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New Browder-Weyl Type Theorems for Direct Sum

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ABSTRACT: In this article, we study the stability of new Browder-Weyl type theorems for orthogonal direct sum $S \oplus T$, where S and T are bounded linear operators acting on Banach spaces. We characterize preservation of properties $(W\Pi)$ and$(UW\Pi_a)$ under direct sum $S \oplus T$. Futhermore, we show if S and T satisfy property (W_E) (resp. (UW_{E_a})) under certain conditions, then $S \oplus T$ satisfies property (W_E) (resp. (UW_{E_a})) if and only if it satisfies generalized Weyl's theorem (resp. generalized a-Weyl's theorem).

Key Words:: Generalized a-Weyl's theorem, Property (UW_{E_a}), SVEP, polaroid, isoloid.

Contents

1. Introduction

Let $T \in L(X)$ be a bounded linear operator defined on an infinite-dimensional complex Banach spaces X, and denote by $\alpha(T)$ and $\beta(T)$ the dimension of the kernel $N(T)$ and the codimension of the range $R(T) = T(X)$, respectively. Denote by $\Phi_+(X) := \{T \in L(X : \alpha(T) < \infty, T(X) \text{ is closed}\}$ the class of all *upper semi-Fredholm operators*, and by $\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$, the class of all *lower semi-Fredholm* operators. If $T \in \Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, the index of T is defined by $ind(T) := \alpha(T) - \beta(T)$. If $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$, denotes the set of all Fredholm operators, the set of *Weyl operators* is defined by

$$W(X) := \{T \in \Phi(X) : ind(T) = 0\},$$

the class of *upper semi-Weyl* operators is defined by

$$W_+(X) := \{T \in \Phi_+(X) : ind(T) \leqslant 0\},$$

and the class of *lower semi-Weyl* operators is defined by

$$W_-(X) := \{T \in \Phi_-(X) : ind(T) \geqslant 0\}.$$

Clearly, $W(X) = W_+(X) \cap W_-(X)$. If T^* denotes the dual of $T \in L(X)$, it is well known that $T \in W_+(X)$ (respectively, $T \in W_-(X)$) if and only if $T^* \in W_-(X^*)$ (respectively, $T^* \in W_+(X^*)$). The above defined classes of operators generate the following spectra: the *Weyl spectrum*, defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin W(X)\},$$

the *upper semi-Weyl spectrum*, defined by

$$\sigma_{uw}(T) := \{\lambda \in \mathbb{C} \mid T - \lambda \notin W_+(X)\},$$

and the *lower semi-Weyl spectrum*, defined by

$$\sigma_{lw}(T) := \{\lambda \in \mathbb{C} \mid T - \lambda \notin W_-(X)\}.$$

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Let $p = p(T)$ be the *ascent* of an operator T, i.e the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$. If such integer does not exist we put $p(T) = \infty$. Analogously, let $q = q(T)$ be the *descent* of T, i.e the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$, and if such integer q does not exist we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 3.3]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent of T, see [20, Proposition 50.2]. The class of all *Browder operators* is defined

$$B(X) := \{T \in \Phi(X) : p(T) = q(T) < \infty, \},$$

while the class of all *upper semi-Browder operators* is defined

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty, \}.$$

Obviously, $B(X) \subseteq W(X)$ and $B_+(X) \subseteq W_+(X)$, see [1, Theorem 3.4]. If $\sigma_b(T)$ and $\sigma_{ub}(T)$ denote the *semi-Browder spectrum* and the *upper semi-Browder spectrum*, respectively, then $\sigma_w(T) \subseteq \sigma_b(T)$ and $\sigma_{uw}(T) \subseteq \sigma_{ub}(T)$.

Semi-Fredholm operators have been generalized by Berkani ([6], [7] and [12]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set T= $T_{[0]}$). $T \in L(X)$ is said to be *semi B-Fredholm*,(respectively, *B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm*) if, for some integer n, the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In the case where $T_{[n]}$ is a semi-Fredholm operator, then $T_{[n]}$ is a semi-Fredholm operator for all $m \geqslant n$ ([12]), with the same index of $T_{[n]}$. This enables one to define the index of a semi-B-Fredholm as $ind(T) = ind(T_{[n]})$. Analogously, a bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, upper semi B-Weyl, lower semi B-Weyl) if for some integer $n \geqslant 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl, lower semi-Weyl). Analogous definitions are given for semi *B-Browder operators*. The B-Weyl spectrum is defined as

$$\sigma_{bw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \; is \; not \; B - Weyl\},$$

and analogously, the upper semi B-Weyl spectrum of T is defined by

$$\sigma_{ubw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \; is \; not \; upper \; semi \; B - Weyl\}.$$

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory. In the case the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* if $p(T) = q(T) < \infty$, and this is equivalent to saying that $T = T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent, see [21, Prop. A] and [22, Corollary 2.2]. Evidently, if T is Drazin invertible then either $\lambda I - T$ is invertible or λ is a pole of the resolvent of T.

The concept of Drazin invertibility suggests the following definition: $T \in L(X)$ is said to be *left Drazin invertible* if $p = p(T) < \infty$ and $R(T^{p+1})$ is closed. The *Drazin spectrum* is then defined as

$$\sigma_d(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\},$$

while the *left Drazin spectrum* is defined as

$$\sigma_{ld}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\}.$$

In the sequel we denote by $\sigma_a(T)$ the *approximate point spectrum*, defined by $\sigma_a(T) := \{\lambda \in \mathbb{C} : \lambda I - T$ is not bounded below$\}$, where an operator T is said to be *bounded below* if it is injective and has closed range. The classical surjective spectrum of T is denoted by $\sigma_s(T)$. We say that $\lambda \in \mathbb{C}$ is a *left pole* of T if $\lambda \in \sigma_a(T)$ and $\lambda I - T$ is left Drazin invertible. Note that every left Drazin invertible operator is upper semi B-Weyl and every Drazin invertible operator is B-Weyl. Indeed, $T \in L(X)$ is left Drazin invertible if and only if is upper semi B-Browder and, analogously, T is Drazin invertible if and only if T is B-Browder, see [6], or [4].

In a recent paper [8] Berkani and the third author of this article, have enriched the family of variants of Browder-Weyl type theorem, by introducing the properties (UW_{Π_a}), (W_{Π}), (UW_{E_a}) and (W_E) for T. In this paper there new properties are investigated by using an important property of direct sum. In the third section, the counterexample which prove that property (UW_{E_a}).

In the present article, we study the preservation under direct sum of the properties (UW_{Π_a}), (W_{Π}), (UW_{E_a}) and (W_E) and the results we get may be summarized as follows. In the second section, we characterizd the presevation of property (UW_{Π_a}), under direct sum via generalize a-Browder's theorem holds for direct sum, and we obtain a similar result for property (W_{Π}). In the third section, we give the counterexample which prove that (UW_{E_a}) is not transferred in general from the direct summands $T \in L(X)$ and $S \in L(Y)$ to direct sum $S \oplus T \in L(X \oplus Y)$. Moreover, we explore certain sufficient condition which property (UW_{E_a}), will be transferred from the direct summands to the direct sum. Similarly, we show in Theorem 3.4 that if T and S both satisfies property (W_E), and are isoloid then direct sum $S \oplus T$ satisfies property (W_E) if and only if it satisfies generalize Weyl's theorem.

2. Properties (W_{Π}) and (UW_{Π_a}) for direct sums

Let $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$ be the set of all poles of T having finite rank, $\Pi(T) := \sigma(T) \setminus \sigma_d(T)$ the set of all poles of T and $p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{ub}(T)$ the set of all left poles of T having finite rank. It is easy to check that $p_{00}(T) \subseteq p_{00}^a(T)$ for all $T \in L(X)$, and that every point of $p_{00}(T)$ is an isolated point of $\sigma(T)$, and hence an isolated point of $\sigma_a(T)$, since every isolated point of the spectrum belongs to $\sigma_a(T)$. Set

$$\Delta_a(T) := \sigma_a(T) \setminus \sigma_{uw}(T) \quad \text{and} \quad \Delta_a^g(T) := \sigma_a(T) \setminus \sigma_{ubw}(T),$$

and let $\Pi_a(T) := \sigma_a(T) \setminus \sigma_{ld}(T)$ be the set of all left poles of the resolvent of T. Obviously, $p_{00}(T) \subseteq \Delta_a(T)$, since each point of $p_{00}(T)$ is an eigenvalue of T and every Browder operator is upper semi-Weyl.

In this section we give several characterizations of properties (W_{Π}), and (UW_{Π_a}). We also describe the structure of B-Weyl, upper B-Weyl spectra of direct sum which are collect by their elementary of operators, and we include some technical such we will need in later sections.

Definition 2.1 *A bounded operator $T \in L(X)$ is said to satisfy:*
- *Generalized Browder's theorem if $\sigma(T) \setminus \sigma_{bw}(T) = \Pi(T)$, [19].*
- *Generalized a-Browder's theorem if $\sigma_a(T) \setminus \sigma_{ubw}(T) = \Pi_a(T)$, [19].*
- *Property (W_{Π}) if $\sigma(T) \setminus \sigma_w(T) = \Pi(T)$, [9].*
- *Property (UW_{Π_a}) if $\sigma_a(T) \setminus \sigma_{uw}(T) = \Pi_a(T)$ [9].*

It is shown in [9] that $T \in L(X)$ satisfies property (W_{Π}) if and only if T satisfies generalized Browder's theorem and $\sigma_w(T) = \sigma_{bw}(T)$, but the converse is not true in general.
It is also proved in [9] that $T \in L(X)$ satisfies property (UW_{Π_a}) if and only if generalized a-Browder's theorem holds for T and $\sigma_{uw}(T) = \sigma_{ubw}(T)$, but not conversely.

Lemma 2.1 *If $T \in L(X)$ the following statements are equivalent:*
(i) T satifies property (W_{Π});
(ii) $\sigma_b(T) = \sigma_w(T) = \sigma_{bw}(T) = \sigma_d(T)$.

Proof: If T satifies property (W_{Π}) then from [9, Theorem 2.4], T satisfies generalized Browder's theorem (i.e $\sigma_{bw}(T) = \sigma_d(T)$) and $\sigma_w(T) = \sigma_{bw}(T)$, and from [1, Theorem 5. 15] we then obtain that the equalities desired. the converse implication is bvious. □

The next theorem describes small preservation and characterization of property (W_{Π}), holds for direct sum.

Theorem 2.1 *Let $S \in L(X)$ and $T \in L(Y)$. If property $(W\pi)$ holds for S and T, then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property $(W\pi)$;
(ii) $S \oplus T$ satisfies generalized Browder's theorem.

Proof: $(i) \Longrightarrow (ii)$ Suppose that $S \oplus T$ satisfies property $(W\pi)$. Then from [9, Theorem 2.4] we obtain that $S \oplus T$ satisfies generalized Browder's theorem.

$(ii) \Longrightarrow (i)$ Suppose now that $S \oplus T$ satisfies generalized Browder's theorem. Since property $(W\pi)$ holds for S and T then from Lemma 2.1 $\sigma_b(S) = \sigma_w(S) = \sigma_{bw}(S) = \sigma_d(S)$, and $\sigma_b(T) = \sigma_w(T) = \sigma_{bw}(T) = \sigma_d(T)$. As we know that $\sigma_d(S \oplus T) = \sigma_d(S) \cup \sigma_d(T)$, and $\sigma_b(S \oplus T) = \sigma_b(S) \cup \sigma_b(T)$, for every pair of operators, then $\sigma_b(S \oplus T) = \sigma_w(S) \cup \sigma_w(T) = \sigma_{bw}(S) \cup \sigma_{bw}(T) = \sigma_d(S \oplus T)$. On the other hand, we have always that $\sigma_w(S \oplus T) \subseteq \sigma_w(S) \cup \sigma_w(T)$ and $\sigma_{bw}(S \oplus T) \subseteq \sigma_{bw}(S) \cup \sigma_{bw}(T)$. Moreover, by assumption, $\sigma_{bw}(S \oplus T) = \sigma_d(S \oplus T)$ it then follows from [1, Theorem 5. 15] that $\sigma_b(S \oplus T) = \sigma_w(S \oplus T) = \sigma_{bw}(S \oplus T) = \sigma_d(S \oplus T)$. Which implies from Lemma 2.1, $S \oplus T$ satisfies property $(UW\pi_a)$. □

Remark 2.1 It follows from Theorem 2.2 that $S \oplus T$ satisfies property $(W\pi)$, if and only if $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$ and if and only if $\sigma_{bw}(S \oplus T) = \sigma_{bw}(S) \cup \sigma_{bw}(T)$.

Generally, if $S \in L(X)$ and $T \in L(Y)$ both satisfy property $(W\pi)$ then it is not guaranteed that their (orthogonal) direct sum $S \oplus T \in L(X \oplus Y)$ satisfies property $(W\pi)$, as proved by the following example.

Example 2.1 Let R be the unilateral right shift operator defined on $l^2(\mathbb{N})$ and L its adjoint, then property $(W\pi)$ holds for R and L, since $\sigma(R) = \sigma(L) = \sigma_w(R) = \sigma_w(R) = D(0,1)$ the closed unit disc in $\mathbb{C}$, and $\Pi(R) = \Pi(L) = \emptyset$, then property $(W\pi)$, holds for R and L. However, the property $(W\pi)$ does not hold for $R \oplus L$. Indeed, we have $0 \notin \sigma_w(R \oplus L)$ since $\alpha(R \oplus L) = \beta(R \oplus L) = 1$, and we know that $R \oplus L$ does not satisfies property $(W\pi)$ because it does not satisfy generalized Browder's theorem. Moreover, obeserve that the inclusion $\sigma_w(R \oplus L) \subset \sigma_w(R) \cup \sigma_w(L)$ is strictly, since $\sigma_w(R) \cup \sigma_w(L) = D(0,1)$ and $\sigma_w(R \oplus L) = C(0,1)$.

The following useful lemma will be needed in the sequel.

Lemma 2.2 *If $T \in L(X)$ the following statements are equivalent:*
(i) T satifies property $(UW\pi_a)$;
(ii) $\sigma_{ub}(T) = \sigma_{uw}(T) = \sigma_{ubw}(T) = \sigma_{ld}(T)$.

Proof: If T satifies property $(UW\pi)$ then from [9, Theorem 2.6], T satisfies generalized Browder's theorem (i.e $\sigma_{ubw}(T) = \sigma_{ld}(T)$) and $\sigma_{uw}(T) = \sigma_{ubw}(T)$, and from [1, Theorem 5. 38] we then obtain that the equalities desired. the converse implication is bvious. □

The precise relationship between property $(UW\pi_a)$ and generalized a-Browder's theorem for the orthogonal direct sum of T is described by the following theorem.

Theorem 2.2 *Let $S \in L(X)$ and $T \in L(Y)$. If property $(UW\pi_a)$ holds for S and T, then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property $(UW\pi_a)$;
(ii) $S \oplus T$ satisfies generalized a-Browder's theorem.

Proof: $(i) \Longrightarrow (ii)$ It is clear from [9, Theorem 2.6], if $S \oplus T$ satisfies property $(UW\pi_a)$ then it satisfies generalized a-Browder's theorem.

$(ii) \Longrightarrow (i)$ Suppose now that $S \oplus T$ satisfies generalized a-Browder's theorem. Since property $(UW\pi_a)$ holds for S and T then from Lemma 2.2 $\sigma_{ub}(S) = \sigma_{uw}(S) = \sigma_{ubw}(S) = \sigma_{ld}(S)$, and $\sigma_{ub}(T) = \sigma_{uw}(T) = \sigma_{ubw}(T) = \sigma_{ld}(T)$. As we know that $\sigma_{ld}(S \oplus T) = \sigma_{ld}(S) \cup \sigma_{ld}(T)$, and $\sigma_{ub}(S \oplus T) = \sigma_{ub}(S) \cup \sigma_{ub}(T)$, for every pair of operators, then $\sigma_{ub}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T) = \sigma_{ubw}(S) \cup \sigma_{ubw}(T) = \sigma_{ld}(S \oplus T)$. On the other hand, we have always that $\sigma_{uw}(S \oplus T) \subseteq \sigma_{uw}(S) \cup \sigma_{uw}(T)$ and $\sigma_{ubw}(S \oplus T) \subseteq \sigma_{ubw}(S) \cup \sigma_{ubw}(T)$.

Moreover, by assumption, $\sigma_{ubw}(S \oplus T) = \sigma_{ld}(S \oplus T)$ we then obtain from [1, Theorem 5. 38] that $\sigma_{ub}(S \oplus T) = \sigma_{uw}(S \oplus T) = \sigma_{ubw}(S \oplus T) = \sigma_{ld}(S \oplus T)$. Therefore, from Lemma 2.2 $S \oplus T$ satisfies property (UW_{Π_a}), □

Remark 2.2 It follows from Theorem 2.2 that $S \oplus T$ satisfies property (UW_{Π_a}), if and only if $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$ if and only if $\sigma_{ubw}(S \oplus T) = \sigma_{ubw}(S) \cup \sigma_{ubw}(T)$.

Example 2.2 Let R and L be the operators defined in Example 2.1, then property (UW_{Π_a}), holds for R and L. Since $\sigma_a(R) = \sigma_{uw}(R) = C(0,1)$ where $C(0,1)$ is the unit circle of $\mathbb{C}$. and $\sigma_a(L) = \sigma_{uw}(L) = D(0,1)$ where $D(0,1)$ is the closed unit disc in $\mathbb{C}$. However, the property (UW_{Π_a}) does not hold for $R \oplus L$, because $\sigma_a(R \oplus L) = D(0,1)$ $\sigma_{uw}(R \oplus L) = C(0,1)$ and $\Pi_a(R \oplus L) = \emptyset$. Observe that the inclusion $\sigma_{uw}(R \oplus L) \subset \sigma_{uw}(R) \cup \sigma_{uw}(L)$ is strictly , since $\sigma_{uw}(R \oplus L) = C(0,1)$ and $\sigma_{uw}(R) \cup \sigma_{uw}(L) = D(0,1)]$

3. Properties (W_E) and (UW_{E_a}) for direct sums

Recall that for every bounded operator $T \in L(X)$ we set:
$E(T)$: eigenvalues of T that are isolated in the spectrum $\sigma(T)$ of T,
$E^0(T)$: eigenvalues of T of finite multiplicity that are isolated in the spectrum $\sigma(T)$ of T,
$E_a(T)$: eigenvalues of T that are isolated in the approximate point spectrum $\sigma_a(T)$ of T,
$E_a^0(T)$: eigenvalues of T of finite multiplicity that are isolated in the spectrum $\sigma_a(T)$ of T,

Obviously, $p_{00}(T) \subseteq E^0(T) \subseteq E_a^0(T) \subseteq E_a(T) \supseteq E(T) \supseteq E^0(T)$ and $E_a(T) \supseteq \Pi_a(T) \supseteq \Pi(T) \subseteq E(T)$.

In this part we consider an operator $T \in L(X)$ and $S \in L(Y)$ are satisfying the properties of Weyl's type theorem and are isoloid or (a- isolatoid, a-polaroid) and we give a necessary and sufficient conditions for S and T which that their sum direct $S \oplus T$ does.

Definition 3.1 *A bounded operator $T \in L(X)$ is said to satisfy:*

- *Weyl's theorem if $\sigma(T) \setminus \sigma_w(T) = E^0(T)$, [19].*
- *Generalized Weyl's theorem if $\sigma(T) \setminus \sigma_{bw}(T) = E(T)$, [19].*
- *a-Weyl's theorem if $\sigma_a(T) \setminus \sigma_{uw}(T) = E_a^0(T)$, [19].*
- *Generalized a-Weyl's theorem if $\sigma_a(T) \setminus \sigma_{ubw}(T) = E_a(T)$, [19].*
- *Property (W_E) if $\sigma(T) \setminus \sigma_w(T) = E(T)$ [8].*
- *Property (UW_{E_a}) if $\sigma_a(T) \setminus \sigma_{uw}(T) = E_a(T)$ [8].*

It is proved in [11] that if $T \in L(X)$ satisfies generalized a-Weyl's theorem, then it satisfies a-Weyl's theorem, but the converse is not true in general.

It is also shown in [11] that if $T \in L(X)$ satisfies generalized Weyl's theorem, then it satisfies Weyl's theorem, but the converse is not true in general. It is well known in [11] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but not conversely. It is shown in [8] that an operator possessing property (W_E) possesses generalized Weyl's theorem, but not conversely, and it is proved also that an operator possessing property (UW_{E_a}) possesses generalized a-Weyl's theorem, but not conversely.

We begin now with the familiar of definition

Definition 3.2 *An operator $T \in L(X)$ is said to be:*

- *isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T i.e $iso\sigma(T) = E(T)$.*
- *a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T i.e $iso\sigma_a(T) = E_a(T)$.*
- *polaroid if every $\lambda \in iso\sigma(T)$ is a pole of the resolvent of T, i.e $iso\sigma(T) = \Pi(T)$.*
- *a-polaroid if every $\lambda \in iso\sigma_a(T)$ is a left pole of the resolvent of T, i.e $iso\sigma_a(T) = \Pi_a(T)$.*

In the sequel we need the following lemma.

Lemma 3.1 *If $S \in L(X)$ and $T \in L(Y)$ are :*
(i) isoloid then $S \oplus T$ is also isoloid.
(ii) polaroid then $S \oplus T$ is also polaroid.
(iii) a-isoloid then $S \oplus T$ is also a-isoloid.
(iv) a-polaroid then $S \oplus T$ is also a-polaroid

Proof: (i) We prove that $S\oplus T$ is islodoid. We know always that $\mathrm{acc}[\sigma(S)\cup\sigma(T)]=\mathrm{acc}\sigma(S)\cup\mathrm{acc}\sigma(T)$. Then

$$\begin{aligned}\mathrm{iso}\sigma(S\oplus T) &= [\mathrm{iso}[\sigma(S)\cup\sigma(T)]\\ &= [\sigma(S)\cup\sigma(T)]\setminus acc[\sigma(S)\cup\sigma(T)]\\ &= [\sigma(S)\cup\sigma(T)]\setminus[\mathrm{acc}\sigma(S)\cup\mathrm{acc}\sigma(T)]\\ &= [\mathrm{iso}\sigma(S)\cap\rho(T)]\cup[\mathrm{iso}\sigma(T)\cap\rho(S)]\cup[\mathrm{iso}\sigma(S)\cap\mathrm{iso}\sigma(T)],\end{aligned}$$

and since S and T are isoloid then $\mathrm{iso}\sigma(S\oplus T)\subseteq\sigma_p(T)\cup\sigma_p(T)$. As we know that $\sigma_p(S\oplus T)=\sigma_p(S)\cup\sigma_p(T)$, it follows that $E(S\oplus T)=\mathrm{iso}\sigma(S\oplus T)$. Therefore $S\oplus T$ is isoloid.

(ii) Suppose that S and T are polaroid. we then have that they are also isoloid such that

$$\begin{aligned}E(S\oplus T) &= \mathrm{iso}\sigma(S\oplus T)\cap\sigma_p(S\oplus T)\\ &= \mathrm{iso}[\sigma(S)\cup\sigma(T)]\cap[\sigma_p(S)\cup\sigma_p(T)]\\ &= [E(S)\cap\rho(T)]\cup[E(T)\cap\rho(S)]\cup[E(S)\cap\mathrm{iso}\sigma(T)]\cup[E(T)\cap\mathrm{iso}\sigma(S)]\\ &= [\Pi(S)\cap\rho(T)]\cup[\Pi(T)\cap\rho(S)]\cup[\Pi(S)\cap\Pi(T)]\\ &= [\sigma(S)\cup\sigma(T)]\setminus[\sigma_d(S)\cup\sigma_d(T)]\\ &= \sigma(S\oplus T)\setminus\sigma_d(S\oplus T)\\ &= \Pi(S\oplus T).\end{aligned}$$

we get from the first part that $E(S\oplus T)=\Pi(S\oplus T)=\mathrm{iso}\sigma(S\oplus T)$. Therefore $S\oplus T$ is polaroid.

(iii) We shall prove that $S\oplus T$ is a-isoloid. We have that

$$\begin{aligned}E_a(S\oplus T) &= \mathrm{iso}\sigma_a(S\oplus T)\cap\sigma_p(S\oplus T)\\ &= \mathrm{iso}[\sigma_a(S)\cup\sigma_a(T)]\cap[\sigma_p(S)\cup\sigma_p(T)]\\ &= [E_a(S)\cap\rho_a(T)]\cup[E_a(T)\cap\rho_a(S)]\cup[E_a(S)\cap\mathrm{iso}\sigma_a(T)]\cup[E_a(T)\cap\mathrm{iso}\sigma_a(S)].\end{aligned}$$

As by hypothesis we have S and T are a-isoloid we then obtain that
$E_a(S\oplus T)=[\mathrm{iso}\sigma_a(S)\cap\rho_a(T)]\cup[\mathrm{iso}\sigma_a(T)\cap\rho_a(S)]\cup[\mathrm{iso}\sigma_a(S)\cap\mathrm{iso}\sigma_a(T)=\mathrm{iso}\sigma_a(S\oplus T)$. It follows that $S\oplus T$ is a-isoloid, as desired.

(iv) We omit the proof, which is very similar to the proof of part (ii). □

A simple example shows that a polaroid or isoloid direct sum operator $S\oplus T$ need not be necessarily hereditarily polaroid or isoloid of S or T. Indeed, let $T:=R\oplus Q$ on $\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N})$, R is the right shift and Q is quasi-nilpotent. Then $\sigma(T)=D(0,1)$, so $\mathrm{iso}\sigma_a(T)$ is empty and hence T is polaroid, since Q is not polaroid and R is polaroid.

Theorem 3.1 *Suppose that generalized Weyl's theorem holds for $S\in L(X)$ and $T\in L(Y)$. If S and T are isoloid then the following insertions are equivalent:*
(i) $S\oplus T$ satisfies generalized Weyl's theorem;
(ii) $S\oplus T$ satisfies Weyl's theorem.

Proof: $(i)\Longrightarrow(ii)$ This implication follows from [11, Theorem 3.9].

$(ii)\Longrightarrow(i)$ Suppose that $S\oplus T$ satisfies Weyl's theorem that's $\sigma(S\oplus T)\setminus\sigma_w(S\oplus T)=E^0(S\oplus T)$. Then $S\oplus T$ satisfies Browder's theorem. As we know from [1, Theorem 5. 15] that Browder's theorem is equivalent to generalized Browder theorem, it follows that $S\oplus T$ satisfies generalized Browder theorem $\sigma(S\oplus T)\setminus\sigma_{bw}(S\oplus T)=\Pi(S\oplus T)$. It is sufficiently to prove that $\Pi(S\oplus T)=E(S\oplus T)$. Indeed, we have always that, $\sigma_d(S\oplus T)=\sigma_d(S)\cup\sigma_d(T)$, then

$$\begin{aligned}\Pi(S\oplus T) &= \sigma(S\oplus T)\setminus\sigma_d(S\oplus T)\\ &= [\sigma(S)\cup\sigma(T)]\setminus[\sigma_d(S)\cup\sigma_d(T)]\\ &= [\Pi(S)\cap\rho(T)]\cup[\Pi(T)\cap\rho(S)]\cup[\Pi(S)\cap\Pi(T)].\end{aligned}$$

As S and T satisfy generalized Weyl's theorem, then $\Pi(T) = E(T)$ and $\Pi(S) = E(S)$. So $\Pi(S \oplus T) = [E(S) \cap \rho(T)] \cup [E(T) \cap \rho(S)] \cup [E(S) \cap E(T)]$. On the other hand, by hypothesis S and T are isoloid, we have that $E(S \oplus T) = [E(S) \cap \rho(T)] \cup [E(T) \cap \rho(S)] \cup [E(S) \cap E(T)]$. Hence $E(S \oplus T) = \Pi(S \oplus T)$. Therefore $S \oplus T$ satisfies generalized Weyl's. □

Theorem 3.2 *Suppose that generalized a-Weyl's theorem holds for $S \in L(X)$ and $T \in L(Y)$. If S and T are a-isoloid then the following statements are equivalent:*
(i) $S \oplus T$ satisfies generalized a-Weyl's theorem;
(ii) $S \oplus T$ satisfies a-Weyl's theorem.

Proof: $(i) \Longrightarrow (ii)$ This implication follows from [11, Theorem 3.11].
$(ii) \Longrightarrow (i)$ Suppose that $S \oplus T$ satisfies a-Weyl's theorem that's $\sigma_a(S \oplus T) \setminus \sigma_{uw}(S \oplus T) = E_a^0(S \oplus T)$. Then $S \oplus T$ satisfies a-Browder's theorem. As we know from [1, theorem 5.38] that a-Browder's theorem is equivalent to generalized a-Browder theorem, it then follows that $S \oplus T$ satisfies generalized a-Browder theorem. Now it is sufficiently to prove the equality $\Pi_a(S \oplus T) = E_a(S \oplus T)$. Indeed, we have always that $\sigma_{ld}(S \oplus T) = \sigma_{ld}(S) \cup \sigma_{ld}(T)$, it then follows that

$$\begin{aligned} \Pi_a(S \oplus T) &= \sigma_a(S \oplus T) \setminus \sigma_{ld}(S \oplus T) \\ &= [\sigma_a(S) \cup \sigma_a(T)] \setminus [\sigma_{ld}(S) \cup \sigma_{ld}(T)], \qquad (1.1) \\ &= [\Pi_a(S) \cap \rho_a(T)] \cup [\Pi_a(T) \cap \rho_a(S)] \cup [\Pi_a(S) \cap \Pi_a(T)]. \end{aligned}$$

As generalized a-Weyl's theorem holds for S and T, which implies that $\Pi_a(T) = E_a(T)$ and $\Pi_a(S) = E_a(S)$. So $\Pi_a(S \oplus T) = [E_(S) \cap \rho_a(T)] \cup [E_a(T) \cap \rho_a(S)] \cup [E_a(S) \cap E_a(T)]$. On the other hand, by hypothesis S and T are a-isoloid, we infer that $E_a(S \oplus T) = [E_a(S) \cap \rho_a(T)] \cup [E_a(T) \cap \rho_a(S)] \cup [E_a(S) \cap E_a(T)]$. Hence $E_a(S \oplus T) = \Pi_a(S \oplus T)$. Therefore $S \oplus T$ satisfies generalized a-Weyl's. □

Example 3.1 Let R denote the right shift on $\ell^2(\mathbb{N})$ defined by $R(x_1, x_2, ...) := (0, x_1, x_2, ...)$ $(x_n) \in \ell^2(\mathbb{N})$, and let Q be the weighted left shift defined by $Q(x_1, x_2, ...) = (x_2/2, x_3/3, ...)(x_n) \in \ell^2(\mathbb{N})$.

We have from Example 2.2, R saisfies property $((UW_{\Pi_a})$. On On the other hand, it is easily seen iso$\sigma(R)$ = iso$\sigma_a(R) = \Pi_a(T) = \emptyset$, hence R is a-polaroid Clearly Q is quasi-nilpotent, $\sigma_a(Q) = \{0\}, \sigma_{uw}(Q)\sigma_{ubw}(Q) = \emptyset$, and $\Pi_a(Q) = E_a(Q) = \{0\}$ it then follows Q property $((UW_{\Pi_a})$ and it is not a-polaroid, since $p(Q) = \infty$, see [1, Example 4.14].

Define now $T = R \oplus Q$ on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Obviously, $\sigma(T) = D(0, 1)$, $\sigma_a(T) = \sigma_{uw}(T) = C(0, 1) \cup \{0\}$, and $E_a(T) = \{0\}$. From this we easily obtain T does not satify property $((UW_{E_a})$, since T does not a-polaroid, because $p(T) = p(R) + p(Q) = \infty$, and $0 \notin \Pi_a(T)$.

The following main result, which will be proved in presence of the polaroid condition entails there are various equivalent forms of these properties hold for the orthogonal direct sum $S \oplus T$.

Theorem 3.3 *Suppose that $S \in L(X)$ and $T \in L(Y)$ are a-polaroid. If property $((UW_{\Pi_a})$, holds for S and T then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property (UW_{E_a});
(ii) $S \oplus T$ satisfies property (UW_{Π_a});
(iii) $S \oplus T$ satisfies generalized a-Weyl's theorem.

Proof: $(i) \Longleftrightarrow (ii)$ Suppose that $S \oplus T$ satisfies property (UW_{Π_a}). As property (UW_{Π_a}) holds for S and T, it then follows from Theorem 2.2 that $S \oplus T$ satisfies generalized a-Browder's theorem. On the other hand, by hypothesis S and T are a-polaroid, then from Lemma 3.1 we obtain that $E_a(S \oplus T) = \Pi_a(S \oplus T)$. This implies by [9, Theorem 2.10], $S \oplus T$ satisfies property (UW_{E_a}). The converse implication follows from [9, Theorem 2.10].

$(ii) \Longleftrightarrow (iii)$ Suppose that $S \oplus T$ satisfies property (UW_{Π_a}). We then have from Theorem 2.1 that $S \oplus T$ satisfies generalized a-Browder's theorem. As S and T are a-polaroid, hence from Lemma 3.1 we infer that $S \oplus T$ is a-polaroid. Therefore, $S \oplus T$ satisfies generalized a-Weyl's theorem. Conversely, suppose now that $S\oplus T$ satisfies generalized a-Weyl's theorem, then $S\oplus T$ satisfies generalized a-Browder's theorem. As property (UW_{Π_a}) holds for S and T, so from Theorem 2.2 we deduce that $S \oplus T$ satisfies property (UW_{Π_a}). □

In the following Theorem, we give certain conditions on S and T to ensure that their orthogonal direct sum $S \oplus T$ obeys property (W_E).

Theorem 3.4 *Suppose that S and T are isoloid. If property $((W_E))$ holds for S and T then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property (W_Π);
(ii) $S \oplus T$ satisfies property (W_E);
(iii) $S \oplus T$ satisfies generalized Weyl's theorem.

Proof: $(i) \Longrightarrow (ii)$ Suppose that $S \oplus T$ satisfies property (W_Π). Then from Theorem 2.1 we have that $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. As S and T are isoloid, hence

$$E(S \oplus T) = [E(S) \cap \rho(T)] \cup [E(T) \cap \rho(S)] \cup [E(S) \cap E(T)].$$

On the other hand, as S and T are satisfying property (W_E), we then obtain

$$\begin{aligned} \sigma(S \oplus T) \setminus \sigma_w(S \oplus T) &= [\sigma(S) \cup \sigma(T)] \setminus [\sigma_w(S) \cup \sigma_w(T)] \\ &= [E(S) \cap \rho(T)] \bigsqcup [E(T) \cap \rho(S)] \bigsqcup [E(S) \cap E(T)]. \end{aligned}$$

Thus $E(S \oplus T) = \sigma(S \oplus T) \setminus \sigma_w(S \oplus T)$, and so $S \oplus T$ satisfies property (W_E).

$(ii) \Longrightarrow (iii)$ It is well known from [8, Theorem 2.3]that if $S \oplus T$ satisfies property (W_E), then $S \oplus T$ satisfies generalized Weyl's theorem and so the proof is complete.

$(iii) \Longrightarrow (i)$ Suppose that $S \oplus T$ satisfies generalized Weyl's theorem. Then $S \oplus T$ satisfies Browder's theorem. As property (W_E) holds for S and T, it follows that property (W_Π), from [9, Theorem 2.3]. We conclude from Theorem 2.1 that $S \oplus T$ satisfies property (W_Π). □

Similarly in Theorem 3.4, we have the following result in the case of property (UW_{E_a}).

Theorem 3.5 *Suppose that S and T are a-isoloid. If property (UW_{E_a}) holds for S and T then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property (UW_{Π_a});
(ii) $S \oplus T$ satisfies property (UW_{E_a});
(iii) $S \oplus T$ satisfies generalized a-Weyl's theorem.

Proof: $(i) \Longleftrightarrow (ii)$ Suppose that $S \oplus T$ satisfies property (UW_{Π_a}). Then from Theorem 2.2 we have that $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$. As S and T are a-isoloid then

$$E_a(S \oplus T) = [E_a(S) \cap \rho_a(T)] \cup [E_a(T) \cap \rho_a(S)] \cup [E_a(S) \cap E_a(T)].$$

On the other hand, since T and S are satisfying property (UW_{E_a}), hence

$$\begin{aligned} \sigma_a(S \oplus T) \setminus \sigma_{uw}(S \oplus T) &= [\sigma_a(S) \cup \sigma_a(T)] \setminus [\sigma_{uw}(S) \cup \sigma_{uw}(T)] \\ &= [E_a(S) \cap \rho_a(T)] \bigsqcup [E_a(T) \cap \rho_a(S)] \bigsqcup [E_a(S) \cap E_a(T)]. \end{aligned}$$

So $E_a(S \oplus T) = \sigma_a(S \oplus T) \setminus \sigma_{uw}(S \oplus T)$, and $S \oplus T$ satisfies property (UW_{E_a}).

Conversely, suppose that $S \oplus T$ satisfies property (UW_{E_a}), then from [9, Theorem 2.6] $S \oplus T$ satisfies property (UW_{Π_a}).
$(i) \Longleftrightarrow (iii)$ Suppose that $S \oplus T$ satisfies generalized a-Weyl's theorem. Since property (UW_{E_a}) holds for S and T it then follows from [9, Theorem 2.6] that $\sigma_{uw}(S) = \sigma_{ubw}(S)$ and $\sigma_{uw}(T) = \sigma_{ubw}(T)$. This implies by Theorem 2.2, that $\sigma_{uw}(S \oplus T) = \sigma_{ubw}(S \oplus T)$. Consequently, $S \oplus T$ satisfies property (W_E). The converse implication follows from [9, Theorem 2.6], and so the statements (i) and (iii) are equivalent. □

If S and T are Banach space operators satisfying property (UW_{E_a}) , then it does not necessarily entail that the direct sum $S \oplus T$ satisfies property (UW_{E_a}).

Example 3.2 Define on $\ell^2(\mathbb{N})$ the null operator $T = 0$ and be the weighted unilateral right shift defined as $S(x_1, x_2, x_3, ...) = (0, x_1/2, x_2/3, ...)$, for all $(x_n) \in \ell^2(\mathbb{N})$. Then $\sigma_a(T) = \{0\}$, $\sigma_{uw}(T) = \emptyset$, and $E_a(T) = \{0\}$. Hence $\sigma_a(T) \setminus \sigma_{uw}(T) = E_a(T)$, which we deduce that T satisfies property (UW_{E_a}). On the other hand, we have $\sigma_a(S) = \{0\}$, $\sigma_{uw}(S) = \{0\}$ and $E_a(S) = \emptyset$. It then follows that $\sigma_a(S) \setminus \sigma_{uw}(S) = E_a(S)$ and hence S satisfies property (UW_{E_a}). We define on the Hilbert space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $U = T \oplus S$. Then $\sigma_a(U) = \{0\}$, $\sigma_{uw}(U) = \{0\}$, and $E_a(U) = \{0\}$. Thus $\sigma_a(U) \setminus \sigma_{uw}(U) = E_a(U)$ and so U does not satisfy property (UW_{E_a}) . Observe that $\sigma_{uw}(U) = \sigma_{uw}(T) \cup \sigma_{uw}(S)$. while S is quasi-nilpotent and hence not polaroid.

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