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Theoretical Analysis of Uniqueness and Stability for Nonlinear Volterra–Fredholm Systems

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ABSTRACT: The primary goal of this research is to investigate the Hyers-Ulam stability (HUS) and Hyers-Ulam-Rassias stability (HURS) for impulsive Volterra-Fredholm system. The uniqueness of solutions for the proposed system are investigated using the fixed-point technique (FPT). We further investigate the HUS and HURS using generalized Gronwall inequality alongside Picard's operator technique of the equation under the Caputo fractional derivative by establishing appropriate conditions. Finally, a numerical application is shown to demonstrate the stability of the proposed system.

Keywords: Volterra-Fredholm system, nonlinear impulsive control system, fixed-point theorem, generalized Gronwall inequality.

Contents

1. Introduction

Integral equations (IEs) and integro-differential equations (IDEs) focuses on integral and derivative operators has a similar mathematical history as conventional differential [2,3,4,5]. IEs are commonly used to depict and understand complex processes in science and engineering. It entails calculating integrals and derivatives with integer orders. It has gained popularity in recent years due to its applications in biology [7], economics [8], control theory [9], and mechanics [10].

Volterra integral equations (IEs) and Volterra integro-differential equations (IDEs) were introduced by Vito Volterra in 1926. Since then, they have been extensively used in research and engineering [13,14]. These equations are used in a variety of physical applications, including nanohydrodynamics, heat transfer, diffusion processes in general neutron diffusion, and the interaction of biological species with varying generation rates.

Ulam addressed an issue in the 1940s concerning the stability of the Cauchy equation, to which Hyers provided a partial solution. In 1978, Rassias expanded on Hyers' notion, determining a more broad bound for the norm of the Cauchy difference. This concept of stability is referred to as Ulam's type stability. Many scholars have since investigated HU stability in ordinary differential equations (DEs), partial DEs, and fractional DEs (see [1,3,6,7,8,11,12,21,28] and references therein).

The HU stability problem is a key issue in classical physical systems, including plasma physics, electrical circuits, aerodynamics, and many other domains. Among these findings, Ulam's type stability of IDEs has piqued the interest of numerous academics; Wang et al. [25] acquired the first IDE-related finding in 2012. They demonstrated Ulam's type stability for first-order nonlinear IDEs on a closed, bounded interval with finite impulse. Furthermore, several generalizations of HU stability have been intensively

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researched, and many elegant conclusions have been produced by employing diverse methodologies; for example, see [7,11,19,20]. Tunç [24] explored the stability of the zero solution and the boundedness of all solutions to the nonlinear Volterra IDE with delay by developing new Lyapunov functions. In 2016, A. Zada et al. [26] investigated the HU stability and HUR stability for first-order IDEs with the following delay:

$$\begin{aligned}
&Q'(q)=G(q,Q(q),Q(h(q))), I=[q_0,\upsilon],\ q\in I'\triangleq I\backslash\{q_1,q_2,\cdots,q_m\}\\
&Q(q)=\alpha(q),\ q\in[q_0-\theta,q_0]\\
&\Delta Q(q_k)=Q(q_k^+)-Q(q_k^-)=Y_k(Q(q_k^-)), k=1,2,\cdots,m
\end{aligned}$$

where $\theta>0, T>q_0\geq 0$ are fixed points, and $G:[q_0,\upsilon]\times\mathbb{R}^2\to\mathbb{R}, Y_k:\mathbb{R}\to\mathbb{R}$ and $\alpha:[q_0-\theta,q_0]\to\mathbb{R}$ are continuous functions. $Q(q_k^+)=\lim_{s\to 0^+}Q(q_k+s)$ and $Q(q_k^-)=\lim_{s\to 0^+}Q(q_k-s)$ are the right and left side limits of $Q(q)$ at $q_k, k=1,2,\cdots,m$, where q_k satisfy $q_0<q_1<q_2<\cdots<q_m<q_{m+1}=T<+\infty$. In 2019, using Gronwall integral inequality, A. Zada et al. [27] investigated the existence and uniqueness theorem for the solutions of a class of nonlinear impulsive IEs with a bounded variable delay. Moreover, the HU stability and HUR stability of the IEs were obtained with the help of open mapping theorem approach.

Aruldass et al. [7] proposed a new method for investigating the Ulam stability of linear differential equations of the form $\varkappa'(\upsilon)+\mu\varkappa(\upsilon)=0$ and the nonhomogeneous linear DE $\varkappa'(\upsilon)+\mu\varkappa(\upsilon)=r(\upsilon)$ by applying Kamal transform method in 2021. In the same year, using FPT in the sense of Cadariu and Radu, Murali et al. [8] proved the HU stability and HUR stability of the n-order DE. Refaai et al. [22] discussed the HU stability of fractional impulsive Volterra delay IDEs of the form:

$$\varkappa_1(\upsilon)=I^{\alpha}_{\omega,\upsilon}f\left(\upsilon,\varkappa_1(\upsilon),\varkappa_1(h(\upsilon)),\int_{\omega}^{\upsilon}g(\upsilon,\theta,\varkappa_1(\theta),\varkappa_1(h(\theta)))\,d\theta\right),\ \upsilon\in I'$$

$$\triangle\varkappa_1(\upsilon_k)=\varkappa_1(\upsilon_k^+)-\varkappa_1(\upsilon_k^-)=\beta_k\int_{\upsilon_k-\theta_k}^{\upsilon_k-e_k}U(\varkappa_1(s))\,ds,\ k=1,2,\cdots,m$$

$$\varkappa_1(\upsilon)=\phi(\upsilon),\quad \upsilon\in[\omega-\theta,\omega]$$

where $\theta>0,\beta_k\geq 0,\ 0\leq e_k\leq\theta_k\leq\upsilon_k-\upsilon_{k-1}$ for $k=1,2,\cdots,m, T>\omega\geq 0, f:[\omega,\upsilon]\times\mathbb{R}^3\to\mathbb{R}$ and $g:[\omega,\upsilon]\times[\omega,\upsilon]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions, $\phi:[\omega-\theta,\omega]\to\mathbb{R}$ is a delay function, and $I^{\alpha}_{\omega,\upsilon}f$ is the Riemann-Liouville fractional integral of order α. Their analysis was based on Pachpatte's inequality and the FPT represented by the Picard operators.

Thereafter, using the fixed point theory and the generalized metric, E. El-hady et al. [12] obtained HUR stability for the following impulsive Volterra IE of second kind

$$\varkappa(\upsilon)=\int_{\omega}^{\upsilon}G(p,\varkappa(p))dp+\sum_{0<\upsilon_k<\upsilon}U(\varkappa(\upsilon_k^-))$$

where $U:\mathbb{C}\to\mathbb{C}, \varkappa(\upsilon_k^-)$ represents the left limit of $\varkappa(\upsilon)$ at $\upsilon=\upsilon_k, k=1,2,\cdots,m$ and G is a continuous function.

Motivated by the above mentioned papers, in this paper, using a novel generalized Gronwall inequality and the FPT, we investigate the HU stability and HUR stability of impulsive delay IDE of the form:

$$\begin{aligned}
&\varkappa'(\upsilon)=G(\upsilon,\varkappa(\upsilon),\varkappa(h(\upsilon)))+\int_{\omega}^{\upsilon}E(\upsilon,s,\varkappa(s),\varkappa(h(s)))ds+\int_{\omega}^{T}\zeta(\upsilon,s,\varkappa(s),\varkappa(h(s)))ds,\ \upsilon\in I'\\
&\varkappa(\upsilon)=\alpha(\upsilon),\ \upsilon\in[\omega-\theta,\omega]\\
&\triangle\varkappa(\upsilon_k)=\varkappa(\upsilon_k^+)-\varkappa(\upsilon_k^-)=\phi_k(\varkappa(\upsilon_k^-)), k=1,2,\cdots,m
\end{aligned}\tag{1.1}$$

where $\theta>0, T>\omega\geq 0, \alpha(\upsilon):[\omega-\theta,T]\to\mathbb{R}$ is a continuous function, $G:[\omega,T]\times\mathbb{R}^2\to\mathbb{R},\ E,\zeta:[\omega,T]\times[\omega,T]\times\mathbb{R}^2\to\mathbb{R}$, and $\phi_k:\mathbb{R}\to\mathbb{R}$. Moreover, we assume that $h:[\omega,T]\to[\omega-\theta,\upsilon]$ is a continuous delay function such that $h(\upsilon)\leq\upsilon$. $\varkappa(\upsilon_k^+)=\lim_{\triangle\upsilon\to 0^+}\varkappa(\upsilon_k+\Delta t)$ and $\varkappa(\upsilon_k^-)=\lim_{\triangle\upsilon\to 0^-}\varkappa(\upsilon_k+\Delta\upsilon)$ are the right and left side limits of $\varkappa(\upsilon)$ at υ_k, where υ_k satisfy $\omega<\upsilon_1<\cdots<\upsilon_{m+1}=T<+\infty$.

The remainder of this paper is organized as follows: Section 2 presents key notations, reviews some concepts and preliminary findings, and proposes a new Gronwall inequality. Section 3 proves the uniqueness, and HU stability of the system (1.1) using the FPT and a novel generalized Gronwall inequality. Using a generalized Gronwall inequality, Section 4 shows that the system (1.1) has HUR stability. Section 5 includes an exemplary case based on the neural Volterra-Fredholm system that demonstrates how our key results can be used.

2. Preliminaries

This section contains preliminary information, such as necessary symbols, definitions, and lemmas [15,16,17,18,19].

Let $C(J)$ be the Banach space of all continuous real valued functions outlined on J with the norm $\|z\|_c = \sup\{|z(\upsilon)| : \upsilon \in J\}$, where J is a compact interval. Let $PC([\omega-\theta,\upsilon])$ be the collection of piecewise continuous functions $z : [\omega-\theta,\upsilon] \to \mathbb{R}$ with discontinuous points υ_k satisfying $\omega < \upsilon_1 < \upsilon_2 < \cdots < \upsilon_m < T \triangleq \upsilon_{m+1}$ and $z\left(\upsilon_k^+\right), z\left(\upsilon_k^-\right)$exist and are finite for $k = 1,2,\cdots,m$. With norm $\|z\|_{PC} = \sup\{|z(\upsilon)| : \upsilon \in [\omega-\theta,\upsilon]\}$, it is easy to see that $PC([\omega-\theta,\upsilon])$ is a Banach space.

In a similar manner we construct a Banach space.

$$PC^1([\omega-\theta,\upsilon]) = \{z \mid z \in PC([\omega-\theta,\upsilon]) \text{ and } z' \in PC([\omega-\theta,\upsilon])\}$$

with norm

$$\|z\|_{PC^1} = \max\{\|z'\|_{PC}, \|z\|_{PC}\}$$

Now, for $\varepsilon > 0$, and an increasing, nonnegative function $\varphi(\upsilon) \in PC([\omega-\theta,\upsilon])$ with $\varphi'(\upsilon) > 0$ and $\chi > 0$ with $\varphi(\upsilon^*) = \chi > 0$ for some $\upsilon^* \in [\omega-\theta,\upsilon]$, we suppose the next inequalities.

$$\begin{aligned}&\left|\varkappa' - G(\upsilon,\varkappa(\upsilon),\varkappa(h(\upsilon))) - \int_\omega^\upsilon E(\upsilon,s,\varkappa(s),\varkappa(h(s)))ds - \int_\omega^T \zeta(\upsilon,s,\varkappa(s),\varkappa(h(s)))ds\right| \le \varepsilon,\ \upsilon \in I' \\ &\left|\Delta\varkappa(\upsilon_k) - \phi_k\left(\varkappa\left(\upsilon_k^-\right)\right)\right| \le \varepsilon,\ k = 1,2,\cdots,m \end{aligned} \tag{2.1}$$

$$\begin{aligned}&\left|\varkappa' - G(\upsilon,\varkappa(\upsilon),\varkappa(h(\upsilon))) - \int_\omega^\upsilon E(\upsilon,s,\varkappa(s),\varkappa(h(s)))ds - \int_\omega^T \zeta(\upsilon,s,\varkappa(s),\varkappa(h(s)))ds\right| \le \varphi(\upsilon),\ \upsilon \in I' \\ &\left|\Delta\varkappa(\upsilon_k) - \phi_k\left(\varkappa\left(\upsilon_k^-\right)\right)\right| \le \chi,\ k = 1,2,\cdots,m. \end{aligned} \tag{2.2}$$

Definition 2.1 *(Picard operator [23]) Let ($Z; d$) be a metric space. An operator $T : Z \to Z$ is said to be a Picard operator if there exists $z^* \in Z$ such that :*
(i) $G_\upsilon = \{z^*\}$, where $G_\upsilon = \{z \in Z : T(z) = z\}$ is the fixed point set of υ;
(ii) The sequence $\{\upsilon^n(z)\}_{n\in N}$ converges to z^* for all $z \in Z$.

Definition 2.2 *The system (1.1) is HU stable on $[\omega-\theta,\upsilon]$ if for each $\varkappa \in PC([\omega-\theta,\upsilon]) \cap PC^1([\omega,\upsilon])$ satisfying (2.1), there exists a solution $\varkappa_0 \in PC([\omega-\theta,\upsilon]) \cap PC^1([\omega,\upsilon])$ of the system (**??**) with $|\varkappa_0(\upsilon) - \varkappa(\upsilon)| \le K\varepsilon$ for all $\upsilon \in [\omega-\theta,\upsilon]$, where $K > 0$ is a constant.*

Definition 2.3 *The system (1.1) is HUR stable on $[\omega-\theta,\upsilon]$ with respect to (φ,χ) if for each $\varkappa \in PC([\omega-\theta,\upsilon]) \cap PC^1([\omega,\upsilon])$ satisfying (2.2), there exists a solution $\varkappa_0 \in PC([\omega-\theta,\upsilon]) \cap PC^1([\omega,\upsilon])$ of the system (1.1) with $|\varkappa_0(\upsilon) - \varkappa(\upsilon)| < M\varphi(\upsilon)$ for all $\upsilon \in [\omega-\theta,\upsilon]$, where $M > 0$ is a constant.*

Lemma 2.1 *(Abstract Gronwall Lemma [26]) Let $(E, d, \le)$ be an ordered metric space and let $Y : E \to E$ be an increasing Picard operator with fixed point q^*. Then for any $q \in E$, $q \le Y(q)$ implies $q \le q^*$ and $q \ge Y(q)$ implies $q \ge q^*$, where q^* is the fixed point of Y in E.*

Lemma 2.2 *(Generalized Gronwall Lemma) If for $\upsilon \ge \omega \ge 0$, we have*

$$z(\upsilon) \le f(\upsilon) + \int_\omega^\upsilon g(r)\left(z(r) + \int_\omega^r c(\theta)z(\theta)d\theta\right)dr + \sum_{\omega<\upsilon_k<\upsilon} z\left(\upsilon_k^-\right)\eta_k, \tag{2.3}$$

where $z, f, g, c \in PC([\omega,\infty))$ are nonnegative functions, $f(\upsilon)$ is nondecreasing and $\eta_k > 0$ for $k = 1,2,\cdots,m$. Then for $\upsilon \geq \omega$, the following inequality holds

$$z(\upsilon) \leq a_k(\upsilon) H(\upsilon_k, \upsilon), \quad \upsilon \in (\upsilon_k, \upsilon_{k+1}], \ k = 0,1,2,\cdots,m, \tag{2.4}$$

with $f_0(\upsilon) = f(\upsilon)$, where

$$f_k(\upsilon) = f(\upsilon)\prod_{i=1}^{k}\left[1 + H(\upsilon_{i-1},\upsilon_i)\left(A(\upsilon_{i-1},\upsilon_i) + \eta_i\right)\right] \tag{2.5}$$

$$A(\upsilon_{i-1},\upsilon) = \int_{\upsilon_{i-1}}^{\upsilon} g(r)\left(1 + \int_{\omega}^{r} c(\theta)d\theta\right)dr \tag{2.6}$$

$$H(\upsilon_k,\upsilon) = 1 + A(\upsilon_k,\upsilon)\left(1 + \int_{\upsilon_k}^{\upsilon} g(r)\exp\left\{\int_{\upsilon_k}^{r}(g(\theta)+c(\theta))d\theta\right\}dr\right) \tag{2.7}$$

Proof: Since z, f, g and c be piecewise nonnegative continuous functions, $f(\upsilon)$ is nondecreasing and $\eta_k > 0$, for $\omega \leq \upsilon \leq \upsilon_1$, we get

$$z(\upsilon) \leq f(\upsilon) + \int_{\omega}^{\upsilon} g(r)z(r)dr + \int_{\omega}^{\upsilon} g(r)\int_{\omega}^{s} c(\theta)z(\theta)d\theta dr$$

Obviously

$$z(\upsilon) \leq f(\upsilon) + w(\upsilon), \tag{2.8}$$

where $w(\upsilon) = \int_{\omega}^{\upsilon} g(r)z(r)dr + \int_{\omega}^{\upsilon} g(r)\int_{\omega}^{r} c(\theta)z(\theta)d\theta dr$ and $w(\omega) = 0$. It is easy to see that

$$\begin{aligned} w(\upsilon) &\leq \int_{\omega}^{\upsilon} g(r)\left(f(r) + w(r) + \int_{\omega}^{r} c(\theta)(f(\theta)+w(\theta))d\theta\right)dr \\ &= \int_{\omega}^{\upsilon} g(r)f(r)dr + \int_{\omega}^{\upsilon} g(r)\int_{\omega}^{r} c(\theta)f(\theta)d\theta dr \\ &\quad + \int_{\omega}^{\upsilon} g(r)\left(w(r) + \int_{\omega}^{r} c(\theta)w(\theta)d\theta\right)dr \end{aligned} \tag{2.9}$$

Let $J(\upsilon) = \int_{\omega}^{\upsilon} g(r)f(r)dr + \int_{\omega}^{\upsilon} g(r)\int_{\omega}^{r} c(\theta)f(\theta)d\theta dr$, we get $J(\omega) = 0$. Since $f(\upsilon)$ is nondecreasing, we get $J(\upsilon) \leq f(\upsilon)A(\omega,\upsilon)$, where $A(\omega,\upsilon)$ is defined by (2.6).

For $\upsilon > \omega$, dividing both sides of (2.9) by $J(\upsilon)$, we have

$$\begin{aligned} \frac{w(\upsilon)}{J(\upsilon)} &\leq 1 + \frac{1}{J(\upsilon)}\int_{\omega}^{\upsilon} f(r)\left(w(r) + \int_{\omega}^{r} c(\theta)w(\theta)d\theta\right)dr \\ &\leq 1 + \int_{\omega}^{\upsilon} g(r)\left(\frac{w(r)}{J(r)} + \int_{\omega}^{r} c(\theta)\frac{w(\theta)}{J(\theta)}d\theta\right)dr \end{aligned}$$

Let

$$Y(\upsilon) = \frac{w(\upsilon)}{J(\upsilon)}, \tag{2.10}$$

we get

$$Y(\upsilon) \leq 1 + \int_{\omega}^{\upsilon} f(r)\left(Y(r) + \int_{\omega}^{r} c(\theta)Y(\theta)d\theta\right)dr$$

Applying Pachpatte's inequalities [?], we obtain

$$Y(\upsilon) \leq 1 + \int_{\omega}^{\upsilon} g(r)\exp\left\{\int_{\omega}^{r}[g(\theta)+c(\theta)]d\theta\right\}dr$$

Hence, (2.10) implies that

$$
\begin{aligned}
w(\upsilon) &\le J(\upsilon)\left(1+\int_{\omega}^{\upsilon} g(r)\exp\left\{\int_{\omega}^{r}[g(\theta)+c(\theta)]d\theta\right\}dr\right) \\
&\le f(\upsilon)A(\omega,\upsilon)\left(1+\int_{\omega}^{\upsilon} g(r)\exp\left\{\int_{\omega}^{r}[g(\theta)+c(\theta)]d\theta\right\}dr\right),
\end{aligned} \tag{2.11}
$$

then, for $\upsilon \in [\omega, \upsilon_1]$, and from (2.8) and (2.11), we have

$$
z(\upsilon) \le f(\upsilon)H(\omega,\upsilon) \tag{2.12}
$$

Specifically, we get

$$
z\left(\upsilon_1^-\right) = z(\upsilon_1) \le f(\upsilon_1)H(\omega,\upsilon_1). \tag{2.13}
$$

For $\upsilon \in (\upsilon_1, \upsilon_2]$, using (2.12) and (2.13), we get

$$
\begin{aligned}
z(\upsilon) &\le f(\upsilon)+\eta_1 z\left(\upsilon_1^-\right)+\int_{\omega}^{\upsilon} g(r)\left(z(r)+\int_{\omega}^{r} c(\theta)z(\theta)d\theta\right)ds \\
&= B_1(\upsilon)+\int_{\upsilon_1}^{\upsilon} g(r)\left(z(r)+\int_{\omega}^{r} c(\theta)z(\theta)d\theta\right)dr
\end{aligned}
$$

where

$$
\begin{aligned}
B_1(\upsilon) &= a(\upsilon)+\eta_1 z\left(\upsilon_1^-\right)+\int_{\omega}^{\upsilon_1} g(r)\left(z(r)+\int_{\omega}^{r} c(\theta)z(\theta)d\theta\right)dr \\
&\le f(\upsilon)+z\left(\upsilon_1^-\right)\left(A(\omega,\upsilon_1)+\eta_1\right) \\
&\le f(\upsilon)\left[1+H(\omega,\upsilon_1)\left(A(\omega,\upsilon_1)+\eta_1\right)\right] \\
&= f_1(\upsilon)
\end{aligned}
$$

thus, we have

$$
z(\upsilon) \le f_1(\upsilon)+\int_{\upsilon_1}^{\upsilon} g(r)\left(z(r)+\int_{\omega}^{r} c(\theta)z(\theta)d\theta\right)dr
$$

Let

$$
Y_1(\upsilon) = \frac{z(\upsilon)}{f_1(\upsilon)} \tag{2.14}
$$

we get

$$
Y_1(\upsilon) \le 1+\int_{\upsilon_1}^{\upsilon} g(r)\left(Y_1(r)+\int_{\omega}^{r} c(\theta)Y_1(\theta)d\theta\right)dr
$$

Hence, we get

$$
Y_1(\upsilon) \le 1+k(\upsilon), \tag{2.15}
$$

where

$$
k(\upsilon) = \int_{\upsilon_1}^{\upsilon} g(r)\left(Y_1(r)+\int_{\omega}^{r} c(\theta)Y_1(\theta)d\theta\right)dr
$$

and $k(\upsilon_1) = 0$. Then we get

$$
\begin{aligned}
k'(\upsilon) &= g(\upsilon)\left(Y_1(r)+\int_{\omega}^{\upsilon} c(r)Y_1(r)dr\right) \\
&\le g(\upsilon)\left(1+\int_{\omega}^{\upsilon} c(r)dr\right)+g(\upsilon)\left(k(\upsilon)+\int_{\omega}^{\upsilon_1} c(r)k(r)dr+\int_{\upsilon_1}^{\upsilon} c(r)k(r)dr\right) \\
&\le g(\upsilon)\left(1+\int_{\omega}^{\upsilon} c(r)ds\right)+g(\upsilon)\left(k(\upsilon)+\int_{\upsilon_1}^{\upsilon} c(r)k(r)dr\right).
\end{aligned} \tag{2.16}
$$

Integrating on both sides of (2.16) from υ_1 to υ, we have

$$k(\upsilon) \leq A(\upsilon_1, \upsilon) + \int_{\upsilon_1}^{\upsilon} g(r)\left(k(r) + \int_{\upsilon_1}^{r} c(\theta)k(\theta)d\theta\right) dr.$$

Since $A(\upsilon_1, \upsilon) > 0$ for $\upsilon > \upsilon_1$, we have

$$\frac{k(\upsilon)}{A(\upsilon_1, \upsilon)} \leq 1 + \int_{\upsilon_1}^{\upsilon} g(r)\left(\frac{k(r)}{A(\upsilon_1, r)} + \int_{\upsilon_1}^{r} c(\theta)\frac{k(\theta)}{A(\upsilon_1, \theta)}d\theta\right) dr.$$

From Pachpatte's inequalities, we have

$$k(\upsilon) \leq A(\upsilon_1, \upsilon)\left(1 + \int_{\upsilon_1}^{\upsilon} g(r) \exp\left\{\int_{\upsilon_1}^{r} (g(\theta) + c(\theta))d\theta\right\} dr\right). \tag{2.17}$$

From (2.15) and (2.17), we get

$$Y_1(\upsilon) \leq 1 + M(\upsilon)\left(1 + \int_{\upsilon_1}^{\upsilon} g(r) \exp\left\{\int_{\upsilon_1}^{r} (g(\theta) + c(\theta))d\theta\right\} dr\right). \tag{2.18}$$

From (2.14) and (2.18), we have

$$z(\upsilon) \leq f_1(\upsilon)Y_1(\upsilon) \leq f_1(\upsilon)(1 + k(\upsilon)) = f_1(\upsilon)H(\upsilon_1, \upsilon)$$

thus, we have

$$z\left(\upsilon_2^-\right) = z(\upsilon_2) \leq f_1(\upsilon_2) H(\upsilon_1, \upsilon_2). \tag{2.19}$$

Suppose for $\upsilon \in (\upsilon_{k-1}, \upsilon_k]$, one has $z(\upsilon) \leq f_{k-1}(\upsilon)H(\upsilon_{k-1}, \upsilon)$ and $z\left(\upsilon_k^-\right) = z(\upsilon_k) \leq f_{k-1}(\upsilon_k) H(\upsilon_{k-1}, \upsilon_k)$, then for $\upsilon \in (\upsilon_k, \upsilon_{k+1}]$, we have

$$\begin{aligned} z(\upsilon) &\leq f(\upsilon) + \int_{\omega}^{\upsilon} g(r)\left(z(r) + \int_{\omega}^{r} c(\theta)z(\theta)d\theta\right) dr + \sum_{\omega<\upsilon_k<\upsilon} z\left(\upsilon_k^-\right)\eta_k \\ &= B_k(\upsilon) + \int_{\upsilon_k}^{\upsilon} g(r)\left(z(r) + \int_{\omega}^{r} c(\theta)z(\theta)d\theta\right) dr \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} B_k(\upsilon) &= f(\upsilon) + \sum_{i=1}^{k}\int_{\upsilon_{i-1}}^{\upsilon_i} g(r)\left(z(s) + \int_{\omega}^{r} c(\theta)z(\theta)d\theta\right) dr + \sum_{i=1}^{k} z\left(\upsilon_i^-\right)\eta_i \\ &\leq f(\upsilon) + \sum_{i=1}^{k}\left[A(\upsilon_{i-1}, \upsilon_i) + \eta_i\right] z\left(\upsilon_i^-\right) \\ &\leq f(\upsilon) + \sum_{i=1}^{k}\left[A(\upsilon_{i-1}, \upsilon_i) + \eta_i\right] f_{i-1}(\upsilon_i) H(\upsilon_{i-1}, \upsilon_i) \\ &\leq f(\upsilon) + \sum_{i=1}^{k}\left[A(\upsilon_{i-1}, \upsilon_i) + \eta_i\right] f_{i-1}(\upsilon)H(\upsilon_{i-1}, \upsilon_i) \\ &\leq f_1(\upsilon) + \sum_{i=2}^{k}\left[A(\upsilon_{i-1}, \upsilon_i) + \eta_i\right] f_{i-1}(\upsilon)H(\upsilon_{i-1}, \upsilon_i) \\ &\leq f_2(\upsilon) + \sum_{i=3}^{k}\left[A(\upsilon_{i-1}, \upsilon_i) + \eta_i\right] f_{i-1}(\upsilon)H(\upsilon_{i-1}, \upsilon_i) \\ &\leq \cdots \\ &\leq f_k(\upsilon) \end{aligned}$$

by (2.20), we get

$$z(\upsilon) \leq f_k(\upsilon) + \int_{\upsilon_k}^{\upsilon} g(r)\left(z(r) + \int_{\omega}^{r} c(\theta)z(\theta)d\theta\right)dr.$$

Using the approach described above, we can conclude that $z(\upsilon) \leq f_k(\upsilon)H(\upsilon_k, \upsilon)$. We use mathematical induction to argue that (2.4) holds for the entire interval I'. This concludes the proof of Lemma. □

Lemma 2.3 *[25] A function $\varkappa \in PC^1([\omega, \upsilon])$ satisfies (2.1) if and only if there is a function $f \in PC([\omega - \theta, \upsilon])$ and a sequence $\{f_k\}$ (which depends on f) such that $|f(\upsilon)| \leq \varepsilon$, $\forall\ \upsilon \in [\omega - \theta, \upsilon], |f_k| \leq \varepsilon$ for all $k = 1, 2, \cdots, m$ and*

$$\varkappa'(\upsilon) = G(\upsilon, \varkappa(\upsilon), \varkappa(h(\upsilon))) + \int_{\omega}^{\upsilon} E(\upsilon, s, \varkappa(h(s)))ds + \int_{\omega}^{T} \zeta(\upsilon, s, \varkappa(h(s)))ds + f(\upsilon), \upsilon \in I' \tag{2.21}$$
$$\triangle\varkappa(\upsilon_k) = \phi_k\left(\varkappa\left(\upsilon_k^-\right)\right) + f_k, k = 1, 2, 3, \cdots, m.$$

Remark 2.1 *A function $\varkappa \in PC^1([\omega, \upsilon])$ satisfies (2.2) if and only if there is a function $f \in PC([\omega - \theta, \upsilon])$ and a sequence $\{f_k\}$ (which depends on f) such that $|f(\upsilon)| \leq \varphi(\upsilon)$, $\forall\ \upsilon \in [\omega - \theta, \upsilon], |f_k| \leq \chi$, $\forall\ k = 1, 2, 3, \cdots, m$ and (2.21) holds.*

Lemma 2.4 *Let $\varkappa \in PC^1([\omega, \upsilon])$ is a solution of (2.1) satisfies the following integral inequality*

$$\begin{aligned}&|\ \varkappa(\upsilon) - \varkappa(\omega) - \sum_{j=1}^{k}\phi_j\left(\varkappa\left(\upsilon_j^-\right)\right) - \int_{\omega}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \\ &- \int_{\omega}^{\upsilon}\int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi - \int_{\omega}^{\upsilon}\int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi\ |\leq (t - \omega + m)\varepsilon,\end{aligned}$$

$\forall\ \upsilon \in (\upsilon_k, \upsilon_{k+1}] \subset [\omega, \upsilon], k = 0, 1, 2, \cdots, m.$

Proof: If $\varkappa \in PC^1([\omega, \upsilon])$ satisfies (2.1), then from Lemma (2.3), we get

$$\varkappa'(\upsilon) = G(\upsilon, \varkappa(\upsilon), \varkappa(h(\upsilon))) + \int_{\omega}^{\upsilon} E(\upsilon, s, \varkappa(s), \varkappa(h(s)))ds + \int_{\omega}^{T} \zeta(\upsilon, s, \varkappa(s), \varkappa(h(s)))ds + f(\upsilon),\ \upsilon \in I' \tag{2.22}$$

$$\triangle\varkappa(\upsilon_k) = \phi_k\left(\varkappa\left(\upsilon_k^-\right)\right) + f_k, k = 1, 2, \cdots, m.$$

For $\upsilon \in [\omega, \upsilon_1]$, integrating (2.22) from ω to υ implies that

$$\begin{aligned}&\int_{\omega}^{\upsilon} \varkappa'(\xi)d\xi \\ &= \int_{\omega}^{\upsilon}\left[G(\xi, \varkappa(\xi), \varkappa(h(\xi))) + \int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s)))ds + \int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))ds + f(\xi)\right]d\xi\end{aligned}$$

thus we get

$$\begin{aligned}&\varkappa(\upsilon) \\ &= \varkappa(\omega) + \int_{\omega}^{\upsilon}\left[G(\xi, \varkappa(\xi), \varkappa(h(\xi))) + \int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s)))ds + \int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))ds + f(\xi)\right]d\xi\end{aligned}$$

and

$$
\begin{aligned}
\varkappa\left(\upsilon_1^-\right) =& \varkappa\left(\upsilon_1\right)\\
=& \varkappa\left(\omega\right) + \int_\omega^{\upsilon_1} G(\xi,\varkappa(\xi),\varkappa(h(\xi)))d\xi + \int_\omega^{\upsilon_1}\int_\omega^{\xi} E(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi\\
&+ \int_\omega^{\upsilon_1}\int_\omega^{T} \zeta(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_\omega^{\upsilon_1} f(\xi)d\xi
\end{aligned}
$$

For $\upsilon \in (\upsilon_1, \upsilon_2]$, integrating (2.22) from υ_1 to υ implies that

$$
\begin{aligned}
&\int_{\upsilon_1}^{\upsilon} \varkappa'(\xi)d\xi\\
&= \int_{\upsilon_1}^{\upsilon}\left[G(\xi,\varkappa(\xi),\varkappa(h(\xi))) + \int_\omega^{\xi} E(\xi,s,\varkappa(s),\varkappa(h(s)))ds + \int_\omega^{T}\zeta(\xi,s,\varkappa(s),\varkappa(h(s)))ds + f(\xi)\right]d\xi
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\varkappa(\upsilon) =& \varkappa\left(\upsilon_1^+\right) + \int_{\upsilon_1}^{\upsilon} G(\xi,\varkappa(\xi),\varkappa(h(\xi)))d\xi + \int_{\upsilon_1}^{\upsilon}\int_\omega^{\xi} E(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi\\
&+ \int_{\upsilon_1}^{\upsilon}\int_\omega^{\upsilon} G(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\upsilon_1}^{\upsilon} f(\xi)d\xi \qquad (2.23)\\
=& \phi_1\left(\varkappa\left(\upsilon_1^-\right)\right) + \varkappa\left(\upsilon_1^-\right) + f_1 + \int_{\upsilon_1}^{\upsilon} G(\xi,\varkappa(\xi),\varkappa(h(\xi)))d\xi\\
&+ \int_{\upsilon_1}^{\upsilon}\int_\omega^{\xi} E(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\upsilon_1}^{\upsilon}\int_\omega^{T} \zeta(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\upsilon_1}^{\upsilon} f(\xi)d\xi\\
=& \phi_1\left(\varkappa\left(\upsilon_1^-\right)\right) + \varkappa\left(\omega\right) + f_1 + \int_{\omega}^{\upsilon} G(\xi,\varkappa(\xi),\varkappa(h(\xi)))d\xi\\
&+ \int_{\omega}^{\upsilon}\int_\omega^{\xi} E(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\omega}^{\upsilon}\int_\omega^{T} \zeta(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\omega}^{\upsilon} f(\xi)d\xi. \qquad (2.24)
\end{aligned}
$$

So we get

$$
\begin{aligned}
\varkappa\left(\upsilon_2^-\right) =& \phi_1\left(\varkappa\left(\upsilon_1^-\right)\right) + \varkappa\left(\omega\right) + f_1 + \int_{\omega}^{\upsilon_2} G(\xi,\varkappa(\xi),\varkappa(h(\xi)))d\xi\\
&+ \int_{\omega}^{\upsilon_2}\int_\omega^{\xi} E(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\omega}^{\upsilon_2}\int_\omega^{T} \zeta(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\omega}^{\upsilon_2} f(\xi)d\xi
\end{aligned}
$$

Now, for $\upsilon \in (\upsilon_{k-1}, \upsilon_k]$, we get

$$
\begin{aligned}
\varkappa(\upsilon) =& \varkappa\left(\omega\right) + \sum_{i=1}^{k-1} f_i + \sum_{j=1}^{k-1}\phi_j\left(\varkappa\left(\upsilon_j^-\right)\right) + \int_{\omega}^{\upsilon} G(\xi,\varkappa(\xi),\varkappa(h(\xi)))d\xi\\
&+ \int_{\omega}^{\upsilon}\int_\omega^{\xi} E(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\omega}^{\upsilon}\int_\omega^{T} \zeta(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\omega}^{\upsilon} f(\xi)d\xi
\end{aligned}
$$

and

$$
\begin{aligned}
\varkappa\left(\upsilon_k^-\right) =& \varkappa\left(\omega\right) + \sum_{i=1}^{k-1} f_i + \sum_{j=1}^{k-1}\phi_j\left(\varkappa\left(\upsilon_j^-\right)\right) + \int_{\omega}^{\upsilon_k} G(\xi,\varkappa(\xi),\varkappa(h(\xi)))d\xi\\
&+ \int_{\omega}^{\upsilon_k}\int_\omega^{\xi} E(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\omega}^{\upsilon_k}\int_\omega^{T} \zeta(\xi,s,\varkappa(s),\varkappa(h(s)))dsd\xi + \int_{\omega}^{\upsilon_k} f(\xi)d\xi.
\end{aligned}
$$

Then for $\upsilon \in (\upsilon_k, \upsilon_{k+1}]$, integrating (2.22) from υ_k to υ implies that

$$\begin{aligned}
\varkappa(\upsilon) =& \varkappa\left(\upsilon_k^+\right) + \int_{\upsilon_k}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \\
&+ \int_{\upsilon_k}^{\upsilon} \int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi + \int_{\upsilon_k}^{\upsilon} \int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi + \int_{\upsilon_k}^{\upsilon} f(\xi)d\xi \\
=& \phi_k\left(\varkappa\left(\upsilon_k^-\right)\right) + \varkappa\left(\upsilon_k^-\right) + f_k + \int_{\upsilon_k}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \\
&+ \int_{\upsilon_k}^{\upsilon} \int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi + \int_{\upsilon_k}^{\upsilon} \int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi + \int_{\upsilon_k}^{\upsilon} f(\xi)d\xi \\
=& \varkappa(\omega) + \sum_{i=1}^{k} f_i + \sum_{j=1}^{k} \phi_j\left(\varkappa\left(\upsilon_j^-\right)\right) + \int_{\omega}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \\
&+ \int_{\omega}^{\upsilon} f(\xi)d\xi + \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi + \int_{\omega}^{\upsilon} \int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi
\end{aligned}$$

Thus, using mathematical induction, we get

$$\begin{aligned}
\varkappa(\upsilon) =& \varkappa(\omega) + \sum_{i=1}^{k} f_i + \sum_{j=1}^{k} \phi_j\left(\varkappa\left(\upsilon_j^-\right)\right) + \int_{\omega}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \\
&+ \int_{\omega}^{\upsilon} f(\xi)d\xi + \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi \\
&+ \int_{\omega}^{\upsilon} \int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi, \ \upsilon \in (\upsilon_k, \upsilon_{k+1}].
\end{aligned} \tag{2.25}$$

It follows by (2.25) that

$$\begin{aligned}
&\left| \varkappa(\upsilon) - \varkappa(\omega) - \sum_{j=1}^{k} \phi_j\left(\varkappa\left(\upsilon_j^-\right)\right) - \int_{\omega}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \right. \\
&\left. - \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi - \int_{\omega}^{\upsilon} \int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi \right| \\
&= \left| \int_{\omega}^{\upsilon} f(\xi)d\xi + \sum_{i=1}^{k} f_i \right| \le \int_{\omega}^{\upsilon} |f(\xi)|d\xi + \sum_{i=1}^{k} |f_i| \\
&\le (t - \omega + k)\,\varepsilon \le (t - \omega + m)\,\varepsilon, \ \upsilon \in (\upsilon_k, \upsilon_{k+1}]
\end{aligned}$$

This completes the proof. □

Remark 2.2 *Each solution $\varkappa \in PC^1([\omega, \upsilon])$ of (2.2) satisfies the following integral inequality*

$$\begin{aligned}
&\left| \varkappa(\upsilon) - \varkappa(\omega) - \sum_{j=1}^{k} \phi_j\left(\varkappa\left(\upsilon_j^-\right)\right) - \int_{\omega}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \right. \\
&\left. - \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi - \int_{\omega}^{\upsilon} \int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi \right| \\
&\le \rho\varphi(\upsilon) + m\chi, \ \textit{for} \ \ \upsilon \in (\upsilon_k, \upsilon_{k+1}] \subset [\omega, \upsilon]
\end{aligned}$$

3. Hyers-Ulam Stability

This section uses Lemma 2.2, Definition 2.2, and Lemma 2.1 to study the HU stability for the given system.

Theorem 3.1 *Assume the following assumptions are true:*
(A1) $F : [\omega, T] \times \mathbb{R}^2 \to \mathbb{R},\ G, \zeta : [\omega, T] \times [\omega, T] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous, and* F, G, ζ *are Lipschitz continuous with respect to the last variables,*

$$|F(\xi, \varkappa_1, \varkappa_2) - F(\xi, \eta_1, \eta_2)| \leq \sum_{i=1}^{2} \Psi_1 |\varkappa_i - \eta_i| \tag{3.1}$$

$$|G(\xi, s, \varkappa_1, \varkappa_2) - G(\xi, s, \eta_1, \eta_2)| \leq \sum_{i=1}^{2} \Psi_1 \Psi_2 |\varkappa_i - \eta_i| \tag{3.2}$$

$$|\zeta(\xi, s, \varkappa_1, \varkappa_2) - \zeta(\xi, s, \eta_1, \eta_2)| \leq \sum_{i=1}^{2} \Psi_1 \Psi_3 |\varkappa_i - \eta_i| \tag{3.3}$$

where $\Psi_1, \Psi_2, \Psi_3 > 0$*, for all* $x, s \in I'$*;*
(A_2) $\phi_j : \mathbb{R} \to \mathbb{R}$ *is such that* $|\phi_j(\varkappa_1) - \phi_j(\varkappa_2)| \leq M_j |\varkappa_1 - \varkappa_2|, M_j > 0$ *for all* $j \in \{1, 2, \cdots, m\}$ *and* $\varkappa_1, \varkappa_2 \in \mathbb{R}$*;*
(A_3) $\sum_{j=1}^{m} M_j + 2\Psi_1 (T - \omega) + (T - \omega)^2 \Psi_1 \Psi_2 + (T - \omega)^2 \Psi_1 \Psi_3 < 1.$
Then there exists a unique solution of the system (1.1) in $PC([\omega - \theta, \upsilon]) \cap PC^1([\omega, \upsilon])$ *and the system (1.1) is HU stable on* $[\omega - \theta, \upsilon]$*.*

Proof: (1) We define an operator $T : PC([\omega - \theta, \upsilon]) \to PC([\omega - \theta, \upsilon])$ as

$$(T\varkappa)(\upsilon) = \begin{cases} \alpha(\upsilon),\ \upsilon \in [\omega - \theta, \omega] \\ \alpha(\omega) + \int_\omega^\upsilon G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \\ + \int_\omega^\upsilon \int_\omega^\xi E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi + \int_\omega^\upsilon \int_\omega^T \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi, \upsilon \in (\omega, \upsilon_1] \\ \alpha(\omega) + \phi_1\left(\varkappa\left(\upsilon_1^-\right)\right) + \int_\omega^\upsilon G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \\ + \int_\omega^\upsilon \int_\omega^\xi E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi + \int_\omega^\upsilon \int_\omega^T \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi, \upsilon \in (\upsilon_1, \upsilon_2] \\ \alpha(\omega) + \sum_{j=1}^{2} \phi_j\left(\varkappa\left(\upsilon_j^-\right)\right) + \int_\omega^\upsilon G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \\ + \int_\omega^\upsilon \int_\omega^\xi E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi + \int_\omega^\upsilon \int_\omega^T \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi, \upsilon \in (\upsilon_2, \upsilon_3] \\ \vdots \\ \alpha(\omega) + \sum_{j=1}^{m} \phi_j\left(\varkappa\left(\upsilon_j^-\right)\right) + \int_\omega^\upsilon G(\xi, \varkappa(\xi), \varkappa(h(\xi)))d\xi \\ + \int_\omega^\upsilon \int_\omega^\xi E(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi + \int_\omega^\upsilon \int_\omega^T \zeta(\xi, s, \varkappa(s), \varkappa(h(s)))dsd\xi, \upsilon \in (\upsilon_m, \upsilon_{m+1}] \end{cases} \tag{3.4}$$

For any $\varkappa_1, \varkappa_2 \in PC([\omega - \theta, \upsilon])$, and for all $\upsilon \in [\omega - \theta, \omega]$, we get

$$|(T\varkappa_1)(\upsilon) - (T\varkappa_2)(\upsilon)| = 0$$

For $\upsilon \in (\upsilon_k, \upsilon_{k+1}]$, we deduce that

$$\begin{aligned} &|(T\varkappa_1)(\upsilon) - (T\varkappa_2)(\upsilon)| \\ \leq &\sum_{j=1}^{k} \left|\phi_j\left(\varkappa_1\left(\upsilon_j^-\right)\right) - \phi_j\left(\varkappa_2\left(\upsilon_j^-\right)\right)\right| \\ &+ \int_\omega^\upsilon |F(\xi, \varkappa_1(\xi), \varkappa_1(h(\xi))) - F(\xi, \varkappa_2(\xi), \varkappa_2(h(\xi)))| d\xi \\ &+ \int_\omega^\upsilon \int_\omega^\xi |\ G(\xi, s, \varkappa_1(s), \varkappa_1(h(s))) - G(\xi, s, \varkappa_2(s), \varkappa_2(h(s)))\ | dsd\xi \end{aligned}$$

$$
\begin{aligned}
&+\int_{\omega}^{\upsilon}\int_{\omega}^{T} \mid \zeta\left(\xi, s, \varkappa_1(s), \varkappa_1(h(s))\right)-\zeta\left(\xi, s, \varkappa_2(s), \varkappa_2(h(s))\right) \mid dsd\xi \\
&\leq \sum_{j=1}^{k} M_j\left|\varkappa_1\left(\upsilon_j^{-}\right)-\varkappa_2\left(\upsilon_j^{-}\right)\right|+\Psi_1 \int_{\omega}^{\upsilon}\left|\varkappa_1(\xi)-\varkappa_2(\xi)\right| d\xi \\
&+\Psi_1 \int_{\omega}^{\upsilon}\left|\varkappa_1(h(\xi))-\varkappa_2(h(\xi))\right| d\xi \\
&+\Psi_1 \Psi_2 \int_{\omega}^{\upsilon}\int_{\omega}^{\xi}\left|\varkappa_1(s)-\varkappa_2(s)\right| dsd\xi+\Psi_1 \Psi_2 \int_{\omega}^{\upsilon}\int_{\omega}^{\xi}\left|\varkappa_1(h(s))-\varkappa_2(h(s))\right| dsd\xi \\
&\leq\left(\sum_{j=1}^{k} M_j+2 \Psi_1(T-\omega)+(T-\omega)^2 \Psi_1 \Psi_2+(T-\omega)^2 \Psi_1 \Psi_3\right) \sup_{\upsilon \in[\omega-\theta, \upsilon]}\left|\varkappa_1(\xi)-\varkappa_2(\xi)\right| \\
&\quad \leq\left(\sum_{j=1}^{m} M_j+2 \Psi_1(T-\omega)+(T-\omega)^2 \Psi_1 \Psi_2+(T-\omega)^2 \Psi_1 \Psi_3\right)\left\|\varkappa_1-\varkappa_2\right\|
\end{aligned}
$$

By (A_3), the operator υ is strictly contractive on $\left(t_k, \upsilon_{k+1}\right], k=0,1,2, \cdots, m$, and hence it is a Picard operator on $PC([\omega-\theta, \upsilon])$. By (3.1) and (3.2), the unique fixed point of this operator is in fact the unique solution of (1.1) in $PC([\omega-\theta, \upsilon]) \cap PC^1([\omega, \upsilon])$.

Next, let $y \in PC([\omega-\theta, \upsilon]) \cap PC^1([\omega, \upsilon])$ be a solution of (2.1). The unique solution $\varkappa \in PC([\omega-\theta, \upsilon]) \cap PC^1([\omega, \upsilon])$ of the following system

$$
\begin{aligned}
&\varkappa'(\upsilon)=G(\upsilon, \varkappa(\upsilon), \varkappa(h(\upsilon))) \\
&\quad+\int_{\omega}^{\upsilon} E(\upsilon, s, \varkappa(s), \varkappa(h(s))) ds+\int_{\omega}^{T} \zeta(\upsilon, s, \varkappa(s), \varkappa(h(s))) ds, \ \upsilon \in I' \qquad (3.5)\\
&\varkappa(\upsilon)=y(\upsilon), \upsilon \in[\omega-\theta, \omega] \qquad (3.6)\\
&\Delta \varkappa\left(\upsilon_k\right)=\phi_k\left(\varkappa\left(\upsilon_k^{-}\right)\right), k=1,2, \cdots, m
\end{aligned}
$$

is given by

$$
u(\upsilon)=\begin{cases}
y(\upsilon), \upsilon \in[\omega-\theta, \omega] \\
y(\omega)+\int_{\omega}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi))) d\xi \\
+\int_{\omega}^{\upsilon}\int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s))) dsd\xi+\int_{\omega}^{\upsilon}\int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s))) dsd\xi, \ \upsilon \in\left(\omega, \upsilon_1\right] \\
y(\omega)+\phi_1\left(\varkappa\left(\upsilon_1^{-}\right)\right)+\int_{\omega}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi))) d\xi \\
+\int_{\omega}^{\upsilon}\int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s))) dsd\xi+\int_{\omega}^{\upsilon}\int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s))) dsd\xi, \ \upsilon \in\left(\upsilon_1, \upsilon_2\right] \\
y(\omega)+\sum_{j=1}^{2} \phi_j\left(\varkappa\left(\upsilon_j^{-}\right)\right)+\int_{\omega}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi))) d\xi \\
+\int_{\omega}^{\upsilon}\int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s))) dsd\xi+\int_{\omega}^{\upsilon}\int_{\omega}^{T} \zeta(\xi, s, \varkappa(s), \varkappa(h(s))) dsd\xi, \ \upsilon \in\left(\upsilon_2, \upsilon_3\right] \\
\vdots \\
y(\omega)+\sum_{j=1}^{m} \phi_j\left(\varkappa\left(\upsilon_j^{-}\right)\right)+\int_{\omega}^{\upsilon} G(\xi, \varkappa(\xi), \varkappa(h(\xi))) d\xi \\
+\int_{\omega}^{\upsilon}\int_{\omega}^{\xi} E(\xi, s, \varkappa(s), \varkappa(h(s))) dsd\xi+\int_{\omega}^{\upsilon}\int_{\omega}^{\upsilon} \zeta(\xi, s, \varkappa(s), \varkappa(h(s))) dsd\xi, \ \upsilon \in\left(\upsilon_m, \upsilon_{m+1}\right]
\end{cases} \qquad (3.7)
$$

We observe that for $\upsilon \in[\omega-\theta, \omega],|y(\upsilon)-\varkappa(\upsilon)|=0$.
For $\upsilon \in\left(\upsilon_k, \upsilon_{k+1}\right]$, using Lemma 2.4, let

$$
\begin{aligned}
B(\upsilon)=&y(\upsilon)-y(\omega)-\sum_{j=1}^{k} \phi_j\left(y\left(\upsilon_j^{-}\right)\right)-\int_{\omega}^{\upsilon} G(\xi, y(\xi), y(h(\xi))) d\xi \\
&-\int_{\omega}^{\upsilon}\int_{\omega}^{\xi} H(\xi, s, y(s), y(h(s))) dsd\xi-\int_{\omega}^{\upsilon}\int_{\omega}^{T} \zeta(\xi, s, y(s), y(h(s))) dsd\xi
\end{aligned}
$$

we get

$$
\begin{aligned}
&|y(\upsilon)-\varkappa(\upsilon)| \\
\leq & |B(\upsilon)|+\int_{\omega}^{\upsilon}|G(\xi, y(\xi), y(h(\xi)))-G(\xi, \varkappa(\xi), \varkappa(h(\xi)))| d \xi \\
&+\int_{\omega}^{\upsilon} \int_{\omega}^{\xi}|E(\xi, s, y(s), y(h(s)))-E(\xi, s, \varkappa(s), \varkappa(h(s)))| d s d \xi \\
&+\int_{\omega}^{\upsilon} \int_{\omega}^{T}|\zeta(\xi, s, y(s), y(h(s)))-\zeta(\xi, s, \varkappa(s), \varkappa(h(s)))| d s d \xi+\sum_{j=1}^{k}\left|\phi_{j}\left(y\left(\upsilon_{j}^{-}\right)\right)-\phi_{j}\left(\varkappa\left(\upsilon_{j}^{-}\right)\right)\right| \\
\leq & (m+t-\omega) \varepsilon+\Psi_{1} \int_{\omega}^{\upsilon}|y(\xi)-\varkappa(\xi)| d \xi+\Psi_{1} \int_{\omega}^{\upsilon}|y(h(\xi))-\varkappa(h(\xi))| d \xi \\
&+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi}|y(s)-\varkappa(s)| d s d \xi+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi}|y(h(s))-\varkappa(h(s))| d s d \xi \\
&+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon}|y(s)-\varkappa(s)| d s d \xi+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon}|y(h(s))-\varkappa(h(s))| d s d \xi \\
&+\sum_{j=1}^{k} M_{j}\left|y\left(\upsilon_{j}^{-}\right)-\varkappa\left(\upsilon_{j}^{-}\right)\right| .
\end{aligned}
$$

Next, we show that the operator $\Lambda: PC([\omega-\theta, \upsilon]) \rightarrow PC([\omega-\theta, \upsilon])$ given below is an increasing Picard operator on $PC([\omega-\theta, \upsilon])$

$$
(\Lambda q)(\upsilon)=\left\{\begin{array}{l}
0, \quad \upsilon \in[\omega-\theta, \omega] \\
(\upsilon-\omega) \varepsilon+\Psi_{1} \int_{\omega}^{\upsilon} q(\xi) d \xi+\Psi_{1} \int_{\omega}^{\upsilon} q(h(\xi)) d \xi \\
+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} q(s) d s d \xi+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} q(h(s)) d s d \xi \\
+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon} q(s) d s d \xi+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon} q(h(s)) d s d \xi, \ \upsilon \in\left(\omega, \upsilon_{1}\right] \\
(1+t-\omega) \varepsilon+M_{1} q\left(\upsilon_{1}^{-}\right)+\Psi_{1} \int_{\omega}^{\upsilon} q(\xi) d \xi+\Psi_{1} \int_{\omega}^{\upsilon} q(h(\xi)) d \xi \\
+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} q(s) d s d \xi+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} q(h(s)) d s d \xi \\
+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon} q(s) d s d \xi+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon} q(h(s)) d s d \xi, \ \upsilon \in\left(\upsilon_{1}, \upsilon_{2}\right] \\
(2+t-\omega) \varepsilon+\Sigma_{j=1}^{2} M_{j} q\left(\upsilon_{j}^{-}\right)+\Psi_{1} \int_{\omega}^{\upsilon} q(\xi) d \xi+\Psi_{1} \int_{\omega}^{\upsilon} q(h(\xi)) d \xi \\
+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} q(s) d s d \xi+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} q(h(s)) d s d \xi \\
+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon} q(s) d s d \xi+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon} q(h(s)) d s d \xi, \ \upsilon \in\left(\upsilon_{2}, \upsilon_{3}\right] \\
\vdots \\
(m+\upsilon-\omega) \varepsilon+\sum_{j=1}^{m} M_{j} q\left(\upsilon_{j}^{-}\right)+\Psi_{1} \int_{\omega}^{\upsilon} q(\xi) d \xi+\Psi_{1} \int_{\omega}^{\upsilon} q(h(\xi)) d \xi \\
+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} q(s) d s d \xi+\Psi_{1} \Psi_{2} \int_{\omega}^{\upsilon} \int_{\omega}^{\xi} q(h(s)) d s d \xi \\
+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon} q(s) d s d \xi+\Psi_{1} \Psi_{3} \int_{\omega}^{\upsilon} \int_{\omega}^{\upsilon} q(h(s)) d s d \xi, \ \upsilon \in\left(\upsilon_{m}, \upsilon_{m+1}\right]
\end{array}\right. \tag{3.8}
$$

For any $q_{1}, q_{2} \in PC([\omega-\theta, \upsilon]),\left|\left(\Lambda q_{1}\right)(\upsilon)-\left(\Lambda q_{2}\right)(\upsilon)\right|=0$ for $\upsilon \in[\omega-\theta, \omega]$. For $\upsilon \in\left(\upsilon_{k}, \upsilon_{k+1}\right]$, we see that

$$\begin{aligned}
&\left|\left(\Lambda q_1\right)(\upsilon)-\left(\Lambda q_2\right)(\upsilon)\right| \\
\leq &\sum_{j=1}^{k} M_j\left|q_1\left(\upsilon_j^{-}\right)-q_2\left(\upsilon_j^{-}\right)\right|+\Psi_1 \int_\omega^\upsilon\left|q_1(\xi)-q_2(\xi)\right| d\xi \\
&+\Psi_1 \int_\omega^\upsilon\left|q_1(h(\xi))-q_2(h(\xi))\right| d\xi \\
&+\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi\left|q_1(s)-q_2(s)\right| ds d\xi+\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi\left|q_1(h(s))-q_2(h(s))\right| ds d\xi \\
&+\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon\left|q_1(s)-q_2(s)\right| ds d\xi+\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon\left|q_1(h(s))-q_2(h(s))\right| ds d\xi \\
\leq &\sum_{j=1}^{k} M_j \sup_{\upsilon \in[\omega-\theta, \upsilon]}\left|q_1(\upsilon)-q_2(\upsilon)\right|+\Psi_1(\upsilon-\omega) \sup_{\upsilon \in[\omega-\theta, \upsilon]}\left|q_1(\upsilon)-q_2(\upsilon)\right| \\
&+\Psi_1(\upsilon-\omega) \sup_{\upsilon \in[\omega-\theta, \upsilon]}\left|q_1(h(\upsilon))-q_2(h(\upsilon))\right| \\
&+\Psi_1 \Psi_2 \int_\omega^\upsilon(\xi-\omega) d\xi \sup_{\upsilon \in[\omega-\theta, \upsilon]}\left|q_1(\upsilon)-q_2(\upsilon)\right| \\
&+\Psi_1 \Psi_2 \int_\omega^\upsilon(\xi-\omega) d\xi \sup_{\upsilon \in[\omega-\theta, \upsilon]}\left|q_1(h(\upsilon))-q_2(h(\upsilon))\right| \\
&+\Psi_1 \Psi_3 \int_\omega^\upsilon(\xi-\omega) d\xi \sup_{\upsilon \in[\omega-\theta, \upsilon]}\left|q_1(\upsilon)-q_2(\upsilon)\right| \\
&+\Psi_1 \Psi_3 \int_\omega^\upsilon(\xi-\omega) d\xi \sup_{\upsilon \in[\omega-\theta, \upsilon]}\left|q_1(h(\upsilon))-q_2(h(\upsilon))\right|
\end{aligned}$$

Then, we have

$$\left|\left(\Lambda q_1\right)(\upsilon)-\left(\Lambda q_2\right)(\upsilon)\right| \leq\left(\sum_{j=1}^{m} M_j+2 \Psi_1(T-\omega)+(T-\omega)^2 \Psi_1 \Psi_2+(T-\omega)^2 \Psi_1 \Psi_3\right)\left\|q_1-q_2\right\|$$

According to (A_3), the operator Λ is contractive on $PC([\omega-\theta, \upsilon])$ in each interval $(\upsilon_k, \upsilon_{k+1}]$, where $k=0,1,2, \cdots, m$. Applying Banach contraction principle, we get Λ is a Picard operator and hence it has a unique fixed point, that is $q^* \in PC([\omega-\theta, \upsilon])$, and

$$\begin{aligned}
q^*(\upsilon)= &(m+\upsilon-\omega) \varepsilon+\sum_{j=1}^{k} M_j q^*\left(\upsilon_j^{-}\right)+\Psi_1 \int_\omega^\upsilon q^*(\xi) d\xi+\Psi_1 \int_\omega^\upsilon q^*(h(\xi)) d\xi \\
&+\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q^*(s) ds d\xi+\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q^*(h(s)) ds d\xi \\
&+\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q^*(s) ds d\xi+\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q^*(h(s)) ds d\xi, \upsilon \in(\upsilon_k, \upsilon_{k+1}]
\end{aligned}$$

q^* is increasing, so $q^*(h(\upsilon)) \leq q^*(\upsilon)$ and then we have
$q^*(\upsilon) \leq (m+T-\omega)\varepsilon+\sum_{j=1}^{k} M_j q^*\left(\upsilon_j^{-}\right)+2\Psi_1 \int_\omega^\upsilon q^*(\xi) d\xi+2\Psi_1\Psi_2 \int_\omega^\upsilon \int_\omega^\xi q^*(s) ds d\xi$
$+2\Psi_1\Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q^*(s) ds d\xi$.

From Lemma 2.2, we have

$$q^*(\upsilon) \leq(m+T-\omega) \varepsilon \prod_{i=1}^{k}\left[1+H\left(\upsilon_{i-1}, \upsilon_i\right)\left(A\left(\upsilon_{i-1}, \upsilon_i\right)+M_i\right)\right] H\left(\upsilon_k, \upsilon\right)$$

where

$$A\left(\upsilon_{i-1}, \upsilon_{i}\right)=2\Psi_{1}\left[1-\Psi_{2}\omega-\Psi_{3}\omega+\frac{\Psi_{2}+\Psi_{3}}{2}\left(\upsilon_{i}+\upsilon_{i-1}\right)\right]\left(\upsilon_{i}-\upsilon_{i-1}\right) \tag{3.9}$$

$$\begin{aligned} H\left(\upsilon_{i-1}, \upsilon_{i}\right)=&1+\frac{2\Psi_{1}}{2\Psi_{1}+\Psi_{2}+\Psi_{3}}\left(\upsilon_{i}-\upsilon_{i-1}\right)\left[1-\Psi_{2}\omega-\Psi_{3}\omega+\frac{\Psi_{2}+\Psi_{3}}{2}\left(\upsilon_{i}+\upsilon_{i-1}\right)\right] \\ &\times\left[\Psi_{2}+\Psi_{3}+2\Psi_{1}\exp\left\{\left(2\Psi_{1}+\Psi_{2}+\Psi_{3}\right)\left(\upsilon_{i}-\upsilon_{i-1}\right)\right\}\right] \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} H\left(\upsilon_{k}, \upsilon\right)=1&+\frac{2\Psi_{1}}{2\Psi_{1}+\Psi_{2}+\Psi_{3}}\left(\upsilon-\upsilon_{k}\right)\left[1-\Psi_{2}\omega-\Psi_{3}\omega+\frac{\Psi_{2}+\Psi_{3}}{2}\left(\upsilon_{k}+\upsilon\right)\right] \\ &\times\left[\Psi_{2}+\Psi_{3}+2\Psi_{1}\exp\left\{\left(2\Psi_{1}+\Psi_{2}+\Psi_{3}\right)\left(\upsilon-\upsilon_{k}\right)\right\}\right] \end{aligned} \tag{3.11}$$

Set $q(\upsilon)=|y(\upsilon)-\varkappa(\upsilon)|$, by (3.8), $q(\upsilon)\leq(\Lambda q)(\upsilon)$, then from Lemma 2.1, we have $q(\upsilon)\leq q^{*}$. Then

$$\begin{aligned} |y(\upsilon)-\varkappa(\upsilon)| &\leq q^{*}(\upsilon) \\ &\leq(m+T-\omega)\,\varepsilon\prod_{i=1}^{k}\left[1+H\left(\upsilon_{i-1}, \upsilon_{i}\right)\left(A\left(\upsilon_{i-1}, \upsilon_{i}\right)+M_{i}\right)\right]H\left(\upsilon_{k}, \upsilon\right) \\ &\leq k\varepsilon, \forall\ \upsilon\in[\omega-\theta, \upsilon], \end{aligned}$$

where

$$k=(m+T-\omega)\prod_{i=1}^{k}\left[1+H\left(\upsilon_{i-1}, \upsilon_{i}\right)\left(A\left(\upsilon_{i-1}, \upsilon_{i}\right)+M_{i}\right)\right]H\left(\upsilon_{k}, \upsilon\right)$$

Consequently, the system (1.1) is HU stable and the proof is completed. □

Remark 3.1 *The system (1.1) has the following two special cases.*

$$\begin{aligned} &\varkappa'(\upsilon)=F(\upsilon, \varkappa(\upsilon), \varkappa(h(\upsilon)))+\int_{\omega}^{\upsilon}G(\upsilon, s, \varkappa(s), \varkappa(h(s)))ds+\int_{\omega}^{T}\zeta(\upsilon, s, \varkappa(s), \varkappa(h(s)))ds, \upsilon\in I \\ &\varkappa(\upsilon)=\alpha(\upsilon), \upsilon\in[\omega-\theta, \omega],\ \theta\geq 0 \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\varkappa'(\upsilon)=F(\upsilon, \varkappa(\upsilon), \varkappa(h(\upsilon)))+\int_{\omega}^{\upsilon}G(\upsilon, s, \varkappa(s), \varkappa(h(s)))ds+\int_{\omega}^{T}\zeta(\upsilon, s, \varkappa(s), \varkappa(h(s)))ds,\ \upsilon\in I' \\ &\varkappa(\omega)=\alpha(\omega), \Delta\varkappa(\upsilon_{k})=\varkappa\left(\upsilon_{k}^{+}\right)-\varkappa\left(\upsilon_{k}^{-}\right)=\phi_{k}\left(\varkappa\left(\upsilon_{k}^{-}\right)\right),\ k=1,2,\cdots,m \end{aligned} \tag{3.13}$$

Theorem 3.1 can be used to establish the following corollaries for these two exceptional situations.

Corollary 3.1 *Suppose that condition (A_1) is satisfied and $2\Psi_{1}(T-\omega)+(T-\omega)^{2}\Psi_{1}\Psi_{2}<1$, then there exists a unique solution of the system (3.12) in $C([\omega-\theta, \upsilon])\cap C^{1}([\omega, \upsilon])$ and the system (3.12) is HU stable on $[\omega-\theta, \upsilon]$.*

Corollary 3.2 *If conditions $(A_1)-(A_3)$ are satisfied, then there exists a unique solution of the system (3.13) in $PC([\omega, \upsilon])$ and the system (3.13) is HU stable on $[\omega, \upsilon]$.*

4. Hyers-Ulam-Rassias Stability

Using Definition 2.3, Lemma 2.2, Remark 2.1, and Lemma 2.1 on $[\omega-\theta, \upsilon]$, we shall demonstrate the HUR stability of the given system in this section.

Theorem 4.1 *Assume the following assumptions are true:*
(A_1') $F:[\omega,T]\times\mathbb{R}^2\to\mathbb{R}, G,\zeta:[\omega,T]\times[\omega,T]\times\mathbb{R}^2\to\mathbb{R}$ *are continuous with the Lipschitz condition:*

$$|F(\xi,\varkappa_1,\varkappa_2)-F(\xi,\eta_1,\eta_2)|\leq\sum_{i=1}^{2}\Psi_1|\varkappa_i-\eta_i| \tag{4.1}$$

$$|G(\xi,s,\varkappa_1,\varkappa_2)-G(\xi,s,\eta_1,\eta_2)|\leq\sum_{i=1}^{2}\Psi_1\Psi_2|\varkappa_i-\eta_i| \tag{4.2}$$

$$|\zeta(\xi,s,\varkappa_1,\varkappa_2)-\zeta(\xi,s,\eta_1,\eta_2)|\leq\sum_{i=1}^{2}\Psi_1\Psi_3|\varkappa_i-\eta_i| \tag{4.3}$$

where $\Psi_1,\Psi_2,\Psi_3>0$ *for all* $x,s\in I'$*;*
(A_2') $\phi_j:\mathbb{R}\to\mathbb{R}$ *is Lipschitz continuous, i.e.,* $|\phi_j(\varkappa_1)-\phi_j(\varkappa_2)|\leq M_j|\varkappa_1-\varkappa_2|$ *for some constant* $M_j>0$, *and for all* $j\in\{1,2,\cdots,m\}$ *and* $\varkappa_1,\varkappa_2\in\mathbb{R}$*;*
(A_3') $\sum_{j=1}^{m}M_j+2\Psi_1(T-\omega)+(T-\omega)^2\Psi_1\Psi_2+(T-\omega)^2\Psi_1\Psi_3<1$;
(A_4') $\varphi(\upsilon):[\omega-\theta,\upsilon]\to\mathbb{R}^+$ *is an increasing function, and* $\int_\omega^\upsilon\varphi(r)dr\leq\rho\varphi(\upsilon)$ *for some constant* $\rho>0$.

Then there exists a unique solution of the system (1.1) *in* $PC([\omega-\theta,\upsilon])\cap PC^1([\omega,\upsilon])$ *and the system* (1.1) *is HUR stable on* $[\omega-\theta,\upsilon]$.

Proof: For given $\varepsilon>0,\varphi(\upsilon)\in PC([\omega-\theta,\upsilon])$, where $\varphi(\upsilon)$ is an increasing and nonnegative functions, $\varphi(\upsilon_1)=\chi>0$ for some $\upsilon_1\in[\omega-\theta,\upsilon]$. Following the same proof steps as Theorem 3.1, we have $|T(\varkappa_1)(\upsilon)-T(\varkappa_2)(\upsilon)|\leq\left(\sum_{j=1}^{m}M_j+2\Psi_1(T-\omega)+(T-\omega)^2\Psi_1\Psi_2+(T-\omega)^2\Psi_1\Psi_3\right)\cdot\|\varkappa_1-\varkappa_2\|$, where the operator υ is defined by (3.4), $\upsilon\in(\upsilon_k,\upsilon_k+1],k=0,1,2,\cdots,m$.
Using (A_3'), the operator υ is strictly contractive on $(\upsilon_k,\upsilon_k+1],k=1,2,\cdots,m$, and υ is a Picard operator on $PC([\omega-\theta,\upsilon])$. Thus, the unique fixed point of this operator is in fact the unique solution of the system (1.1) in $PC([\omega-\theta,\upsilon])\cap PC^1([\omega,\upsilon])$.

Let $y\in PC([\omega-\theta,\upsilon])\cap PC^1([\omega,\upsilon])$ be a solution to the system (2.2). The unique solution $\varkappa\in PC([\omega-\theta,\upsilon])\cap PC^1([\omega,\upsilon])$ of the system (3.5) is given by (3.7). Following the proof Theorem 3.1, we get

$$\begin{aligned}|y(\upsilon)-\varkappa(\upsilon)|\leq&\int_\omega^\upsilon\varphi(\xi)d\xi+k\chi+\Psi_1\int_\omega^\upsilon|y(\xi)-\varkappa(\xi)|d\xi\\&+\Psi_1\int_\omega^\upsilon|y(h(\xi))-\varkappa(h(\xi))|d\xi+\Psi_1\Psi_2\int_\omega^\upsilon\int_\omega^\xi|y(s)-\varkappa(s)|dsd\xi\\&+\Psi_1\Psi_2\int_\omega^\upsilon\int_\omega^\xi|y(h(s))-\varkappa(h(s))|dsd\xi+\Psi_1\Psi_3\int_\omega^\upsilon\int_\omega^\upsilon|y(s)-\varkappa(s)|dsd\xi\\&+\Psi_1\Psi_3\int_\omega^\upsilon\int_\omega^\upsilon|y(h(s))-\varkappa(h(s))|dsd\xi+\sum_{j=1}^{k}M_j\left|y\left(\upsilon_j^-\right)-\varkappa\left(\upsilon_j^-\right)\right|.\end{aligned} \tag{4.4}$$

Next we show that operator $\Lambda_1:PC([\omega-\theta,\upsilon])\to PC([\omega-\theta,\upsilon])$ given below is an increasing Picard operator on $PC([\omega-\theta,\upsilon])$:

$$
\left(\Lambda_1 q\right)(\upsilon)=\begin{cases} 0, \quad \upsilon \in[\omega-\theta, \omega] \\ \rho \varphi(\upsilon)+\Psi_1 \int_\omega^\upsilon q(\xi) d \xi+\Psi_1 \int_\omega^\upsilon q(h(\xi)) d \xi+\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q(s) d s d \xi \\ +\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q(h(s)) d s d \xi+\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q(s) d s d \xi \\ +\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q(h(s)) d s d \xi, \quad \upsilon \in\left(\omega, \upsilon_1\right] \\ \rho \varphi(\upsilon)+\chi+M_1 q\left(\upsilon_1^{-}\right)+\Psi_1 \int_\omega^\upsilon q(\xi) d \xi+\Psi_1 \int_\omega^\upsilon q(h(\xi)) d \xi \\ +\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q(s) d s d \xi+\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q(h(s)) d s d \xi \\ +\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q(s) d s d \xi+\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q(h(s)) d s d \xi, \quad \upsilon \in\left(\upsilon_1, \upsilon_2\right] \\ \rho \varphi(\upsilon)+2 \chi+\Sigma_{j=1}^2 M_j q\left(\upsilon_j^{-}\right)+\Psi_1 \int_\omega^\upsilon q(\xi) d \xi+\Psi_1 \int_\omega^\upsilon q(h(\xi)) d \xi \\ +\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q(s) d s d \xi+\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q(h(s)) d s d \xi \\ +\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q(s) d s d \xi+\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q(h(s)) d s d \xi, \quad \upsilon \in\left(\upsilon_2, \upsilon_3\right] \\ \vdots \\ \rho \varphi(\upsilon)+m \chi+\sum_{j=1}^m M_j q\left(\upsilon_j^{-}\right)+\Psi_1 \int_\omega^\upsilon q(\xi) d \xi+\Psi_1 \int_\omega^\upsilon q(h(\xi)) d \xi \\ +\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q(s) d s d \xi+\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q(h(s)) d s d \xi \\ +\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q(s) d s d \xi+\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q(h(s)) d s d \xi, \quad \upsilon \in\left(\upsilon_m, \upsilon_{m+1}\right]\end{cases}
$$

Using the same proof as Theorem 3.1, we get that the operator Λ_1 is contractive on PC $([\omega-\theta, \upsilon])$ for $\upsilon \in\left(\upsilon_k, \upsilon_k+1\right]$ where $k=0,1,2, \cdots, m$. Applying Banach contraction principle, we get Λ_1 is a Picard operator with unique fixed $q^* \in PC([\omega-\ \theta, T]\ )$, that is

$$
\begin{aligned}
q^*(\upsilon)= & \rho \varphi(\upsilon)+k \chi+\sum_{j=1}^k M_j q^*\left(\upsilon_j^{-}\right)+\Psi_1 \int_\omega^\upsilon q^*(\xi) d \xi+\Psi_1 \int_\omega^\upsilon q^*(h(\xi)) d \xi \\
& +\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q^*(s) d s d \xi+\Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q^*(h(s)) d s d \xi \\
& +\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q^*(s) d s d \xi+\Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q^*(h(s)) d s d \xi, \ \upsilon \in\left(\upsilon_k, \upsilon_{k+1}\right]
\end{aligned} \tag{4.5}
$$

Since q^* is increasing, $q^*(h(\upsilon)) \leq q^*(\upsilon)$ and then we have by (4.5)that $q^*(\upsilon) \leq \rho \varphi(\upsilon)+k \chi+\sum_{j=1}^k M_j q^*\left(\upsilon_j^{-}\right)+2 \Psi_1 \int_\omega^\upsilon q^*(\xi) d \xi+2 \Psi_1 \Psi_2 \int_\omega^\upsilon \int_\omega^\xi q^*(s) d s d \xi+2 \Psi_1 \Psi_3 \int_\omega^\upsilon \int_\omega^\upsilon q^*(s) d s d \xi$. Using Lemma 2.2, we get

$$
q^*(\upsilon) \leq(k \chi+\rho \varphi(\upsilon)) \varepsilon \prod_{i=1}^k\left[1+H\left(\upsilon_{i-1}, \upsilon_i\right)\left(A\left(\upsilon_{i-1}, \upsilon_i\right)+M_i\right)\right] H\left(\upsilon_k, \upsilon\right)
$$

where $A\left(\upsilon_{i-1}, \upsilon_i\right)$ is defined by (2.6)(, and $H\left(\upsilon_k, \upsilon\right)$ is defined by (2.7). If we set $q(\upsilon)=|y(\upsilon)-\varkappa(\upsilon)|$, then by (3.8), $q(\upsilon) \leq\left(\Lambda_1 q\right)(\upsilon)$ and using the abstract Gronwall lemma, it follows that $q(\upsilon) \leq q^*$. Then

$$
|y(\upsilon)-\varkappa(\upsilon)| \leq(k \chi+\rho \varphi(\upsilon)) \varepsilon \prod_{i=1}^k\left[1+H\left(\upsilon_{i-1}, \upsilon_i\right)\left(A\left(\upsilon_{i-1}, \upsilon_i\right)+M_i\right)\right] H\left(\upsilon_k, \upsilon\right)
$$

Hence, the system (1.1) is HUR stable, and the proof is completed. □

Corollary 4.1 *If conditions (A_1') and (A_4') are satisfied and* $2 \Psi_1(T-\omega)+(T-\omega)^2 \Psi_1 \Psi_2+(T-\omega)^2 \Psi_1 \Psi_3<1$, *then there exists a unique solution of the system* (3.12) *in* $C([\omega-\theta, \upsilon]) \cap C^1([\omega, \upsilon])$ *and the system* (3.12) *is HUR stable on* $[\omega-\theta, \upsilon]$.

Corollary 4.2 *If conditions* $(A_1')-(A_4')$ *are satisfied, then there exists a unique solution of the system* (3.13) *in* $PC^1([\omega, \upsilon])$ *and the system* (3.13) *is HUR stable on* $[\omega, \upsilon]$.

5. Example

Consider the following system of neural mixed Volterra–Fredholm integro–differential equations:

$$\begin{cases} \varkappa_1'(\upsilon) = -\dfrac{|\varkappa_1(\upsilon)|}{300\big(1+|\varkappa_1(\upsilon)|\big)} + \displaystyle\int_0^{\upsilon} \frac{e^{-(\upsilon-s)}}{1000}\Big(\sin|\varkappa_1(s)| + \frac{|\varkappa_2(s)|}{1+|\varkappa_2(s)|}\Big)\,ds \\ \qquad\qquad + \displaystyle\int_0^{100} \frac{1}{2000}\Big(\sin|\varkappa_1(s)| + \frac{|\varkappa_2(s)|}{1+|\varkappa_2(s)|}\Big)\,ds, \\ \varkappa_2'(\upsilon) = -\dfrac{|\varkappa_2(\upsilon)|}{300\big(1+|\varkappa_2(\upsilon)|\big)} + \displaystyle\int_0^{\upsilon} \frac{e^{-(\upsilon-s)}}{1000}\Big(\sin|\varkappa_1(s)| + \frac{|\varkappa_2(s)|}{1+|\varkappa_2(s)|}\Big)\,ds \\ \qquad\qquad + \displaystyle\int_0^{100} \frac{1}{2000}\Big(\sin|\varkappa_1(s)| + \frac{|\varkappa_2(s)|}{1+|\varkappa_2(s)|}\Big)\,ds. \end{cases} \tag{5.1}$$

Here

$$\upsilon \in [0,100]\setminus\{25,50\}, \qquad \omega = 0, \qquad \upsilon_1 = 25, \qquad \upsilon_2 = 50.$$

Let

$$U = (\varkappa_1, \varkappa_2)^{\upsilon} \in \mathbb{R}^2, \qquad \|U\| = \sum_{i=1}^{2} |\varkappa_i|.$$

Then the system (5.1) can be written in the vector form

$$\varkappa'(\upsilon) = G(\upsilon, U(\upsilon)) + \int_0^{\upsilon} E(\upsilon, s, U(s))\,ds + \int_0^{100} \zeta(\upsilon, s, U(s))\,ds,$$

where

$$G(\upsilon, U) = \begin{pmatrix} -\dfrac{|\varkappa_1|}{300(1+|\varkappa_1|)} \\ -\dfrac{|\varkappa_2|}{300(1+|\varkappa_2|)} \end{pmatrix},$$

$$E(\upsilon, s, U) = K_1(\upsilon, s) f(U), \qquad \zeta(\upsilon, s, U) = K_2(\upsilon, s) f(U),$$

$$K_1(\upsilon, s) = \begin{pmatrix} \dfrac{e^{-(\upsilon-s)}}{1000} & \dfrac{e^{-(\upsilon-s)}}{1000} \\ \dfrac{e^{-(\upsilon-s)}}{1000} & \dfrac{e^{-(\upsilon-s)}}{1000} \end{pmatrix}, \quad K_2(\upsilon, s) = \begin{pmatrix} \dfrac{1}{2000} & \dfrac{1}{2000} \\ \dfrac{1}{2000} & \dfrac{1}{2000} \end{pmatrix},$$

and

$$f(U) = \begin{pmatrix} \sin|\varkappa_1| \\ \dfrac{|\varkappa_2|}{1+|\varkappa_2|} \end{pmatrix}.$$

Let

$$U = (\varkappa_1, \varkappa_2)^{\upsilon}, \qquad \widetilde{U} = (\widetilde{\varkappa}_1, \widetilde{\varkappa}_2)^{\upsilon}.$$

$$\|G(\upsilon, U) - G(\upsilon, \widetilde{U})\| \le \frac{1}{300}\|U - \widetilde{U}\|.$$

Hence,

$$M_1 = \frac{1}{300}.$$

$$\|f(U) - f(\widetilde{U})\| \le \|U - \widetilde{U}\|, \qquad \Psi_1 = 1.$$

$$\|K_1(\upsilon,s)\| \le \frac{1}{1000}, \qquad M_2 = \frac{1}{1000}.$$

$$\|K_2(\upsilon,s)\| \le \frac{1}{2000}, \qquad \Psi_2 = \frac{1}{2000}.$$

$$\Psi_3 = M_2 + \Psi_2 = \frac{3}{2000}.$$

Let $T = 100$ and $\omega = 0$. Then

$$\begin{aligned} & M_1 + M_2 + 2\Psi_1(T-\omega) + \Psi_1\Psi_2(T-\omega)^2 + \Psi_1\Psi_3(T-\omega)^2 \\ &= \frac{1}{300} + \frac{1}{1000} + 2(1)(100) + \frac{1}{2000}(100)^2 + \frac{3}{2000}(100)^2 \\ &< 1. \end{aligned}$$

Lastly, the system (5.1) has a unique solution in $PC^1[0,100]$ according to Theorem 3.1, and all of its requirements are met. As a result, on $[0,100]$, the system (5.1) is Hyers-Ulam stable.

6. Conclusion

A new generation of information technology uses Volterra-Fredholm equations to solve continuous deep learning models through neural network learning dynamics. This study uses a novel extended Gronwall inequality, fixed-point approach, and Picard's operator technique to achieve the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of impulsive delay ordinary differential equations. There are very few results on Ulam-type stability for fractional differential equations and third-order and higher-order differential equations; these can be studied in the future. Furthermore, the use of differential equations and differential-integral equations in deep learning and artificial intelligence has only recently begun, and there are still a lot of intriguing and new issues to be resolved.

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