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New Modular Identities and Partition Interpretations For Septic Rogers-Ramanujan Functions

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ABSTRACT: The septic Rogers–Ramanujan functions $A(q)$, $B(q)$, and $C(q)$ serve as natural counterparts of the classical Rogers–Ramanujan functions and are significant in the study of theta functions and modular identities. Building upon Hahn's ground breaking research, other modular relations concerning these functions have been derived using Ramanujan's theory of theta functions. This work presents an extensive array of novel modular relations pertaining to the septic Rogers–Ramanujan functions. Through the methodical application of identities related to Ramanujan's general theta function, alongside transformation formulae and decomposition methods, we construct several families of modular relations of differing degrees. These findings substantially enhance the current literature and provide a cohesive framework for deriving further identities related to septic Rogers–Ramanujan functions.

Keywords: Septic Rogers–Ramanujan functions, Ramanujan theta functions, modular relations.

Contents

1. Introduction

The Rogers–Ramanujan functions and their generalisations have a vital role in the theory of partitions, q-series, and modular forms. In the last hundred years, these functions have been thoroughly examined because of their profound relationships with combinatorics, number theory, and special functions. The septic Rogers–Ramanujan functions are a significant category among its higher-order equivalents, demonstrating intricate modular properties [7,11]. The septic Rogers–Ramanujan functions $A(q)$, $B(q)$, and $C(q)$ were first presented via product and series representations identified by Rogers and then examined by Slater. These functions fulfil notable characteristics like the classical Rogers–Ramanujan identities and may be articulated inherently in relation to Ramanujan's generic theta function. Hahn's subsequent research yielded several septic analogues of Ramanujan's forty identities, uncovering complex modular relationships among $A(q)$, $B(q)$, and $C(q)$. Certain identities also possess partition-theoretic interpretations, underscoring their combinatorial importance [8,10]. Recently, more advancements have been achieved using other methods, including integer matrix exact covering systems and theta function transformations. These methodologies have resulted in novel modular relationships and product identities pertaining to septic Rogers–Ramanujan functions. Nonetheless, despite these advancements, the current findings remain fragmented and mostly confined to discrete identities or certain parameter selections [9]. The main aim of this study is to methodically expand the established set of modular connections with septic Rogers–Ramanujan functions. Employing Ramanujan's universal theta function and its essential transformation features, we establish many novel families of modular relations concerning products and shifts of. Our methodology consolidates and extends previous findings by situating them within a comprehensive theta-function framework. Specifically, we derive modular relations of elevated degrees

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and diverse moduli that are not clearly shown in previous studies. Define the septic Rogers-Ramanujan functions

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty(q^3;q^7)_\infty(q^4;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{1.1}$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty(q^2;q^7)_\infty(q^5;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{1.2}$$

$$C(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{(q^7;q^7)_\infty(q;q^7)_\infty(q^6;q^7)_\infty}{(q^2;q^2)_\infty}. \tag{1.3}$$

where as usual, for any complex number a,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), n \geq 1, \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), |q| < 1$$

which are analogues of Roger-Ramanujan identities [21, pp. 214-215]

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}, \tag{1.4}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{1.5}$$

In his Lost Notebook [20], Ramanujan Published forty indentities involving Roger-Ramanujan function. For example, one of them is

$$H(q)G(q^{11}) - qG(q)H(q^{11}) = 1.$$

In 2003, H. Hahn [16] established several analogues of Ramanujan's forty identities involving $A(q)$, $B(q)$ and $C(q)$. Some of these identities have partition theoretic interpretations. For example

$$A(q)B(q^3) - qB(q)C(q^3) - C(q)A(q^3) = 0. \tag{1.6}$$

Cao [14] proved the above result using the technique called integer matrix exact covering system. Motivated by the above work we extend the list of modular relations involving Septic Roger Ramanujan functions in this paper.

Theorem 1.1 *We have*

$$\frac{1}{A_*(q^2)}\left[\frac{1}{A^*(q)A_*(q)} + \frac{1}{B^*(q)C_*(q)}\right] + \frac{1}{B_*(q^2)}\left[\frac{q}{C^*(q)A_*(q)} + \frac{1}{B^*(q)B_*(q)}\right] + \frac{1}{C_*(q^2)}\left[\frac{q}{A^*(q)B_*(q)} + \frac{q^2}{C^*(q)C_*(q)}\right] = 3\frac{f_7f_2f_8^3f_4^3}{f_1f_{14}^5f_{28}^2} \tag{1.7}$$

$$\frac{1}{A_*(q^2)}\left[\frac{A(q)}{A^*(q)} + \frac{C(q)}{B^*(q)}\right] + \frac{1}{B_*(q^2)}\left[\frac{-qA(q)}{C^*(q)} + \frac{B(q)}{B^*(q)}\right] - \frac{1}{C_*(q^2)}\left[\frac{qB(q)}{A^*(q)} + \frac{q^2C(q)}{C^*(q)}\right] = 3\frac{f_1f_8f_7^3}{f_{14}f_4f_2} \tag{1.8}$$

For simplicity,we use the following notations: $A_*(q^\alpha) := A(q^\alpha)B(q^{2\alpha})C(q^{2\alpha})$, $B_*(q^\alpha) := B(q^\alpha)A(q^{2\alpha})C(q^{2\alpha})$, $C_*(q^\alpha) := C(q^\alpha)B(q^{2\alpha})A(q^{2\alpha})$, $A^*(q^\alpha) = A(q^\alpha)B(q^{4\alpha})C(q^{2\alpha})$, $B_*(q^\alpha) = B(q^\alpha)A(q^{2\alpha})C(q^{4\alpha})$ and $C_*(q^\alpha) = C(q^\alpha)A(q^{4\alpha})B(q^{2\alpha})$

This work's originality is rooted on the methodical derivation of unique modular relations for the septic Rogers–Ramanujan functions $A(q)$, $B(q)$, and $C(q)$, which considerably broaden previously established findings. Previous research, particularly that of Hahn and later scholars, concentrated on singular identities or modular relations; this study, however, constructs many families of modular relations with diverse degrees and moduli inside a cohesive theta-function framework. This paper's primary contributions may be succinctly summarised as follows:
(i) New families of modular relations concerning products and shifts of the septic Rogers–Ramanujan functions have been derived, many of which are absent from the current literature.
(ii) The findings generalise and expand Hahn's septic analogues of the Rogers–Ramanujan identities by methodically using Ramanujan's general theta function identities and transformation formulae. (iii) A consistent technique using theta-function decompositions is used, facilitating the derivation of many identities as specific instances of more extensive structural links.
(iv) The derived identities enhance the algebraic and modular framework of classical Rogers–Ramanujan functions and provide a basis for future research, including higher-order analogues and possible combinatorial interpretations.
(v) These contributions enhance the theory of septic Rogers–Ramanujan functions by integrating established findings and significantly broadening the repertoire of existing modular linkages.

2. Definition and Preliminary Results

In this section, we recall some fundamental definitions and preliminary results on Ramanujan's theta functions. The general theta function as defined by Ramanujan is [15, p. 34, Eq. 18.1]

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, |ab| < 1. \tag{2.1}$$

The theta function $f(a,b)$ satisfy the following simple properties [15, p. 34, Entry 18] :

$$f(a,b) = f(b,a), \tag{2.2}$$
$$f(1,a) = 2f(a,a^3), \tag{2.3}$$
$$f(-1,a) = 0, \tag{2.4}$$
$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{2.5}$$

These properties are used frequently in this paper without citing them. Setting $n = 1, a = q^r, b = q^s$ in (2.5), we get

$$f(q^r, q^s) = q^r f(q^{2r+s}, q^{-r}). \tag{2.6}$$

This identity is often used whenever negative powers of argument occur in $f(a,b)$. The Jacobi triple product identity [15, p. 35, Entry 19] in Ramanujan's notation takes the following form:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The three special cases of (2.1) as recorded by Ramanujan are [15, p. 36, Entry 22]

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{2.7}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.8}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{2.9}$$

Ramanujan also defines another important function $\chi(q) := (-q;q^2)_\infty$. For sake of convenience, define $f_n := f(-q^n) = (q^n;q^n)_\infty$, for $n \geq 1$. Setting $n=2$ in [15, p. 48, Entry 31], we get

$$f(a,b) = f(ab^3, a^3b) + af(b/a, a^5b^3). \tag{2.10}$$

From (2.7) and (2.8), it is easy to obtain the following lemma:

Lemma 2.1 *We have*

$$\phi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \psi(q) = \frac{f_2^2}{f_1}, \phi(-q) = \frac{f_1^2}{f_2}, \psi(-q) = \frac{f_1 f_4}{f_2}, f(q) = \frac{f_2^3}{f_1 f_4}.$$

The septic Rogers-Ramanujan functions in terms of $f(a,b)$ can be expressed as

$$A(q) = \frac{f(-q^3,-q^4)}{f_2}, B(q) = \frac{f(-q^2,-q^5)}{f_2}, C(q) = \frac{f(-q,-q^6)}{f_2}. \tag{2.11}$$

It can be easily seen that $A(q)B(q)C(q) = \frac{f_1 f_7^2}{f_2^3}$. We need the following functional relations of $A(q)$, $B(q)$ and $C(q)$:

Lemma 2.2 *We have*

$$f(q,q^6) = \frac{f_4}{f_2}\frac{C(q^2)}{C(q)}\phi(-q^7), \tag{2.12}$$

$$f(q^2,q^5) = \frac{f_4}{f_2}\frac{B(q^2)}{B(q)}\phi(-q^7), \tag{2.13}$$

$$f(q^3,q^4) = \frac{f_4}{f_2}\frac{A(q^2)}{A(q)}\phi(-q^7), \tag{2.14}$$

$$f(q^3,q^{11}) = \frac{f_7 f_{28} f_{14}}{f_4 f_8}\frac{1}{A(q)B(q^4)C(q^2)}, \tag{2.15}$$

$$f(q^5,q^9) = \frac{f_7 f_{28} f_{14}}{f_4 f_8}\frac{1}{B(q)C(q^4)A(q^2)}, \tag{2.16}$$

$$f(q,q^{13}) = \frac{f_7 f_{28} f_{14}}{f_4 f_8}\frac{1}{C(q)A(q^4)B(q^2)}. \tag{2.17}$$

Proof: Two identities in (2.14) can be obtained by setting $(a,b) := (q,q^6), (q^2,q^5)$ and (q^3,q^4) in $f(a,b)f(-a,-b) = f(-a^2,-b^2)\phi(-ab)$. □

Lemma 2.3 *[15, p. 45, Entry 29] If $ab = cd$, then we have*

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af(\frac{b}{c}, c^2ad)f(\frac{b}{d}, d^2ac). \tag{2.18}$$

Lemma 2.4 *[15, p. 68, Eq. 36.3] We have*

$$\phi(q^{\mu+\nu})\phi(q^{\mu-\nu}) + \phi(-q^{\mu+\nu})\phi(-q^{\mu-\nu}) =$$
$$2\sum_{m=0}^{\mu-1} q^{2\mu m^2} f\left(q^{(2\mu+4m)(\mu^2-\nu^2)}, q^{(2\mu-4m)(\mu^2-\nu^2)}\right) f\left(q^{2\mu+4\nu m}, q^{2\mu-4\nu m}\right). \tag{2.19}$$

Lemma 2.5 *[15, p. 69, Eq. 36.9] If m is odd, then we have*

$$\psi(q^{m+n})\psi(q^{m-n}) = q^{\frac{m^3-m}{4}}\psi(q^{2m(m^2-n^2)})f(q^{m-mn}, q^{m+mn})$$
$$+ \sum_{k=0}^{\frac{m-3}{2}} q^{mk(k1)} f(q^{(m+2k+1)(m^2-n^2)}, q^{(m-2k-1)(m^2-n^2)}) f(q^{m+n+2kn}, q^{m-n-2kn}). \tag{2.20}$$

Lemma 2.6 *[15, p. 69, Eq. 36.8] If m is even, then we have*

$$\psi(q^{m+n})\psi(q^{m-n}) = \phi(q^{m(m^2-n^2)})\psi(q^{2m}) + q^{\frac{m^3}{4}-\frac{1}{2}mn}\psi(q^{2m(m^2-n^2)})f(q^{mn}, q^{2m-mn})$$
$$+ \sum_{k=1}^{\frac{m}{2}-1} q^{k^2m-nk} f(q^{(m+2k)(m^2-n^2)}, q^{(m-2k)(m^2-n^2)})f(q^{2nk}, q^{2m-2nk}). \tag{2.21}$$

3. Proof of Main Results

Theorem 3.1 *We have*

$$\frac{1}{A_*(q^2)}\left[\frac{1}{A^*(q)A_*(q)} + \frac{1}{B^*(q)C_*(q)}\right] + \frac{1}{B_*(q^2)}\left[\frac{q}{C^*(q)A_*(q)} + \frac{1}{B^*(q)B_*(q)}\right] + \frac{1}{C_*(q^2)}\left[\frac{q}{A^*(q)B_*(q)} + \frac{q^2}{C^*(q)C_*(q)}\right] = 3\frac{f_7f_2f_8^3f_4^3}{f_1f_{14}^5f_{28}^2} \tag{3.1}$$

$$\frac{1}{A_*(q^2)}\left[\frac{A(q)}{A^*(q)} + \frac{C(q)}{B^*(q)}\right] + \frac{1}{B_*(q^2)}\left[\frac{-qA(q)}{C^*(q)} + \frac{B(q)}{B^*(q)}\right] - \frac{1}{C_*(q^2)}\left[\frac{qB(q)}{A^*(q)} + \frac{q^2C(q)}{C^*(q)}\right] = 3\frac{f_1f_8f_7^3}{f_{14}f_4f_2}. \tag{3.2}$$

For simplicity,we use the following notations:
$A_*(q^\alpha) := A(q^\alpha)B(q^{2\alpha})C(q^{2\alpha})$,
$B_*(q^\alpha) := B(q^\alpha)A(q^{2\alpha})C(q^{2\alpha})$,
$C_*(q^\alpha) := C(q^\alpha)B(q^{2\alpha})A(q^{2\alpha})$,
$A^*(q^\alpha) = A(q^\alpha)B(q^{4\alpha})C(q^{2\alpha})$,
$B_*(q^\alpha) = B(q^\alpha)A(q^{2\alpha})C(q^{4\alpha})$ and
$C_*(q^\alpha) = C(q^\alpha)A(q^{4\alpha})B(q^{2\alpha})$

Proof: Setting $(a, b, c, d) := (q, q^6, q^2, q^5), (q^2, q^5, q^3, q^4)$ and (q, q^6, q^3, q^4) in (2.18), we get the following equations respectively

$$f(q, q^6)f(q^2, q^5) = f(q^3, q^{11})f(q^6, q^8) + qf(q^4, q^{10})f(q, q^{13}), \tag{3.3}$$
$$f(q^2, q^5)f(q^3, q^4) = f(q^5, q^9)f(q^6, q^8) + q^2f(q^2, q^{12})f(q, q^{13}), \tag{3.4}$$
$$f(q, q^6)f(q^3, q^4) = f(q^5, q^9)f(q^4, q^{10}) + qf(q^2, q^{12})f(q^3, q^{11}). \tag{3.5}$$

Now multiplying (3.3) by $f(q^3, q^4)$, (3.4) by $f(q, q^6)$, (3.5) by $f(q^2, q^5)$ then adding and noting $f(q, q^6)f(q^2, q^5)f(q^3, q^4) = \frac{f_2f_7^4}{f_1f_{14}}$ we obtain

$$f(q^3, q^4)\left[f(q^3, q^{11})f(q^6, q^8) + qf(q^4, q^{10})f(q, q^{13})\right] + f(q, q^6)\left[f(q^5, q^9)f(q^6, q^8) + q^2f(q^2, q^{12})f(q, q^{13})\right] + f(q^2, q^5)\left[f(q^5, q^9)f(q^4, q^{10}) + qf(q^2, q^{12})f(q^3, q^{11})\right] = 3\frac{f_2f_7^4}{f_1f_{14}}. \tag{3.6}$$

Applying (2.12) - (2.14) in the above, we obtain

$$\frac{1}{A_*(q)}\left[f(q^3, q^{11})f(q^6, q^8) + qf(q^4, q^{10})f(q, q^{13})\right] + \frac{1}{C_*(q)}\left[f(q^5, q^9)f(q^6, q^8) + q^2f(q^2, q^{12})f(q, q^{13})\right] + \frac{1}{B_*(q)}\left[f(q^5, q^9)f(q^4, q^{10}) + qf(q^2, q^{12})f(q^3, q^{11})\right] = 3\frac{f_2f_4^2f_7^2}{f_1f_{14}^2}. \tag{3.7}$$

Rearranging the terms of the above equation, we obtain

$$\begin{aligned} f(q^3,q^{11})\left[\frac{1}{A_*(q)}f(q^6,q^8)+q\frac{1}{B_*(q)}f(q^2,q^{12})\right]+\\ f(q,q^{13})\left[q\frac{1}{A_*(q)}f(q^4,q^{10})+q^2\frac{1}{C_*(q)}f(q^2,q^{12})\right]+\\ f(q^5,q^9)\left[\frac{1}{C_*(q)}f(q^6,q^8)+\frac{1}{B_*(q)}f(q^4,q^{10})\right]=3\frac{f_2f_4^2f_7^2}{f_1f_{14}^2}. \end{aligned} \tag{3.8}$$

Applying (2.15) - (2.17) in the above, we obtain

$$\begin{aligned} \frac{1}{A^*(q)}\left[\frac{1}{A_*(q)}f(q^6,q^8)+q\frac{1}{B_*(q)}f(q^2,q^{12})\right]+\\ \frac{1}{C^*(q)}\left[q\frac{1}{A_*(q)}f(q^4,q^{10})+q^2\frac{1}{C_*(q)}f(q^2,q^{12})\right]+\\ \frac{1}{B^*(q)}\left[\frac{1}{C_*(q)}f(q^6,q^8)+\frac{1}{B_*(q)}f(q^4,q^{10})\right]=3\frac{f_2f_4^3f_7f_8}{f_1f_{14}^3f_{28}}. \end{aligned} \tag{3.9}$$

Rearranging the terms of the above equation, we obtain

$$\begin{aligned} f(q^6,q^8)\left[\frac{1}{A^*(q)A_*(q)}+\frac{1}{B^*(q)C_*(q)}\right]+f(q^4,q^{10})\left[\frac{q}{C^*(q)A_*(q)}+\frac{1}{B^*(q)B_*(q)}\right]+\\ f(q^2,q^{12})\left[\frac{q}{A^*(q)B_*(q)}+\frac{q^2}{C^*(q)C_*(q)}\right]=3\frac{f_2f_4^3f_7f_8}{f_1f_{14}^3f_{28}}. \end{aligned} \tag{3.10}$$

Applying (2.12) - (2.14) in the above, we obtain (3.1). The proof of (3.2) is of the same nature. So, we omit the proof here. □

Theorem 3.2 *We have*

$$\frac{1}{B^*(q)B^*(q^6)}+\frac{q^3}{A^*(q)A^*(q^6)}+\frac{q^8}{C^*(q)C^*(q^6)}=\frac{f_4f_8f_{24}f_{48}}{2qf_7f_{14}f_{28}f_{42}f_{84}f_{168}}\left(\frac{f_2^5}{f_1^2f_4^2}\frac{f_{12}^5}{f_6^2f_{24}^2}-\frac{f_{14}^5}{f_7^2f_{28}^2}\frac{f_{84}^5}{f_{42}^2f_{168}^2}\right), \tag{3.11}$$

$$\frac{1}{B^*(q^{10})A^*(q)}+\frac{q^5}{A^*(q^{10})C^*(q)}+\frac{q^{11}}{C^*(q^{10})B^*(q)}=\frac{f_4f_8f_{40}f_{80}}{2q^2f_7f_{14}f_{28}f_{70}f_{140}f_{280}}\left(\frac{f_4^5}{f_2^2f_8^2}\frac{f_{10}^5}{f_5^2f_{20}^2}-\frac{f_{14}^5}{f_7^2f_{28}^2}\frac{f_{140}^5}{f_{70}^2f_{280}^2}\right), \tag{3.12}$$

$$\frac{1}{B^*(q^{12})C^*(q)}+\frac{q^4}{A^*(q^{12})B^*(q)}+\frac{q^{13}}{C^*(q^{12})A^*(q)}=\frac{f_4f_8f_{48}f_{96}}{2q^3f_7f_{14}f_{28}f_{84}f_{168}f_{336}}\left(\frac{f_6^5}{f_3^2f_{12}^2}\frac{f_8^5}{f_4^2f_{16}^2}-\frac{f_{14}^5}{f_7^2f_{28}^2}\frac{f_{168}^5}{f_{84}^2f_{336}^2}\right), \tag{3.13}$$

$$\frac{1}{B^*(q^5)B^*(q^2)}+\frac{q^3}{A^*(q^5)A^*(q^2)}+\frac{q^8}{C^*(q^5)C^*(q^2)}=\frac{f_8f_{16}f_{20}f_{40}}{2qf_{14}f_{28}f_{56}f_{35}f_{70}f_{140}}\left(\frac{f_4^5}{f_2^2f_8^2}\frac{f_{10}^5}{f_5^2f_{20}^2}-\frac{f_{28}^5}{f_{14}^2f_{56}^2}\frac{f_{70}^5}{f_{35}^2f_{140}^2}\right), \tag{3.14}$$

$$\frac{1}{B^*(q^3)B^*(q^4)}+\frac{q^3}{A^*(q^3)A^*(q^4)}+\frac{q^8}{C^*(q^3)C^*(q^4)}=\frac{f_{12}f_{24}f_{16}f_{32}}{2qf_{21}f_{42}f_{84}f_{28}f_{56}f_{112}}\left(\frac{f_6^5}{f_3^2f_{12}^2}\frac{f_8^5}{f_4^2f_{16}^2}-\frac{f_{42}^5}{f_{21}^2f_{84}^2}\frac{f_{56}^5}{f_{28}^2f_{112}^2}\right), \tag{3.15}$$

$$\frac{q}{B^*(q^3)C^*(q^2)}+\frac{q^4}{A^*(q^3)B^*(q^2)}+\frac{q^3}{C^*(q^3)A^*(q^2)}=\frac{f_8f_{16}f_{12}f_{24}}{2q^2f_{14}f_{28}^2f_{56}f_{21}f_{42}^2f_{84}}\left(\frac{f_6^5}{f_3^2f_{12}^2}\frac{f_4^5}{f_2^2f_8^2}-\frac{f_{28}^5}{f_{14}^2f_{56}^2}\frac{f_{42}^5}{f_{21}^2f_{84}^2}\right). \tag{3.16}$$

Proof: With the help of Lemma (2.1), Lemma (2.2) we can rewrite the identity (3.11) in the following form:

$$\phi(q)\phi(q^6)=\phi(q^7)\phi(q^{42})+2qf(q^9,q^5)f(q^{54},q^{30})+2q^4f(q^{11},q^3)f(q^{66},q^{18})+2q^9f(q^{13},q)f(q^{78},q^6). \tag{3.17}$$

To prove (3.11), it is enough to prove (3.17). Consider,

$$\phi(q)\phi(q^6)=\sum_{m,n=-\infty}^{\infty} q^{m^2+6n^2} \tag{3.18}$$

Let $m=6M+N+a$ and $n=M-N$, where $0\le a\le 6$ and $M,N\in Z$. Since there is a bijective relation between the sets $A=\{(m,n)|m,n\in Z\}$ and $B=\{(M,N,a)|M,N\in Z, a=0,1,2,\dots,6\}$ from (3.18) , we find that

$$\begin{aligned}\phi(q)\phi(q^6)&=\sum_{a=0}^{6} q^{a^2}\sum_{M=-\infty}^{\infty} q^{42M^2+12aM}\sum_{N=-\infty}^{\infty} q^{7N^2+2aN}\\&=\sum_{a=0}^{6} q^{a^2} f(q^{42+12a},q^{42-12a})f(q^{7+2a},q^{7-2a})\end{aligned}$$

$$=\phi(q^7)\phi(q^{42})+2qf(q^9,q^5)f(q^{54},q^{30})+2q^4f(q^{11},q^3)f(q^{66},q^{18})+2q^9f(q^{13},q)f(q^{78},q^6).$$

which is the same as (3.17). Hence, (3.11) is proved. The proof of (3.12) - (3.16) is of the same nature. So, we omit the proof here.

□

Theorem 3.3 *We have*

$$\frac{1}{A_*(q)A_*(q^6)}+\frac{q}{B_*(q)B_*(q^6)}+\frac{q^3}{C_*(q)C_*(q^6)}=\frac{f_4^2f_{24}^2f_{84}^2}{f_7^2f_{14}f_{42}^2f_{84}}\left(\frac{f_6^5f_2^2}{f_3^2f_{12}^2f_1}-2q^6\frac{f_{14}^2}{f_7f_{42}}\right), \tag{3.19}$$

$$\frac{q}{C_*(q^2)B_*(q^3)}+\frac{1}{B_*(q^2)A_*(q^3)}+\frac{q}{A_*(q^2)C_*(q^3)}=\frac{f_8^2f_{12}}{f_{14}^2f_{28}f_{21}^2f_{42}}\left(\frac{f_2^5f_6^2}{f_1^2f_4^2f_3}-2q^4\frac{f_{42}^2f_{28}^2}{f_{21}f_{14}}\right), \tag{3.20}$$

$$\frac{1}{A_*(q^2)A_*(q^5)}+\frac{q}{B_*(q^2)B_*(q^5)}+\frac{q^3}{C_*(q)C_*(q^5)}=\frac{f_8^2f_{20}^2}{f_{14}^2f_{28}f_{35}^2f_{70}}\left(\frac{f_{10}^5f_2^2}{f_5^2f_{20}^2f_1}-2q^6\frac{f_{70}^2f_{28}^2}{f_{35}f_{14}}\right), \tag{3.21}$$

$$\frac{1}{A_*(q^{10})C_*(q)}+\frac{q}{B_*(q^{10})A_*(q)}+\frac{q^4}{C_*(q^{10})B_*(q)}=\frac{f_4^2f_{40}^2}{f_7^2f_{14}f_{70}^2f_{140}}\left(\frac{f_2^5f_{10}^2}{f_1^2f_4^2f_5}-2q^9\frac{f_{14}^2f_{140}^2}{f_7f_{70}}\right), \tag{3.22}$$

$$\frac{1}{B^*(q)A_*(q^6)}+\frac{q^2}{C^*(q)B_*(q^6)}+\frac{q^3}{A^*(q)C_*(q^6)}=\frac{f_4f_8f_{24}^2}{f_7f_{28}f_{14}f_{42}^2f_{84}}\left(\frac{f_6^5f_4^2}{f_3^2f_{12}^2f_2}-q^5\frac{f_{84}^2f_{14}^5}{f_{42}f_7^2f_{28}^2}\right), \tag{3.23}$$

$$\frac{1}{B^*(q^3)B_*(q^2)}+\frac{q}{A^*(q^3)A_*(q^2)}+\frac{q^4}{C^*(q^3)C_*(q^2)}=\frac{f_{12}f_{24}f_8^2}{f_{21}f_{84}f_{42}f_{14}^2f_{28}}\left(\frac{f_2^5f_{12}^2}{f_1^2f_4^2f_6}-q\frac{f_{28}^2f_{42}^5}{f_{14}f_{21}^2f_{84}^2}\right). \tag{3.24}$$

Proof: With the help of Lemma (2.1) and Lemma (2.2) , (3.19) takes the following form:

$$\phi(q^3)\psi(q)=2q^6\psi(q^7)\psi(q^{42})+f(q^{18},q^{24})f(q^3,q^4)+qf(q^{12},q^{30})f(q^2,q^5)+q^3f(q^{36},q^6)f(q^6,q). \tag{3.25}$$

Setting $(a,b)=(q,q^6)$, (q^2,q^5) and (q^3,q^4) in (2.10) , we obtain

$$\begin{aligned}f(q,q^6)&=f(q^9,q^{19})+qf(q^5,q^{23}),\\f(q^2,q^5)&=f(q^{11},q^{17})+q^2f(q^3,q^{25}),\\f(q^3,q^4)&=f(q^{15},q^{13})+q^3f(q^1,q^{27}).\end{aligned}$$

Employing the above three identities in (3.25), we arrive at

$$\begin{aligned}\phi(q^3)\psi(q)&=f(q^{15},q^{13})f(q^{18},q^{24})+q^3f(q^{27},q^1)f(q^{18},q^{24})+qf(q^{11},q^{17})f(q^{12},q^{30})\\&+q^4f(q^5,q^{23})f(q^{36},q^6)+q^6f(q^7,q^{21})f(1,q^{42})+q^3f(q^9,q^{19})f(q^{36},q^6)+q^3f(q^{25},q^3)f(q^{30},q^{12}).\end{aligned} \tag{3.26}$$

To prove (3.19), it suffices to prove (3.26). Consider

$$\phi(q^3)\psi(q) = f(q^3, q^3)f(q, q^3) = \sum_{m,n=-\infty}^{\infty} q^{3m^2+2n^2-n} \tag{3.27}$$

Let $m = 2M + N + a$ and $n = -M + 3N$, where $0 \leq a \leq 6$ and $M, N \in Z$. Since there is a bijective relation between the sets $A = \{(m, n)|m, n \in Z\}$ and $B = \{(M, N, a)|M, N \in Z, a = 0, 1, 2, \ldots, 6\}$ from (3.27) , we find that

$$\begin{aligned}\phi(q^3)\psi(q) &= \sum_{a=0}^{6} q^{a^2} \sum_{M=-\infty}^{\infty} q^{14M^2+12aM+M} \sum_{N=-\infty}^{\infty} q^{21N^2+6aN-3N} \\ &= \sum_{a=0}^{6} q^{3a^2} f(q^{15+12a}, q^{13-12a}) f(q^{18+6a}, q^{24-6a})\end{aligned}$$

$$\begin{aligned}= \phi(q^3)\psi(q) = f(q^{15}, q^{13})f(q^{18}, q^{24}) + q^3 f(q^{27}, q^1)f(q^{18}, q^{24}) + qf(q^{11}, q^{17})f(q^{12}, q^{30}) \\ +q^4 f(q^5, q^{23})f(q^{36}, q^6) + q^6 f(q^7, q^{21})f(1, q^{42}) + q^3 f(q^9, q^{19})f(q^{36}, q^6) + q^3 f(q^{25}, q^3)f(q^{30}, q^{12}).\end{aligned}$$

which is the same as (3.26). The proofs of (3.20) - (3.24) can be done similarly. □

Theorem 3.4 *We have*

$$\begin{aligned}
&\frac{1}{B^*(-q^{26})B^*(-q^2)}+\frac{q^{12}}{A^*(-q^{26})A^*(-q^2)}+\frac{q^{32}}{C^*(-q^{26})C^*(-q^2)}\\
&=\frac{f_8f_{16}f_{14}f_{104}f_{208}f_{182}}{4q^4f_{28}^4f_{364}^4}\left(\frac{f_1^2f_{26}^5}{f_2f_{13}^2f_{52}^2}+\frac{f_2^5f_{13}^2}{f_1^2f_4^2f_{26}}-2\frac{f_{14}^2f_{182}^2}{f_{28}f_{364}}\right), \quad (3.28)\\
&\frac{1}{B^*(-q^{22})B^*(-q^6)}+\frac{q^{12}}{A^*(-q^{22})A^*(-q^6)}+\frac{q^{32}}{C^*(-q^{22})C^*(-q^6)}\\
&=\frac{f_{24}f_{48}f_{42}f_{88}f_{176}f_{154}}{4q^4f_{84}^4f_{308}^4}\left(\frac{f_1^2f_{66}^5}{f_2f_{33}^2f_{132}^2}+\frac{f_2^5f_{33}^2}{f_1^2f_4^2f_{66}}-2\frac{f_{42}^2f_{154}^2}{f_{84}f_{308}}\right), \quad (3.29)\\
&\frac{1}{B^*(-q^{18})B^*(-q^{10})}+\frac{q^{12}}{A^*(-q^{18})A^*(-q^{10})}+\frac{q^{32}}{C^*(-q^{18})C^*(-q^{10})}\\
&=\frac{f_{40}f_{80}f_{70}f_{72}f_{144}f_{126}}{4q^4f_{140}^4f_{252}^4}\left(\frac{f_1^2f_{90}^5}{f_2f_{45}^2f_{180}^2}+\frac{f_2^5f_{45}^2}{f_1^2f_4^2f_{90}}-2\frac{f_{70}^2f_{126}^2}{f_{140}f_{252}}\right), \quad (3.30)\\
&\frac{1}{B^*(-q^{24})B^*(-q^4)}+\frac{q^{12}}{A^*(-q^{24})A^*(-q^4)}+\frac{q^{32}}{C^*(-q^{24})C^*(-q^4)}\\
&=\frac{f_{16}f_{32}f_{28}f_{96}f_{192}f_{168}}{4q^4f_{56}^4f_{336}^4}\left(\frac{f_1^2f_{24}^2}{f_2f_{48}}+\frac{f_2^5f_{24}^2}{f_1^2f_4^2f_{48}}-2\frac{f_{168}^2f_{28}^2}{f_{336}f_{56}}\right), \quad (3.31)\\
&\frac{1}{B^*(-q^{20})B^*(-q^8)}+\frac{q^{12}}{A^*(-q^{20})A^*(-q^8)}+\frac{q^{32}}{C^*(-q^{20})C^*(-q^8)}\\
&=\frac{f_{32}f_{64}f_{56}f_{80}f_{160}f_{140}}{4q^4f_{112}^4f_{280}^4}\left(\frac{f_1^2f_{40}^2}{f_2f_{80}}+\frac{f_2^5f_{40}^2}{f_1^2f_4^2f_{80}}-2\frac{f_{56}^2f_{140}^2}{f_{112}f_{280}}\right), \quad (3.32)\\
&\frac{1}{B^*(-q^{16})B^*(-q^{12})}+\frac{q^{12}}{A^*(-q^{16})A^*(-q^{12})}+\frac{q^{32}}{C^*(-q^{16})C^*(-q^{12})}\\
&=\frac{f_{48}f_{96}f_{84}f_{64}f_{128}f_{102}}{4q^4f_{168}^4f_{204}^4}\left(\frac{f_1^2f_{48}^2}{f_2f_{96}}+\frac{f_2^5f_{48}^2}{f_1^2f_4^2f_{96}}-2\frac{f_{102}^2f_{84}^2}{f_{204}f_{168}}\right), \quad (3.33)\\
&\frac{B(q^{12})}{B^*(-q^4)}+\frac{A(q^{12})}{A^*(-q^4)}+\frac{q^8C(q^{12})}{C^*(-q^4)}\\
&=\frac{f_{16}f_{32}f_{28}}{2qf_{56}^4f_{24}}\left(\frac{f_2^5f_{12}f_{48}}{f_1^2f_4^2f_{24}}-\frac{f_1^2f_{12}f_{48}}{f_2f_{24}}-2q^8\frac{f_{28}^2f_{84}f_{336}}{f_{56}f_{168}}\right), \quad (3.34)\\
&\frac{B(q^4)}{B^*(-q^{20})}+\frac{q^8A(q^4)}{A^*(-q^{20})}+\frac{q^{24}C(q^4)}{C^*(-q^{20})}\\
&=\frac{f_{16}f_{32}f_{28}}{2qf_{56}^4f_8}\left(\frac{f_2^5f_{20}f_{80}}{f_1^2f_4^2f_{40}}-\frac{f_1^2f_{20}f_{80}}{f_2f_{40}}-2q\frac{f_{140}^2f_{28}f_{112}}{f_{280}f_{56}}\right), \quad (3.35)\\
&\frac{B(q^8)}{B^*(-q^{12})}+\frac{q^4A(q^8)}{A^*(-q^{12})}+\frac{q^{16}C(q^8)}{C^*(-q^{12})}\\
&=\frac{f_{48}f_{96}f_{84}}{2f_{168}^4f_{16}}\left(\frac{f_2^5f_{24}f_{96}}{f_1^2f_4^2f_{48}}+\frac{f_1^2f_{24}f_{96}}{f_2f_{48}}-2q^4\frac{f_{84}^2f_{56}f_{224}}{f_{112}f_{168}}\right), \quad (3.36)\\
&\frac{1}{B^*(-q^{14})^2}+\frac{q^{12}}{A^*(-q^{14})^2}+\frac{q^{32}}{C^*(-q^{14})^2}\\
&=\frac{f_{56}^2f_{116}^2f_{98}^2}{2q^4f_{196}^8}\left(\frac{f_1^2f_{98}^5}{f_2f_{49}^2f_{196}^2}+\frac{f_2^5f_{49}^2}{f_1^2f_4^2f_{98}}-2\frac{f_{98}^4}{f_{196}^2}\right). \quad (3.37)
\end{aligned}$$

Proof: Using Lemma (2.1) and Lemma (2.2), the identity (3.28) can be rewritten as follows:

$$\phi(-q)\phi(q^{13})+\phi(q)\phi(-q^{13})=2f(-q^{14},-q^{14})f(-q^{182},-q^{182})+4q^4f(-q^{10},-q^{18})f(-q^{130},-q^{234})$$
$$+4q^{16}f(-q^6,-q^{22})f(-q^{78},-q^{286})+4q^{36}f(-q^2,-q^{26})f(-q^{26},-q^{338}). \tag{3.38}$$

To prove (3.28), it is enough to prove (3.38). Consider

$$\phi(-q)\phi(q^{13})=f(-q,-q)f(q^{13},q^{13})=\sum_{m,n=-\infty}^{\infty}(-1)^m q^{m^2+13n^2} \tag{3.39}$$

Let $m=13M+N+a$ and $n=-M-N$, where $-1\le a\le 7$ and $M,N\in Z$. Since there is a bijective relation between the sets $A=\{(m,n)|m,n\in Z\}$ and $B=\{(M,N,a)|M,N\in Z, a=-1,0,1,\dots,7\}$ from (3.39) , we find that

$$\phi(-q)\phi(q^{13})=\sum_{a=-1}^{7}(-1)^a(-1)^{M+N}q^{a^2}\sum_{M=-\infty}^{\infty}q^{182M^2+26aM}\sum_{N=-\infty}^{\infty}q^{14N^2+2aN}$$
$$=\sum_{a=-1}^{7}(-1)^a q^{3a^2}f(-q^{182+26a},-q^{182-26a})f(-q^{14+2a},-q^{14-2a})$$

$$=f(-q^{14},-q^{14})f(-q^{182},-q^{182})-2qf(-q^{12},-q^{16})f(-q^{156},-q^{208})+2q^4f(-q^{10},-q^{18})f(-q^{130},-q^{234})$$
$$-2q^9f(-q^8,-q^{20})f(-q^{104},-q^{260})+2q^{16}f(-q^6,-q^{22})f(-q^{78},-q^{286})-2q^{25}f(-q^4,-q^{24})f(-q^{52},-q^{312})$$
$$+2q^{36}f(-q^2,-q^{26})f(-q^{26},-q^{338}). \tag{3.40}$$

Changing q by $-q$ in (3.40), and then adding the resulting identity with (3.40), we obtain

$$\phi(-q)\phi(q^{13})+\phi(q)\phi(-q^{13})=2f(-q^{14},-q^{14})f(-q^{182},-q^{182})+4q^4f(-q^{10},-q^{18})f(-q^{130},-q^{234})$$
$$+4q^{16}f(-q^6,-q^{22})f(-q^{78},-q^{286})+4q^{36}f(-q^2,-q^{26})f(-q^{26},-q^{338}). \tag{3.41}$$

which is equivalent to (3.38). This completes the proof of (3.28). The proofs of (3.42) - (3.48) are similar. So, we omit the proofs. □

Theorem 3.5 *We have*

$$\frac{1}{B^*(q^{48})B^*(q)}+\frac{q^{21}}{A^*(q^{48})A^*(q)}+\frac{q^{56}}{C^*(q^{48})C^*(q)}$$
$$=\frac{f_4f_8f_{192}f_{384}}{f_7f_{14}f_{28}f_{336}f_{672}f_{1344}}\left(\frac{f_8^5f_6^5}{f_4^2f_{16}^2f_3^2f_{12}^2}-\frac{f_4^2f_3^2}{f_8f_6}-2\frac{f_{14}^5f_{672}^5}{f_7^2f_{28}^2f_{336}^2f_{1344}^2}\right), \tag{3.42}$$

$$\frac{1}{B^*(q^{90})A^*(q^2)}+\frac{q^{40}}{A^*(q^{90})C^*(q^2)}+\frac{q^{102}}{C^*(q^{90})B^*(q^2)}$$
$$=\frac{f_8f_{16}f_{360}f_{720}}{4q^{14}f_{14}f_{28}f_{56}f_{630}f_{1260}f_{2520}}\left(\frac{f_{10}^5f_{18}^5}{f_5^2f_{20}^2f_9^2f_{36}^2}-\frac{f_5^2f_9^2}{f_{18}f_{10}}-2\frac{f_{28}^5f_{1260}^5}{f_{14}^2f_{56}^2f_{630}^2f_{2520}^2}\right), \tag{3.43}$$

$$\frac{1}{B^*(q^{40})B^*(q)}+\frac{q^{16}}{A^*(q^{40})A^*(q)}+\frac{q^{45}}{C^*(q^{40})C^*(q)}$$
$$=\frac{f_4f_8f_{160}f_{320}}{4q^7f_7f_{14}f_{28}f_{280}f_{560}f_{1120}}\left(\frac{f_4^5f_{10}^5}{f_2^2f_8^2f_5^2f_{20}^2}-\frac{f_2^2f_5^2}{f_4f_{10}}-2\frac{f_{14}^5f_{560}^5}{f_7^2f_{28}^2f_{280}^2f_{1120}^2}\right), \tag{3.44}$$

$$\frac{1}{B^*(q^{66})C^*(q^2)}+\frac{q^{26}}{A^*(q^{66})B^*(q^2)}+\frac{q^{74}}{C^*(q^{66})A^*(q^2)}$$
$$=\frac{f_8f_{16}f_{264}f_{528}}{4q^{12}f_{14}f_{28}f_{56}f_{462}f_{924}f_{1848}}\left(\frac{f_{22}^5f_6^5}{f_{11}^2f_{44}^2f_3^2f_{12}^2}-\frac{f_{11}^2f_3^2}{f_{22}f_6}-2\frac{f_{28}^5f_{924}^5}{f_{14}^2f_{56}^2f_{462}^2f_{1848}^2}\right), \tag{3.45}$$

$$\frac{1}{B^*(q^{24})A^*(q)}+\frac{q^{11}}{A^*(q^{24})C^*(q)}+\frac{q^{27}}{C^*(q^{24})B^*(q)}$$
$$=\frac{f_4f_8f_{96}f_{192}}{4q^4f_7f_{14}f_{28}f_{168}f_{336}f_{672}}\left(\frac{f_{12}^5f_2^5}{f_6^2f_{24}^2f_1^2f_4^2}-\frac{f_6^2f_1^2}{f_{12}f_2}-2\frac{f_{14}^5f_{336}^5}{f_7^2f_{28}^2f_{168}^2f_{672}^2}\right), \tag{3.46}$$

$$\frac{1}{B^*(q^{26})B^*(q^2)}+\frac{q^{12}}{A^*(q^{26})A^*(q^2)}+\frac{q^{32}}{C^*(q^{26})C^*(q^2)}$$
$$=\frac{f_8f_{16}f_{102}f_{204}}{4q^4f_{14}f_{28}f_{56}f_{182}f_{368}f_{736}}\left(\frac{f_{26}^5f_2^5}{f_{13}^2f_{52}^2f_1^2f_4^2}-\frac{f_{13}^2f_1^2}{f_{26}f_2}-2\frac{f_{28}^5f_{368}^5}{f_{14}^2f_{56}^2f_{182}^2f_{736}^2}\right). \tag{3.47}$$

Proof: In view of Lemma (2.1) and Lemma (2.2), the identity (3.42) takes the following form:

$$f(q^{432},q^{240})f(q^9,q^5)+q^{21}f(q^{528},q^{144})f(q^{11},q^3)+q^{56}f(q^{624},q^{48})f(q^{13},q)$$
$$=\frac{1}{4q^7}\{\phi(q^4)\phi(q^3)-\phi(-q^4)\phi(-q^3)-2\phi(q^{336})\phi(q^7)\}. \tag{3.48}$$

Now, we will prove (3.48). Set $(m,n)=(7,1)$ in (2.19), we obtain

$$\phi(q^8)\phi(q^6)-\phi(-q^8)\phi(-q^6)=2f(q^{672},q^{672})f(q^{14},q^{14})+2q^{14}f(q^{864},q^{480})f(q^{18},q^{10})$$
$$+2q^{56}f(q^{1056},q^{288})f(q^{22},q^6)+2q^{126}f(q^{1248},q^{96})f(q^{26},q^2)+2q^{224}f(q^{1440},q^{-96})f(q^{30},q^{-2})$$
$$+2q^{350}f(q^{1632},q^{-288})f(q^{34},q^{-6})+2q^{504}f(q^{1824},q^{-480})f(q^{38},q^{-10}).$$

Applying (2.6) in the above and then replacing q by $q^{\frac{1}{2}}$ in the resulting identity, we obtain (3.48). This completes the proof of (3.42). The proofs of (3.43) − (3.47) can be obtained in a similar fashion by setting $(m,n)=(7,2),(7,3),(7,4),(7,5),(7,6)$ in (2.19), respectively. □

Theorem 3.6 *We have*

$$\frac{1}{A_*(q^{48})A_*(q)}+\frac{q^7}{B_*(q^{48})B_*(q)}+\frac{q^{21}}{C_*(q^{48})C_*(q)} = \frac{f_4^2f_{192}^2}{f_7^2f_{14}f_{336}^2f_{672}}\left(\frac{f_8^2f_6^2}{f_4^2f_3^2}-2q^{42}\frac{f_{672}^2f_{14}^2}{f_{336}f_7}\right), \tag{3.49}$$

$$\frac{1}{A_*(q^{40})B_*(q)}+\frac{q^6}{B_*(q^{40})C_*(q)}+\frac{q^{17}}{C_*(q^{40})A_*(q)} = \frac{f_4^2f_{160}^2}{f_7^2f_{14}f_{280}^2f_{560}}\left(\frac{f_{10}^2f_4^2}{f_5^2f_2^2}-2q^{35}\frac{f_{560}^2f_{14}^2}{f_{280}f_7}\right), \tag{3.50}$$

$$\frac{1}{A_*(q^{24})C_*(q)}+\frac{q^3}{B_*(q^{24})A_*(q)}+\frac{q^{10}}{C_*(q^{24})B_*(q)} = \frac{f_4^2f_{96}^2}{f_7^2f_{14}f_{168}^2f_{336}}\left(\frac{f_{12}^2f_2^2}{f_6^2f_1^2}-2q^{21}\frac{f_{336}^2f_{14}^2}{f_{168}f_7}\right), \tag{3.51}$$

$$\frac{1}{A_*(q^{90})B^*(q)}+\frac{q^{14}}{B_*(q^{90})C^*(q)}+\frac{q^{39}}{C_*(q^{90})A^*(q)} = \frac{f_4f_8f_{360}^2}{f_7f_{14}f_{28}f_{630}^2f_{1260}}\left(\frac{f_{18}^2f_{10}^2}{f_9f_5}-q^{77}\frac{f_{1260}^2f_{14}^5}{f_{630}f_7^2f_{28}^2}\right), \tag{3.52}$$

$$\frac{1}{A_*(q^{99})A^*(q)}+\frac{q^9}{B_*(q^{99})B^*(q)}+\frac{q^{29}}{C_*(q^{99})C^*(q)} = \frac{f_4f_8f_{264}^2}{f_7f_{14}f_{28}f_{462}^2f_{924}}\left(\frac{f_{22}^2f_6^2}{f_{11}f_3}-q^{56}\frac{f_{924}^2f_{14}^5}{f_{462}f_7^2f_{28}^2}\right), \tag{3.53}$$

$$\frac{1}{A_*(q^{26})C^*(q)}+\frac{q^3}{B_*(q^{26})A^*(q)}+\frac{q^{10}}{C_*(q^{26})B^*(q)} = \frac{f_4f_8f_{104}^2}{f_7f_{14}f_{28}f_{182}^2f_{364}}\left(\frac{f_{26}^2f_2^2}{f_{13}f_1}-q^{21}\frac{f_{364}^2f_{14}^5}{f_{182}f_7^2f_{28}^2}\right). \tag{3.54}$$

Proof: With the aid of Lemma (2.1) and Lemma (2.2), the identity (3.49) can be written in the following form:

$$\psi(q^4)\psi(q^4) = f(q^{192},q^{144})f(q^4,q^3)+q^7f(q^{240},q^{96})f(q^5,q^2)+q^{21}f(q^{288},q^{48})f(q^6,q) +2q^{42}\psi(q^7)\psi(q^{336}). \tag{3.55}$$

To prove (3.49), it is enough to prove (3.55). Setting $(m,n)=(7,1)$ in (2.20) and then replacing q by $q^{\frac{1}{2}}$ in the resulting identity, we obtain (3.55). This completes the proof of proves (3.49). The proofs of (3.50) – (3.54) can be obtained in a similar way. □

Theorem 3.7 *We have*

$$\frac{1}{B^*(q^{390})C_*(q^4)}+\frac{q^{168}}{A^*(q^{390})B_*(q^4)}+\frac{q^{444}}{C^*(q^{390})A_*(q^4)}$$
$$=\frac{f_{16}^2f_{1560}f_{3120}}{2q^{54}f_{28}^2f_{56}f_{2730}f_{5460}f_{10920}}\left(\frac{f_{30}^2f_{26}^2}{f_{15}f_{13}}+\frac{f_{15}f_{60}f_{13}f_{52}}{f_{30}f_{26}}-2\frac{f_{5460}^5f_{56}^2}{f_{2730}^2f_{10920}^2f_{28}}\right), \tag{3.56}$$

$$\frac{1}{B^*(q^{374})A_*(q^4)}+\frac{q^{162}}{A^*(q^{374})C_*(q^4)}+\frac{q^{428}}{C^*(q^{374})B_*(q^4)}$$
$$=\frac{f_{16}^2f_{1496}f_{2992}}{2q^{50}f_{28}^2f_{56}f_{2618}f_{5236}f_{10472}}\left(\frac{f_{34}^2f_{22}^2}{f_{17}f_{11}}+\frac{f_{17}f_{68}f_{111}f_{44}}{f_{34}f_{22}}-2\frac{f_{5236}^5f_{56}^2}{f_{2618}^2f_{10472}^2f_{28}}\right), \tag{3.57}$$

$$\frac{1}{B^*(q^{342})B_*(q^4)}+\frac{q^{146}}{A^*(q^{342})A_*(q^4)}+\frac{q^{392}}{C^*(q^{342})C_*(q^4)}$$
$$=\frac{f_{16}^2f_{1368}f_{2736}}{2q^{46}f_{28}^2f_{56}f_{2394}f_{4788}f_{9576}}\left(\frac{f_{38}^2f_{18}^2}{f_{19}f_9}+\frac{f_{19}f_{76}f_9f_{36}}{f_{38}f_{18}}-2\frac{f_{4788}^5f_{56}^2}{f_{2394}^2f_{9576}^2f_{28}}\right), \tag{3.58}$$

$$\frac{1}{B^*(q^{230})B_*(q^4)}+\frac{q^{98}}{A^*(q^{230})A_*(q^4)}+\frac{q^{264}}{C^*(q^{230})C_*(q^4)}$$
$$=\frac{f_{16}^2f_{920}f_{1840}}{2q^{30}f_{28}^2f_{56}f_{1610}f_{3220}f_{6440}}\left(\frac{f_{46}^2f_{10}^2}{f_{23}f_5}+\frac{f_{23}f_{92}f_5f_{20}}{f_{46}f_{10}}-2\frac{f_{3220}^5f_{56}^2}{f_{1610}^2f_{6440}^2f_{28}}\right), \tag{3.59}$$

$$\frac{1}{B^*(q^{150})A_*(q^4)}+\frac{q^{66}}{A^*(q^{150})C_*(q^4)}+\frac{q^{172}}{C^*(q^{150})B_*(q^4)}$$
$$=\frac{f_{16}^2f_{600}f_{1200}}{2q^{18}f_{28}^2f_{56}f_{1050}f_{2100}f_{4200}}\left(\frac{f_{50}^2f_6^2}{f_{25}f_3}+\frac{f_{25}f_{100}f_3f_{12}}{f_{50}f_6}-2\frac{f_{2100}^5f_{56}^2}{f_{1050}^2f_{4200}^2f_{28}}\right), \tag{3.60}$$

$$\frac{1}{B^*(q^{54})C_*(q^4)}+\frac{q^{22}}{A^*(q^{54})B_*(q^4)}+\frac{q^{60}}{C^*(q^{54})A_*(q^4)}$$
$$=\frac{f_{16}^2f_{216}f_{432}}{2q^{6}f_{28}^2f_{56}f_{756}f_{1512}f_{3024}}\left(\frac{f_{54}^2f_2^2}{f_{27}f_1}+\frac{f_{27}f_{108}f_1f_4}{f_{54}f_2}-2\frac{f_{1512}^5f_{56}^2}{f_{756}^2f_{3024}^2f_{28}}\right). \tag{3.61}$$

Proof: The identity (3.56) can be rewritten with the help of Lemma (2.1) and Lemma (2.2) as follows:

$$\psi(q^{15})\psi(q^{13})+\psi(-q^{15})\psi(-q^{13})=2\phi(q^{2730})\psi(q^{28})+2q^{54}f(q^{3510},q^{1950})f(q^4,q^{24})$$
$$+2q^{220}f(q^{4290},q^{1170})f(q^8,q^{20})+2q^{498}f(q^{5070},q^{390})f(q^{12},q^{16}) \tag{3.62}$$

Setting $(m,n)=(14,1)$ in (2.21) we obtain

$$\psi(q^{15})\psi(q^{13})=\phi(q^{2730})\psi(q^{28})+q^{679}\psi(q^{5460})\phi(q^{14})+q^{13}f(q^{3120},q^{2340})f(q^2,q^{26})$$
$$+q^{54}f(q^{3510},q^{1950})f(q^4,q^{24})+q^{123}f(q^{3900},q^{1560})f(q^6,q^{22})+q^{220}f(q^{4290},q^{1170})f(q^8,q^{20})$$
$$+q^{345}f(q^{4680},q^{780})f(q^{10},q^{18})+q^{498}f(q^{5070},q^{390})f(q^{12},q^{16}). \tag{3.63}$$

Replacing q by $-q$ in (3.63) and then adding the resulting identity with (3.63), we obtain (3.62). This completes the proof of (3.56). The proofs of (3.57) – (3.61) follow in a similar way. □

4. Applications to Theory of Partitions

In this section, we extract the partition theoretic interpretations for some of the modular relations proved in section 3. We use the standard notation of q-products

$$(a_1,a_2,\ldots,a_n;q)_\infty:=\prod_{i=1}^{n}(a_i;q)_\infty$$

and we define

$$(q^{\pm r};q^s)_\infty:=(q^r,q^{s-r};q^s)_\infty$$

where r and s are positive integers $(r < s)$.
Definition 4.1 A positive integer n is said to have k colors if there are k copies of n available and all of them are viewed as distinct objects.
For example, if 2 is allowed to have three colors (say, red (r), green (g), blue (b)), then $2_r, 2_g, 2_b$ are treated as distinct parts. Partitions of a positive integer with parts in colors are called colored partitions. It can be seen that, $\frac{1}{(q^a;q^b)_\infty^k}$ is a generating function for number of partitions of n with parts congruent to a (mod b) having k colors. For example, identities (3.19), (3.20), (3.23) and (3.24) can be read as Theorem 4.1, 4.2, 4.3 and 4.4 respectively stated below and the proofs of these theorems are left to the readers. Four examples illustrating the two theorems are also given below.

Theorem 4.1 *Let $P_1(n)$ denotes the number of partitions of n into parts congruent to $\pm 2, \pm 3, \pm 11, \pm 17, \pm 16, \pm 25, \pm 26, \pm 31, \pm 39, \pm 40, \pm 42 (mod\, 84)$, parts congruent to $\pm 4, \pm 10, \pm 12, \pm 32, \pm 38 (mod\, 84)$ in two colors, parts congruent to $\pm 6 (mod\, 84)$ in three colors, parts congruent to $\pm 24, \pm 30 (mod\, 84)$ in four colors and parts congruent to $\pm 18 (mod\, 84)$ in six colors. Let $P_2(n)$ denotes the number of partitions of n into parts congruent to $\pm 5, \pm 8, \pm 9, \pm 19, \pm 20, \pm 22, \pm 23, \pm 33, \pm 34, \pm 37, \pm 42 (mod\, 84)$, parts congruent to $\pm 2, \pm 16, \pm 18, \pm 26, \pm 36, \pm 40 (mod\, 84)$ in two colors, parts congruent to $\pm 6, \pm 12 (mod\, 84)$ in four colors and parts congruent to $\pm 30 (mod\, 84)$ in six colors. Let $P_3(n)$ denotes the number of partitions of n into parts congruent to $\pm 1, \pm 4, \pm 10, \pm 13 \pm 15, \pm 27, \pm 29, \pm 32, \pm 38, \pm 41, \pm 42 (mod\, 84)$, parts congruent to $\pm 8, \pm 20, \pm 22, \pm 24 (mod\, 84)$ in two colors, parts congruent to $\pm 30 (mod\, 84)$ in three colors, parts congruent to $\pm 18, \pm 36 (mod\, 84)$ in four colors and parts congruent to $\pm 6 (mod\, 84)$in six colors. Let $P_4(n)$ denotes the number of partitions of n into parts congruent to $\pm 1, \pm 5, \pm 11, \pm 13, \pm 17, \pm 19, \pm 23, \pm 25, \pm 29, \pm 31, \pm 37, \pm 41 (mod\, 84)$, parts congruent to $\pm 7, \pm 35 (mod\, 84)$ in two colors, parts congruent to $\pm 3, \pm 9, \pm 15, \pm 27, \pm 33, \pm 39 (mod\, 84)$ in threer colors and parts congruent to $\pm 21 (mod\, 84)$ in four colors. Let $P_5(n)$ denotes the number of partitions of n into parts congruent to $\pm 2, \pm 4, \pm 8, \pm 10, \pm 16, \pm 20, \pm 22, \pm 26, \pm 32, \pm 34, \pm 38, \pm 40 (mod\, 84)$, parts congruent to $\pm 7, \pm 12, \pm 21, \pm 24, \pm 35, \pm 36 (mod\, 84)$ in two colors and parts congruent to $\pm 6, \pm 18, \pm 30, \pm 42 (mod\, 84)$ in four colors.*
Then, for any positive integer $n \geq 6$, we have

$$P_1(n) + P_2(n-1) + P_3(n-3) - P_4(n) + 2P_5(n-6) = 0. \tag{4.1}$$

Example 4.1 Let red (r), green (g), blue (b), and pink (p) be the available colors. The following table gives an illustration of Theorem (4.1) for the case n = 7.

$P_1(7) = 3$	$4_r + 3, 4_b + 3, 3 + 2 + 2.$
$P_4(7) = 13$	$7_r, 7_b, 3_r + 3_r + 1, 3_b + 3_b + 1, 3_r + 3_b + 1, 3_g + 3_g + 1, 3_r + 3_g + 1, 3_b + 3_g + 1, 3_r + 1 + 1 + 1 + 1$
	$3_b + 1 + 1 + 1 + 1, 3_g + 1 + 1 + 1 + 1, 5 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1 + 1$
$P_2(6) = 8$	$6_r, 6_b, 6_g, 6_p, 2_r + 2_r + 2_r, 2_r + 2_r + 2_b, 2_r + 2_b + 2_b, 2_b + 2_b + 2_b$
$P_3(4) = 2$	$4, 1 + 1 + 1 + 1$
$P_5(1) = 0$	

Theorem 4.2 *Let $P_1(n)$ denotes the number of partitions of n into parts congruent to $\pm 8, \pm 15, \pm 20, \pm 24, \pm 27, \pm 42 (mod\, 84)$, parts congruent to $\pm 12, \pm 14, \pm 16, \pm 40 (mod\, 84)$ in two colors, parts congruent to $\pm 10, \pm 22, \pm 34, \pm 36, \pm 38 (mod\, 84)$ in three colors, parts congruent to $\pm 2, \pm 18, \pm 26, \pm 30 (mod\, 84)$ in four colors and parts congruent to $\pm 6 (mod\, 84)$ in five colors. Let $P_2(n)$ denotes the number of partitions of n into parts congruent to $\pm 9, \pm 16, \pm 33, \pm 36, \pm 40, \pm 42 (mod\, 84)$, parts congruent to $\pm 4, \pm 14, \pm 24, \pm 32 (mod\, 84)$ in two colors, parts congruent to $\pm 2, \pm 12, \pm 22, \pm 26, \pm 34 (mod\, 84)$ in three colors, parts congruent to $\pm 6, \pm 10, \pm 18, \pm 38 (mod\, 84)$ in four colors and parts congruent to $\pm 30 (mod\, 84)$ in five colors. Let $P_3(n)$ denotes the number of partitions of n into parts congruent to $\pm 3, \pm 4, \pm 12, \pm 32 \pm 39, \pm 42 (mod\, 84)$, parts congruent to $\pm 8, \pm 14, \pm 20, \pm 36 (mod\, 84)$ in two colors, parts congruent to $\pm 2, \pm 10, \pm 24, \pm 26, \pm 38 (mod\, 84)$ in three colors, parts congruent to $\pm 6, \pm 22, \pm 30, \pm 34 (mod\, 84)$ in four colors and parts congruent to $\pm 18 (mod\, 84)$in five colors. Let $P_4(n)$ denotes the number of partitions of n into parts congruent to $\pm 1, \pm 5, \pm 7, \pm 11, \pm 13, \pm 17, \pm 19, \pm 23, \pm 25, \pm 29, \pm 31,$*

$\pm\, 35, \pm 37, \pm 41 (mod\, 84)$ *in two colors, parts congruent to* $\pm 3, \pm 9, \pm 15, \pm 27, \pm 33, \pm 39 (mod\, 84)$ *in threer colors and parts congruent to* $\pm 21 (mod\, 84)$ *in four colors. Let* $P_5(n)$ *denotes the number of partitions of n into parts congruent to* $\pm 4, \pm 8, \pm 16, \pm 32, \pm 40 (mod\, 84)$*, parts congruent to* $\pm 12, \pm 21, \pm 24, \pm 36, \pm 42 (mod\, 84)$ *in two colors, parts congruent to* $\pm 2, \pm 10, \pm 22, \pm 26, \pm 34, \pm 38 (mod\, 84)$ *in threer colors and parts congruent to* $\pm 6, \pm 14, \pm 18, \pm 30 (mod\, 84)$ *in four colors.*

Then, for any positive integer $n \geq 4$*, we have*

$$P_1(n-1) + P_2(n) + P_3(n-1) - P_4(n) + 2P_5(n-4) = 0. \tag{4.2}$$

Example 4.2 Let red (r), green (g), blue (b), and pink (p) be the available colors. The following table gives an illustration of Theorem 4.1 (4.1) for the case n = 5.

$P_1(4) = 10$	$2_r + 2_r, 2_b + 2_b, 2_g + 2_g, 2_p + 2_p, 2_r + 2_b, 2_r + 2_g, 2_r + 2_p, 2_b + 2_g, 2_b + 2_P, 2_g + 2_p$
$P_2(5) = 0$	
$P_4(5) = 17$	$5_r, 5_b, 3_r + 1_r + 1_r, 3_r + 1_r + 1_b, 3_r + 1_b + 1_b, 3_b + 1_r + 1_r, 3_b + 1_r + 1_b, 3_b + 1_b + 1_b$ $3_g + 1_r + 1_r, 3_g + 1_r + 1_b, 3_g + 1_b + 1_b, 1_r + 1_r + 1_r + 1_r + 1_r, 1_r + 1_r + 1_r + 1_r + 1_b,$ $1_r + 1_r + 1_r + 1_b + 1_b, 1_r + 1_r + 1_b + 1_b + 1_b, 1_r + 1_b + 1_b + 1_b + 1_b, 1_b + 1_b + 1_b + 1_b + 1_b$
$P_3(4) = 7$	$4, 2_r + 2_r, 2_b + 2_b, 2_g + 2_g, 2_r + 2_b, 2_r + 2_g, 2_b + 2_g$
$P_5(1) = 0$	

Theorem 4.3 *Let* $P_1(n)$ *denotes the number of partitions of n into parts congruent to* $\pm 2, \pm 4, \pm 5, \pm 8, \pm 9, \pm 16, \pm 19, \pm 20, \pm 22, \pm 23, \pm 26, \pm 32, \pm 33, \pm 34, \pm 36, \pm 37, \pm 40 (mod\, 84)$*, parts congruent to* $\pm 6, \pm 12, \pm 14, \pm 18, \pm 30 (mod\, 84)$ *in two colors and parts congruent to* $\pm 24 (mod\, 84)$ *in three colors. Let* $P_2(n)$ *denotes the number of partitions of n into parts congruent to* $\pm 1, \pm 4, \pm 8, \pm 10, \pm 13, \pm 15, \pm 16, \pm 20, \pm 22, \pm 24, \pm 27, \pm 29, \pm 32, \pm 34, \pm 38, \pm 40, \pm 41 (mod\, 84)$*, parts congruent to* $\pm 6, \pm 14, \pm 18, \pm 30, \pm 36 (mod\, 84)$ *in two colors and parts congruent to* $\pm 12 (mod\, 84)$ *in three colors. Let* $P_3(n)$ *denotes the number of partitions of n into parts congruent to* $\pm 2, \pm 3, \pm 4, \pm 8, \pm 10 \pm 11, \pm 12, \pm 16, \pm 17, \pm 20, \pm 25, \pm 26, \pm 31, \pm 32, \pm 38, \pm 39, \pm 40 (mod\, 84)$*, parts congruent to* $\pm 6, \pm 14, \pm 18, \pm 24, \pm 30 (mod\, 84)$ *in two colors and parts congruent to* $\pm 36 (mod\, 84)$ *in three colors. Let* $P_4(n)$ *denotes the number of partitions of n into parts congruent to* $\pm 2, \pm 3, \pm 10, \pm 15, \pm 21, \pm 22, \pm 26, \pm 27, \pm 33, \pm 34, \pm 38, \pm 39 (mod\, 84)$ *in two colors and parts congruent to* $\pm 14 (mod 84)$ *in four colors. Let* $P_5(n)$ *denotes the number of partitions of n into parts congruent to* $\pm 2, \pm 4, \pm 8, \pm 16, \pm 20, \pm 22, \pm 26, \pm 32, \pm 34, \pm 38, \pm 40 (mod\, 84)$ *and parts congruent to* $\pm 6, \pm 7, \pm 10, \pm 12, \pm 18, \pm 21, \pm 24, \pm 30, \pm 35, \pm 36 (mod\, 84)$ *in two colors.*

Then, for any positive integer $n \geq 5$*, we have*

$$P_1(n) + P_2(n-2) + P_3(n-3) - P_4(n) + P_5(n-5) = 0. \tag{4.3}$$

Example 4.1 Let red (r), green (g), blue (b), and pink (p) be the available colors. The following table gives an illustration of Theorem (4.3) for the case n = 6.

$P_1(6) = 4$	$6_r, 6_b, 4 + 2, 2 + 2 + 2$
$P_2(4) = 2$	4, 1+1+1+1
$P_3(3) = 1$	3
$P_4(6) = 7$	$3_r + 3_r, 3_b + 3_b, 3_r + 3_b, 2_r + 2_r + 2_r, 2_b + 2_b + 2_b, 2_r + 2_r + 2_b, 2_r + 2_b + 2_b$
$P_5(1) = 0$	

Theorem 4.4 *Let* $P_1(n)$ *denotes the number of partitions of n into parts congruent to* $\pm 15, \pm 16, \pm 27, \pm 30, \pm 36, \pm 40 (mod\, 84)$*, parts congruent to* $\pm 4, \pm 6, \pm 12, \pm 14, \pm 32 (mod\, 84)$ *in two colors, parts congruent to* $\pm 2, \pm 18, \pm 22, \pm 24, \pm 26, \pm 34 (mod\, 84)$ *in three colors and parts congruent to* $\pm 10, \pm 38 (mod\, 84)$ *in four*

colors. Let $P_2(n)$ denotes the number of partitions of n into parts congruent to $\pm4, \pm9, \pm12, \pm18, \pm32, \pm33 (mod\, 84)$*, parts congruent to* $\pm8, \pm14, \pm20, \pm24, \pm30 (mod\, 84)$ *in two colors, parts congruent to* $\pm2, \pm6, \pm10, \pm26, \pm36, \pm38 (mod\, 84)$ *in three colors and parts congruent to* $\pm22, \pm34 (mod\, 84)$ *in four colors. Let $P_3(n)$ denotes the number of partitions of n into parts congruent to* $\pm3, \pm6, \pm8, \pm20, \pm24, \pm39 (mod\, 84)$*, parts congruent to* $\pm14, \pm16, \pm18, \pm36, \pm40 (mod\, 84)$ *in two colors, parts congruent to* $\pm10, \pm12, \pm22, \pm30, \pm34, \pm38 (mod\, 84)$ *in three colors and parts congruent to* $\pm2, \pm26 (mod\, 84)$ *in four colors. Let $P_4(n)$ denotes the number of partitions of n into parts congruent to* $\pm1, \pm3, \pm5, \pm7, \pm9, \pm11, \pm13, \pm15, \pm17, \pm19, \pm21, \pm23, \pm25, \pm27, \pm29, \pm31, \pm33, \pm35, \pm37, \pm39 (mod\, 84)$ ±41 *in two colors. Let $P_5(n)$ denotes the number of partitions of n into parts congruent to* $\pm4, \pm8, \pm16, \pm20, \pm32, \pm40 (mod\, 84)$*, parts congruent to* $\pm6, \pm12, \pm18, \pm21, \pm24, \pm30, \pm36 (mod\, 84)$ *in two colors, parts congruent to* $\pm2, \pm10, \pm22, \pm26, \pm34, \pm38 (mod\, 84)$ *in three colors and parts congruent to* $\pm14 (mod\, 84)$ *in four colors.*

Then, for any positive integer $n \geq 5$, we have

$$P_1(n) + P_2(n-1) + P_3(n-4) - P_4(n) + P_5(n-1) = 0. \tag{4.4}$$

$P_4(6) = 22$	$5_r + 1_r, 5_b + 1_b, 5_r + 1_b, 5_b + 1_r, 3_r + 3_r, 3_b + 3_b, 3_r + 3_b, 3_r + 1_r + 1_r + 1_r,$ $3_r + 1_r + 1_r + 1_b, 3_r + 1_r + 1_b + 1_b, 3_r + 1_b + 1_b + 1_b, 3_b + 1_r + 1_r + 1_r,$ $3_b + 1_r + 1_r + 1_b, 3_b + 1_r + 1_b + 1_b, 3_b + 1_r + 1_b + 1_b, 1_r + 1_r + 1_r + 1_r + 1_r + 1_r,$ $1_r + 1_r + 1_r + 1_r + 1_r + 1_b, 1_r + 1_r + 1_r + 1_r + 1_b + 1_b, 1_r + 1_r + 1_r + 1_b + 1_b + 1_b,$ $1_r + 1_r + 1_b + 1_b + 1_b + 1_b, 1_r + 1_b + 1_b + 1_b + 1_b + 1_b, 1_b + 1_b + 1_b + 1_b + 1_b + 1_b$
$P_1(6) = 18$	$6_r, 6_b, 4_r + 2_r, 4_r + 2_b, 4_r + 2_g, 4_b + 2_r, 4_b + 2_b, 4_b + 2_g, 2_r + 2_r + 2_r,$ $2_r + 2_r + 2_b, 2_r + 2_r + 2_g, 2_b + 2_b + 2_b, 2_b + 2_b + 2_r, 2_b + 2_b + 2_g, 2_g + 2_g + 2_g,$ $2_g + 2_g + 2_r, 2_g + 2_g + 2_b, 2_r + 2_b + 2_g$
$P_2(5) = 0$	
$P_3(2) = 4$	$2_b, 2_r, 2_g, 2_p$
$P_5(15 = 0$	

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