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Existence of Positive Solutions to a System of Two Coupled Second Order Dynamic Equations with Nonlocal Boundary Conditions on Time Scales

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ABSTRACT: The main aim of this paper is to show some existence results for a system of coupled second order dynamic equations with non local boundary conditions on time scales .The important key tool used in our proofs is the Guo-krasnoselskii's fixed point theorem in a cone.

Key Words: Positive solution, time scale, Guo-Krasnoselskii fixed point theorem, cone, fixed point, non local conditions.

Contents

1. Introduction

In the present paper; we are interested in the study of the existence of positive solutions, in the sense of time scale theory, of the following dynamic system :

$$\begin{cases} x^{\Delta\nabla}(t) + \lambda_1 w_1(t) f_1(x(t), y(t)) = 0 & ; t \in (0,T)_{\mathbb{T}} \\ y^{\Delta\nabla}(t) + \lambda_2 w_2(t) f_2(x(t), y(t)) = 0 & ; t \in (0,T)_{\mathbb{T}} \\ x(0) = \beta_1 x(\theta), \quad x(T) = \alpha_1 x(\theta) \\ y(0) = \beta_2 y(\theta), \quad y(T) = \alpha_2 y(\theta) \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale with $0, T \in \mathbb{T}$;$\lambda_1, \lambda_2 > 0$ are positive parameters ,$\alpha_1, \alpha_2 > 0$,$\beta_1, \beta_2 > 0$, $0 < \theta < T$, $\alpha_i \theta < T$ for $i = 1, 2$.By a positive solution to problem (1.1), we mean a vector-valued function $(x, y) \in C^1_{ld}([0,T]_{\mathbb{T}}, \mathbb{R}^+) \times C^1_{ld}([0,T]_{\mathbb{T}}, \mathbb{R}^+)$ satisfying (1.1) with $x + y > 0$ in $[0,T]_{\mathbb{T}}$
The study of the existence of positive solutions to nonlinear differential equations under various boundary conditions has attracted considerable interest in recent years, because these boundary value problems arise in applications where only positive solutions make sense.For example,Ma[13] considered the problem

$$\begin{cases} u''(t) + \lambda a(t) f(u(t)) = 0 & ; t \in (0,1) \\ u(0) = \beta u(\eta) & ; u(1) = \alpha u(\eta) \end{cases}$$

with $0 < \eta < 1$ and $0 < \alpha < \dfrac{1}{\eta}$ and gave intervals on λ in which this problem had at least one positive solution u(t). when $\beta = 0$ this problem was studied by Ma[14] and Raffoul[16]; for more details; we refer the reader to the monograph by Agarwal;O'Regan and Wong [1].More recently, interest in obtaining the existence of positive solutions for systems has grown, and we cite as recent references the papers of Benchohra el al.[2],Henderson and Ntouyas[4,5,6],Henderson,Ntouyas and Purnaras[7],Henderson and Wang [8,9], Hu and Wang [10],Liu,Liu and Wu [11],Luo and Ma [12], R.Ma [15],Wang [17] ;Zhou and Xu [18].For instance,Henderson et al [9] considerd the four-point problem :

$$\begin{cases} u''(t) + \lambda a(t) f(v(t)) = 0 & ; t \in (0,1) \\ v''(t) + \mu b(t) g(u(t)) = 0 & ; t \in (0,1) \\ u(0) = \alpha u(\xi) & ; u(1) = \beta u(\eta) \\ v(0) = \alpha v(\xi) & ; v(1) = \beta v(\eta) \end{cases}$$

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with $0 < \xi < \eta < 1$ and $0 \leq \alpha, \beta < 1$; they too obtained the existence of at least one positive solution $(u(t), v(t))$ of this problem.
Our interest in this paper is to extend the results of Ma [13] and Henderson [9] to dynamic system of the form (1.1) on time scales .
Now,we make the following assumptions:
$(C_1) : w_1, w_2 \in C_{ld}\left([0,T]_{\mathbb{T}}, \mathbb{R}^+\right)$ and there exits $t_0, t_1 \in [0,T]_{\mathbb{T}}$ such that $w_1(t_0) > 0$ and $w_2(t_1) > 0$.
(C_2) : $f_1, f_2 \in C_{ld}\left([0,\infty)_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}}, [0,\infty)\right)$ *and there exist non − negative constants* $f_1^0, f_2^0, F_1^\infty; F_2^\infty$ *such that*

$$f_1^0 = \lim_{x+y \longrightarrow 0^+} \frac{f_1(x,y)}{x+y}, \quad F_1^\infty = \lim_{x+y \longrightarrow +\infty} \frac{f_1(x,y)}{x+y}$$

$$f_2^0 = \lim_{x+y \longrightarrow 0^+} \frac{f_2(x,y)}{x+y}, \quad F_2^\infty = \lim_{x+y \longrightarrow +\infty} \frac{f_2(x,y)}{x+y}$$

Our approach is based on the following theorem: Guo-Kranoselskii fixed point theorem in a cone.

Theorem 1.1 *([3]): Let E be a Banach space and $K \subset E$ a cone in E.Assume that Ω_1 and Ω_2 are two open bounded subsets of E such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$.Let $L : K \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow K$ be a completely continuous operator such that either :*
(i) $\| Lu \| \leq \| u \|$ for $u \in K \cap \partial\Omega_1$, and $\| Lu \| \geq \| u \|$ for $u \in K \cap \partial\Omega_2$ or
(ii) $\| Lu \| \geq \| u \|$ for $u \in K \cap \partial\Omega_1$, and $\| Lu \| \leq \| u \|$ for $u \in K \cap \partial\Omega_2$.
Then L has at least one fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

This article is arranged as follows. In section 2, we introduced some basic time scale definitions and state and prove several preliminary lemmas needed in later sections , in section 3, by using the Guo-Krasnoselskii fixed point theorem in a cone, we prove the existence of at least one positive solution for the problem (1.1)

2. Preliminaries and Basic Lemmas

In this section, we shall recall some basic definitions and lemmas which are used in what follows. Let $\mathbb{T}$ be a non-empty closed subset (time scale) of $\mathbb{R}$. the forward and backward jump operators $\sigma, \rho : \mathbb{T} \longrightarrow \mathbb{T}$ and the forward and backward graininess functions $\mu, \nu : \mathbb{T} \longrightarrow \mathbb{R}^+$ are defined, respectively,

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

$$\mu(t) = \sigma(t) - t,$$

$$\nu(t) = t - \rho(t).$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$.If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$ and If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is right-dense continuous (rd-continuous) provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$ and f is left-dense continuous(ld-continuous)if f is continuous at such lefft-dense point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. If f is continuous at each right-dense points and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$ and the set of continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted by $C(\mathbb{T}, \mathbb{R})$).The set of ld-continuous (rd-continuous) functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$ $(C_{rd}(\mathbb{T}, \mathbb{R})))$.
Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$,then the Δ-derivative of f at the point t is defined to be the number $f^\Delta(t)$, provided it exists, with the property that for each $\epsilon > 0$ there is a neighbour V of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)\,(\sigma(t) - s)\,| \leqslant |\sigma(t) - s|, \forall s \in V$$

Similarly,for $t \in \mathbb{T}_k$,the ∇-derivative of f at the point t is defined to be the number $f^{\nabla}(t)$,provided it exists, with property that for each $\epsilon > 0$ there is a neighbourd U of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)\left(\rho(t) - s\right) \right| \leqslant |\rho(t) - s|, \forall s \in U$$

if f is ld-continous, then there exist a function F such that $F^{\nabla}(t) = f(t)$ and in this case, it is defined that

$$\int_a^b f(t)\nabla t = F(b) - F(a)$$

Similarly, if f is rd-continous, then there exist a function G such that $G^{\Delta}(t) = f(t)$ .In this case, it is defined that

$$\int_a^b f(t)\Delta t = G(b) - G(a)$$

The set $C_{ld}\left([0,T]_{\mathbb{T}}, \mathbb{R}\right)$ is a Banach space with the sup norm :

$$\| f \| = \sup_{t \in [0,T]_{\mathbb{T}}} |f(t)|$$

Now, we state and prove several lemmas that will be used in the proofs of our main results .

Lemma 2.1 *([12,13]): Let* $\beta \neq \dfrac{T - \alpha\eta}{T - \eta}$*.Then for* $x \in C_{ld}\left([0,T]_{\mathbb{T}}, \mathbb{R}\right)$*, the problem*

$$\begin{cases} y^{\Delta\nabla}(t) + x(t) & = 0; t \in [0,T]_{\mathbb{T}} \\ y(0) & = \beta y(\eta) \\ y(T) & = \alpha y(\eta) \end{cases} \tag{2.1}$$

has a unique solution

$$y(t) = -\int_0^t (t-s)x(s)\nabla s + \frac{(\beta - \alpha)t - \beta T}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^\eta (\eta - s)x(s)\nabla s + \frac{(1-\beta)t + \beta\eta}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^T (T-s)x(s)\nabla s$$

Proof: From (2.1), we have

$$y(t) = y(0) + y^{\Delta}(0)t - \int_0^t (t-s)x(s)\nabla s$$

since $y(0) = \beta y(\eta)$ and $y(T) = \alpha y(\eta)$ we obtain

$$\begin{cases} (1-\beta)y(0) - \beta\eta y^{\Delta}(0) & = -\beta\int_0^\eta (\eta - s)x(s)\nabla s \\ (1-\alpha)y(0) - (T - \alpha\eta)y^{\Delta}(0) & = \int_0^T (T-s)x(s)\nabla s - \alpha\int_0^\eta (\eta - s)x(s)\nabla s \end{cases}$$

therefore

$$\begin{cases} y(0) & = \dfrac{\beta\eta}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^T (T-s)x(s)\nabla s - \dfrac{\beta T}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^\eta (\eta - s)x(s)\nabla s \\ y^{\Delta}(0) & = \dfrac{(1-\beta)}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^T (T-s)x(s)\nabla s - \dfrac{(\alpha - \beta)}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^\eta (\eta - s)x(s)\nabla s \end{cases}$$

From which it follow that

$$\begin{aligned} y(t) &= \frac{\beta\eta}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^T (T-s)x(s)\nabla s - \frac{\beta T}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^\eta (\eta - s)x(s)\nabla s \\ &\quad + \frac{(1-\beta)t}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^T (T-s)x(s)\nabla s - \frac{(\alpha - \beta)t}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^\eta (\eta - s)x(s)\nabla s \\ &\quad - \int_0^t (t-s)x(s)\nabla s \\ &= -\int_0^t (t-s)x(s)\nabla s + \frac{(\beta - \alpha)t - \beta T}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^\eta (\eta - s)x(s)\nabla s + \frac{(1-\beta)t - \beta\eta}{(T - \alpha\eta) - \beta(T - \eta)}\int_0^T (T-s)x(s)\nabla s. \end{aligned}$$

The function x presented above is a solution to the problem (2.1), and the uniqueness of x is obvious. □

Lemma 2.2 *([12,13]): Let* $0 < \alpha < \dfrac{T}{\eta}, \quad 0 \le \beta < \dfrac{T - \alpha\eta}{T - \eta}$
If $x \in C_{ld}\left([0,T]_{\mathbb{T}}, [0,\infty)\right)$*, then the unique solution* y *of the problem (2.1)satisfies*

$$y(t) \ge 0, \quad t \in [0,T]_{\mathbb{T}}$$

Proof: Since $y^{\Delta\nabla}(t) = -x(t) \le 0$, then the graph of y is concave down on $[0,T]_{\mathbb{T}}$ so

$$\frac{y(\eta) - y(0)}{\eta} \ge \frac{y(T) - y(0)}{T}$$

Since $y(0) = \beta y(\eta)$ and $y(T) = \alpha y(\eta)$, we obtain

$$\frac{1-\beta}{\eta} y(\eta) \ge \frac{\alpha - \beta}{T} y(\eta).$$

If $y(0) < 0$, then $y(\eta) < 0$, it implies that $\beta \ge \dfrac{T - \alpha\eta}{T - \eta}$, contradiction to $\beta \le \dfrac{T - \alpha\eta}{T - \eta}$.
If $y(T) < 0$, then $y(\eta) < 0$, and same contradiction emerges.Thus, it is true that $y(0) \ge 0, y(T) \ge 0$; and since y is concave, we have $y(t) \ge 0, \quad t \in [0,T]_{\mathbb{T}}$ as required □

Lemma 2.3 *([12,13]): Let* $0 < \alpha < \dfrac{T}{\eta}, \quad 0 \le \beta < \dfrac{T - \alpha\eta}{T - \eta}$
If $x \in C_{ld}\left([0,T]_{\mathbb{T}}, [0,\infty)\right)$*, then the unique solution* y *of the problem (2.1)satisfies*

$$\min_{t \in [0,T]_{\mathbb{T}}} y(t) \ge \gamma \parallel y \parallel$$

where

$$\gamma = \min\left\{ \frac{\alpha(T-\eta)}{T - \alpha\eta}; \frac{\alpha\eta}{T}; \frac{\beta\eta}{T}; \frac{\beta(T-\eta)}{T} \right\}$$

Proof: Since $y^{\Delta\nabla}(t) = -x(t) \le 0$, then the graph of y is concave down on $[0,T]_{\mathbb{T}}$.We divide the proof into two cases.
Case 1: $0 < \alpha < 1$
then $\dfrac{T - \alpha\eta}{T - \eta} > \alpha$,for $y(0) = \beta y(\eta) = \dfrac{\beta}{\alpha} y(T)$; it may develop in the following two possible directions.
(i) $0 < \alpha \le \beta$.It implies that $y(0) \ge y(T)$, so

$$\min_{t \in [0,T]_{\mathbb{T}}} y(t) = y(T)$$

Assume $\parallel y \parallel = y(t_1); t_1 \in [0,T)_{\mathbb{T}}$, then either $0 \le t_1 \le \eta < \rho(T)$, or $0 < \eta < t_1 < T$.
If $0 \le t_1 \le \eta < \rho(T)$, then

$$\begin{aligned} y(t_1) &\le y(T) + \frac{y(T) - y(\eta)}{T - \eta}(t_1 - T) \\ &\le y(T) - \frac{y(T) - y(\eta)}{T - \eta} T \\ &= \frac{T - \alpha\eta}{\alpha(T-\eta)} y(T), \end{aligned}$$

from wich it follow that

$$\min_{t \in [0,T]_{\mathbb{T}}} y(t) \ge \frac{\alpha(T-\eta)}{T - \alpha\eta} \parallel y \parallel$$

If $0 < \eta < t_1 < T$, from $\frac{y(\eta)}{\eta} \geq \frac{y(t_1)}{t_1} \geq \frac{y(t_1)}{T}$,together with $y(T) = \alpha y(\eta)$,we have $y(T) \geq \frac{\alpha\eta}{T} y(t_1)$ so that,

$$\min_{t\in[0,T]_{\mathbb{T}}} y(t) \geq \frac{\alpha\eta}{T} \parallel y \parallel .$$

(ii) $0 < \beta \leq \alpha$.It implies that $y(0) \leq y(T)$, so

$$\min_{t\in[0,T]_{\mathbb{T}}} y(t) = y(0)$$

Assume $\parallel y \parallel = y(t_2); t_2 \in]0, T]_{\mathbb{T}}$, then either $0 \leq t_2 < \eta < \rho(T)$, or $0 < \eta \leq t_2 \leq T$.
If $0 \leq t_2 < \eta < \rho(T)$, from $\frac{y(\eta)}{T-\eta} \geq \frac{y(t_2)}{T-t_2} \geq \frac{y(t_2)}{T}$,together with $y(0) = \beta y(\eta)$,we have $y(0) \geq \frac{\beta(T-\eta)}{T} y(t_2)$, hence

$$\min_{t\in[0,T]_{\mathbb{T}}} y(t) \geq \frac{\beta(T-\eta)}{T} \parallel y \parallel$$

If $0 < \eta \leq t_2 \leq T$, from

$$\frac{y(\eta)}{\eta} \geq \frac{y(t_2)}{t_2} \geq \frac{y(t_2)}{T}$$

,together with $y(0) = \beta y(\eta)$,we have $y(0) \geq \frac{\beta\eta}{T} y(t_2)$ so that,

$$\min_{t\in[0,T]_{\mathbb{T}}} y(t) \geq \frac{\beta\eta}{T} \parallel y \parallel .$$

Case 2: $\frac{T}{\eta} > \alpha \geq 1$
then $\frac{T-\alpha\eta}{T-\eta} \leq \alpha$.In this case ; $\beta < \alpha$ is true .It implies that $y(0) \leq y(T)$.So, $\min_{t\in[0,T]_{\mathbb{T}}} y(t) = y(0)$.Assume $\parallel y \parallel = y(t_2); t_2 \in]0, T]_{\mathbb{T}}$ again .Since $\alpha \geq 1$, it is know that $y(\eta) \leq y(T)$; together with the concavity of y, we have $0 < \eta \leq t_2 \leq T$.Similar to the above discussion ,

$$\min_{t\in[0,T]_{\mathbb{T}}} y(t) \geq \frac{\beta\eta}{T} \parallel y \parallel .$$

Summing up, we have

$$\min_{t\in[0,T]_{\mathbb{T}}} y(t) \geq \gamma \parallel y \parallel$$

where

$$0 < \gamma = \min\left\{\frac{\alpha(T-\eta)}{T-\alpha\eta}; \frac{\alpha\eta}{T}; \frac{\beta\eta}{T}; \frac{\beta(T-\eta)}{T}\right\} < 1$$

□

Lemma 2.4 *([12,13]): Let $\alpha\eta \neq T; \beta > max\left\{\frac{T-\alpha\eta}{T-\eta}; 0\right\}$*
If $x \in C_{ld}([0, T]_{\mathbb{T}}; \mathbb{R}^+)$, then problem (2.1) has a non negative solution

Proof: Suppose that problem(2.1) has a non negative solution y satisfiying $y(t) \geqslant> 0; t \in [0, T]_{\mathbb{T}}$ and there is a $t_0 \in (0, T)_{\mathbb{T}}$ such that $y(t_0) > 0$.
If $y(T) > 0$, then $y(\eta) > 0$ and then we obtain

$$y(0) = \beta y(\eta) > \frac{T-\alpha\eta}{T-\eta} y(\eta) = \frac{Ty(\eta) - \eta y(T)}{T-\eta}$$

that is

$$\frac{y(T)-y(0)}{T} > \frac{y(\eta)-y(0)}{\eta}$$

which is a contradiction to the concavity of y.
If $y(T)=0$,then $y(\eta)=0$,when $t_0 \in (0,\eta)_{\mathbb{T}}$ we have $y(t_0) > y(\eta) = y(T)$, a violation of the concavit of y.When $t_0 \in (\eta,T)_{\mathbb{T}}$ we get $y(0)=\beta y(\eta) = 0 = y(\eta) < y(t_0)$, another violation of the concavit of y. therefore non negative solutions exist.

□

Note that (x,y) is solution of (1.1), if and only if

$$\begin{aligned} x(t) &= \lambda_1\Big[-\int_0^t (t-s)a_1(s)f_1(x(s),y(s))\nabla s + \frac{(\beta_1-\alpha_1)t-\beta_1 T}{(T-\alpha_1\eta)-\beta_1(T-\eta)}\int_0^\eta (\eta-s)a_1(s)f_1(x(s),y(s))\nabla s \\ &\quad + \frac{(1-\beta_1)t-\beta_1\eta}{(T-\alpha_1\eta)-\beta_1(T-\eta)}\int_0^T (T-s)a_1(s)f_1(x(s),y(s))\nabla s\Big] \\ &= : L_1(x,y)(t) \end{aligned}$$

and

$$\begin{aligned} y(t) &= \lambda_2\Big[-\int_0^t (t-s)a_2(s)f_2(x(s),y(s))\nabla s + \frac{(\beta_2-\alpha_2)t-\beta_2 T}{(T-\alpha_2\eta)-\beta_2(T-\eta)}\int_0^\eta (\eta-s)a_2(s)f_2(x(s),y(s))\nabla s \\ &\quad + \frac{(1-\beta_2)t+\beta_2\eta}{(T-\alpha_2\eta)-\beta_2(T-\eta)}\int_0^T (T-s)a_2(s)f_2(x(s),y(s))\nabla s\Big] \\ &= : L_2(x,y)(t) \end{aligned}$$

We will take E to be the Banach space $C_{ld}([0,T]_{\mathbb{T}},\mathbb{R}) \times C_{ld}([0,T]_{\mathbb{T}},\mathbb{R})$ endowed with the norm $\|(x,y)\| = \|x\| + \|x\|$.Let

$$\gamma_3 = \min\{\gamma_1,\gamma_2\}$$

where

$$\gamma_i = \min\Big\{\frac{\alpha_i(T-\eta)}{T-\alpha_i\eta};\frac{\alpha_i\eta}{T};\frac{\beta_i\eta}{T};\frac{\beta_i(T-\eta)}{T}\Big\}\ for\ i=1,2$$

and define a cone in E as $P_1 \times P_2$ where

$$P_i = \{v : v \in C_{ld}([0,T]_{\mathbb{T}},[0,\infty)), \quad \min_{t\in[0,T]_{\mathbb{T}}} v(t) \geq \gamma_i \|v\|\} \quad i=1,2$$

Then ,(x,y) is a positive solution of (1.1) if and only if it is a fixed point of the operator

$$L : P_1 \times P_2 \longrightarrow E, \quad L = (L_1,L_2)$$

By lemmas 2.2 and 2.3, we know that

$$L(P_1\times P_2) = \big(L_1(P_1\times P_2), L_2(P_1\times P_2)\big) \subset P_1 \times P_2.$$

and it is easy to verify that $L : P_1\times P_2 \longrightarrow P_1\times P_2$ is completely continuous since $L_1 : P_1\times P_2 \longrightarrow P_1$ and $L_2 : P_1 \times P_2 \longrightarrow P_2$ are completely continuous.

3. Main Results

Throughout this paper, we denote:

$$\begin{aligned} \Lambda_1 &= \frac{T+\beta_1(T+\theta)}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^T (T-s)a_1(s)\nabla s, \\ \Lambda_2 &= \frac{T+\beta_2(T+\theta)}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^T (T-s)a_2(s)\nabla s, \\ \widetilde{\Lambda_1} &= \frac{\beta_1(T-\theta)}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^\theta s a_1(s)\nabla s, \\ \widetilde{\Lambda_2} &= \frac{\beta_2(T-\theta)}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^\theta s a_2(s)\nabla s. \end{aligned}$$

Theorem 3.1 *Suppose that* $(C_1)-(C_2)$ *hold ,* $0<\alpha_i<\dfrac{T}{\theta}$*, and* $0\le\beta_i<\dfrac{T-\alpha_i\theta}{T-\theta}$ *for* $i=1,2$ *and let* p *and* q *be two positive numbers such that*

$$\frac{1}{p}+\frac{1}{q}=1$$

(a) If $\Lambda_1 f_1^0<\gamma_3\widetilde{\Lambda_1}F_1^\infty$ *and* $\Lambda_2 f_2^0<\gamma_3\widetilde{\Lambda_2}F_2^\infty$*, then for each*

$$(\lambda_1,\lambda_2)\in\Big(\frac{1}{p\gamma_3\widetilde{\Lambda_1}F_1^\infty},\frac{1}{p\Lambda_1 f_1^0}\Big)\times\Big(\frac{1}{q\gamma_3\widetilde{\Lambda_2}F_2^\infty},\frac{1}{q\Lambda_2 f_2^0}\Big)$$

then the system (1.1) has at least one positive solution.
(b) If $f_1^0=f_2^0=0$ *and* $0<F_1^\infty,F_2^\infty<\infty$*, then for each*

$$(\lambda_1,\lambda_2)\in\Big(\frac{1}{p\gamma_3\widetilde{\Lambda_1}F_1^\infty},\infty\Big)\times\Big(\frac{1}{q\gamma_3\widetilde{\Lambda_2}F_2^\infty},\infty\Big)$$

then the system (1.1) has at least one positive solution.
(c) If $f_1^0=f_2^0=0$ *and* $F_1^\infty=F_2^\infty=\infty$*, then for each*

$$(\lambda_1,\lambda_2)\in\Big(0,\infty\Big)\times\Big(0,\infty\Big)$$

then the system (1.1) has at least one positive solution.
(d) If $F_1^\infty=F_2^\infty=\infty$ *and* $0<f_1^0,f_2^0<\infty$*, then for each*

$$(\lambda_1,\lambda_2)\in\Big(0,\frac{1}{p\Lambda_1 f_1^0}\Big)\times\Big(0,\frac{1}{q\Lambda_2 f_2^0}\Big)$$

then the system (1.1) has at least one positive solution.

Proof: the proof of (b),(c)and (d) is similar to the proof of (a) ; so we omit the details ; we only prove (a).
Let $(\lambda_1,\lambda_2)\in\Big(\frac{1}{p\gamma_3\widetilde{\Lambda_1}F_1^\infty},\frac{1}{p\Lambda_1 f_1^0}\Big)\times\Big(\frac{1}{q\gamma_3\widetilde{\Lambda_2}F_2^\infty},\frac{1}{q\Lambda_2 f_2^0}\Big)$ and choose $\epsilon>0$ such that

$$\frac{1}{p\gamma_3\widetilde{\Lambda_1}(F_1^\infty-\epsilon)}<\lambda_1<\frac{1}{p\Lambda_1(f_1^0+\epsilon)} \tag{3.1}$$

$$\frac{1}{q\gamma_3\widetilde{\Lambda_2}(F_2^\infty-\epsilon)}<\lambda_2<\frac{1}{q\Lambda_2(f_2^0+\epsilon)} \tag{3.2}$$

By definition of f_1^0,f_2^0, there existes $R_1>0$ such that

$$f_1(x,y)\le(f_1^0+\epsilon)(x+y)\quad\textbf{for } x,y\ge0\quad\textbf{with}\quad x+y\in[0,R_1]$$

$$f_2(x,y)\le(f_2^0+\epsilon)(x+y)\quad\textbf{for } x,y\ge0\quad\textbf{with}\quad x+y\in[0,R_1]$$

Let $(x,y)\in P_1\times P_2$ with $\| (x,y) \|= R_1$; according to the two aboe inequalities (3.1) and (3.2), we obtain

$$\begin{aligned}
L_1(x,y)(t) &= \lambda_1\Big[-\int_0^t (t-s)a_1(s)f_1(x(s),y(s))\nabla s+\frac{(\beta_1-\alpha_1)t-\beta_1 T}{(T-\alpha_1\eta)-\beta_1(T-\theta)}\int_0^\theta(\theta-s)a_1(s)f_1(x(s),y(s))\nabla s\\
&\quad+\frac{(1-\beta_1)t-\beta_1\theta}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^T (T-s)a_1(s)f_1(x(s),y(s))\nabla s\Big]\\
&\le \frac{\lambda_1\beta_1 t}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^\theta(\theta-s)a_1(s)f_1(x(s),y(s))\nabla s\\
&\quad+\frac{\lambda_1(t+\beta_1\theta)}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^T (T-s)a_1(s)f_1(x(s),y(s))\nabla s\\
&\le \frac{\lambda_1\beta_1 T}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^T (T-s)a_1(s)f_1(x(s),y(s))\nabla s\\
&\quad+\frac{\lambda_1(T+\beta_1\theta)}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^T (T-s)a_1(s)f_1(x(s),y(s))\nabla s\\
&= \frac{\lambda_1(T+\beta_1 T+\beta_1\theta)}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^T (T-s)a_1(s)f_1(x(s),y(s))\nabla s\\
&\le \frac{\lambda_1(T+\beta_1 T+\beta_1\theta)}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^T (T-s)a_1(s)(f_1^0+\epsilon)(x(s)+y(s))\nabla s\\
&\le \lambda_1\Lambda_1(f_1^0+\epsilon)\,\| (x,y) \|\\
&\le \frac{1}{p}\,\| (x,y) \|
\end{aligned}$$

As result

$$\| L_1(x,y) \|\le \frac{1}{p}\,\| (x,y) \| \tag{3.3}$$

Similarly, we have

$$\begin{aligned}
L_2(x,y)(t) &= \lambda_2\Big[-\int_0^t (t-s)a_2(s)f_2(x(s),y(s))\nabla s+\frac{(\beta_2-\alpha_2)t-\beta_2 T}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^\theta(\theta-s)a_2(s)f_2(x(s),y(s))\nabla s\\
&\quad+\frac{(1-\beta_2)t-\beta_2\theta}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^T (T-s)a_2(s)f_2(x(s),y(s))\nabla s\Big]\\
&\le \frac{\lambda_2\beta_2 t}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^\theta(\theta-s)a_2(s)f_2(x(s),y(s))\nabla s\\
&\quad+\frac{\lambda_2(t+\beta_2\theta)}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^T (T-s)a_2(s)f_2(x(s),y(s))\nabla s\\
&\le \frac{\lambda_2\beta_2 T}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^T (T-s)a_2(s)f_2(x(s),y(s))\nabla s\\
&\quad+\frac{\lambda_2(T+\beta_2\theta)}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^T (T-s)a_2(s)f_2(x(s),y(s))\nabla s\\
&= \frac{\lambda_2(T+\beta_2 T+\beta_2\theta)}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^T (T-s)a_2(s)f_2(x(s),y(s))\nabla s\\
&\le \frac{\lambda_2(T+\beta_2 T+\beta_2\theta)}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^T (T-s)a_2(s)(f_2^0+\epsilon)(x(s)+y(s))\nabla s\\
&\le \lambda_2\Lambda_2(f_2^0+\epsilon)\,\| (x,y) \|\\
&\le \frac{1}{q}\,\| (x,y) \|
\end{aligned}$$

Therefore

$$\| L_2(x,y) \|\le \frac{1}{q}\,\| (x,y) \| \tag{3.4}$$

Combining the two above inequalities (3.3) and (3.4), we obtain

$$\begin{aligned} \| L(x,y) \| &= \| L_1(x,y) \| + \| L_2(x,y) \| \\ &\leq \left(\frac{1}{p}+\frac{1}{q}\right) \| (x,y) \| \\ &= \| (x,y) \| . \end{aligned}$$

Let

$$\Omega_1 = \{(x,y) \in E : \| (x,y) \| < R_1\}$$

Then

$$\| L(x,y) \| \leq \| (x,y) \|, \quad \text{for} \quad (x,y) \in (P_1 \times P_2) \cap \partial\Omega_1.$$

Also by the definitions of F_1^∞ and F_2^∞, there exist a $R_2 > 0$ such that

$$f_1(x,y) \geq (F_1^\infty - \epsilon)(x+y) \quad \textbf{for } x,y \geq 0 \quad \textbf{with} \quad x+y \geq R_2$$

$$f_2(x,y) \geq (F_2^\infty - \epsilon)(x+y) \quad \textbf{for } x,y \geq 0 \quad \textbf{with} \quad x+y \geq R_2$$

Set

$$R_3 = \max\{2R_1, \frac{R_2}{\gamma_3}\}$$

If $(x,y) \in P_1 \times P_2$ with $\| (x,y) \| = R_3$, then

$$\min_{t\in[0,T]_{\mathbb{T}}} \{x(s)+y(s)\} \geq \gamma_1 \| x \| + \gamma_2 \| y \| \geq \gamma_3 \| x \| + \gamma_3 \| y \| = \gamma_3 \| (x,y) \| \geq R_2.$$

Then , we have

$$\begin{aligned} L_1(x,y)(0) &= \frac{-\lambda_1\beta_1 T}{(T-\alpha_1\theta)-\beta_1(T-\theta)} \int_0^\eta (\theta-s)a_1(s)f_1(x(s),y(s))\nabla s \\ &\quad + \frac{\lambda_1\beta_1\theta}{(T-\alpha_1\theta)-\beta_1(T-\theta)} \int_0^T (T-s)a_1(s)f_1(x(s),y(s))\nabla s \\ &\geq \frac{-\lambda_1\beta_1 T}{(T-\alpha_1\theta)-\beta_1(T-\theta)} \int_0^\theta (\theta-s)a_1(s)f_1(x(s),y(s))\nabla s \\ &\quad + \frac{\lambda_1\beta_1\theta}{(T-\alpha_1\theta)-\beta_1(T-\theta)} \int_0^\theta (T-s)a_1(s)f_1(x(s),y(s))\nabla s \\ &= \frac{\lambda_1\beta_1(T-\theta)}{(T-\alpha_1\theta)-\beta_1(T-\theta)} \int_0^\theta s a_1(s)f_1(x(s),y(s))\nabla s \\ &\geq \frac{\lambda_1\beta_1(T-\theta)}{(T-\alpha_1\theta)-\beta_1(T-\theta)} \int_0^\theta s a_1(s)(F_1^\infty-\epsilon)(x(s)+y(s))\nabla s \\ &\geq \lambda_1\gamma_3\widetilde{\Lambda_1}(F_1^\infty-\epsilon) \| (x,y) \| \\ &\geq \frac{1}{p} \| (x,y) \| \end{aligned}$$

Consequently,

$$\| L_1(x,y) \| \geq \frac{1}{p} \| (x,y) \|, \quad \text{for} \quad (x,y) \in (P_1 \times P_2) \cap \partial\Omega_2. \tag{3.5}$$

Where

$$\Omega_2 = \{(x,y) \in E : \| (x,y) \| < R_3\}$$

Similarly, we have

$$\begin{aligned} L_2(x,y)(0) &= \frac{-\lambda_2\beta_2 T}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^\theta(\theta-s)a_2(s)f_2(x(s),y(s))\nabla s \\ &\quad +\frac{\lambda_2\beta_2\theta}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^T(T-s)a_2(s)f_2(x(s),y(s))\nabla s \\ &\geq \frac{-\lambda_1\beta_1 T}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^\theta(\theta-s)a_1(s)f_1(x(s),y(s))\nabla s \\ &\quad +\frac{\lambda_1\beta_1\theta}{(T-\alpha_1\theta)-\beta_1(T-\theta)}\int_0^\theta(T-s)a_1(s)f_1(x(s),y(s))\nabla s \\ &= \frac{\lambda_2\beta_2(T-\theta)}{(T-\alpha_2\theta)-\beta_1(T-\theta)}\int_0^\theta s a_2(s)f_2(x(s),y(s))\nabla s \\ &\geq \frac{\lambda_2\beta_2(T-\theta)}{(T-\alpha_2\theta)-\beta_2(T-\theta)}\int_0^\theta s a_2(s)(F_2^\infty-\epsilon)(x(s)+y(s))\nabla s \\ &\geq \lambda_2\gamma_3\widetilde{\Lambda_2}(F_2^\infty-\epsilon)\parallel (x,y)\parallel \\ &\geq \frac{1}{q}\parallel (x,y)\parallel \end{aligned}$$

Consequently,

$$\parallel L_2(x,y)\parallel \geq \frac{1}{q}\parallel (x,y)\parallel, \quad \text{for} \quad (x,y)\in(P_1\times P_2)\cap\partial\Omega_2. \tag{3.6}$$

Combining the two above inequalities (3.5) and (3.6), we have

$$\begin{aligned} \parallel L(x,y)\parallel &= \parallel L_1(x,y)\parallel + \parallel L_2(x,y)\parallel \\ &\geq \Big(\frac{1}{p}+\frac{1}{q}\Big)\parallel (x,y)\parallel \\ &= \parallel (x,y)\parallel . \end{aligned}$$

It follows from part (i) of Guo-Krasnoselskii's fixed point theorem in a cone (theorem 1.1) :
L has a fixed point (x,y) with $R_1\leq\parallel x\parallel+\parallel y\parallel\leq R_3$ in $(P_1\times P_2)\cap(\overline{\Omega}_2\setminus\ \Omega_1)$ then (1.1)has at least one positive solution □

Theorem 3.2 *Suppose that $(C_1)-(C_2)$ hold , $0<\alpha_i<\frac{T}{\theta}$, and $0\leq\beta_i<\frac{T-\alpha_i\theta}{T-\theta}$ for $i=1,2$ and let p and q be two positive numbers such that*

$$\frac{1}{p}+\frac{1}{q}=1$$

(a) If $\Lambda_1 F_1^\infty<\gamma_3\widetilde{\Lambda_1}f_1^0$ and $\Lambda_2 F_2^\infty<\gamma_3\widetilde{\Lambda_2}f_2^0$, then for each

$$(\lambda_1,\lambda_2)\in\Big(\frac{1}{p\gamma_3\widetilde{\Lambda_1}f_1^0},\frac{1}{p\Lambda_1 F_1^\infty}\Big)\times\Big(\frac{1}{q\gamma_3\widetilde{\Lambda_2}f_2^0},\frac{1}{q\Lambda_2 F_2^\infty}\Big)$$

then the problem (1.1) has at least one positive solution.
(b) If $F_1^\infty=F_2^\infty=0$ and $0<f_1^0,f_2^0<\infty$, then for each

$$(\lambda_1,\lambda_2)\in\Big(\frac{1}{p\gamma_3\widetilde{\Lambda_1}f_1^0},\infty\Big)\times\Big(\frac{1}{q\gamma_3\widetilde{\Lambda_2}f_2^0},\infty\Big)$$

then the system (1.1) has at least one positive solution.
(c) If $f_1^0=f_2^0=\infty$ and $0<F_1^\infty,F_2^\infty<\infty$, then for each

$$(\lambda_1,\lambda_2)\in\Big(0,\frac{1}{p\Lambda_1 F_1^\infty}\Big)\times\Big(0,\frac{1}{q\Lambda_2 F_2^\infty}\Big)$$

then the system (1.1) has at least one positive solution.

(d) *If* $f_1^0 = f_2^0 = \infty$ *and* $F_1^\infty = F_2^\infty = 0$*, then for each*

$$(\lambda_1, \lambda_2) \in \Big(0, \infty\Big) \times \Big(0, \infty\Big)$$

then the system (1.1) has at least one positive solution.

Proof: the proof is the same as that of theorem (3.1) ; so we omit it . □

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