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A Course on Derived Categories

Edson Ribeiro Alvares

Centro Politécnico, Departamento de Matemática, Universidade Federal do Paraná, Caixa Postal 019081, Curitiba, Paraná, 81531-990, Brasil.

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Edson Ribeiro Alvares

Centro Politécnico, Departamento de Matemática,
Universidade Federal do Paraná, Caixa Postal 019081,
Curitiba, Paraná, 81531-990, Brasil.
E-mail address: rolo1rolo@gmail.com

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Centro Politécnico, Departamento de Matemática,
Universidade Federal do Paraná, Caixa Postal 019081, Curitiba-PR
81531-990, Brasil.
E-mail address: rolo1rolo@gmail.com

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Introduction

In the forty years of existence of derived category, it was first thought as a tool in algebraic geometry, especially in the development of duality theories that were done by Hartshorne and others. After these first moments, the theory provided a powerful homological tool for the study of linear differential equations. The basic example in the literature that can be found about this is the Riemann-Hilbert problem of associating suitable regular systems of differential equations to constructible sheaves. This studies can be found in the work of Kashiwara and Schapira. See M. Kashiwara, P. Schapira "Sheaves on manifolds" ([**15**]).

To understand the structure of the derived category is necessary to study the axioms of triangulated categories that were introduced in the mid 1960's by J.L.Verdier in his thesis "Des catégories dérivées des catégories abéliennes" ([**21**]). The role of the triangles in the derived category is a similar role of the exact sequence in the abelian category. But it is important to remember that these axioms had their origins in algebraic geometry and algebraic topology. Nowadays there are important applications of triangulated categories in areas like algebraic geometry, algebraic topology (stable homotopy theory), commutative algebra, differential geometry and representation theory of artin algebras. See, for instance, the book of D. Happel- "Triangulated categories and the representation theory of finite dimensional algebras" ([**11**]).

The objective of this notes is to present an introdutory material to the undergraduate and graduate students that would like to know some ideas about the derived category.

These are the notes a one week series of introductory lectures which I gave in the XXIII-Escola de Álgebra, in Maringá, Paraná, Brazil. Firstly we introduced the concepts of additive and abelian category to show the axioms of triangulated category that are our main objective. The triangulated category obey four axioms. We first introduced the first three axioms and their consequences on chapter one and then the octahedral axioms in various equivalent forms in a separate section of the first chapter.

The objective of this section is to give a model capable of making this axiom more palatable since, in general, the form that it is presented in the literature does not remind the reader of any similar structure in other fields of mathematics. So, we make the necessary efforts here to present another form of this axiom that is similar to other tools that could be seen in the abelian categories.

We present in chapter one the main example of triangulated category, the homotopy category of complexes. Secondly, to understand the morphisms in the derived category I introduced the concept of localization in chapter two. To those that are starting to study localization, we present the necessary background to understand the localization of non commutative ring. We believe that with this model in mind the student will profit more from the study of localization of categories.

On chapter two, the student will find the necessary information and exercises to begin to manipulate morphisms in the derived category. So, on chapter three we introduce the definition of derived category of an abelian category and we explain how one sees the original abelian category as a subcategory of its derived category.

After having done all this work, it is natural to have many questions about the behavior of the derived category or its applications. Therefore, we present here a bibliography in portuguese and in english that will help the students to make further investigations.

The reader that whishes to know the history and the motivation of the begining of the derived category with many details, should read the introduction of the book "Sheaves on Manifolds - M. Kashiwara and M. Schapira ([**15**]).

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CHAPTER 1

Triangulated Categories

In this chapter we quickly present the concepts of additive and abelian categories. We are mainly interested in the defining axioms of abelian category. We give some of the imediatly properties of these categories, which came from the refered axioms. Next we define the notion of pre-triangulated categories, which are the triangulated categories that do not necessarily satisfy the octahedral axiom.

The axioms of pre-triangulate category will be given in a separate section. Pre-triangulated structures have a lot of similarity with the additive category.

An important example of triangulated category will be presented, the homotopy category of complexes. Most of the topic compiled here can be found in the book "Lectures on Derived Categories", by Milicic ([18]).

1. Additive Category

In this section we present the basic definitions of additive and abelian categories. It is possible to find many good books about this topic. But nowadays we have on the internet many thesis (or TCC in Portuguese) about this. I strongly suggest the work of the student Valente Santiago Vargas from Unam - Mexico "Elementos de álgebra homológica en categorías abelianas y el Teorema de Inmersión en la categoría de grupos abelianos" ([20]).

Definition 1.1. A category $\mathcal{A}$ is called an **additive category** if the following conditions hold:

(A1) For every $X, Y \in Obj\mathcal{A}$, $Hom_{\mathcal{A}}(X,Y)$ is an abelian group and the composition of morphisms is bilinear over the integers.

(A2) The category $\mathcal{A}$ contains a zero object 0 (ie, for an object $X \in Obj\mathcal{A}$, such that the sets $Hom_{\mathcal{A}}(X,0)$ and $Hom_{\mathcal{A}}(0,X)$ has precisely one element).

(A3) For every pair of objects $X, Y \in \mathcal{A}$ there exists a coproduct $X \oplus Y$ in $\mathcal{A}$.

Definition 1.2. Let $(M_\lambda)_{\lambda\in\Lambda}$ be a family of objects in $\mathcal{A}$. A **coproduct**

$$(M, (\iota_\lambda)_{\lambda\in\Lambda})$$

of this family is an object M in $\mathcal{A}$ and a family of morphisms $\iota_\lambda : M_\lambda \to M$ such that, if $(M', (\iota'_\lambda)_{\lambda\in\Lambda})$ is a pair with $\iota'_\lambda : M_\lambda \to M'$ then, there exists a unique morphism $f : M \to M'$ such that $f\iota_\lambda = \iota'_\lambda$.

Example 1.1. (a) Let R be a ring and consider R as a category: $Obj\mathcal{A} = *$ and $Hom(*,*) = R$. The composition of morphisms is given by ring multiplication. Then this category satisfies $A1$. But this category is not additive in general because there is no zero object or coproduct. The coproduct of $*$ with $*$ would have to be $*$ together with fixed morphisms

$$g_1 : * \to *$$

and

$$g_2 : * \to *.$$

Then the universal property would mean that for an arbitrary ring, any element of R, $f_1 : * \to *$ and $f_2 : * \to *$ there exists a unique element f factoring them as $f_1 = fg_1$ and $f_2 = fg_2$.

(b) Let R be an associative ring with unit element. Then the category $Mod\ R$ of all right R-modules is additive. Similarly, the category $mod\ R$ of finitely generated R-modules is additive. In particular, the categories Ab of abelian groups and Vec_K of vector spaces over a field K are additives.

(c) For a ring R the full subcategory R-Proj of projective right R-modules is additive; similarly for R-proj, the category of finitely generated projective right R-modules.

The **zero morphism** in $Hom(A, B)$ is a morphism that factors through the object zero.

Lemma 1.1. *Let $\mathcal{A}$ be a category with zero object. Then the set $Hom(X, Y)$ has precisely one zero morphism.*

Proof: We know that a zero morphism $\alpha : A \to B$ is a morphism that factors through the object zero. Let $A \xrightarrow{\alpha} 0$ and $0 \xrightarrow{\beta} B$ morphisms in $\mathcal{A}$ and let $0_{AB} = \beta\alpha$ be a zero morphism. Suppose that $0'$ is another zero object in $\mathcal{A}$ and let $0'_{AB} = \beta'\alpha'$, $\alpha' : A \to 0'$, $\beta' : 0' \to B$. How 0 and $0'$ are zero objects, then there is an unique morphism $h : 0' \to 0$ so the composition $h\alpha'$ is another morphism from A to 0, so $h\alpha' = \alpha$. In the same way $\beta h = \beta'$. Then $0_{AB} = \beta\alpha = \beta h\alpha' = \beta'\alpha' = 0'_{AB}$. □

Lemma 1.2. *Let $\alpha : A \to B$ and 0 the zero morphism $0 : C \to A$. Then $\alpha 0 = 0$ where the last zero is the unique morphism from $C \to B$ that factors through the zero object.*

Proof: We have the following

$$\begin{tikzcd} C \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[r, "\alpha"] & B \\ & 0 \arrow[u, "0"'] & \end{tikzcd}$$

So $\alpha 0 = (\alpha 0)0$. Then $\alpha 0 = 0$. □

Lemma 1.3. *Let X, Y be objects in $\mathcal{A}$ and $\iota_X : X \to X \oplus Y$, $\iota_Y : Y \to X \oplus Y$ the morphisms of the coproduct. Then there exist $p_X : X \oplus Y \to X$ and $p_Y : X \oplus Y \to Y$ satisfying $p_X\iota_X = 1_X$, $p_X\iota_Y = 0$ and $p_Y\iota_Y = 1_Y, p_Y\iota_X = 0$ and $\iota_X p_X + \iota_Y p_Y = 1_{X\oplus Y}$.*

Proof: The following diagram

$$\begin{tikzcd} & X & \\ X \arrow[ur, "1_X"] \arrow[r, "\iota_X"] & X \oplus Y & Y \arrow[l, "\iota_Y"'] \arrow[ul, "0"'] \end{tikzcd}$$

allows us to say that there exist an unique $p_X : X \oplus Y \to X$ such that $p_X\iota_X = 1_X$ and $p_X\iota_Y = 0$. In the similar way we have the other equations. And finally we have

$$(\iota_X p_X + \iota_Y p_Y)\iota_X = \iota_X p_X \iota_X + \iota_Y p_Y \iota_X = \iota_X 1_X = \iota_X$$

and
$$(\iota_X p_X + \iota_Y p_Y)\iota_Y = \iota_Y.$$

Then we have two morphisms doing the following diagram commutative:

$$\begin{tikzcd}
 & X \oplus Y & \\
X \arrow[ur, "\iota_X"] \arrow[r, "\iota_X"] & X \oplus Y \arrow[u] & Y \arrow[ul, "\iota_Y"'] \arrow[l, "\iota_Y"']
\end{tikzcd}$$

Then $\iota_X p_X + \iota_Y p_Y = 1_{X\oplus Y}$. □

We introduced next the category of complexes over an aditive category. This category has a crucial hole in this work.

Let $\mathcal{A}$ be an additive category. A **complex** over $\mathcal{A}$ is a family
$$X^\bullet = (X^n, d_X^n)_{n\in\mathbb{Z}}$$
of objects X^n of $\mathcal{A}$ and morphisms $d_X^n : X^n \to X^{n+1}$ such that
$$d_X^n d_X^{n-1} = 0.$$
The object X^n is called homogeneous component of degree n of $X^\bullet$. Let $X^\bullet$ and $Y^\bullet$ be complexes over $\mathcal{A}$. A morphism of complexes $f^\bullet : X^\bullet \to Y^\bullet$ is a family of morphisms $f = (f^n : X^n \to Y^n)$ satisfying
$$f^n d_X^{n-1} = d_Y^{n-1} f^{n-1}.$$

The complexes over an additive category $\mathcal{A}$ together with the morphism of complexes form a category $\mathcal{C}(\mathcal{A})$**, the category of complexes over** $\mathcal{A}$.

PROPOSITION 1.4. *Let $\mathcal{A}$ be an additive category. The category of complexes $\mathcal{C}(\mathcal{A})$ is again additive.*

Proof: The addition of morphism of complexes is defined in the following way
$$f + g = (f^n + g^n)_{n\in\mathbb{Z}}.$$
It is easy to check that $(A1)$ in the definition of additive category holds. The zero object in $\mathcal{C}(\mathcal{A})$ is the complex $(0_{\mathcal{A}}, d)$, where $0_{\mathcal{A}}$ is the zero object of the additive category $\mathcal{A}$ and all differentials are the unique zero morphism on the zero object.

Let $X^\bullet$ and $Y^\bullet$ be two complexes in $\mathcal{C}(\mathcal{A})$. We define $X^\bullet \oplus Y^\bullet = (X^n \oplus Y^n, d^n)_{n\in\mathbb{Z}}$, where d^n is obtained in the following way:

Let $\iota_X^n : X^n \to X^n \oplus Y^n$ and $\iota_Y^n : Y^n \to X^n \oplus Y^n$ be morphisms from the coproduct, then we have the following diagram

$$\begin{tikzcd}
 & X^{n+1} \oplus Y^{n+1} & \\
X^n \arrow[ur, "\iota_X^{n+1} d_X^n"] \arrow[r, "\iota_X^n"] & X^n \oplus Y^n & Y^n \arrow[ul, "\iota_Y^{n+1} d_Y^n"'] \arrow[l, "\iota_Y^n"']
\end{tikzcd}$$

From the above diagram and from uniqueness in the universal property, there exists a unique morphism d and we have the following commutative diagram:

$$\begin{tikzcd}
 & X^{n+1} \oplus Y^{n+1} & \\
X^n \arrow[ur, "\iota_X^{n+1} d_X^n"] \arrow[r, "\iota_X^n"] & X^n \oplus Y^n \arrow[u, "d^n"] & Y^n \arrow[ul, "\iota_Y^{n+1} d_Y^n"'] \arrow[l, "\iota_Y^n"']
\end{tikzcd}$$

From uniqueness in the universal property applied to the following diagram

$$\begin{tikzcd}
 & X^{n+2}\oplus Y^{n+2} & \\
X^n \arrow[ur, "0"] \arrow[r, "\iota_X^n"] & X^n\oplus Y^n \arrow[u, "d^{n+1}d^n"'] & Y^n \arrow[l, "\iota_Y^n"'] \arrow[ul, "0"']
\end{tikzcd}$$

we have that $d^{n+1}d^n = 0$.

Therefore, we have the complex $X^\bullet \oplus Y^\bullet = (X^n \oplus Y^n, d^n)_{n\in\mathbb{Z}}$ and we have to prove that this complex satisfies the properties of a coproduct in the category of complex $\mathcal{C}(\mathcal{A})$.

Let $\iota_X = (\iota_X^n : X^n \to X^n \oplus Y^n)_{n\in\mathbb{Z}}$ and $\iota_Y = (\iota_Y^n : Y^n \to X^n \oplus Y^n)_{n\in\mathbb{Z}}$ morphisms of complexes. Let $Z^\bullet$ be an arbitrary complex and let $f_{X^\bullet} : X^\bullet \to Z^\bullet$ and $f_{Y^\bullet} : Y^\bullet \to Z^\bullet$ be arbitrary morphisms.

So there exists a family of morphisms $(f^n)_{n\in\mathbb{Z}}$, $f^n : X^n \oplus Y^n \to Z^n$ satisfying

$$f^n\iota_X^n = f^n_{X^\bullet}$$
$$f^n\iota_Y^n = f^n_{Y^\bullet}.$$

We have to prove that $f^\bullet = (f^n)_{n\in\mathbb{Z}}$ is a morphism of complexes.

$$\begin{tikzcd}
 & X^n \arrow[dl, "\iota_X^n"'] \arrow[dr, "f^n_{X^\bullet}"] \arrow[ddd, dashed, "d_X^n"] & \\
X^n\oplus Y^n \arrow[rr, dashed, "f^n"] \arrow[ddd, "d^n"] & & Z^n \arrow[ddd, "d_Z^n"] \\
 & Y^n \arrow[ul, "\iota_{Y^n}"] \arrow[ur, "f^n_{Y^\bullet}"'] & \\
 & X^{n+1} \arrow[dl, "\iota_X^{n+1}"'] \arrow[dr, "f^{n+1}_{X^\bullet}"] & \\
X^{n+1}\oplus Y^{n+1} \arrow[rr, dashed, "f^{n+1}"] & & Z^{n+1} \\
 & Y^{n+1} \arrow[ul, "\iota_{Y^{n+1}}"] \arrow[ur, "f^{n+1}_{Y^\bullet}"'] &
\end{tikzcd}$$

For this it is enough to prove that $d_Z^n f^n = f^{n+1}d^n$. We have the following

$$d^n\iota_X^n = \iota_X^{n+1}d_X^n$$
$$f^{n+1}_{X^\bullet}d_X^n = d_Z^n f^n_{X^\bullet}$$
$$d^n\iota_Y^n = \iota_Y^{n+1}d_Y^n$$
$$f^{n+1}_{Y^\bullet}d_Y^n = d_Z^n f^n_{Y^\bullet}.$$

So

$$f^{n+1}d^n = f^{n+1}d^n(\iota_X^n p_X^n + \iota_Y^n p_Y^n) =$$
$$f^{n+1}\iota_X^{n+1}d_X^n p_X^n + f^{n+1}\iota_Y^{n+1}d_Y^n p_Y^n = f^{n+1}_{X^\bullet}d_X^n p_X^n + f^{n+1}_{Y^\bullet}d_Y^n p_Y^n =$$
$$d_Z^n f^n_{X^\bullet}p_X^n + d_Z^n f^n_{Y^\bullet}p_Y^n = d_Z^n(f^n_{X^\bullet}p_X^n + f^n_{Y^\bullet}p_Y^n)$$
$$d_Z^n(f^n\iota_X^n p_X^n + f^n\iota_Y^n p_Y^n) = d_Z^n f^n. \square$$

2. Triangulated Category

2.1. Axioms. In order for an aditive category to be triangulated it needs a new structure called a triangulated structure. In some sence this new structure is similar to the notions of exact sequences, which are necessary for an additive category to be an abelian category (Definition 3.1). For an additive category to be triangulated it needs first an autoequivalence, which is called a suspention functor then a set of sequences of morphism between some objects which is called exact triangles, obeying some axioms, as we explain in this section.

The work of Paul Balmer ("Triangulated Categories with Several Triangulations", [4]) explain what we mean in the begining of this section. If an aditive category is abelian, the aditive structure already detemines the short exact sequences, so the abelian structure. That is not the same, a priori for a triangulate structure.

In the following definition, the additive functor T is an automorphism, that is, it is invertible, thus there exists a functor T^{-1} on the additive category $\mathcal{T}$ such that $T \circ T^{-1}$ and $T^{-1} \circ T$ are the identity functors. Sometimes we use the following notation $T^n X = X[n], T^n(f) = f[n], \quad \forall \quad n \in \mathbb{Z}$.

A **triangle** in $\mathcal{T}$ is a sequence of objects and morphisms in $\mathcal{T}$ of the form

$$X \to Y \to Z \to TX.$$

A **morphism of triangles** is a triple (f, g, h) of morphisms such that the following diagram is commutative in $\mathcal{T}$:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX'
\end{tikzcd}$$

If the morphisms f, g and h are isomorphisms in $\mathcal{T}$, then the morphism of triangles is called **isomorphism of triangles**.

DEFINITION 2.1. Let $\mathcal{T}$ be an additive category. The structure of a **pre-triangu lated category** on $\mathcal{T}$ is given by an additive automorphism

$$T : \mathcal{T} \to \mathcal{T}$$

called **shift** functor and a set of triangles called **exact triangles**

$$A \to B \to C \to TA$$

subject to the axioms $TR1, TR2, TR3$.

(TR1) Any triangle isomorphic to an exact triangle is again an exact triangle. For every object X in $\mathcal{T}$, the triangle

$$X \xrightarrow{Id_X} X \longrightarrow 0 \longrightarrow TX$$

is an exact triangle. Any morphism $u : X \to Y \in \mathcal{T}$ can be completed to an exact triangle

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow TX.$$

(TR2) A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

is an exact triangle if and only if the triangle

$$Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$$

is an exact triangle.

(TR3) Given exact triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ and $A \xrightarrow{u'} B \xrightarrow{v'} C \xrightarrow{w'} TA$, then each commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & TX \arrow[d, "Tf"] \\ A \arrow[r, "u'"] & B \arrow[r, "v'"] & C \arrow[r, "w'"] & TA \end{tikzcd}$$

can be completed to a morphism of triangles (but not necessarily uniquely).

Lemma 2.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ be an exact triangle. Then the composition of any two consecutive morphisms in the triangle is equal to zero.*

Proof : Consider the commutative diagram, which can be completed.

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed] & TX \arrow[d, "T1"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & TX \end{tikzcd}$$

So $gf = 0$. □

Proposition 2.2. *Let $A \to B \to C \to TA$ be an exact triangle in a triangulated category $\mathcal{T}$. Then for any object $A_0 \in \mathcal{T}$, there is the following induced exact sequences:*

$$Hom(A_0, A) \to Hom(A_0, B) \to Hom(A_0, C)$$
$$Hom(C, A_0) \to Hom(B, A_0) \to Hom(A, A_0).$$

Proof:

$$\begin{tikzcd} A_0 \arrow[r, "Id"] \arrow[d, dashed] & A_0 \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

This allows us to lift f to $A_0 \to A$. This implies that the first sequence is exact. That the second sequence is exact is similar. Due to $TR2$,

$$Hom(A_0, B) \to Hom(A_0, C) \to Hom(A_0, TA)$$

is exact. □

With this proposition one obtains in fact a long exact sequence

$$\cdots \to Hom(A_0, B) \to Hom(A_0, C) \to Hom(A_0, TA) \to Hom(A_0, TB) \to Hom(A_0, TC) \to \cdots$$

is exact.

We give now a series of statements whose proof we left to the reader.

PROPOSITION 2.3. *Let*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\ A & \xrightarrow{u'} & B & \xrightarrow{v'} & C & \xrightarrow{w'} & TA \end{array}$$

be a morphism of two exact triangles. If two among the morphisms f, g and h are isomorphisms, then the third one is also an isomorphism.

Therefore, the third vertex in an exact triangle is determined up to isomorphism.

COROLLARY 2.4. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ be an exact triangle in $\mathcal{T}$. If two of its vertices are isomorphic to 0, then the third one is isomorphic to 0.*

COROLLARY 2.5. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ be an exact triangle. Then the following statements are equivalent:*
(i) f is an isomorphism;
(ii) $Z \simeq 0$.

PROPOSITION 2.6. *The following are equivalent for an exact triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$$

(a) f is a split monomorphism.
(b) g is a split epimorphism.
(c) $h = 0$.

PROPOSITION 2.7. *Let $X \xrightarrow{\iota_X} X \oplus Y$ be the natural inclusion and let $p : X \oplus Y \xrightarrow{p} Y$ be the natural projection. Then*

$$X \xrightarrow{\iota} X \oplus Y \xrightarrow{p} Y \xrightarrow{0} TX$$

is an exact triangle.

PROPOSITION 2.8. *Let $X \xrightarrow{u} Z \xrightarrow{v} Y \xrightarrow{0} TX$ be an exact triangle. Then there exists a morphism*
$\Phi : X \oplus Y \to Z$ such that the following diagram is an isomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{\iota} & X \oplus Y & \xrightarrow{p} & Y & \longrightarrow & TX \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle T1} \\ X & \xrightarrow{u} & Z & \xrightarrow{v} & Y & \xrightarrow{0} & TX \end{array}$$

DEFINITION 2.2. Given a morphism u in a pre-triangulated category $\mathcal{T}$, we have then a triangle $A \xrightarrow{u} B \to C \to TA$. We call C the **cone of** u.

EXERCISE: 1. *If $A \xrightarrow{u} B \to C \to TA$ is an exact triangle, show that $A \to B$ is an isomorphism if and only if $C \simeq 0$.*

3. Abelian Category

In this section we present a canonical way to construct a pre-triangulated category out of an abelian category.

DEFINITION 3.1. A category $\mathcal{A}$ is an **abelian category** if it is an additive category, have kernel and cokernel, every monomorphism α is the kernel of some morphism in $\mathcal{A}$, every epimorphism is the cokernel of some morphism in $\mathcal{A}$ and every morphism $A \xrightarrow{\alpha} B$ can be written as $A \xrightarrow{u} I \xrightarrow{v} B$ such that u is an epimorphism and v is a monomorphism.

As we said before, the fact that an additive category to be abelian does not depend on any aditional structure. This is already done with the additive structure.

Let R be a ring. The category $Mod\ R$ of all R-modules is an abelian category. However, $mod\ R$, the subcategory of $Mod\ R$ of finitely generated modules is, in general, not an abelian category. We have that $mod\ R$ is an abelian category if and only if R is a noetherian ring. So, a consequence of this is that the category of finite dimensional vector spaces over a field is abelian and the category of finitely generated abelian groups is abelian. More examples of interesting, of non abelian category can be found in the book "Fourier-Mukai Transforms in Algebraic Geometry - D. Huybrechts" ([14]).

PROPOSITION 3.1. *Let $\mathcal{A}$ be an abelian category. Then the category of complexes $\mathcal{C}(\mathcal{A})$ is an abelian category.*

We have seen that the category of complexes is an additive category. We leave to the reader the following exercise.

EXERCISE: 2. *Prove the Proposition above.*

An abelian category $\mathcal{A}$ is semisimple if any short exact sequence in $\mathcal{A}$ splits.

THEOREM 3.2. *Let $\mathcal{A}$ be a pre-triangulated category which is an abelian category. Then every monomorphism and epimorphism in $\mathcal{A}$ splits.*

Proof: Let $f : X \to Y$ be a monomorphism in $\mathcal{A}$. Then, by $(TR1)$, there exists an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{v} TX$$

and by $(TR2)$ there exists an exact triangle

$$Y \to Z \xrightarrow{v} TX \xrightarrow{-Tf} TY.$$

Therefore $-Tf \circ v = 0$ and Tf is a monomorphism. Thus $v = 0$ and so g is a split epimorphism and f is a split monomorphism. □

This Theorem shows that if an abelian category has a structure of pre-triangulated category then it is unique and the category is what is called semisimple.

4. Pre-triangulated structure on the Homotopic Category of Complexes

We shall see in this chapter that the homotopy category of complexes over an additive category is a pre-triangulated category. We would like to emphasize the fact that in general, the homotopy category is not abelian even if the original one is abelian (see "Teorema de Morita para Categoria Derivada"-M.R. Fidelis ([7]). On the other hand the homotopy category has the structure of a pre-triangulated category.

We can identify the category $\mathcal{A}$ with a full subcategory of $\mathcal{C}(\mathcal{A})$ in the following proposition.

PROPOSITION 4.1. *Let $\mathcal{A}$ be an abelian category. The functor $C : \mathcal{A} \to \mathcal{C}(\mathcal{A})$ mapping $X \in \mathcal{A}$ to the complex $X^\bullet$ with $X^0 = X$ and $X^i = 0$ for $i \neq 0$ is fully and faithful.*

Any object X of an abelian category $\mathcal{A}$, can be considered as a complex

$$\cdots \to 0 \to 0 \to X \to 0 \to 0 \cdots$$

with X in the 0-th place. This complex will be called **0-complex or stalk complex at zero**. A stalk complex is a complex where there is only one non zero entry.

EXAMPLE 4.1. Now we give an example of an automorphism in $\mathcal{C}(\mathcal{A})$, the category of complex, which will be of interess. Let $T : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ be the functor defined as follows: if X is a complex in $\mathcal{A}$, then

$$(TX^\bullet)^n = X^{n+1}$$

$$d^n_{TX} = -d^{n+1}_X,$$

for each $n \in \mathbb{Z}$, and if $f^\bullet : X^\bullet \to Y^\bullet$ is a morphism of complexes, then $Tf^\bullet : TX^\bullet \to TY^\bullet$ is given by $(Tf^\bullet)^n = f^{n+1}$ for all $n \in \mathbb{Z}$.

DEFINITION 4.1. We say that a **complex $X^\bullet$ is bounded below** if there exists $n_0 \in \mathbb{Z}$ such that $X^n = 0$ for $n < n_0$. Similarly we can define a **complex bounded above**. The complex $X^\bullet$ is a **bounded complex** if it is bounded above and below. We denote $\mathcal{C}^-(\mathcal{A})$ the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of bounded above complexes and we denote $\mathcal{C}^+(\mathcal{A})$ the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of the bounded below complexes. We denote $\mathcal{C}^b(\mathcal{A})$ the full subcategory of $\mathcal{C}(\mathcal{A})$ of the bounded complexes.

The subcategories $\mathcal{C}^{-,+,b}(\mathcal{A})$ of $\mathcal{C}(\mathcal{A})$ are invariant for action of the translation functor T and are additive if $\mathcal{A}$ is additive.

DEFINITION 4.2. Let $\mathcal{A}$ be an additive category. Two morphisms $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ in the category $\mathcal{C}(\mathcal{A})$ of complexes are called **homotopic**, denoted by $f^\bullet \sim g^\bullet$, if there exists a family $(s^n)_{n\in\mathbb{Z}}$ of morphisms $s^n : X^n \to Y^{n-1}$ in $\mathcal{A}$ satisfying

$$f^n - g^n = d^{n-1}_Y s^n + s^{n+1} d^n_X.$$

In particular, setting g to be zero morphism, we can speak of morphisms being **homotopic to zero**.

EXERCISE: 3. *Verify that $\sim$ is an equivalence relation. If $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ are morphism of complexes and $f^\bullet \sim g^\bullet$ and $\alpha^\bullet : W^\bullet \to X^\bullet$ is an arbitrary morphism of complexes, then also the composition $f^\bullet\alpha^\bullet \sim g^\bullet\alpha^\bullet$ are homotopic. Similary if $\beta^\bullet : Y^\bullet \to W^\bullet$ then $\beta^\bullet f^\bullet \sim \beta^\bullet g^\bullet$ are homotopic.*

In particular if a morphism of complex is homotopic to zero then any composition on the left or in the right is also. Moreover the sum of two morphism homotopic to zero is homotopic to zero. That is what it usually abreviate by saying the morphisms which are homotopic to zero form an ideal in the category of complexes.

From these properties in the exercise, we have a well-defined composition of equivalence classes of morphisms modulo homotopy by defining the composition of representatives.

Definition 4.3. Let $\mathcal{A}$ be an additive category. The **homotopy category** $\mathcal{K}(\mathcal{A})$ has the same objects as the category $\mathcal{C}(\mathcal{A})$ of complexes over $\mathcal{A}$. The morphisms in the homotopy category are the equivalence classes of morphisms in $\mathcal{C}(\mathcal{A})$ modulo homotopy.

Proposition 4.2. *Let $\mathcal{A}$ be an additive category. Then the homotopy category $\mathcal{K}(\mathcal{A})$ is again an additive category.*

In the last section we defined pre-triangulated category and we shall see that the homotopy category $\mathcal{K}(\mathcal{A})$ of complexes over an additive category $\mathcal{A}$ is a pre-triangulated category. This result will then give a more structural explanation of the following observation that $\mathcal{K}(\mathcal{A})$ is not abelian. We have proved that if an abelian category is triangulated, then every exact sequence splits. The category of complex $\mathcal{C}(\mathcal{A})$ endowed with the shift functor (translation functor) does not define a triangulated category, since it is not in general semisimple.

Exercise: 4. *Shows that, if $\mathcal{A}$ is a semisimple category then $\mathcal{C}(\mathcal{A})$ is semisimple and therefore triangulated.*

Exercise: 5. *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism of complex. Prove that the following statements are equivalent:*
(a) $f^\bullet$ is homotopic to zero;
(b) $T(f^\bullet)$ is homotopic to zero.

Using this exercise we conclude that the translation functor T induces an isomorphism of

$$Hom_{\mathcal{K}(\mathcal{A})}(X^\bullet, Y^\bullet) \simeq Hom_{\mathcal{K}(\mathcal{A})}(TX^\bullet, TY^\bullet).$$

It follows that T induces an automorphism of the additive category $\mathcal{K}(\mathcal{A})$ and as before we define the full subcategories $\mathcal{K}^+(\mathcal{A}), \mathcal{K}^-(\mathcal{A}), \mathcal{K}^b(\mathcal{A})$ of the bounded below complexes , bounded above complexes and bounded complexes.

Let

$$H : \mathcal{C}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$$

be the natural functor which is the identity on objects and maps morphisms of complexes into their homotopy classes. This functor is an additive functor which commutes with the translation functor. We also have the additive functor $K = H \circ C : \mathcal{A} \to \mathcal{K}(\mathcal{A})$ where C is the functor that maps each objects $X \in \mathcal{A}$ in the 0-complex induced by X (Proposition (4.1)).

Proposition 4.3. *The functor $K : \mathcal{A} \to \mathcal{K}(\mathcal{A})$ is fully faithful.*

Now we talk about the cohomology. Assume that $\mathcal{A}$ is an abelian category. For $p \in \mathbb{Z}$ and any complex $X^\bullet \in \mathcal{C}(\mathcal{A})$, we define

$$H^p(X^\bullet) = Kerd_X^p / Imd_X^{p-1}$$

in $\mathcal{A}$. If $f : X^\bullet \to Y^\bullet$ is a morphism of complexes,

$$f^p(Kerd_X^p) \subset Kerd_Y^p$$

and

$$f^p(Imd_X^{p-1}) \subset Imd_Y^{p-1},$$

and f induces a morphism

$$H^p(f) : H^p(X^\bullet) \to H^p(Y^\bullet).$$

Therefore, $H^p : \mathcal{C}(\mathcal{A}) \to \mathcal{A}$ is a functor. These functors are additive and are called **cohomological functors**.

EXERCISE: 6. *Given $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ two homotopic morphisms of complexes, prove that $H^0(f) = H^0(g)$. Conclude that $H^p(f) = H^p(g)$ for all $p \in \mathbb{Z}$.*

These functors have the following property:

$$H^p(T(X^\bullet)) = H^{p+1}(X^\bullet)$$

and

$$H^p(T(f)) = H^{p+1}(f).$$

Therefore,

$$H^p = H^0 \circ T^p$$

for any $p \in \mathbb{Z}$.

We have seen that given a morphism $f : X \to Y$ in a pre-triangulated category $\mathcal{A}$, then the third vertex of the triangle is well determined. Now we give the construction of an object in the category of complex that helps to determine this "third object" in the pre-tringulated category $\mathcal{K}(\mathcal{A})$.

Let $\mathcal{A}$ be an additive category and $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism of complex in $\mathcal{C}^\bullet(\mathcal{A})$. We define a graded object $C_f^\bullet$ by $C_f^n = X^{n+1} \oplus Y^n$ for all $n \in \mathbb{Z}$. We also define $d_{C_f}^n : C_f^n \to C_f^{n+1}$ by

$$d_{C_f^\bullet}^n = \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix}$$

EXERCISE: 7. *Prove that $d_{C_f^\bullet}^{n+1} \circ d_{C_f^\bullet}^n = 0$ and then conclude that $(C_f^\bullet, d_{C_f})$ is a complex in $\mathcal{C}^\bullet(\mathcal{A})$.*

We call the complex $(C_f^\bullet, d_{C_f})$ the **mapping cone of f**. Note, however, that $\mathcal{C}^\bullet(\mathcal{A})$ endowed with the shift functor T does not define a pre-triangulated category. For example, the cone of the morphism $X^\bullet \xrightarrow{1_X} X^\bullet$ is not zero, however is homotopic to zero.

Now we define some important morphisms to construct the exact triangle in the homotopic category. Let

$$\iota_f : Y^\bullet \to C_f^\bullet$$

be the morphism of complex given by $\iota_f^n = (0 \ 1_{Y^n})^t$ for all $n \in Z$. It is easy to verify that $d_{C_f^\bullet}^n \circ \iota_f^n = \iota_f^{n+1} \circ d_Y^n$ for all $n \in \mathbb{Z}$, so ι_f is a morphism of complexes in $\mathcal{C}^\bullet(\mathcal{A})$. Now let

$$p_f : C_f^\bullet \to T(X^\bullet)$$

given by $p_f^n = (1_{X^{n+1}} \ 0)$. We have $p_f^{n+1} \circ d_{C_f^\bullet}^n = d_{T(X^\bullet)}^n \circ p_f^n$ for all $n \in \mathbb{Z}$, so p_f is a morphism of complexes in $\mathcal{C}^\bullet(\mathcal{A})$. From the construction, we always have

$$p_f \circ \iota_f = 0.$$

PROPOSITION 4.4. *Let $\mathcal{A}$ be an abelian category. Then we have an exact sequence of complexes*

$$0^\bullet \to Y^\bullet \xrightarrow{\iota_f} C_f^\bullet \xrightarrow{p_f} T(X^\bullet) \to 0^\bullet.$$

The proof is left to the reader.

Let $f^\bullet : X^\bullet \to Y^\bullet$ a morphism of complexes. The above short exact sequence splits, if an only if $f^\bullet$ is homotopic to zero. In fact, if the short exact sequence splits, then there exist $\sigma : T(X^\bullet) \to C_f^\bullet$ such that $p_f \circ \sigma = 1_{TX^\bullet}$. So by definition of p_f is possible to say that $\sigma(x) = (x, -s(x))$ for some morphism s, that is, $\sigma^n(x) = (x, -s_{n+1}(x))$. From $d^n_{C_f^\bullet} \circ \sigma^n(x) = \sigma^{n+1}(-d^{n+1}_{X^\bullet}(x)) = (-d^{n+1}_{X^\bullet}(x), f^{n+1}(x) - d^n_{Y^\bullet}(s^{n+1}(x))) = (-d^{n+1}_{X^\bullet}(x), s^{n+2}d^{n+1}_{X^\bullet}(x))$. So $f^{n+1}(x) = d^n_{Y^\bullet}s^{n+1}(x) + s^{n+2}d^{n+1}_{X^\bullet}(x)$. Now, if we suppose that $f^\bullet$ is homotopic to zero, then this last equation gives us that p_f is an split epimorphism.

Definition 4.4. Let $\mathcal{A}$ be an additive category. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $\mathcal{C}^\bullet(\mathcal{A})$. Then the diagram

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\iota_f} C_f^\bullet \xrightarrow{p_f} T(X^\bullet)$$

is called the **standard triangle** in $\mathcal{C}^\bullet(\mathcal{A})$ attached to $f^\bullet$.

Then, we can conclude that for every standard triangle in $\mathcal{K}^\bullet(\mathcal{A})$ there is a corresponding short exact sequence in $\mathcal{C}^\bullet(\mathcal{A})$

$$0^\bullet \to Y^\bullet \xrightarrow{\iota_f} C_f^\bullet \xrightarrow{p_f} T(X^\bullet) \to 0^\bullet.$$

On the other hand, it is not true that any short exact sequence in $\mathcal{C}(\mathcal{A})$ would lead to an exact triangle in the homotopy category $\mathcal{K}^\bullet(\mathcal{A})$. But it is crucial to understand that a short exact sequence in $\mathcal{C}^\bullet(\mathcal{A})$ induces an exact triangle in the localization of the $\mathcal{K}^\bullet(\mathcal{A})$. This explanation will be done in the next section.

We present now one important proposition about morphisms between standard triangles.

Proposition 4.5. *Let*

$$\begin{tikzcd} X^\bullet \arrow[r, "f"] \arrow[d, "u"] & Y^\bullet \arrow[d, "v"] \\ X_1^\bullet \arrow[r, "g"] & Y_1^\bullet \end{tikzcd}$$

be a diagram in $\mathcal{C}^\bullet(\mathcal{A})$ which commutes up to homotopy ($v \circ f \sim g \circ u$). Then there exists a morphism $w : C_f^\bullet \to C_g^\bullet$ such that the diagram

$$\begin{tikzcd} X^\bullet \arrow[r, "f"] \arrow[d, "u"] & Y^\bullet \arrow[r, "\iota_f"] \arrow[d, "v"] & C_f^\bullet \arrow[r, "p_f"] \arrow[d, "w"] & T(X^\bullet) \arrow[d, "T(u)"] \\ X_1^\bullet \arrow[r, "g"] & Y_1^\bullet \arrow[r, "\iota_g"] & C^\bullet{}_g \arrow[r, "p_g"] & T(X_1^\bullet) \end{tikzcd}$$

commutes up to homotopy. If the first diagram commutes in $\mathcal{C}^\bullet(\mathcal{A})$, then the second diagram commutes in $\mathcal{C}^\bullet(\mathcal{A})$.

Proof : By hypothesis $v \circ f \sim g \circ u$, ie,

$$g \circ u - v \circ f = d_{Y_1^\bullet} \circ h + h \circ d_{X^\bullet}$$

for some h.

If we define

$$w^n = \begin{bmatrix} u^{n+1} & 0 \\ -h^{n+1} & v^n \end{bmatrix}$$

for all $n \in \mathbb{Z}$, then it is easy to see that each square in the following diagram is commutative:

$$\begin{tikzcd}
 & & {} & & {} & & {} & \\
 & X^{n+1} \arrow[rr, "f^{n+1}"] \arrow[ur, dashed] \arrow[dd, dashed, "u^{n+1}"'] & & Y^{n+1} \arrow[rr, "\iota_f^{n+1}"] \arrow[ur, dashed] \arrow[dd, dashed, "v^{n+1}"] & & X^{n+2} \oplus Y^{n+1} \arrow[rr, "(1\ 0)^t"] \arrow[ur, dashed] \arrow[dd, "w^{n+1}"] & & X^{n+2} \arrow[dd] \\
X^n \arrow[ur, "d^n_{X^\bullet}"] \arrow[rr, "f^n"'] \arrow[dd, "u^n"'] & & Y^n \arrow[ur, "d^n_{Y^\bullet}"] \arrow[rr, "\iota_f^n"'] \arrow[dd, "v^n"] & & X^{n+1} \oplus Y^n \arrow[ur, "d^n_{C_f^\bullet}"] \arrow[rr, "(1\ 0)^t"'] \arrow[dd, "w^n"] & & X^{n+1} \arrow[ur] \arrow[dd] & \\
 & X_1^{n+1} \arrow[rr, "g^{n+1}"] & & Y_1^{n+1} \arrow[rr, "\iota_g^{n+1}"] & & X_1^{n+2} \oplus Y_1^{n+1} \arrow[rr, "(1\ 0)^t"] & & X_1^{n+2} \\
X_1^n \arrow[ur, "d^n_{X_1^\bullet}"] \arrow[rr, "g^n"'] & & Y_1^n \arrow[ur, "d^n_{Y_1^\bullet}"] \arrow[rr, "\iota_g^n"] & & X_1^{n+1} \oplus Y_1^n \arrow[ur, "d^n_{C_g^\bullet}"] \arrow[rr, "(1\ 0)^t"] & & X_1^{n+1} \arrow[ur] &
\end{tikzcd}$$

and $d^n_{C_g^\bullet} \circ w^n = w^{n+1} \circ d^n_{C_f^\bullet}$ because $v \circ f \sim g \circ u$. The other squares are commutative without the homotopy. □

Remark:

(a) Doing the composition $\iota_f \circ f^\bullet$, we have $(0\ 1)^t f^n = (0\ f^n)^t$. So this composition is not zero in $\mathcal{C}^\bullet(\mathcal{A})$. But, taking $s^n = (1\ 0)^t$ we have

$$(0\ f^n)^t = s^{n+1} \circ d^n_{X^\bullet} + d^{n-1}_{C_f^\bullet} \circ s^n$$

so $(0\ f^\bullet) \sim 0^\bullet$.

Therefore $\iota_f \circ f^\bullet = 0$ in $\mathcal{K}^\bullet(\mathcal{A})$.

(b) Let $X^\bullet$ be a complex in $\mathcal{C}^\bullet(\mathcal{A})$ and $X^\bullet \xrightarrow{1_{X^\bullet}} X^\bullet$ the identity morphism. The mapping cone of the identity morphism in degree n is $C^n_{1_{X^\bullet}} = X^{n+1} \oplus X^n$. The differential is

$$\begin{bmatrix} -d^{n+1}_{X^\bullet} & 0 \\ 1_{X_{n+1}} & d^n_{X^\bullet} \end{bmatrix}$$

and it is possible to prove that the identity morphism $1_{C^\bullet_{1_{X^\bullet}}} : C^\bullet_{1_{X^\bullet}} \to C^\bullet_{1_{X^\bullet}}$ is homotopic to zero via the map $s = (s_n)_{n\in\mathbb{Z}}$ where

$$s_n = \begin{bmatrix} 0 & 1_{X^n} \\ 0 & 0 \end{bmatrix}$$

Thus, in the homotopy category $\mathcal{K}(\mathcal{A})$, the identity morphism $1_{C^\bullet_{1_{X^\bullet}}} : C^\bullet_{1_{X^\bullet}} \to C^\bullet_{1_{X^\bullet}}$ is equal to the zero map. So, in the homotopy category the mapping cone $C^\bullet_{1_{X^\bullet}}$ is isomorphic to the zero complex.

Now we have the necessary elements to define the exact triangles in the homotopic category of complexes. So, let $\mathcal{A}$ be an additive category and $\mathcal{K}^\bullet(\mathcal{A})$ the corresponding homotopic category of complexes with the translation functor T. We say that $X \xrightarrow{f}$

$Y \xrightarrow{g} Z \xrightarrow{h} TX$ is **an exact triangle in** $\mathcal{K}^\bullet(\mathcal{A})$ if it is isomorphic to the image of a standard triangle in $\mathcal{K}^\bullet(\mathcal{A})$, ie, it is isomorphic to a triangle

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\iota_f} C_f^\bullet \xrightarrow{p_f} T(X^\bullet).$$

THEOREM 4.6. *Let $\mathcal{A}$ be an additive category and $\mathcal{K}^\bullet(\mathcal{A})$ the homotopic category of complexes equipped with the translation functor T and the classes of exact triangles. Then $\mathcal{K}^\bullet(\mathcal{A})$ is a pre-triangulated category.*

Proof : We give now a sketch of the proof of the validity of $TR1$. The proof of $TR2$ and $TR3$ can be found in the book of D.Milicic-Lectures on Derived Categories [18].

We have that any triangle isomorphic to an exact triangle is an exact triangle and for any $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathcal{K}^\bullet(\mathcal{A})$, there exists an exact triangle $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \to Z^\bullet \to TX^\bullet$.

We have shown that the cone $C^\bullet_{1_{X^\bullet}}$ of the identity morphism on $X^\bullet$ is isomorphic to 0 in $\mathcal{K}^\bullet(\mathcal{A})$. Therefore, the diagram

$$\begin{tikzcd}
X^\bullet \arrow[r, "1_{X^\bullet}"] \arrow[d, "1_{X^\bullet}"] & X^\bullet \arrow[r] \arrow[d, "1_{X^\bullet}"] & 0 \arrow[r] \arrow[d, "0"] & TX^\bullet \arrow[d, "1_{TX^\bullet}"] \\
X^\bullet \arrow[r, "1_{X^\bullet}"] & X^\bullet \arrow[r, "\iota_{1_{X^\bullet}}"] & C^\bullet_{1_{X^\bullet}} \arrow[r, "p_{1_{X^\bullet}}"] & TX^\bullet
\end{tikzcd}$$

is commutative in $\mathcal{K}^\bullet(\mathcal{A})$ and the vertical morphisms are isomorphisms. Since the bottom row is the image of a standard triangle, then the top row is an exact triangle. So the axiom $TR1$ is satisfied. □

5. Octahedral Axiom

A triangulated category is a pre-triangulated category whose pre-triangulated structure obeys one axiom more $TR4$, called the octahedral axiom. This axiom is called the "octahedral axiom" because its representation looks like the skeleton of an octahedron. The definition of a triangulated category was given by Jean-Louis Verdier (1963) in his posthumous published Ph.D. thesis, which was based on the ideas of Grothendieck.

(TR4) Given exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1\alpha_1$, there exists an exact triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, "1"] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, dashed, "\delta_1"] & TX \arrow[d, "T1"] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\gamma_3"] \arrow[d, dashed, "\delta_2"] & TX \arrow[d, "T\alpha_1"] \\
 & W \arrow[r, "1"] \arrow[d, "\beta_3"] & W \arrow[r, "\beta_3"] \arrow[d, dashed, "\delta_3"] & TY \\
 & TY \arrow[r, "T\alpha_2"] & TU &
\end{tikzcd}$$

DEFINITION 5.1. We call **triangulated category** a pre-triangulated category which satisfies the axiom $TR4$.

The first contact with this axiom causes an odd feeling. This feeling is a necessary ingredient to profit from the explanation we try to give in the next pages.

In general the $TR4$ axiom is presented in the fashion showed above. But we would like to give an equivalent form of presentation that was suggested by V. Dlab. He suggested that the octahedral axiom should be equivalent to a simpler "push-out" property. The first formulation and also the proof of this observation was given by Parshall and Scott in "Derived Categories, Quasi-Hereditary Algebras and Algebraic Groups" ([5]) and reformulated recently by Henning Krause in "Derived categories, resolutions, and Brown representability" ([17]). This equivalent form has some similarities to other things that happen in abelian categories. Then, for the readers who are familiar with abelian categories, this new presentation will clarify the axiom $TR4$ and the strangeness will be replaced by beauty.

We would like to remind you of some properties of an abelian category in order to present this equivalent definition. Those properties give us the existence of an exact sequence and specific pull-back and push-out.

A commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f'"] & B \arrow[d, "g"] \\
B' \arrow[r, "g'"] & C
\end{tikzcd}$$

is a pull-back and a push-out if and only if we can dispose the objects and morphisms of the square in an exact sequence:

$$0 \to A \xrightarrow{(f\ \ -f')^t} B \oplus B' \xrightarrow{(g\ \ g')} C \to 0.$$

One of the conditions for this sequence to be exact is that $(g\ g')(f\ \ -f')^t = 0$, and this happens if and only if the above square is commutative. Now, to prove that

$$A \xrightarrow{(f\ -f')^t} B \oplus B'$$

is the kernel of $(g\ g')$, lets suppose that the square is a pull-back. Let $u : X \to B \oplus B'$ be a morphism such that $(g\ g')u = 0$. Then, rewriting $u = (u_1\ u_2)^t$, from the equality

$$(g\ g')(u_1\ u_2)^t = 0$$

we can conclude that $gu_1 = -g'u_2$. From the pull-back diagram

$$\begin{tikzcd}
Z \arrow[dr, "\eta"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "0"'] & & \\
& A \arrow[r, "f"] \arrow[d, "f'"] & B \arrow[d, "g"] \\
& C \arrow[r, "g'"] & C
\end{tikzcd}$$

there exists a unique $\theta : X \to A$ such that $f\theta = u_1$ and $f'\theta = -u_2$, that is,

$$(f \;\; -f')^t\theta = (u_1 \; u_2)^t,$$

showing us that u factors through $(f \;\; -f')^t$. This finishes the proof that $(f \;\; -f')^t$ is the kernel of $(g \; g')$.

Another step which is necessary to prove the exactness of the sequence above is to show that $(f \;\; -f')^t$ is a monomorphism. For this, consider a morphism $\eta : Z \to A$ such that $(f \;\; -f')^t\eta = 0$. Then, $f\eta = 0$ and $f'\eta = 0$. These two equalities give us to the following picture:

$$\begin{tikzcd}
Z \arrow[dr, "\eta"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "0"'] & & \\
& A \arrow[r, "f"] \arrow[d, "f'"] & B \arrow[d, "g"] \\
& B' \arrow[r, "g'"] & C
\end{tikzcd}$$

which will be explained now. From the equality $g0 = g'0$ and the uniqueness property of the pull-back, there exists a unique $h : Z \to A$ such that $fh = 0$ and $f'h = 0$. Besides, η and the zero morphism satisfy the property of h. Therefore, by uniqueness, $\eta = 0$ and so $(f \;\; -f')^t$ is a monomorphism.

Then we have that the following sequence of morphisms

$$0 \to A \overset{(f \;\; -f')^t}{\longrightarrow} B \oplus B' \overset{(g \;\; g')}{\longrightarrow} C$$

is an exact sequence.

It remain us to prove that $(g \; g')$ is an epimorphism and for this purpose we will use the push-out's property of the square. Let $\eta : C \to X$ be a morphism such that $\eta(g \; g') = 0$. Then $\eta g = 0 = \eta g'$, which leads us to the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f'"] & B \arrow[d, "g"] \arrow[ddr, bend left, "0"] & \\
B' \arrow[r, "g'"] \arrow[drr, bend right, "0"'] & C \arrow[dr, "\eta"] & \\
& & X
\end{tikzcd}$$

By uniqueness we have $\eta = 0$.

Conversely, suppose that

$$0 \to A \overset{(f \;\; -f')^t}{\longrightarrow} B \oplus B' \overset{(g \;\; g')}{\longrightarrow} C \to 0$$

is an exact sequence. Let $u_1 : X \to B$ and $u_2 : X \to B'$ be morphisms such that $gu_1 = g'u_2$. Then $(g \; g')(u_1 \;\; -u_2)^t = 0$. By the property of the kernel, there exists a unique $\theta : X \to A$ such that $(f \ \ -f')^t\theta = (u_1 \ \ -u_2)^t$. Then, $f\theta = u_1, f'\theta = u_2$. So the square is a pull-back diagram and in a similar way, using the property of the cokernel, we can obtain that the square is a push-out, which finishes the proof.

We know that in a triangulated category we do not have at our disposal an exact sequence, because in this kind of categories, every exact sequence is a split sequence. On the other hand, the triangles constitute the structure of this category.

We will now present a type of square which is not necessarily a pull-back and a push-out square as the one we considered before, but which has another similar property in the context of pre-triangulated category. The square

$$\begin{tikzcd} X \arrow[r, "\alpha'"] \arrow[d, "\alpha''"] & Y' \arrow[d, "\beta'"] \\ Y'' \arrow[r, "\beta''"] & Z \end{tikzcd}$$

is called **homotopy cartesian** if there exists an exact triangle

$$X \overset{(\alpha' \ \alpha'')^t}{\longrightarrow} Y' \oplus Y'' \overset{(\beta' \ -\beta'')}{\longrightarrow} Z \overset{\gamma}{\longrightarrow} TX.$$

The map γ is called a differential of the homotopy cartesian square. In this definition we can see the similarity with the problem presented before. We will introduce now an axiom called $TR4'$ and we will prove that $TR4$ and $TR4'$ are equivalent. We call $TR4'$ the following axiom:

(TR4') Every pair of morphisms $X \to Y$ and $X \to X'$ can be completed to a morphism between exact triangles:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "1"] & TX \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z \arrow[r] & TX' \end{tikzcd}$$

such that the left hand square is a homotopy cartesian square whose differential is the compositon $Y' \to Z \to TX$.

We should prove now that $TR4$ is equivalent to $TR4'$. But first we introduce another formulation of $TR4'$.

(TR4") The diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & TX \\ X' \arrow[r] & Y' & & \end{tikzcd}$$

consisting of a homotopy cartesian square with differential $\delta : Y' \to TX$ and an exact triangle can be completed to a morphism of exact triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "1"] & TX \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z \arrow[r] & TX' \end{tikzcd}$$

such that the composition $Y' \to Z \to TX$ is equal to δ.

THEOREM 5.1. *Let $\mathcal{T}$ be a pre-triangulated category. Then the axioms $TR4$, $TR4'$ and $TR4''$ are equivalent.*

Proof: First we would like to show that the axiom $TR4'$ implies the axiom $TR4''$. Then, we have to suppose that there is a diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "\phi"] & V \arrow[r, "\beta"] \arrow[d, "\Psi"] & W \arrow[r, "\gamma"] & TU \\ X \arrow[r, "\kappa"] & Y & & \end{tikzcd} \tag{1}$$

consisting of a homotopy cartesian square with differential $\delta : Y \to TU$ and an exact triangle. We have to construct the second triangle which is necessary for the diagram $TR4''$.

We are going to use the axiom $TR4'$ which gives us a triangle. Then, by applying the axiom $TR4'$ on α and ϕ, we obtain a morphism between exact triangles such that

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "\phi"] & V \arrow[r, "\beta'"] \arrow[d, "\Psi'"] & W' \arrow[r, "\gamma'"] \arrow[d, "1"] & TU \arrow[d, "T\phi"] \\ X \arrow[r, "\kappa'"] & Y' \arrow[r, "\lambda'"] & W' \arrow[r, "\mu'"] & TX \end{tikzcd} \tag{2}$$

the left hand square is homotopy cartesian with differential $\gamma'\lambda'$. However, this diagram is not yet the diagram $TR4''$ that we would like to have. We can use the following strategies to obtain the necessary diagram.

Putting together the triangle (1) and the diagram (2) we obtain the diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "1"] & V \arrow[r, "\beta"] \arrow[d, "1"] & W \arrow[r, "\gamma"] \arrow[d, "\sigma"] & TU \arrow[d, "1"] \\ U \arrow[r, "\alpha"] \arrow[d, "\phi"] & V \arrow[r, "\beta'"] \arrow[d, "\Psi'"] & W' \arrow[r, "\gamma'"] \arrow[d, "1"] & TU \arrow[d, "T\phi"] \\ X \arrow[r, "\kappa'"] & Y' \arrow[r, "\lambda'"] & W' \arrow[r, "\mu'"] & TX \end{tikzcd} \tag{3}$$

The morphism σ is an isomorphism and from the commutative square we have $\sigma\beta = \lambda'\Psi'$. Thus $\beta = \sigma^{-1}\lambda'\Psi'$. We also have $T\phi\gamma = \mu'\sigma$. These commutative squares can help to construct the following morphism between exact triangles

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "\phi"] & V \arrow[r, "\beta"] \arrow[d, "\Psi'"] & W \arrow[r, "\gamma"] \arrow[d, "1"] & TU \arrow[d, "T\phi"] \\ X \arrow[r, "\kappa'"] & Y' \arrow[r, "\sigma^{-1}\lambda'"] & W \arrow[r, "\mu'\sigma"] & TX \end{tikzcd} \tag{4}$$

In order to have the $TR4''$ diagram, we still have to change the object Y' in (4), and introduce the necessary morphism to have the correct differential.

In the first paragraphs we had organized some homotopic squares and we also had some triangles with the respective differential. It remains us to find the most suitable triangle to finish the proof of $TR4''$.

The diagram (1) gives us the triangle

$$U \xrightarrow{(\alpha\ \ \phi)^t} V \oplus X \xrightarrow{(\Psi\ \ -\kappa)} Y \xrightarrow{\delta} TU \tag{5}$$

and the diagram (2) gives us the triangle

(6)
$$U \xrightarrow{(\alpha\ \phi)^t} V \oplus X \xrightarrow{(\Psi'\ -\kappa')} Y' \xrightarrow{\gamma'\lambda'} TU.$$

We can put the last two diagrams together to obtain the following morphism of triangles

(7)
$$\begin{tikzcd} U \arrow[r, "(\alpha\ \phi)^t"] \arrow[d, "1"] & V \oplus X \arrow[r, "(\Psi\ -\kappa)"] \arrow[d, "1"] & Y \arrow[r, "\delta"] \arrow[d, "\tau"] & TU \arrow[d, "1"] \\ U \arrow[r, "(\alpha\ \phi)^t"] & V \oplus X \arrow[r, "(\Psi'\ -\kappa')"] & Y' \arrow[r, "\gamma'\lambda'"] & TU \end{tikzcd}$$

where τ is an isomorphism and $\tau\Psi = \Psi'$. From (1) we have the following:

(8)
$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "\phi"] & V \arrow[r, "\beta"] \arrow[d, "\Psi"] & W \arrow[r, "\gamma"] \arrow[d, "1"] & TU \arrow[d, "T\phi"] \\ X \arrow[r, "\kappa"] & Y \arrow[r, "\sigma^{-1}\lambda'\tau"] & W \arrow[r, "\mu'\sigma"] & TX \end{tikzcd}$$

Each square is commutative due to the following:

1) The first square is commutative because of (1).
2) The second one is commutative because of $\sigma^{-1}\lambda'\tau\Psi = \sigma^{-1}\lambda'\Psi'$ due to (7) and $\sigma^{-1}\lambda'\Psi' = \beta$ due to (4).
3) The third one is commutative because of $\mu'\sigma = T\phi\gamma$ due to (4).

The last necessary observation is about the differential. We have to verify if the composition $\gamma\sigma^{-1}\lambda'\tau$ is the differential δ of the first homotopic square. From (3)

$$\gamma\sigma^{-1}\lambda'\tau = \gamma'\lambda'\tau$$

and from (7) we have

$$\gamma'\lambda'\tau = \delta.$$

So, we have proved the axiom $TR4''$.

Lets prove that $TR4''$ implies $TR4$. Suppose that there are exact triangles$(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1\alpha_1$. This can be seen in the diagram below:

$$\begin{tikzcd} X \arrow[r, "\alpha_1"] \arrow[d, "1"] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] & TX \\ X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\gamma_3"] & TX \\ & W \arrow[d, "\beta_3"] & & \\ & TY & & \end{tikzcd}$$

Now considering the morphism $Y \xrightarrow{(\alpha_2\ \beta_1)^t} U \oplus Z$ and completing it to an exact triangle $Y \xrightarrow{(\alpha_2\ \beta_1)^t} U \oplus Z \xrightarrow{(\delta_1'\ -\gamma_2')^t} V' \xrightarrow{\delta} TY$, we obtain that the square

$$\begin{tikzcd} Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[d, "\delta_1'"] \\ Z \arrow[r, "\gamma_2'"] & V' \end{tikzcd}$$

is a homotopic square with differential $\delta : V' \to TY$. By applying $TR4''$ to the following situation

$$\begin{tikzcd} Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, "\delta_1'"] & TX \arrow[r, "-T\alpha_1"] & TY \\ Z \arrow[r, "\gamma_2'"] & V' \end{tikzcd}$$

we obtain the diagram

(9)
$$\begin{tikzcd} Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, "\delta_1'"] & TX \arrow[r, "-T\alpha_1"] \arrow[d, "1"] & TY \arrow[d, "T\beta_1"] \\ Z \arrow[r, "\gamma_2'"] & V' \arrow[r, "\gamma_3'"] & TX \arrow[r, "u"] & TZ \end{tikzcd}$$

where $(-T\alpha_1)\gamma_3' = \delta$. Note that from the square

$$\begin{tikzcd} X \arrow[r, "\alpha_1"] \arrow[d, "1"] & Y \arrow[d, "\beta_1"] \\ X \arrow[r, "\gamma_1"] & Z \end{tikzcd}$$

we can say that $u = -T\gamma_1$. We also have the following triangle

$$Y \xrightarrow{\beta_1} Z \xrightarrow{\beta_2} W \xrightarrow{\beta_3} TY$$

and the following homotopic square

$$\begin{tikzcd} Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[d, "\delta_1'"] \\ Z \arrow[r, "\gamma_2'"] & V' \end{tikzcd}$$

with differential $\delta : V' \to TY$. So we can apply $TR4''$ and we have

$$\begin{tikzcd} Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[d, "\delta_1'"] \\ Z \arrow[r, "\gamma_2'"] \arrow[d, "\beta_2"] & V' \\ W \arrow[d, "\beta_3"] & \\ TY & \end{tikzcd}$$

which gives us

$$
\begin{tikzcd}
Y \arrow[r, "\beta_1"] \arrow[d, "\alpha_2"] & Z \arrow[r, "\beta_2"] \arrow[d, "\gamma_2'"] & W \arrow[r, "\beta_3"] \arrow[d, "1"] & TY \arrow[d, "T\alpha_2"] \\
U \arrow[r, "\delta_1'"] & V' \arrow[r, "\delta_2'"] & W \arrow[r, "\delta_3'"] & TU
\end{tikzcd}
\tag{10}
$$

and $\beta_3\delta_2' = \delta$. Therefore we have the diagram

$$
\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, "1"] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, "\delta_1'"] & TX \arrow[r, "-T\alpha_1"] \arrow[d, "1"] & TY \arrow[d, "T\beta_1"] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2'"] \arrow[d, "\beta_2"] & V' \arrow[r, "\gamma_3'"] \arrow[d, "\delta_2'"] & TX \arrow[r, "-T\gamma_1"] & TZ \\
& W \arrow[r, "1"] \arrow[d, "\beta_3"] & W \arrow[d, "\delta_3'"] & & \\
& TY \arrow[r, "T\alpha_2"] & TU & &
\end{tikzcd}
$$

and $(-T\alpha_1)\gamma_3' = \delta = \beta_3\delta_2'$. Now we have the triangles

$$
\begin{tikzcd}
X \arrow[r, "\gamma_1"] \arrow[d, "1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "1"] & V \arrow[r, "\gamma_3"] & TX \arrow[d, "1"] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2'"] & V' \arrow[r, "\gamma_3'"] & TX
\end{tikzcd}
$$

From the diagram above, there exists an isomorphism ϕ such that the following diagram is commutative

$$
\begin{tikzcd}
X \arrow[r, "\gamma_1"] \arrow[d, "1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "1"] & V \arrow[r, "\gamma_3"] \arrow[d, dashed, "\phi"] & TX \arrow[d, "1"] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2'"] & V' \arrow[r, "\gamma_3'"] & TX
\end{tikzcd}
\tag{11}
$$

Then we can complete the diagram below

$$
\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, "1"] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] & TX \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\gamma_3"] & TX \\
& W \arrow[d, "\beta_3"] & & \\
& TY & &
\end{tikzcd}
$$

with the following choices: first defining $\delta_1 = \phi^{-1}\delta_1'$, and using that $\phi\gamma_2 = \gamma_2'$ and $\gamma_3 = \gamma_3'\phi$ (11), then $\delta_1\alpha_2 = \phi^{-1}\delta_1'\alpha_2 = \phi^{-1}\gamma_2'\beta_1 = \gamma_2\beta_1$ (9) and $\gamma_3\delta_1 = \gamma_3\phi^{-1}\delta_1' = \gamma_3'\delta_1' = \alpha_3$ (9). Completing the diagram above we have the following:

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, "1"] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, "\delta_1"] & TX \arrow[d, "T1"] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\gamma_3"] & TX \\
& W \arrow[d, "\beta_3"] & & \\
& TY & &
\end{tikzcd}$$

Defining $\delta_2'' = \delta_2'\phi$ then $\delta_2''\gamma_2 = \delta_2'\phi\gamma_2 = \delta_2'\gamma_2' = \beta_2$ (10). We know that $T\alpha_2\beta_3 = \delta_3'$ (10) and $\beta_3\delta_2'' = \beta_3\delta_2'\phi = -T\alpha_1\gamma_3'\phi = -T\alpha_1\gamma_3$. So the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, "1"] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, "\delta_1"] & TX \arrow[d, "T1"] \\
X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\gamma_3"] \arrow[d, "\delta_2''"] & TX \arrow[d, "T\alpha_1"] \\
& W \arrow[r, "1"] \arrow[d, "\beta_3"] & W \arrow[r, "\beta_3"] \arrow[d, "\delta_3'"] & TY \\
& TY \arrow[r, "T\alpha_2"] & TU &
\end{tikzcd}$$

We still have to prove that the following sequence

$$U \xrightarrow{\delta_1} V \xrightarrow{\delta_2''} W \xrightarrow{\delta_3'} TU$$

is an exact triangle. From (10) we have the exact triangle

$$U \xrightarrow{\delta_1'} V' \xrightarrow{\delta_2'} W \xrightarrow{\delta_3'} TU$$

and the commutative diagram:

(12)
$$\begin{tikzcd}
U \arrow[r, "\delta_1'"] \arrow[d, "1"] & V' \arrow[r, "\delta_2'"] \arrow[d, "\phi^{-1}"] & W \arrow[r, "\delta_3'"] \arrow[d, "1"] & TU \arrow[d, "1"] \\
U \arrow[r, "\delta_1"] & V \arrow[r, "\delta_2''"] & W \arrow[r, "\delta_3'"] & TU
\end{tikzcd}$$

So axiom $TR4$ is proven.

Let's show show now that $TR4$ implies $TR4''$. Suppose that there exists a diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\phi_1"] & Y \arrow[r, "\beta"] \arrow[d, "\phi_2"] & Z \arrow[r, "\gamma"] & TX \\
X' \arrow[r, "\alpha'"] & Y' & &
\end{tikzcd}$$

consisting of a homotopy cartesian square with differential δ. So we have the exact triangle below

$$X \xrightarrow{(\alpha\ \phi_1)^t} Y \oplus X' \xrightarrow{(\phi_2\ -\alpha')} Y' \xrightarrow{\delta} TX$$

.

By applying $TR4$ we obtain the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "(\alpha\ \phi_1)^t"] \arrow[d, "1"] & Y \oplus X' \arrow[r, "(\phi_2\ -\alpha')"] \arrow[d, "(1\ 0)"] & Y' \arrow[r, "\delta"] \arrow[d, "\beta'"] & TX \arrow[d, "T1"] \\
X \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] \arrow[d, "0"] & Z \arrow[r, "\gamma"] \arrow[d, "\gamma'"] & TX \arrow[d, "T(\alpha_1\ \phi_1)^t"] \\
 & TX' \arrow[r, "1"] \arrow[d, "(0\ 1)^t"] & TX' \arrow[r, "(0\ 1)^t"] \arrow[d, "-T\alpha'"] & TY \oplus TX' \\
 & TY \oplus TX' \arrow[r, "T(\phi_2\ -\alpha')"'] & TY' &
\end{tikzcd}$$

in which the third column $(\beta', \gamma', -T\alpha')$ is an exact triangle. This gives us the following morphism of triangles

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\phi_1"] & Y \arrow[r, "\beta"] \arrow[d, "\phi_2"] & Z \arrow[r, "\gamma"] \arrow[d, "1"] & TX \arrow[d, "T\phi_1"] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z \arrow[r, "\gamma'"] & TX'
\end{tikzcd}$$

where $\delta = \gamma\beta'$ is the differential of the homotopy cartesian square. So axiom $TR4''$ is proven.

Finally, lets show that $TR4''$ implies $TR4'$. Given morphisms $X \xrightarrow{\alpha} Y$ and $X \xrightarrow{\phi_1} X'$, and applying $TR1$ we have the exact triangle

$$X \xrightarrow{(\alpha\ \phi_1)^t} Y \oplus X' \xrightarrow{(\beta\ \mu)} Z \xrightarrow{\delta} TX.$$

By applying $TR1$ to the morphism $X \xrightarrow{\alpha} Y$ we obtain the following exact triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta'} W \xrightarrow{\gamma} TX.$$

Thus we have the following diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\phi_1"] & Y \arrow[r, "\beta'"] \arrow[d, "\beta"] & W \arrow[r, "\gamma"] & TX \\
X' \arrow[r, "-\mu"] & Z & &
\end{tikzcd}$$

consisting of a homotopy cartesian square with differential δ and an exact triangle. Applying $TR4''$, this diagram can be completed to

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\phi_1"] & Y \arrow[r, "\beta'"] \arrow[d, "\beta"] & W \arrow[r, "\gamma"] \arrow[d, "1"] & TX \arrow[d, "T\phi_1"] \\
X' \arrow[r, "-\mu"] & Z \arrow[r, "\epsilon"] & W \arrow[r, "\theta"] & TX
\end{tikzcd}$$

such that $\gamma\epsilon = \delta$. Thus axiom $TR4'$ is proven. $\square$

EXERCISE: 8. *Consider the commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\alpha_1"] \arrow[d, "1"] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, "\epsilon"] & TX \arrow[d, "1"] \\ X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] & V \arrow[r, "\gamma_3"] & TX \end{tikzcd}$$

where the rows are exact triangles. Prove that:

(a) *there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\alpha_1"] \arrow[d, "1"] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, "\omega"] & TX \arrow[d, "1"] \\ X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2'"] & V' \arrow[r, "\gamma_3'"] & TX \end{tikzcd}$$

such that the middle square is a homotopic square and there exists an isomorphism $\eta : V' \to V$ and a morphism $g : U \to Z$ such that $\gamma_2 = \eta\gamma_2', \gamma_3' = \gamma_3\eta^{-1}$ and $\epsilon = \eta\omega + \gamma_2 g$.

(b) *there exists an exact triangle*

$$Y \xrightarrow{(\alpha_2\ \ \beta_2)^t} U \oplus Z \xrightarrow{(\epsilon - \gamma_2 g - \gamma_2)} U \xrightarrow{-T(\alpha_1)\gamma_3} TX$$

in such a way that if $g = 0$, then the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[d, "\epsilon"] \\ Z \arrow[r, "\gamma_2"] & V \end{tikzcd}$$

is a homotopic square.

THEOREM 5.2. *Let $\mathcal{A}$ be an additive category and $\mathcal{K}^\bullet(\mathcal{A})$ the homotopic category of complexes equipped with the translation functor T and the classes of exact triangles. Then $\mathcal{K}^\bullet(\mathcal{A})$ is a triangulated category.*

CHAPTER 2

Localization

Before introduce the definition of localization of categories, we give a motivation defining localization in rings. This material is inspired in the work of Michael Artin which can be found online. Remember that a ring can be understood as a category with one object, so this is really a particular case of the general theory, but it easy to grasp. More details can be found in the work of M. Artin [**3**]. First of all, we would like to explain localization of non-commutative rings. We would like to use fractions in noncommutative algebra, i.e, we would like to embed a domain A in a skew field of fractions.

The idea from an Ore set, is quite similar to the one used in the construction of rational numbers from integers. However it is a little more sophisticated in this case because we do not have commutativity.

Then we define the Ore set, that will be the set of denominators of our fractions. The Ore condition that we put in this definition is a condition that allows us to multiplicate two fractions and obtain a new one. To do so, we need a relation between left fractions and right fractions. So, this axiom reflects this idea very well. Another property is that two elements of the set of denominators have common multipliers. Because of this it is possible to give a condition that relates two fractions and this condition will allow us to define an equivalence class in this set of fractions. Given these properties we will construct the ring of right fractions.

But we would like to emphasize that the idea of building the ring of fractions here is exclusively to give a quick insight to the main idea of the next section, that is, the localizing of category. We hope it will be easier to understand the idea of localizing a category, because you will have a model of reference.

The main theorem of this chapter (Theorem (1.5)) is completely translated to the language of category. Another similarity that we would like to call your attention to is when S, is both a left and a right Ore set. Then the rings of left fractions and of right fractions are isomorphic. In the next section, we define a "localizing class of morphism in a category and we set both properties to be a localizing class, i.e, S is a kind of right and left Ore set. Therefore we can establish a bijection between the equivalence classes of left fractions and right fractions between two objects of a certain category.

1. Ore Set

An Ore set gives us the main ingredients to build fractions and to multiply fractions in such a way that the product of two fractions will be another fraction.

To explain these ideas, we need some definitions. Let A be a ring and $s \in A$. We say that s is **regular** if it is neither left nor right zero divisor, i.e., if $as = 0$ implies $a = 0$ and also $sa = 0$ implies $a = 0$. If s is regular, then we can cancel: $sa = sb$ implies that $a = b$ and $as = bs$ implies $a = b$.

Another important aspect of regular elements is the following: If a regular element s has a right inverse, then it is invertible. It is easy to see this fact. Suppose that s is a regular element with a right inverse t: $st = 1$. Then, from the equation $sts = s$, we have $s(ts - 1) = 0$. So $ts = 1$.

We will work with **right fractions**

$$as^{-1}$$

where $a, s \in A$. The first question that we have to think is: Is the formal product of two right fractions

$$(bs^{-1})(at^{-1})$$

a right fraction? We can see that we need to rewrite a left fraction as a right fraction.

DEFINITION 1.1. Let A be a ring. We call a subset $S \subseteq A$ a **right Ore set** if it has the following properties:

(a) $st \in S$ for all $s, t \in S$ and $1 \in S$.
(b) The elements of S are regular.
(c) For all $a \in A$ and $s \in S$, there exist $b \in A$ and $t \in S$ such that

$$sb = at.$$

The condition (c) is called **right Ore condition**.

The following results give us an idea about the ring of fractions that we have talked about previously.

THEOREM 1.1. *Let $S \subset A$ be a right Ore set. There is a ring $A[S^{-1}]$ of right fractions and an injective ring homomorphism $A \to A[S^{-1}]$ such that the ring $A[S^{-1}]$ is determined by the following universal properties:*

(a) *the image of every element $s \in S$ is invertible in $A[S^{-1}]$, and*
(b) *every element of $A[S^{-1}]$ can be written as a product as^{-1}.*

Moreover, any homomorphism $f : A \to R$ such that the images of elements of S are invertible in R factors uniquely throught $A[S^{-1}]$.

When S is both a left and a right Ore set, then the rings of left fractions and of right fractions are isomorphic. Now we will prove some properties of Ore sets. The second property in the following lemma says that if we have fractions $1/s$ and $1/t$ then it is possible to transform them in fractions with a common denominator.

LEMMA 1.2. *Let S be a right Ore set of a ring A.*

(i) *Suppose that $s, t \in S$ and there exists $x \in A$ such that $sx = t$. Then x is regular.*
(ii) *Let $s, t \in S$, then there exists a common multiple $u \in S$, i.e., $u = sx$ and $u = ty$ for some regular elements $x, y \in A$.*
(iii) *With the notation of the Ore condition, suppose that $sb_1 = at_1$ and also $sb_2 = at_2$, with $s, t_1, t_2 \in S$. Then, there are regular elements $x_1, x_2 \in A$ such that $t_1x_1 = t_2x_2$ and $b_1x_1 = b_2x_2$.*

Proof: (i) If $xa = 0$, then $ta = 0$. This implies that $a = 0$. Now we suppose that $bx = 0$. Applying (c) to s, t in the definition of Ore set, we have u, v, with $u \in S$ such that $su = tv$. Then, we have $sxv = tv = su$. Then $xv = u$ and with $bu = 0$ and $u \in S$, follows that $b = 0$. So x is regular.

(ii) Let $s, t \in S$. By the right Ore condition there exist $x \in A$ and $y \in S$ such that

$$sx = ty.$$

Since S is closed under multiplication $u = ty \in S$.

(iii) We choose a common multiple $t_1x_1 = t_2x_2 \in S$ with x_1, x_2 regular. Then $sb_1x_1 = at_1x_1 = at_2x_2 = sb_2x_2$. Since s is regular, then $b_1x_1 = b_2x_2$. □

The part *(iii)* of this lemma is establishing a relation between two fractions.

EXERCISE: 9. *We can define the following relation: two fractions $a_1s_1^{-1}$ and $a_2s_2^{-1}$ are related if there are regular elements x_1, x_2 such that $s_1x_1 \in S$, and $s_1x_1 = s_2x_2$ and $a_1x_1 = a_2x_2$. Verify that we have an equivalence relation.*

Now we describe the equivalence classes as elements of a direct limit. Then, first we define the following category S: the set of objects S_0 is the set S. Given $s, t \in S$, $Hom_S(s, t) = \{x \in A | sx = t\}$.

We have the following properties to this category:

(a) Given $s, t \in S$, there is at most one element $x \in A$ such that $sx = t$.
(b) Given $s, t \in S$, there exist an element $u \in S$ such that $Hom_S(s, u) \neq \emptyset$ and $Hom_S(t, u) \neq \emptyset$.

This two properties allow us to say that S is a filtered category. The first property can be proved in the following way: $sx = t$ and $sy = t$ implies $sx = sy$ and since s is regular follows the prove. The second property: From Lemma (1.2) (ii), given $s, t \in S$, there exist $x, y \in A$ such that $sx = ty$. Then, $u = sx$ and $x \in Hom_S(s, u)$ and $y \in Hom_S(t, u)$.

Now we define the following left A-module generated freely by formal fractions as^{-1}.

EXERCISE: 10. *Prove that the left A-module ${}_AA$ is canonically isomorphic to As^{-1}.*

EXERCISE: 11. *Let S and $A - Mod$ be two categories and $F : S \to A - Mod$ defined by:*

$$F(s) = As^{-1}$$

and for each morphism $s \to t$ (i.e, $sx = t$),

$$F(s \to t) = As^{-1} \to At^{-1} : as^{-1} \mapsto (ax)t^{-1}.$$

Show that F is a functor.

Let $B = \varinjlim As^{-1}$. The elements of B can be represented by formal fractions as^{-1}. If as^{-1} and bt^{-1} represent the same element in B, there exist $x, x' \in A$ such that, $sx = tx' \in S$ and $ax = bx'$.

EXERCISE: 12. *Show that*

(a) *B is a left A-module*
(b) *there are canonical injective linear maps $As^{-1} \to B$*
(c) *There is a injective linear map ${}_AA \to B$.*
(d) *The left multiplication by $s \in S$ on B is injective.*

EXERCISE: 13. *Since B is a left A-module, show that there is a canonical bijection of left A-modules*

$${}_AB \to Hom_A({}_AA, {}_AB).$$

Proposition 1.3. *For every $\beta \in B$, the right multiplication by β on A extends uniquely to an endomorphism $\varphi_\beta :_A B \to_A B$. This extension provides a bijection $B \to End_A B$.*

Proof Case 1: $\beta \in A$.

So we can define $\phi_\beta : B \to B$ by

$$as^{-1} \mapsto a\alpha t^{-1}.$$

The element α and t are such that $s\alpha = \beta t$ (Ore condiditon).

Now we have to prove that ϕ_β is well defined, verifying independence of the two choices made. The first was the choice of α, t in the Ore condition, and the second was the choice of the fraction as^{-1}.

Suppose that we have another choice $s\alpha' = \beta t'$. We have from Lemma (1.2) (iii) that there are x_1, x_2 such that $tx_1 = t'x_2$ and $\alpha x_1 = \alpha' x_2$. Then the fraction are equivalent: $a\alpha' t'^{-1} = a\alpha t^{-1}$.

Now we discuss the other choice that we had done. Suppose that we have two fractions that are equivalent: $as^{-1} = a's'^{-1}$. Then, there are x, y such that $sx = s'y$ and $ax = a'y$. We also have the following equalities:

$$as^{-1} = ax(sx)^{-1}$$

$$(ax)(sx)^{-1} = (a'y)(s'y)^{-1}$$

$$(a'y)(s'y)^{-1} = a's'^{-1}.$$

Then, it suffices to treat the case that the second fraction is such that $a' = ax$ and $s' = sx$.

We have

$$\phi_\beta(as^{-1}) = a\alpha_1 s_1^{-1}$$

$(s\alpha_1 = \beta s_1)$ and

$$\phi_\beta((ax)(sx)^{-1}) = (ax)\alpha_2 s_2^{-1}$$

$((sx)\alpha_2 = \beta s_2)$.

Since there is δ such that $s_2\delta = s_1\delta$, then

$$s\alpha_1\delta = \beta s_1\delta$$

and

$$(sx)\alpha_2\delta = \beta s_2\delta.$$

Then $s\alpha_1\delta = (sx)\alpha_2\delta$. Since s is regular, $\alpha_1\delta = x\alpha_2\delta$. Then $(ax)\alpha_2 s_2^{-1} = (ax)\alpha_2\delta(s_2\delta)^{-1} = a\alpha_1\delta(s_2\delta)^{-1} = a\alpha_1\delta(s_1\delta)^{-1} = a\alpha_1 s_1^{-1}$.

It is easy to see that ϕ_β is A-linear.

Case 2: $\beta = \sigma^{-1}$, $\sigma \in S$. By definition we have $\phi_\beta(as^{-1}) = as^{-1}\sigma^{-1}$. We identify the formal product $as^{-1}\sigma^{-1} = a(\sigma s)^{-1}$. It is easy to check the independence of the choice of fraction representating and A-linearity.

Case 3:

Let $\beta \in B$. We represent $\beta = \alpha\sigma^{-1}$. Then right multiplication by β is the composition of two maps $A \xrightarrow{\alpha} A \xrightarrow{\sigma^{-1}} B$. This composition extends to B in two steps that we can see in the following diagram:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & A \arrow[r, "\sigma^{-1}"] \arrow[d] & B \arrow[d] \\ A \arrow[r, "\alpha"] \arrow[d] & B \arrow[r, "\phi_{\sigma^{-1}}"] \arrow[d] & B \arrow[d] \\ B \arrow[r, "\phi_\alpha"] & B \arrow[r, "\phi_{\sigma^{-1}}"] & B \end{tikzcd} .$$

□

Now we check that ϕ_β is the only A-linear extension of $A \to B$.

LEMMA 1.4. *An endomorphism $\phi :_A B \to_A B$ of is determined uniquely by the element $\phi(1_A) = \beta$.*

Proof Let $\phi : B \to B$ an endomorphism. Then $\phi(a) = a\phi(1)$. Let $\beta = \phi(1)$. Since $\phi : A \to B$, is the restriction of ϕ, then it can be extended to B. We will prove that the extension is uniquely determined. Since ϕ is defined on A, then to each $s \in S$, ϕ is defined on s. So, we have only one way to define $\phi(s^{-1})$ because

$$s\phi(s^{-1}) = \phi(ss^{-1}) = \phi(1_B) = 1_B.$$

So the extension of $\phi : A \to B$ is uniquely determined. □

THEOREM 1.5. *Let $S \subset A$ be a right Ore set. There is a ring $A[S^{-1}]$ of right fractions and an injective ring homomorphism $A \to A[S^{-1}]$ such that*

(a) *the image of every element $s \in S$ is invertible in $A[S^{-1}]$, and*
(b) *every element of $A[S^{-1}]$ can be written as a product as^{-1}.*

Moreover, any homomorphism $f : A \to R$ such that the images of elements of S are invertible in R factors uniquely throught $A[S^{-1}]$.

Proof Let $E = End_A B$ be a ring. Since $Hom_A({}_AA,{}_A B) \simeq B$, and we have the bijection $\Phi : B \to Hom_A({}_AA,{}_A B), b \mapsto \phi_b$ that gives B the ring structure and we know that the multiplication on B corresponds to a composition in $Hom_A(A, B)$, then a composition on $Hom_A(B, B)$.

We see that the bijection $B \to End\ B$ is an isomorphism of A-modules. Let $\beta_3 = \alpha_1\beta_1 + \alpha_2\beta_2 \in B$. Since $\phi_{\beta_3} = \alpha_1\phi_{\beta_1} + \alpha_2\phi_{\beta_2}$ we can prove that the bijection is an homomorphism of ring. □

2. Localization of Categories

The derived category of an abelian category is constructed from the homotopy category of complexes. This construction is done by localizing that category with respect to the class of quasi-isomorphisms. We call attention to the fact that the class of quasi-isomorphisms usually does not satisfy the conditions for being a localizing class. Therefore it is necessary to consider the homotopy category. The conditions which are necessary to localize it (similar to those ones given by Ore sets in the last section) will be presented in this chapter.

We should prove that the class of quasi-isomorphisms in the homotopy category is a localizing class. However, because of the limit of space in this course, we will only give the main notations and some necessary calculations to understand and to manipulate the localizing category.

It is also important to ask if the category that will be localized is a small category or not. We know that if S is a set, then the localization exists. This topic discussion and its bibliography along with the one about the set of equivalence class of left fraction being a set or not, can be found in the book "An Introduction to Homological Algebra", by C. A. Weibel ([22]). The reader can also find more details about localization of categories in the book "A Course on Derived Categories", by Amnon Yekutieli ([23]).

2.1. Left and Right Fractions. Let $\mathcal{C}$ be a category and let S be an arbitrary class of morphisms in $\mathcal{C}$. The idea of localization of a category $\mathcal{C}$ with respect to S is to create a new category $\mathcal{C}[S^{-1}]$ together with a functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ satisfying

(L1) $Q(s)$ is an isomorphism for all $s \in S$, and

(L2) any functor $F : \mathcal{C} \to \mathcal{T}$ such that $F(s)$ is an isomorphism for all $s \in S$, factors uniquely through Q, ie, there exists $G : \mathcal{C}[S^{-1}] \to \mathcal{T}$ such that $G \circ Q = F$.

The above mentioned idea is our objective in the next pages, that is, allow the reader the possibility to understand this new category.

The calculus of fractions which appears in the description of the localization of a category was developed by P. Gabriel and M. Zisman in "Calculus of Fractions and Homotopy Theory" ([8]).

Before begining to give the necessary conditions to make the localizing category, we would like to present what is the main set of morphisms that we would like to localize. In the next sections, we will present the category of complex and the homotopy category of complex. In this category the set of quasi-isomorphisms, ie, the set of morphisms of complex such that the induced morphism in the homology are isomorphisms, will be localized, because we need this kind of complex to be isomorphic.

But not only this. Another important property we need is that the left and right functors defined for example in categories that have enough projectives or enough injectives could be well defined in the homotopy category.

Definition 2.1. Let $\mathcal{C}$ be a category and S be a set of morphisms in $\mathcal{C}$. We say that S **admits left fractions** if the following axioms hold:

(LF1) If $s, t \in S$ are composable morphisms in S, then $st \in S$. The identity $1_X \in S$ for all $X \in \mathcal{C}$.

(LF2) Each pair of morphisms $X' \xrightarrow{f'} Y' \xleftarrow{s'} Y$, with $s' \in S$, can be completed to a commutative square

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "s"] & Y \arrow[d, "s'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

with $s \in S$.

(LF3) Let $f, g : X \to Y$ be morphisms in $\mathcal{C}$. If there is a morphism $s : Y \to Y'$ in S such that $sf = sg$, then there exists a morphism $t : X' \to X$ in S such that $ft = gt$.

The first axiom is a simple condition to have a closed set, and this condition is analogous to that multiplicative system we have in commutative algebra. The second and third ones give us the possibility of changing the order of the morphisms when we need it. The role of the last axiom will be clear in the proofs we will present. This axiom helps us to construct a commutative square from a non commutative one. A more thorough explanation of how reasonable these axioms are is given by Yannick Delbecque in his thesis "Les catégories dérivées" ([6]).

It is not only possible to localize the category with respect to a class of morphisms but also with respect to a class of objects in order to obtain similar results. This work was done in the book "Derived Equivalence for Group Rings", by Steffen Konig and Alexander Zimmermann ([16]).

The dual of left fraction is called **right fraction**. So we can define the dual of Definition (2.1)

Definition 2.2. Let $\mathcal{C}$ be a category and let S be a set of morphisms in $\mathcal{C}$. We say that S **admits right fractions** if the following axioms holds:

(RF1) If $s, t \in S$ are composable morphisms in S, then $st \in S$. The identity $1_X \in S$ for all $X \in \mathcal{C}$.

(RF2) Each pair of morphisms $X' \xleftarrow{s} X \xrightarrow{f} Y$, with $s \in S$, can be completed to a commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"] & Y \arrow[d, dashed, "s'"] \\
X' \arrow[r, dashed, "f'"] & Y'
\end{tikzcd}$$

with $s' \in S$.

(RF3) Let $f, g : X \to Y$ morphisms in $\mathcal{C}$. If there is a morphism $s : X' \to X$ in S such that $fs = gs$, then there exists a morphism $t : Y \to Y'$ in S such that $tf = tg$.

We would like to define the category of left fractions ($\mathcal{C}[S^{-1}]$) as the category whose class of objects is the class of objects of $\mathcal{C}$ and the morphisms are the equivalence class of fractions. So, in order to do that, we need first to define fractions.

Given objects $X, Y \in (\mathcal{C}[S^{-1}])_0$, we call

$$\{(s, f) | s : X \to X' \in S, f : X \to Y\}$$

a set of **left fractions from** X **to** Y. Two left fractions (s, f) and (t, g) from X to Y are called **equivalent left fractions** if there exists morphisms r and h making the following diagram commutative

$$\begin{tikzcd}
 & & X''' \arrow[dl, "r"'] \arrow[d, "h"] & \\
 & X' \arrow[dl, "s"'] \arrow[drr, "f"] & X'' \arrow[dll, "t"] \arrow[dr, "g"] & \\
X & & & Y
\end{tikzcd}$$

in such a way that $sr \in S$.

Lemma 2.1. *(a) The equivalence of left fractions is an equivalence relation.*
(b) The equivalence of right fractions is an equivalence relation.

Proof: Since (b) can be obtained dually, we will only prove (a). We need to verify that the relation is transitive. The other properties are quite simple. Let (s, f), (t, g) and (u, h) three left fractions.

$$\begin{tikzcd} & F \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd} & G \arrow[dl, "t"'] \arrow[dr, "g"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd} & H \arrow[dl, "u"'] \arrow[dr, "h"] & \\ M & & N \end{tikzcd}$$

Suppose (s, f) equivalent to (t, g) and (t, g) equivalent to (u, h). Thus, there exists morphisms p, q and r, v such that

$$\begin{tikzcd} & & P \arrow[dl, "p"] \arrow[d, "q"] & \\ & F \arrow[dl, "s"'] \arrow[drr, "f"' near end] & G \arrow[dll, "t" near end] \arrow[dr, "g"] & \\ M & & & N \end{tikzcd}$$

and

$$\begin{tikzcd} & & Q \arrow[dl, "r"] \arrow[d, "v"] & \\ & G \arrow[dl, "t"'] \arrow[drr, "g"' near end] & H \arrow[dll, "u" near end] \arrow[dr, "h"] & \\ M & & & N \end{tikzcd}$$

are commutative diagrams and $sp \in S$ and $tr \in S$.

Consider now the morphisms

$$\begin{tikzcd} & Q \arrow[d, "r"] \\ P \arrow[r, "q"] & G \end{tikzcd}$$

where $r \in S$. According to $(LF2)$, there exists morphisms γ and β, with $\beta \in S$, such that the following diagram is commutative:

$$\begin{tikzcd} R \arrow[r, "\gamma"] \arrow[d, "\beta"] & Q \arrow[d, "r"] \\ P \arrow[r, "q"] & G \end{tikzcd}$$

Since $sp \in S$, then $sp\beta \in S$ and

$$s(p\beta) = (tq)\beta = t(r\gamma) = (uv)\gamma.$$

Also

$$f(p\beta) = (gq)\beta = g(r\gamma) = h(v\gamma).$$

Thus $p\beta$ and $v\gamma$ give us the equivalence between the left fractions (s, f) and (u, h).
□

DEFINITION 2.3. Let $\mathcal{C}$ be a category and let S be a set of morphisms in $\mathcal{C}$ that admits left fractions. We call $\mathcal{C}[S^{-1}]$ **class of left fractions**:

(a) *Obj* $(\mathcal{C}[S^{-1}]) = Obj\ \mathcal{C}$.
(b) Given objects X, Y in $\mathcal{C}$, the set of morphisms from X to Y coincides with the equivalence classes of left fractions.
(c) Composition: Let $(s, f) : X \to Y$ and $(t, g) : Y \to Z$ be two classes of left fractions with $s : X' \to X$ and $t : Y' \to Y \in S$. By applying $(LF2)$ to $X' \xrightarrow{f} Y \xleftarrow{t} Y'$, there exist a left fraction $X' \xleftarrow{u} U \xrightarrow{h} Y'$ such that $fu = th$. So we define

$$(t, g) \circ (s, f) = (su, gh).$$

In the same way we can define the class of right fractions that will be denoted by $[S^{-1}]\mathcal{C}$.

Exercise: 14. *The composition in the definition above determines the left fraction (su, gh), which depends on a choice of U, u and h. Prove that its equivalence class is independent of these choices.*

Lemma 2.2. *The composition of equivalence classes of left fractions is associative.*

Proof: Exercise. □

Lemma 2.3. *Giving the objects of an additive category $\mathcal{A}$ and considering the equivalence classes of left fractions as morphisms, we can obtain a category $\mathcal{A}[S^{-1}]$, that will be called the category of left fractions.*

Exercise: 15. *Given a left fraction (s, f)*

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

applying $RF2$, suppose that we can complete the diagrams in the following ways:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"] & Y \arrow[d, "t"] \\ X' \arrow[r, "g"] & Y' \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"] & Y \arrow[d, "t'"] \\ X' \arrow[r, "g'"] & Y'' \end{tikzcd}$$

Prove that (t, g) and (t', g') are equivalent right fractions.

Theorem 2.4. *Let $\mathcal{C}$ be a category and let S be a set of morphisms in $\mathcal{C}$ that admits left and right fractions. Then there exists an isomorphism of categories $\mathcal{C}[S^{-1}] \simeq [S^{-1}]\mathcal{C}$.*

Proof: Exercise.

Theorem 2.5. *Let $\mathcal{A}$ be an additive category and let $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ be the application which is the identity on the objects and associates a morphism $X \xrightarrow{f} Y$ to the left fraction $(1_X, f)$*

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

Then this application defines a functor and we can see in this way $\mathcal{A}$ as a full subcategory of $\mathcal{A}[S^{-1}]$, since this functor is fully faithfull.

Proof : Exercise.

Given two equivalent left fractions (s, f) and (t, g) from X to Y, then there exists morphisms r and h making the following diagram commutative

$$\begin{tikzcd} & & X''' \arrow[dl, "r"'] \arrow[d, "h"] & \\ & X' \arrow[dl, "s"'] \arrow[drr, "f"] & X'' \arrow[dll, "t"'] \arrow[dr, "g"] & \\ X & & & Y \end{tikzcd}$$

in such a way that $sr \in S$. So $Q(sr) = Q(s)Q(r)$. Since $Q(s)$ is an isomorphism, so is $Q(r)$. Similarly $Q(h)$ is an isomorphism. So

$$Q(f)Q(s)^{-1} = Q(f)Q(r)Q(r)^{-1}Q(s)^{-1} =$$

$$Q(fr)Q(sr)^{-1} = Q(gh)Q(th)^{-1} =$$

$$Q(g)Q(h)Q(h)^{-1}Q(t)^{-1} = Q(g)Q(t)^{-1}$$

and this shows that these morphisms are equal in the category $\mathcal{C}[S^{-1}]$.

Exercise: 16. *Prove the previous theorem.*

Theorem 2.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor such that $F(s)$ is an isomorphism for any $s \in S$. Then there exists a unique functor $G : \mathcal{A}[S^{-1}] \to \mathcal{B}$ such that $G \circ Q = F$.*

Exercise: 17. *Prove the previous theorem.*

2.2. Localization of an additive category. Assuming that $\mathcal{A}$ is an additive category and S is a class of morphisms that admits left fractions and right fractions, we call fractions the morphism in the localized category.

In this case, for example, the axiom $(LF3)$ can be replaced by the property: let $f : M \to N$ be a morphism. Then there exists $s \in S$ such that $sf = 0$ if and only if there exists $t \in S$ such that $ft = 0$. Actually, since $Hom_{\mathcal{A}}(M, N)$ is an abelian group, then $sf = sg$ is equivalent to $s(f - g) = 0$ and $ft = gt$ is equivalent to $(f - g)t = 0$.

Another important aspect about the sum of fractions is that, given a finite family of left fractions, that is, a family of class of fractions, we can choose another family of fractions with the same denominator to replace it. We discuss this property in the next lemma.

LEMMA 2.7. *Let*

$$\begin{tikzcd} & L_i \arrow[dl, "s_i"'] \arrow[dr, "f_i"] & \\ M & & N \end{tikzcd}$$

be left fractions representing morphisms $\varphi_i : M \to N$ for all $1 \leq i \leq n$ in $\mathcal{A}[S^{-1}]$. Then there exists an object L in $\mathcal{A}$, $s \in S$ and morphisms $g_i : L \to N$ in $\mathcal{A}$ such that the left fractions

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "g_i"] & \\ M & & N \end{tikzcd}$$

represent φ_i for all $1 \leq i \leq n$.

Proof : If $n = 1$, there is nothing to prove. Assume that $n > 1$ and that there exists $k, t \in S$ and morphisms h_i, $1 \leq i \leq n-1$, such that

$$\begin{tikzcd} & K \arrow[dl, "t"'] \arrow[dr, "h_i"] & \\ M & & N \end{tikzcd}$$

represents φ_i for $1 \leq i \leq n-1$. By $LF2$, there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "u'"] \arrow[d, "u"] & L_n \arrow[d, "s_n"] \\ K \arrow[r, "t"] & M \end{tikzcd}$$

where $u \in S$ and $tu = s_n u' \in S$. Then the diagram

$$\begin{tikzcd} & & U \arrow[dl, "u"'] \arrow[d, "1_U"] & \\ & K \arrow[dl, "t"'] \arrow[drr, "h_i"] & U \arrow[dll, "tu"] \arrow[dr, "h_iu"] & \\ M & & & N \end{tikzcd}$$

shows that the left fraction

$$\begin{tikzcd} & U \arrow[dl, "tu"'] \arrow[dr, "h_iu"] & \\ M & & N \end{tikzcd}$$

represents φ_i, $1 \leq i \leq n-1$, and the diagram

$$\begin{tikzcd}
 & & U \arrow[dl, "u'"'] \arrow[d, "1_U"] & \\
 & L_n \arrow[dl, "s_n"'] \arrow[drr, "f_n"] & U \arrow[dll, "tu"] \arrow[dr, "f_n u'"] & \\
M & & & N
\end{tikzcd}$$

shows that the left fraction

$$\begin{tikzcd}
 & U \arrow[dl, "tu"'] \arrow[dr, "f_n u'"] & \\
M & & N
\end{tikzcd}$$

represents φ_n. □

Let φ and Ψ be the following morphisms

$$\begin{tikzcd}
 & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
 & L \arrow[dl, "s"'] \arrow[dr, "g"] & \\
M & & N
\end{tikzcd}$$

in $Hom_{\mathcal{A}[S^{-1}]}(M, N)$. Then is possible to prove that the morphism

$$\begin{tikzcd}
 & L \arrow[dl, "s"'] \arrow[dr, "f+g"] & \\
M & & N
\end{tikzcd}$$

depends only on φ, Ψ, that is, it is dependent of the choice of L, s, f and g.

So, we can give a structure of abelian group to the set $Hom_{\mathcal{A}[S^{-1}]}(M, N)$ with the operation defined above.

We denote the zero morphism in $Hom_{\mathcal{A}[S^{-1}]}(M, N)$ by 0. It is represented by the left fraction

$$\begin{tikzcd}
 & M \arrow[dl, "Id_M"'] \arrow[dr, "0"] & \\
M & & N
\end{tikzcd}$$

If $s : L \to M \in S$, then we have the following commutative diagram

$$\begin{tikzcd}
 & & L \arrow[dl, "s"'] \arrow[d, "1_L"] & \\
 & M \arrow[dl, "1_M"'] \arrow[drr, "0"] & L \arrow[dll, "s"] \arrow[dr, "0"] & \\
M & & & N
\end{tikzcd}$$

Therefore

$$\begin{tikzcd} & M \arrow[dl, "Id_M"'] \arrow[dr, "0"] & \\ M & & N \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "0"] & \\ M & & N \end{tikzcd}$$

are equivalent fractions.

We would like to emphasize that if M, N and P are three objects in $\mathcal{A}$, since the addition in $Hom_{\mathcal{A}}(X, Y)$ is associative for all $X, Y \in \mathcal{A}$, then the binary operation on $Hom_{\mathcal{A}[S^{-1}]}(M, N)$ is also associative, $Hom_{\mathcal{A}[S^{-1}]}(M, N)$ is an abelian group and then the composition of morphisms

$$Hom_{\mathcal{A}[S^{-1}]}(M, N) \times Hom_{\mathcal{A}[S^{-1}]}(N, P) \longrightarrow Hom_{\mathcal{A}[S^{-1}]}(M, P)$$

is biadditive.

Lemma 2.8. *Considering $\mathcal{A}$ as a full subcategory of $\mathcal{A}[S^{-1}]$, the zero object in $\mathcal{A}[S^{-1}]$ is the zero object of $\mathcal{A}$.*

Proof : Consider a zero endomorphism in $\mathcal{A}[S^{-1}]$

$$\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr, "0"] & \\ 0 & & 0 \end{tikzcd}$$

Then we have the commutative diagram

$$\begin{tikzcd} & & M \arrow[dl, "1_M"'] \arrow[d, "s"] & \\ & M \arrow[dl, "s"'] \arrow[drr, "0"] & 0 \arrow[dll, "Id_0"] \arrow[dr, "Id_0"] & \\ 0 & & & 0 \end{tikzcd}$$

So the left fraction

$$\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr, "0"] & \\ 0 & & 0 \end{tikzcd}$$

can be represented by

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 0 & & 0 \end{tikzcd}$$

This fraction also represents the zero morphism. Therefore, the only zero endomorphism is the zero morphism.

Let

$$\begin{tikzcd} & L_1 \arrow[dl] \arrow[dr] & \\ 0 & & N \end{tikzcd}$$

be a morphism of 0 to N and suppose that there exists another morphism from 0 to N:

$$\begin{tikzcd} & L_2 \arrow[dl] \arrow[dr] & \\ 0 & & N \end{tikzcd}$$

The diagram

$$\begin{tikzcd} & & 0 \arrow[dl, "u"'] \arrow[d, "v"] & \\ & L_1 \arrow[dl, "s"'] \arrow[drr, "f"] & L_2 \arrow[dll, "t"'] \arrow[dr, "g"] & \\ 0 & & & N \end{tikzcd}$$

shows us that $su = tv = 1_0 \in S$ (where 1_0 is the identity of the zero object) and $fu = gv$. Therefore these two morphisms are equal and so there exists exactly one morphism from 0 to N. We leave to the reader the proof that there is only one morphism from M to 0. $\square$

If M, $N \in \mathcal{A}[S^{-1}]$, then there exists a coproduct of M and N in $\mathcal{A}[S^{-1}]$. It is possible to prove that $\mathcal{A}[S^{-1}]$ becomes an additive category.

We also can prove that the functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ is additive. Indeed, let $f, g : M \to N$ be morphisms in $\mathcal{A}$. Then the morphisms $Q(f)$, $Q(g) \in \mathcal{A}[S^{-1}]$ are represented by the left fractions

$$\begin{tikzcd} & M \arrow[dl, "Id_M"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & M \arrow[dl, "Id_M"'] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

Hence, $Q(f) + Q(g)$ is represented by

$$\begin{tikzcd} & M \arrow[dl, "Id_M"'] \arrow[dr, "f+g"] & \\ M & & N \end{tikzcd}$$

thus $Q(f + g) = Q(f) + Q(g)$.

So, if $\mathcal{A}$ is an additive category, then $\mathcal{A}[S^{-1}]$ is an additive category and the quotient functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ is additive.

Many properties about the localizing category are pointed in the next theorem. One of them is that $Q(s)$ is an isomorphism in this category. That means that $Q(s) = (1, s)$ is invertible in the localizing category. It is easy to verify that $(s, 1)$ is the inverse of $(1, s)$.

EXERCISE: 18. *Let $s, t \in S$ be morphisms and (s, t) a fraction. Prove the following properties:*
(a) $(s, 1) \circ (1, t) = (s, t)$
(b) $(s, t) \circ (t, s) = (s, s)$
(c) $(t, s) \circ (s, t) = (t, t)$
(d) Prove that for t and $s \in \mathcal{A}$, (s, s) and (t, t) are the identity in $\mathcal{A}[S^{-1}]$.

THEOREM 2.9. *Let $\mathcal{A}$ be an additive category and let S be a class of left fractions. Then $\mathcal{A}[S^{-1}]$ is an additive category, unique up to isomorphism, and the functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ is an additive functor such that:*
(a) $Q(s)$ is an isomorphism for all $s \in S$;
(b) if $\mathcal{B}$ is an additive category and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor such that $F(s)$ is an isomorphism for all $s \in S$, then there exists an unique additive functor $G : \mathcal{A}[S^{-1}] \to \mathcal{B}$ such that $F = G \circ Q$, that is, making the following diagram commutative

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{B} \\ \mathcal{A}[S^{-1}] \arrow[ur, "G"'] & \end{tikzcd}$$

We start now a discussion about an important aspect of morphisms in the localized category. We usually have some difficulties to verify when a morphism in this type of category is a zero morphism or not. So the next lemma will shed some light on these characteristics. We have an equivalence that can help us identify these morphisms.

PROPOSITION 2.10. *Let $\varphi : M \to N$ be a morphism in $\mathcal{A}[S^{-1}]$ represented by a left fraction (s, f). Then the following conditions are equivalent:*
(a) $\varphi = 0$
(b) There exists $t \in S$ such that $ft = 0$
(c) There exists $t \in S$ such that $tf = 0$.

Proof : We first prove that (a) implies (b). Let $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ be the additive functor which comes from localization. We are supposing that $\varphi = 0$, hence

$$Q(f) \circ Q(s)^{-1} =$$

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, "1_L"'] \arrow[dr, "f"] & \\ L & & N \end{tikzcd} \circ \begin{tikzcd}[column sep=small] & L \arrow[dl, "s"'] \arrow[dr, "1"] & \\ M & & L \end{tikzcd} = \begin{tikzcd}[column sep=small] & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

Therefore, from $Q(f) \circ Q(s)^{-1} = \varphi = 0$, we have $Q(f) = 0$ and so

$$Q(f) \quad = \begin{tikzcd}[column sep=small] & L \arrow[dl, "Id_L"'] \arrow[dr, "f"] & \\ L & & N \end{tikzcd}$$

represents the zero morphism in $Hom_{\mathcal{A}[S^{-}]}(L, N)$ and the zero morphism from L to N is represented by

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, "Id_L"'] \arrow[dr, "0"] & \\ L & & N \end{tikzcd}$$

Hence, these two left fractions are equivalent, that is, there exists u, $v : U \to L$ in $\mathcal{A}$ such that $v = u \in S$ and $fu = 0$.

$$\begin{tikzcd}
 & & U \arrow[dl, "u"'] \arrow[d, "v"] & \\
 & L \arrow[dl, "1_L"'] \arrow[drr, "f"' near end] & L \arrow[dll, "1_L" near end] \arrow[dr, "0"] & \\
L & & & N
\end{tikzcd}$$

Now we prove that (b) implies (a). We are supposing that there exists $u : U \to L$ in S such that $fu = 0$. Then $Q(f)Q(u) = 0$. Hence $Q(f) = 0$. Therefore,

$$\varphi \quad = \quad \begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ L & & N \end{tikzcd} \quad = \quad Q(f)Q(s)^{-1} = 0$$

□

This last result allows us to say that if $f : M \to N$ is a morphism in $\mathcal{A}$, then we have that the statement $Q(f) = 0$ is equivalent to $tf = 0$, for some $t \in S$, which is also equivalent to $ft = 0$, for some $t \in S$.

In the next result, we have another consequence of the previous proposition. This curious consequence comes from the following reasons. We know that the localizing functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ maps each object $M \in \mathcal{A}$ to $M \in \mathcal{A}[S^{-1}]$. So, if we say that $Q(M) = 0$, then immediately we want say that $M = 0$ in the category $\mathcal{A}$. But this obvious mistake is due to our tendency to forget that the definition of zero object depends on the set of morphisms in the category (an object 0 is zero if for each object in this category, there exists only one morphism from this object to zero and only one morphism from 0 to this object). So, the aim of the next result is to present the necessary and sufficient conditions to an object in $\mathcal{A}$ to be zero in $\mathcal{A}[S^{-1}]$.

PROPOSITION 2.11. *Let X be an object in $\mathcal{A}$. Then the following statements are equivalent:*

(a) $Q(X) = 0$*;*
(b) *There exists an object $Y \in \mathcal{A}$ such that the zero morphism in $Hom_{\mathcal{A}}(Y, X)$ is in S;*
(c) *There exists an object $Y \in \mathcal{A}$ such that the zero morphism in $Hom_{\mathcal{A}}(X, Y)$ is in S.*

Proof : Suppose that $Q(X) = 0$. We know that $Q(X) = X$. Then $X = 0$ means that X is the zero object in $\mathcal{A}[S^{-1}]$, that is, every morphism $Y \to X$ or $X \to Y$ in $\mathcal{A}[S^{-1}]$ is zero. So, in particular, the identity in $\mathcal{A}[S^{-1}]$ from X to X is the zero morphism, that is, the morphism

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & \\ X & & X \end{tikzcd}$$

is equivalent to

$$\begin{tikzcd}
& X \arrow[dl, "1_X"'] \arrow[dr, "0"] & \\
X & & X
\end{tikzcd}$$

Therefore, there exists $U \in \mathcal{A}$ and $u : U \to X$ in S such that $u = 0$ and the following diagram is commutative

$$\begin{tikzcd}
& & U \arrow[dl, "u"'] \arrow[d, "u"] & \\
& X \arrow[dl, "1_X"'] \arrow[drr, "1_X"] & X \arrow[dll, "1_X"] \arrow[dr, "0"] & \\
X & & & X
\end{tikzcd}$$

Now, supposing that there exists $u : U \to X$ in S a zero morphism, we have to prove that there exists only one morphism in $Hom_{\mathcal{A}[S^{-1}]}(X, Y)$ and $Hom_{\mathcal{A}[S^{-1}]}(Y, X)$, for each $Y \in \mathcal{A}$. Let

$$\begin{tikzcd}
& L \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

be a morphism in $Hom_{\mathcal{A}[S^{-1}]}(X, Y)$. We have $u = u \circ 1_U \in S$ and since $u = 0$, $u \circ 1_U = s \circ 0$ and $f \circ 0 = 0 \circ 1_U$, we obtain the following commutative diagram

$$\begin{tikzcd}
& & U \arrow[dl, "0"'] \arrow[d, "1_U"] & \\
& L \arrow[dl, "s"'] \arrow[drr, "f"] & U \arrow[dll, "u"] \arrow[dr, "0"] & \\
X & & & Y
\end{tikzcd}$$

So we have proved that any given morphism (s, f) in $Hom_{\mathcal{A}[S^{-1}]}(X, Y)$ is the zero morphism.

We leave to the reader the proof that there exists only one morphism in $Hom_{\mathcal{A}[S^{-1}]}(Y, X)$.

Exercise: 19. *Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Prove that $Q(f)$ is a monomorphism (epimorphism) if f is a monomorphism (epimorphism). Observe that the former observations shows that the reverse implication is, in general, false.*

CHAPTER 3

Derived Category

Let $\mathcal{A}$ be an abelian category and let $\mathcal{K}^\bullet(\mathcal{A})$ be the corresponding homotopy category of complexes with the triangulated structure. A morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathcal{C}^\bullet(\mathcal{A})$ is called a quasi-isomorphism if $H^p(f) : H^p(X^\bullet) \to H^p(Y^\bullet)$ are isomorphisms for all $p \in \mathbb{Z}$, where H^p is the cohomology defined in (4).

Exercise: 20. *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a quasi-isomorphism and let $g^\bullet : X^\bullet \to Y^\bullet$ be homotopic to $f^\bullet$. Prove that $g^\bullet$ is also a quasi-isomorphism.*

By abuse of language we say that a morphism in $\mathcal{K}^\bullet(\mathcal{A})$ is a quasi-isomorphism if all of its representatives are quasi-isomorphism.

Example 0.1. Let $X \in ModR$ and let

$$\cdots \to P^n \to \cdots P^1 \to P^0 \to X \to 0$$

be a projective resolution of X. Then the 0-complex with X concentrated in degree zero and the complex

$$\cdots \to P^n \to \cdots P^1 \to P^0 \to 0 \to \cdots 0$$

are quasi-isomorphic. A similar statement holds if we take injective resolutions of X.

Let S be the class of all quasi-isomorphisms in $\mathcal{K}^\bullet(\mathcal{A})$. An object $X^\bullet \in \mathcal{K}^\bullet(\mathcal{A})$ is called acyclic if $H^p(X^\bullet) = 0$ for all $p \in \mathbb{Z}$.

Exercise: 21. *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $\mathcal{K}^\bullet(\mathcal{A})$. Prove that the following conditions are equivalent:*
(i) The morphism $f^\bullet$ is a quasi-isomorphism.
(ii) The cone of $f^\bullet$ is acyclic.

Definition 0.1. Let $\mathcal{C}$ be a triangulated category. A class of morphisms S that admits left and right fractions is **compatible with triangulation** if it satisfies
(i) For any morphism $s \in S$, $Ts \in S$.
(ii) Given a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"] & Y \arrow[r] \arrow[d, "t"] & Z \arrow[r] & TX \arrow[d, "Ts"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd}$$

where the rows are exact triangles and the first square is commutative, and $s, t \in S$, then it can be completed to a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"] & Y \arrow[r] \arrow[d, "t"] & Z \arrow[r] \arrow[d, "u"] & TX \arrow[d, "Ts"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd}$$

where $u \in S$.

Proposition 0.1. *The class S of all quasi-isomorphism in $\mathcal{K}^\bullet(\mathcal{A})$ is a class of morphisms that admits left and right fractions and is compatible with the tringulation.*

In general, if $\mathcal{C}$ is a triangulated category and S is a localizing class in $\mathcal{C}$ compatible with the triangulation, then the category $\mathcal{C}[S^{-1}]$ is triangulated and the natural functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is an exact functor.

Definition 0.2. Let $\mathcal{A}$ be an abelian category, $\mathcal{C}^\bullet(\mathcal{A})$ the corresponding category of complexes and $\mathcal{K}^\bullet(\mathcal{A})$ the homotopic category of complexes which is a triangulated category. Let S be the class of quasi-isomorphism in $\mathcal{K}^\bullet(\mathcal{A})$. We know that S is a localizing class compatible with the triangulation of $\mathcal{K}^\bullet(\mathcal{A})$. The localization of the category $\mathcal{K}^\bullet(\mathcal{A})$ with respect to the class S of all quasi-isomorphisms will be denoted by $\mathcal{D}^\bullet(\mathcal{A})$ and will be called **derived category of $\mathcal{A}$**.

Now we have a concrete example of a class of morphisms that admits right and left fractions, that is, the set of quasi-isomorphism. We know that this class of morphisms is compatible with the triangulation. So, we can use this class of morphisms to illustrate the Proposition (2.10) with an example.

Example 0.2. Let $X, Y \in \mathcal{A}$ and $X[0]$ the 0-complex and $Y[i]$ the complex with cohomology zero for $j \neq i$ and cohomology Y in i. Let

$$\begin{tikzcd} & K^\bullet \arrow[dl, "s^\bullet"'] \arrow[dr, "f^\bullet"] & \\ X[0] & & Y[i] \end{tikzcd}$$

be a morphism in $Hom_{\mathcal{D}^\bullet(\mathcal{A})}(X, Y[i])$ with $s^\bullet$ a quasi-isomorphism. So, since $X[0]$ is a 0-complex, with cohomology zero except in degree zero, then $K^\bullet$ has cohomology zero, except in degree zero, whose cohomology is $H^0(K^\bullet) = X$.

Now, consider the following morphism of complexes $t^\bullet : L^\bullet \to K^\bullet$ with $L^\bullet$ defined in the first row of the following diagram

$$\begin{tikzcd} \cdots \arrow[r] & K^{i-3} \arrow[r, "d^{i-3}"] \arrow[d, "1"] & K^{i-2} \arrow[r, "d^{i-2}"] \arrow[d, "1"] & Ker\, d^{i-1} \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & K^{i-3} \arrow[r, "d^{i-3}"] & K^{i-2} \arrow[r, "d^{i-2}"] & K^{i-1} \arrow[r, "d^{i-1}"] & K^i \arrow[r] & \cdots \end{tikzcd}$$

It is a quasi-isomorphism if $i \neq 0$. If $i < 0$, the composition below

$$\begin{tikzcd} \cdots \arrow[r] & K^{i-3} \arrow[r, "d^{i-3}"] \arrow[d, "1"] & K^{i-2} \arrow[r, "d^{i-2}"] \arrow[d, "1"] & Ker\, d^{i-1} \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & K^{i-3} \arrow[r, "d^{i-3}"] \arrow[d] & K^{i-2} \arrow[r, "d^{i-2}"] \arrow[d] & K^{i-1} \arrow[r, "d^{i-1}"] \arrow[d] & K^i \arrow[r] \arrow[d] & \cdots \arrow[r] & K^0 \arrow[r] \arrow[d, "f^0"] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & X \arrow[r] & \cdots \end{tikzcd}$$

$L^\bullet \xrightarrow{t^\bullet} K^\bullet \xrightarrow{f^\bullet} Y[i]$ is zero and since $i < 0$, $t^\bullet$ is a quasi-isomorphism. Therefore, by Proposition (2.10), the fraction

$$\begin{tikzcd} & K^\bullet \arrow[dl, "s^\bullet"'] \arrow[dr, "f^\bullet"] & \\ X[0] & & Y[i] \end{tikzcd}$$

is zero if $i < 0$.

THEOREM 0.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories and $F : \mathcal{C} \to \mathcal{D}$ an exact functor. Let S be a localizing class in $\mathcal{C}$ compatible with the triangulation such that $s \in S$ implies that $F(s)$ is an isomorphism in $\mathcal{D}$. Then there exists a unique functor $F_S : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F = F_S \circ Q$ and the functor F_S is exact.*

EXAMPLE 0.3. The first ideas about the structure of the derived category can be obtained from the following construction. Let $\mathcal{C}(\mathcal{A})$ be the category of complexes in $\mathcal{A}$. And $\mathcal{C}_0(\mathcal{A})$ is the category where all its differentials are zero (cyclic complexes) $(\mathcal{C}(\mathcal{A})_0 \subset \mathcal{C}(\mathcal{A}))$. This category is isomorphic to the category

$$\prod_{n=-\infty}^{\infty} \mathcal{A}[n]$$

where $\mathcal{A}[n]$ is the n-th copy of $\mathcal{A}$.

We would like to give a relation between this category and the derived category in the case of $\mathcal{A}$ is a semisimple category. First we define a functor

$$F : \mathcal{C}(\mathcal{A}) \to \mathcal{C}_0(\mathcal{A}).$$

We use now the following notation $(X^\bullet, d^\bullet)$ to a complex $X^\bullet$ with the respective differential $d^\bullet$. Then $F((X^\bullet, d^\bullet)) = (H^n(X^\bullet, d^\bullet), 0)$ is a cyclic complex and $F(f : (X^\bullet, d^\bullet_{X^\bullet}) \to (Y^\bullet, d^\bullet_{Y^\bullet})) = (H^\bullet(f), 0)$. Since the functor F transforms quasi-isomorphism in isomorphism, it can be factors through $\mathcal{D}(\mathcal{A})$. So we have for any category $\mathcal{A}$ a functor $G : \mathcal{D}(\mathcal{A}) \to \mathcal{C}_0(\mathcal{A})$. If $\mathcal{A}$ is a semisimple abelian category, then this functor is an equivalence of categories. The proof can be read in the book Methods of Homological Algebra -S.I. Gelfand and Y.I. Manin ([10]).

Another reasonable functor to define between $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$ is a functor that maps objects of $\mathcal{A}$ in stalk complex at zero, ie, complex concentrated in degree zero. This simple functor gives us a kind of embedding of $\mathcal{A}$ in $\mathcal{D}(\mathcal{A})$. The comprehension of this embedding sheds light on the structure of the derived category.

1. The behavior of the category $\mathcal{A}$ inside the derived category $\mathcal{D}(\mathcal{A})$

A complex $X^\bullet$ will be called a **H^0-complex** if $H^n(X^\bullet) = 0$ for $n \neq 0$. This notion is a generalization of the notion of 0-complex.

We have seen that the functor $K : \mathcal{A} \to \mathcal{K}^\bullet(\mathcal{A})$ defined by $K(X) = X^\bullet$, where $X^\bullet$ is the 0-complex, and morphisms of complexes are associated to their homotopy classes is a full and faithful functor. Considering $Q : \mathcal{K}^\bullet(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ the localizing functor defined in (2.9), we would like to prove that the functor $Q \circ K$ gives us an equivalence between $\mathcal{A}$ and the full subcategory of $\mathcal{D}(\mathcal{A})$ given by H^0-complex.

THEOREM 1.1. *Let $\mathcal{A}$ be an abelian category. The functor $Q \circ K : \mathcal{A} \to \mathcal{D}(\mathcal{A})$ yields an equivalence of the category $\mathcal{A}$ and the full subcategory of $\mathcal{D}(\mathcal{A})$ formed by H^0-complex.*

Proof : The equivalence is given by $Q \circ K$ and the functor

$$H^0 : \mathcal{D}^\bullet(\mathcal{A}) \to \mathcal{A}$$

defined by $H^0(X^\bullet) = Kerd^0_{X^\bullet}/Im\ d^{-1}_{X^\bullet}$ and if $(s,f) : X^\bullet \to Y^\bullet$ is a fraction, then $H^0(s^\bullet, f^\bullet) = H^0(f^\bullet) \circ H^0(s^\bullet)^{-1}$.

First we need to prove that $H^0 \circ Q \circ K \simeq 1$ and $Q \circ K \circ H^0 \simeq 1$ on the objects. For this purpose, let $X \in \mathcal{A}$. Then

$$H^0(Q \circ K(X)) = H^0(X^\bullet),$$

where $X^\bullet$ is the 0-complex defined by the object $X \in \mathcal{A}$. Therefore

$$H^0(Q \circ K(X)) = X.$$

Let $Z^\bullet$ be a H^0-complex. We would like to prove that

$$Q \circ K \circ H^0(Z^\bullet) \simeq Z^\bullet.$$

We know that $Q \circ K \circ H^0(Z^\bullet)$ is the 0-complex with the homology $H^0(Z^\bullet)$ concentrated in the degree zero. So, we have to prove that this 0-complex is isomorphic to the H^0-complex $Z^\bullet$ in the derived category. Let $V^\bullet$ be the complex

$$\cdots \longrightarrow Z^{-i} \xrightarrow{d^{-i}} \cdots \longrightarrow Z^{-1} \longrightarrow Kerd^0_{Z^\bullet} \longrightarrow 0 \longrightarrow \cdots$$

with the differential induced by the differentials of $Z^\bullet$. Then we have a quasi-isomorphism $V^\bullet \xrightarrow{s^\bullet} Z^\bullet$

$$\begin{tikzcd}
\cdots \arrow[r] & Z^{-1} \arrow[r] \arrow[d, "1"] & Kerd^0_{Z^\bullet} \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Z^{-1} \arrow[r, "d^{-1}_{Z^\bullet}"] & Z^0 \arrow[r] & Z^1 \arrow[r] & \cdots
\end{tikzcd}$$

and the quasi-isomorphism $V^\bullet \xrightarrow{t^\bullet} Q \circ K \circ H^0(Z^\bullet)$

$$\begin{tikzcd}
\cdots \arrow[r] & Z^{-1} \arrow[r] \arrow[d] & Kerd^0_{Z^\bullet} \arrow[r] \arrow[d, "p"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & H^0(Z^\bullet) \arrow[r] & {} \arrow[r] & \cdots
\end{tikzcd}$$

where $Q \circ K \circ H^0(Z^\bullet)$ is the 0-complex with the homology $H^0(Z^\bullet)$ concentrated in the degree zero and $p : Kerd^0_{Z^\bullet} \to Kerd^0_{Z^\bullet}/Im\ d^{-1}_{Z^\bullet}$ is the natural projection. Thus we have a morphism in $\mathcal{D}^\bullet(\mathcal{A})$ given by the fraction

$$\begin{tikzcd}
& V^\bullet \arrow[dl, dash, "s^\bullet"'] \arrow[dr, "t^\bullet"] & \\
Z^\bullet & & H^0(Z^\bullet)
\end{tikzcd}$$

By exercise (18), we have that this last morphism is an isomorphism in $\mathcal{D}^\bullet(\mathcal{A})$.

Now we would like to prove that the morphism

$$\varphi : Hom_{\mathcal{A}}(X,Y) \to Hom_{\mathcal{D}^\bullet(\mathcal{A})}(Q \circ K(X), Q \circ K(Y))$$

is an isomorphism. Considering $f : X \to Y$ a morphism in $\mathcal{A}$, then $Q \circ K(f) = Q(f^\bullet)$, where $f^\bullet : X^\bullet \to Y^\bullet$ is a morphism from the 0-complex $X^\bullet$ to the 0-complex $Y^\bullet$ defined in the degree zero by f. The morphism $Q(f^\bullet)$ is the fraction $(1_{X^\bullet}, f^\bullet)$

$$\begin{tikzcd} & X^\bullet \arrow[dl, dash, "1_{X^\bullet}"'] \arrow[dr, "f^\bullet"] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

Thus $H^0(Q \circ K(f)) = H^0(1_{X^\bullet}, f^\bullet) = H^0(f^\bullet) \circ H^0(1_{X^\bullet})^{-1}$. In this case $H^0(f^\bullet) = f$ because $f^\bullet$ is the morphism induced by f on 0-complexes. Analogously, $H^0(1_{X^\bullet}) = 1_X$. Therefore

$$H^0(Q \circ K(f)) = f.$$

Then φ is injective. Now, to prove the surjectivity we take $\Phi : Q \circ K(X) \to Q \circ K(Y)$ a morphism in $\mathcal{D}^\bullet(\mathcal{A})$, where $Q \circ K(X)$ and $Q \circ K(Y)$ are 0-complexes with X concentrated in degreee zero and Y concentrated in degree zero respectively. So Φ can be represented by the fraction:

$$\begin{tikzcd} & Z^\bullet \arrow[dl, dash, "s^\bullet"'] \arrow[dr, "f^\bullet"] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

where $X^\bullet$ and $Y^\bullet$ are 0-complexes with X and Y concentrated in degree zero respectively. Since $s^\bullet$ is a quasi-isomorphism so $Z^\bullet$ is an H^0-complex and $H^0(Z^\bullet) \simeq X$.

Consider the following morphism in $\mathcal{A}$:

$$\phi : X \overset{H^0(s^\bullet)^{-1}}{\longrightarrow} H^0(Z^\bullet) \overset{H^0(f^\bullet)}{\longrightarrow} Y.$$

So, we would like to prove that $Q \circ K(\phi) = \Phi$. We know that $Q \circ K(\phi) = Q(\phi^\bullet)$ where $\phi^0 = \phi$ and $\phi^i = 0$ for $i \neq 0$ and $Q(\phi^\bullet)$ is the fraction

$$\begin{tikzcd} & X^\bullet \arrow[dl, dash, "1^\bullet"'] \arrow[dr, "\phi^\bullet"] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

where each complex $X^\bullet, Y^\bullet$ are 0-complex with X and Y concentrated in degree zero respectively. To prove that $Q \circ K(\phi)$ and Φ are equal, we have to prove that both fractions are in the same equivalent class. We must construct the commutative diagram

$$\begin{tikzcd} & & V^\bullet \arrow[dl, "r^\bullet"'] \arrow[d, "h^\bullet"] & \\ & Z^\bullet \arrow[dl, "s^\bullet"'] \arrow[drr, "f^\bullet"' near start] & X^\bullet \arrow[dll, "1^\bullet" near end] \arrow[dr, "\phi^\bullet"] & \\ X^\bullet & & & Y^\bullet \end{tikzcd}$$

such that $s^\bullet \circ r^\bullet$ is a quasi-isomorphism.

Let $V^\bullet$ the following complex:

$$\cdots \to Z^{-2} \xrightarrow{d^{-2}_{Z^\bullet}} Z^{-1} \xrightarrow{d^{-1}_{Z^\bullet}} Ker\ d^0_{Z^\bullet} \longrightarrow 0 \longrightarrow \cdots$$

Therefore, we can define the quasi-isomorphism $r^\bullet : V^\bullet \to Z^\bullet$:

$$\begin{tikzcd}
\cdots \arrow[r] & Z^{-2} \arrow[r, "d^{-2}_{Z^\bullet}"] \arrow[d, "1"] & Z^{-1} \arrow[r, "d^{-1}_{Z^\bullet}"] \arrow[d, "1"] & Kerd^0_{Z^\bullet} \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Z^{-2} \arrow[r, "d^{-2}_{Z^\bullet}"] & Z^{-1} \arrow[r, "d^{-1}_{Z^\bullet}"] & Z^0 \arrow[r, "d^0_{Z^\bullet}"] & Z^1 \arrow[r] & \cdots
\end{tikzcd}$$

and the morphism $h^\bullet : V^\bullet \to X^\bullet$:

$$\begin{tikzcd}
\cdots \arrow[r] & Z^{-2} \arrow[r, "d^{-2}_{Z^\bullet}"] \arrow[d] & Z^{-1} \arrow[r, "d^{-1}_{Z^\bullet}"] \arrow[d] & Kerd^0_{Z^\bullet} \arrow[r] \arrow[d, "H^0(s^\bullet)\circ p"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where $p : Ker\ d^0_{Z^\bullet} \to H^0(Z^\bullet)$ is the natural projection. Therefore $s^\bullet \circ r^\bullet = 1^\bullet \circ h^\bullet$ since if $x \in Ker\ d^0_{Z^\bullet}$, then $s^0 \circ r^0(x) = s^0(x)$ and $H^0(s^\bullet)(p(x)) = H^0(s^\bullet)(\overline{x}) = s^0(x)$.

Now we would like to prove that $\phi^\bullet \circ h^\bullet = f^\bullet \circ r^\bullet$. We need only to prove in degree zero. So, $\phi^0 \circ h^0 = \phi^0 \circ H^0(s^\bullet) \circ p = \phi \circ H^0(s^\bullet) \circ p = H^0(f^\bullet) \circ H^0(s^\bullet)^{-1} \circ H^0(s^\bullet) \circ p = H^0(f^\bullet) \circ p = H^0(f^\bullet)$. And is easy to see that $f^0 \circ r^0 = H^0(f^\bullet)$.

Therefore, the fractions $(s^\bullet, f^\bullet)$ and $(1^\bullet, \phi^\bullet)$ are equivalent and this allows us to say that $Q \circ K(\phi) = \Phi$. $\square$

2. The connection between Extensions and Fractions

In this section we denote to the composition of the translation functor T^i by $[i]$. So, the object $T^i(X)$ will be denoted by the notation $X[i]$ and a morphism $T^i(f)$ by $f[i]$.

Let $X, Y \in \mathcal{A}$. We would like to present a close connection between the morphisms $Hom_{\mathcal{D}^\bullet(\mathcal{A})}(X, Y[i])$ in the derived category and the extensions $Ext^i(X, Y)$, that is, the set of exact sequences

$$\mathcal{E} : 0 \to Y \to K^{-i+1} \to \cdots \to K^0 \xrightarrow{d^0} X \to 0$$

in $\mathcal{A}$ module the usual equivalence relation. We would like to define a map

$$Ext^i(X, Y) \longrightarrow Hom_{\mathcal{D}^\bullet(\mathcal{A})}(X, Y[i]).$$

So we take the extension $\mathcal{E}$ and we map this extension to a fraction that will be constructed now.

Let $K^\bullet$ the following complex:

$$\cdots \to 0 \to Y \to K^{-i+1} \to \cdots \to K^0 \to 0 \to 0 \cdots$$

and let $s^\bullet : K^\bullet \to X[0]$ the following quasi-isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & Y \arrow[r] \arrow[d] & K^{-i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & K^{-1} \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d, "d^0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

and let $f^\bullet : K^\bullet \to Y[i]$ the following morphism of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & Y \arrow[r] \arrow[d, "1"] & K^{-i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & K^{-1} \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Y \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Therefore with $\mathcal{E}$ we can associate the following fraction on $\mathcal{D}^\bullet(\mathcal{A})$:

$$\begin{tikzcd}
 & K^\bullet \arrow[dl, "s^\bullet"'] \arrow[dr, "f^\bullet"] & \\
X[0] & & Y[i]
\end{tikzcd}$$

We have seen that if $i < 0$, then the morphisms in $Hom_{\mathcal{D}^\bullet(\mathcal{A})}(X, Y[i])$ are zero morphisms (Example (0.2)). If $i > 0$, is possible to map each fraction in $Hom_{\mathcal{D}^\bullet(\mathcal{A})}(X, Y[i])$ to an extension in $Ext^i(X, Y)$. These computation can be found on page 167 in the book "Methods of Homological Algebra", by S. I. Gelfand and Y. I. Manin ([10]).

2.1. Exact Sequence of Complexes and Triangles. Now we would like to explain the remark that a short exact sequence in $\mathcal{C}^\bullet(\mathcal{A})$ induces an exact triangle in $\mathcal{D}^\bullet(\mathcal{A})$ but not necessarily in $\mathcal{K}^\bullet(\mathcal{A})$ (See "Triangulated Categories: Definitions, Properties and Examples - Thorsten Holm and Peter Jϕrgensen ([13]).

Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism of complex in $\mathcal{C}^\bullet(\mathcal{A})$. The mapping cylinder of $f^\bullet$ is the complex $Cyl(f)$ such that

$$Cyl(f)^n = X^n \oplus X^{n+1} \oplus Y^n$$

and

$$d^n_{Cyl(f)}(x, x', y) = (d^n_{X^\bullet}x - x', -d^{n+1}_{X^\bullet}x', f^{n+1}x' + d^n_{Y^\bullet}y).$$

EXERCISE: 22. *Check that $Cyl(f)$ is indeed a complex, that is,*

$$d^{n-1}_{Cyl(f)} \circ d^n_{Cyl(f)} = 0.$$

Consider the following morphism of complexes

$$\iota : X^\bullet \to Cyl(f)$$

given by $\iota^n = (1_{X^n}, 0, 0)$ and

$$\pi : Cyl(f) \to C^\bullet_f$$

given by

$$\pi_n = \begin{bmatrix} 0 & 1_{X^{n+1}} & 0 \\ 0 & 0 & 1_{Y^n} \end{bmatrix}$$

These maps indeed commute with the differentials and we have a short exact sequence in $\mathcal{C}(\mathcal{A})$

$$0 \to X^\bullet \xrightarrow{\iota} Cyl(f) \xrightarrow{\pi} C^\bullet_f \to 0$$

Proposition 2.1. *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism of complexes. Then there are morphisms of complexes $\sigma : Y^\bullet \to Cyl(f)$ with $\sigma^n = (0, 0, 1_{Y^n})$ and $\tau : Cyl(f) \to Y^\bullet$ with $\tau^n = (f^n, 0, 1_{Y^n})$ such that the following holds:*
(i) The following diagram with exact rows is commutative in the category $\mathcal{C}^\bullet(\mathcal{A})$ of complexes

$$\begin{tikzcd}
 & 0 \arrow[r] & Y^\bullet \arrow[r, "\iota_f"] \arrow[d, "\sigma"] & C_f^\bullet \arrow[r, "p_f"] \arrow[d, "1"] & TX^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^\bullet \arrow[r] \arrow[d, "1"] & Cyl(f) \arrow[r] \arrow[d, "\tau"] & C_f^\bullet \arrow[r] & 0 & \\
 & X^\bullet \arrow[r] & Y^\bullet & & &
\end{tikzcd}$$

(ii) $\tau \circ \sigma = 1_{Y^\bullet}$ and $\sigma \circ \tau$ is homotopic to the identity $1_{Cyl(f)}$, ie, $Y^\bullet$ and $Cyl(f)$ are isomorphic in the homotopy category, and hence also in the derived category $\mathcal{D}^\bullet(\mathcal{A})$.
(iii) σ and τ are quasi-isomorphism.

Corollary 2.2. *Given any short exact sequence $0 \to X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \to 0$ in $\mathcal{C}(\mathcal{A})$ there exists a corresponding exact triangle $\mathcal{D}^\bullet(\mathcal{A})$ of the form*

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \to TX^\bullet$$

Proof : Consider the following diagram of exact sequences in the row

$$\begin{tikzcd}
0 \arrow[r] & X^\bullet \arrow[r, "\iota"] \arrow[d, "1"] & Cyl(f^\bullet) \arrow[r, "\pi"] \arrow[d, "\tau"] & C_f^\bullet \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X^\bullet \arrow[r, "f^\bullet"] & Y^\bullet \arrow[r, "g^\bullet"] & Z^\bullet \arrow[r] & 0
\end{tikzcd}$$

It is possible to define γ by $\gamma_n(x, y) = g_n(y)$ and check that this is a morphism of complexes. Since 1 and τ are quasi-isomorphism, using the long exact sequence of cohomology, it is possible to prove that γ is a quasi-isomorphism. So, we have the following isomorphism of triangles in the derived category

$$\begin{tikzcd}
X^\bullet \arrow[r, "f^\bullet"] \arrow[d, "1"] & Y^\bullet \arrow[r, "\iota_{f^\bullet}"] \arrow[d, "1"] & C_f^\bullet \arrow[r, "p_{f^\bullet}"] \arrow[d, "\gamma"] & TX^\bullet \arrow[d, "1"] \\
X^\bullet \arrow[r, "f^\bullet"] & Y^\bullet \arrow[r, "g^\bullet"] & Z^\bullet \arrow[r, "p_{f^\bullet} \circ \gamma^{-1}"] & TX^\bullet
\end{tikzcd}$$

but not necessarily in the homotopy category. □

CHAPTER 4

Hereditary Categories

If a category is hereditary then its derived category can be easily described. Therefore we finish these notes describing the derived category of a hereditary category.

Hereditary categories are important examples for many phenomena in the Representation Theory. For instance, in the context of the Representation Theory we can use Auslander-Reiten Theory to present the structure of some hereditary categories, for example module category over an hereditary algebra, or the category of coherent sheaves on a weighted projective line, introduced by W. Geigle and H. Lenzing in "A class of weighted projective lines arising in representation theory of finite dimensional algebras" ([9]). This kind of presentation uses Auslander-Reiten quiver, whose presentation illustrate very well some concepts of representation finite, tame and wild algebras and the concepts of wild, tame tubular, tame domestic coherent sheaves.

An investigation of the class of algebras that are derived equivalent to an hereditary category can be found in "The Strong Global Dimension of Piecewise Hereditary Algebras", by E. R. Alvares, P. Le Meur and E. N. Marcos ([1]).

1. Derived Category of an Hereditary Category

Let $\mathcal{H}$ be a small, abelian and connected k-category, where k is a field. We are supposing that $Ext^i(X,Y)$ is finite dimensional over k for every $i \in \mathbb{Z}$. The category $\mathcal{H}$ is called **hereditary** if $Ext^2(-,-) = 0$.

If k is an algebraically closed field and the hereditary category $\mathcal{H}$ has a tilting object, then Happel's theorem ("A characterization of hereditary categories with tilting object", by D. Happel [12] and "Um estudo sobre categorias hereditárias com objeto inclinante" - C. Schmidt [19]) states that, up to derived equivalence, there are only two standard types of hereditary category: one of them is *mod* H of finite dimensional modules over a finite dimensional hereditary k-algebra H, and the other one is the category $Coh\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$. We suggest the book "Handbook of Tilting Theory", edited by L. Angeleri Hugel, D. Happel and H. Krause [2]. In this book there is a good compendium of hereditary categories, written by Helmut Lenzing.

We would like to show that if $\mathcal{H}$ is hereditary, then $\mathcal{D}^b(\mathcal{H})$ is the additive closure of the disjoint union

$$\bigvee_{i\in\mathbb{Z}} \mathcal{H}[i]$$

where each $\mathcal{H}[n]$ is a copy of $\mathcal{H}$ with objects $X[n]$, $X \in \mathcal{H}$ is a complex with cohomology concentrated in degree n. In this case, morphisms are determined by $Hom_{\mathcal{D}^b(\mathcal{H})}(X[m], Y[n]) = Ext^{n-m}_{\mathcal{H}}(X,Y)$, and the composition is given by Yoneda composition of Ext. The notation $\bigvee_{i\in\mathbb{Z}} \mathcal{H}[i]$ indicates that there are no non-zero morphisms backwards, that is, from $\mathcal{H}[n]$ to $\mathcal{H}[m]$ for $n > m$. We would like to prove the following:

THEOREM 1.1. *Let $\mathcal{H}$ be an abelian category. Then $\mathcal{D}^b(\mathcal{H}) \simeq \bigvee_{i\in\mathbb{Z}} \mathcal{H}[i]$*

Proof : It is easy to see that $\bigvee_{i\in\mathbb{Z}} \mathcal{H}[i]$ is a full subcategory of the bounded derived category $\mathcal{D}^b(\mathcal{H})$. Each object X in the full subcategory $\bigvee_{i\in\mathbb{Z}} \mathcal{H}[i]$ has the shape $X = \oplus_{n\in Z} X_n[n]$, where $X_n \in \mathcal{H}$ and only finitely many X_n are non-zero. Let $X^\bullet$ be a complex in $\mathcal{C}(\mathcal{H})$ with $X^i = 0$ for degree $i > n$

$$\cdots \to X^{n-1} \overset{d^{n-1}}{\to} X^n \to 0 \to \cdots$$

Denoting $B^i = Im\ d^{i-1}$ and $Z^i = Ker\ d^i$, we have the following exact sequences

$$0 \to B^n \overset{f}{\to} X^n \overset{h}{\to} H^n(X^\bullet) \to 0$$

$$0 \to Z^{n-1} \overset{b}{\to} X^{n-1} \overset{c}{\to} B^n \to 0$$

where f and b are the canonical inclusion maps, h is the projection and c is induced by d^{n-1}. Since $\mathcal{H}$ is an hereditary category, this latter exact sequence gives us the epimorphism

$$Ext^1(H^n(X^\bullet), X^{n-1}) \to Ext^1(H^n(X^\bullet), B^n) \to 0.$$

The diagram below shows better the relation between the morphisms involved in this computation.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 \\
 & & & B^n \arrow[ur] \arrow[dr, "f"] & \\
X^{n-2} \arrow[rr, "d^{n-2}"] \arrow[dr, "a"] & & X^{n-1} \arrow[ur, "c"] \arrow[rr, "d^{n-1}"] & & X^n \\
 & Z^{n-1} \arrow[ur, "b"] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Therefore, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Z^{n-1} \arrow[r, "1"] \arrow[d, "b"] & Z^{n-1} \arrow[d, "\overline{b}"] & 0 \arrow[d] & \\
0 \arrow[r] & X^{n-1} \arrow[r, "e"] \arrow[d, "c"] & \overline{X}^n \arrow[r, "g"] \arrow[d, "\overline{c}"] & H^n(X^\bullet) \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & B^n \arrow[r, "f"] \arrow[d] & X^n \arrow[r, "h"] \arrow[d] & H^n(X^\bullet) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact sequences in each rows and columns. So we have the following quasi-isomorphism $U^\bullet \stackrel{\alpha}{\to} X^\bullet$, where $U^\bullet$ is the first complex in the diagram below

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-3} \arrow[r] \arrow[d, "1"] & X^{n-2} \arrow[r, "(0\ a)^t"] \arrow[d, "1"] & X^{n-1} \oplus Z^{n-1} \arrow[r, "(e\ 0)"] \arrow[d, "(1\ b)"] & \overline{X}^n \arrow[r] \arrow[d, "\overline{c}"] & 0 \\
\cdots \arrow[r] & X^{n-3} \arrow[r, "d^{n-3}"] & X^{n-2} \arrow[r, "d^{n-2}"] & X^{n-1} \arrow[r, "d^{n-1}"] & X^n \arrow[r] & 0
\end{tikzcd}$$

Consider now $V^\bullet$ to be the following complex

$$\cdots \to X^{n-3} \stackrel{d^{n-3}}{\to} X^{n-2} \stackrel{a}{\to} Z^{n-1} \to 0 \to \cdots .$$

We also have the quasi-isomorphism $U^\bullet \to V^\bullet \oplus H^n(X^\bullet)[n]$:

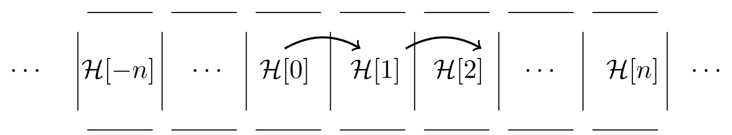
$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-3} \arrow[r] \arrow[d, "1"] & X^{n-2} \arrow[r, "(0\ a)^t"] \arrow[d, "1"] & X^{n-1} \oplus Z^{n-1} \arrow[r, "(e\ 0)"] \arrow[d, "(0\ 1)"] & \overline{X}^n \arrow[r] \arrow[d, "g"] & 0 \\
\cdots \arrow[r] & X^{n-3} \arrow[r, "d^{n-3}"] & X^{n-2} \arrow[r, "a"] & Z^{n-1} \arrow[r, "0"] & H^n(X^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Therefore we have an isomorphism in the derived category

$$\begin{tikzcd}
 & U^\bullet \arrow[dl] \arrow[dr] & \\
X^\bullet & & V^\bullet \oplus H^n(X^\bullet)[n]
\end{tikzcd}$$

We can apply again this procedure to obtain the result. □

In the picture below we have an idea of the structure of the derived category of an abelian hereditary category:



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