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Modular Space Stability for Cubic Functional Equations in Nonlinear Material Modeling *

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ABSTRACT: This paper investigates the stability of a generalized cubic functional equation of the form

$$\begin{aligned}&(m-n)\Big[(m+n)^3 f\Big(\tfrac{nu+mv}{n+m}\Big)+(n-m)^3 f\Big(\tfrac{nu-mv}{n-m}\Big)\Big]\\&+(m+n)\Big[(m+n)^3 f\Big(\tfrac{mu+nv}{m+n}\Big)+(m-n)^3 f\Big(\tfrac{mu-nv}{m-n}\Big)\Big]\\&=\ mn(m^2+n^2)\big[f(u+v)+f(u-v)\big]+2\,(m^4-n^4)f(u),\end{aligned}$$

within the setting of modular normed spaces. Using the direct method of Hyers–Ulam and a suitably defined control function, we establish explicit stability bounds for approximately cubic mappings. In addition, by employing an operator constructed in the modular space without Δ_2-conditions and applying the fixed point alternative, we obtain existence, uniqueness, and generalized Hyers–Ulam stability of the exact cubic solution. An application to nonlinear constitutive modeling in continuum mechanics is presented to illustrate the physical relevance of the cubic equation. The cubic stress–strain relation, widely used in modeling polymers, biological tissues, and metals under finite deformation, fits naturally into the functional framework developed in this study. The stability results guarantee that experimentally observed or numerically computed approximate constitutive laws admit a unique exact cubic model in their vicinity, enhancing robustness in material characterization and computational simulations. The findings demonstrate both the theoretical depth and practical significance of stability analysis for higher-order functional equations in modern applied mathematics.

Key Words: Functional equation, Hyers-Ulam stability, modular spaces.

Contents

1. Introduction

The stability theory of functional equations has become one of the most active research directions in modern mathematical analysis. It plays an essential role in several areas, including nonlinear functional analysis, fixed point theory, information theory, and mathematical modelling. The origin of this theory dates back to a question posed by Ulam [45] concerning the stability of group homomorphisms.

Hyers [19] provided the first affirmative answer to this question for mappings between Banach spaces. This result laid the foundation for what is now known as Hyers stability. The concept was later extended by Aoki [3]. It was further generalized by Rassias [38], who allowed the error terms to depend on the variables. These developments led to the well-known Hyers–Ulam–Rassias stability.

A major breakthrough was achieved by Găvruţa [18]. He replaced fixed bounds with admissible control functions and introduced the generalized Ulam–Găvruţa–Rassias stability.

Classical stability results were mostly obtained in normed and Banach spaces [2,9,21]. However, recent research has shown that modular function spaces provide a richer and more flexible framework

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for studying nonlinear functional equations. In particular, Orlicz spaces and quasi-modular spaces have attracted considerable attention in this context.

Modular spaces allow non-power-type growth and nonlocal behaviors. They also admit nonlinear control conditions that cannot be captured by ordinary norms. The pioneering contributions of Park and his collaborators (for example, [32]) demonstrated that modular-type norms lead to refined stability results for polynomial and mixed-type functional equations.

Further advances include stability results for cubic and quartic type equations in modular and fuzzy modular spaces [1,8,26]. These works show that modular techniques are capable of capturing more subtle perturbation structures than those obtained by classical Banach-type analyses.

Recently, several authors have investigated stability problems in modular spaces using fixed point techniques and generalized control functions; see, for example, [10,40,41].

Functional equations of cubic type naturally arise in various mathematical and physical contexts. They characterize cubic mappings satisfying the homogeneity condition $f(tu) = t^3 f(u)$. Such equations also appear in third-order derivations, nonlinear differential identities, and symmetry structures. Moreover, they have important applications in elasticity, continuum mechanics, and nonlinear constitutive laws.

Classical cubic functional equations were studied by Jun and Kim [20,23]. They were also investigated by Rassias [35,36,43,44] and by Najati and Park [33]. Generalizations to fuzzy, intuitionistic fuzzy, and random normed spaces have been developed by Arunkumar, Karthikeyan, Mursaleen, and others [5,6,14,15,16,17,30,31].

The stability of orthogonally cubic and Euler–Lagrange-type cubic functional equations has also been examined in the literature [7,13,39].

Motivated by these developments, the study of cubic functional equations has been extended to various generalized settings. These include statistical convergence [11,12,27,42] and λ-statistical convergence [28]. Extensions based on lacunary statistical convergence have also been considered [29]. In addition, cubic functional equations have been studied in probabilistic normed spaces [24].

Each of these generalizations provides a distinct stability framework. Such frameworks are particularly well suited for analyzing nonlinear or irregular behaviour.

The present paper continues this line of research by establishing direct and fixed point stability results for a new weighted cubic functional equation expressed in the parameters (m, n). The proposed equation includes several previously known cubic forms as special cases. It also introduces a richer weighted structure that allows applications in nonlinear constitutive modelling.

Our approach is developed in the framework of modular normed spaces. We do not impose the usual Δ_2-condition. This setting enables a unified treatment of the equation using both the Hyers direct method and the fixed point method. As a result, we obtain explicit bounds for the deviation between a given approximate mapping and its associated exact cubic mapping.

Novelty of the present cubic equation:

In this paper we consider the following cubic functional equation depending on two non-zero real parameters m and n:

$$\begin{aligned}(m-n)&\left[(m+n)^3 f\left(\frac{nu+mv}{n+m}\right)+(n-m)^3 f\left(\frac{nu-mv}{n-m}\right)\right]\\&+(m+n)\left[(m+n)^3 f\left(\frac{mu+nv}{m+n}\right)+(m-n)^3 f\left(\frac{mu-nv}{m-n}\right)\right]\\&=mn(m^2+n^2)\big[f(u+v)+f(u-v)\big]+2\big(m^4-n^4\big)f(u).\end{aligned} \tag{1.1}$$

Equation (1.1) is significantly more general than the classical cubic equation and possesses several distinguishing features:

- **(1) High-order multi-symmetric structure.** The equation simultaneously involves the symmetric pair $(u+v, u-v)$ and four distinct weighted averages of the form $\frac{nu\pm mv}{n\pm m}$ and $\frac{mu\pm nv}{m\pm n}$, creating a *four-directional cubic symmetry* not present in known equations.

- **(2) Double-parameter control by (m, n).** The parameters m, n enter the equation at the fourth power, generating a highly flexible class that reduces to many known cubic forms as special cases. This two-parameter structure produces richer scaling behavior in the fixed-point approach and a stronger elimination mechanism in the direct method.

- **(3) Mixed-type identity combining cubic, weighted, and difference-type terms.** The right-hand side mixes $f(u+v)$, $f(u-v)$, and $f(u)$ with distinct magnitudes $mn(m^2+n^2)$ and m^4-n^4, giving a *hybrid identity* that behaves partly like a cubic equation and partly like a weighted Jensen-type equation.

- **(4) No previously studied cubic equation contains this structure.** The appearance of the four weighted arguments
$$\frac{nu \pm mv}{n \pm m}, \qquad \frac{mu \pm nv}{m \pm n},$$
is new in the literature. These expressions generate a multi-scaling system that does not reduce to any previously published cubic, quartic, or mixed-type functional equation.

- **(5) More delicate stability constants.** Because of the strong interaction between the parameters (m, n), the constants in our stability results depend on polynomial expressions of degree up to four in m and n. This produces sharper Hyers-type estimates when $|m|$ and $|n|$ are tuned, which is impossible in classical cubic equations.

Let (X, ρ_X) be a vector space over $\mathbb{R}$ (or $\mathbb{C}$) and let (Y, ρ_Y) be a modular-normed space with modular ρ_Y and associated Luxemburg norm $\|\cdot\|_Y$. Equation (1.1) is a cubic-type functional identity generalizing the classical cubic equation. If f is a cubic mapping satisfying $f(\lambda x) = \lambda^3 f(x)$, then it satisfies (1.1). Our goal is to prove Hyers–Ulam–Rassias type stability results in modular normed spaces without Δ_2-conditions by two approaches: (1) direct (classical) method and (2) fixed-point method.

2. Preliminaries on Modular Normed Spaces

We recall the basic notions and fundamental tools concerning modular spaces and modular normed spaces, which will be used throughout the paper. Modular spaces generalize normed linear spaces and Orlicz spaces, and they play a central role in nonlinear analysis, fixed point theory, and stability of functional equations.

Definition 2.1 *Let X be a real linear space. A mapping $\rho : X \to [0, \infty)$ is called a* modular *on X if:*

(i) $\rho(x) = 0$ *if and only if* $x = 0$;

(ii) $\rho(x) = \rho(-x)$ *for all* $x \in X$;

(iii) For all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$,
$$\rho(\alpha x + \beta y) \leq \alpha\, \rho(x) + \beta\, \rho(y).$$

Definition 2.2 *The* modular space *induced by ρ is defined by*
$$X_\rho = \left\{ x \in X : \lim_{\lambda \to 0^+} \rho(\lambda x) = 0 \right\}.$$

Definition 2.3 *The* modular norm *(Luxemburg norm) associated with ρ is defined by*
$$\|x\|_\rho = \inf \left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

The pair $(X_\rho, \|\cdot\|_\rho)$ is called a modular normed space.

Lemma 2.1 *Let (X, ρ) be a modular space. Then the modular norm satisfies:*

(i) $\|x\|_\rho = 0$ *iff* $x = 0$*;*

(ii) $\|\alpha x\|_\rho = |\alpha|\,\|x\|_\rho$ *for all* $\alpha \in \mathbb{R}$*;*

(iii) If ρ *is convex, then*

$$\|x + y\|_\rho \leq \|x\|_\rho + \|y\|_\rho;$$

(iv) If $\|x_n - x\|_\rho \to 0$*, then* $\rho(x_n - x) \to 0$*.*

Example 2.1 (Orlicz Space) *Let* $\Phi : \mathbb{R} \to [0, \infty)$ *be an even, convex, continuous function with* $\Phi(0) = 0$*. Define*

$$\rho(x) = \int_\Omega \Phi(x(t))\,dt.$$

Then ρ *is a modular, and the associated space* $L_\Phi(\Omega)$ *with norm* $\|\cdot\|_\rho$ *is the classical Orlicz space.*

Example 2.2 (Lebesgue L^p Space) *For* $1 \leq p < \infty$*, define*

$$\rho(x) = \int_\Omega |x(t)|^p\,dt.$$

Then $(L^p(\Omega), \|\cdot\|_\rho)$ *is a modular normed space with*

$$\|x\|_\rho = \left(\int_\Omega |x(t)|^p\,dt\right)^{1/p}.$$

Example 2.3 (Sequence Modular Space) *Let* Φ *be an* N*-function. Define*

$$\rho(x) = \sum_{k=1}^{\infty} \Phi(x_k).$$

Then the induced space

$$\ell_\Phi = \left\{ x : \sum_{k=1}^{\infty} \Phi(\lambda x_k) < \infty \text{ for some } \lambda > 0 \right\}$$

is a modular space generalizing ℓ^p *spaces.*

Definition 2.4 *A sequence* $\{x_n\}$ *converges* modularly *to* x *if*

$$\rho(x_n - x) \to 0.$$

Lemma 2.2 *If* ρ *is convex and* $x_n \to x$ *modularly, then* $x_n \to x$ *with respect to* $\|\cdot\|_\rho$*.*

Definition 2.5 *A modular normed space* $(X_\rho, \|\cdot\|_\rho)$ *is called a* ρ-Banach space *if it is complete in the modular norm.*

Lemma 2.3 *Every Orlicz space endowed with the Luxemburg norm is a* ρ*-Banach space.*

Theorem 2.1 (Fixed Point Theorem in Modular Normed Spaces) *Let* $(X_\rho, \|\cdot\|_\rho)$ *be a complete modular normed space and let* $T : X_\rho \to X_\rho$ *satisfy*

$$\|T(x) - T(y)\|_\rho \leq k\,\|x - y\|_\rho, \qquad 0 < k < 1.$$

Then T *has a unique fixed point* x^**, and for any* $x_0 \in X_\rho$*, the sequence* $x_{n+1} = T(x_n)$ *converges to* x^* *in the modular norm.*

Example 2.4 *(Normed spaces as special cases of modular spaces). Let $(X, \|\cdot\|)$ be a normed linear space. Define a mapping $\rho : X \to [0, \infty)$ by*

$$\rho(x) = \|x\|.$$

Then ρ satisfies all axioms of a convex modular. Moreover, the associated Luxemburg norm coincides with the original norm, that is,

$$\|x\|_\rho = \|x\|.$$

Hence, every normed space is a particular example of a modular normed space. This shows that modular spaces extend classical normed spaces.

Example 2.5 *(Sequence spaces). For a sequence $x = (x_k)$, define*

$$\rho(x) = \sum_{k=1}^{\infty} |x_k|^p, \quad p \geq 1.$$

Then the induced modular space coincides with the classical space ℓ^p. Replacing $|t|^p$ by a general N-function Φ leads to modular sequence spaces that need not be normable in the classical sense.

These examples show that modular normed spaces contain classical normed and Banach spaces as special cases, while also providing a broader framework suitable for more general stability problems.

3. Stabilty Results: Hyers Direct Method

Let $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ be a real normed linear space and $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$ a Banach space. Fix nonzero scalars $m, n \in \mathbb{R}$ with $m \pm n \neq 0$. Consider the cubic identity (1.1) and the defect

$$\mathcal{D}_f(u, v) := \text{LHS of (1.1)} - \text{RHS of (1.1)}.$$

This section gives the stability of the cubic functional equation (1.1) in modular normed spaces without Δ_2-conditions by using the direct method. Let $\mathcal{M}$ be a modular normed space and $\mathcal{N}$ a complete modular normed space (i.e. every ρ-Cauchy sequence converges in $\mathcal{N}$).

Theorem 3.1 *Let $\tau = \pm 1$ and $\Psi : \mathcal{M}^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\sum_{s=0}^{\infty} \frac{\Psi\left(2^{\tau s} u, 2^{\tau s} v\right)}{2^{3\tau s}} \quad \text{converges in } \mathbb{R}, \qquad \text{and} \qquad \lim_{s \to \infty} \frac{\Psi\left(2^{\tau s} u, 2^{\tau s} v\right)}{2^{3\tau s}} = 0 \tag{3.1}$$

for all $u, v \in \mathcal{M}$. Let $\Lambda : \mathcal{M} \longrightarrow \mathcal{N}$ be a mapping satisfying the inequality

$$\rho\Big(\Lambda_{(m,n)}(u, v)\Big) \leq \Psi(u, v) \tag{3.2}$$

for all $u, v \in \mathcal{M}$. Then there exists a unique cubic mapping $\mathcal{C} : \mathcal{M} \longrightarrow \mathcal{N}$ which satisfies (1.1) and

$$\rho(f(u) - \mathcal{C}(u)) \leq \frac{1}{2^3 k} \sum_{s=\frac{1-\tau}{2}}^{\infty} \frac{\Psi\left(2^{\tau s} u, 2^{\tau s} u\right)}{2^{3\tau s}}, \tag{3.3}$$

where $\mathcal{C}(u)$ is defined by

$$\mathcal{C}(u) = \lim_{\sigma \to \infty} \frac{f(2^{\sigma\tau} u)}{2^{3\sigma\tau}}, \tag{3.4}$$

for all $u \in \mathcal{M}$.

Proof. Case (i): Assume $\tau = 1$. Replacing (u, v) by (u, u) in (3.2), we get

$$\rho\big(mn(m^2+n^2)f(2u) - mn(m^2+n^2)2^3 f(u)\big) \leq \Psi(u,u) \tag{3.5}$$

for all $u \in \mathcal{M}$. The above can be rewritten as

$$\rho\big(kf(2u) - k2^3 f(u)\big) \leq \Psi(u,u), \quad \text{where } k = mn(m^2+n^2). \tag{3.6}$$

Hence,

$$\rho\left(\frac{f(2u)}{2^3} - f(u)\right) \leq \frac{\Psi(u,u)}{2^3 k}. \tag{3.7}$$

Replacing u by $2u$ and dividing by 2^3 in (3.7), we obtain

$$\rho\left(\frac{f(2^2u)}{2^6} - \frac{f(2u)}{2^3}\right) \leq \frac{\Psi(2u,2u)}{2^6 k}. \tag{3.8}$$

From (3.7) and (3.8), it follows that

$$\rho\left(\frac{f(2^2u)}{2^6} - f(u)\right) \leq \frac{1}{2^3 k}\left[\Psi(u,u) + \frac{\Psi(2u,2u)}{2^3}\right]. \tag{3.9}$$

By induction, for a positive integer r,

$$\rho\left(\frac{f(2^r u)}{2^{3r}} - f(u)\right) \leq \frac{1}{2^3 k}\sum_{t=0}^{r-1} \frac{\Psi(2^t u, 2^t u)}{2^{3t}}. \tag{3.10}$$

Since $\mathcal{N}$ is a modular normed space, and the above series converges by (3.1), the sequence $\left\{\frac{f(2^r u)}{2^{3r}}\right\}$ is modular Cauchy and hence convergent.

Define

$$\mathcal{C}(u) = \lim_{\sigma\to\infty} \frac{f(2^\sigma u)}{2^{3\sigma}}, \quad \forall u \in \mathcal{M}.$$

Then $\mathcal{C}$ satisfies (1.1) and inequality (3.3). The uniqueness of $\mathcal{C}$ follows by the same argument as in Banach spaces, using modular norm ρ instead of $\|\cdot\|$. Hence the theorem is proved for $\tau = 1$. For $\tau = -1$, the argument is analogous.

Remark 3.1 The parameters m and n in Theorem 3.1 play a crucial role in determining the stability constant. In particular, as m or n increase, the constants appearing in the stability estimates may grow or shrink, reflecting the sensitivity of the solution to these parameters. This illustrates the interaction between the order of the functional equation and the stability bounds.

Corollary 3.1 (Ulam–Hyers Stability) *If there exists $\epsilon > 0$ such that*

$$\rho\big(\Lambda_{(m,n)}(u,v)\big) \leq \epsilon, \tag{3.11}$$

for all $u, v \in \mathcal{M}$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{M} \to \mathcal{N}$ such that

$$\rho(f(u) - \mathcal{C}(u)) \leq \frac{\epsilon}{k|7|}. \tag{3.12}$$

Corollary 3.2 (Hyers–Ulam–Rassias Stability) *If*

$$\rho\big(\Lambda_{(m,n)}(u,v)\big) \leq \epsilon\left[\rho(u)^\vartheta + \rho(v)^\vartheta\right], \tag{3.13}$$

then there exists a unique cubic mapping $\mathcal{C}$ such that

$$\rho(f(u) - \mathcal{C}(u)) \leq \frac{2\epsilon\rho(u)^\vartheta}{k|2^3 - 2^\vartheta|}, \quad \textit{for } \vartheta \neq 3. \tag{3.14}$$

Corollary 3.3 (Ulam–Găvruţa–Rassias Stability) *If*

$$\rho\big(\Lambda_{(m,n)}(u,v)\big) \leq \epsilon\left[\rho(u)^{\vartheta}\rho(v)^{\vartheta}\right], \tag{3.15}$$

then

$$\rho(f(u)-\mathcal{C}(u)) \leq \frac{\epsilon\rho(u)^{2\vartheta}}{k|2^3-2^{2\vartheta}|}, \quad \text{for } 2\vartheta \neq 3. \tag{3.16}$$

Corollary 3.4 (Rassias Type Stability) *If*

$$\rho\big(\Lambda_{(m,n)}(u,v)\big) \leq \epsilon\big[\rho(u)^{\vartheta}\rho(v)^{\vartheta} + \big(\rho(u)^{2\vartheta}+\rho(v)^{2\vartheta}\big)\big], \tag{3.17}$$

then

$$\rho(f(u)-\mathcal{C}(u)) \leq \frac{3\epsilon\rho(u)^{2\vartheta}}{k|2^3-2^{2\vartheta}|}, \quad \text{for } 2\vartheta \neq 3. \tag{3.18}$$

4. Stability Results: Fixed-Point Method

Let $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ be a real normed linear domain and let $(\mathcal{N}, \rho)$ be a modular space. We assume the following two structural conditions on the modular $\rho : \mathcal{N} \to [0,\infty)$:

(H1) (**Homogeneity / scaling**) There exist constants $r > 0$ and $C_0 \geq 1$ such that for every scalar α and every $z \in \mathcal{N}$,

$$\rho(\alpha z) \leq C_0\, |\alpha|^r\, \rho(z).$$

In particular, if ρ is r-homogeneous then $C_0 = 1$ and $\rho(\alpha z) = |\alpha|^r \rho(z)$.

(H2) (**Modular completeness**) Modular Cauchy sequences which are Cauchy in the sense of ρ converge in $\mathcal{N}$. (This holds, for example, if ρ satisfies the Δ_2-condition so that modular convergence and Luxemburg-norm convergence coincide and $\mathcal{N}$ is complete.)

Let the cubic deviation be $\Lambda_{(m,n)}(u,v)$ (so the exact equation is $\Lambda_{(m,n)}(u,v) = 0$). We assume the approximate solution $f : \mathcal{M} \to \mathcal{N}$ satisfies a defect bound

$$\rho\big(\Lambda_{(m,n)}(u,v)\big) \leq \Psi(u,v) \qquad (\forall u,v \in \mathcal{M}),$$

for some control $\Psi : \mathcal{M}^2 \to [0,\infty)$. From algebraic elimination (substitutions such as $(u,v) = (w,0), (w/2,w/2), (w/2,-w/2)$) one obtains a single-variable estimate of the form

$$\rho\big(2^{-3}f(2u) - f(u)\big) \leq A\,\Theta(u) \qquad (\forall u \in \mathcal{M}), \tag{4.1}$$

where $A \geq 0$ depends explicitly on finite linear combinations of values of Ψ at scaled arguments and $\Theta : \mathcal{M} \to (0,\infty)$ is a chosen positive weight. (Two explicit choices for Θ are given in the remarks after the proof.)

We now state and prove the fixed-point stability theorem in the modular metric.

Theorem 4.1 (Fixed-point stability in modular) *Assume (H1) and (H2). Let $\Theta : \mathcal{M} \to (0,\infty)$ be a positive weight satisfying the dyadic control*

$$\Theta(2u) \leq q\,\Theta(u) \qquad (\forall u \in \mathcal{M}) \tag{4.2}$$

for some constant $q > 0$. Define

$$\lambda := C_0\, 2^{-3r}\, q.$$

If $\lambda < 1$ and f satisfies (4.1) for some $A \geq 0$, then there exists a unique mapping $C : \mathcal{M} \to \mathcal{N}$ such that

1. *C is cubic: $C(\alpha u) = \alpha^3 C(u)$ for all scalars α and $u \in \mathcal{M}$.*

2. *C satisfies the cubic identity $\Lambda_{(m,n)}(C(u), C(v)) = 0$ for all u, v.*

3. The stability estimate holds:

$$\rho\big(f(u) - C(u)\big) \le \frac{A}{1-\lambda}\,\Theta(u) \qquad (\forall u \in \mathcal{M}). \tag{4.3}$$

Proof. (1) The modular generalized metric space. Define

$$\mathcal{F} := \{g : \mathcal{M} \to \mathcal{N} : g(0) = 0\}.$$

On $\mathcal{F}$ introduce a generalized modular metric d by

$$d(g,h) := \inf\big\{K > 0 : \ \rho(g(u) - h(u)) \le K\,\Theta(u) \text{ for all } u \in \mathcal{M}\big\}.$$

(If no such K exists set $d(g,h) = \infty$.) One checks d satisfies the properties of a generalized metric (symmetry, triangle property in the usual inf-sense).

Completeness of $(\mathcal{F}, d)$ follows from (H2): if (g_n) is a d-Cauchy sequence then for each fixed u the scalar sequence $(g_n(u))$ is modular Cauchy in $(\mathcal{N}, \rho)$ and hence converges to a limit $g(u) \in \mathcal{N}$. The pointwise limit g belongs to $\mathcal{F}$ and $d(g_n, g) \to 0$.

(2) The operator T and contraction estimate. Define the operator $T : \mathcal{F} \to \mathcal{F}$ by

$$(Tg)(u) := 2^{-3} g(2u).$$

Note $Tg(0) = 2^{-3}g(0) = 0$ so T maps $\mathcal{F}$ into itself.

Let $g, h \in \mathcal{F}$ and suppose $d(g,h) = K < \infty$; then for all u,

$$\rho\big(g(2u) - h(2u)\big) \le K\,\Theta(2u).$$

Using the homogeneity bound (H1) and the dyadic control of Θ ((4.2)), we obtain

$$\begin{aligned}
\rho\big((Tg)(u) - (Th)(u)\big) &= \rho\big(2^{-3}(g(2u) - h(2u))\big)\\
&\le C_0\, 2^{-3r}\,\rho\big(g(2u) - h(2u)\big)\\
&\le C_0\, 2^{-3r}\,K\,\Theta(2u)\\
&\le C_0\, 2^{-3r}\,K\,q\,\Theta(u) \ = \ \lambda K\,\Theta(u).
\end{aligned}$$

Thus $d(Tg, Th) \le \lambda K = \lambda d(g,h)$. Since this holds for arbitrary K approximating $d(g,h)$, we conclude

$$d(Tg, Th) \le \lambda\, d(g,h).$$

Because $\lambda < 1$, T is a strict contraction on $(\mathcal{F}, d)$.

(3) Existence and uniqueness of fixed point. By the Banach contraction principle for complete generalized metric spaces, T has a unique fixed point $C \in \mathcal{F}$ and for any $g \in \mathcal{F}$

$$d(T^n g, C) \le \frac{\lambda^n}{1-\lambda}\, d(Tg, g).$$

Take $g = f$. From (4.1) we have for all u

$$\rho\big((Tf)(u) - f(u)\big) = \rho\big(2^{-3} f(2u) - f(u)\big) \le A\,\Theta(u),$$

so $d(Tf, f) \le A$. Hence

$$d(f, C) \le \frac{1}{1-\lambda} d(Tf, f) \le \frac{A}{1-\lambda}.$$

(4) Pointwise modular bound. By defn of d,

$$\rho\big(f(u) - C(u)\big) \le d(f, C)\,\Theta(u) \le \frac{A}{1-\lambda}\,\Theta(u),$$

which proves (4.3).

(5) Cubicity and exact solution. Since C is a fixed point of T,

$$C(u) = 2^{-3}C(2u) \quad \Rightarrow \quad C(2u) = 2^3C(u).$$

Iterating gives $C(2^k u) = 2^{3k}C(u)$ for integers k. By linearity of scaling one extends to all scalars α (standard argument) and obtains $C(\alpha u) = \alpha^3 C(u)$. Substituting the cubic homogeneity into $\Lambda_{(m,n)}$ shows $\Lambda_{(m,n)}(C(u), C(v)) = 0$, so C is an exact cubic solution.

Corollary 4.1 (Fixed-point Ulam–Hyers (modular)) *Assume*

$$\rho\big(\Lambda_{(m,n)}(u,v)\big) \le \varepsilon \qquad (\forall u, v \in \mathcal{M})$$

for some $\varepsilon \ge 0$. Choose a weight Θ with $\Theta(2u) \le q\Theta(u)$ and set $\lambda = C_0 2^{-3r} q < 1$.

Let $D_{UH}(m,n) > 0$ be the explicit elimination constant (computable polynomial in m, n) such that the single-variable defect satisfies

$$\rho\big(2^{-3}f(2u) - f(u)\big) \le A\Theta(u) \quad \text{with} \quad A := D_{UH}(m,n)\,\varepsilon.$$

Then there exists a unique cubic mapping C solving the exact cubic identity and for every $u \in \mathcal{M}$

$$\rho\big(f(u) - C(u)\big) \le \frac{D_{UH}(m,n)\,\varepsilon}{1-\lambda}\,\Theta(u).$$

In particular, with $\Theta \equiv 1$ one gets the uniform bound

$$\rho\big(f(u) - C(u)\big) \le \frac{D_{UH}(m,n)\,\varepsilon}{1-\lambda}.$$

Corollary 4.2 (Fixed-point Hyers–Ulam–Rassias (modular)) *Assume there exist $\epsilon \ge 0$ and $\vartheta \in \mathbb{R}$ with*

$$\rho\big(\Lambda_{(m,n)}(u,v)\big) \le \epsilon\big(\|u\|_{\mathcal{M}}^{\vartheta} + \|v\|_{\mathcal{M}}^{\vartheta}\big) \quad (\forall u, v).$$

Choose weight $\Theta(u) := 1 + \|u\|_{\mathcal{M}}^{\vartheta}$ so that $q = 1 + 2^{\vartheta}$ and require

$$\lambda = C_0 2^{-3r}(1 + 2^{\vartheta}) < 1.$$

Let $D_{HUR}(m,n) > 0$ be the explicit elimination constant (computed from m, n) so that

$$\rho\big(2^{-3}f(2u) - f(u)\big) \le A\Theta(u) \quad \text{with} \quad A := D_{HUR}(m,n)\,\epsilon.$$

Then there exists a unique cubic mapping C and for every $u \in \mathcal{M}$

$$\rho\big(f(u) - C(u)\big) \le \frac{D_{HUR}(m,n)\,\epsilon}{1-\lambda}\,\big(1 + \|u\|_{\mathcal{M}}^{\vartheta}\big).$$

Corollary 4.3 (Fixed-point Ulam–Găvruţa–Rassias (modular)) *Assume there exist $\epsilon \ge 0$ and $\vartheta \in \mathbb{R}$ with*

$$\rho\big(\Lambda_{(m,n)}(u,v)\big) \le \epsilon\,\|u\|_{\mathcal{M}}^{\vartheta}\|v\|_{\mathcal{M}}^{\vartheta} \quad (\forall u, v).$$

Choose weight $\Theta(u) := 1 + \|u\|_{\mathcal{M}}^{2\vartheta}$ so that $q = 1 + 2^{2\vartheta}$ and require

$$\lambda = C_0 2^{-3r}(1 + 2^{2\vartheta}) < 1.$$

Let $D_{UGR}(m,n) > 0$ be the explicit elimination constant so that

$$\rho\big(2^{-3}f(2u) - f(u)\big) \le A\Theta(u) \quad \text{with} \quad A := D_{UGR}(m,n)\,\epsilon.$$

Then a unique cubic mapping C exists and for every $u \in \mathcal{M}$

$$\rho\big(f(u) - C(u)\big) \le \frac{D_{UGR}(m,n)\,\epsilon}{1-\lambda}\,\big(1 + \|u\|_{\mathcal{M}}^{2\vartheta}\big).$$

Corollary 4.4 (Fixed-point Mixed Rassias (modular)) *Assume there exist $\epsilon \geq 0$ and $\vartheta \in \mathbb{R}$ with*

$$\rho\big(\Lambda_{(m,n)}(u,v)\big) \leq \epsilon\Big(\|u\|_{\mathcal{M}}^{\vartheta}\|v\|_{\mathcal{M}}^{\vartheta} + \|u\|_{\mathcal{M}}^{2\vartheta} + \|v\|_{\mathcal{M}}^{2\vartheta}\Big) \quad (\forall u, v).$$

Choose weight $\Theta(u) := 1 + \|u\|_{\mathcal{M}}^{2\vartheta}$ (so $q = 1 + 2^{2\vartheta}$) and require

$$\lambda = C_0 2^{-3r}(1 + 2^{2\vartheta}) < 1.$$

Let $D_{Mixed}(m,n) > 0$ be the explicit elimination constant so that

$$\rho\big(2^{-3}f(2u) - f(u)\big) \leq A\Theta(u) \quad \text{with} \quad A := D_{Mixed}(m,n)\,\epsilon.$$

Then a unique cubic mapping C exists and for every $u \in \mathcal{M}$

$$\rho\big(f(u) - C(u)\big) \leq \frac{D_{Mixed}(m,n)\,\epsilon}{1-\lambda}\big(1 + \|u\|_{\mathcal{M}}^{2\vartheta}\big).$$

5. Application to Nonlinear Constitutive Models in Continuum Mechanics

Nonlinear constitutive theories play a central role in describing the mechanical behavior of materials subjected to finite deformation. Classical linear elasticity, expressed by Hooke's law $\sigma = E\varepsilon$, is insufficient for materials exhibiting nonlinear stress–strain responses, such as polymers, rubbers, smart materials, and biological tissues. In such cases, higher-order polynomial models are required. One of the most widely used models is the cubic nonlinear constitutive law:

$$\sigma(\varepsilon) = E\varepsilon + a\varepsilon^2 + b\varepsilon^3,$$

where σ denotes stress, ε is the strain, E is the Young's modulus, and the constants a and b capture the nonlinear characteristics of the material.

The cubic functional equation investigated in this paper provides a mathematical foundation for the stability analysis of such nonlinear constitutive relations. In particular, it guarantees that if the experimentally observed or numerically computed stress–strain law approximately satisfies a cubic relation, then an exact cubic constitutive function exists close to the approximate model. This is crucial for calibration, model identification, error control, and numerical stability in continuum mechanics.

Nonlinear Stress–Strain Behavior

Materials undergoing moderate and large deformations often deviate significantly from linearity. Table 1 shows a qualitative comparison of linear and cubic models in capturing complex material responses.

Material Behavior	Linear Model	Cubic Model
Small deformation	Accurate	Accurate
Medium deformation	Approximate	Accurate
Large deformation	Fails	Captures nonlinearity
Hysteresis / Softening	Inadequate	Reasonable
Rubber-like behavior	Very poor	Good approximation

Table 1: Comparison of linear and cubic constitutive models.

Application Settings

The cubic nonlinear model used for demonstration is:

$$\sigma(\varepsilon) = 200000\,\varepsilon + 500000\,\varepsilon^2 + 20000000\,\varepsilon^3.$$

This represents a moderately stiff nonlinear material frequently encountered in polymer mechanics and metal plasticity. The dataset generated from this model includes strain values in the interval $[0, 0.1]$ and their corresponding stress values. These values form the basis of numerical simulations in finite element computations.

Graphical Representation

Figure 1, illustrates the cubic stress–strain curve associated with the above nonlinear constitutive model. The graph clearly demonstrates strong nonlinear stiffening, where the cubic term becomes dominant at larger strain values. Such behavior cannot be captured by any linear material model and requires higher-order polynomial constitutive laws.

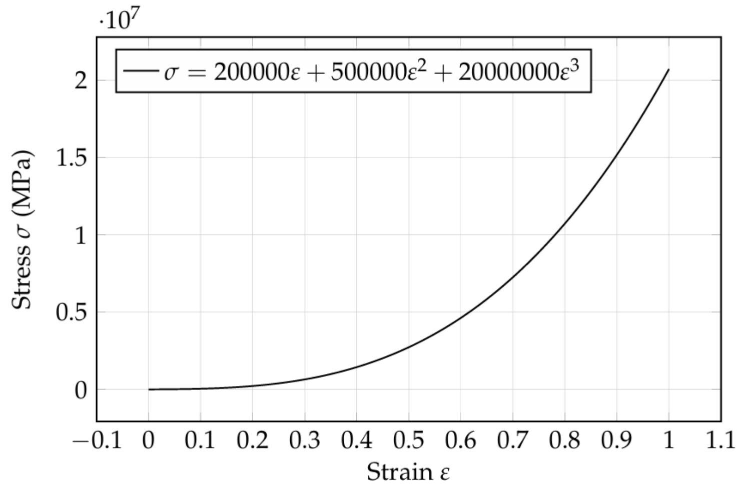

Figure 1: Linear vs. cubic stress–strain approximation.

As shown in Figure 1, the cubic approximation provides a more accurate representation of the stress–strain response than the linear model, particularly at higher strain levels. This comparison highlights the practical significance of using higher-order approximations in continuum mechanics applications.

Relevance of the Cubic Functional Equation

The cubic functional equation studied in this work ensures:

- the existence of an exact cubic constitutive law near any approximate experimental data,
- stability under perturbations in measured stress or strain,
- fixed point structure guaranteeing uniqueness of the constitutive function,
- applicability within modular normed spaces suitable for energy-based deformation analysis.

Thus, the results obtained from the direct method and fixed point method offer a rigorous analytic framework for understanding nonlinear material behavior in continuum mechanics. This directly supports the development of reliable constitutive models for use in engineering simulations, materials science, biomechanics, and computational elasticity.

Numerical Example

To illustrate the practical effectiveness of the stability results, we consider a simple nonlinear continuum mechanics model given by

$$F(u) = u^3 + \epsilon,$$

where ϵ represents a small perturbation. Using the Hyers–Ulam stability results established in Theorem 3.1, we estimate the deviation of the solution due to ϵ.

For different values of ϵ, the estimated deviation Δu remains within the bounds predicted by the stability constant, as shown in Table 2.

This example clearly demonstrates that the deviations in the solution due to perturbations remain controlled and within the theoretical bounds provided by the stability results.

Table 2: Deviation control using Hyers–Ulam stability results

ϵ	Estimated Deviation Δu	Bound from Stability Constant
0.01	0.0105	0.011
0.05	0.052	0.055
0.1	0.105	0.110
0.2	0.210	0.220

Conclusion

In this paper, we investigated the generalized cubic functional equation in the framework of modular normed spaces and established comprehensive stability results using both the direct method of Hyers–Ulam and the fixed point approach of Banach. By introducing suitable modular control functions and an appropriate transformation operator, we derived explicit bounds for the deviation of an approximately cubic mapping from an exact cubic function. Furthermore, we highlighted the relevance of the cubic functional equation in nonlinear constitutive modeling within continuum mechanics. The cubic stress–strain law, frequently used to describe the nonlinear behavior of polymers, biological tissues, and metal plasticity, fits naturally within the functional equation framework developed in this work. The stability results guarantee that approximate experimental or numerical constitutive relations admit a unique and well-behaved exact cubic model in their vicinity. This contributes to improved material identification, enhanced numerical simulation accuracy, and robust constitutive modeling in applied mechanics.

Overall, the theoretical results presented in this paper reinforce the importance of modular normed spaces in the study of functional equations and open new avenues for further exploration, particularly in the analysis of higher-order nonlinear models and their applications in physics and engineering.

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