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Regularization of Nonlinear Volterra Integral Equations of the First Kind in a Generalized Sense

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ABSTRACT: This paper investigates an ill-posed nonlinear Volterra integral equation of the first kind. In order to study the original integral equation and to prove its regularizability in the class of generalized functions, the method of integral operators with the use of auxiliary functions is applied, as well as an asymptotic regularization algorithm containing a singular function with respect to a small parameter. It should be noted that the considered Volterra integral equation of the first kind degenerates in inverse problems arising in the theory of moisture transfer, hereditary media, and other applied fields, which determines the actuality of this paper.

Keywords: Volterra integral equation of the first kind (VIE-1), regularization method (RM), uniqueness of solution, regularizability in the generalized sense.

Contents

1. Introduction

It is known that in the theory of constructing singular solutions of linear and nonlinear VIE-1, there exist a number of works [1,5, 7], where the regularizability of these equations is proved in certain functional spaces related to generalized functions. However, depending on the solution of a VIE-1, the introduced spaces may differ, and therefore the proposed variants of the RM may also vary.

In this regard, the present paper investigates an ill-posed nonlinear VIE-1, namely:

$$\int_0^t K(t,s)\theta^3(s)ds = F(t) \tag{1.1}$$

subject to the conditions:

a) $K(t,s) \in C(D_1) \cap Lip(t\,|L_K), 0 < L_K = const; 0 \le K(t,t), K(0,0) \ne 0; |K(.)| \le C_{01}$, $D_1 = \{(t,s) : 0 \le s \le t \le T\}$,

b) $F(t) \in C[0,T] \cap Lip(t\,|L_F), 0 < L_F; F(t) \ge \alpha > 0, |F(t)| \le C_{02}, \forall t \in [0,T]$.

2. Regularizability of the VIE-1

To prove the regularizability of equation (1.1) in the generalized sense, we first transform this VIE-1 into the form:

$$\begin{cases} \int_0^t h(\tau)\theta(\tau)d\tau = (\Phi\theta)(t) + F(t), \\ \Phi\theta \equiv \int_0^t h_0(\tau)\theta(\tau)(J\theta)(\tau)d\tau - (J\theta)(t), \end{cases} \tag{2.1}$$

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where the following conditions are assumed:

$$\begin{cases} h(t) \equiv [\gamma + \frac{1}{\alpha}\mu(t)]F(t) \geq m > 0, (1 < \gamma = const); h_0(t) \equiv \gamma + \frac{1}{\alpha}\mu(t), \\ 0 \leq \mu(t) \in L^1(0,T); h_0(t) \leq a^{-1}h(t); F_0(t) \equiv F(t) - F(0), (F_0(0) = 0), \\ \phi_0(t) = \int\limits_0^t [\gamma + \frac{1}{\alpha}\mu(\tau)]F(\tau)d\tau = \int\limits_0^t h(\tau)d\tau, \\ |F_0(t) - F_0(\tau)| \leq L_{F_0} M_0(\phi_0(t) - \phi_0(\tau)), (\tau \leq t); \gamma > 1; M_0 = \frac{1}{\gamma\alpha}, \\ 0 < \max C_{0j} = C_0, (j = 1,2);\ t \in [0,T] : \\ t = (t^{\frac{2}{9}})^{\frac{9}{2}} \leq (\phi_0(t))^{\frac{9}{2}}, (\lambda(t) = \frac{2}{9\sqrt[9]{t^7}}); t \leq M_1(\phi_0(t))^2, (M_1 = \sup\limits_{[0,T]} (\phi_0(t))^{\frac{5}{2}}). \end{cases} \tag{2.2}$$

Based on these conditions along with equation (2.1), we obtain a VIE-1 with a small parameter:

$$\begin{cases} \varepsilon\theta_\varepsilon(t) + \int\limits_0^t h(\tau)\theta_\varepsilon(\tau)d\tau = (\Phi\theta_\varepsilon)(t) + F_\varepsilon(t), \\ (\Phi_0\theta_\varepsilon)(t) \equiv \int\limits_0^t h(\tau)\theta_\varepsilon(\tau)d\tau - (\Phi\theta_\varepsilon)(t), \end{cases} \tag{2.3}$$

and seek its solution according to the rule:

$$\begin{cases} \theta_\varepsilon(t) = \frac{1}{\varepsilon}\Pi_\varepsilon(t) + \upsilon(t) + \xi_\varepsilon(t), \\ \Pi_\varepsilon(0) = F(0),\ \upsilon(0) = 0,\ \xi_\varepsilon(0) = 0. \end{cases} \tag{2.4}$$

As a result, with respect to the unknown functions, we obtain the system:

$$\begin{cases} \Pi_\varepsilon(t) = -\frac{1}{\varepsilon}\int\limits_0^t h(s)\,\Pi_\varepsilon(s)\,ds + F(0), \\ \int\limits_0^t h(s)\,\upsilon(s)ds = (\Phi\upsilon)(t) + F_0(t), \\ \varepsilon\xi_\varepsilon(t) + \int\limits_0^t h(s)\,\xi_\varepsilon(s)\,ds = (\Phi[\frac{1}{\varepsilon}\Pi_\varepsilon + \upsilon + \xi_\varepsilon])(t) - (\Phi\upsilon)(t) - \varepsilon\upsilon(t) + F_\varepsilon(t) - F(t), \end{cases} \tag{2.5}$$

Taking into account the parameterized equation:

$$\delta\upsilon_\delta(t) + \int_0^t h(s)\,\upsilon_\delta(s)ds = (\Phi\upsilon_\delta)(t) + F_0(t) \tag{2.6}$$

and the resolvent:

$$R \equiv -\frac{1}{\varepsilon}h(s)\exp\big(-\frac{1}{\varepsilon}\int_s^t h(\eta)\,d\,\eta\big) \tag{2.7}$$

we formulate the following lemma:

Lemma 2.1 *Under conditions (a,b) and (2.2), the following statements hold for system (2.5):*

1)

$$|\Pi_\varepsilon(t)| \leq C_0 \exp\left(-\frac{1}{\varepsilon}\phi_0(t)\right) \tag{2.8}$$

2) The solution of the parameterized equation (2.6) converges uniformly to the solution of the second equation of system (2.5) as $\delta \to 0$, since this IE has a solution in $C[0,T]$;

3) The function $\xi_\varepsilon(t)$, as the unique solution of the third equation of system (6) in $C[0,T]$, and converges uniformly to zero as $\varepsilon \to 0$.

Proof: 1) Taking into account the resolvent (2.7) of the first equation of system (2.7), we obtain:

$$\Pi_\varepsilon\left(t\right) = F\left(0\right)\exp\left(-\frac{1}{\varepsilon}\int\limits_0^t h\left(s\right)d\,s\right) \tag{2.9}$$

which implies estimate (9), where $|F(0)| \le C_0$.

2) Equation (2.6), under condition (2.2), can be rewritten in the form:

$$\begin{cases} \upsilon_\delta = -\frac{1}{\delta^2}\int\limits_0^t h(\tau)\exp(-\frac{1}{\delta}(\phi_0(t)-\phi_0(\tau))\{\int\limits_0^\tau h_0(\tilde\tau)\upsilon_\delta\left(\tilde\tau\right)\int\limits_0^{\tilde\tau} K(\tilde\tau,\bar\tau)\upsilon_\delta^3\left(\bar\tau\right)d\bar\tau d\tilde\tau- \\ -\int\limits_0^t h_0(\tilde\tau)\upsilon_\delta\left(\tilde\tau\right)\int\limits_0^{\tilde\tau} K(\tilde\tau,\bar\tau)\upsilon_\delta^3\left(\bar\tau\right)d\bar\tau d\tilde\tau - \int\limits_0^\tau K(\tau,\bar\tau)\upsilon_\delta^3\left(\bar\tau\right)d\bar\tau+ \\ +\int\limits_0^t K(t,\bar\tau)\upsilon_\delta^3\left(\bar\tau\right)d\bar\tau\}d\tau + \frac{1}{\delta}\{\int\limits_0^t h_0(\tau)\upsilon_\delta\left(\tau\right)\int\limits_0^\tau K(\tau,\bar\tau)\upsilon_\delta^3\left(\bar\tau\right)d\bar\tau d\tau- \\ -\int\limits_0^t K(t,\tau)\upsilon_\delta^3\left(\tau\right)d\tau\}\exp(-\frac{1}{\delta}\phi_0(t)) + \Delta_1(F_0,\delta) \equiv (P_1\upsilon_\delta)(x), \\ \Delta_1(F_0,\delta) \equiv -\frac{1}{\delta^2}\int\limits_0^t h(s)\exp(-\frac{1}{\delta}(\phi_0(t)-\phi_0(s))\{F_0(s)-F_0(t)\}ds+ \\ +\frac{1}{\delta}F_0(t)\exp(-\frac{1}{\delta}\phi_0(t)). \end{cases} \tag{2.10}$$

Assuming conditions

$$\begin{cases} |\Delta_1(F_0,\delta)| \le L_{F_0}[\frac{1}{\alpha\gamma}+2^2e^{-2}M_1\delta] \le L_0, \\ \int\limits_0^\infty e^{-z}zdz = 1; \rho \equiv \frac{1}{\delta}\phi_0(t); \chi(\rho) \equiv \rho^k\exp(-\rho); \sup\limits_{\rho\ge 0}\chi(\rho) = k^k\exp(-k), \\ \rho = 0 : \chi(0) = 0; \rho\to\infty : \chi\to 0, (k=1,2), \\ 0 < L_{P_1} = C_0[6r_1^2M_0 + \frac{1}{\alpha}8r_1^3T] + 3r_1^2L_KM_0T < 1, \\ P_1 : S_{r_1}(0)\to S_{r_1}(0) = \left\{\upsilon_\delta\left(t\right)\in C[0,T] : |\upsilon_\delta\left(t\right)| \le r_1, \forall t\in[0,T]\right\}, \end{cases} \tag{2.11}$$

from (2.10) we obtain the estimate:

$$\|\upsilon_\delta\left(t\right)\|_C \le \left(1-L_{P_1}\right)^{-1}\|\Delta_1\left(F_0,\delta\right)\|_C \le \left(1-L_{P_1}\right)^{-1}L_0 = r_1, \tag{2.12}$$

where $L_{P_1}-$ is the Lipschitz constant of the operator P_1, Under condition (2.11), the Banach fixed-point principle applies, and equation (2.10) has a unique solution in $C[0,T]$.

Using the substitution

$$\upsilon_\delta\left(t\right) = \upsilon\left(t\right) + \eta_\delta\left(t\right) \tag{2.13}$$

and the resolvent (2.7), we obtain the IE:

$$\begin{cases} \eta_\delta = -\frac{1}{\delta^2}\int\limits_0^t h(s)\exp(-\frac{1}{\delta}\int\limits_s^t h(s')ds')\{-\int\limits_0^s K(s,s')[3(\upsilon(s'))^2\eta_\delta(s')+3\upsilon(s')(\eta_\delta(s'))^2+ \\ +(\eta_\delta(s'))^3]ds'+\int\limits_0^t K(t,s')[3(\upsilon(s'))^2\eta_\delta(s')+3\upsilon(s')(\eta_\delta(s'))^2+(\eta_\delta(s'))^3]ds'+ \\ +\int\limits_0^s h_0(s')\eta_\delta(s')\int\limits_0^{s'} K(s',\bar{s})(\upsilon(\bar{s})+\eta_\delta(\bar{s}))^3 d\bar{s}ds'+\int\limits_0^s h_0(s')\upsilon(s')\int\limits_0^{s'} K(s',\bar{s})[3(\upsilon(\bar{s}))^2\eta_\delta(\bar{s})+ \\ +3\upsilon(\bar{s})(\eta_\delta(\bar{s}))^2+(\eta_\delta(\bar{s}))^3]d\bar{s}ds'-\int\limits_0^t h_0(s')\eta_\delta(s')\int\limits_0^{s'} K(s',\bar{s})(\upsilon(\bar{s})+\eta_\delta(\bar{s}))^3 d\bar{s}ds'- \\ -\int\limits_0^t h_0(s')\upsilon(s')\int\limits_0^{s'} K(s',\bar{s})[3(\upsilon(\bar{s}))^2\eta_\delta(\bar{s})+3\upsilon(\bar{s})(\eta_\delta(\bar{s}))^2+(\eta_\delta(\bar{s}))^3]d\bar{s}ds'\}ds+ \\ +\frac{1}{\delta}\exp(-\frac{1}{\delta}\phi_0(t))\{-\int\limits_0^t K(t,s')[3(\upsilon(s'))^2\eta_\delta(s')+3\upsilon(s')(\eta_\delta(s'))^2+(\eta_\delta(s'))^3]ds'+ \\ +\int\limits_0^t h_0(s')\eta_\delta(s')\int\limits_0^{s'} K(s',\bar{s})(\upsilon(\bar{s})+\eta_\delta(\bar{s}))^3 d\bar{s}ds'+\int\limits_0^t h_0(s')\upsilon(s')\int\limits_0^{s'} K(s',\bar{s})\times \\ \times[3(\upsilon(\bar{s}))^2\eta_\delta(\bar{s})+3\upsilon(\bar{s})(\eta_\delta(\bar{s}))^2+(\eta_\delta(\bar{s}))^3]d\bar{s}ds'\}+\Delta(\upsilon,\delta)\equiv(P_2\eta_\delta)(t), \\ \Delta(\upsilon,\delta)\equiv-\frac{1}{\delta}\int\limits_0^t h(s)\exp(-\frac{1}{\delta}\int\limits_s^t h(s')ds')(-\upsilon(s)+\upsilon(t))ds-\upsilon(t)\exp(-\frac{1}{\delta}\int\limits_0^t h(s')ds), \\ \eta_\delta\left(t\right)\in S_{\tilde{r}_1}(0)=\{\eta_\delta\left(t\right):|\eta_\delta\left(t\right)|\le\tilde{r}_1,\forall t\in[0,T]\}. \end{cases} \tag{2.14}$$

Let:

$$\left\|\Delta\left(\upsilon,\delta\right)\right\|_C\le 3\left\|\upsilon\left(t\right)\right\|_C\exp\left(-\frac{1}{\delta^{1-\beta}}\right)+\omega_{\overline{\upsilon}}\left(\delta^\beta\right),\left(0<\beta<1\right) \tag{2.15}$$

where $\omega_{\overline{\upsilon}}\left(\delta^\beta\right)=\sup\left\{\left|\upsilon\left({\phi_0}^{-1}\left(t\right)\right)-\upsilon\left({\phi_0}^{-1}\left(z\right)\right)\right|;\left|t-z\right|\le\delta^\beta\right\}$ – is the modulus of continuity and ${\phi_0}^{-1}\left(t\right)$ – is the inverse function to $\phi_0\left(t\right)=\int\limits_0^t h\left(s\right)ds$ Taking estimate (2.15) into account, the following inequality holds:

$$\begin{cases} a_1)\,|\,\frac{1}{\delta}\exp(-\frac{1}{\delta}\phi_0(t))\{-\int\limits_0^t K(t,s')[3(\upsilon(s'))^2\eta_\delta(s')+3\upsilon(s')(\eta_\delta(s'))^2+(\eta_\delta(s'))^3]ds'+ \\ +\int\limits_0^t h_0(s')\eta_\delta(s')\int\limits_0^{s'} K(s',\bar{s})(\upsilon(\bar{s})+\eta_\delta(\bar{s}))^3 d\bar{s}ds'+\int\limits_0^t h_0(s')\upsilon(s')\int\limits_0^{s'} K(s',\bar{s})\times \\ \times[3(\upsilon(\bar{s}))^2\eta_\delta(\bar{s})+3\upsilon(\bar{s})(\eta_\delta(\bar{s}))^2+(\eta_\delta(\bar{s}))^3]d\bar{s}ds'\}\,|\le\{C_0e^{-1}[(r_1+\tilde{r}_1)^2+r_1^2+ \\ +r_1(r_1+\tilde{r}_1)](M_0+\frac{1}{\alpha}r_1T)+\frac{1}{\alpha}C_0T(r_1+\tilde{r}_1)^3\}\|\eta_\delta\|_C\le d_1\|\eta_\delta\|_C,(e^{-1}<1), \end{cases}$$

and a similar inequality is also valid:

$$\begin{cases} a_2)\,|-\frac{1}{\delta^2}\int\limits_0^t h(s)\exp(-\frac{1}{\delta}\int\limits_s^t h(s')ds')\{-\int\limits_0^s K(s,s')[3(\upsilon(s'))^2\eta_\delta(s')+3\upsilon(s')(\eta_\delta(s'))^2+ \\ +(\eta_\delta(s'))^3]ds'+\int\limits_0^t K(t,s')[3(\upsilon(s'))^2\eta_\delta(s')+3\upsilon(s')(\eta_\delta(s'))^2+(\eta_\delta(s'))^3]ds'+ \\ +\int\limits_0^s h_0(s')\eta_\delta(s')\int\limits_0^{s'} K(s',\bar{s})(\upsilon(\bar{s})+\eta_\delta(\bar{s}))^3 d\bar{s}ds'+\int\limits_0^s h_0(s')\upsilon(s')\int\limits_0^{s'} K(s',\bar{s})[3(\upsilon(\bar{s}))^2\eta_\delta(\bar{s})+ \\ +3\upsilon(\bar{s})(\eta_\delta(\bar{s}))^2+(\eta_\delta(\bar{s}))^3]d\bar{s}ds'-\int\limits_0^t h_0(s')\eta_\delta(s')\int\limits_0^{s'} K(s',\bar{s})(\upsilon(\bar{s})+\eta_\delta(\bar{s}))^3 d\bar{s}ds'- \\ -\int\limits_0^t h_0(s')\upsilon(s')\int\limits_0^{s'} K(s',\bar{s})[3(\upsilon(\bar{s}))^2\eta_\delta(\bar{s})+3\upsilon(\bar{s})(\eta_\delta(\bar{s}))^2+(\eta_\delta(\bar{s}))^3]d\bar{s}ds'\}ds\,|\le d_2\|\eta_\delta\|_C. \end{cases}$$

Estimation of the equation (2.14) gives:

$$\begin{cases} \|\eta_\delta\left(t\right)\|_C \leq \left(1-L_{P_2}\right)^{-1}[3\|\upsilon\left(t\right)\|_C \exp\left(-\frac{1}{\delta^{1-\beta}}\right) + \omega_{\overline{\upsilon}}\left(\delta^\beta\right)] \xrightarrow[\delta\to 0]{} 0, \\ L_{P_2} = d_1 + d_2 < 1, \end{cases} \tag{2.16}$$

Moreover, the function η_δ is uniquely determined in $C[0,T]$. Therefore, based on (2.13), as $\delta \to 0$, it follows that $\upsilon_\delta\left(t\right) \to \upsilon\left(t\right), \forall t \in [0,T]$. This proves the second statement of Lemma 2.1.

3) Since the function $\xi_\varepsilon\left(t\right)$, is determined from the third IE of system (2.5), taking into account the resolvent (2.7) and partially inverting this IE, we obtain:

$$\begin{cases} \xi_\varepsilon = -\frac{1}{\varepsilon^2}\int\limits_0^t h(s)\exp(-\frac{1}{\varepsilon}\int\limits_s^t h(s')ds')\{-\int\limits_0^s K(s,s')[\frac{1}{\varepsilon^3}(\Pi_\varepsilon(s'))^3 + (\xi_\varepsilon(s'))^3 + \\ +3(\upsilon(s'))^2\xi_\varepsilon(s') + 3\upsilon(s')(\xi_\varepsilon(s'))^2 + 3(\upsilon(s'))^2\frac{1}{\varepsilon}\Pi_\varepsilon(s') + 3\frac{1}{\varepsilon^2}\upsilon(s')(\Pi_\varepsilon(s'))^2 + \\ +3\frac{1}{\varepsilon}(\xi_\varepsilon(s'))^2\Pi_\varepsilon(s') + 3\frac{1}{\varepsilon^2}\xi_\varepsilon(s')(\Pi_\varepsilon(s'))^2 + 6\frac{1}{\varepsilon}\upsilon(s')\xi_\varepsilon(s')\Pi_\varepsilon(s')]ds' + \\ +\int\limits_0^t K(t,s')[\frac{1}{\varepsilon^3}(\Pi_\varepsilon(s'))^3 + (\xi_\varepsilon(s'))^3 + 3(\upsilon(s'))^2\xi_\varepsilon(s') + 3\upsilon(s')(\xi_\varepsilon(s'))^2 + \\ +3(\upsilon(s'))^2\frac{1}{\varepsilon}\Pi_\varepsilon(s') + 3\frac{1}{\varepsilon^2}\upsilon(s')(\Pi_\varepsilon(s'))^2 + 3\frac{1}{\varepsilon}(\xi_\varepsilon(s'))^2\Pi_\varepsilon(s') + 3\frac{1}{\varepsilon^2}\xi_\varepsilon(s')(\Pi_\varepsilon(s'))^2 + \\ +6\frac{1}{\varepsilon}\upsilon(s')\xi_\varepsilon(s')\Pi_\varepsilon(s')]ds' + \int\limits_s^t h_0(s')(\eta_\delta(s') + \frac{1}{\varepsilon}\Pi_\varepsilon(s'))\int\limits_0^{s'} K(s',\bar{s})(\upsilon(\bar{s}) + \eta_\delta(\bar{s}) + \\ +\frac{1}{\varepsilon}\Pi_\varepsilon(\bar{s}))^3 d\bar{s}ds' + \int\limits_s^t h_0(s')\upsilon(s')\int\limits_0^{s'} K(s',\bar{s})[\frac{1}{\varepsilon^3}(\Pi_\varepsilon(\bar{s}))^3 + (\xi_\varepsilon(\bar{s}))^3 + 3(\upsilon(\bar{s}))^2\xi_\varepsilon(\bar{s}) + \\ +3\upsilon(\bar{s})(\xi_\varepsilon(\bar{s}))^2 + 3(\upsilon(\bar{s}))^2\frac{1}{\varepsilon}\Pi_\varepsilon(\bar{s}) + 3\frac{1}{\varepsilon^2}\upsilon(\bar{s})(\Pi_\varepsilon(\bar{s}))^2 + 3\frac{1}{\varepsilon}(\xi_\varepsilon(\bar{s}))^2\Pi_\varepsilon(\bar{s}) + \\ +3\frac{1}{\varepsilon^2}\xi_\varepsilon(\bar{s})(\Pi_\varepsilon(\bar{s}))^2 + 6\frac{1}{\varepsilon}\upsilon(\bar{s})\xi_\varepsilon(\bar{s})\Pi_\varepsilon(\bar{s})]d\bar{s}ds'\}ds + \frac{1}{\varepsilon}\exp(-\frac{1}{\varepsilon}\phi_0(t)) \times \\ \times\{-\int\limits_0^t K(t,s')[\frac{1}{\varepsilon^3}(\Pi_\varepsilon(s'))^3 + (\xi_\varepsilon(s'))^3 + 3(\upsilon(s'))^2\xi_\varepsilon(s') + 3\upsilon(s')(\xi_\varepsilon(s'))^2 + \\ +3(\upsilon(s'))^2\frac{1}{\varepsilon}\Pi_\varepsilon(s') + 3\frac{1}{\varepsilon^2}\upsilon(s')(\Pi_\varepsilon(s'))^2 + 3\frac{1}{\varepsilon}(\xi_\varepsilon(s'))^2\Pi_\varepsilon(s') + 3\frac{1}{\varepsilon^2}\xi_\varepsilon(s')(\Pi_\varepsilon(s'))^2 + \\ +6\frac{1}{\varepsilon}\upsilon(s')\xi_\varepsilon(s')\Pi_\varepsilon(s')]ds' + \int\limits_0^t h_0(s')(\eta_\delta(s') + \frac{1}{\varepsilon}\Pi_\varepsilon(s'))\int\limits_0^{s'} K(s',\bar{s})(\upsilon(\bar{s}) + \eta_\delta(\bar{s}) + \\ +\frac{1}{\varepsilon}\Pi_\varepsilon(\bar{s}))^3 d\bar{s}ds' + \int\limits_0^t h_0(s')\upsilon(s')\int\limits_0^{s'} K(s',\bar{s})[\frac{1}{\varepsilon^3}(\Pi_\varepsilon(\bar{s}))^3 + (\xi_\varepsilon(\bar{s}))^3 + 3(\upsilon(\bar{s}))^2\xi_\varepsilon(\bar{s}) + \\ +3\upsilon(\bar{s})(\xi_\varepsilon(\bar{s}))^2 + 3(\upsilon(\bar{s}))^2\frac{1}{\varepsilon}\Pi_\varepsilon(\bar{s}) + 3\frac{1}{\varepsilon^2}\upsilon(\bar{s})(\Pi_\varepsilon(\bar{s}))^2 + 3\frac{1}{\varepsilon}(\xi_\varepsilon(\bar{s}))^2\Pi_\varepsilon(\bar{s}) + \\ +3\frac{1}{\varepsilon^2}\xi_\varepsilon(\bar{s})(\Pi_\varepsilon(\bar{s}))^2 + 6\frac{1}{\varepsilon}\upsilon(\bar{s})\xi_\varepsilon(\bar{s})\Pi_\varepsilon(\bar{s})]d\bar{s}ds'\} + \Delta_2(F_\varepsilon, F, \varepsilon) + \Delta_*(\upsilon,\varepsilon) \equiv (P_3\xi_\varepsilon)(t), \end{cases} \tag{2.17}$$

where

$$\begin{cases} \Delta_*(\upsilon,\varepsilon) \equiv -\frac{1}{\varepsilon}\int\limits_0^t h(s)\exp(-\frac{1}{\varepsilon}\int\limits_s^t h(s')ds')(-\upsilon(s) + \upsilon(t))ds - \upsilon(t)\exp(-\frac{1}{\varepsilon}\phi_0(t)), \\ \xi_\varepsilon\left(t\right) \in S_{\tilde{r}_2}(0) = \{\xi_\varepsilon\left(t\right) : |\xi_\varepsilon\left(t\right)| \leq \tilde{r}_2, \forall t \in [0,T]\}. \\ \Delta_2(F_\varepsilon, F, \varepsilon) \equiv -\frac{1}{\varepsilon^2}\int\limits_0^t h(s)(\exp(-\frac{1}{\varepsilon}\int\limits_s^t h(s')ds'))(F_\varepsilon(s) - F(s))ds + \frac{1}{\varepsilon}(F_\varepsilon(t) - F(t)), \\ \|\Delta_*\left(\upsilon,\varepsilon\right)\|_C \leq 3\|\upsilon\left(t\right)\|_C \exp\left(-\frac{1}{\varepsilon^{1-\beta}}\right) + \omega_{\overline{\upsilon}}\left(\varepsilon^\beta\right), (0<\beta<1), \\ |F_\varepsilon\ (t) - F(t\ )| \leq \Delta_0(\varepsilon), \forall t \in [0,T], (\frac{1}{\varepsilon}\Delta_0(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0), \\ |\Delta_2(F_\varepsilon, F, \varepsilon)| \leq \frac{1}{\varepsilon^2}\int\limits_0^t h(\tau)\exp(-\frac{1}{\varepsilon}(\phi_0(t) - \phi_0(\tau))\Delta_0(\varepsilon)d\tau + \frac{1}{\varepsilon}\Delta_0(\varepsilon) \leq \frac{2}{\varepsilon}\Delta_0(\varepsilon). \end{cases} \tag{2.18}$$

Estimating equation (2.17), we obtain:

$$\begin{cases} \|\xi_\varepsilon\left(t\right)\|_C \leq \left(1-L_{P_3}\right)^{-1}\left[\frac{2}{\varepsilon}\Delta_0(\varepsilon) + \|\Delta_*\left(\upsilon,\varepsilon\right)\|_C\right] = \Delta_3(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0, (\frac{1}{\varepsilon}\Delta_0(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0), \\ 0 < L_{P_3} < 1, \\ P_3 : S_{\tilde{r}_2}(0) \to S_{\tilde{r}_2}(0). \end{cases} \tag{2.19}$$

Here, in deriving estimate (2.19), the following facts concerning the terms containing the function $\xi_\varepsilon\,(t)$, are taken into account, for example:

$$\begin{cases} 1)\,|-\frac{1}{\varepsilon^2}\int\limits_0^t h(s)\exp(-\frac{1}{\varepsilon}\int\limits_s^t h(s')ds')[\int\limits_s^t h_0(\bar{s})\xi_\varepsilon(\bar{s})\int\limits_0^{\bar{s}} K(\bar{s},\tilde{s})\frac{1}{\varepsilon^3}(\Pi_\varepsilon(\tilde{s}))^3 d\tilde{s}d\bar{s}]ds\,| \leq \\ \leq \frac{1}{\alpha}C_0^4(\int\limits_0^\infty e^{-z}zdz)\int\limits_0^t \frac{1}{\varepsilon^3}\exp(-\frac{3}{\varepsilon}\phi_0(\tilde{s}))d\tilde{s}\|\xi_\varepsilon\|_C \leq \frac{1}{\alpha}C_0^4 3^{-\frac{9}{2}}\sqrt{\varepsilon^3}[(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}}+\int\limits_0^\infty \rho^{\frac{9}{2}}e^{-\rho}d\rho]\times \\ \times\|\xi_\varepsilon\|_C = \frac{1}{\alpha}C_0^4 3^{-\frac{9}{2}}\sqrt{\varepsilon^3}[(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}}+\frac{945}{32}\sqrt{\pi}]\|\xi_\varepsilon\|_C = \gamma_0\sqrt{\varepsilon^3}\|\xi_\varepsilon\|_C, \\ 2)\,|\,\frac{1}{\varepsilon^3}\exp(-\frac{1}{\varepsilon}\phi_0(t))\int\limits_0^t K(t,s)(\Pi_\varepsilon(s))^2\xi_\varepsilon(s)ds\,| \leq C_0^3\frac{1}{\varepsilon^3}t\exp(-\frac{1}{\varepsilon}\phi_0(t))\|\xi_\varepsilon\|_C \leq \\ \leq C_0^3\sqrt{\varepsilon^3}(\frac{1}{\varepsilon}\phi_0(t))^{\frac{9}{2}}\exp(-\frac{1}{\varepsilon}\phi_0(t))\|\xi_\varepsilon\|_C \leq C_0^3\sqrt{\varepsilon^3}(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}}\|\xi_\varepsilon\|_C = \gamma_1\sqrt{\varepsilon^3}\|\xi_\varepsilon\|_C, \\ \gamma_0 = \frac{1}{\alpha}C_0^4 3^{-\frac{9}{2}}[(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}}+\frac{945}{32}\sqrt{\pi}];\gamma_1 = C_0^3(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}};\ \rho\equiv\frac{1}{\varepsilon}\phi_0(t);\chi(\rho)\equiv\rho^k\exp(-\rho), \\ \sup\limits_{\rho\geq 0}\chi(\rho) = k^k\exp(-k),(k=1,...,\frac{9}{2});\rho=0:\chi(0)=0;\rho\to\infty:\chi\to 0. \end{cases} \tag{2.20}$$

Similar regularity estimates with respect to the small parameter are obtained for the remaining terms containing the function $\xi_\varepsilon\,(t)$.

In addition, analogous estimates can be given for the terms that do not contain the function $\xi_\varepsilon\,(t)$, for example:

$$\begin{cases} 3)\,|-\frac{1}{\varepsilon^2}\int\limits_0^t h(s)\exp(-\frac{1}{\varepsilon}\int\limits_s^t h(s')ds')[\int\limits_s^t h_0(\bar{s})\frac{1}{\varepsilon}\Pi_\varepsilon(\bar{s})\int\limits_0^{\bar{s}} K(\bar{s},\tilde{s})\frac{1}{\varepsilon^3}(\Pi_\varepsilon(\tilde{s}))^3 d\tilde{s}d\bar{s}]ds\,| \leq \\ \leq \frac{1}{\alpha}C_0^5(\int\limits_0^\infty e^{-z}zdz)\int\limits_0^t \frac{1}{\varepsilon^4}\exp(-\frac{3}{\varepsilon}\phi_0(\tilde{s}))d\tilde{s} \leq \frac{1}{\alpha}C_0^5 3^{-\frac{9}{2}}\sqrt{\varepsilon}[(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}}+\int\limits_0^\infty \rho^{\frac{9}{2}}e^{-\rho}d\rho] = \\ = \frac{1}{\alpha}C_0^5 3^{-\frac{9}{2}}\sqrt{\varepsilon}[(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}}+\frac{945}{32}\sqrt{\pi}] = \gamma_2\sqrt{\varepsilon},(\gamma_2 = \frac{1}{\alpha}C_0^5 3^{-\frac{9}{2}}[(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}}+\frac{945}{32}\sqrt{\pi}]), \\ 4)\,|\,\frac{1}{\varepsilon^4}\exp(-\frac{1}{\varepsilon}\phi_0(t))\int\limits_0^t K(t,s)(\Pi_\varepsilon(s))^3 ds\,| \leq C_0^4\sqrt{\varepsilon}(\frac{1}{\varepsilon}\phi_0(t))^{\frac{9}{2}}\exp(-\frac{1}{\varepsilon}\phi_0(t)) \leq \\ \leq C_0^4\sqrt{\varepsilon}(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}} = \gamma_3\sqrt{\varepsilon},(\gamma_3 = C_0^4(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}}), \end{cases}$$

and so on.

On the other hand, it follows from (2.19) that the Banach conditions are satisfied for IE (2.17). Consequently, equation (2.17) is uniquely solvable in $C[0,T]$, and, moreover, the function $\xi_\varepsilon\,(t)$ converges to zero as ?¿0 for any $t\in[0,T]$. Lemma 2.1 is proved. □

Remark 2.1 *The results obtained in the above statements do not provide a complete proof of the regularizability of the VIE-1. Therefore, we prove the following theorem which establishes regularizability in the generalized sense in* $Z^3(0,T)$.

Theorem 2.1 *Under the conditions of Lemma 2.1, the following statements hold:*

1.

$$\begin{cases} \|\Pi_\varepsilon\|_{Z^3(0,T)} \leq \gamma_4\varepsilon^{\frac{9}{6}},(\gamma_4 = C_0[3^{\frac{9}{2}}(2e)^{-\frac{9}{2}}+\frac{945}{32}\sqrt{\pi}]^{\frac{1}{3}}), \\ \|\Omega_\varepsilon\|_{Z^3(0,T)} \leq \gamma_4\varepsilon^{\frac{1}{2}}, \end{cases} \tag{2.21}$$

2.

$$\begin{cases} ||\theta_\varepsilon - \upsilon||_{Z^3(0,T)} \leq \tilde{M}_0(\varepsilon),(\tilde{M}_0(\varepsilon) = 2[\Delta_3(\varepsilon)\sqrt[3]{T}+\gamma_4\varepsilon^{\frac{1}{2}}]), \\ ||\theta_\varepsilon||_{Z^3(0,T)} \leq r_* = const, \end{cases} \tag{2.22}$$

3.

$$\|(\Phi_0\theta_\varepsilon)\,(t) - F\,(t)\|_{Z^3(0,T)} \leq \tilde{M}\,(\varepsilon)\,,\left(\tilde{M}_0\,(\varepsilon)\,,\tilde{M}\,(\varepsilon)\to 0\ \varepsilon\to 0\right). \tag{2.23}$$

Proof: Consider inequality (2.8) of Lemma 1. Raising it to the third power and integrating with respect to the variable t, we obtain:

$$\int\limits_0^t |\Pi_\varepsilon(\tau)|^3 d\tau \le C_0^3 \int\limits_0^t \exp(-\frac{3}{\varepsilon}\phi_0(\tau))d\tau = C_0^3[\tau\exp(-\frac{3}{\varepsilon}\phi_0(\tau))\,|_0^t +$$

$$+\int\limits_0^t \tau\exp(-\frac{3}{\varepsilon}\phi_0(\tau))d(\frac{3}{\varepsilon}\phi_0(\tau))] = C_0^3[t\exp(-\frac{3}{\varepsilon}\phi_0(t)) + \int\limits_0^t 3^{-\frac{9}{2}}\varepsilon^{\frac{9}{2}}(\frac{3}{\varepsilon}\phi_0(\tau))^{\frac{9}{2}}\times$$

$$\times\exp(-\frac{3}{\varepsilon}\phi_0(\tau))d(\frac{3}{\varepsilon}\phi_0(\tau))] \le C_0^3 3^{-\frac{9}{2}}\varepsilon^{\frac{9}{2}}[(\frac{9}{2})^{\frac{9}{2}}e^{-\frac{9}{2}} + \frac{945}{32}\sqrt{\pi}] = C_0^3\varepsilon^{\frac{9}{2}}[3^{\frac{9}{2}}(2e)^{-\frac{9}{2}} + \frac{945}{32}\sqrt{\pi}],$$

or, in terms of the norm $Z^3(0,T)$, we arrive at the estimate:

$$\begin{cases} ||\theta_\varepsilon - \upsilon||_{Z^3} \le 2[\Delta_3(\varepsilon)\sqrt[3]{T} + \gamma_4\varepsilon^{\frac{1}{2}}] = \tilde{M}_0(\varepsilon), \\ ||\theta_\varepsilon||_{Z^3(0,T)} \le 4[\sqrt[3]{T}(r + \Delta_3(\varepsilon)) + \gamma_4\varepsilon^{\frac{1}{2}}] \le r_*, (|\upsilon| \le r, \forall t \in [0,T]). \end{cases}$$

This implies that inequality (2.22) is satisfied.

Finally, to prove (2.23), we first take into account the estimate:

$$|(\Phi_0\theta_\varepsilon)(t) - F(t)| = |\varepsilon\theta_\varepsilon + (\Phi_0\theta_\varepsilon)(t) - F_\varepsilon(t) - \varepsilon(\theta_\varepsilon - \upsilon + \upsilon) + F_\varepsilon(t) - F(t)|\,, \tag{2.24}$$

where the operator $(\Phi_0\theta_\varepsilon)(t)$ is defined by (2.3). Passing from (2.24) to the norm $Z^3(0,T)$, we obtain:

$$\|(\Phi_0\theta_\varepsilon)(t) - F(t)\|_{Z^3(0,T)} \le 4[\|F_\varepsilon(t) - F(t)\|_{Z^3} + \varepsilon\|\theta_\varepsilon(t) - \upsilon(t)\|_{Z^3} + \varepsilon r\sqrt[3]{T}] \le$$

$$\le 4[\Delta_0(\varepsilon)\sqrt[3]{T} + \varepsilon\tilde{M}_0(\varepsilon) + \varepsilon r\sqrt[3]{T}] = \tilde{M}(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0.$$

Q.E.D. □

Statement 2.1. Under the conditions of Theorem 2.1, VIE-1 (1.1) is regularizable in $Z^3(0,T)$. in the generalized sense.

3. Conclusion

The results obtained in this work make it possible to carry out an analysis of VIE-1 in the space $Z^3(0,T)$, when the original IE is transformed into form (2.2). Therefore, the investigation of such IEs and their regularization methods in the generalized sense is of scientific interest and is necessary for understanding many important issues related to ill-posed nonlinear VIE-1 that degenerate in inverse problems of mathematical physics, which determines the relevance of the results obtained in this paper.

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