



A Nonhomogeneous Steklov Problem with (p, q) -Laplace Differential Operator

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ABSTRACT: We consider a nonlinear Steklov problem driven by the (p, q) -Laplacian operator with a concave parametric term and an asymmetric perturbation. We prove a multiplicity theorem producing three non-trivial solutions all with sign information (two positive and one negative), when the parameter is sufficiently small. Under an oddness condition near the origin for the perturbation.

Key Words: (p, q) -Laplacian, Steklov problem, Mountain pass theorem.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper we study the following nonlinear, nonhomogeneous Steklov eigenvalue problem:

$$(S_{p,q}) \begin{cases} \Delta_p u + \mu \Delta_q u &= |u|^{p-2}u + \mu |u|^{q-2}u & \text{in } \Omega, \\ \langle |\nabla u|^{p-2} \nabla u + \mu |\nabla u|^{q-2} \nabla u, \nu \rangle &= \lambda |u|^{s-2}u + f(x, u) & \text{on } \partial\Omega, \end{cases}$$

where $1 < s < q \leq p < \infty$, $\mu \in \mathbb{R}_0^+$ and $\Delta_r u := \operatorname{div}(|\nabla u|^{r-2} \nabla u)$ is the r -Laplacian, $r > 1$. ν is the outward unit normal vector on $\partial\Omega$, $\langle \cdot, \cdot \rangle$ is the scalar product of \mathbb{R}^N , λ is a real parameter, while the reaction term $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function.

Elliptic equations involving differential operators of the form

$$Au := \operatorname{div}(D(u) \nabla u) = \Delta_p u + \Delta_q u,$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$, usually called (p, q) -Laplacian, occur in many important concrete situations. For instance, this happens when one seeks stationary solutions to the reaction-diffusion system.

$$u_t = Au + c(x, u), \tag{1.1}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences like chemical reaction design [4], biophysics [11] and plasma physics [23]. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.1) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x; u)$ has a polynomial form with respect to the concentration.

In the last few years, the (p, q) -Laplace attracts a lot of attention and has been studied in [6,7,8,25,26,9].

In the past, equations with 'concave terms' driven by the p -Laplace differential operator were investigated by Garcia Azorero et al. in [15] for $p > 1$ and by Guo and Zhang in [16] for $p \geq 2$. In [19], Martínez and Rossi, proved the existence of weak solutions of the following classical Steklov problem:

$$(S_p) \begin{cases} \Delta_p u &= |u|^{p-2}u + f(x, u) & \text{in } \Omega, \\ \langle |\nabla u|^{p-2} \nabla u, \nu \rangle &= \lambda |u|^{p-2}u + h(x, u) & \text{on } \partial\Omega, \end{cases}$$

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where the perturbations $h : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are bounded Carathéodory functions satisfying an integral condition of Landesmann-Lazer type. To the best of our knowledge, few articles treat equations driven by the (p, q) -Laplace operator with an asymmetric nonlinearity we mention the works of Papageorgiou and Radulescu in [21] and Marano et al. in [20].

Our purpose of this work is to extend some of the known results with Dirichlet boundary conditions on bounded domain (see, [20]). Our main results, stated in theorem 3.1 and theorem 3.2 present the existence of at least three distinct non-trivial smooth solutions provided that λ is a sufficiently small positive constant. Moreover, we provide precise sign information for all these solutions: two are positive, one is negative.

In the next section we review the main mathematical tools that will be required in the rest of the paper. In section 3, we obtain the existence of three constant sign solutions (two positive and one negative).

2. Preliminaries

Let X be a Banach space and X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X) . An operator $A : X \rightarrow X^*$ is called of type $(S)_+$ provided

$$x_n \rightharpoonup \text{ in } X, \quad \limsup_{n \rightarrow +\infty} \langle A(x_n), x_n - x \rangle \leq 0 \quad \Rightarrow \quad x_n \rightarrow x \text{ in } X.$$

Let $\varphi \in C^1(X)$. A point $\bar{x} \in X$ is called a critical point of φ if $\varphi'(\bar{x}) = 0$. A number $c \in \mathbb{R}$ is said to be a critical value of φ if there exists a critical point $\bar{x} \in X$ such that $\varphi(\bar{x}) = c$. We say that $\varphi \in C^1(X)$ satisfies the Cerami condition when

every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n \geq 1}$ is bounded and

$$\lim_{n \rightarrow +\infty} (1 + \|x_n\|)\varphi'(x_n) = 0 \quad \text{in } X^*$$

admits a strongly convergent subsequence. This condition is more general than the Palais-Smale (PS) condition which is common in critical point theory. However, it can be shown (see, for example, [13]) that the minimax theory of critical values remains valid if the (PS) condition is replaced by the Cerami condition. The next result is known as the a 'mountain-pass theorem'(see, [[13], p.648]).

Theorem 2.1 *If $\varphi \in C^1(X)$ and satisfies the Cerami condition, $x_0, x_1 \in X, \|x_1 - x_0\| > \rho > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} =: m_\rho,$$

and $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$, where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$, then $m_\rho \geq c$ and c is a critical value of φ .

In the analysis of problem $(S_{p,q})$ we will use the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\bar{\Omega})$ as well as the order cone

$$C^1(\bar{\Omega})_+ = \{u \in C^1(\bar{\Omega}) : u(x) \geq 0 \text{ for every } x \in \bar{\Omega}\}.$$

This cone has a non-empty interior described as follows:

$$\text{int}C^1(\bar{\Omega})_+ = \left\{ u \in C^1(\bar{\Omega}) : u(x) > 0 \text{ for all } x \in \bar{\Omega} \right\}.$$

Let $u, v : \Omega \rightarrow \mathbb{R}$ be measurable and let $t \in \mathbb{R}$. The symbol $u \leq v$ means $u(x) \leq v(x)$ for almost every $x \in \bar{\Omega}$, $t^\pm := \max\{\pm t, 0\}$. If $p \in [1, +\infty)$ then $p' := p/(p-1)$ is the conjugate exponent of p and p^* indicates the Sobolev conjugate in dimension N , namely

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{when } p < N, \\ \text{any } q > 1 & \text{for } p = N, \\ +\infty & \text{otherwise.} \end{cases}$$

Set, provided $r \in [1, +\infty]$,

$$L_+^r(\partial\Omega) = \{u \in L^r(\partial\Omega) : u \geq 0 \text{ a.e. in } \partial\Omega\}.$$

If $r < +\infty$ then, as usual, $\|u\|_{1,r} := \left(\int_{\Omega} (|\nabla u|^r + |u|^r) dx \right)^{1/r}$, $u \in W^{1,r}(\Omega)$.

$W^{1,r}(\Omega)^*$ denotes the dual space of $W^{1,r}(\Omega)$ while $A_r : W^{1,r}(\Omega) \rightarrow W^{1,r}(\Omega)^*$ is the nonlinear operator stemming from the negative r -Laplacian, i.e.

$$\langle A_r(u), v \rangle := \int_{\Omega} (|\nabla u|^{r-2} \nabla u \cdot \nabla v + |u|^{r-2} uv) dx \quad \forall u, v \in W^{1,p}(\Omega). \quad (2.1)$$

The next proposition gives the main properties of A_r (see, for example, Gasiński Papageorgiou [[14], p. 562]).

Proposition 2.1 *The map $A_r : W^{1,r}(\Omega) \rightarrow W^{1,r}(\Omega)^*$ defined by (2.1) is continuous, strictly monotone, bounded and of type $(S)_+$, i.e., if $u_n \rightharpoonup u$ in $W^{1,r}(\Omega)$ and $\limsup_{n \rightarrow +\infty} \langle A(u_n), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in $W^{1,r}(\Omega)$.*

Given $\theta \in L_+^\infty(\partial\Omega) \setminus \{0\}$, we define

$$\lambda_{1,r}(\theta) := \inf \left\{ \frac{\|u\|_{1,r}^r}{\int_{\partial\Omega} \theta |u|^r d\sigma} : u \in W^{1,r}(\Omega), u \neq 0 \right\}. \quad (2.2)$$

When no confusion can arise, simply write $\lambda_{1,r} := \lambda_{1,r}(1)$. Some basic properties of $\lambda_{1,r}(\theta)$ and its eigenfunctions are listed below (see, [2,3,5]).

Proposition 2.2 *Let $1 < r < +\infty$ and $\theta \in L_+^\infty(\partial\Omega) \setminus \{0\}$. Then:*

1) $\lambda_{1,r}(\theta)$ is positive and attained on a positive function $\hat{u}_{1,r} \in W^{1,r}(\Omega)$, which fulfills $\|\hat{u}_{1,r}\|_{L^r(\partial\Omega)} = 1$ as well as

$$A_r(u) = \lambda_{1,r}(\theta) \theta |u|^{r-2} u. \quad (2.3)$$

2) Solutions to (2.3) coincide with minima of (2.2) and form a one-dimensional linear space.

3) The function $\theta \mapsto \lambda_{1,r}(\theta)$ is monotone (strictly) decreasing with respect to the a.e. ordering of $L_+^\infty(\partial\Omega)$.

4) $\lambda_{1,r} \|u\|_{L^r(\partial\Omega)}^r \leq \|u\|_{1,r}^r$.

Proposition 2.3 *If $\theta \in L_+^\infty(\partial\Omega) \setminus \{\lambda_{1,r}\}$ and $\theta \leq \lambda_{1,r}$ then there exists a constant $c(\theta) > 0$ such that*

$$\|u\|_{1,r}^r - \int_{\partial\Omega} \theta |u|^r d\sigma \geq c(\theta) \|u\|_{1,r}^r \quad \forall u \in W^{1,r}(\Omega).$$

Proof: Let $\psi(u) = \|u\|_{1,r}^r - \int_{\partial\Omega} \theta |u|^r d\sigma$. It is clear that $\psi \geq 0$. Suppose that the lemma is not true. Since ψ is r -homogeneous, we can find a sequence $\{u_n\} \subset W^{1,r}(\Omega)$ such that

$$\|u_n\|_{1,r} = 1 \text{ and } \psi(u_n) \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (2.4)$$

Its clear that (2.4) implies

$$\int_{\partial\Omega} \theta |u_n|^r d\sigma \rightarrow 1, \text{ as } n \rightarrow \infty. \quad (2.5)$$

As u_n is bounded, by passing to a suitable subsequence if necessarily, we may affirm that

$$u_n \rightharpoonup u \text{ in } W^{1,r}(\Omega), \quad u_n \rightarrow u \text{ in } L^r(\partial\Omega). \quad (2.6)$$

Thus, we have

$$\int_{\partial\Omega} \theta |u_n|^r d\sigma \rightarrow \int_{\partial\Omega} \theta |u|^r d\sigma = 1 \text{ and } \psi(u) \leq \liminf_{n \rightarrow \infty} \psi(u_n) = 0.$$

Consequently,

$$u \not\equiv 0 \text{ and } \|u\|_{1,r}^r \leq \int_{\partial\Omega} \theta |u|^r d\sigma \leq \lambda_{1,r} \|u\|_{L^r(\partial\Omega)}^r. \quad (2.7)$$

From (2.7) and by item 4) of proposition 2.2, we have

$$\|u\|_{1,r}^r = \int_{\partial\Omega} \theta |u|^r d\sigma = \lambda_{1,r} \|u\|_{L^r(\partial\Omega)}^r.$$

Its follows that

$$\int_{\partial\Omega} (\theta - \lambda_{1,r}) |u|^r d\sigma = 0.$$

Taking into account $u \not\equiv 0$, we conclude that $\theta = \lambda_{1,r}$. Which is absurd and the lemma is thus proved. \square

3. Main Results

In this section we will construct three nontrivial constant-sign solutions to problem $(S_{p,q})$ provided the parameter is small enough. For this purpose, we state the following hypotheses on the nonlinearity $f(x, t)$:

$H_1(f)$ $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function with $f(x, 0) = 0$ for a.e. $x \in \partial\Omega$ and there is $C > 0$ such that

$$|f(x, t)| \leq C(1 + |t|^{p-1}) \quad \forall (x, t) \in \partial\Omega \times \mathbb{R}.$$

$H_2(f)$ There exists $\xi_1 \in L_+^\infty(\partial\Omega) \setminus \{\lambda_{1,p}\}$ satisfying

$$\lambda_{1,p} \leq \xi_1(x) \leq \liminf_{t \rightarrow +\infty} \frac{f(x, t)}{t^{p-1}} \quad \text{uniformly with respect to } x \in \partial\Omega.$$

$H_3(f)$ There exists $\xi_2 \in L_+^\infty(\partial\Omega) \setminus \{\lambda_{1,p}\}$ satisfying

$$\limsup_{t \rightarrow -\infty} \frac{f(x, t)}{|t|^{p-2}t} \leq \xi_2(x) \leq \lambda_{1,p} \quad \text{uniformly with respect to } x \in \partial\Omega.$$

$H_4(f)$ There exist $\delta_0, \theta_0 \in (0, 1)$ fulfilling

$$0 \leq f(x, t)t \leq \theta_0 \lambda_{1,p} |t|^p + \mu \lambda_{1,q} |t|^q \quad \forall (x, t) \in \partial\Omega \times [-\delta_0, \delta_0].$$

Remark 3.1 *It should be noted that $H(f)_1$ - $H(f)_3$ entail*

$$-C \leq \liminf_{t \rightarrow -\infty} \frac{f(x, t)}{|t|^{p-2}t} \leq \xi_2(x) \leq \lambda_{1,p} \leq \xi_1(x) \leq \limsup_{t \rightarrow +\infty} \frac{f(x, t)}{t^{p-1}} \leq C.$$

If $1 < s < q$ and $\lambda \in \mathbb{R}^+$, we put

$$\hat{f}_\lambda(x, t) := \lambda |t|^{s-2}t + f(x, t),$$

which still satisfies a growth condition like $H_1(f)$, but with a different positive constant depending on λ , say C_λ . We use truncation techniques, specifically, we introduce the truncations of the forcing terms

$$\hat{f}_{\lambda,+}(x, t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \lambda |t|^{s-2}t + f(x, t) & \text{if } t > 0, \end{cases}$$

and

$$\hat{f}_{\lambda,-}(x, t) = \begin{cases} 0 & \text{if } t \geq 0 \\ \lambda |t|^{s-2}t + f(x, t) & \text{if } t < 0, \end{cases}$$

which are a Carathéodory functions. Setting

$$\hat{F}_{\lambda, \mp}(x, t) := \int_0^t \hat{f}_{\lambda, \mp}(x, \tau) d\tau.$$

The energy functional $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$ stemming from problem $(S_{p,q})$ is defined by

$$\varphi_\lambda(u) := \frac{1}{p} \|u\|_{1,p}^p + \frac{\mu}{q} \|u\|_{1,q}^q - \int_{\partial\Omega} \int_0^t \hat{f}_\lambda(x, \tau) d\tau d\sigma \quad \forall u \in W^{1,p}(\Omega),$$

while the associated truncated functionals

$$\varphi_{\lambda, \mp}(u) := \frac{1}{p} \|u\|_{1,p}^p + \frac{\mu}{q} \|u\|_{1,q}^q - \int_{\partial\Omega} \hat{F}_{\lambda, \mp}(x, u(x)) d\sigma \quad \forall u \in W^{1,p}(\Omega),$$

turn out to be C^1 as well. Using the nonlinear regularity theory developed in [17,18], the strong maximum principle, and the Hopf boundary point lemma [[22], pp. 111 and 120], yields

Proposition 3.1 *Under $H_1(f)$ and $H_3(f)$, nontrivial critical points for $\varphi_{\lambda,+}$ (resp., $\varphi_{\lambda,-}$) actually are critical points of φ_λ and belong to $\text{int}C^1(\bar{\Omega})_+$ (resp., $-\text{int}C^1(\bar{\Omega})_+$).*

Our first aim is to check the Cerami condition for the functionals.

Lemma 3.1 *If $H_1(f)$ - $H_3(f)$ hold, then*

1. $\varphi_{\lambda,+}$ satisfies the Cerami condition.
2. $\varphi_{\lambda,-}$ is coercive (hence it fulfills the Cerami condition).
3. φ_λ satisfies the Cerami condition for all $\lambda > 0$.

Proof: 1. Consider a sequence $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ be such that $\{\varphi_{\lambda,+}(u_n)\}_{n \geq 1}$ is bounded and

$$(1 + \|u_n\|_{1,p}) \varphi'_{\lambda,+}(u_n) \rightarrow 0 \quad \text{in } W^{1,p}(\Omega)^*. \quad (3.1)$$

From (3.1) we have

$$\left| \langle A_p(u_n), v \rangle + \mu \langle A_q(u_n), v \rangle - \int_{\partial\Omega} \hat{f}_{\lambda,+}(x, u_n) v d\sigma \right| \leq \frac{\varepsilon_n}{1 + \|u_n\|_{1,p}} \|v\|_{1,p} \quad (3.2)$$

for all $v \in W^{1,p}(\Omega)$, all $n \geq 1$, with $\varepsilon_n \downarrow 0$. [where A_r is defined in (2.1), with $r = p, q$]. Choosing $v := -u_n^-$ we derive that

$$\|u_n^-\|_{1,p} + \mu \|u_n^-\|_{1,q} \leq \varepsilon_n,$$

which shows that $u_n^- \rightarrow 0$ and

$$\left| \langle A_p(u_n^+), v \rangle + \mu \langle A_q(u_n^+), v \rangle - \int_{\partial\Omega} \hat{f}_{\lambda,+}(x, u_n^+) v d\sigma \right| \leq \varepsilon'_n \|v\|_{1,p} \quad (3.3)$$

for some $\varepsilon'_n \downarrow 0$. Since the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ is compact, while $A_p + \mu A_q$ enjoys property $(S)_+$, it suffices to show that $\{u_n\}_{n \geq 1}$ is bounded in $W^{1,p}(\Omega)$. Arguing by contradiction, suppose that, along a relabeled subsequence, $\|u_n^+\|_{1,p} \rightarrow +\infty$ and put $w_n := \frac{u_n^+}{\|u_n^+\|_{1,p}}$. Then $\|w_n\|_{1,p} = 1$ and $w_n \geq 0$. We may assume that

$$w_n \rightharpoonup w \quad \text{in } W^{1,p}(\Omega), \quad w_n \rightarrow w \quad \text{in } L^p(\partial\Omega). \quad (3.4)$$

Hypothesis $H_1(f)$ implies that $\left\{ \frac{\hat{f}_{\lambda,+}(\cdot, u_n^+)}{\|u_n^+\|_{1,p}^{p-1}} \right\}_{n \geq 1}$ is bounded in $L^{p'}(\partial\Omega)$. Hence we may assume that

$$\frac{\hat{f}_{\lambda,+}(\cdot, u_n^+)}{\|u_n^+\|_{1,p}^{p-1}} \rightharpoonup h \quad \text{in } L^{p'}(\partial\Omega). \quad (3.5)$$

Recalling that $s < p$, namely $H_2(f)$ holds true for $\hat{f}_{\lambda,+}$. By $H_1(f)$ - $H_3(f)$ we have

$$\xi_1(x)w^{p-1} \leq \frac{\hat{f}_{\lambda,+}(x, u_n^+)}{\|u_n^+\|_{1,p}^{p-1}} \leq Cw^{p-1} + \frac{C}{\|u_n^+\|_{1,p}^{p-1}},$$

for a.e $x \in \partial\Omega$, all $n \geq 1$. Invoking Mazur's Theorem (e.g., Brezis [[10], p. 61]) and using (3.4) (3.5), we deduce $\xi_1(x)w^{p-1} \leq h \leq Cw^{p-1}$ for a.e $x \in \partial\Omega$. Hence there exists $\eta \in L^\infty(\partial\Omega)$ with $\xi_1(x) \leq \eta(x) \leq C$ such that

$$h = \eta w^{p-1}. \quad (3.6)$$

Through (3.3) we than have

$$\left| \langle A_p(w_n), v \rangle + \mu \|u_n^+\|_{1,p}^{q-p} \langle A_q(w_n), v \rangle - \int_{\partial\Omega} \frac{\hat{f}_{\lambda,+}(x, u_n^+)}{\|u_n^+\|_{1,p}^{p-1}} v d\sigma \right| \leq \frac{\varepsilon'_n}{\|u_n^+\|_{1,p}^{p-1}} \|v\|_{1,p}. \quad (3.7)$$

Choosing $v := w_n - w$ in (3.7), passing to the limit as $n \rightarrow \infty$ and using (3.4), we obtain

$$\lim_{n \rightarrow +\infty} \langle A_p(w_n), w_n - w \rangle = 0.$$

Since A_p is an $(S)_+$ map, we get

$$w_n \rightarrow w \quad \text{in } W^{1,p}(\Omega) \quad \text{as } n \rightarrow +\infty,$$

so, in particular $\|w\|_{1,p} = 1$, $w_n \geq 0$. Via (3.7) we thus obtain, letting $n \rightarrow +\infty$,

$$\langle A_p(w_n), v \rangle = \int_{\partial\Omega} \eta w^{p-2} v d\sigma. \quad \forall v \in W^{1,p}(\Omega),$$

i.e., w is an eigenfunction of the following p -Laplacian Steklov problem, with respect to the weight η ,

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda \eta |u|^{p-2} u & \text{on } \partial\Omega, \end{cases}$$

corresponding to the eigenvalue 1 (see, [5]). Due to (Item 3) of Proposition 2.2, $H_2(f)$, and (3.6),

$$1 = \lambda_{1,p}(\lambda_{1,p}) > \lambda_{1,p}(\xi_1) \geq \lambda_{1,p}(\eta).$$

Invoking point 2) in the same result ensures that w must be sign changing, a contradiction of the fact that $w \geq 0$.

2. By $H_1(f)$ and $H_3(f)$, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$F(x, t) \leq \frac{1}{p} (\xi_2(x) + \varepsilon) |t|^p + C_\varepsilon \quad \forall (x, t) \in \partial\Omega \times (-\infty, 0].$$

Thus, on account of Proposition 2.3,

$$\begin{aligned} \varphi_{\lambda,-}(u) &\geq \frac{1}{p} \|u\|_{1,p}^p + \frac{\mu}{q} \|u\|_{1,q}^q - \frac{\lambda}{s} \|u\|_{L^s(\partial\Omega)}^s - \int_{\partial\Omega} \frac{1}{p} (\xi_2 + \varepsilon) |u^-|^p d\sigma - C_\varepsilon |\partial\Omega| \\ &\geq \frac{1}{p} \|u^+\|_{1,p}^p + \frac{1}{p} \left(\|u^-\|_{1,p}^p - \int_{\partial\Omega} \xi_2 |u^-|^p d\sigma \right) - \frac{\varepsilon}{p} \|u^-\|_{L^p(\partial\Omega)}^p \\ &\quad - \frac{\lambda}{s} \|u\|_{L^s(\partial\Omega)}^s - C_\varepsilon |\partial\Omega| \\ &\geq \frac{1}{p} \|u^+\|_{1,p}^p + \frac{1}{p} \left(c(\xi_2) - \frac{\varepsilon}{\lambda_{1,p}} \right) \|u^-\|_{1,p}^p - \frac{\lambda}{s} \|u\|_{L^s(\partial\Omega)}^s - C_\varepsilon |\partial\Omega|. \end{aligned}$$

Choosing $\varepsilon := \frac{\lambda_{1,p}c(\xi_2)}{2}$ and recalling that $s < p$ finally provides the desired coercivity property.

3. Fix $\lambda > 0$. Let $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ be such that $\{\varphi_\lambda(u_n)\}_{n \geq 1}$ is bounded and

$$(1 + \|u_n\|_{1,p})\varphi'_\lambda(u_n) \rightarrow 0 \quad \text{in } W^{1,p}(\Omega)^*. \quad (3.8)$$

Then 3.2 holds with \hat{f}_λ instead of $\hat{f}_{\lambda,+}$. Choosing $v := -u_n^-$, it furnishes

$$\|u_n^-\|_{1,p}^p + \mu \|u_n^-\|_{1,q}^q - \lambda \|u_n^-\|_{L^s(\partial\Omega)}^s - \int_{\partial\Omega} f(x, -u_n^-)(-u_n^-) d\sigma \leq \varepsilon_n \quad n \in \mathbb{N}, \quad (3.9)$$

where $\varepsilon_n \downarrow 0^+$. Thanks to $H_1(f)$ and $H_3(f)$, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$f(x, t)t \leq (\xi_2(x) + \varepsilon)|t|^p + C_\varepsilon \quad \forall (x, t) \in \partial\Omega \times]-\infty, 0].$$

So, the proof on conclusion 2 in the previous lemma carries over, giving the coerciveness of the functional

$$u \mapsto \|u^-\|_{1,p}^p + \mu \|u^-\|_{1,q}^q - \lambda \|u^-\|_{L^s(\partial\Omega)}^s - \int_{\partial\Omega} f(x, -u^-)(-u^-) d\sigma \quad u \in W^{1,p}(\Omega).$$

Hence, due to (3.9), the sequence $\{u_n^-\}$ has to be bounded. To check that the same holds for $\{u_n^+\}$, suppose on the contrary $\|u_n^+\|_{1,p} \rightarrow +\infty$ and put $w_n := u_n / \|u_n^+\|_{1,p}$. Obviously, $\{w_n\}$ turns out to be bounded, because so is $\{u_n^-\}$. Moreover, $w_n^- \rightarrow 0$ while, along a subsequence when necessary

$$w_n \rightharpoonup w \quad \text{in } W^{1,p}(\Omega), \quad w_n \rightarrow w \quad \text{in } L^p(\partial\Omega).$$

As before, via $H_1(f)$ we see that $\left\{ \frac{\hat{f}_\lambda(\cdot, u_n^+)}{\|u_n^+\|_{1,p}^{p-1}} \right\}_{n \geq 1}$ is bounded in $L^{p'}(\partial\Omega)$. Now, divide the present version of (3.2) by $\|u_n^+\|_{1,p}^{p-1}$, test with $v := w_n - w$, use the inequality $q < p$, and let $n \rightarrow +\infty$ achieve

$$\lim_{n \rightarrow +\infty} \langle A_p(w_n), w_n - w \rangle = 0,$$

which implies $w_n \rightarrow w$ in $W^{1,p}(\Omega)$. Consequently, $w \geq 0$ and $\|w\|_{1,p} \geq 1$. Since

$$\int_{\partial\Omega} \left| \frac{\hat{f}_\lambda(\cdot, -u_n^-)}{\|u_n^+\|_{1,p}^{p-1}} \right|^{p'} \leq C_\lambda^{1/p'} \left(\frac{|\partial\Omega|}{\|u_n^+\|_{1,p}^p} + \frac{1}{\lambda_{1,p}} \frac{\|u_n^-\|_{1,p}^p}{\|u_n^+\|_{1,p}^p} \right)^{1/p'} \rightarrow 0,$$

we have

$$\frac{\hat{f}_\lambda(\cdot, u_n)}{\|u_n^+\|_{1,p}^{p-1}} - \frac{\hat{f}_\lambda(\cdot, u_n^+)}{\|u_n^+\|_{1,p}^{p-1}} \rightarrow 0 \quad \text{in } L^{p'}(\partial\Omega). \quad (3.10)$$

used the same argument as above (3.5) yield here

$$\frac{\hat{f}_\lambda(\cdot, u_n^+)}{\|u_n^+\|_{1,p}^{p-1}} \rightharpoonup \eta w^{p-1} \quad \text{in } L^{p'}(\partial\Omega),$$

with appropriate $\eta \in L^\infty(\partial\Omega)$ fulfilling $\xi_1(x) \leq \eta(x) \leq C$. Thanks to (3.10), this holds true also for $\left\{ \frac{\hat{f}_\lambda(\cdot, u_n)}{\|u_n^+\|_{1,p}^{p-1}} \right\}_{n \geq 1}$. Hence, from

$$\left| \langle A_p(w_n), v \rangle + \mu \|u_n^+\|_{1,p}^{q-p} \langle A_p(w_n), v \rangle - \int_{\partial\Omega} \frac{\hat{f}_\lambda(x, u_n)}{\|u_n^+\|_{1,p}^{p-1}} v d\sigma \right| \leq \frac{\varepsilon'_n}{\|u_n^+\|_{1,p}^{p-1}} \|v\|_{1,p}$$

(cf. (3.7)) it follows, when $n \rightarrow +\infty$,

$$\langle A_p(w), v \rangle = \int_{\partial\Omega} \eta w^{p-1} v d\sigma \quad \forall v \in W^{1,p}(\Omega).$$

Now the proof goes on exactly as the one of Item 1 in Lemma 3.1. \square

Lemma 3.2 *If $H_1(f)$ and $H_4(f)$ are satisfied then there exists a constant $\lambda^* > 0$ such that to every $\lambda \in (0, \lambda^*)$ we can find a $\rho_\lambda > 0$, for which we have*

$$m_\lambda := \inf\{\varphi_{\lambda,+}(u) : \|u\|_{1,p} = \rho_\lambda\} > 0.$$

Proof: Fix any $r \in (p, p^*)$. By $H_1(f)$ and $H_4(f)$ we have

$$f(x, t) \leq \theta_0 \lambda_{1,p} t^{p-1} + \mu \lambda_{1,q} t^{q-1} + C_r t^{r-1}, \quad \forall (x, t) \in \partial\Omega \times [0, +\infty).$$

which, when integrated, entails

$$F(x, t) \leq \theta_0 \frac{\lambda_{1,p}}{p} t^p + \mu \frac{\lambda_{1,q}}{q} t^q + \frac{C_r}{r} t^r \quad \text{in } \partial\Omega \times [0, +\infty).$$

Where $C_r > 0$. By the Sobolev, Hölder one has

$$\begin{aligned} \varphi_{\lambda,+}(u) &\geq \frac{1}{p} \|u\|_{1,p}^p + \frac{\mu}{p} \|u\|_{1,q}^q - \frac{\lambda}{s} \|u^+\|_{L^s(\partial\Omega)}^s - \theta_0 \frac{\lambda_{1,p}}{p} \|u\|_{L^p(\partial\Omega)}^p \\ &\quad - \mu \frac{\lambda_{1,q}}{q} \|u\|_{L^q(\partial\Omega)}^q - \frac{C_r}{r} \|u\|_{L^r(\partial\Omega)}^r \\ &\geq \frac{1-\theta_0}{p} \|u\|_{1,p}^p - \frac{\lambda}{s} \|u^+\|_{L^s(\partial\Omega)}^s - \frac{C_r}{r} \|u\|_{L^r(\partial\Omega)}^r \\ &\geq \frac{1-\theta_0}{p} \|u\|_{1,p}^p - C_1(\partial\Omega) \frac{\lambda}{s} \|u^+\|_{L^{p^*}(\partial\Omega)}^s - \frac{C_2(\partial\Omega)C_r}{r} \|u\|_{L^{p^*}(\partial\Omega)}^r \\ &\geq \left[\frac{1-\theta_0}{p} - C'_1 \lambda \|u\|_{1,p}^{s-p} - C'_2 \|u\|_{1,p}^{r-p} \right] \|u\|_{1,p}^p \end{aligned}$$

for appropriate positive constants C'_1, C'_2 . Letting $\|u\|_{1,p} = \lambda^{1/r-s}$ yields

$$\varphi_{\lambda,+}(u) \geq \left[\frac{1-\theta_0}{p} - C'_1 \lambda^{1-\frac{p-s}{r-s}} - C'_2 \lambda^{\frac{r-p}{r-s}} \right] \lambda^{\frac{1}{r-s}} = \left[\frac{1-\theta_0}{p} - (C'_1 + C'_2) \lambda^{\frac{r-p}{r-s}} \right] \lambda^{\frac{1}{r-s}}.$$

This immediately brings the conclusion, because $s < p < r$ and $0 < \theta_0 < 1$. □

From now on, λ^* will denote the real number found in Lemma 3.2.

Lemma 3.3 *Suppose $H_1(f)$ and $H_2(f)$ hold true. Then*

$$\lim_{\tau \rightarrow +\infty} \varphi_{\lambda,+}(\tau \hat{u}_{1,p}) = -\infty,$$

with $\hat{u}_{1,p}$ as in Proposition 2.2.

Proof: In view of $H_1(f)$ and $H_2(f)$, given $\varepsilon > 0$ we can find a constant M_ε such that

$$F(x, t) \geq \frac{\xi_1(x) - \varepsilon}{p} t^p - M_\varepsilon \quad \forall (x, t) \in \partial\Omega \times [0, +\infty). \quad (3.11)$$

The properties of $\hat{u}_{1,p}$ and ξ_1 produce

$$\int_{\partial\Omega} (\xi_1 - \lambda_{1,p}) \hat{u}_{1,p}^p d\sigma > 0.$$

Choose $\varepsilon > 0$ fulfilling

$$\theta := \int_{\partial\Omega} (\xi_1 - \lambda_{1,p}) \hat{u}_{1,p}^p d\sigma - \varepsilon \int_{\partial\Omega} \hat{u}_{1,p}^p d\sigma > 0.$$

Since $\|\hat{u}_{1,p}\|_{1,p}^p = \lambda_{1,p}\|\hat{u}_{1,p}\|_{L^p(\partial\Omega)}^p = \lambda_{1,p}$, via (3.11) we get

$$\begin{aligned} \varphi_{\lambda,+}(\tau\hat{u}_{1,p}) &\leq \frac{\tau^p}{p}\|\hat{u}_{1,p}\|_{1,p}^p + \mu\frac{\tau^q}{q}\|\hat{u}_{1,p}\|_{1,q}^q - \lambda\frac{\tau^s}{s}\|\hat{u}_{1,p}\|_{L^s(\partial\Omega)}^s \\ &\quad - \frac{\tau^p}{p}\int_{\partial\Omega}(\xi_1 - \varepsilon)\hat{u}_{1,p}^p d\sigma + M_\varepsilon|\partial\Omega| \\ &\leq -\theta\frac{\tau^p}{p} + \mu\frac{\tau^q}{q}\|\hat{u}_{1,p}\|_{1,q}^q - \lambda\frac{\tau^s}{s}\|\hat{u}_{1,p}\|_{L^s(\partial\Omega)}^s + M_\varepsilon|\partial\Omega| \end{aligned}$$

for all $\tau > 0$. The conclusion follows from $q < p$. \square

A further nontrivial smooth solution to $(S_{p,q})$ can now be found.

Theorem 3.1 *If hypotheses $H_1(f) - H_4(f)$ hold, then for all $\lambda \in (0, \lambda^*)$, problem $(S_{p,q})$ admits two positive solutions $u_1, u_2 \in \text{int}C^1(\bar{\Omega})_+$.*

Proof: Pick $\lambda \in (0, \lambda^*)$. Lemma 3.3 gives a $\tau > 0$ so large that $\varphi_{\lambda,+}(\tau\hat{u}_{1,p}) < 0$. On account of Lemma 3.1 and Lemma 3.2, Theorem 2.1 applies to $\varphi_{\lambda,+}$. Thus, there is $u_1 \in W^{1,p}(\Omega)$ fulfilling

$$\varphi_{\lambda,+}(u_1) \geq m_\lambda > 0 \quad (3.12)$$

and

$$\varphi'_{\lambda,+}(u_1) = 0. \quad (3.13)$$

From (3.12) we see that $u_1 \neq 0$, while (3.13) yields

$$A_p(u_1) + \mu A_q(u_1) = N_{f_+}(u_1), \quad (3.14)$$

where $N_{f_+}(u)(\cdot) = f_+(\cdot, u(\cdot))$ for all $u \in W^{1,p}(\Omega)$. On (3.14) we act with $-u_1^- \in W^{1,p}(\Omega)$ and obtain $u_1 \geq 0$, a fortiori, the function u_1 solves $(S_{p,q})$. By the regularity result in [18], $u_1 \in C^{1,\alpha}(\bar{\Omega})$. From the first equation of $(S_{p,q})$, we conclude

$$\begin{aligned} \Delta_p u_1 + \mu \Delta_q u_1 &= |u_1|^{p-2} u_1 + \mu |u_1|^{q-2} u_1 \\ &\leq u_1^{p-1} + \mu u_1^{q-1} \\ &\leq \mu' u_1^{q-1}. \end{aligned}$$

Setting $\beta(s) = \mu' s^{q-1}$ for $s > 0$ and note that $\int_{0+} 1/(s\beta(s))^{\frac{1}{q}} ds = +\infty$. Hence, the assumptions of Vázquez's strong maximum principle [24] are satisfied and we obtain $u_1 > 0$ for a.e. $x \in \Omega$. In order to prove that u is strictly positive in the closure of Ω , we suppose there exists $x_0 \in \partial\Omega$ such that $u_1(x_0) = 0$. Applying again the maximum principle yields $\frac{\partial u_1}{\partial\nu}(x_0) < 0$. However, we know that $0 \in \partial F(x_0, u(x_0)) = \partial F(x_0, 0)$ which leads to a contradiction in view of problem $(S_{p,q})$ because in this case we have $\frac{\partial u_1}{\partial\nu}(x_0) = 0$. Therefore, it holds $u_1(x) > 0$ in $\bar{\Omega}$, which implies $u_1 \in \text{int}(C^1(\bar{\Omega})_+)$.

Next, define

$$B_{\rho\lambda} := \{u \in W^{1,p}(\Omega) : \|u\|_{1,p} < \rho\lambda\},$$

where $\rho\lambda$ comes from Lemma 3.2. A standard procedure based on the weak sequential lower semicontinuity of $\varphi_{\lambda,+}$ ensures that this functional attains its minimum at some $u_2 \in \bar{B}_{\rho\lambda}$. Fix $w \in \text{int}C^1(\bar{\Omega})_+$ and choose $\tau_1 > 0$ complying with

$$\|\tau_1 w\|_{1,p} < \rho\lambda, \quad \tau_1 \sup_{x \in \bar{\Omega}} w(x) \leq \delta_0.$$

Thanks to $H_4(f)$ we have

$$f(x, \tau w(x)) \geq 0 \quad \forall \tau \in (0, \tau_1),$$

which easily entails

$$\varphi_{\lambda,+}(\tau w) \leq \frac{\tau^p}{p} \|w\|_{1,p}^p + \mu \frac{\tau^q}{q} \|w\|_{1,q}^q - \lambda \frac{\tau^s}{s} \|w\|_{L^s(\partial\Omega)}^s < 0$$

provided τ is sufficiently small (recall that $s < q < p$). Hence, a fortiori,

$$\varphi_{\lambda,+}(u_2) < 0.$$

Due to (3.12), we have

$$m := m_\lambda - \inf_{\overline{B}_{\rho\lambda}} \varphi_{\lambda,+} > 0,$$

where $\overline{B}_{\rho\lambda} := \{u \in W^{1,p}(\Omega) : \|u\|_{1,p} \leq \rho\lambda\}$. Let $\varepsilon \in (0, m)$. By Ekeland's variational principle [12], there exists $u_\varepsilon \in \overline{B}_{\rho\lambda}$ such that

$$\varphi_{\lambda,+}(u_\varepsilon) \leq \inf_{\overline{B}_{\rho\lambda}} \varphi_{\lambda,+} + \varepsilon \quad (3.15)$$

and

$$\varphi_{\lambda,+}(u_\varepsilon) \leq \varphi_{\lambda,+}(y) + \varepsilon \|y - u_\varepsilon\|_{1,p} \quad \text{for all } y \in \overline{B}_{\rho\lambda}. \quad (3.16)$$

From inequality (3.15) and since $\varepsilon < m$, we have $\varphi_{\lambda,+}(u_\varepsilon) < \inf_{\partial B_{\rho\lambda}} \varphi_{\lambda,+}$. It follows that $u_\varepsilon \in B_{\rho\lambda}$. This ensures that, for every $h \in W^{1,p}(\Omega)$, we have $u_\varepsilon + th \in B_{\rho\lambda}$ whenever $t > 0$ is sufficiently small. Inserting $y = u_\varepsilon + th$ in (3.16) dividing by t and then letting $t \rightarrow 0$ leads to

$$-\varepsilon \|h\|_{1,p} \leq \langle \varphi'_{\lambda,+}(u_\varepsilon), h \rangle.$$

Taking into account the fact that $h \in W^{1,p}(\Omega)$ was arbitrary, we arrive at

$$\|\varphi'_{\lambda,+}(u_\varepsilon)\|_{1,p} \leq \varepsilon.$$

Fix a sequence $\varepsilon_n \downarrow 0$ and denote $u_n = u_{\varepsilon_n}$. Then one has $\varphi'_{\lambda,+}(u_n) \rightarrow 0$ and also $(1 + \|u_n\|_{1,p})\varphi'_{\lambda,+}(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Applying lemma 3.1 guarantees the existence of $u_2 \in W^{1,p}(\Omega)$ such that, along a relabeled subsequence, $u_n \rightarrow u_2 \in W^{1,p}(\Omega)$. It enables us to pass to the limite in (3.15) and, by (3.16), we obtain

$$\varphi_{\lambda,+}(u_2) = \inf_{\partial B_{\rho\lambda}} \varphi_{\lambda,+} < 0 = \varphi_{\lambda,+}(0),$$

and therefore $u_2 \neq 0$. Moreover, comparing this with (3.16), we see that $u_2 \neq u_1$. Since $\varphi'_{\lambda,+}(u_n) \rightarrow 0$ in $W^{1,p}(\Omega)^*$ and $u_n \rightarrow u_2$ in $W^{1,p}(\Omega)$ as $n \rightarrow \infty$, we infer that $\varphi'_{\lambda,+}(u_2) = 0$. Finally, du to the Proposition 3.1, the function u_2 solves $(S_{p,q})$ and lies in $u_2 \in \text{int}(C^1(\overline{\Omega})_+)$. This completes the proof. \square

Theorem 3.2 *Let $H_1(f) - H_4(f)$ be satisfied. Then: For every $\lambda > 0$ there exists a negative solution $u_3 \in -\text{int}C^1(\overline{\Omega})_+$ to $(S_{p,q})$.*

Proof: $\varphi_{\lambda,-}$ is coercive (cf. Lemma 3.1) and weakly sequentially lower semicontinuous. So, it attains its minimum at some $u_3 \in W^{1,p}(\Omega)$. As before, we see that $\varphi_{\lambda,-}(u_3) < 0$, whence $u_3 \neq 0$. $\varphi'_{\lambda,-}(u_3) = 0$. Finally, using similar arguments as for u_1 , we deduce that u_3 solves $(S_{p,q})$ and $u_3 \in -\text{int}(C^1(\overline{\Omega})_+)$. \square

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