Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 5** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.81376

Ward Continuity in Topological Vector Spaces

Yasemin Şimşek* and Hüseyin Çakallı

ABSTRACT: In this paper, we investigate the concepts of quasi-Cauchyness of sequences and ward continuity of functions in topological vector spaces. We prove that totally boundedness is equivalent to ward compactness and ward continuity on a totally bounded subset E of topological vector space X coincides with uniform continuity. We also prove some other interesting theorems.

Keywords: Quasi Cauchy sequence, compactness, totally boundedness, continuity.

Contents

1. Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, economics, and biological science. Using the idea of sequential continuity, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: ward continuity ([12,11]), slowly oscillating continuity ([26]), statistical ward continuity ([15], [14]), lacunary statistical ward continuity ([6]), λ statistical ward continuity ([19]), ρ statistical ward continuity ([4], [16]).

Let X be a vector space, and τ be a topology on X. If the addition function $+ : X \times X \to X$, $+(x, y) = x + y$, and the scalar product function $\cdot : \mathbb{R} \times X \to X$, $\lambda \cdot x = \lambda x$ are continuous where $X \times X$ and $\mathbb{R} \times X$ have product topologies, then X is called a topological vector space.

Recently, quasi-Cauchy sequences have been extensively studied by many authors in the real line, in metric spaces [3], [20], in 2-normed spaces [23], and in asymmetric metric spaces [21].

Throughout this paper, X will denote a T_2 topological vector space which satisfies the first axiom of countability, altough most of the notions and results are valid in topological vector spaces which do not necesserily satisfy the first axiom of countbility. A sequence (x_n) in X is called convergent to a point $\ell \in X$ if there exists an integer $n_0 > 0$ such that $x_n - \ell \in U$ for $n \geq n_0$, for each neighbourhood U of 0. A sequence (x_n) in X is said to be slowly oscillating if there exist a real number $\delta = \delta(U) > 0$ and a natural number $N = N(U)$ such that $x_k - x_n \in U$ whenever $n \geq N$ and $n \leq k \leq (1 + \delta)n$. A collection $\{U_j : j \in I\}$ is called an open cover of $A \subseteq X$ if $A \subseteq \bigcup_{j \in I} U_j$ where each U_j is an open subsets of X for an index set I. If this collection contains only finitely many elements, then it is called a finite cover of the set A. A subset E of X is called totally bounded if for every neighbourhood V of 0 in X corresponds a finite set F such that $E \subset F + V$.

If every open cover of a subset $A \subseteq X$ has a finite subcover, then the set A is called a compact set. Similarly, $A \subseteq X$ is called sequentially compact if every sequence in A has a subsequence that converges to an element of the set A.

* Corresponding author.
2020 *Mathematics Subject Classification:* 54A20.
Submitted January 15, 2026. Published June 05, 2026

2. Quasi Cauchy Sequences in Topological Vector Spaces

The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. Quasi Cauchy sequence concept was first introduced in [11] and further studied in [3] for sequences of real numbers, and in [15] for sequences in metric spaces. Now we give the definition in topological vector spaces in the following.

Definition 2.1 *A sequence (x_n) in X is called quasi-Cauchy if for each neighbourhood U of 0, there exists a positive integer n_0 such that $x_n - x_{n+1} \in U$ for $n \geq n_0$. In other words, a sequence (x_n) is quasi-Cauchy if the sequence $(x_n - x_{n+1})$ converges to 0.*

Any slowly oscillating sequence is quasi-Cauchy, so any Cauchy sequence is, so any convergent sequence is. We prove in the following that the sum of two quasi-Cauchy sequences is quasi Cauchy, and (λx_n) is quasi-Cauchy whenever (x_n) is for every scalar λ.

Theorem 2.1 *Sum of two quasi-Cauchy sequences are also quasi-Cauchy, and for every scalar λ, (λx_n) is quasi-Cauchy whenever (x_n) is.*

Proof: Let (x_n) and (y_n) be two quasi-Cauchy sequences in X, and λ be any scalar. Let U be any neighborhood of 0. Then there is a neighborhood V of 0 such that $V + V \subseteq U$. As (x_n) is quasi-Cauchy, there exists a positive integer n_1 such that $x_n - x_{n+1} \in V$ for $n \geq n_1$. Similarly, there exists a positive integer $n_2 \geq n_1$ such that $y_n - y_{n+1} \in V$ for $n \geq n_2$.
This implies that

$$(x_n + y_n) - (x_{n+1} + y_{n+1}) = (x_n - x_{n+1}) + (y_n - y_{n+1}) \in V + V \subseteq U.$$

It is not difficult to show that (λx_n) is quasi-Cauchy whenever (x_n) is for every scalar λ. □

Corollary 2.1 *The set of all quasi-Cauchy sequences is a vector subspace of the vector space of all sequences in X.*

We see that the set of convergent sequences is a subset of the vector space of quasi Cauchy sequences, so is the set of all Cauchy sequences, so is the set of all slowly oscillating sequences.

Corollary 2.2 *Every convergent sequence is quasi-Cauchy. The converse is not always true.*

Example 2.1 *For a fixed element x of X, the sequence $(\sqrt{n}x)$ is quasi-Cauchy, but not convergent.*

Using the main idea in the sequential compactness new types of compactness are introduced, not all but some of them are, slowly oscillating compactness ([13], [18]), quai slowly oscillating compactness ([9]), statistical ward compactness ([15]), ρ statistical ward compactness ([5]).

Using the main idea in the definition of sequential compactness the concept of ward compactness is initially introduced for real sequences with the term "forward compactness" in [11], and investigated in [12]. We extend the concept to topological vector spaces in the following.

Definition 2.2 *A subset E of X is called ward compact if any sequence of points in E has a quasi-Cauchy subsequence.*

It is clear that every finite subset of X is ward compact, the union of two ward compact subsets of X is ward compact, so the union of finite number of ward compact subsets of X is ward compact, the intersection of any family of subsets of ward compact subsets of X is ward compact, any subset of a ward compact subset of ward compact subset of X is ward compact. The sum of two ward compact subsets of X is ward compact, i.e. $A+B$ is ward compact when A and B are. αA is ward compact for any constant scalar α when A is ward compact. Every compact subset of X is ward compact.

We note that a subset E of X is totally bounded as a subset of X if and only if it is totally bounded as a subspace of X, i.e. a subset E of X is totally bounded if for every neighborhood U of 0 in X corresponds a finite set $F \subseteq E$ such that $E = F + E \cap U$.

Theorem 2.2 *A subset E of X is ward compact if and only if it is totally bounded, i.e. any sequence of vectors in E has a quasi-Cauchy subsequence if and only if E is totally bounded.*

Proof: First, assume that E is totally bounded. Let $\{U_n\}$ be a base of nested symetric neighborhoods of 0 such that $U_k + U_k \subseteq U_{k-1}$ for each positive integer k, and (x_n) be any sequence in E. Since E can be covered by a finite U_1-net, one of the sets forming the finite U_1-net , which we denote by B_1, must contain x_n for infinitely many values of n. Choose a positive integer n_1 such that $x_{n_1} \in B_1$. B_1 is totally bounded, and hence can be covered by a finite U_2-net, one of sets forming the finite U_2-net , which we denote by B_2, must contain x_n for infinitely many values of n. Choose positive integer n_2 such that $x_{n_2} \in B_2$ satisfying $n_2 > n_1$. Since $U_2 \subset U_1$. $x_{n_2} \in B_1$. Continuing in this way, we obtain for any positive integer k, a subset B_k of B_{k-1} with B_k U_k-net. We can construct a subsequence (x_{n_k}) inductively as above. Then for all k,

$$x_{n_{k+1}} - x_{n_k} \in U_k,$$

so (x_{n_k}) is a quasi-Cauchy subsequence of (x_n). Thus, E is ward compact.
Now, assume that E is ward compact, Let if possible, E be not totally bounded. Then there exists a symmetric neighborhood U of 0 such that there does not exists a finite U-net. Let x_1 be an element of E. Then it is clear that $E \nsubseteq x_1 + U$ for otherwise $\{x_1\}$ would be a finite U-net for E. Let x_2 be an element such that $x_2 \notin \{x_1\} + U$, i.e. $x_1 - x_2 \notin U$. Then $E \nsubseteq (\{x_1\} + U) \cup (\{x_2\} + U)$ because otherwise $\{x_1, x_2\}$ would be a finite U -net. Let $x_3 \in E$ such that $x_3 \notin (\{x_1\} + U) \cup (\{x_2\} + U)$, i.e. $x_1 - x_2 \notin U$, $x_1 - x_3 \notin U$ and $x_2 - x_3 \notin U$. Continuing the process in this manner, we obtain a sequence (x_n) of points in E such that $x_n \notin \bigcup_{i=1}^{n-1} (\{x_i\} \ + U)$, $(n = 2, 3, \dots)$, i.e., $x_i - x_n \notin U$, $(i = 1, 2, \dots, n-1$ and $n = 1, 2, \dots), (n \neq i)$. Consequently $x_n - x_m \notin U$, for all n and m, $n \neq m$. Thus the sequence (x_n) of points in E cannot have any quasi-Cauchy subsequence. This is a contradiction. Hence E should be totally bounded.

□

3. Ward Continuity in Topological Vector Spaces

A function f defined on a subset of X is continuous if and only if, for each point ℓ in the domain, $\lim_{n\to\infty} f(x_n) = f(\ell)$ whenever $\lim_{n\to\infty} x_n = \ell$. This is equivalent to the statement that $(f(x_n))$ is a convergent sequence whenever (x_n) is. This is also equivalent to the statement that $(f(x_n))$ is a Cauchy sequence whenever (x_n) is Cauchy provided that the domain of the function is complete. These well known results for ordinary continuity for functions in terms of sequences suggested to us introducing a new type of continuity, namely, ward continuity. The concept of ward continuity was introduced for real functions in [11], [12] as forward continuity and was studied in [23] in 2-normed spaces. Now we give the definition of ward continuity in topological vector spaces in the following.

Definition 3.1 *A function from a subset E of X into X, $f : E \to X$ is called ward continuous if $(f(x_n))$ is a quasi-Cauchy sequence for each quasi-Cauchy sequence (x_n) in E. In other words, f is ward continuous if it preserves quasi Cauchy sequences, i.e. f is ward continuous if $(f(x_n) - f(x_{n+1}))$ converges to 0 for every quasi Cauchy sequence (x_n).*

The composite of two ward continuous functions is ward continuous, the sum of two ward continuous functions is ward continuous, and if f is a ward continuous function, then cf is also ward continuous for any constant c. Thus the set of all ward continuous functions on a subset E of X is a subspace of the vector space of all continuous functions on E,

Theorem 3.1 *Ward continuous image of any ward compact subset of X is ward compact.*

Proof: Let f be any ward continuous function from X into X and E be any ward compact subset of X. Take any sequence (y_n) of points in $f(E)$. Then there exists x_n such that $y_n = f(x_n) \in E$ for every positive integer n. Since E is ward compact, there exist a quasi-Cauchy subsequence (x_{n_k}) of the sequence (x_n). By the assumption that f is ward continuous, the sequence $(f(x_{n_k}))$ is a quasi-Cauchy sequence. Now we found that (y_n) has a quasi-Cauchy subsequence (y_{n_k}). This completes of the proof of the theorem. □

Theorem 3.2 *Let f be a function from a subset E of X into X. If f is ward continuous, then f is sequentially continuous.*

Proof: Let f be a ward continuous function on E, and let (x_n) be a convergent sequence in E with the limit l. Then, the sequence

$$(x_1, l, x_2, l, \ldots, x_n, l, \ldots)$$

also converges to l. Moreover, the sequence

$$(x_1, l, x_2, l, \ldots, x_n, l, \ldots)$$

is a quasi-Cauchy sequence. Since f is ward continuous, the sequence

$$(f(x_1), f(l), f(x_2), f(l), \ldots, f(x_n), f(l), \ldots)$$

is also a quasi-Cauchy sequence. Hence, it follows that the sequence $(f(x_n))$ converges to $f(l)$. Therefore, the function f is continuous on E. □

Corollary 3.1 *Ward continuous image of any compact subset of X is ward compact.*

The proof follows from the preceding theorem.

We recall the definition of uniform continuity in topological vector spaces. Let E be a subset of X. A function $f : E \to X$ is said to be uniformly continuous on E if for every neighborhood V of 0 in X, there exists a neighborhood U of 0 in X such that for all $x, y \in E$,

$$x - y \in U \quad \Rightarrow \quad f(x) - f(y) \in V.$$

Theorem 3.3 *Any ward continuous function on a ward compact subset of X is uniformly continuous.*

Proof: Let $\{U_n\}$ be a base of nested symetric neighborhoods of 0. Suppose that f is not uniformly continuous on E so that there exists a neighborhood U of 0 such that $x - y \in U$ but $f(x) - f(y) \notin U_n$ for each $n \in \mathbb{N}$. For each positive integer n, there are x_n and y_n such that $x_n - y_n \in U$, and $f(x_n) - f(y_n) \notin U_n$. Since E is ward compact, there exists a quasi-Cauchy subsequence (x_{n_k}) of the sequence (x_n). Then, the corresponding subsequence (y_{n_k}) of the sequence (y_n) is also quasi-Cauchy, since $(y_{n_{k+1}} - y_{n_k})$ is a sum of three null sequences, i.e.

$$y_{n_{k+1}} - y_{n_k} = (y_{n_{k+1}} - x_{n_{k+1}}) + (x_{n_{k+1}} - x_{n_k}) + (x_{n_k} - y_{n_k}).$$

Then the sequence

$$(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, ..., x_{n_k}, y_{n_k}, ...)$$

is quasi-Cauchy since the sequence $(x_{n_k} - y_{n_k})$ and $(y_{n_k} - x_{n_{k+1}})$ are convergent to 0. But the transformed sequence is not quasi-Cauchy. Thus f does not preserve quasi-Cauchy sequences. This contradiction completes the proof of the theorem. □

The concept of closedness of a set is an important notion that a closed set keeps limit points inside of the set.

Theorem 3.4 *The set of all ward continuous functions on a subset E of X is a closed subset of the set of all continuous functions on E, i.e. $\overline{\Delta WC(E)} = \Delta WC(E)$ where $\Delta WC(E)$ is the set of all ward continuous functions on E, $\overline{\Delta WC(E)}$ denotes the set of all cluster points of $\Delta WC(E)$.*

Proof: f be any element in $\overline{\Delta WC(E)}$. Then there exists a sequence of points in $\Delta WC(E)$ such that $\lim_{k\to\infty} f_k = f$. To show that f is ward continuous, take any quasi-Cauchy sequence (x_k) of points in E. Let U be a neighborhood of 0, and V be a symmetric neighborhood of 0 such that $V+V+V \subset U$. Since (f_k) converges to f, for each neighborhood U of 0, there exists an N such that for all $x \in E$ and for all $n \geq N$, $f_n(x) - f(x) \in V$. In particular, for $n = N$ we have

$$f_N(x_{k+1}) - f(x_{k+1}) \in V \quad \text{and} \quad f_N(x_k) - f(x_k) \in V.$$

As f_N is ward continuous, the image of the quasi-Cauchy sequence (x_k) under f_N is also quasi-Cauchy. Therefore, there exists a natural number $K \geq N$ such that for all $k \geq K$,

$$f_N(x_{k+1}) - f_N(x_k) \in V.$$

Now consider

$$\begin{aligned} f(x_k) - f(x_{k+1}) = &\big[f(x_k) - f_N(x_k)\big] + \big[f_N(x_k) - f_N(x_{k+1})\big] \\ &+ \big[f_N(x_{k+1}) - f(x_{k+1})\big]. \end{aligned}$$

Each term on the right-hand side belongs to V for all $k \geq K$. Hence,

$$f(x_k) - f(x_{k+1}) \in V + V + V \subset U.$$

Since U is an arbitrary neighborhood of 0, this shows that $(f(x_k))$ is a quasi-Cauchy. Therefore, f preserves quasi-Cauchy sequences, that is, f is ward continuous.
Consequently, $\Delta WC(E)$ is closed.

□

Corollary 3.2 *Ward continuous image of any totally bounded subset of X is totally bounded.*

The proof follows from Theorem 3.4 and Theorem 5 in [8].

Corollary 3.3 *Ward continuous image of a G-sequentially compact subset of X is ward compact for any subsequential regular method G.*

It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case of ward continuity, i.e. uniform limit of a sequence of ward continuous functions is ward continuous.

Theorem 3.5 *Uniform limit of ward continuous functions on X is ward continuous.*

Proof: Let (x_k) be a quasi-Cauchy sequence in X, i.e.,

$$x_k - x_{k+1} \to 0 \text{ when } k \to \infty.$$

We want to prove that $(f(x_k))$ is also quasi-Cauchy, i.e.,

$$f(x_k) - f(x_{k+1}) \to 0.$$

Let U be a neighborhood of zero in X. Since addition is continuous in X, there exists a symmetric neighborhood V of zero such that:

$$V + V + V \subseteq U.$$

Since $f_n \to f$ uniformly, there exists $n_0 \in \mathbb{N}$ such that:

$$\forall x \in X, \quad f(x) - f_{n_0}(x) \in V.$$

Since f_{n_0} is ward continuous and (x_k) is quasi-Cauchy, we have:

$$f_{n_0}(x_k) - f_{n_0}(x_{k+1}) \to 0.$$

So there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$f_{n_0}(x_k) - f_{n_0}(x_{k+1}) \in V.$$

Now for all $k \geq K$, consider:

$$\begin{aligned} f(x_k) - f(x_{k+1}) &= [f(x_k) - f_{n_0}(x_k)] + [f_{n_0}(x_k) - f_{n_0}(x_{k+1})] + [f_{n_0}(x_{k+1}) - f(x_{k+1})] \\ &\in V + V + V \subseteq U. \end{aligned}$$

Thus, for all $k \geq K$, we have:

$$f(x_k) - f(x_{k+1}) \in U.$$

Therefore, we conclude that

$$f(x_k) - f(x_{k+1}) \to 0, \text{when } k \to \infty,$$

i.e., $(f(x_k))$ is quasi-Cauchy. Hence, f is ward continuous.

□

Corollary 3.4 *The set of all ward continuous functions on a subset E of X is a complete subspace of the space of all continuous functions on E.*

Proof: The proof follows straightforward from the preceding theorem. □

4. Conclusion

In this paper, we focus on quasi-Cauchy sequences and present theorems related to quasi-Cauchyness, ward compactness, some other kinds of compactness, and ward continuity in topological vector spaces. It is expected that this investigation will be a practical tool in topology, facilitating the modelling of problems across diverse scientific fields, including computer science, biology, dynamical systems, and information theory. On the other side, we recommend investigating the quasi-Cauchyness of the sequences consisting of fuzzy points, soft points, or neutrosophic points in topological vector spaces (see [17], [24] for related concepts in fuzzy setting, see [10], [2] for definitions and relevant notions in soft setting, and see [1] for the definitions in neutrosophic setting). Nevertheless, the definitions and proof techniques may differ, due to the variations in settings. An investigation of ward compactness can be conducted for double sequences in topological vector spaces(see [7], [25] and [22] for main concepts in the case of double sequences).

References

1. A. Acikgoz, H. Çakalli, F. Esenbel, Ljubisa DR Kocinac, A quest of G-continuity in neutrosophic spaces, Math. Methods Appl. Sci. **44** 9 (2021) 7834-7844. https://doi.org/10.1002/mma.7113
2. C.G. Aras, A. Sonmez, H. Cakalli, *An approach to soft functions*, J. Math. Anal. **8**, 2, 129-138, (2017)
3. D. Burton, and J. Coleman, Quasi-Cauchy Sequences, Amer. Math. Monthly. **117** 4 (2010), 328-333. https://doi.org/10.4169/000298910X480793
4. H. Cakalli, A Variation on Statistical Ward Continuity. Bull. Malays. Math. Sci. Soc. 40 (2017), 1701–1710. https://doi.org/10.1007/s40840-015-0195-0 .
5. H. Cakalli, Variations on statistical quasi Cauchy sequences, Bol. Soc. Paran. Mat. (3s.) **26** 2 (2019), 1-10. https://periodicos.uem.br/ojs/index.php/BSocParanMat/article/download/39991/751375146545/
6. H. Cakalli, C.G. Aras, A. Sonmez, Lacunary statistical ward continuity, AIP Conf. Proc. **1676**, **2015** 020042 https://doi.org/10.1063/1.4930468
7. H Cakalli and R.F. Patterson, Functions preserving slowly oscillating double sequences, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **62** 1,2 (2016), 531-536.
8. Huseyin Cakalli, Yasemin Simsek; Quasi Cauchy sequences in topological vector spaces. AIP Conf. Proc. 4 August 2025; 3431 (1): 020004. https://doi.org/10.1063/5.0290216
9. I. Canak and M. Dik, New types of continuities, Abstr. Appl. Anal. **2010** (2010) Article ID 258980, 6 pages. https://doi.org/10.1155/2010/258980

10. A. Coskun, C. Aras, H. Cakalli, A. Sonmez, "Soft matrices on soft multisets in an optimal decision process," 3rd International Conference on Analysis and Applied Mathematics (ICAAM) , vol.1759, Almaty, Kazakhstan, 2016

11. H.Çakallı, Forward compactness, Conference on Summability and Applications, Shawnee State University, November 6-November 8, (2009). ttps://webpages.math.luc.edu/ mgb/ShawneeConference/Articles/HuseyinCakalliOhio.pdf .

12. H. Çakallı, Forward continuity, J. Comput. Anal. Appl. **13** 2 (2011), 225-230.

13. H. Çakalli, Slowly oscillating continuity, Abstr. Appl. Anal. **2008** (2008) Article ID 485706 5 pages. https://doi.org/10.1155/2008/485706

14. H. Çakalli, Statistical ward continuity. Appl. Math. Lett. **24** (2011), 1724-1728. https://doi.org/10.1016/j.aml.2011.04.029

15. H. Çakalli, Statistical-quasi-Cauchy sequences, Math. Comput. Modelling 54 (2011), 1620-1624. https://doi.org/10.1016/j.mcm.2011.04.037

16. H. Çakalli, A new approach to statistically quasi Cauchy sequences, Maltepe J. Math., **1** 1 (2019), 1-8. https://dergipark.org.tr/en/pub/mjm/issue/41581/477328

17. H. Çakalli, and Pratulananda Das, Fuzzy compactness via summability, Appl. Math. Lett. **22** (2009), 1665-1669. https://doi.org/10.1016/j.aml.2009.05.015

18. H. Çakalli and A. Sonmez, Slowly oscillating continuity in abstract metric spaces, Filomat **27** (2013), 925-930. https://doi.org/10.2298/FIL1305925C

19. H. Çakalli, A. Sonmez, and C.G. Aras, λ-statistical ward continuity, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **63** 2 (2017), 313-321.

20. Fikriye Ince Dagci, Huseyin Cakalli; On variations on quasi Cauchy sequences in metric spaces. AIP Conf. Proc. 2 April 2019; 2086 (1): 030012. https://doi.org/10.1063/1.5095097

21. F. Ince Dağcı and H. Cakalli, Quasi-Cauchy sequences on asymmetric metric spaces, Filomat **38** 1–7 (2024), ISSN 0354-5180. https://doi.org/10.2298/FIL2425917D

22. D. Djurcic, Ljubisa D. R. Kocinac, M. R. Zizovic, Double Sequences and Selections, Abstr. Appl. Anal. Hindawi Publ. Corp., New York, **2012** (2012), Article ID 497594 6 pages. http://www.hindawi.com/journals/aaa/2012/497594/cta/ https://doi.org/10.1155/2012/497594.

23. S. Ersan and H. Cakalli, Ward Continuity in 2-Normed Spaces, Filomat **29** 7 (2015), 1507-1513. DOI 10.2298/FIL1507507E

24. Ljubisa D.R. Kocinac, Selection properties in fuzzy metric spaces, Filomat. **26** 2 (2012), 305-312. https://doi.org/10.2298/FIL1202305K

25. R.F. Patterson and H. Cakalli, Quasi Cauchy double sequences, Tbilisi Mathematical Journal **8** 2 (2015), 211-219. https://projecteuclid.org/journals/tbilisi-mathematical-journal/volume-8/issue-2/Quasi-Cauchy-double-sequences/10.1515/tmj-2015-0023.pdf

26. R.W. Vallin, Creating slowly oscillating sequences and slowly oscillating continuous functions (with an appendix by Vallin and H. Çakallı, Acta Math. Univ. Comenianae **25** (2011), 71-78.

Yasemin Şimşek,
Department of Mathematics,
Maltepe University,
Türkiye.
E-mail address: `yaseminsimsek@maltepe.edu.tr`

and

Hüseyin Çakallı,
MALTEPE / İSTANBUL
Istanbul University, Department of Mathematics, Emeritus, Istanbul, Türkiye.
E-mail address: `hcakalli@gmail.com ; hcakalli@istanbul.edu.tr`