



Existence of a Radial Solution for a Kirchhoff Type Problem in an Annulus

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ABSTRACT: In this paper, by using a minimization principle, we study the existence of a radial solution for the following Kirchhoff type problems:

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda f(|x|, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter, $\Omega = \{x \in \mathbb{R}^N : \alpha < |x| < \beta\}$, $a, b, \alpha, \beta > 0$, $N \geq 2$, Δ is the Laplacian operator and $f : [\alpha, \beta] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. We will prove the existence of a radial solution for large values of λ .

Keywords: Kirchhoff problem, Laplacian, radial solution, variational methods.

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1. Introduction

The purpose of this work is to investigate the existence of a radial solution for the following Kirchhoff type problems :

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda f(|x|, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\lambda > 0$ is a parameter, $\Omega = \{x \in \mathbb{R}^N : \alpha < |x| < \beta\}$, $a, b, \alpha, \beta > 0$, $N \geq 2$ and $f : [\alpha, \beta] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying :

- (F1) there exists r in $[\alpha, \beta]$ such that $f(r, 0) \neq 0$,
- (F2) there exists $\tilde{s} > 0$ such that $\forall r \in [\alpha, \beta]$, $F(r, \tilde{s}) > 0$,
- (F3) $\lim_{s \rightarrow \pm\infty} \frac{F(r, s)}{s^2} = 0$ uniformly with respect to r ,

where $F(r, s) = \int_0^s f(r, \sigma) d\sigma$.

This problem is related to the stationary analogue of the Kirchhoff equation (see [10]),

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

proposed by Kirchhoff as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings.

Laplacian problems with Dirichlet boundary conditions have been treated in many papers; see for example ([3], [4], [8], [13]). About existence of radial solutions for a class of Laplacian or p-Laplacian problems, we can cite [5], [9] and [12]. In [5] and [12], the considered domain is the exterior of a ball and in [9] the problem is posed on an annulus.

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Concerning Kirchhoff type problems, many researchers are interested in them; see for example [1], [11] and [14]. In [1], by using truncations and Mountain pass theorem, the authors proved, under some hypotheses on f and M , the existence of solutions to the problem:

$$\begin{cases} -M(\int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of \mathbb{R} ; in [11], the authors treat, by using minimax methods, the following problem

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with f satisfying the 4-superlinear conditions and in [14], Li Wang et al., by considering Ω also a bounded and smooth domain of \mathbb{R}^N , treat a problem of the form

$$\begin{cases} -M(\int_{\Omega} |\nabla u|^p dx) \Delta_p u = \lambda f(x, u) + |u|^{p^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

($p^* = \frac{Np}{N-p}$). Additional recent result on Kirchhoff type problems can be found in [6] where $f \in C([r_0, +\infty) \times [0, +\infty), \mathbb{R})$ is a non-decreasing function with respect to the second variable.

Extending ideas from [9], where the equation is of the form $-\Delta u = \lambda f(u)$, we establish our result by considering, instead of Δu , a Kirchhoff type operator and generalize the term $f(u)$ to $f(|x|, u)$.

The change of variables $r = |x|$ transforms (1.1) to the boundary-value problem

$$\begin{cases} -(a + b_1 \|u\|^2)(u'' + \frac{N-1}{r} u') = \lambda f(r, u), & r \in (\alpha, \beta), \\ u(\alpha) = u(\beta) = 0, \end{cases} \quad (1.2)$$

with $b_1 = bN\omega_N$ (ω_N is the volume of the unit ball in \mathbb{R}^N) and

$$\|u\| = \left(\int_{\alpha}^{\beta} |u'|^2 r^{N-1} dr \right)^{\frac{1}{2}}.$$

Remark 1.1 The function $\|\cdot\|$ defined on $H_0^1(\alpha, \beta)$ by $\|u\| = \left(\int_{\alpha}^{\beta} |u'|^2 r^{N-1} dr \right)^{\frac{1}{2}}$ is a norm on the same space which is equivalent to the standard norm that we denote by $\|\cdot\|_{1,2}$.

Remark 1.2 Consider the function $f(r, s) = r(|s|^{\frac{1}{2}} - 1)$, then f satisfies all the hypotheses above.

Next, we define what is meant by a solution of our problem.

Definition 1.1 We say that $u \in H_0^1(\alpha, \beta)$ is a weak solution of problem (1.2) if

$$(a + b_1 \|u\|^2) \int_{\alpha}^{\beta} u' v' r^{N-1} dr = \lambda \int_{\alpha}^{\beta} f(r, u) v r^{N-1} dr,$$

for all $v \in H_0^1(\alpha, \beta)$.

We will establish the following theorem, which is our main result in this paper.

Theorem 1.1 Assume that **(F1)**-**(F3)** hold, then there exists $\Lambda > 0$ such that the problem (1.2) admits a solution for all $\lambda \geq \Lambda$.

2. Preliminaries

Remark 2.1 *If f satisfies (F3) then, for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $|F(r, s)| \leq \epsilon s^2 + C_\epsilon$ for all $(r, s) \in [\alpha, \beta] \times \mathbb{R}$.*

Lemma 2.1 *Let $J_\lambda : H_0^1(\alpha, \beta) \rightarrow \mathbb{R}$ be defined by*

$$J_\lambda(u) = \frac{1}{2} \hat{M}(\|u\|^2) - \lambda I(u)$$

where $I(u) = \int_\alpha^\beta F(r, u(r)) r^{N-1} dr$ and

$$\hat{M}(t) = \int_0^t M(\sigma) d\sigma, \quad M(t) = a + b_1 t.$$

Then J_λ is well defined, Gâteaux differentiable on $H_0^1(\alpha, \beta)$ and for all $v \in H_0^1(\alpha, \beta)$,

$$J'_{\lambda_G}(u)v = M(\|u\|^2) \int_\alpha^\beta u'v' r^{N-1} dr - \lambda \int_\alpha^\beta f(r, u)vr^{N-1} dr.$$

Proof: Let $u \in H_0^1(\alpha, \beta)$ then $\hat{M}(\|u\|^2)$ is finite and due to Remark 2.1, for $\epsilon = 1$

$$\int_\alpha^\beta F(r, u(r)) r^{N-1} dr \leq \beta^{N-1} \int_\alpha^\beta [(u(r))^2 + C] dr < \infty.$$

We put $I_1(u) = \|u\|^2$, then $\hat{M} \circ I_1$ is differentiable and for all $v \in H_0^1(\alpha, \beta)$

$$\frac{1}{2}(\hat{M} \circ I_1)'(u)(v) = M(\|u\|^2) \int_\alpha^\beta u'v' r^{N-1} dr.$$

From the continuity of f and the fact that $H_0^1(\alpha, \beta)$ is embedded in $C([\alpha, \beta])$, we deduce that I is Gâteaux differentiable and

$$I'_G(u)v = \int_\alpha^\beta f(r, u)vr^{N-1} dr, \quad v \in H_0^1(\alpha, \beta).$$

So, J_λ is Gâteaux differentiable on $H_0^1(\alpha, \beta)$ and

$$J'_{\lambda_G}(u)v = M(\|u\|^2) \int_\alpha^\beta u'v' r^{N-1} dr - \lambda \int_\alpha^\beta f(r, u)vr^{N-1} dr, \quad \forall v \in H_0^1(\alpha, \beta).$$

□

3. Proof of Theorem 1.1

Note that proving Theorem 1.1 is equivalent to prove that the functional J_λ , defined above, admits a critical point for all λ greater or equal to some positive value Λ (see [2]).

To do this, we wish to apply a theorem which is stated below.

Theorem 3.1 ([7]) *Let X be a real Banach space and J a functional defined on a region $\mathcal{U} \subset X$, and let u_0 be an interior point of \mathcal{U} . Suppose J is Gâteaux differentiable at u_0 . Then, if u_0 is an extremum of J , it is a critical point of J .*

Let's show that the functional J has an extremum.

Lemma 3.1 *For any $\lambda > 0$, the functional J_λ is bounded from below in $H_0^1(\alpha, \beta)$.*

Proof: For $u \in H_0^1(\alpha, \beta)$, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2} \hat{M}(\|u\|^2) - \lambda \int_\alpha^\beta F(r, u(r)) r^{N-1} dr \\ &= \frac{1}{2} a \|u\|^2 + \frac{b_1}{4} \|u\|^4 - \lambda \int_\alpha^\beta F(r, u(r)) r^{N-1} dr, \end{aligned}$$

and for $\epsilon > 0$, (see Remark 2.1)

$$\begin{aligned} \int_\alpha^\beta F(r, u(r)) r^{N-1} dr &\leq \beta^{N-1} \int_\alpha^\beta (\epsilon u^2(r) + C_\epsilon) dr \\ &\leq \beta^{N-1} (\epsilon \|u\|_{L^2(\alpha, \beta)}^2 + C_\epsilon(\beta - \alpha)). \end{aligned}$$

But, there exists a constant $\mu > 0$ such that

$$\|u\|_{L^2(\alpha, \beta)}^2 \leq \mu \|u'\|_{L^2(\alpha, \beta)}^2 \leq \frac{\mu}{\alpha^{N-1}} \|u\|^2, \quad \forall u \in H_0^1(\alpha, \beta)$$

then,

$$J_\lambda(u) \geq \left(\frac{1}{2} a - \left(\frac{\beta}{\alpha} \right)^{N-1} \lambda \epsilon \mu \right) \|u\|^2 + \frac{b_1}{4} \|u\|^4 - \lambda \beta^{N-1} C_\epsilon (\beta - \alpha). \quad (3.1)$$

Choosing $0 < \epsilon < \frac{a}{2\mu\lambda} \left(\frac{\alpha}{\beta} \right)^{N-1}$, we deduce that

$$J_\lambda(u) \geq -\lambda \beta^{N-1} C_\epsilon (\beta - \alpha).$$

Thus, J_λ is bounded from below. \square

Remark 3.1 Note that $J_\lambda(0) = 0$. Thus, the infimum of J_λ is nonpositive. If the infimum is 0, the minimizer may be equal to zero function, but this is not a solution of (1.2). Therefore, we are interested in the case where the infimum is negative.

Lemma 3.2 If f satisfies all the hypotheses cited above, there exists $\Lambda > 0$ such that for all $\lambda \geq \Lambda$, $j(\lambda) = \inf_{w \in H_0^1(\alpha, \beta)} J_\lambda(w) < 0$.

In order to prove Lemma 3.2, we recall the following theorem.

Theorem 3.2 [15] Let $\Omega \subset \mathbb{R}^N$ be an open set, $K \subset \Omega$ a compact and \mathcal{O} an open set such that $K \subset \mathcal{O}$ and $\overline{\mathcal{O}} \subset \Omega$. Then there exists a function $\varphi \in D(\Omega)$ such that $\varphi = 1$ on K , $\varphi = 0$ in $\Omega \setminus \mathcal{O}$ and $0 \leq \varphi \leq 1$.

Proof: Let $\tilde{s} > 0$ such that $\forall r \in [\alpha, \beta]$, $F(r, \tilde{s}) > 0$ (see **(F2)**). For $n \in \mathbb{N}$ and $n > \frac{2}{\beta - \alpha}$, we put $K = [\alpha + \frac{1}{n}, \beta - \frac{1}{n}]$ and $\mathcal{O} =]\alpha, +\infty[$, due to Theorem 3.2, there exists a function $\varphi_n \in D(\mathbb{R})$, $\varphi_n = 1$ on $[\alpha + \frac{1}{n}, \beta - \frac{1}{n}]$, $\varphi_n = 0$ in $]-\infty, \alpha]$ and $0 \leq \varphi_n \leq 1$.

And for $\mathcal{O} =]-\infty, \beta[$, there exists a function $\psi_n \in D(\mathbb{R})$, $\psi_n = 1$ on $[\alpha + \frac{1}{n}, \beta - \frac{1}{n}]$, $\psi_n = 0$ in $[\beta, +\infty[$ and $0 \leq \psi_n \leq 1$.

We put

$$\forall r \in \mathbb{R}, u_n(r) = \begin{cases} \tilde{s} \varphi_n(r), & r \in]-\infty, \alpha + \frac{1}{n}[, \\ \tilde{s} = \tilde{s} \varphi_n(r) = \tilde{s} \psi_n(r), & r \in [\alpha + \frac{1}{n}, \beta - \frac{1}{n}], \\ \tilde{s} \psi_n(r), & r \in]\beta - \frac{1}{n}, +\infty[. \end{cases}$$

The sequence $(u_n)_n$ is in $C^1(\alpha, \beta) \cap H_0^1(\alpha, \beta)$, $0 \leq u_n \leq \tilde{s}$ and converges almost everywhere in (α, β) to \tilde{s} .

Since (see Remark 2.1),

$$|F(r, \bar{u}_n) r^{N-1}| \leq ((\bar{u}_n)^2 + C) r^{N-1} \leq (\tilde{s}^2 + C) r^{N-1},$$

the Lebesgue dominated convergence theorem implies that

$$\lim_{n \rightarrow +\infty} \int_\alpha^\beta F(r, \bar{u}_n(r)) r^{N-1} dr = \int_\alpha^\beta F(r, \tilde{s}) r^{N-1} dr > 0.$$

We choose n_0 so large such that $\int_{\alpha}^{\beta} F(r, \bar{u}_{n_0}(r)) r^{N-1} dr > 0$. Then, taking $\Lambda > \frac{\hat{M}(\|\bar{u}_{n_0}\|)}{2 \int_{\alpha}^{\beta} F(r, \bar{u}_{n_0}(r)) r^{N-1} dr}$ we have, for all $\lambda \geq \Lambda$, $J_{\lambda}(\bar{u}_{n_0}) < 0$. Thus, $j(\lambda) < 0$, for all $\lambda \geq \Lambda$. \square

Theorem 3.3 *For all $\lambda \geq \Lambda$, there exists u_{λ} from $H_0^1(\alpha, \beta)$ such that $J_{\lambda}(u_{\lambda}) = \inf_{w \in H_0^1(\alpha, \beta)} J_{\lambda}(w)$.*

Proof: Let $\lambda \geq \Lambda$, since $j(\lambda) = \inf_{w \in H_0^1(\alpha, \beta)} J_{\lambda}(w)$, there exists a sequence $(w_n)_n$ from $H_0^1(\alpha, \beta)$ that verifies $\lim_{n \rightarrow +\infty} J_{\lambda}(w_n) = j(\lambda)$.

Taking $u = w_n$ in (3.1) with ϵ from $(0, \frac{a}{2\mu\lambda}(\frac{\alpha}{\beta})^{N-1})$ and knowing that $(J_{\lambda}(w_n))_n$ converges, we deduce that $(w_n)_n$ is bounded in $H_0^1(\alpha, \beta)$.

Thereby, there exists a subsequence, call it again $(w_n)_n$, which converges weakly in $H_0^1(\alpha, \beta)$ to a certain u_{λ} . Since $H_0^1(\alpha, \beta)$ is compactly embedded in $C([\alpha, \beta])$, $(w_n)_n$ converges strongly to u_{λ} in $C([\alpha, \beta])$. This implies the existence of a constant K such that for all n , $|w_n| \leq K$.

From (Remark 2.1), we have

$$|F(r, w_n) r^{N-1}| \leq (K^2 + C) r^{N-1}, \quad \text{for all } r \in [\alpha, \beta] \text{ and for all } n,$$

the Lebesgue dominated convergence theorem implies that

$$\lim_{n \rightarrow +\infty} \int_{\alpha}^{\beta} F(r, w_n(r)) r^{N-1} dr = \int_{\alpha}^{\beta} F(r, u_{\lambda}) r^{N-1} dr.$$

But, $\lim_{n \rightarrow +\infty} J_{\lambda}(w_n) = j(\lambda)$, then

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{2} \hat{M}(\|w_n\|^2) &= j(\lambda) + \lambda \int_{\alpha}^{\beta} F(r, u_{\lambda}) r^{N-1} dr \\ &\leq J_{\lambda}(u_{\lambda}) + \lambda \int_{\alpha}^{\beta} F(r, u_{\lambda}) r^{N-1} dr = \frac{1}{2} \hat{M}(\|u_{\lambda}\|^2), \end{aligned}$$

this implies that

$$\limsup_{n \rightarrow +\infty} \hat{M}(\|w_n\|^2) \leq \hat{M}(\|u_{\lambda}\|^2), \quad (3.2)$$

this on the one hand.

On the other,

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \hat{M}(\|w_n\|^2) &\geq a \liminf_{n \rightarrow +\infty} (\|w_n\|^2) + \frac{b_1}{2} \liminf_{n \rightarrow +\infty} (\|w_n\|^4) \\ &\geq a \liminf_{n \rightarrow +\infty} (\|w_n\|^2) + \frac{b_1}{2} (\liminf_{n \rightarrow +\infty} (\|w_n\|^2))^2 \\ &= \hat{M}(\liminf_{n \rightarrow +\infty} (\|w_n\|^2)). \end{aligned}$$

But, $(w_n)_n$ converges weakly to u_{λ} then,

$$\|u_{\lambda}\| \leq \liminf_{n \rightarrow +\infty} \|w_n\|$$

and therefore

$$\hat{M}(\|u_{\lambda}\|^2) \leq \hat{M}(\liminf_{n \rightarrow +\infty} (\|w_n\|^2)) \leq \liminf_{n \rightarrow +\infty} \hat{M}(\|w_n\|^2). \quad (3.3)$$

From (3.2) and (3.3), we deduce that

$$\lim_{n \rightarrow +\infty} \hat{M}(\|w_n\|^2) = \hat{M}(\|u_{\lambda}\|^2),$$

hence,

$$J_{\lambda}(u_{\lambda}) = \lim_{n \rightarrow +\infty} J_{\lambda}(w_n) = j(\lambda).$$

\square

From Theorem 3.3 and Theorem 3.1 we deduce that the functional J_{λ} admits a critical value for λ great or equal to some positive value Λ , this achieves the proof of Theorem 1.1.

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