous and coercive function, by the Lax-Milgram lemma, there exists a unique $p \in \mathcal{V}$ such that

$$\langle q, \gamma_0 v \rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} = \varphi(v) = a(p, v) = (\nabla p, \nabla v)_{L^2(\Omega)}, \quad \forall\ v \in \mathcal{V}. \tag{5.11.392}$$

*In particular, for $v \in C_0^\infty(\Omega)$, from (5.11.392) and (**??**) we obtain*

$$(\nabla p, \nabla v) = 0, \ \forall\ v \in C_0^\infty(\Omega), \tag{5.11.393}$$

By 5.11.393, since $(-\Delta p, v) = (\nabla p, \nabla v)$, then $-\Delta p = 0$ in $\mathcal{D}'(\Omega)$, and hence, $\Delta p = 0 \in L^2(\Omega)$. With this, the second generalized Green's formula and an argument analogous to that used in Claim 2, we obtain

$$(\Delta p, v) + (\nabla p, \nabla v) = \langle \gamma_1 p, \gamma_0 v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \langle \gamma_1 p, \gamma_0 v \rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)}, \quad \forall v \in \mathcal{V},$$

that is,

$$(\nabla p, \nabla v) = \langle \gamma_1 p, \gamma_0 v \rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)}, \quad \forall v \in \mathcal{V}.$$

From (5.11.392) it follows that

$$\langle q, \gamma_0 v \rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} = \langle \gamma_1 p, \gamma_0 v \rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)}, \quad \forall v \in \mathcal{V}.$$

By Claim 3, it follows that $q = \gamma_1 p$ in $H^{-1/2}(\Gamma_1)$.

Now, let us prove that N is a continuous operator.

Initially, let us prove that N is a closed operator. For this, let us consider $\{q_n\} \subset D(N)$ and

$q \in H^{-1/2}(\Gamma_1)$ *such that* $q_n \to q$ *in* $H^{-1/2}(\Gamma_1)$, *and let us assume that there exists* $f \in \mathcal{V}$ *such that* $Nq_n \to f$ *in* $\mathcal{V}$. *To obtain that* N *is a closed operator, we must prove that* $q \in D(N)$ *and* $f = Nq$. *Now, but it is clear that* $q \in D(N)$, *since* $D(N) = H^{-1/2}(\Gamma_1)$. *It remains to prove that* $f = Nq$.

For each $n \in \mathbb{N}$, *letting* $p_n = Nq_n$, *we have*

$$\begin{cases} -\Delta p_n &= 0 \quad \text{in} \quad \Omega; \\ \gamma_0 p_n &= 0 \quad \text{on} \quad \Gamma_0; \\ \gamma_1 p_n &= q_n \quad \text{on} \quad \Gamma_1. \end{cases}$$

Since $p_n \to f$ *in* $\mathcal{V}$, *it follows that* $\Delta p_n \to \Delta f$ *in* $\mathcal{D}'(\Omega)$, *whence* $\Delta f = 0 \in L^2(\Omega)$, *and hence,* $f \in \mathcal{H}^1(\Omega)$, *with* $p_n \to f$ *in* $\mathcal{H}^1(\Omega)$. *Since* $\gamma_1 : \mathcal{H}^1(\Omega) \to H^{-1/2}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma_1)$ *is continuous, (the last embedding holds by arguments analogous to those of Claim 1) it follows that* $\gamma_1 f = q$ *in* $H^{-1/2}(\Gamma_1)$. *Moreover, by the continuity of* $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma_1)$, *we have* $\gamma_0 f = 0$ *on* Γ_0. *Therefore, by the definition of the operator* N, *we obtain* $Nq = f$.

Since $D(N) = H^{-1/2}(\Gamma_1)$, *by Theorem 1, it follows that* N *and* N^* *are continuous, with* $D(N^*) = \mathcal{V}'$.

Now, let us prove that $N^*(-\Delta v) = \gamma_0 v$, *for all* $v \in \mathcal{V}$. *It suffices to prove the equality for* $v \in D(-\Delta)$, *since* $D(-\Delta)$ *is dense in* $\mathcal{V}$.

Since

$$D(-\Delta) = \left\{ u \in H^2(\Omega) \cap \mathcal{V}; \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1 \right\},$$

then

$$-\Delta v \in L^2(\Omega) \hookrightarrow \mathcal{V}' \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_1, \text{ for } v \in D(-\Delta). \tag{5.11.394}$$

From the adjoint property, it follows that

$$\langle N^*(-\Delta v), q \rangle_{H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1)} = \langle -\Delta v, Nq \rangle_{\mathcal{V}', \mathcal{V}}, \tag{5.11.395}$$

for $q \in H^{-1/2}(\Gamma_1)$.

Now, let p *satisfying*

$$\begin{cases} \Delta p &= 0 \text{ in } \Omega \\ \quad p &= 0 \text{ on } \Gamma_0 \\ \dfrac{\partial p}{\partial \nu} &= q \text{ on } \Gamma_1 \end{cases} \quad \Leftrightarrow \quad Nq = p. \tag{5.11.396}$$

From (5.11.394)-(5.11.396), the first generalized Green's formula and an argument analogous to that of Claim 2,

$$\begin{aligned} \langle N^*(-\Delta v), q \rangle_{H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1)} &= \langle -\Delta v, Nq \rangle_{\mathcal{V}', \mathcal{V}} \\ &= \langle -\Delta v, p \rangle_{\mathcal{V}', \mathcal{V}} \\ &= (-\Delta v, p) \\ &= (\Delta p, v) - (p, \Delta v) \\ &= \langle \gamma_1 p, \gamma_0 v \rangle_{H^{-3/2}(\Gamma), H^{3/2}(\Gamma)} - \langle \gamma_0 p, \gamma_1 v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \\ &= \langle \gamma_1 p, \gamma_0 v \rangle_{H^{-3/2}(\Gamma), H^{3/2}(\Gamma)} \\ &= \langle \gamma_1 p, \gamma_0 v \rangle_{H^{-3/2}(\Gamma_1), H^{3/2}(\Gamma_1)}. \end{aligned} \tag{5.11.397}$$

But $v \in H^1(\Omega)$ *implies* $\gamma_0 v \in H^{1/2}(\Gamma_1)$ *and* $p \in \mathcal{H}^1(\Omega)$ *implies* $\gamma_1 \in H^{-1/2}(\Gamma_1)$, *and thus,*

$$\langle N^*(-\Delta v), q \rangle_{H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1)} = \langle \gamma_0 v, \gamma_1 p, \rangle_{H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1)} = \langle \gamma_0 v, q \rangle_{H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1)},$$

as we wanted.

The Operator A

Consider the phase space $\mathcal{H} = \mathcal{V} \times L^2(\Omega)$, *endowed with the inner product*

$$\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_{\mathcal{H}} = (\nabla u_1, \nabla u_2) + (v_1, v_2).$$

Consider also the operator A *given by*

$$A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ -\Delta(u + Ng(\gamma_0 v)) \end{pmatrix},$$

with

$$D(A) = \{(u, v) \in \mathcal{V} \times \mathcal{V}; u + Ng(\gamma_0 v) \in D(-\Delta)\}.$$

Initially, let us verify that $Ng(\gamma_0 v)$ *is well defined, that is, that* $g(\gamma_0 v) \in H^{-1/2}(\Gamma_1)$, *for all* $v \in \mathcal{V}$. *Indeed, by Claim 1,* $\gamma_0 v \in H^{1/2}(\Gamma_1)$ *for* $v \in \mathcal{V}$. *Then, to conclude the desired result, it suffices to prove that* $g \circ w \in L^2(\Gamma_1)$, *for all* $w \in H^{1/2}(\Gamma_1)$. *By the continuity of* g *and by the growth hypothesis at infinity, it holds that*

$$|g(s)| \leq \max_{-1 \leq r \leq 1} |g(r)| + \max\{k, K\}|s|, \quad \forall r \in \mathbb{R},$$

that is, there exists $c_1 > 0$ *such that*

$$|g(s)| \leq c_1 + c_1|s|, \quad \forall s \in \mathbb{R},$$

whence,

$$\int_{\Gamma_1} |g(w(x))|^2 \, d\Gamma \leq \int_{\Gamma_1} (c_1 + c_1|w(x)|)^2 \, d\Gamma < \infty,$$

since $w \in H^{1/2}(\Gamma_1) \hookrightarrow L^2(\Gamma_1) \hookrightarrow L^1(\Gamma_1)$. *Thus,* $g \circ w \in L^2(\Gamma_1) \hookrightarrow H^{-1/2}(\Gamma_1)$, *for all* $w \in H^{1/2}(\Gamma_1)$.

Now, let us prove that A *is monotone. For this, considering* $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in D(A)$, *we have*

$$\begin{aligned} \left(A \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - A \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_{\mathcal{H}} &= -(\nabla(v_1 - v_2), \nabla(u_1 - u_2)) \\ &- (\Delta(u_1 + Ng(\gamma_0 v_1)) - \Delta(u_2 + Ng(\gamma_0 v_2)), v_1 - v_2). \end{aligned}$$

By the definition of the operator N,

$$\Delta Ng(\gamma_0 v_1) = \Delta Ng(\gamma_0 v_2) = 0 \in L^2(\Omega)$$

and since

$$\Delta(u_1 + Ng(\gamma_0 v_1)), \ \Delta(u_2 + Ng(\gamma_0 v_2)) \in L^2(\Omega)$$

it follows that $\Delta u_1, \ \Delta u_2 \in L^2(\Omega)$, *and thus,*

$$u_1, \ u_2, \ Ng(\gamma_0 v_1), \ Ng(\gamma_0 v_2) \in \mathcal{H}^1(\Omega).$$

We have that

$$\begin{aligned}
& - \quad (\Delta(u_1 + Ng(\gamma_0 v_1)) - \Delta(u_2 + Ng(\gamma_0 v_2)), v_1 - v_2) \\
& = \quad (\nabla(u_1 + Ng(\gamma_0 v_1)) - \nabla(u_2 + Ng(\gamma_0 v_2)), \nabla v_1 - \nabla v_2) \\
& - \quad \langle \underbrace{\gamma_1(u_1 + Ng(\gamma_0 v_1)) - \gamma_1(u_2 + Ng(\gamma_0 v_2))}_{=0 \ on\ \Gamma_1,\ since\ u_i + Ng(\gamma_0 v_i) \in D(-\Delta)}, \quad \underbrace{\gamma_0 v_1 - \gamma_0 v_2}_{=0\ on\ \Gamma_0,\ since\ v_i \in \mathcal{V}} \quad \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \\
& = \quad (\nabla(u_1 + Ng(\gamma_0 v_1)) - \nabla(u_2 + Ng(\gamma_0 v_2)), \nabla v_1 - \nabla v_2)
\end{aligned}$$

By an argument analogous to that of Claim 2,

$$\begin{aligned}
& \quad (\nabla Ng(\gamma_0 v_1) - \nabla Ng(\gamma_0 v_2), \nabla v_1 - \nabla v_2) \\
= & \quad -(\Delta Ng(\gamma_0 v_1) - \Delta Ng(\gamma_0 v_2), v_1 - v_2) \\
+ & \quad \langle \gamma_1(Ng(\gamma_0 v_1)) - \gamma_1(Ng(\gamma_0 v_2)), \gamma_0 v_1 - \gamma_0 v_2 \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \\
= & \quad \langle g(\gamma_0 v_1) - g(\gamma_0) v_2, \gamma_0 v_1 - \gamma_0 v_2 \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \\
= & \quad \langle g(\gamma_0 v_1) - g(\gamma_0) v_2, \gamma_0 v_1 - \gamma_0 v_2 \rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} \\
= & \quad (g(\gamma_0 v_1) - g(\gamma_0 v_2), \gamma_0 v_1 - \gamma_0 v_2)_{L^2(\Gamma_1)},
\end{aligned}$$

whence we conclude that

$$\left(A \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - A \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_{\mathcal{H}} = (g(\gamma_0 v_1) - g(\gamma_0 v_2), \gamma_0 v_1 - \gamma_0 v_2)_{L^2(\Gamma_1)} \geq 0,$$

since g is monotone increasing. Thus, A is monotone.

Now, let us prove that A is maximal monotone, that is, $Im(I + A) = \mathcal{H}$.

Given $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{H} = \mathcal{V} \times L^2(\Omega)$, we must exhibit $\begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$ such that

$$\begin{cases} u - v = h_1 \\ v - \Delta u - \Delta Ng(\gamma_0 v) = h_2, \end{cases}$$

that is,

$$\begin{cases} u - v = h_1 \\ v - \Delta u - \underbrace{\Delta Ng N^*(-\Delta v)}_{N^*(-\Delta v) = \gamma_0 v} = h_2. \end{cases}$$

Since $u = v + h_1$, we obtain

$$v - \Delta v - \Delta h_1 - \Delta Ng N^*(-\Delta v) = h_2,$$

or equivalently,

$$-\Delta v + v - \Delta Ng N^*(-\Delta v) = \Delta h_1 + h_2 \in \mathcal{V}'. \tag{5.11.398}$$

Define

$$B = (-\Delta) \circ N \circ g \circ N^* \circ (-\Delta).$$

Let us consider the duality map $F : \mathcal{V} \to \mathcal{V}'$ and the extension of the Laplacian operator $-\Delta : \mathcal{V} \to \mathcal{V}'$. Then, given $v \in \mathcal{V}$, there exists $v' \in \mathcal{V}'$ such that $F(v) = v'$. Moreover,

$$\langle v', v \rangle_{\mathcal{V}', \mathcal{V}} = \|v\|_{\mathcal{V}}^2 = (\nabla v, \nabla v) = -\langle \Delta v, v \rangle_{\mathcal{V}', \mathcal{V}},$$

which implies $v' = Fv = -\Delta v$. Thus, $-\Delta : \mathcal{V} \to \mathcal{V}'$ is the duality map.

To prove that A is maximal monotone, we will prove that there exists $v \in \mathcal{V}$ satisfying (5.11.398).

This is equivalent to demonstrating the surjectivity of the operator

$$-\Delta + (I + B) : \mathcal{V} \to \mathcal{V}'.$$

We know that $-\Delta + (I + B)$ is surjective if, and only if, $I + B$ is maximal monotone in $\mathcal{V} \times \mathcal{V}'$. So, initially, let us prove that $I + B$ is maximal monotone in $\mathcal{V} \times \mathcal{V}'$. Let us note that $I : \mathcal{V} \to \mathcal{V} \hookrightarrow \mathcal{V}'$ is continuous, monotone and bounded. We will prove that B is maximal monotone to guarantee that $I + B$ is maximal monotone.

Identifying $L^2(\Gamma_1) \equiv (L^2(\Gamma_1))'$, consider the operator $G : H^{1/2}(\Gamma_1) \to (L^2(\Gamma_1))' \hookrightarrow H^{-1/2}(\Gamma_1)$ given by

$$(Gu, v)_{L^2(\Gamma_1)} = \int_{\Gamma_1} g(u)v \; d\Gamma, \quad \forall v \in L^2(\Gamma_1).$$

G is well defined since we showed that $g \circ u \in L^2(\Gamma_1)$, for all $u \in H^{1/2}(\Gamma_1)$, and moreover, we have $Gz = g \circ z$, for all $z \in H^{1/2}(\Gamma_1)$.

Let us also take the functional $\phi : H^{1/2}(\Gamma_1) \to \mathbb{R}$ given by

$$\phi(u) = \int_{\Gamma_1} \int_{[0,u(x)]} g(\tau) \; d\tau \; d\Gamma,$$

where $[0, u(x)]$ denotes the interval with endpoints 0 and $u(x)$. We will prove that

$$\partial\phi(u) = \phi'(u) = Gu.$$

First, let us see that ϕ is well defined.

$$|\phi(u)| \leq \int_{\Gamma_1} \int_{[0,u(x)]} |g(\tau)| \; d\tau \; d\Gamma = \int_{\{x\in\Gamma_1; |u(x)|>1\}} \int_{[0,u(x)]} |g(\tau)| \; d\tau \; d\Gamma + \int_{\{x\in\Gamma_1; |u(x)|\leq 1\}} \int_{[0,u(x)]} |g(\tau)| \; d\tau \; d\Gamma.$$

When $u(x) < -1$, we have

$$\int_{u(x)}^{0} -g(\tau)d\tau \leq -\int_{u(x)}^{-1} g(\tau)d\tau - \int_{-1}^{0} g(\tau)d\tau \leq \|g\|_{L^1(-1,1)} + (-k)\int_{u(x)}^{-1} \tau d\tau = \|g\|_{L^1(-1,1)} + \frac{k}{2}(u^2(x) - 1).$$

When $u(x) > 1$, we have

$$\int_{0}^{u(x)} g(\tau)d\tau = \int_{0}^{1} g(\tau)d\tau + \int_{1}^{u(x)} g(\tau)d\tau \leq \|g\|_{L^1(-1,1)} + K\int_{1}^{u(x)} \tau d\tau = \|g\|_{L^1(-1,1)} + \frac{K}{2}(u^2(x) - 1).$$

Thus, setting $c_2 = \max\{k, K\}$, it follows

$$\begin{aligned}\int_{\{x\in\Gamma_1; |u(x)|>1\}} \int_{[0,u(x)]} |g(\tau)| \; d\tau \; d\Gamma &\leq |\Gamma_1|\|g\|_{L^1(-1,1)} + c_2 \int_{\{x\in\Gamma_1; |u(x)|>1\}} (u^2(x) - 1)d\Gamma \\ &\leq |\Gamma_1|\|g\|_{L^1(-1,1)} + c_2\|u\|^2_{L^2(\Gamma_1)}.\end{aligned}$$

Also,

$$\int_{\{x\in\Gamma_1; |u(x)|\leq 1\}} \int_{[0,u(x)]} |g(\tau)| \; d\tau \; d\Gamma \leq \int_{\{x\in\Gamma_1; |u(x)|\leq 1\}} \int_{-1}^{1} (c_1 + c_1|\tau|) \; d\tau \; d\Gamma \leq 3c_1|\Gamma_1|,$$

and with this we conclude that $|\phi(u)| < \infty$.

Now, let us prove that ϕ is continuous. Let $\{u_n\} \subset H^{1/2}(\Gamma_1)$ and $u \in H^{1/2}(\Gamma_1)$ such that $u_n \to u$

in $H^{1/2}(\Gamma_1)$*. We have*

$$\begin{aligned}
|\phi(u_n)-\phi(u)| &= \left|\int_{\Gamma_1}\int_{[0,u_n(x)]} g(\tau)d\tau d\Gamma - \int_{\Gamma_1}\int_{[0,u(x)]} g(\tau)d\tau d\Gamma\right| \\
&= \left|\int_{\Gamma_1}\int_{[u_n(x),u(x)]} g(\tau)d\tau d\Gamma\right| \\
&\leq \int_{\Gamma_1}\int_{[u_n(x),u(x)]} |g(\tau)|d\tau d\Gamma \\
&\leq c_1\int_{\Gamma_1}\int_{[u_n(x),u(x)]} (1+|\tau|)d\tau d\Gamma \\
&\leq c_1\int_{\Gamma_1} |u_n(x)-u(x)|d\Gamma + c_1\int_{\Gamma_1} ||u_n(x)|^2-|u(x)|^2|d\Gamma \\
&\leq c_1\int_{\Gamma_1} |u_n(x)-u(x)|d\Gamma + c_1\int_{\Gamma_1} (|u_n(x)|+|u(x)|)|u_n(x)-u(x)|d\Gamma.
\end{aligned}$$

Since $\{u_n\}$ *is bounded in* $L^1(\Gamma_1)$ *and converges to* u *in* $L^2(\Gamma_1)$*, it follows that* $\phi(u_n)\to\phi(u)$*, as we wanted.*

Now, let us prove that ϕ *is Gateaux differentiable. Let* $u,v\in H^{1/2}(\Gamma_1)$*,* $\delta\in\mathbb{R}$ *and* $x\in\Gamma_1$ *be given. Take* $\lambda=\delta v(x)$*, we have*

$$\frac{1}{\delta}\left[\int_{[0,u(x)+\delta v(x)]} g(s)ds - \int_{[0,u(x)]} g(s)ds\right] = \frac{1}{\delta}\left[\int_{[u(x),u(x)+\delta v(x)]} g(s)ds\right] = \frac{1}{\lambda}\left[\int_{[u(x),u(x)+\lambda]} g(s)ds\right]v(x).$$

By the Mean Value Theorem,

$$\lim_{\delta\to 0}\frac{1}{\delta}\left[\int_{[0,u(x)+\delta v(x)]} g(s)ds - \int_{[0,u(x)]} g(s)ds\right] = \lim_{\lambda\to 0}\frac{1}{\lambda}\left[\int_{[u(x),u(x)+\lambda]} g(s)ds\right]v(x) = g(u(x))v(x).$$

Moreover,

$$\begin{aligned}
\left|\frac{1}{\lambda}\int_{[u(x),u(x)+\lambda]} g(s)dsv(x)\right| &\leq \frac{1}{|\lambda|}|v(x)|\{|\lambda| + \left||u(x)|^2-|u(x)+\lambda|^2\right|\} \\
&= \frac{1}{|\lambda|}|v(x)|\{|\lambda|+|\lambda||2u(x)+\lambda|\} \\
&= |v(x)|+|v(x)||2u(x)+\lambda|,
\end{aligned}$$

which is integrable on Γ_1*. Then, by the Lebesgue Dominated Convergence Theorem,*

$$\lim_{\delta\to 0}\frac{\phi(u+\delta v)-\phi(u)}{\delta} = \int_{\Gamma_1} g(u)v\ d\Gamma = (Gu,v)_{L^2(\Gamma_1)},$$

whence it follows that ϕ *is Gateaux differentiable, with* $\phi'(u)=Gu$*.*

Note that $\phi'(u)\in H^{-1/2}(\Gamma_1)$*, since*

$$|\langle\phi'(u),v\rangle| \leq \int_{\Gamma_1}|g(u)||v|d\Gamma \leq \|g\circ u\|_{L^2(\Gamma_1)}\|v\|_{L^2(\Gamma_1)} \leq c_3\|g\circ u\|_{L^2(\Gamma_1)}\|v\|_{H^{1/2}(\Gamma_1)},$$

where $c_3>0$ *is the constant of the embedding* $H^{1/2}(\Gamma_1)\hookrightarrow L^2(\Gamma_1)$*.*

To prove that ϕ *is convex, it suffices to prove that*

$$\langle\phi'(u)-\phi'(v),u-v\rangle_{H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1)} \geq 0, \quad \forall u,v\in H^{1/2}(\Gamma_1).$$

Indeed, since g is monotone increasing,

$$\langle \phi'(u) - \phi'(v), u - v \rangle_{H^{-1/2}(\Gamma_1), H^{1/2}(\Gamma_1)} = (Gu - Gv, u - v)_{\Gamma_1} = \int_{\Gamma_1} [g(u) - g(v)][u - v] d\Gamma \geq 0.$$

Thus, since ϕ is convex, the subdifferential is unitary and consists of the Gateaux derivative, that is,

$$\phi'(u) = Gu = g \circ u = \partial \phi(u), \quad \forall u \in H^{1/2}(\Gamma_1).$$

Since $N : H^{-1/2}(\Gamma_1) \to \mathcal{V}$ is continuous, $N^{**} = N$ and then, defining $\Lambda = N^* \circ (-\Delta)$, we obtain $\Lambda^* = (-\Delta) \circ N$, and thus,

$$B = (-\Delta) \circ N \circ g \circ N^* \circ (-\Delta) = \Lambda^* \circ g \circ \Lambda.$$

Since $-\Delta : \mathcal{V} \to \mathcal{V}'$, $N : H^{-1/2}(\Gamma_1) \to \mathcal{V}$ and $N^* : \mathcal{V}' \to H^{1/2}(\Gamma_1)$ are continuous, it follows that $\Lambda : \mathcal{V} \to H^{1/2}(\Gamma_1)$ is continuous. Recall that $\phi : H^{1/2}(\Gamma_1) \to \mathbb{R}$ is also continuous. Then, it follows that

$$\partial(\phi \circ \Lambda) = \Lambda^* \circ \partial\phi \circ \Lambda,$$

that is,

$$\partial(\phi \circ N^* \circ (-\Delta)) = B.$$

Since ϕ is convex and $\Lambda = N^* \circ (-\Delta)$ is linear, then $\phi \circ \Lambda$ is convex and also continuous, from where it follows that $B = \partial(\phi \circ \Lambda)$ is maximal monotone.

Knowing that $I : \mathcal{V} \to \mathcal{V} \hookrightarrow \mathcal{V}'$ is continuous, bounded, monotone, and B is maximal monotone, it follows that $I + B$ is maximal monotone. Therefore, $-\Delta + I + B$ is surjective. Thus, given $h = h_2 + \Delta h_1 \in \mathcal{V}'$, there exists $v \in \mathcal{V}$ satisfying (5.11.398).

Since

$$u = v + h_1 \in \mathcal{V}$$

we obtain

$$-\Delta u - \Delta N g(\gamma_0 v) = h_2 - v \in L^2(\Omega).$$

Bearing in mind that

$$D(-\Delta) = \{w \in \mathcal{V}; \exists f_w \in L^2(\Omega) \text{ such that } (\nabla w, \nabla v) = (f_w, v) \quad \forall v \in \mathcal{V}\},$$

it results $\begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$ and thus,

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = (I + A) \begin{pmatrix} u \\ v \end{pmatrix}.$$

Therefore, A is maximal monotone.

Existence of solution

Consider $U(t) = \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix}$ and the abstract formulation of the problem:

$$\begin{cases} U_t(t) = -AU(t), \ t > 0 \\ U(0) = U_0. \end{cases}$$

Since A is a maximal monotone operator, $-A$ generates a nonlinear semigroup. Then, if $U_0 =$

$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in D(A)$, *the problem has a unique strong solution, with* $U(t) \in D(A)$ *for all* $t \geq 0$ *and, for each* $T > 0$ *given,* $U \in C([0,T];\mathcal{H})$, *that is,*

$$u \in C([0,T];\mathcal{V}) \cap C^1([0,T];L^2(\Omega)).$$

Moreover, $U \in W^{1,\infty}([0,\infty);\mathcal{H})$. *In this case, we have*

$$\begin{gathered} u \in C([0,T];\mathcal{V}) \cap L^\infty(0,\infty;\mathcal{V}), \\ u_t \in C([0,T];L^2(\Omega)) \cap L^\infty(0,\infty;\mathcal{V}) \cap L^\infty(0,\infty;L^2(\Omega)), \\ u_{tt} \in L^\infty(0,\infty;L^2(\Omega)). \end{gathered} \tag{5.11.399}$$

Therefore,

i) $u_{tt} - \Delta u = 0$ *in* $L^\infty(0,\infty;L^2(\Omega))$;

ii) $\gamma_0 u \in L^\infty(0,\infty;H^{1/2}(\Gamma_1))$ *and* $\gamma_0 u = 0$ *on* Γ_0;

iii) *Since* $u(t) + Ng(\gamma_0 u_t(t)) \in D(-\Delta)$, *for all* $t \geq 0$, *then*

$$\underbrace{\gamma_1[u(t) + Ng(\gamma_0 u_t(t))]}_{\in H^{1/2}(\Gamma_1)} = 0 \text{ on } \Gamma_1, \text{ for all } t \geq 0,$$

whence

$$\gamma_1 u(t) + g(\gamma_0 u_t(t)) = 0 \text{ on } \Gamma_1, \text{ for all } t \geq 0.$$

To obtain the regularity of each term of the sum above, let us consider, for each $t \in [0,T]$,

$$z(t) = \int_0^t u(s)ds.$$

We have $z \in C([0,T];\mathcal{V})$ *and*

$$\Delta z(t) = \int_0^t \Delta u(s)ds = u_t(t) - u_t(0), \text{ for each } t \in [0,T].$$

Thus, $\Delta z \in C([0,T],L^2(\Omega))$. *Denoting* $\mathcal{V}^1 = \{v \in \mathcal{V};\ \Delta v \in L^2(\Omega)\}$, *we see that* $z \in C([0,T];\mathcal{V}^1)$, *whence* $u = z_t \in H^{-1}(0,T;\mathcal{V}^1)$, *and then,* $\gamma_1 u \in H^{-1}(0,T;H^{-1/2}(\Gamma_1))$.

Furthermore, by the regularity obtained in (5.11.399), it follows that $u_t \in L^\infty(0,T;\mathcal{V})$, *which implies* $\gamma_0 u_t \in L^\infty(0,T;L^2(\Gamma_1))$, *and since* g *is increasing by hypothesis,* $g(\gamma_0 u_t) \in L^\infty(0,T;L^2(\Gamma_1))$. *Therefore,*

$$\gamma_1 u + g(\gamma_0 u_t) = 0 \text{ in } L^\infty(0,T;L^2(\Gamma_1)).$$

Observe also that, since $U \in W^{1,\infty}([0,\infty);\mathcal{H})$, *then* $AU = -U_t \in L^\infty(0,\infty;\mathcal{H})$. *Thus,*

$$\|AU(t)\|_{\mathcal{H}} = \|v(t)\|_{\mathcal{V}} + \|-\Delta[u(t) + Ng(\gamma_0 u_t(t))]\| \leq c_4 \quad \forall t \geq 0,$$

for some $c_4 > 0$. *But* $(u(t), u_t(t)) \in D(A)$, *for all* $t \geq 0$, *and with this it follows that*

$$-\Delta[u + Ng(\gamma_0 u_t(t))] = -\Delta u(t) \in L^2(\Omega).$$

Thus, $u \in H^2(\Omega)$, *and so,*

$$\|AU(t)\|_{\mathcal{H}} = \|\nabla v(t)\| + \|\Delta u(t)\| \leq c_4, \quad \forall t \geq 0.$$

Therefore,

$$u \in L^\infty(0,\infty; H^2(\Omega)\cap\mathcal{V}); \;\; u_t \in L^\infty(0,\infty;\mathcal{V})\cap W^{1,\infty}(0,\infty;L^2(\Omega)).$$

When $U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{H}$, *the given problem has a unique generalized solution* U *which, for each* $T>0$ *satisfies* $U \in C([0,T];\mathcal{H})$, *by the Crandall-Liggett Theorem.*

Let $U_0^{(n)} = \begin{pmatrix} u_0^{(n)} \\ u_1^{(n)} \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ *be a sequence of initial data such that* $U_0^{(n)} \to U_0$ *in* $\mathcal{H}$. *Let* U_n *be the strong solution of (5.11.391) with initial data* $U_0^{(n)}$. *Composing* (5.11.391) *with* $u_t^{(n)}$, *integrating on* $(0,t)$, *for* $t>0$ *and using the second generalized Green's formula, we obtain*

$$\|u_t^{(n)}(t)\|^2 + \|\nabla u^{(n)}(t)\|^2 + 2\int_0^t (g(\gamma_0 u_t^{(n)}), \gamma_0 u_t^{(n)})_{L^2(\Gamma_1)}\, dt = \|u_1^{(n)}\|^2 + \|\nabla u_0^{(n)}\|^2. \tag{5.11.400}$$

By the linearity of (5.11.391), we obtain that

$$u_{tt} - \Delta u = 0 \text{ in } H^{-1}(0,T;\mathcal{V}'), \text{ for any } T>0 \text{ given.}$$

Since $u \in C([0,T];\mathcal{V})$, *then* $\gamma_0 u = 0$ *on* $\Gamma_0 \times (0,T)$.

Finally, if we consider that there exists $\alpha > 0$ *such that*

$$g(s_1) - g(s_2) \geq \alpha(s_1 - s_2), \quad \forall s_1 - s_2 \geq 0,$$

then (5.11.400) gives us

$$\|u_t^{(n)}(t)\|^2 + \|\nabla u^{(n)}(t)\|^2 + 2\alpha\int_0^t \|\gamma_0 u_t^{(n)}\|_{\Gamma_1}\, dt \leq \|u_1^{(n)}\|^2 + \|\nabla u_0^{(n)}\|^2.$$

From the last inequality and the growth condition of g, *it follows that* $\{g(u_t^{(n)})\}$ *is bounded in* $L^2(0,T;L^2(\Gamma_1))$. *Then,* $\{g(u_t^{(n)})\}$ *converges weakly to* $g(u_t)$ *in* $L^2(0,T;L^2(\Gamma_1))$. *On the other hand, using the continuity of the trace of order one, we obtain that*

$$g(u_t^{(n)}) = -\gamma_1 u^{(n)} \to -\gamma_1 u \text{ in } L^2(0,T;H^{-1/2}(\Gamma_1)).$$

By the uniqueness of the weak limit in $L^2(0,T;H^{-1/2}(\Gamma_1))$, *we obtain that* $g(u_t) = -\gamma_1 u$ *in* $L^2(0,T;H^{-1/2}(\Gamma_1))$, *but by the regularity of* $g(u_t)$, *we conclude*

$$\gamma_1 u + g(u_t) = 0 \text{ in } L^2(0,T;L^2(\Gamma_1)).$$

Example 5.287 Let us show that the following problem has a regular solution

$$\begin{cases} u'' - \Delta u = 0 & \textbf{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial \nu} = -g(u') - f(u) & \textbf{on } \Gamma_1\times(0,\infty), \\ u = 0 & \textbf{on } \Gamma_0\times(0,\infty), \\ u(0) = u^0 \in \mathcal{V}, \;\; u'(0) = u^1 \in L^2(\Omega), & \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ **is an open, bounded set with sufficiently smooth boundary,** g **is assumed as in the previous exercise and satisfies**

$$(g(s) - g(t))(s-t) \geq (s-t)^2$$

and f is a locally Lipschitz function satisfying

$$|f(s)| \le C|s|^{k_0}, k_0 < \frac{n-1}{n-2}.$$

Solution: *Initially, let us consider f Lipschitz with constant L and consider the operator N as in the previous exercise. In this case, our problem is of the form*

$$U_t = AU$$

where, for $U = (u, v)$,

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ -\Delta(u + N(g(\gamma_0 v) + f(\gamma_0 u)) \end{pmatrix},$$

with

$$D(A) = \{(u, v) \in \mathcal{V} \times \mathcal{V}; u + N(g(\gamma_0 v) + f(\gamma_0 u)) \in D(-\Delta).$$

From the previous exercise, we have that $N(g(\gamma_0 v)$ is well defined and, since f is Lipschitz, we have that $N(f(\gamma_0 u)$ is also well-posed. Let us show then, that $A + \omega I$ is monotone for some ω. Indeed, given $(u_1, v_1), (u_2, v_2) \in D(A)$, we have

$$\begin{aligned}(A(u_1, v_1) \quad &- \quad A(u_2, v_2), (u_1, v_1) - (u_2, v_2)) = -((v_1 - v_2), (u_1 - u_2))_{\mathcal{V}} \\ &- \quad (\Delta(u_1 + N(g(\gamma_0 v_1) + f(\gamma_0 u_1))) - \Delta(u_2 + N(g(\gamma_0 v_2) + f(\gamma_0 u_2))), v_1 - v_2).\end{aligned}$$

Now,

$$\begin{aligned}-(\Delta(u_1 \quad &+ \quad N(g(\gamma_0 v_1) + f(\gamma_0 u_1))) - \Delta(u_2 + N(g(\gamma_0 v_2) + f(\gamma_0 u_2))), v_1 - v_2) \\ &= \quad (\nabla(u_1 + N(g(\gamma_0 v_1) + f(\gamma_0 u_1))) - \nabla(u_2 + N(g(\gamma_0 v_2) + f(\gamma_0 u_1))), \nabla v_1 - \nabla v_2) \\ &- \quad \langle \gamma_1(u_1 + N(g(\gamma_0 v_1) + f(\gamma_0 u_1)) - \gamma_1(u_2 + N(g(\gamma_0 v_2) + f(\gamma_0 u_2)), \gamma_0 v_1 - \gamma_0 v_2\rangle \\ &= \quad (\nabla(u_1 + N(g(\gamma_0 v_1) + f(\gamma_0 u_1))) - \nabla(u_2 + N(g(\gamma_0 v_2) + f(\gamma_0 u_2))), \nabla v_1 - \nabla v_2) \\ &= \quad ((u_1 - u_2), (v_1 - v_2))_{\mathcal{V}} + (N(g(\gamma_0 v_1) + f(\gamma_0 u_1) - g(\gamma_0 v_2) - f(\gamma_0 u_2)), -\Delta(v_1 - v_2)) \\ &= \quad ((u_1 - u_2), (v_1 - v_2))_{\mathcal{V}} + \langle g(\gamma_0 v_1) + f(\gamma_0 u_1) - g(\gamma_0 v_2) - f(\gamma_0 u_2), -N * \Delta(v_1 - v_2)\rangle\end{aligned}$$

where the duality is in $H^{-1/2}(\Gamma_1) \times H^{1/2}(\Gamma_1)$. Since $-N * \Delta(v_1 - v_2) = \gamma_0(v_1 - v_2)$, then

$$\begin{aligned}\langle g(\gamma_0 v_1) \quad &+ \quad f(\gamma_0 u_1) - g(\gamma_0 v_2) - f(\gamma_0 u_2), -N * \Delta(v_1 - v_2)\rangle \\ &= \quad \langle g(\gamma_0 v_1) - g(\gamma_0 v_2), \gamma_0(v_1 - v_2)\rangle + \langle f(\gamma_0 u_1) - f(\gamma_0 u_2), \gamma_0(v_1 - v_2)\rangle \\ &\ge \quad m_0\|\gamma_0(v_1 - v_2)\|_{\Gamma_1}^2 - L\|\gamma_0(u_1 - u_2)\|_{\Gamma_1}\|\gamma_0(v_1 - v_2)\|_{\Gamma_1} \\ &\ge \quad (m_0 - \varepsilon)\|\gamma_0(v_1 - v_2)\|_{\Gamma_1}^2 - \frac{L}{4\varepsilon}\|\gamma_0(u_1 - u_2)\|_{\Gamma_1}^2.\end{aligned}$$

Thus,

$$\begin{aligned}(A(u_1, v_1) - A(u_2, v_2), (u_1, v_1) - (u_2, v_2)) + \omega\|u_1 - u_2\|_{\mathcal{V}}^2 + \omega\|v_1 - v_2\|_{L^2(\Omega)} \\ \ge (m_0 - \varepsilon)\|\gamma_0(v_1 - v_2)\|_{\Gamma_1}^2 - \frac{L}{4\varepsilon}\|\gamma_0(u_1 - u_2)\|_{\Gamma_1}^2 + \omega\|u_1 - u_2\|_V^2 \\ \ge (m_0 - \varepsilon)\|\gamma_0(v_1 - v_2)\|_{\Gamma_1}^2 + \left(\omega - \frac{L + C}{4\varepsilon}\right)\|u_1 - u_2\|_V^2\end{aligned}$$

where C is the continuity constant of γ_0 in $\mathcal{V}$. Hence, choosing $\varepsilon < m_0$ and $\omega > \dfrac{L + C}{4\varepsilon}$, we have the desired result.

It remains to show the maximality of $A + \omega I$, that is, there exists $\widetilde{\lambda} > 0$ such that $(A + \omega I + \widetilde{\lambda} I)(D(A)) = \mathcal{V} \times L^2(\Omega)$. Let us denote $\lambda = \omega + \widetilde{\lambda}$, thus we must find $\lambda > \omega$ such that $(A + \lambda I)$ is surjective.

Given $(h_1, h_2) \in \mathcal{V} \times L^2(\Omega)$, *we must exhibit* $(u, v) \in D(A)$ *such that*

$$\begin{cases} \lambda u - v = h_1 \\ \\ \lambda v - \Delta u - \Delta(N(g(\gamma_0 v) + f(\gamma_0 u))) = h_2, \end{cases}$$

Writing

$$u = \frac{1}{\lambda}(v + h_1)$$

we obtain

$$\lambda v - \frac{1}{\lambda}\Delta v - \Delta N(g\gamma_0 v) - \Delta N\left(f\gamma_0\left(\frac{1}{\lambda}(v + h_1)\right)\right) = h_2 + \frac{1}{\lambda}\Delta h_1.$$

In this case, our problem reduces to showing the surjectivity of the operator $T : \mathcal{V} \longrightarrow \mathcal{V}'$

$$Tv := \lambda v - \frac{1}{\lambda}\Delta v - \Delta N(g\gamma_0 v) - \Delta N\left(f\gamma_0\left(\frac{1}{\lambda}(v + h_1)\right)\right).$$

or even,

$$T = -\frac{1}{\lambda}\Delta + \lambda I + B$$

where

$$Bv = -\Delta N(g\gamma_0 v) - \Delta N\left(f\gamma_0\left(\frac{1}{\lambda}(v + h_1)\right)\right)$$

As argued in the previous exercise, it suffices to show that $\lambda I + B$ *is maximal monotone.*

Now, observe that, since $-\Delta N : H^{-1/2}(\Gamma_1) \to \mathcal{V}'$ *is bounded and* f *is Lipschitz, then*

$$B_1 := -\Delta N\left(f\gamma_0\left(\frac{1}{\lambda}(v + h_1)\right)\right) \text{ is Lipschitz from } \mathcal{V} \text{ into } \mathcal{V}'.$$

Indeed, let v_1 *and* $v_2 \in \mathcal{V}$, *then*

$$\begin{aligned} & \left\|-\Delta N\left(f\gamma_0\left(\frac{1}{\lambda}(v_1 + h_1)\right) - f\gamma_0\left(\frac{1}{\lambda}(v_2 + h_1)\right)\right)\right\|_{\mathcal{V}'} \\ \leq\ & C\left\|f\gamma_0\left(\frac{1}{\lambda}(v_1 + h_1)\right) - f\gamma_0\left(\frac{1}{\lambda}(v_2 + h_1)\|_{H^{-1/2}(\Gamma_1)}\right)\right\|_{H^{-1/2}(\Gamma_1)} \\ \leq\ & \frac{CL}{\lambda}\|v_1 - v_2\|_{H^{-1/2}(\Gamma_1)} \\ \leq\ & \frac{CC_1L}{\lambda}\|v_1 - v_2\|_{H^{1/2}(\Gamma_1)} \\ \leq\ & \frac{CC_1C_2L}{\lambda}\|v_1 - v_2\|_{\mathcal{V}} = \frac{C_3L}{\lambda}\|v_1 - v_2\|_{\mathcal{V}} \end{aligned}$$

where C, C_1 *and* C_2 *are the continuity constants of* $-\Delta N$, *of the embedding* $H^{1/2} \hookrightarrow H^{-1/2}(\Gamma_1)$ *and of the continuity of the trace map* γ_0, *respectively. Furthermore,* $B_1 + I$ *is maximal monotone for* $\lambda > C_3^2 L$, *since, if* v_1 *and* $v_2 \in \mathcal{V}$, *then*

$$\begin{aligned} \Big\langle -\Delta N\left(f\gamma_0\left(\frac{1}{\lambda}(v_1 + h_1)\right) - f\gamma_0\left(\frac{1}{\lambda}(v_2 + h_1)\right)\right) \quad , \quad v_1 - v_2\Big\rangle_{\mathcal{V}',\mathcal{V}} & \\ \geq\ -\frac{C_3^2 L}{\lambda}\|v_1 - v_2\|_{\mathcal{V}}^2 & \end{aligned}$$

Thus, $B_1 + I$ *is monotone and continuous and, therefore, by Theorem 1.3, page 40, Barbu, we have* $B_1 + I$ *maximal monotone. Now, if* $B_2 := -\Delta N(g\gamma_0 v)$ *then,* $B_2 + I$ *is maximal monotone by the previous exercise and, thus, by Corollary 1.1, page 39, Barbu, we have that* $B_1 + B_2 + 2I = B + 2I$ *is maximal monotone. Therefore, if* $\lambda > max\{2, C_3^2 L\}$ *we have* $B + \lambda I$ *maximal monotone, as we wanted. Whence it follows that* T *is surjective and* $\omega I + A$ *is maximal monotone.*

Assuming that f is Lipschitz, we know that there exists an $\omega > 0$ sufficiently large such that $A+\omega I$ is maximal monotone. Thus, by the Crandall-Liggett Theorem, we conclude that there exists a unique solution

$$u \in C([0,T];\mathcal{V}) \cap C^1([0,T];L^2(\Omega)),$$

of (??) for any finite $T > 0$.

To prove that $u', \dfrac{\partial u}{\partial \nu} \in L^2(0,T;L^2(\Gamma_1))$, observe that if $(u^0,u^1) \in D(A)$, we have that $\gamma_0(u^1) \in H^{1/2}(\Gamma_1)$, whence

$$\frac{\partial u^0}{\partial \nu},\ g(\gamma_0 u^1) \in L^2(\Gamma_1).$$

If $(u(t),u'(t))$ is a solution of the problem for the initial data $(u^0,u^1) \in D(A)$, then, by the semi-group property, we have $(u(t),u'(t)) \in D(A)$ and, consequently,

$$u' \in L^\infty(0,T;L^2(\Gamma_1)) \ \ \text{and} \ \ \frac{\partial u}{\partial \nu} \in L^\infty(0,T;L^2(\Gamma_1)).$$

Now, consider the following energy identity

$$E(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}\|\nabla u^0\|^2 + \int_0^t \int_{\Gamma_1} \frac{\partial u}{\partial \nu}\ u' d\Gamma ds$$

Thus, we have

$$\begin{aligned}
\|u'(t)\|^2 + \|\nabla u(t)\|^2 &= \|u^1\|^2 + \|\nabla u^0\|^2 + 2\int_0^t \int_{\Gamma_1} \frac{\partial u}{\partial \nu}\ u' d\Gamma ds \\
&= 2\cdot E(0) - 2\int_0^t \int_{\Gamma_1} g(u')u'\ d\Gamma ds - 2\int_0^t \int_{\Gamma_1} f(u)u'\ d\Gamma ds \\
&\leq 2\cdot E(0) - 2\alpha \int_0^t \int_{\Gamma_1} |u'|^2\ d\Gamma ds + 2L\int_0^t \int_{\Gamma_1} |u||u'|\ d\Gamma ds \\
&\leq 2\cdot E(0) - 2\alpha \int_0^t \int_{\Gamma_1} |u'|^2\ d\Gamma ds \\
&\quad +2L\left\{\frac{1}{4\epsilon}\int_0^t \int_{\Gamma_1} |u|^2 d\Gamma ds + \epsilon\int_0^t \int_{\Gamma_1} |u'|^2 d\Gamma ds\right\}.
\end{aligned} \tag{5.11.401}$$

Choose $\epsilon = \dfrac{\alpha}{2L}$ and we obtain

$$\|u'(t)\|^2 + \|\nabla u(t)\|^2 + \alpha\int_0^t \|u'(t)\|^2_{L^2(\Gamma_1)} \leq C\left\{\|\nabla u^0\|^2 + \|u^0\|^2 + \|u^1\|^2\right\}. \tag{5.11.402}$$

By the density of $D(A)$ in $\mathcal{H}$, we can extend the previous inequality to all $\mathcal{H}$. From the hypotheses on g it is concluded that $g(u') \in L^2(0,\infty;L^2(\Gamma_1))$ and, in this case, it makes sense to speak of the normal derivative $\dfrac{\partial u}{\partial \nu}$ in the space $H^{-1}(0,T;H^{-1/2}(\Gamma_1))$. But, by the equality

$$\frac{\partial u}{\partial \nu} = -g(u') - f(u)$$

and, from the regularity of the functions in question, we obtain $\dfrac{\partial u}{\partial \nu} \in L^2(0,\infty;L^2(\Gamma_1))$.

When f is only Lipschitz we can consider the following approximation of our problem

$$\begin{cases} u_l'' - \Delta u_l = 0 & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u_l}{\partial \nu} = -g(u_l') - f_l(u_l) & \text{on } \Gamma_1 \times (0,\infty), \\ u_l = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ u_l(0) = u^0 \in \mathcal{V}, \;\; u_l'(0) = u^1 \in L^2(\Omega), \end{cases} \tag{5.11.403}$$

where the functions f_l are defined by

$$f_l(s) = \begin{cases} f(s), & |s| \leq l; \\ f(l), & s > l; \qquad i = 0,1 \\ f(-l), & s \leq -l. \end{cases}$$

We have that f_l is Lipschitz for each l and $f_l(s) \to f(s)$ for all s. Thus, there exists a solution

$$u_l \in C([0,T];\mathcal{V}) \cap C^1([0,T];L^2(\Omega)),$$

with

$$\frac{\partial u_l}{\partial \nu}, \;\; u_l', \;\; g(u_l') \in L^2(0,\infty;L^2(\Gamma_1)).$$

Let us prove that this sequence of solutions has a convergent subsequence, whose limit is a solution of our problem. For that, we first claim that

$$\int_{\Gamma_1} F_l(u)d\Gamma \leq C, \tag{5.11.404}$$

where $F_l(t) = \displaystyle\int_0^t f_l(s)\; ds$ and $C = C(\|u\|_{\mathcal{V}})$. Furthermore,

$$f_l(u_l) \longrightarrow f(u) \;\;\text{ in }\;\; L^2(\Gamma_1). \tag{5.11.405}$$

Indeed, since $|f_l(s)| \leq C|s|^{k_0}$, then for $u \in \mathcal{V}$,

$$\left| \int_{\Gamma_1} F_l(u(x))\; dx \right| \leq \int_{\Gamma_0} \left(C_1 |u(x)|^{k_0} \right) dx \leq C_0,$$

by the embeddings $H^{1/2}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$ and $H^{1/2}(\Gamma_1) \hookrightarrow L^{\frac{2n-2}{n-2}}(\Gamma_1) \hookrightarrow L^{k_0}(\Gamma_1)$ and using the continuity of the trace.

Thus, (5.11.404) is proved. To prove (5.11.405), let us set $\Gamma_l = \{x \in \Gamma_1 : \; |u_l(x)| > l\}$.

$$\begin{aligned} &\int_{\Gamma_1} |f_l(u_l(x)) - f(u(x))|^2 d\Gamma \\ &\leq 2 \left\{ \int_{\Gamma_1} |f_l(u_l(x)) - f(u_l(x))|^2 d\Gamma + \int_{\Gamma_1} |f(u_l(x)) - f(u(x))|^2 d\Gamma \right\}. \end{aligned}$$

In view of the compact embedding $H^{1/2}(\Gamma_1) \overset{c}{\hookrightarrow} L^2(\Gamma_1)$ and the continuity of the function f, by the Lebesgue dominated convergence theorem the second integral on the right side of the inequality above tends to zero. Let us analyze then the first integral:

Since $f_l(s) = f(s)$ for $|s| \leq l$, then

$$\int_{\Gamma_1} |f_l(u_l(x)) - f(u_l(x))|^2 d\Gamma$$

$$\leq 2\left\{\int_{\Gamma_l} |f(u_l(x))|^2 d\Gamma + \int_{\Gamma_l} \left(|f(l)|^2 + |f(-l)|^2\right) d\Gamma\right\}.$$

One has

$$\left(\int_{\Gamma_l} l^{\frac{2n-2}{n-2}}\right)^{\frac{n-2}{2n-2}} \leq \left(\int_{\Gamma} |u_l(x)|^{\frac{2n-2}{n-2}}\right)^{\frac{n-2}{2n-2}} \leq C = C(\|u_l\|_{\mathcal{V}})$$

by the embedding $H^{1/2}(\Gamma_1) \overset{c}{\hookrightarrow} L^{\frac{2n-2}{n-2}}(\Gamma_1)$ and by the continuity of the trace map of order 0. From this it follows that meas$(\Gamma_l) \leq C\cdot l^{\frac{-2n+2}{n-2}}$.

$$\begin{aligned}\int_{\Gamma_l} |f(u_l(x))|^2 d\Gamma &\leq C\int_{\Gamma_l} |u_l(x)|^{2k_1} d\Gamma \\ &\leq C\left[\int_{\Gamma_l} |u_l(x)|^{\frac{2n-2}{n-2}}\right]^{\frac{k_1(n-2)}{n-1}} \cdot (meas\Gamma_l)^{1-\frac{k_1(n-2)}{n-1}} \longrightarrow 0,\end{aligned}$$

since $k_1(n-2) < n-1$.

$$\int_{\Gamma_l} |f_1(\pm l)|^2 d\Gamma \leq C\cdot l^{2k_1}\cdot meas\Gamma_l \leq C\cdot l^{2k_1 - \frac{2n-2}{n-2}} \longrightarrow 0.$$

Now, if

$$E_l(t) = \frac{1}{2}\left(\|u_l'(t)\|^2 + \|\nabla u_l(t)\|^2\right) + \int_{\Gamma_1} F_l(u_l) d\Gamma,$$

by the energy identity we have

$$E_l(t) + \int_0^t \int_{\Gamma_1} g_l(u_l')\ u_l'\ d\Gamma\ ds = E_l(0) \leq C\left(\|u^0\|_{\mathcal{V}}; \|u^1\|_{L^2(\Omega)}\right),$$

where the inequality follows from the claim.

From (5.11.402), it follows that

$$\begin{aligned}&\|u_l'\|^2_{L^2(\Sigma_1)} \leq C\left(\|u^0\|_{\mathcal{V}}; \|u^1\|_{L^2(\Omega)}\right)\\ &\|u_l\|_{C([0,T];\mathcal{V})} + \|u_l'\|_{C([0,T;L^2(\Omega)])} \leq C.\end{aligned}$$

From the bounds above, it follows that

$$\begin{aligned}&\gamma_0 u_l \longrightarrow \gamma_0 u \quad in \quad L^2(\Sigma_1)\\ &\gamma_0 u_l' \rightharpoonup \gamma_0 u' \quad in \quad L^2(\Sigma_1).\end{aligned}$$

Since the Sobolev embeddings $H^1(\Omega) \overset{c}{\hookrightarrow} L^{2k_0}(\Omega)$ and $H^{1/2}(\Gamma) \overset{c}{\hookrightarrow} L^{2k_1}(\Gamma)$ hold, together with the convergence of $\{f_l\}$ we conclude that

$$f_l(u_l) \rightharpoonup f(u) \quad in \quad L^2(\Sigma_1) \tag{5.11.406}$$

The convergences above allow us to pass to the limit in (5.11.403), completing the proof of the exercise.

Example 5.288 *Let us determine the existence of weak solutions (in $H_0^1(\Omega)$) for the problem below:*

$$\begin{cases} i\,\partial_t\, u - \Delta\, u + |u|^2\, u = 0 & \textit{in } \Omega \times (0,\infty) \\ u = 0 & \textit{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \textit{in } x \in \Omega \end{cases} \tag{5.11.407}$$

where $\Omega \subset \mathbb{R}^n$ $(n = 1,2,3)$ *is a bounded open set with smooth boundary.*

Definition 5.289 *Let* $T > 0$ *and given* $u_0 \in \mathcal{X} := H^1(\Omega) \cap L^4(\Omega)$. *A weak solution of problem* (5.11.407) *in* $[0,T]$ *is a function* u *in the class* $L^\infty(0,T;\mathcal{X}) \cap C([0,T];L^2(\Omega))$ *that satisfies the identity*

$$\int_0^T -(u(t), \partial_t\, \varphi(t))_{L^2(\Omega)} + i\,(\nabla\, u(t), \nabla\varphi(t))_{L^2(\Omega)}\, dt \tag{5.11.408}$$
$$+ \int_0^T i\langle|\, u(t)\,|^2\, u(t), \varphi(t)\rangle_{L^{\frac{4}{3}}(\Omega),\, L^4(\Omega)}\, dt = 0$$

for all $\varphi \in C_0^\infty(0,T;H_0^1(\Omega) \cap L^4(\Omega))$ *and for almost every* $t \in [0,T]$.

Theorem 5.290 *If* $u_0 \in \mathcal{X} = H^1(\Omega) \cap L^4(\Omega)$. *Then problem* **??** *has a weak solution in the sense of definition (5.11.408).*

Proof: *Let* $\psi : L^2(\Omega) \to (-\infty,\infty]$ *be a convex, proper and lower semi-continuous function. Then, the subdifferential of* $\psi(u)$ *where* $u \in D(\psi)$ *is defined as the set of all* $g \in L^2(\Omega)$ *such that*

$$\psi(u) \le Re(g, u - z)_{L^2(\Omega)} + \psi(z), \quad \forall\, z \in L^2(\Omega)\,. \tag{5.11.409}$$

and denoted by $\partial\,\psi(u)$.

Consider the nonlinear operator $\mathcal{B}$ *in* $L^2(\Omega)$ *defined by*

$$D(\mathcal{B}) = \{u \in L^2(\Omega);\, |u|^2\, u \in L^2(\Omega)\}, \tag{5.11.410}$$
$$\mathcal{B}u = |u|^2\, u,\ \forall\, u \in D(\mathcal{B})\,. \tag{5.11.411}$$

By the next lemma it follows that $\mathcal{B}$ *is m-accretive, its proof can be found in Okazawa and Yokota [[72], Lemma 3.1, page 258].*

Lemma 5.291 *Let* $\mathcal{B}$ *be defined as above. Then,* $\mathcal{B}$ *is m-accretive.*

Now, we can define the Yosida approximations (which are Lipschitz continuous) $\mathcal{B}_n$ *of* $\mathcal{B}$ *in terms of the resolvent* J_n,

$$J_n = \left(1 + \frac{1}{n}\,\mathcal{B}\right)^{-1} \tag{5.11.412}$$

and

$$\mathcal{B}_n := n\,(I - J_n) = \mathcal{B}\, J_n\,. \tag{5.11.413}$$

Moreover, from the general theory of monotone operators we know that we can represent the operators $\mathcal{B}$ *and* $\mathcal{B}_n$ *by subdifferentials of* ψ *and* ψ_n *given by*

$$\psi(z) := \begin{cases} \frac{1}{4}\,||z||^4_{L^4(\Omega)} & \textit{for } z \in L^4(\Omega) \\ \infty & \textit{otherwise} \end{cases} \tag{5.11.414}$$

and

$$\psi_n(z) := \min_{v\in L^2(\Omega)} \left\{ \frac{n}{2} \, ||v - z||^2_{L^2(\Omega)} + \psi(v) \right\} = \frac{1}{2n} \, \|\mathcal{B}_n(z)\|^2_{L^2(\Omega)} + \psi(J_n(z)), \;\; z \in \, L^2(\Omega) \qquad (5.11.415)$$

so that

$$\mathcal{B} = \partial \, \psi \qquad \text{and} \qquad \mathcal{B}_n = \partial \, \psi_n \, .$$

Furthermore,

$$\psi(J_n(z)) \leq \psi_n(z) \leq \psi(z) \, . \qquad (5.11.416)$$

On the other hand, given $u_0 \in \mathcal{X}$, *then there exists* $\{u_{n,0}\} \subset H^1_0(\Omega) \, \cap \, H^2(\Omega)$ *such that*

$$u_{n,0} \to u_0 \quad \text{in} \quad \mathcal{X} \, . \qquad (5.11.417)$$

Let us consider now the following approximate problem:

$$\begin{cases} i \, \partial_t \, u_n - \Delta \, u_n + \mathcal{B}_n(u_n) = 0 & \text{in } \Omega \, \times \, (0, \infty) \\ u_n = 0 & \text{on } \partial\Omega \, \times \, (0, \infty) \\ u_n(x, 0) = u_{n,0}(x) & \text{in } x \in \Omega \end{cases} \qquad (5.11.418)$$

Let us prove that problem (5.11.418) is well-posed for each n. *For this, observe that problem 5.11.418 can be rewritten as the following Cauchy problem:*

$$\begin{cases} \dfrac{d \, u_n}{dt} + A \, u_n = F_n(u_n) \\ u_n(0) = u_{n,0} \end{cases} \qquad (5.11.419)$$

where

$$\begin{aligned} A : D(A) &\to L^2(\Omega) \\ z \quad &\mapsto \; Az = i\Delta z \end{aligned} \qquad \text{and} \qquad \begin{aligned} F_n : L^2(\Omega) &\to L^2(\Omega) \\ w \quad &\mapsto \; F_n(w) := - \mathcal{B}_n(w) \end{aligned}$$

where $D(A) = H^1_0(\Omega) \, \cap \, H^2(\Omega)$.

Note that A *is a skew-adjoint operator in* Ω. *Thus, from [[15], proposition 1, page 13], we know that* A *is a maximal monotone operator in* Ω. *Moreover, since* $\mathcal{B}_n$ *is Lipschitz continuous for each* n, *we have that for each* n, *the operator* F_n *is Lipschitz continuous in* $L^2(\Omega)$.

Thus, from [[15], theorem 1, page 18], for each n, *given* $u_{n,0} \in H^1_0(\Omega) \cap H^2(\Omega)$, *there exists a unique solution* u_n *for problem (5.11.418) in the class*

$$C \left([0, \infty); H^1_0(\Omega) \, \cap \, H^2(\Omega) \right) \, \cap \, C^1 \left([0, \infty); L^2(\Omega) \right) \, . \qquad (5.11.420)$$

*We observe that problem (5.11.418) is an approximation of the original problem (**??**).*

Now, taking the inner product in L^2 *of (5.11.418) with* u_n, *the imaginary part becomes*

$$Re(\partial_t \, u_n, u_n)_{L^2(\Omega)} + \underbrace{Im(\nabla u_n, \nabla u_n)_{L^2(\Omega)}}_{=0} + \underbrace{Im(\mathcal{B}_n(u_n), u_n)_{L^2(\Omega)}}_{=0} = 0,$$

from (5.11.413), *we have*

$$
\begin{aligned}
(\mathcal{B}_n(u_n), u_n)_{L^2(\Omega)} &= \left(\mathcal{B}_n(u_n), \frac{1}{n}\mathcal{B}_n(u_n) + J_n(u_n)\right)_{L^2(\Omega)} \\
&= \frac{1}{n}\,\|\mathcal{B}_n(u_n)\|^2_{L^2(\Omega)} + (\mathcal{B}_n(u_n), J_n(u_n))_{L^2(\Omega)} \\
&= \frac{1}{n}\,\|\mathcal{B}_n(u_n)\|^2_{L^2(\Omega)} + \|J_n(u_n)\|^4_{L^4(\Omega)}\,.
\end{aligned}
\tag{5.11.421}
$$

Therefore, we obtain the identity

$$
\frac{1}{2}\,\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} = 0. \tag{5.11.422}
$$

Thus, taking the inner product in $(L^2(\Omega))$ *of (5.11.418) with* $\partial_t\, u_n$ *and looking at the real parts, we have:*

$$
\underbrace{Re(i\,\partial_t\, u_n, \partial_t\, u_n)_{L^2(\Omega)}}_{=0} \;+\; Re(\nabla u_n, \nabla\partial_t\, u_n)_{L^2(\Omega)} + Re(\mathcal{B}_n(u_n), \partial_t\, u_n)_{L^2(\Omega)} = 0. \tag{5.11.423}
$$

Now, using the following classical lemma given in Showalter's book [[87], chapter IV, lemma 4.3, page 186] for functions ψ_n, u_n *and* $\partial_t\, u_n$ *both with* $\mathcal{B}_n(u_n) = \partial\,\psi_n(u_n)$, *it follows that*

$$
\frac{d}{dt}\psi_n(u_n) = Re\,(\mathcal{B}_n(u_n), \partial_t\, u_n)_{L^2((\Omega))} \tag{5.11.424}
$$

Combining (5.11.422), *(5.11.423) and (5.11.424), we conclude that*

$$
\frac{d}{dt}\left[\frac{1}{2}\,\|u_n(t)\|^2_{H^1(\Omega)} + \psi_n(u_n)\right] = 0. \tag{5.11.425}
$$

From (5.11.416) *and integrating* (5.11.425) *from* 0 *to* t, *we observe that*

$$
\begin{aligned}
\frac{1}{2}\,\|u_n(t)\|^2_{H^1(\Omega)} + \psi_n(u_n) &= \frac{1}{2}\,\|u_{n,0}\|^2_{H^1(\Omega)} + \psi_n(u_{n,0}) \\
&\le C\,\|u_{n,0}\|^2_{\mathcal{X}}\,.
\end{aligned}
\tag{5.11.426}
$$

Moreover, from (5.11.416), *it derives*

$$
\|u_n(t)\|^2_{H^1(\Omega)} + \|J_n(u_n)\|^4_{L^4(\Omega)} \quad \le C\|u_{n,0}\|^2_{\mathcal{X}}\,. \tag{5.11.427}
$$

Inequality (5.11.427) and the boundedness of the sequence $(u_{n,0})$ *in* $\mathcal{X}$, *allow us to conclude that*

$$
\{u_n\} \qquad \textit{is bounded in} \qquad L^\infty(0,T;H^1(\Omega)) \tag{5.11.428}
$$

$$
\{J_n(u_n)\} \qquad \textit{is bounded in} \qquad L^\infty(0,T;L^4(\Omega)) \hookrightarrow L^4(0,T;L^4(\Omega))\,. \tag{5.11.429}
$$

Let us note that

$$
\mathcal{B}_n(u_n) = \mathcal{B}(J_n(u_n)) = |\,J_n(u_n)\,|^2\,J_n(u_n) \tag{5.11.430}
$$

Thus, from (5.11.429) and (5.11.430), it follows that

$$
\{\mathcal{B}_n\,u_n\} \quad \textit{is bounded in} \quad L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega)). \tag{5.11.431}
$$

On the other hand, from (5.11.428) and (5.11.431) we have:

$$
\begin{aligned}
\|\partial_t u_n(t)\|_{\mathcal{X}'} &= \sup_{\|\varphi\|_{\mathcal{X}}=1} \left\{ (\partial_t u_n(t), \varphi)_{L^2(\Omega)} \right\} \\
&= \sup_{\|\varphi\|_{\mathcal{X}}=1} \{ (-i\,\Delta\, u_n(t), \varphi)_{L^2(\Omega)} + (i\,\mathcal{B}_n(u_n), \varphi)_{L^2(\Omega)} \} \\
&\leq \sup_{\|\varphi\|_{\mathcal{X}}=1} \left\{ \|\nabla\, u_n(t)\|_{L^2(\Omega)} \, \|\nabla\, \varphi\|_{L^2(\Omega)} + \|\mathcal{B}_n\, u_n(t)\|_{L^{\frac{4}{3}}(\Omega)} \, \|\varphi\|_{L^4(\Omega)} \right. \\
&< +\infty,
\end{aligned}
$$

thus,

$$
\{\partial_t\, u_n\} \quad \textit{is bounded in} \quad L^\infty(0,T;\mathcal{X}')\,. \tag{5.11.432}
$$

Combining (5.11.428), (5.11.429), (5.11.431) *and* (5.11.432), *it follows that* $\{u_n\}$ *has a subsequence (still denoted by* $\{u_n\}$*) such that*

$$
\begin{aligned}
u_n &\overset{*}{\rightharpoonup} u \quad \textit{in} \quad L^\infty(0,T;H^1(\Omega))\,. && (5.11.433)\\
J_n(u_n) &\overset{*}{\rightharpoonup} \mathcal{U} \quad \textit{in} \quad L^\infty(0,T;L^4(\Omega))\,. && (5.11.434)\\
\mathcal{B}_n(u_n) &\rightharpoonup \mathcal{Z} \quad \textit{in} \quad L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))\,. && (5.11.435)\\
\partial_t\, u_n &\overset{*}{\rightharpoonup} \partial_t\, u \quad \textit{in} \quad L^\infty(0,T;\mathcal{X}')\,. && (5.11.436)
\end{aligned}
$$

The next step is to prove that $\mathcal{U} = u$ *and* $\mathcal{Z} = |u|^2\, u$*. Indeed, from (5.11.433) and (5.11.436) it follows by the Aubin-Lions Theorem that there exists a* $\tilde{u} \in L^2(0,T;L^2(\Omega))$ *and a subsequence of* u_n *(still denoted by* u_n*) such that*

$$
u_n \to \tilde{u} \quad \textit{in} \quad L^2(0,T;L^2(\Omega)) \tag{5.11.437}
$$

On the other hand, from (5.11.433) and (5.11.437), by the uniqueness of the weak limit in $L^2(0,T;L^2(\Omega))$*, we infer that* $\tilde{u} = u$ *a.e. in* $\Omega \times (0,\infty)$*. Therefore, again from (5.11.437), we have*

$$
\begin{aligned}
u_n &\to u \quad \textit{in} \quad L^2(0,T;L^2(\Omega)) && (5.11.438)\\
u_n &\to u \quad \textit{a.e. in} \quad \Omega \times (0,T)\,. && (5.11.439)
\end{aligned}
$$

Next, let us consider the fact that the operator $\mathcal{B}$ *is also accretive in* $\mathbb{C}$*. Thus, from Showalter, [[87], page 211], we know that the resolvents* J_n *given in* (5.11.412) *are contractions in* $\mathbb{C}$*, that is,*

$$
|J_n(z) - J_n(w)| \leq |z - w|,\ \forall\, z, w \in \mathbb{C}, \tag{5.11.440}
$$

note that $\mathcal{B}_n$ *and* J_n *are essentially the same operators given at the beginning of the section, except that we are considering them in* $\mathbb{C}$ *instead of* $L^2(\Omega)$*.*

From this, we define

$$
|||\, C\, ||| = \inf\{|x| : x \in C\}.
$$

Again, from Showalter [[87], proposition 7.1, item c, page 211], we obtain

$$
|\mathcal{B}_n(w)| \leq |||\mathcal{B}(w)||| = |\mathcal{B}(w)|,\ \forall\, w \in \mathbb{C}, \tag{5.11.441}
$$

where the equality on the right side of (5.11.441) is from the fact that the operator $\mathcal{B}$ *given in (5.11.410) is single-valued in* $\mathbb{C}$*.*

On the other hand, from (5.11.413)*, we have* $\omega - J_n(\omega) = \dfrac{1}{n} B_n(\omega)$*. Therefore, combining* (5.11.440) *and* (5.11.441)*, we obtain*

$$\begin{aligned} |J_n(z) - w| &\le |J_n(z) - J_n(w)| + |J_n(w) - w| \\ &\le |z - w| + \frac{1}{n}\,|\mathcal{B}_n(w)| \\ &\le |z - w| + \frac{1}{n}\,|\mathcal{B}(w)|,\ \forall\, w, z \in \mathbb{C}\,. \end{aligned} \tag{5.11.442}$$

It follows from (5.11.439)*:*

$$|u_n - u| \to 0 \qquad a.e.\ \ in\ \Omega \times (0,T)\,. \tag{5.11.443}$$

Now, let $(x,t) \in \Omega \times (0,T)$ *such that the convergence* (5.11.443) *holds, and let* $z = u_n(x,t)$ *and* $w = u(x,t)$ *in* (5.11.442) *and letting* $n \to \infty$*, considering* (5.11.443)*, it follows that*

$$J_n(u_n) \quad \to \quad u \qquad a.e.\ \ in \quad \Omega\,\times\,(0,T)\,. \tag{5.11.444}$$

Moreover, from (5.11.444) and since $\mathcal{B}(z) = |z|^2\, z$ *is continuous, it follows that*

$$\mathcal{B}(J_n(u_n)) \quad \to \quad \mathcal{B}(u) = |u|^2\, u \quad a.e.\ \ in \quad \Omega\,\times\,(0,T)\,.$$

Using the definition of the Yosida approximations $\mathcal{B}_n$ *given in (5.11.430), it results that*

$$\mathcal{B}_n(u_n) \quad \to \quad |u|^2\, u \quad a.e.\ \ in \quad \Omega\,\times\,(0,T)\,. \tag{5.11.445}$$

Now, combining (5.11.429)*, (5.11.444) and (5.11.431), (5.11.445), we have by Lions' lemma [J. L. Lions, [64], lemma 1.3, page 12] the following convergences:*

$$J_n(u_n) \quad \rightharpoonup \quad u \qquad in \quad L^4(0,T;L^4(B))\,. \tag{5.11.446}$$

$$\mathcal{B}_n(u_n) \quad \rightharpoonup \quad |u|^2\, u \quad in \quad L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(B))\,. \tag{5.11.447}$$

Thus, by the convergences (5.11.434)*,* (5.11.435) (5.11.446) *and* (5.11.447)*, we have that* $\mathcal{U} = u$ *and* $\mathcal{Z} = |u|^2\, u$ *almost everywhere in* $\Omega \times (0,T)$*. Moreover, from the convergence* (5.11.434) *together with (5.11.433) we infer that*

$$u \in\, L^\infty(0,T;\mathcal{X}). \tag{5.11.448}$$

Finally, let $\varphi \in\, C_0^\infty([0,T); H_0^1(\Omega)\,\cap\, L^4(\Omega))$*. Then, from (5.11.418), we have*

$$\begin{aligned} &\int_0^T -(u_n(t), \partial_t\, \varphi(t))_{L^2(\Omega)} - i\,(\nabla\, u_n(t), \nabla\varphi(t))_{L^2(\Omega)}\, dt \\ &-i\int_0^T \langle |\, u_n(t)\,|^p\, u_n(t), \varphi(t)\rangle_{L^{\frac{4}{3}}(\Omega),\, L^4(\Omega}\, dt = 0 \end{aligned} \tag{5.11.449}$$

From (5.11.433) and (5.11.434) taking the limit as $n \to \infty$*, we obtain the variational formula given in (5.11.408).*
Finally, from (5.11.408), it follows that u *belongs to the space*

$$\mathcal{W} = \{u \in\, L^2(0,T;\mathcal{X})\ \ such\ that\ \ \partial_t\, u \in\, L^2(0,T;\mathcal{X}')\}.$$

Then, employing Showalter [[87], proposition 1.2, page 106], we have that $\mathcal{W}$ can be continuously embedded into the space $C([0,T];L^2(\Omega))$ and, therefore, combining this fact with (5.11.448), we obtain that u has the regularity given in definition 5.289 and, thus, the proof of the theorem is concluded. $\square$

Example 5.292 *Let us determine the existence of solutions for the problem*

$$\begin{cases} \partial_t \beta(u) - \Delta u = 0, & \text{in } \Omega \times (0,+\infty); \\ u = 0, & \text{on } \partial\Omega \times (0,+\infty); \\ u(x,0) = u^0(x) = 0, & x \in \Omega, \end{cases} \tag{5.11.450}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ and $\beta : \mathbb{R} \to \mathbb{R}$ is a monotone increasing map such that $\beta(0) = 0$. We assume that $|\beta|$ has polynomial growth, for example, $\beta(s) = |s|^{p-2}s$, $p > 2$. We will use two results that can be found in [87] and which are stated below.

Proposition II.9.1: *Let $D(A_1) = \{u \in W_0^{1,1}(G);\ A_1 u \in L^1(G)\}$ where $A_1 u = f \in L^1(G)$ means*

$$u \in W_0^{1,1}(G);\quad \int_G \left(\sum_{i,j=1}^n a_{ij}\partial_i u \partial_j v - \sum_{i=1}^n a_i u \partial_i v + a u v \right) = \int_G f v,\ \forall v \in W_0^{1,\infty}(G).$$

a) $D(A_1)$ is dense and $(I+\lambda A_1)^{-1}$ is a contraction in L^1 for each $\lambda > 0$;

b) $D(A_1) \subset W_0^{1,q}$ for $1 \le q < \frac{n}{n-1}$ and there exists $c(q) > 0$ such that

$$c(q)\|u\|_{W^{1,q}} \le \|A_1 u\|_{L^1} \text{ for } u \in D(A_1);$$

c) A_1 is the L^1-closure $\overline{A_2}$ of A_2;

d) $\sup_G (I+\lambda A_1)^{-1} f \le \max\{0, \sup_G f\}$ for each $\lambda > 0$ and $f \in L^1$, that is,

$$\|(I+\lambda A_1)^{-1} f\|_{L^\infty(G)} \le \|f^+\|_{L^\infty(G)},$$

where $x^+ = \max\{0,x\}$ denotes the positive part of $x \in \mathbb{R}$.

Theorem II.9.2 *(Brezis-Strauss): Let α be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and $0 \in \alpha(0)$. Let $A : D(A) \to L^1(G)$ be linear and satisfying*

(i) $D(A)$ is dense and $(I+\lambda A)^{-1}$ is a contraction in L^1 for each $\lambda > 0$;

(ii) $\sup_G (I+\lambda A)^{-1} f \le (\sup_G f)^+ = \|f^+\|_{L^\infty}$ for $f \in L^1$ and $\lambda > 0$;

(iii) There exists $c > 0$ such that $c\|u\|_{L^1} \le \|Au\|_{L^1}$ for $u \in D(A)$.

Then, for each $f \in L^1$ there exists a unique pair $u \in L^1$, $v \in D(A)$ such that

$$u + Av = f \text{ and } v(x) \in \alpha(u(x)) \text{ a.e. } x \in G.$$

If u_1, v_1 and u_2, v_2 are solutions corresponding to f_1, f_2 as above, then

$$\|(u_1-u_2)^+\|_{L^1} \le \|(f_1-f_2)^+\|_{L^1},\quad \|(u_1-u_2)^-\|_{L^1} \le \|(f_1-f_2)^-\|_{L^1},$$

and, therefore,

$$\|u_1 - u_2)\|_{L^1} \le \|f_1 - f_2\|_{L^1}.$$

If $f_1 \ge f_2$ a.e. then $u_1 \ge u_2$ a.e. in G.

Initially, let us make a change of variables. Let $u = \phi(w)$ *and consider* $\beta(\cdot) = \phi(\cdot)^{-1}$*, then the equation of problem 5.11.450 becomes*

$$\partial_t w - \Delta\phi(w) = 0,$$

which is known as the generalized porous medium equation (GPME). Let us define, $D(A \circ \phi) = \{w \in W_0^{1,1}(G);\ (A \circ \phi)(w) \in L^1(G)\}$ *where* $(A \circ \phi)(w) = 0 \in L^1(G)$ *means*

$$w \in W_0^{1,1}(G); \quad \int_G \nabla\phi(w)\nabla v = 0, \ \forall v \in W_0^{1,\infty}(G).$$

Therefore, $(A \circ \phi)(w) = -\Delta\phi(w) \in L^1(G)$*. Applying Proposition II.9.1 and Theorem II.9.2, we have that for each* $\lambda > 0$ *there exists a unique pair*

$$w \in L^1(G), \quad \phi(w) \in W_0^{1,1}(G)$$

for which $\Delta\phi(w) \in L^1(G)$ *and*

$$w(x) - \Delta\phi(w(x)) = 0 \quad a.e.\ x \in G.$$

Thus, $A \circ \phi$ *is m-acretive. Thus, from Corollary 5.88 we have that* $Im(I + \lambda A) = L^1(G),\ \forall\lambda > 0$*. In this way, the operator* $A \circ \phi$ *satisfies the hypotheses of the Crandall-Liggett Theorem and according to Remark 5.260 we have that problem 5.11.450 admits a generalized solution. That is, there exists a unique solution* w *of the problem and the solution is continuous.*

Example 5.293 *Determine the existence of weak solutions in* $(L^2(\Omega))$ *for the problem below:*

$$\begin{cases} iu_t + \Delta u + ig(u) = 0, & in \quad \Omega \times (0,\infty) \\ u = 0, & on \quad \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), & if \quad x \in \Omega \end{cases} \tag{5.11.451}$$

where Ω *is a bounded open set with smooth boundary and*

$$g : \mathbb{C} \to \mathbb{C} \quad \text{is a continuous function satisfying:} \tag{5.11.452}$$

(*i*) $Re[g(z) - g(w)](\bar{z} - \bar{w}) \geq 0\ \forall\ z, w \in \mathbb{C}$.

(*ii*) $Im(g(z)\bar{z}) = 0\ \forall z \in \mathbb{C}$.

(*iii*) *There exist positive constants* c_1, c_2 *such that* $c_1|z|^2 \leq |g(z)\bar{z}|^2 \leq c_2|z|^2 \quad \forall z \in \mathbb{C}$ *with* $|z| \geq 1$.

We will use the following results:

Let us consider the following problem

$$\begin{cases} u_t(t) = Tu(t) + Su(t), & t \in (0,\infty) \\ u(0) = u_0 \end{cases} \tag{5.11.453}$$

posed in a Banach space X.

Definition 5.294 *A map* $u : [0,\infty) \to X$ *is called a weak solution of problem (5.11.453) if* u *is continuous on* $[0,\infty), u(0) = 0$ *and satisfies the inequality for each* $T > 0$

$$\|u(t) - v\|_X^2 \leq \|u(s) - v\|_X^2 + 2\int_s^t < Tv + Su(\tau), u(\tau) - v >_s d\tau. \tag{5.11.454}$$

Theorem 5.295 *Let* H *be a real Hilbert space and* $T : H \to H$ *an m-dissipative operator and let* $S : H \to H$ *be continuous such that* $D(S) = H$*. Then for each* $u_0 \in \overline{D(T)}$ *there exists a unique map*

$u : [0, \infty) \to H$ weak solution of problem (5.11.453).

Proof: *See Barbu([7], Theorem 3.1, p.152)* □

It is important to observe that we will work with complex-valued functions, so that, in order for the spaces $L^2(\Omega)$, as well as, $H^m(\Omega)$, $m \in \mathbb{N}$, to become real Hilbert spaces, we define

$$(u, v)_{L^2(\Omega)} = Re \int_\Omega w\bar{v} dx.$$

Finally, we will denote by $H_0^1(\Omega)$ the Hilbert space

$$H_0^1(\Omega) = \{w \in H^1(\Omega); w|_{\partial\Omega} = 0\}.$$

Let us now prove the following existence theorem of weak solution for problem (5.11.451).

Theorem 5.296 *Under the hypotheses of the function g given in (5.11.452), we have: problem (5.11.451) is well-posed in the space $L^2(\Omega)$, that is, for each initial value $u_0 \in L^2(\Omega)$, there exists a unique weak solution of (5.11.451).*

Proof: *Problem (5.11.451) can be rewritten as*

$$\begin{cases} u_t - i\Delta u + g(u) = 0, & in \quad \Omega \times (0, \infty) \\ u = 0, & on \quad \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x), & if \quad x \in \Omega \end{cases} \tag{5.11.455}$$

Define the following operators:

$$\begin{array}{rcccc} A & : & D(A) \subset L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & & u & \longmapsto & Au = -i\Delta u \end{array}$$

and

$$\begin{array}{rcccc} B & : & D(B) \subset L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & & u & \longmapsto & Bu = g(u) \end{array}$$

Then $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ and $D(B) = L^2(\Omega)$.
Our goal is to show that $A + B$ is a maximal monotone operator. First, observe that using Green's Theorem and the Lax-Milgram Theorem, we have that A is maximal monotone. Next, we will show some properties associated with the operator B.

- ***B maps bounded sets into bounded sets***

Indeed, let $u \in L^2(\Omega)$ such that $\|u\|^2_{L^2(\Omega)} \leq R$. Thus, by hypothesis (iii) of function g we obtain

$$\begin{aligned} \|Bu\|^2_{L^2(\Omega)} &= \int_\Omega |g(u(x))|^2 dx \\ &\leq c_3 \int_\Omega |u(x)|^2 dx \\ &\leq Rc_3. \end{aligned}$$

- ***B is monotone***

Indeed, let $u_1, u_2 \in L^2(\Omega)$. Then by hypothesis (i) of function g, we obtain:

$$\begin{aligned}(Bu_1 - Bu_2, u_1 - u_2)_{L^2(\Omega)} &= Re\int_\Omega (Bu_1 - Bu_2)\overline{(u_1 - u_2)}dx \\ &= \int_\Omega Re\{(g(u_1) - g(u_2))(\overline{u_1} - \overline{u_2})\}dx \geq 0\end{aligned}$$

- ***B is hemicontinuous***

Indeed, we have to prove that given any sequence $t_n \subset \mathbb{R}$ such that $t_n \to 0$ then $\lim_{n\to\infty}(B(u + t_n v), w)_{L^2(\Omega)} = (Bu, w)_{L^2(\Omega)}$ for all $u, v \in L^2(\Omega)$.
For this purpose, define $f_n = g(u + t_n v)\overline{w}$. Thus,

$$\begin{aligned}|f_n(x)| &= |g(u(x) + t_n v(x))||w(x)| \\ &\leq c_2|u(x) + t_n v(x)||w(x)| \\ &\leq c_2|u(x)||w(x)| + c_4|v(x)||w(x)| \quad \textit{almost everywhere in } \Omega\end{aligned}$$

where c_4 is such that $|t_n| \leq c_4$.
Since $u, v, w \in L^2(\Omega)$ then $f_n \in L^1(\Omega)$ $\forall$ $n \in \mathbb{N}$. Moreover, if h is the function defined by $h(x) = c_2|u(x)||w(x)| + c_4|v(x)||w(x)|$, it follows that $h \in L^1(\Omega)$ and $|f_n| \leq h(x)$ almost everywhere in Ω.
Note that due to the continuity of g, we deduce that $\lim_{n\to\infty} g(u(x) + t_n v(x))\overline{w(x)} = g(u(x))\overline{w(x)}$. Thus, by the Lebesgue Dominated Convergence Theorem, we conclude that

$$\int_\Omega |g(u(x) + t_n v(x))\overline{w(x)} - g(u(x))\overline{w(x)}|dx \to 0.$$

Then

$$\left|\int_\Omega g(u(x) + t_n v(x))\overline{w(x)} - g(u(x))\overline{w(x)}dx\right| \to 0$$

and, consequently,

$$Re\int_\Omega g(u(x) + t_n v(x))\overline{w(x)}dx \to Re\int_\Omega g(u(x))\overline{w(x)}dx$$

that is, $\lim_{n\to\infty}(B(u + t_n v), w)_{L^2(\Omega)} = (Bu, w)_{L^2(\Omega)}$.

Therefore, B is a monotone map, maps bounded sets into bounded sets and is hemicontinuous and, since A is a maximal monotone operator, then by (5.59) [Linear and nonlinear semigroups lecture notes. Cavalcanti, Marcelo.] we conclude that $T \equiv A + B : D(A + B) \subset L^2(\Omega) \to L^2(\Omega)$ is maximal monotone in $L^2(\Omega)$. Thus, assuming $S \equiv 0$ in problem (5.11.453), then according to Theorem 5.295 for each $u_0 \in \overline{D(A + B)} = \overline{H_0^1(\Omega) \cap H^2(\Omega)} = L^2(\Omega)$ there exists a unique map $u : [0, \infty) \to L^2(\Omega)$ weak solution of problem (5.11.451). □

Chapter 6

Appendix

6.1 Properties

Results used in Example 5.286.

TECHNICAL RESULTS ON TRACE OPERATORS

Claim 1: If $v \in \mathcal{V}$ then $\gamma_0 v \in H^{1/2}(\Gamma_1)$.

Proof of Claim 1: Let $\{(U_1, \varphi_1), ..., (U_k, \varphi_k), (U_{k+1}, \varphi_{k+1}), ..., (U_{k+l}, \varphi_{k+l})\}$ be a system of local charts for $\Gamma_0 \cup \Gamma_1$, such that

$$\{(U_1, \varphi_1), ..., (U_k, \varphi_k)\} \text{ is a system of local charts for } \Gamma_0,$$

and

$$\{(U_{k+1}, \varphi_{k+1}), ..., (U_{k+l}, \varphi_{k+l})\} \text{ is a system of local charts for } \Gamma_1,$$

such that

$$\overline{U}_m \cap \overline{U}_n = \emptyset, \quad \forall m = 1, ..., k; \qquad \forall n = k+1, ..., k+l.$$

Such consideration is possible because, by hypothesis, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$.

Consider also C^∞ partitions,

$$\theta_0, \theta_1, ..., \theta_k \text{ and } \theta_0', \theta_{k+1}, ..., \theta_{k+l}$$

subordinate to the respective open covers

$$\Omega, U_1, ..., U_k \text{ and } \Omega, U_{k+1}, ..., U_{k+l}.$$

Then

- $supp\left(\frac{\theta_0}{2} + \frac{\theta_0'}{2}\right) \subset \Omega, \;\; supp\left(\frac{\theta_i}{2}\right) \subset U_i, \quad \forall i = 1, ..., k+l;$

- $\frac{1}{2}[\theta_0(x) + \theta_0'(x)] + \frac{1}{2}\sum_{i=1}^{k+l} \theta_i(x) = 1, \quad \forall x \in \overline{\Omega};$

- $0 \leq \theta_0' \leq 1, \;\; 0 \leq \theta_i \leq 1, \quad \forall i = 0, 1, ..., k+l.$

We know that $\gamma_0 v \in H^{1/2}(\Gamma_0 \cup \Gamma_1)$. Then, letting $\phi_i(u) = \widetilde{(u\theta_i) \circ \varphi_i^{-1}}$ be the null extension of $(u\theta_i) \circ \varphi_i^{-1}$, we obtain

$$\begin{aligned}
\|\gamma_0 v\|^2_{H^{1/2}(\Gamma_1)} &= \sum_{i=k+1}^{k+l} \|\phi_i(\gamma_0 v)\|^2_{H^{1/2}(\mathbb{R}^{n-1})} \\
&= \sum_{i=1}^{k+l} \|\phi(\gamma_0 v)\|^2_{H^{1/2}(\mathbb{R}^{n-1})} \text{ (since } \gamma_0 v = 0 \text{ on } \Gamma_0 \text{)} \\
&= \sqrt{2} \sum_{i=1}^{k+l} \left\| \frac{\phi_i}{2}(\gamma_0 v) \right\|^2_{H^{1/2}(\mathbb{R}^{n-1})} = \sqrt{2} \|\gamma_0 v\|^2_{H^{1/2}(\Gamma)} < \infty,
\end{aligned}$$

which gives us $\gamma_0 v \in H^{1/2}(\Gamma_1)$.

Claim 2: If $u \in \mathcal{V} \cap H^2(\Omega)$ with $\dfrac{\partial u}{\partial \nu} = 0$ on Γ_1 and $v \in \mathcal{V}$ then $\langle \gamma_1 u, \gamma_0 v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 0$.

Proof of Claim 2: Consider the system of local charts for $\Gamma_0 \cap \Gamma_1$ and $u\widetilde{\theta_i \circ \varphi_i^{-1}}$ as in claim 1. Define, here, $\phi_i(u) = \dfrac{\widetilde{u\theta_i}}{2} \circ \varphi_i^{-1}$.

Now, by the Riesz Representation Theorem, there exists a unique $w \in H^{1/2}(\Gamma)$ such that

$$\langle \gamma_1 u, \gamma_0 v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = (w, \gamma_0 v)_{H^{1/2}(\Gamma)},$$

and then

$$\langle \gamma_1 u, \gamma_0 v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = (w, \gamma_0 v)_{H^{1/2}(\Gamma)} = \sum_{i=1}^{k+l} (\phi_i(w), \phi_i(\gamma_0 v))_{H^{1/2}(\mathbb{R}^{n-1})}.$$

But $w = 0$ in $H^{1/2}(\Gamma_1)$, whence

$$\phi_{k+1}(w) = ... = \phi_{k+l}(w) = 0 \text{ on } \Gamma_1,$$

and thus,

$$\langle \gamma_1 u, \gamma_0 v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \sum_{i=1}^{k} (\phi_i(w), \phi_i(\gamma_0 v))_{H^{1/2}(\mathbb{R}^{n-1})} = \frac{1}{2}(w, \gamma_0 v)_{H^{1/2}(\Gamma_0)}.$$

Since $v \in \mathcal{V}$, then $\gamma_0 v = 0$ on Γ_0, and with this, we conclude that

$$\langle \gamma_1 u, \gamma_0 v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \frac{1}{2}(w, \gamma_0 v)_{H^{1/2}(\Gamma_0)} = 0.$$

(For more details or better understanding of the arguments used in the proof of claims 1 and 2, consult (MC-S), page 277.)

Claim 3: The map $\gamma_0 : \mathcal{V} \to H^{1/2}(\Gamma_1)$ is surjective.

Proof of Claim 3: Given $z \in H^{1/2}(\Gamma_1)$, we can consider the extension $\widetilde{z} \in H^{1/2}(\Gamma)$ of z on Γ given by

$$\widetilde{z}(x) = \begin{cases} z(x), & \text{if } x \in \Gamma_1, \\ 0, & \text{if } x \in \Gamma_0. \end{cases}$$

Since the trace $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ is surjective, there exists $y \in H^1(\Omega)$ such that $\gamma_0 y = \widetilde{z}$. But

$$\gamma_0 y = \widetilde{z}|_{\Gamma_0} = 0,$$

whence it follows $y \in \mathcal{V}$.

AUXILIARY RESULTS

Theorem 1: Let E and F be Banach spaces and $A : D(A) \subset E \to F$ a linear, closed operator with $D(A) = E$. Then,

i) A is bounded;

ii) $D(A^*) = F'$;

iii) A^* is bounded.

Proof: See Theorem 2.50, page 95 of (MC-A).

Theorem 2: Let X and X' be reflexive and strictly convex. Let $F : X \to X'$ be the duality map of X. Let A be a monotone subset of $X \times X'$. Then A is maximal monotone in $X \times X'$ if, and only if, for some $\lambda > 0$ (equivalently, for all $\lambda > 0$), $Im(A + \lambda F) = X'$.

Proof: See Theorem 1.2, page 39 of (VB).

Corollary 3: Let X be reflexive and B be monotone, hemicontinuous and bounded from X into X'. Let A be a maximal monotone operator on $X \times X'$. Then $A + B$ is maximal monotone.

Proof: See Corollary 1.1, page 39 of (VB).

Theorem 4 (Kachurovskii): Let K be a convex set in V and let $\varphi : V \to (-\infty, +\infty]$ be Gateaux differentiable at each $u \in K$, where $K = D(\varphi)$. The following statements are equivalent:

i) φ is convex;

ii) $\varphi'(u)(v - u) \leq \varphi(v) - \varphi(u)$, for all $u, v \in K$;

iii) $\langle \varphi'(u) - \varphi'(v), u - v \rangle_{V',V} \geq 0, \quad \forall u, v \in K$.

Proof: See Proposition 7.4, page 80 of (RS).

Theorem 5: Let $\varphi : W \to (-\infty, +\infty]$ be convex, $\Lambda : V \to W$ continuous and linear, and assume that φ is continuous at some point of $Im(\Lambda)$. Then $\partial(\varphi \circ \Lambda) = \Lambda' \circ \partial\varphi \circ \Lambda$.

Proof: See Proposition 7.8, page 82 of (RS).

Theorem 6: Let X be a real Banach space. If φ is a proper, convex and lower semicontinuous function on X, then $\partial\varphi$ is a maximal monotone operator from X to X'

Proof: See Theorem 2.1, page 54 of (VB).

Theorem 7: Let $\varphi : V \to (-\infty, +\infty]$ be convex and proper. If φ is Gateaux-differentiable at $u \in int(dom(\varphi))$, then $\partial\varphi(u) = \{\varphi'(u)\}$. If φ is continuous and $\partial\varphi(u)$ has a unique element, then φ is Gateaux-differentiable at u.

Proof: See Proposition 7.6, page 81 of (RS).

Theorem 8: Let A be an $\omega-$accretive operator, closed in a Banach space X, and satisfying

$$\overline{D(A)} \subset Im(I + \lambda A), \text{ for all } \lambda > 0 \text{ small.}$$

Let X be reflexive and $y_0 \in D(A)$. Then there exists a unique $y \in W^{1,\infty}([0, \infty); X)$ satisfying

$$\frac{dy}{dt} + Ay \ni 0, \quad t > 0$$
$$y(0) = y_0.$$

Proof: See Theorem 1.5, page 216 of (VB2).

BIBLIOGRAPHIC REFERENCES

(VB) BARBU, V. **Nonlinear Semigroups and Differential Equations in Banach Spaces.** Romania, Bucuresti: Noordhoff International Publishing, 1976.

(VB2) BARBU, V. **Analysis and Control of Nonlinear Infinite Dimensional Systems.** Boston, Academic Press, 1993.

(AHC) CAIXETA, A. H. **Uniform Stabilization on the Boundary of the Semilinear Wave Equation with Nonlinear Damping on the Boundary.** Dissertation (Master's in Mathematics) - State University of Maringá, Maringá, 2012.

(MC-S) CAVALCANTI, M. M.; DOMINGOS CAVALCANTI, V. N. **Introduction to Distribution Theory and Sobolev Spaces.** Maringá. Eduem, 2009.

(MC-A) CAVALCANTI, M. M.; DOMINGOS CAVALCANTI, V. N. **Introduction to Functional Analysis.** Maringá. Eduem, 2011.

(EM) MONTEIRO, E. **Existence and Asymptotic Behavior of Solutions of a Wave Equation with Nonlinear Damping on the Boundary and Source Term.** Dissertation (Master's in Mathematics) - State University of Maringá, 2006.

(RS) SHOWALTER, R. **Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations.** AMS, Providence, 1997.

Bibliography

[1] Adams, R. A., *Sobolev Spaces.* Academic Press, 1975.

[2] Agmon, S.; Douglis, A.; Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary value conditions I. *Comm. Pure Appl. Math.* **12** (1959), 623–727.

[3] Asplund, E., Average Norms. *Isr. J. Math.* **5** (1967), 227–233.

[4] Aubin, J. P., *Mathematical Methods of Game and Economic Theory.* North-Holland, 1979.

[5] Ball, J. M., Strongly continuous semigroups, weak solutions, and the variation of constants formula. *Proc. Amer. Math. Soc.* **63** (1977), 370–373.

[6] Balakrishnan, A., *Applied Functional Analysis.* Springer, 1976.

[7] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff, 1976.

[8] Bachman, G.; Narici, L., *Functional Analysis.* Academic Press, New York, 1972.

[9] Bénilan, P., Solutions faibles d'équations d'évolution dans un espace réflexif. Séminaire Deny sur les semi-groupes non linéaires, Orsay (1970–1971).

[10] Bénilan, P.; Brézis, H., Solutions faibles d'équations d'évolution dans les espaces de Hilbert. *Ann. Inst. Fourier* **22** (1972), 311–329.

[11] Bénilan, P.; Brézis, H.; Crandall, M. G., A semilinear equation in $L^1(\mathbb{R}^N)$. *Ann. Scuola Norm. Sup. Pisa* **2** (1975), 523–555.

[12] Bourbaki, N., *Topologie Générale*, Livre III, Ch. 1, 2 et 9. Herman, Paris, 1953–1961.

[13] Bourbaki, N., *Espaces Vectoriels Topologiques*, Livre V, Ch. 1–5. Herman, Paris, 1953–1961.

[14] Brézis, H., *Analyse Fonctionnelle, Théorie et Applications.* Masson, 1987.

[15] Brézis, H., Nonlinear Evolution Equations. Autumn Course on Semigroups, Theory and Applications. International Centre for Theoretical Physics, Trieste, 1984.

[16] Brézis, H.; Browder, F., Partial differential equations in the 20th century. *Adv. Math.* **135** (1998), 76–144.

[17] Brézis, H., *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert.* North-Holland, Amsterdam, 1973.

[18] Brézis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, 2011.

[19] Brézis, H.; Crandall, M. G.; Pazy, A., Perturbations of nonlinear maximal monotone sets in Banach space. *Comm. Pure Appl. Math.* **23** (1970), 123–144.

[20] Caffarelli, L. A.; Salsa, A., *A Geometric Approach to Free Boundary Problems.* American Mathematical Society, 2005.

[21] Calvert, B., Maximal accretive is not m-accretive. *Boll. Un. Mat. Ital.* **3** (1970), 1042–1044.

[22] Cavalcanti, M. M.; Domingos Cavalcanti, V. N., *Introdução à Teoria das Distribuições e aos Espaços de Sobolev.* Eduem, Maringá, 2009.

[23] Cavalcanti, M. M.; Domingos Cavalcanti, V. N.; Komornik, V., *Introdução à Análise Funcional.* Eduem, Maringá, 2011.

[24] Cazenave, T.; Haraux, A., *An Introduction to Semilinear Evolution Equations.* Oxford University Press, 1998.

[25] Chernoff, P., Note on product formulas for operator semigroups. *J. Funct. Anal.* **2** (1968), 238–242.

[26] Choquet, G., *Lectures On Analysis, Volume I: Integration And Topological Vector Spaces.* W. A. Benjamin, 1969. ISBN: 0-8053-6955-4.

[27] Choquet, G., *Lectures On Analysis, Volume II: Representation Theory.* W. A. Benjamin, 1969. ISBN: 0-8053-6957-0.

[28] Choquet, G., *Lectures On Analysis, Volume III: Infinite Dimensional Measures And Problem Solutions.* W. A. Benjamin, 1969. ISBN: 0-8053-6959-7.

[29] Coifman, R.; Meyer, Y., *Wavelets: Calderón-Zygmund and Multilinear Operators.* Cambridge University Press, 1997.

[30] Croke, C. B.; Lasiecka, I.; Uhlmann, G.; Vogelius, M.; eds, *Geometric Methods in Inverse Problems and PDE Control.* Springer, 2004.

[31] Crandall, M. G., Differential equations on convex sets. *J. math. Soc. Japan*, **22** (1970), 396–414.

[32] Dacorogna, B., Weak continuity and weak lower semicontinuity of nonlinear functionals. *Lec. Notes in Math*, No**992**, Springer-Verlag, 1982.

[33] Dieudonné,J., *Foundations of Modern Analysis.* Academis Press, 1960.

[34] Dieudonné, J., Recent Developments in the Theory of Locally Convex Vector Spaces. *Bull. Amer. Math. Soc*, **59** (1953), 495–512.

[35] Dieudonné, J.; Schwartz, L., La Dualité des Espaces $\mathcal{F}$ et $\mathcal{LF}$. *Ann. de L'Inst. Fourier I*, (1949), 61–101.

[36] Diestel, J., Geometry of Banach Spaces-Selected Topics. *Lectures Notes in Math.*, 488, Springer Verlag (1975).

[37] Dautray, R.; Lions, J. L., *Mathematical Analysis and Numerical Methods for Science and Technology*(6 volumes), Springer, 1988.

[38] Duvaut, G.; Lions,J.L., *Inequalities in Mechanics and Phisics.* Springer Verlag, New York, 1976.

[39] DiBenedetto, E., *Partial Differential Equations (2nd ed.)*, Birkhäuser, 2009.

[40] Evans, L. C., *Weak convergence methods for nonlinear partial differential equations.* Regional Conference Series in Mathematics, No **74**, AMS, 1990.

[41] Edwards,R., *Functional Analysis*, Holt-Rinehart-Winston, 1965.

[42] Ekeland, I., *Convexity Methods in Hamiltonian mechanics*, Springer, 1990.

[43] Evans, L. C., *Partial Differential Equations*, American Mathematical Society, 1998.

[44] Fernandez, C. S.; Bernardes Jr, N.C., *Introdução às Funções de uma Variável Complexa*, 2a edição, Coleção Textos Universitários, SBM, 2008.

[45] Friedman,A., [1] *Partial Differential Equations of Parabolic Type*, Prentice Hall, 1964, [2] *Partial Differential Equations*, Holt-Rinehart-Winston, 1969, [3] *Foundations of Modern Analysis*, Holt-Rinehart-Winston, 1970, [4] *Variational Principles and Free Boundary Problems*, Wiley, 1982.

[46] Goldstein, J., *Semigroups of Operators and Applications*, Oxford University Press, 1985.

[47] Gomes, A. M., *Semigrupos Não Lineares e Equações Diferenciais nos Espaços de Banach.* Textos Matemáticos do IM-UFRJ, UFRJ, 2003.

[48] Gilbert, R. P.; Shi, P.; Shillor, M., A quasistatic contact problem in linear thermoelasticity. *Rendiconti di mat.*, **10** (1990), 785–808.

[49] Granas, A.; Dugundji J., *Fixed Point Theory*, Springer, 2003.

[50] Grisvard, P., Équations différentielles abstraites, *Ann. Sci. ENS*, **2** (1969), 311–395.

[51] Hille, E.E., *Methods in Classical and Functional Analysis*, Addison-Wesley, 1972.

[52] Hörmander,L., [1] *Linear Partial Differential Operators*, Springer, 1963, [2] *The Analysis of Linear Partial Differenetial Operators*, (4 volumes), Springer, 1983-1985, [3] *Notions of Convexity*, Birkhäuser, 1994.

[53] Horváth,J., *Topological Vector Spaces and Distributions, Vol. I.* Adilson-Wesley, reading, Massachusetts, 1966.

[54] Kato, T., *Perturbation Theory for Linear Operators*, Springer, 1976.

[55] Kinderlehrer, D.; Stampacchia,G., *An Introduction to Variational Inequalities and Their Applications*, Academic Press, 1980.

[56] Kim, J. U., A boundary thin obstacle problem for a wave equation. *Commun. in Partial Differential Equations*, 14 (8-9), 1011–1026, 1989.

[57] Klainerman, S., *Introduction to Analysis*, Lecture Notes, Princeton University, 2008.

[58] Kolmogorov, A. N.; Fomin, S. V., *Elementos de la teoria de funcionales y del Analysis Funcional.* Editorial MIR - Moscow, 1978.

[59] Komornik, V., *Exact controllability and stabilization. The multiplier method. RAM: Research in Applied Mathematics.* Masson, Paris; John Wiley and Sons, Ltd., Chichester, 1994. viii+156 pp. ISBN: 2-225-84612-X.

[60] Kreyszig, E., *Introductory Functional Analysis With Applications.* John Wiley and Sons, Inc., Canada, 1989.

[61] Lima, E. L., *Curso de Análise, volume 2.* IMPA, CNPq, PROJETO EUCLIDES - Rio de Janeiro, 2014.

[62] Lima, E. L., *Espaços Métricos.* IMPA, CNPq, PROJETO EUCLIDES - Rio de Janeiro, 1983.

[63] Lions, J. L., *Contrôlabilité exacte perturbations et stabilisation de systèmes distribués.* Collection RMA, Masson Paris (tome 1), 1988.

[64] Lions, J. L., [1] *Problèmes aux limites dans les équations aux dérivées partielles*, Presses de l'Université de Montreal, 1965, [2] *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, 1971, [3] *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod-Gauthier Villars, 1969.

[65] Lions, J. L.; Magenes, E., *Non-homogeneous Boundary Value Problems and Applications (3 volumes)*, Springer, 1972.

[66] Liu, Z.Y.; Renardy, M., A note on the equations of thermoelastic plate. *Appl. Math. Lett.*, Vol. **8** (1995), N 3, pp. 1–6.

[67] Magenes,E., *Topics in parabolic equations: some typical free boundary problems, in Boundary Value Problems for Linear Evolution Partial Differential Equations.* (Garnir, H. G., ed.), Reidel, 1977.

[68] Maz'ja, V. G., *Sobolev Spaces*, Springer, 1985.

[69] Miranda, M. M., Traço para o dual dos espaços de Sobolev, *Bol. Soc. Paran. Matemática* (2ª série) 11(2) (1990), p.131–157.

[70] Miyadera, I.; Oharu, S., Approximation of semigroups of nonlinear operators. *Tohoku Math. J.* **22** (1970), 24–47.

[71] Nachbin, L., *Lecture on the Theory of Distributions, Lectures Notes* Rochester (1963) e Textos de Matemática, Recife (1965).

[72] Okazawa, N.; Yokota, Y., Monotonicity Method Applied to the Complex Ginzburg-Landau and Related Equations. *J. Math. Anal. Appl.* 267 no. **2** (2002), 247–263.

[73] Raviart, P. A.; Thomas, J. M., *Introduction à l'Analyse Numérique des équations aux Dérivées Partielles*, Masson, Paris, 1983.

[74] Rockafeller, R. T., Characterization of the subdifferential of convex functions. *Pacific J. Math.*, **17** (1966), pp 497–510.

[75] Ruiz,A., Unique continuation for weak solutions of the wave equation plus a potential. *J. Math. Pures Appl.*, **71** (1992), p. 455–467.

[76] Shi, P.; Shillor, M., Existence of a solution to the n-dimensional problem of thermoelastic contact. *Commun. in Partial Diff. Equations*, **17**(9-10), 1597–1618, 1992.

[77] Shi, P.; Shillor, M., Uniqueness and stability of the solution to a thermoelastic contact problem. *Euro. J. Appl. Math.*, **1** (1990), 371–387.

[78] Shi, P.; Shillor, M., A quasistatic contact problem in thermoelasticity with a radiation condition for the temperature. *J. Math. Anal. Appl.*, 172, No **1** (1993), 147–165.

[79] Shi,P.; Shillor, M.; Zou, X. L., Numerical solution to the one dimensional problems of thermoelastic contact. *Comput. Math. Appl.*, **22**(10), 65-78, 1991.

[80] Temam, R., *Navier-Stokes Equations: Theory and numerical analysis.* North-Holland, 1979.

[81] Rudin, W., *Principles of Mathematical Analysis.* McGRAW-HILL International Book Company, pp 1-339, 1976.

[82] Riesz, F.; Nagy, B. S., *Functional Analysis.* Ungar, New York, 1955.

[83] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.

[84] Pazy, A., *Semigroups of nonlinear contractions and their asymptotic behavior.* - Nonlinear Analysis and Mechanics: Heriot - Watt Symposium Vol. III. R. J. Knops (editor) Pitman Research Notes in Math. 30. London, 1978.

[85] Schwartz, L., *Théorie das Distributions, Tome I et II.* Actualites Scientifiques et Industrielles 1091, Herman, Paris (1957).

[86] Schwartz,J. T., *Nonlinear Functional Analysis*, Gordon Breach, 1969.

[87] Showalter, R. E., *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations*, American Mathematical Society, United States of America, 1997.

[88] Strauss,W., *Partial Differential Equations: An Introduction*, Wiley, 1992.

[89] Struwe, M., *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, 1990.

[90] Treves, F.; Figueiredo, D. G., *Espaços Vetoriais Topológicos e Distribuições.* Notas de Matemática no.41, Rio de Janeiro, 1965.

[91] Yosida, K., *Functional Analysis, Die Grundlehrender* Mathematishen Wissenschaften, Bd. 123, Springer-Verlag, Berlin, 1965.

[92] Zeidler, E., *Nonlinear Functional Analysis and Its Applications*, Springer, 1988.