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Viability for Volterra-Type Integral Equations

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ABSTRACT: We study viability of closed state constraints for nonlinear Volterra-type integral equations modeling hereditary (memory) effects. We introduce a history-aware Euler–Volterra approximation that preserves the nonlocal structure of the dynamics and use it to derive a tangency criterion ensuring viability of cylindrical constraint sets. Under continuity assumptions, the tangency condition admits a checkable form through the diagonal kernel term $F(t, t, \cdot)$.

Keywords: Volterra equations, viability, tangency condition.

Contents

1. Introduction

We address the viability problem for the Volterra-type integral equation

$$y(t) = x_0 + \int_{t_0}^{t} F(t, s, y(s))\, ds, \tag{1.1}$$

where $t_0 \in [a, b)$ and $F : [a, b) \times [a, b) \times \mathbb{R}^n \to \mathbb{R}^n$ is given for $a \le t_0 \le s \le t < b$.

Given a constraint $\mathcal{G} = [a, b) \times G$, where $G \subset \mathbb{R}^n$ is closed, we say that $\mathcal{G}$ is viable with respect to (1.1) if, for every initial condition $(t_0, x_0) \in \mathcal{G}$ and every $T > t_0$ with $[t_0, T] \subset [a, b)$, there exists at least one solution $y(\cdot)$ of (1.1) on $[t_0, T]$ such that $y(t) \in G$ for all $t \in [t_0, T]$.

In contrast with the ODE framework, the explicit dependence of F on the current time t makes (1.1) genuinely nonautonomous. Moreover, the Volterra structure introduces a memory effect through the integral term: the value $y(t)$ depends on the past trajectory $\{y(s) : s \le t\}$.

In this paper we develop a viability approach for (1.1) based on: (i) a tangency condition adapted to the Volterra structure, and (ii) a constructive Euler–Volterra approximation scheme that preserves the nonlocal kernel dependence.

The paper is organized as follows. Section 2 introduces notation and tangency. Section 3 constructs Euler–Volterra approximations and proves convergence. Section 4 establishes viability for cylindrical constraints under a (strong) tangency hypothesis. An application section may be added in Section 5.

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2. Tangency Conditions and Related Notions

The Euclidean space $\mathbb{R}^n$ is endowed with its usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. For a closed set $G \subset \mathbb{R}^n$ we define the distance

$$\operatorname{dist}(v, G) := \inf_{g \in G} \|v - g\|.$$

For $x \in \mathbb{R}^n$ and $\rho > 0$, $B(x, \rho) := \{z \in \mathbb{R}^n : \|z - x\| \le \rho\}$.

If G is nonempty, closed and convex, the metric projection $P_G : \mathbb{R}^n \to G$ is well-defined and satisfies the variational inequality

$$\langle x - P_G(x),\, g - P_G(x) \rangle \le 0 \qquad \forall g \in G.$$

Let $G \subset \mathbb{R}^n$ be closed and set $\mathcal{G} = [a, b) \times G$.

Definition 2.1 *We say that F satisfies the Volterra tangency condition to $\mathcal{G}$ at $(t, x) \in \mathcal{G}$ if*

$$\liminf_{h \to 0^+} \frac{1}{h} \operatorname{dist}\left(x + \int_t^{t+h} F(t + h, s, x)\, ds,\ G \right) = 0.$$

Remark 2.1 *For viability proofs based on* arbitrary *fine partitions, one typically needs a* strong *version (a true limit, or a uniform form on compact sets), not only a* lim inf *along subsequences. In Section 4 we will assume the* strong *version, which is still checkable via $F(t, t, \cdot)$.*

Example 2.1 (Trivial constraint) *If $G = \mathbb{R}^n$, then $\operatorname{dist}(\cdot, G) \equiv 0$, hence the tangency condition holds at every $(t, x) \in [a, b) \times \mathbb{R}^n$.*

Example 2.2 (ODE case) *Assume $F(t, s, x) = f(x)$ (no (t, s)-dependence). Let*

$$G = \{x \in \mathbb{R}^n :\ \langle \nu, x \rangle \le c\}$$

be a closed half-space and take $x \in \partial G$ (i.e. $\langle \nu, x \rangle = c$). Then

$$x + \int_t^{t+h} F(t + h, s, x)\, ds = x + \int_t^{t+h} f(x)\, ds = x + h f(x),$$

and

$$\operatorname{dist}(x + h f(x), G) = \max\{0, \langle \nu, x + h f(x) \rangle - c\} = \max\{0, h \langle \nu, f(x) \rangle\}.$$

Hence the tangency condition is equivalent to

$$\langle \nu, f(x) \rangle \le 0 \qquad \text{for all } x \in \partial G.$$

Example 2.3 ($F(t, s, x) = A(t)x$ and a ball constraint) *Assume $F(t, s, x) = A(t)x$, where $A : [a, b) \to \mathbb{R}^{n \times n}$ is continuous. Let $G = \overline{B(0, R)} = \{x \in \mathbb{R}^n : \|x\| \le R\}$ and take $\|x\| = R$. Then*

$$x + \int_t^{t+h} F(t + h, s, x)\, ds = x + \int_t^{t+h} A(t + h)x\, ds = x + hA(t + h)x.$$

Using $\|x + hA(t + h)x\|^2 = \|x\|^2 + 2h\langle x, A(t + h)x \rangle + O(h^2)$ gives

$$\operatorname{dist}\big(x + hA(t + h)x, G\big) = \big(\|x + hA(t + h)x\| - R\big)_+ = h\Big(\frac{\langle x, A(t)x \rangle}{\|x\|}\Big)_+ + o(h),$$

so tangency holds whenever

$$\langle x, A(t)x \rangle \le 0 \qquad \text{for all } \|x\| = R.$$

Example 2.4 (Genuine Volterra dependence: $F(t,s,x) = K(t,s)v$ and a half-space) *Let $v \in \mathbb{R}^n$ be fixed and let K be continuous with $K(t,s) \geq 0$. Set*

$$F(t,s,x) = K(t,s)\,v, \qquad G = \{x \in \mathbb{R}^n : \ \langle \nu, x\rangle \leq 0\}.$$

For $x \in \partial G$ we have $\langle \nu, x\rangle = 0$ and

$$x + \int_t^{t+h} F(t+h,s,x)\,ds = x + \Big(\int_t^{t+h} K(t+h,s)\,ds\Big)v.$$

Let $\alpha(h) := \int_t^{t+h} K(t+h,s)\,ds$. Then $\alpha(h) = hK(t,t) + o(h)$ as $h \to 0^+$, and

$$\operatorname{dist}(x + \alpha(h)v, G) = \max\{0, \alpha(h)\langle \nu, v\rangle\}.$$

Hence

$$\liminf_{h\to 0^+} \frac{1}{h} \operatorname{dist}\left(x + \int_t^{t+h} F(t+h,s,x)\,ds,\ G\right) = \max\{0, K(t,t)\langle \nu, v\rangle\}.$$

If $K(t,t) > 0$, the tangency condition is equivalent to $\langle \nu, v\rangle \leq 0$.

Proposition 2.1 (Sequential characterization) *Let $\Phi(h) := x + \int_t^{t+h} F(t+h,s,x)\,ds$. Then*

$$\liminf_{h\to 0^+} \frac{1}{h} \operatorname{dist}(\Phi(h), G) = 0$$

is equivalent to each of:

1. *For every $\varepsilon > 0$ and every $\delta > 0$, there exists $h \in (0,\delta)$ such that*
$$\operatorname{dist}(\Phi(h), G) \leq \varepsilon h.$$
2. *There exists $h_n \downarrow 0$ such that $\operatorname{dist}(\Phi(h_n), G)/h_n \to 0$.*
3. *There exist $h_n \downarrow 0$ and $g_n \in G$ such that $\|\Phi(h_n) - g_n\| = o(h_n)$.*

Proof: The equivalence between (1) and (2) is the standard ε–δ/subsequence characterization of $\liminf$. The equivalence between (2) and (3) follows from the definition of distance: choose $g_n \in G$ such that $\|\Phi(h_n) - g_n\| \leq \operatorname{dist}(\Phi(h_n), G) + \frac{1}{n}h_n$. □

3. A Memory-Aware Euler Scheme for Volterra Integral Equations

Fix $[t_0, T] \subset [a,b)$. Let $\pi = \{t_0 < t_1 < \cdots < t_N = T\}$ be a partition of $[t_0, T]$. Its mesh is

$$\mu_\pi := \max\{t_i - t_{i-1} : 1 \leq i \leq N\}.$$

We define the Euler–Volterra interpolant y_π as follows.
For $t \in [t_0, t_1]$ set

$$y_\pi(t) := x_0 + \int_{t_0}^{t} F(t,s,x_0)\,ds, \qquad x_1 := y_\pi(t_1).$$

For $t \in [t_1, t_2]$ set

$$y_\pi(t) := x_0 + \int_{t_0}^{t_1} F(t,s,x_0)\,ds + \int_{t_1}^{t} F(t,s,x_1)\,ds, \qquad x_2 := y_\pi(t_2).$$

In general, assume $x_0, \ldots, x_i$ are constructed. For $t \in [t_i, t_{i+1}]$ set (empty sums are 0)

$$y_\pi(t) := x_0 + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} F(t,s,x_j)\,ds + \int_{t_i}^{t} F(t,s,x_i)\,ds, \qquad t \in [t_i, t_{i+1}], \tag{3.1}$$

and define the node

$$x_{i+1} := y_\pi(t_{i+1}) = x_0 + \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} F(t_{i+1},s,x_j)\,ds. \tag{3.2}$$

Definition 3.1 *1. An* Euler–Volterra approximate solution *to* (1.1) *on* $[t_0,T]$ *is a continuous function* $y_\pi : [t_0,T] \to \mathbb{R}^n$ *built by* (3.1)–(3.2) *for some partition* π*.*

2. A function $y : [t_0,T] \to \mathbb{R}^n$ *is an* Euler–Volterra solution *to* (1.1) *if there exists a sequence* $(y_{\pi_k})_k$ *of Euler–Volterra approximate solutions such that* $\mu_{\pi_k} \to 0$ *and* $y_{\pi_k} \to y$ *uniformly on* $[t_0,T]$*.*

Proposition 3.1 (Relative compactness) *Assume that* F *is continuous and satisfies a linear growth condition: there exist constants* $c_0, c_1 \ge 0$ *such that*

$$\|F(t,s,x)\| \le c_0 + c_1\|x\|, \qquad \forall\, t_0 \le s \le t \le T,\ \forall\, x \in \mathbb{R}^n.$$

Then the family of Euler–Volterra approximate solutions $\{y_\pi\}$ *is relatively compact in* $C([t_0,T];\mathbb{R}^n)$*. In particular, any sequence* (y_{π_k}) *with* $\mu_{\pi_k} \to 0$ *admits a uniformly convergent subsequence in* $C([t_0,T];\mathbb{R}^n)$*.*

Proof: Let y_π be an Euler–Volterra approximant and define the step function

$$\bar{y}_\pi(s) := x_j \quad \text{for } s \in [t_j, t_{j+1}), \quad j = 0,\dots,N-1.$$

Then for every $t \in [t_0,T]$,

$$y_\pi(t) = x_0 + \int_{t_0}^{t} F\big(t,s,\bar{y}_\pi(s)\big)\,ds. \tag{3.3}$$

Step 1: uniform boundedness. Let $m_\pi(t) := \sup_{r\in[t_0,t]} \|y_\pi(r)\|$. Using (3.3) and the growth condition,

$$\|y_\pi(t)\| \le \|x_0\| + \int_{t_0}^{t} \big(c_0 + c_1\|\bar{y}_\pi(s)\|\big)\,ds \le \|x_0\| + c_0(T-t_0) + c_1\int_{t_0}^{t} m_\pi(s)\,ds.$$

Hence

$$m_\pi(t) \le \|x_0\| + c_0(T-t_0) + c_1\int_{t_0}^{t} m_\pi(s)\,ds.$$

By Grönwall,

$$\|y_\pi\|_\infty \le M := \big(\|x_0\| + c_0(T-t_0)\big)\,e^{c_1(T-t_0)}, \tag{3.4}$$

uniformly in π.

Step 2: equicontinuity. Set

$$\mathcal{K} := \{(t,s,x) : t_0 \le s \le t \le T,\ \|x\| \le M\}.$$

By continuity, F is uniformly continuous and bounded on $\mathcal{K}$; let $C_F := \sup_{\mathcal{K}}\|F\| < \infty$. For $t \ge t'$ we write

$$y_\pi(t) - y_\pi(t') = \int_{t'}^{t} F(t,s,\bar{y}_\pi(s))\,ds + \int_{t_0}^{t'} \Big(F(t,s,\bar{y}_\pi(s)) - F(t',s,\bar{y}_\pi(s))\Big)\,ds,$$

and deduce

$$\|y_\pi(t) - y_\pi(t')\| \le C_F|t-t'| + (T-t_0)\,\omega(|t-t'|),$$

where ω is a modulus of continuity of F in t on $\mathcal{K}$. Hence $\{y_\pi\}$ is equicontinuous. Arzelà–Ascoli yields relative compactness in $C([t_0,T];\mathbb{R}^n)$. □

Theorem 3.1 *Assume that* F *is continuous. Let* (y_{π_k}) *be Euler–Volterra approximate solutions with* $\mu_{\pi_k} \to 0$ *and* $y_{\pi_k} \to y$ *uniformly on* $[t_0,T]$*. Then* y *is a (classical) solution of* (1.1)*.*

Proof: For a fixed partition π, define $\bar{y}_\pi$ as above. By construction,

$$y_\pi(t) = x_0 + \int_{t_0}^{t} F(t, s, \bar{y}_\pi(s))\, ds, \qquad t \in [t_0, T]. \tag{3.5}$$

Step 1: $\|y_\pi - \bar{y}_\pi\|_\infty \to 0$ as $\mu_\pi \to 0$. Let $s \in [t_j, t_{j+1})$. Using (3.5) at times s and t_j,

$$y_\pi(s) - x_j = \int_{t_0}^{t_j} \Big(F(s, \tau, \bar{y}_\pi(\tau)) - F(t_j, \tau, \bar{y}_\pi(\tau)) \Big)\, d\tau + \int_{t_j}^{s} F(s, \tau, \bar{y}_\pi(\tau))\, d\tau.$$

Uniform continuity of F on a compact set containing the trajectories yields $\|y_\pi(s) - x_j\| \le (T - t_0)\omega(\mu_\pi) + C_F \mu_\pi$, hence $\|y_\pi - \bar{y}_\pi\|_\infty \to 0$.

Step 2: $\bar{y}_{\pi_k} \to y$ uniformly. Since $y_{\pi_k} \to y$ uniformly and $\|y_{\pi_k} - \bar{y}_{\pi_k}\|_\infty \to 0$, we obtain $\|\bar{y}_{\pi_k} - y\|_\infty \to 0$.

Step 3: pass to the limit in (3.5). Uniform continuity of F on a suitable compact set implies

$$\int_{t_0}^{t} \Big(F(t, s, \bar{y}_{\pi_k}(s)) - F(t, s, y(s)) \Big)\, ds \to 0 \quad \text{uniformly in } t.$$

Taking limits in (3.5) yields $y(t) = x_0 + \int_{t_0}^{t} F(t, s, y(s))\, ds$ for all $t \in [t_0, T]$. □

4. Viability and Tangency

Definition 4.1 (Viability) *Let $G \subset \mathbb{R}^n$ be closed and set $\mathcal{G} := [a, b) \times G$. We say that $\mathcal{G}$ is* viable *with respect to (1.1) if for every $(t_0, x_0) \in \mathcal{G}$ and every $T > t_0$ with $[t_0, T] \subset [a, b)$, there exists at least one solution $y : [t_0, T] \to \mathbb{R}^n$ of (1.1) such that $y(t_0) = x_0$ and $y(t) \in G$ for all $t \in [t_0, T]$.*

4.1. Standing assumptions

Fix $[t_0, T] \subset [a, b)$. We assume:

(H1) (Linear growth) $\exists\, c_0, c_1 > 0$ such that

$$\|F(t, s, x)\| \le c_0 + c_1 \|x\|, \qquad \forall\, t_0 \le s \le t \le T,\ \forall\, x \in \mathbb{R}^n.$$

(H2) (Lipschitz in x) $\exists\, L > 0$ such that

$$\|F(t, s, x) - F(t, s, z)\| \le L \|x - z\|, \qquad \forall\, t_0 \le s \le t \le T,\ \forall\, x, z \in \mathbb{R}^n.$$

(H3) (Lipschitz in t on bounded sets) $\exists\, K \ge 0$ such that

$$\|F(t', s, x) - F(t, s, x)\| \le K |t' - t|\, (1 + \|x\|), \qquad \forall\, t, t', s \in [t_0, T],\ \forall\, x \in \mathbb{R}^n.$$

(H4) (Strong Volterra tangency on G) for every $(t, x) \in [t_0, T] \times G$,

$$\lim_{h \to 0^+} \frac{1}{h}\, \mathrm{dist}\left(x + \int_{t}^{t+h} F(t + h, s, x)\, ds,\ G \right) = 0. \tag{4.1}$$

Remark 4.1 *Assumption (4.1) strengthens Definition 2.1 by requiring a true limit. This allows us to obtain estimates that hold for* all sufficiently small *step sizes, which is needed to treat arbitrary fine partitions.*

4.2. A checkable form via the diagonal kernel

Lemma 4.1 (Reduction to the diagonal term) *Assume F is continuous. Then (4.1) at $(t,x) \in [t_0,T] \times G$ is equivalent to*

$$\lim_{h\to 0^+} \frac{1}{h}\,\mathrm{dist}\big(x + h\,F(t,t,x),\ G\big) = 0. \tag{4.2}$$

If G is closed and convex, (4.2) is equivalent to

$$\langle p,\, F(t,t,x)\rangle \le 0 \qquad \forall\, p \in N_G(x),\ \forall\, x \in \partial G,$$

where $N_G(x)$ is the (convex) normal cone to G at x.

Proof: By continuity of F,

$$\frac{1}{h}\int_t^{t+h} F(t+h,s,x)\,ds \to F(t,t,x) \quad (h \to 0^+),$$

so

$$x + \int_t^{t+h} F(t+h,s,x)\,ds = x + hF(t,t,x) + o(h).$$

Using that $\mathrm{dist}(\cdot, G)$ is 1-Lipschitz yields

$$\left|\,\mathrm{dist}\Big(x + \int_t^{t+h} F(t+h,s,x)\,ds, G\Big) - \mathrm{dist}(x + hF(t,t,x), G)\right| \le o(h),$$

and dividing by h gives the equivalence. For convex G, $F(t,t,x) \in T_G(x)$ is equivalent to the normal cone inequality. □

4.3. Uniform tangency on bounded sets

Lemma 4.2 (Uniform tangency on a compact set) *Assume (H4). Let $M > 0$ and set*

$$\mathcal{K}_M := \{(t,x) : t \in [t_0,T],\ x \in G \cap \overline{B(0,M)}\}.$$

Then for every $\varepsilon > 0$ there exists $\mu_0 > 0$ such that for all $(t,x) \in \mathcal{K}_M$ and all $h \in (0,\mu_0]$,

$$\mathrm{dist}\left(x + \int_t^{t+h} F(t+h,s,x)\,ds,\ G\right) \le \varepsilon h.$$

Proof: For each $(t,x) \in \mathcal{K}_M$, by (4.1) there exists $\delta(t,x) > 0$ such that the inequality holds for all $h \in (0, \delta(t,x)]$. The family of open neighborhoods provided by these radii covers $\mathcal{K}_M$. By compactness, extract a finite subcover and take μ_0 as the minimum of the corresponding radii. □

4.4. Discrete distance estimate for Euler–Volterra nodes

Let $\pi = \{t_0 < t_1 < \cdots < t_N = T\}$ and let y_π be the Euler–Volterra interpolant, with nodes $x_i := y_\pi(t_i)$.

Lemma 4.3 *Let $G \subset \mathbb{R}^n$ be nonempty, closed and convex. Assume (H1)–(H4). Then there exist constants $\mu_* > 0$ and $\alpha > 0$ such that for every partition with $\mu_\pi \le \mu_*$ and every node $i = 0,\ldots,N$,*

$$\mathrm{dist}(x_i, G)^2 \le e^{C(T-t_0)}\,\mathrm{dist}(x_0, G)^2 + \alpha\,\mu_\pi, \tag{4.3}$$

where $C > 0$ depends only on $(L, K, c_0, c_1, T - t_0)$ (not on π). In particular, if $x_0 \in G$, then $\mathrm{dist}(x_i, G)^2 \le \alpha\mu_\pi$ for all i.

Proof: Step 1: uniform bound. By Proposition 3.1, there exists $M > 0$ such that $\|y_\pi\|_\infty \le M$ for all partitions, hence $\|x_i\| \le M$.

Step 2: node decomposition. Let $h_i := t_{i+1} - t_i$ and $\bar{y}_\pi$ be the associated step function. Evaluating (3.3) at t_{i+1} and t_i gives

$$x_{i+1} = x_i + \int_{t_i}^{t_{i+1}} F(t_{i+1}, s, x_i)\, ds + R_i,$$

where

$$R_i := \int_{t_0}^{t_i} \Big(F(t_{i+1}, s, \bar{y}_\pi(s)) - F(t_i, s, \bar{y}_\pi(s)) \Big)\, ds.$$

By (H3) and $\|\bar{y}_\pi\| \le M$,

$$\|R_i\| \le \int_{t_0}^{t_i} K h_i (1 + \|\bar{y}_\pi(s)\|)\, ds \le K h_i (T - t_0)(1 + M) =: C_R h_i.$$

Step 3: projection and uniform tangency. Let $\xi_i := P_G(x_i)$ and $d_i := \|x_i - \xi_i\| = \operatorname{dist}(x_i, G)$. Define

$$w_i := \xi_i + \int_{t_i}^{t_{i+1}} F(t_{i+1}, s, \xi_i)\, ds.$$

Since $\|\xi_i\| \le \|x_i\| \le M$, we have $(t_i, \xi_i) \in \mathcal{K}_M$. Apply Lemma 4.2 with $\varepsilon = 1$ to get $\mu_* > 0$ such that if $\mu_\pi \le \mu_*$ then $\operatorname{dist}(w_i, G) \le h_i$. Choose $\zeta_i \in G$ with $\|w_i - \zeta_i\| \le h_i$. Then

$$d_{i+1} \le \|x_{i+1} - \zeta_i\| \le \|x_{i+1} - w_i\| + h_i.$$

Moreover,

$$x_{i+1} - w_i = (x_i - \xi_i) + \int_{t_i}^{t_{i+1}} \Big(F(t_{i+1}, s, x_i) - F(t_{i+1}, s, \xi_i) \Big)\, ds + R_i.$$

Using (H2) and $\|x_i - \xi_i\| = d_i$,

$$\left\| \int_{t_i}^{t_{i+1}} \Big(F(t_{i+1}, s, x_i) - F(t_{i+1}, s, \xi_i) \Big)\, ds \right\| \le \int_{t_i}^{t_{i+1}} L d_i\, ds = L h_i d_i.$$

Hence

$$d_{i+1} \le (1 + L h_i) d_i + \|R_i\| + h_i \le (1 + L h_i) d_i + (C_R + 1) h_i.$$

Step 4: square and discrete Grönwall. A standard estimate yields

$$d_{i+1}^2 \le (1 + C_1 h_i) d_i^2 + C_2 h_i^2,$$

with constants $C_1, C_2 > 0$ independent of π. Iteration and discrete Grönwall give

$$d_i^2 \le e^{C_1 (t_i - t_0)} d_0^2 + C_2 e^{C_1 (T - t_0)} \sum_{j=0}^{i-1} h_j^2.$$

Since $\sum h_j^2 \le \mu_\pi \sum h_j \le \mu_\pi (T - t_0)$, we obtain (4.3). □

Theorem 4.1 (Viability of cylindrical constraints) *Let $G \subset \mathbb{R}^n$ be nonempty, closed and convex. Assume (H1)–(H4). Then $\mathcal{G} = [a, b) \times G$ is viable with respect to (1.1).*

Proof: Fix $(t_0, x_0) \in [a, b) \times G$ and $T > t_0$ with $[t_0, T] \subset [a, b)$. Choose partitions π_k with $\mu_{\pi_k} \to 0$ and $\mu_{\pi_k} \le \mu_*$, and let y_{π_k} be the corresponding Euler–Volterra approximants. By Proposition 3.1, extract a subsequence (still denoted y_{π_k}) converging uniformly to some $y \in C([t_0, T]; \mathbb{R}^n)$. By Theorem 3.1, y solves (1.1).

Since $x_0 \in G$, Lemma 4.3 gives $\operatorname{dist}(x_i^{(k)}, G) \to 0$ uniformly in nodes as $k \to \infty$. Using the definition of y_{π_k} and continuity, one also obtains $\sup_{t \in [t_0, T]} \operatorname{dist}(y_{\pi_k}(t), G) \to 0$. Passing to the limit and using that G is closed yields $\operatorname{dist}(y(t), G) = 0$ for all $t \in [t_0, T]$, i.e. $y(t) \in G$ on $[t_0, T]$. □

5. Application

We illustrate how the diagonal tangency criterion of Lemma 4.1 yields a *checkable design rule* for keeping a system inside a safe set under a bounded (saturated) input.

Safe set. Fix a unit vector $\nu \in \mathbb{R}^n$ ($\|\nu\| = 1$) and a threshold $R > 0$, and consider the closed half–space

$$G := \{x \in \mathbb{R}^n : \ \langle \nu, x\rangle \le R\}.$$

This models a safety requirement of the form "the ν–component of the state must not exceed R" (e.g., temperature, pressure, concentration, voltage, etc.).

Hereditary (Volterra) dynamics with saturation. Let $k : [a, b) \times [a, b) \to [0, \infty)$ be continuous, with $k(t, s) = 0$ for $s > t$, and assume that $k(\cdot, s)$ is Lipschitz on $[t_0, T]$ uniformly in s (e.g. the fading memory kernel $k(t, s) = e^{-\lambda(t-s)}$, $\lambda > 0$). Fix constants $a_0 \ge 0$ (damping), $c \ge 0$ (bias pushing in the ν direction), an input bound $U > 0$, and a gain $\kappa > 0$. Define the saturation map

$$\mathrm{sat}_U(r) := \max\{-U, \min\{U, r\}\},$$

and the kernel

$$F(t, s, x) := k(t, s)\Big(- a_0 x + c\,\nu - \mathrm{sat}_U\left(\kappa\langle \nu, x\rangle\right) \nu\Big), \qquad (t_0 \le s \le t \le T). \tag{5.1}$$

Then (1.1) becomes

$$y(t) = x_0 + \int_{t_0}^{t} k(t, s)\Big(- a_0 y(s) + c\,\nu - \mathrm{sat}_U\left(\kappa\langle \nu, y(s)\rangle\right) \nu\Big)\, ds. \tag{5.2}$$

The term $-\mathrm{sat}_U(\kappa\langle \nu, y(s)\rangle)\nu$ represents a *bounded corrective action* along the direction ν, while the t–dependence in $k(t, s)$ encodes memory.

Proposition 5.1 (Verification of (H1)–(H3)) *Assume k is bounded on $\{(t, s) : t_0 \le s \le t \le T\}$ and Lipschitz in t on $[t_0, T]$ uniformly in s. Then the kernel F defined by (5.1) satisfies the growth and Lipschitz hypotheses (H1)–(H3) on $[t_0, T]$.*

Proof: Since $|\,\mathrm{sat}_U(\cdot)| \le U$ and $\|\nu\| = 1$, we have

$$\|F(t, s, x)\| \le k(t, s)\big(a_0\|x\| + c + U\big),$$

hence linear growth (H1) follows from boundedness of k. Moreover, sat_U is 1–Lipschitz, so

$$\|F(t, s, x) - F(t, s, z)\| \le k(t, s)\Big(a_0\|x - z\| + \kappa|\langle \nu, x - z\rangle|\Big) \le k(t, s)(a_0 + \kappa)\|x - z\|,$$

yielding (H2) since k is bounded. Finally, (H3) follows from the Lipschitz dependence of k in t and the estimate $\| - a_0x + c\nu - \mathrm{sat}_U(\kappa\langle \nu, x\rangle)\nu\| \le a_0\|x\| + c + U$. □

Diagonal tangency and a design inequality. For the half–space $G = \{x : \langle \nu, x\rangle \le R\}$, the (convex) normal cone at $x \in \partial G$ is $N_G(x) = \{\lambda\nu : \lambda \ge 0\}$. Hence, by Lemma 4.1, the tangency condition reduces to the scalar inequality

$$\langle \nu,\ F(t, t, x)\rangle \le 0 \qquad \text{for all } t \in [t_0, T] \text{ and all } x \in \partial G. \tag{5.3}$$

From (5.1), using $k(t, t) \ge 0$ and $\langle \nu, x\rangle = R$ on ∂G, we compute

$$\langle \nu,\ F(t, t, x)\rangle = k(t, t)\Big(- a_0R + c - \mathrm{sat}_U(\kappa R)\Big).$$

Therefore, (5.3) is ensured by the *explicit* condition

$$\operatorname{sat}_U(\kappa R) \;\geq\; c - a_0 R. \tag{5.4}$$

Since $\operatorname{sat}_U(\kappa R) = \min\{U, \kappa R\}$ for $R > 0$, (5.4) is equivalent to

$$U \;\geq\; (c - a_0 R)_+, \qquad \kappa \;\geq\; \frac{(c - a_0 R)_+}{R}. \tag{5.5}$$

This gives a transparent *actuator sizing rule*: the maximal admissible input U and the gain κ must be large enough to compensate the outward bias $c - a_0 R$ at the boundary.

Theorem 5.1 (Safe design under bounded input) *Assume G is the half–space above, $k \geq 0$ is continuous and satisfies the regularity of Proposition 5.1. If the design inequality (5.5) holds, then the set $\mathcal{G} = [a, b) \times G$ is viable with respect to (5.2) on $[t_0, T]$: for every $x_0 \in G$ there exists a solution $y(\cdot)$ of (5.2) on $[t_0, T]$ such that*

$$\langle \nu, y(t) \rangle \leq R \qquad \forall\, t \in [t_0, T].$$

Proof: By Proposition 5.1, assumptions (H1)–(H3) hold. Condition (5.5) implies $\langle \nu, F(t, t, x) \rangle \leq 0$ for all $x \in \partial G$ and $t \in [t_0, T]$, hence (H4) holds by Lemma 4.1. The conclusion follows from Theorem 4.1. □

Remark 5.1 (Interpretation)

- *If $c \leq a_0 R$, then the uncontrolled drift is already inward (or neutral) at the boundary, and any $U > 0$ works.*
- *If $c > a_0 R$, then the boundary is outward–pushing. Inequality (5.5) quantifies the minimum control authority U (and/or gain κ) needed to keep the state safe.*
- *The memory kernel $k(t, s)$ affects transient behavior but not the* sign condition *at the boundary, since the tangency test depends only on the diagonal $k(t, t)$ (here $k(t, t) \geq 0$).*

Conclusion

We proposed a memory-consistent Euler–Volterra construction and derived a practical tangency condition, based on the diagonal kernel term $F(t, t, \cdot)$, ensuring viability of closed cylindrical constraints for Volterra integral equations. This provides a checkable tool for constrained hereditary models.

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