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A Nonlocal Inverse Problem of Parabolic Type in a Bounded Domain

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ABSTRACT: This paper examines issues related to the method of integration based on the Green's function and the uniqueness of the solution of a nonlocal inverse parabolic problem in a bounded domain. When integral equations of the first kind degenerate in inverse problems, various regularization methods can be applied in specific spaces to solve these equations. Many classes of inverse problems of parabolic type belong to the class of conditionally well-posed problems, and in our case, the studied problem falls into this category, which underlines its relevance.

Keywords: Partial differential equation, inverse problem, method of integral transformations, regularization method, uniqueness of solution.

Contents

1. Introduction

Parabolic differential equations in bounded and unbounded domains are widely used in applied problems. Several problems lead to parabolic-type equations, such as wave theory problems and the problem of periodic changes in groundwater levels [5,9], among others. If these problems contain unknown functions apart from the solution itself, such as a free function (or parameters), a coefficient of the desired function, or an unknown function in the boundary conditions, then they form inverse problems of a specific class, namely coefficient and boundary inverse problems [2,3,8,12], among others. If the studied inverse problems degenerate into an integral equation of the first kind, as mentioned earlier, specific regularization methods applicable in various spaces are used to study these equations [3,7,8,10], etc. Thus, this paper investigates a boundary inverse problem of the form:

$$L_0 W = \frac{\partial}{\partial t}(W_t + W) - \frac{\partial^2}{\partial x^2}(W_t + W) = f(x,t), (x,t) \in \overline{D}, (D = (0,X) \times (0,T)), \tag{1.1}$$

$$\begin{cases} W(x,0) = g(x), \forall x \in [0,X]; AW|_{t=0} = 0, \forall \in [0,X], \\ (\frac{\partial}{\partial x} AW)|_{x=0} = (Hz)(t) \equiv \psi(t), \forall t \in [0,T], (\psi(0) = 0), AW|_{x=X} = 0, \forall t \in [0,T], \\ p(t)(\frac{\partial}{\partial x} AW)|_{x=0} = \sum\limits_{j=1}^{n} \alpha_j\,(t)\, AW|_{x=x_j} + \varphi_0\,(t)\,;\;\; (\varphi_0(0) = 0; x_j \in (0,X)), \\ AW = W_t + W, \forall (x,t) \in \bar{D}; Hz \equiv \int\limits_0^t K(t,\tau)\gamma(\tau)z(\tau)d\tau, (z(0) = 0), \end{cases} \tag{1.2}$$

It is necessary to find a continuous vector function (W, z) in the domain $\overline{D}$ such that this function is a regular solution to the inverse problem under study in the domain D, where $\overline{D}$ is the closure of D in the topology of the space of continuous functions [1], [6].

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2. Integrability of the inverse problem (1.1), (1.2)

To represent the studied inverse problem in an integral form, we use a transformation of the form:

$$W = g(x)e^{(-t)} + \int\limits_0^t e^{(-(t-s)}u(x,s)ds \equiv (Au)(x,t), \tag{2.1}$$

where is the new unknown function. Then equation (1.1) can be rewritten as:

$$Lu \equiv u_t - u_{x^2} = f \tag{2.2}$$

with condition

$$\begin{cases} u(x,0) = 0, \forall x \in [0,X], \ (\frac{\partial}{\partial x}u)\,|_{x=0} = \psi(t), \forall t \in [0,T], \\ u(X,t) = 0, \forall t \in [0,T], \ p(t)(\frac{\partial}{\partial x}u)\,|_{x=0} = \sum\limits_{j=1}^{n} a_j(t)u(x^j,t) + \varphi_0(t), \end{cases} \tag{2.3}$$

i.e. we obtain the inverse problem (2.3), (2.4) with differential the second order operator, where (u,z) is unknown vector function (here for convenience we have specified this vector function instead of (u,ψ), since the functions and are related to the integral Hz, (see (1.2)). Here, in fact, (2.4) (or (1.2)) is an analogue of the Bitsadze – Samarskii problem [7, 8], etc. It is known that the Green function for problem (2.3), (2.4) is given by the formula [4]:

$$\begin{cases} G(\tau,s;x,t) \equiv \sqrt{(t-s)^{-1}}G_0(\tau,s;x,t), \\ G_0(\tau,s;x,t) = \frac{1}{2\sqrt{\pi}} \sum\limits_{n=-\infty}^{+\infty} \Big\{ \exp\frac{-(x-\tau+2n)^2}{4(t-s)} + \exp\frac{-(x+\tau+2n)^2}{4(t-s)} - \exp\frac{-(x-\tau-2X+2n)^2}{4(t-s)} - \\ - \exp\frac{-(x+\tau-2X+2n)^2}{4(t-s)} \Big\}. \end{cases} \tag{2.4}$$

Then the solution u(x, t) is presented in domain D as

$$u(x,t) = \int\limits_0^t \psi(s)G(0,s;x,t)ds - \int\limits_0^t \int\limits_0^X G(\tau,s;x,t)f(\tau,s)d\tau ds \equiv (B\psi)(x,t), \tag{2.5}$$

Substituting (2.6) into the third relation of (2.4) and considering (2.5), we obtain

$$\begin{cases} \theta(t) = \sum\limits_{j=1}^{n} \alpha_j\,(t)\,[\int\limits_0^t \frac{1}{\sqrt{t-s}}(p(s))^{-1}\theta(s)G_0\,(0,s;x_j,t)\,ds - \int\limits_0^t \int\limits_0^X G\,(\tau,s;x_j,t)\,f(\tau,s)d\tau ds] + \\ +\varphi_0(t) \equiv (H\theta)(t), \\ p(t)\psi(t) = \theta(t), \ \psi(0) = 0, \ \varphi_0(0) = 0, \ \theta(0) = 0. \end{cases} \tag{2.6}$$

Since the first integral equation in system (2.7) is a Volterra-Abel integral equation of the second kind with respect to the variable , following lemma holds:

Lemma 1. Under conditions

$$\begin{cases} d < 1, \\ \|H\theta_0 - \theta_0\| \leq (1-d)\,r \end{cases} \tag{2.7}$$

the first equation of (2.7) is unique solvable in and this solution is constructed according to the Picard rule in: $S_r\,(\theta_0) = \{\theta \in C\,[0,T] : |\theta - \theta_0| \leq r, \forall t \in [0,T]\}$. Then the second equation of system (2.7) is uniquely defined in the class of continuous functions. Therefore, the inverse problem (2.3), (2.4) also has a unique continuous solution according to the rule (2.6).

Proof is evident. Indeed, under conditions (2.8) with respect to the first equation of system (2.7), the conditions of the Banach principle hold, it means that is uniquely solvable in , moreover, this solution can be found by the Picard method: $\theta_{n+1} = H\theta_n, (n = 0, 1, ...)$. Hence, taking into account the conclusions of this method, we obtain the estimate

$$\begin{cases} \|\theta_{n+1} - \vartheta\| \leq d^{n+1} r \xrightarrow[n\to\infty]{} 0, \\ |\theta| \leq r_0, \forall t \in [0, T], \end{cases} \tag{2.8}$$

θ_0 is an initial approximation.

Next, we study the second equation of system (2.7). For this, if we allow the denotation: $\theta(t)(p(t))^{-1} = \omega(t)$, and:

$$\omega(t) = \begin{cases} \theta(t)(p(t))^{-1}, t \neq 0, \\ 0, t = 0, \end{cases} \tag{2.9}$$

$$\lim_{t\to 0} \theta(t)(p(t))^{-1} = 0. \tag{2.10}$$

Then the function $\omega(t)$ is redefined at the point $t = 0$, i.e. it is continuous function. So, $\psi(t)$ is continuous and

$$\psi(t) = \omega(t), \forall t \in [0, T] \tag{2.11}$$

Therefore, substituting the found value of (2.11) into (2.6), we obtain the conclusions of Lemma 1.

Definition 1. A generalized solution of equation (2.3) in a domain D is any continuous $\bar{D}$ solution of integral equation (2.6).

3. Regularization of the first kind Volterra integral equation in a space with a uniform metric

Since by the conditions of Lemma 1 the function is unique, then the right side of the integral equation

$$\int_0^t K(t, s)\gamma(s)z(s)ds = \psi(t), \tag{3.1}$$

is a known function that satisfies the conditions specified in (2.1). If relatively integral equation (3.2) we carry out some mathematical transformations, then as a result we get the system:

$$\begin{cases} \vartheta(t) = (K(t, t))^{-1}[\int_0^t K_s(t, s)\vartheta(s)ds + \psi(t)] \equiv (P\vartheta)(t), \\ \int_0^t \gamma(s)z(s)ds = \vartheta(t), (\vartheta(0) = 0; \vartheta \in^1_\phi [0, T]), \end{cases} \tag{3.2}$$

where the first equation satisfies to Lemma 1 because it is the second kind Volterra integral equation, i.e. for operator P the conditions of the Banach principle hold. Therefore we can say it has a unique solution in C[0,T]. Hence, by means of it the Volterra type integral equation follows:

$$\int_0^t \gamma(s)z(s)ds = \vartheta(t) \tag{3.3}$$

where conditions:

$$\begin{cases} \vartheta(t) \in C^1_\phi[0, T]; \vartheta(0) = 0, \ |\vartheta(t) - \vartheta(s)| \leq C_0 |\phi(t) - \phi(s)|, (0 < C_0 = const), \\ \vartheta_\delta(0) = 0, \vartheta_\delta(t) : |\vartheta_\delta(t) - \vartheta(t)| \leq \Delta(\delta), \forall t \in [0, T], (\frac{\Delta(\delta_\varepsilon)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0), \\ z(0) = 0; z(t) \in C^\beta_\phi[0, T], (0 < \beta < 1) : |z(t) - z(s)| \leq \\ \leq C_1 \mid \phi(t) - \phi(s) \mid^\beta, \forall t \in [0, T]. \end{cases} \tag{3.4}$$

Then equation (3.3) is regularizable in the space it is also required that (3.3) has a solution in the space under consideration. Indeed, to prove the above, first, taking into account the small parameter, we introduce the integral equation:

$$\varepsilon z_\varepsilon(t) + \int_0^t \gamma(s) z_\varepsilon(s) ds = \vartheta_\delta(t), \tag{3.5}$$

or, by the resolving theory, the integral equation (3.6) is equivalently transformed to the form:

$$\begin{aligned} & z_\varepsilon(t) = -\tfrac{1}{\varepsilon^2} \int_0^t \gamma(s) exp(-\tfrac{1}{\varepsilon}(\phi(t) - \phi(s)))[\vartheta_\delta(s) - \vartheta_\delta(t)]ds + \\ & + \tfrac{1}{\varepsilon}\vartheta_\delta(t)\exp(-\tfrac{1}{\varepsilon}\phi(t)), \end{aligned} \tag{3.6}$$

where $(0,1)\varepsilon$ - a small parameter. Hence, estimating (3.6) we have:

$$\begin{cases} |z_\varepsilon(t)| = | -\tfrac{1}{\varepsilon^2} \int_0^t \gamma(s) exp(-\tfrac{1}{\varepsilon}(\phi(t) - \phi(s)))[\vartheta_\delta(s) - \vartheta_\delta(t)]ds + \tfrac{1}{\varepsilon}\vartheta_\delta(t) \times \\ \times \exp(-\tfrac{1}{\varepsilon}\phi(t)) \mid \le \tfrac{1}{\varepsilon^2} \int_0^t \gamma(s) exp(-\tfrac{1}{\varepsilon}(\phi(t) - \phi(s)))[\,|\vartheta_\delta(s) - \vartheta(s)| + |\vartheta(s) - \vartheta(t)| + \\ + |\vartheta(t) - \vartheta_\delta(t)|\,]ds + \tfrac{1}{\varepsilon}(\mid \vartheta_\delta(t) - \vartheta(t) \mid + \mid \vartheta(t) \mid) \exp(-\tfrac{1}{\varepsilon}\phi(t)) \le N_0, \forall t \in [0, \mathrm{T}], \\ 3\tfrac{\Delta(\delta)}{\varepsilon} + 2C_0 \le N_0, (\tfrac{\Delta(\delta_\varepsilon)}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0), \end{cases} \tag{3.7}$$

or after making an estimate in the sense of the norm in the class of continuous functions, from (3.7) we have

$$\|z_\varepsilon(t)\|_\le 3\frac{\Delta(\delta)}{\varepsilon} + 2C_0 \le N_0. \tag{3.8}$$

Further, taking into account transformations of the form

$$z_\varepsilon(t) = z(t) + \xi_\varepsilon(t), \forall t \in C[0, T], \tag{3.9}$$

we obtain $\xi_\varepsilon(t) = -\tfrac{1}{\varepsilon^2} \int_0^t \gamma(s) exp(-\tfrac{1}{\varepsilon}(\phi(t) - \phi(s)))[\vartheta_\delta(s) - \vartheta(s)]ds + \tfrac{1}{\varepsilon}(\vartheta_\delta(t) - \vartheta(t)) -$

$$-\frac{1}{\varepsilon} \int_0^t \gamma(s) exp(-\frac{1}{\varepsilon}(\phi(t) - \phi(s)))[z(t) - z(s)]ds - z(t)\exp(-\frac{1}{\varepsilon}\phi(t)), \tag{3.10}$$

where (3.10) is Volterra integral equation with respect to the residual function. In addition, estimating (3.10) we have:

$$\begin{cases} |\xi_\varepsilon(t)| \le \tfrac{1}{\varepsilon^2} \int_0^t \gamma(s) exp(-\tfrac{1}{\varepsilon}(\phi(t) - \phi(s)))\, |\vartheta_\delta(s) - \vartheta(s)|\, ds + \tfrac{1}{\varepsilon}\, |\vartheta_\delta(t) - \vartheta(t)| + \\ + |\Delta_0(\varepsilon, z)| \le 2\tfrac{\Delta(\delta)}{\varepsilon} + |\Delta_0(\varepsilon, z)| \le 2\tfrac{\Delta(\delta)}{\varepsilon} + 2C_1\varepsilon^\beta, \\ \Delta_0(\varepsilon, z) \equiv -\tfrac{1}{\varepsilon} \int_0^t \gamma(s) exp(-\tfrac{1}{\varepsilon}(\phi(t) - \phi(s)))(z(t) - z(s))ds - z(t)\exp(-\tfrac{1}{\varepsilon}\phi(t)). \end{cases} \tag{3.11}$$

Therefore, from estimation (3.11) passing to the norm of the space of continuous functions it follows:

$$\|\xi_\varepsilon(t)\|_C \le 2\frac{\Delta(\delta_\varepsilon)}{\varepsilon} + 2C_1\varepsilon^\beta = \Delta_1(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0, (\frac{\Delta(\delta_\varepsilon)}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0). \tag{3.12}$$

From these estimates, it follows that, first, integral equation (3.10) is regular with respect to the small parameter, and second, it uniformly converges to zero as the small parameter approaches zero. Consequently, taking (3.9) and (3.12) into account, we obtain:

$$z_\varepsilon(t) z(t), \forall t \in [0, T]. \tag{3.13}$$

From the obtained results we have the following conclusions:

Lemma 2. Under conditions (1.2), (3.4) and (3.12), the permissible error between the solutions of equations (3.3), (3.5) will be of the order $(\Delta_1(\varepsilon))$ in $C[0,T]$.

Statement 1. Under the conditions of Lemma 2 the first kind Volterra integral equation of (3.4) is uniquely regularizable in C[0,T].

Theorem 1. Let the conditions of Lemmas 1 and 2 be satisfied. Then the integral equation (2.2) has a unique solution in $C(\overline{D})$.

Definition 2. Under the conditions of Theorem 1, the function constructed by the rule (2.1), is said to be the general solution of the original problem (1.1), (1.2) in domain D.

Conclusion

This work deals with issues related to the method of integration and the proof of the unique solvability of a parabolic type nonlocal inverse problem in a bounded domain. At the same time, since the first kind Volterra integral equation degenerates from the problem under study, the study of this problem is related to regularization algorithms in the introduced spaces too. The results obtained are also applicable to nonlinear inverse problems in a limited region, where the first kind Volterra integral equation also can be nonlinear.

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