



G-Sequential Convergences in Submethods

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ABSTRACT: In a topological space X , limits of sequences give us a set valued function defined for convergent sequences and taking subsets of X as values. Then sequential versions of some topological notions are generalised to G -methods by replacing \lim function with any map G . A G -method enables us to define a variety of convergent method G_s called G -sequentially convergence and gives rise to a method G_Y on a subset $Y \subseteq X$ called submethod. In this paper, we consider G -sequentially methods or G_s -methods created by G -methods and search the reducibility of such methods to subsets with some properties and characterisations.

Keywords: G -convergence, G -submethod, G_s -sequential convergence, G_Y -closed subset.

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1. Introduction

Convergences are among the main concepts of some branches of mathematics such as analysis, topology and many others. We have many generalisations of classical convergences of sequences to different directions by means of filters, nets, statistical, and matrix methods.

A main property of a metric space or Hausdorff space X is that any convergent sequence has a limit, which defines a function \lim with domain of convergent sequences and range X . As a generalisation of \lim function, a G -method, which is originally appeared in [1], is defined to be a function with domain of sequences and range X . In the context of G -methods, recently many works have been carried out by different authors. Our main references are: [2,3,4] about G -continuity, [5,6] for other continuities, [7,8,9,10] as interested in G -compactness, [11,12,13,14,15] on G -connectedness (see [16] for non-connected topological groups), [4,17] related to the extended G -methods to arbitrary sets G -hulls, G -closures, G -kernels and G -interiors, [18] and [19] on various counterexamples of some G -convergences taken into account (see [20] statistical convergence), [14,21,22] for G -topological groups, [23] the quotient space of G -method as well as [24,25] interested in an extensions of some of these work to neutrosophic topological spaces.

Varieties of limits of a sequence $\mathbf{x} = (x_n)$ for topological space X help us to have a set valued function \lim such that $\lim(\mathbf{x})$ is set of limit points of the sequence. A G -method is considered in [26] as a function from set of some sequences in X to $\mathcal{P}(X)$.

Since intersection of G -open subsets does not usually hold property of G -openness. Hence collection of G -open subsets does not constitute a topology. Upon a question that which types of G -open subsets define a topology, in [27] and [28] the idea of G -sequentially convergence for sequences was developed and then defining G -sequentially open subsets, it was proved that G -sequentially open subsets define a topology called G -sequential topology. Then the latter paper has been recently taken into account in [26] to establish more relationships between G -convergence and G -sequentially convergence.

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In this paper, we consider submethods on subsets induced by G -methods on sets and G -sequentially sub-convergences associated with G -sequentially methods. Then we analyse and investigate the behaviours and properties of G -sequentially convergent submethods.

2. Preliminaries

Throughout the paper, $s(X)$ denotes set of all sequences in set X , while $c(X)$ is set of convergent sequences in topological space X . Here by convergence sequences we mean not only sequences with unique limit but also those with many limits.

When X is topological space, we have a set valued function $\lim: c(X) \rightarrow \mathcal{P}(X)$ such that any convergent sequence is assigned to set of its limits. Similarly for set X not necessarily topological space, G -method is defined to be a map $G: G(X) \rightarrow \mathcal{P}(X)$, where $G(X)$ is some particular set of sequences for which G is defined.

For given G -method on any set X and subset $A \subseteq X$, a point $u \in X$ is said to be G -hull point of A and written by $u \in [A]^G$ subject to the existence of a sequence $\mathbf{a} = (a_n)$ in A such that \mathbf{a} is in domain of G and $u \in G(\mathbf{a})$. A G -closed subset is defined to be one with property $[A]^G \subseteq A$. In other words A is G -closed if whenever $u \in G(\mathbf{a})$ for a sequence, then $u \in A$. Hence obviously subset A with $[A]^G = \emptyset$ is G -closed as well as X . It is not necessarily that neither $[A]^G$ is G -closed nor $A \subseteq [A]^G$. Moreover unlike intersection, the union of G -closed subsets does not necessarily retain G -closedness. G -open subset $U \subseteq X$ is one with G -closed complement $X \setminus U$. Eventually unlike union, G -openness of intersection of G -open subsets is not necessarily essential. In this context, the references [18] and [19] for various examples of G -methods are cited.

If G is a method on X given by $G: G(X) \rightarrow \mathcal{P}(X)$ and Y is subset of X subject to property $G(\mathbf{x}) \subseteq Y$ for any sequence $\mathbf{x} = (x_n)$ in Y with $\mathbf{x} \in G(X)$, then we also have a method $G_Y: G(Y) \rightarrow \mathcal{P}(Y)$ on Y defined by $G_Y(\mathbf{x}) = G(\mathbf{x})$ for $\mathbf{x} \in G(Y)$, where $G(Y) = G(X) \cap s(Y)$. Such a method G_Y is called *submethod* on Y induced by G and G is *reducible* to Y .

In topological spaces, convergence of a sequence $\mathbf{x} = (x_n)$ to $u \in X$ is equivalent to case that any open neighbourhood of u includes a tail of the sequence. In this context, a variety of G -convergence comes to our mind. Hence we define G -sequentially convergence of any sequence $\mathbf{x} = (x_n)$ to $u \in X$ to be that any G -open neighbourhood of u contains a tail and in that case u is also called G -sequential limit of the sequence or G_s -limit of $\mathbf{x} = (x_n)$. That idea paves way to a method $G_s: G_s(X) \rightarrow \mathcal{P}(X)$ in which any sequence $\mathbf{x} = (x_n)$ is assigned to set of G -sequentially limits. We call such a method G_s as G -sequentially convergent method or briefly as G_s -method.

Parallel to G -methods, G_s -hulls of subsets and G_s -closed subsets associated with G -method can be defined. In other words, G_s -hull of A is set of G_s -limit points of sequences in A and A is G_s -closed whenever it contains G_s -hull $[A]^{G_s}$, which means G_s -limits of any sequence with terms of A are contained by A as well. Then G_s -open subsets are defined to be complements of G_s -closed subsets. As it is proved in [28, Theorem 2.1], G_s -openness of subset A is the same as that A includes a tail of any sequence which is G_s -convergent to a point of A . G_s -openness of G -open subset; and therefore G_s -closedness of G -closed subset are immediate by the definitions of G -open and G_s -open subsets. By the proof in [28, Corollary 6.2], unlike G -open subset, G_s -open subsets define a topology.

To distinguish between G -closed and G_s -closed subsets we call the following counterexample with a revision of [28, Example 3.1].

Example 2.1 *In the cited example, a G -method on \mathbb{R} is defined by $G(\mathbf{x}) = \lim \frac{x_n + x_{n+1}}{2}$ on \mathbb{R} . Then subsets \mathbb{R} , \emptyset , $(-\infty, a]$, $[a, b]$, $\{a\}$ and $[b, \infty)$ are detected to be G -closed and their complements are to be G -open subsets. Here we note that singleton subsets $\{a\}$ and $\{b\}$ are G -closed but union $\{a, b\}$ is not G -closed because $\mathbf{x} = (a, b, a, b, \dots)$ is a sequence with terms in $\{a, b\}$ but G -limit is $\frac{a+b}{2}$ which is not in $\{a, b\}$. This example helps us to see that union of G -closed subsets does not usually preserve G -closedness and also indicates that unlike G -closed subsets, finite subsets are G_s -closed and therefore G -closed and G_s -closed subsets differ. Such defined G -method is not reducible to some subsets. For example G is not reducible to subset $Y = \{a, b\}$, since the sequence $\mathbf{x} = (a, b, a, b, \dots)$ is G -convergent to $\frac{a+b}{2}$ but*

$\frac{a+b}{2} \notin Y$. However according to the G -method finite subsets are G_s -closed as any sequence $\mathbf{y} = (y_n)$ in finite subset $Y \subseteq X$ has a term y_{n_0} which appears infinitely many times. Then G -open subset $\mathbb{R} \setminus \{a_{n_0}\}$ does not include a tail of the sequence and therefore any point x in $\mathbb{R} \setminus Y$ can not be in G_s -hull $[Y]^{G_s}$ of Y , since $\mathbb{R} \setminus \{a_{n_0}\}$ is G -open neighbourhood of any point $x \in \mathbb{R} \setminus Y$, which proves that $[Y]^{G_s} = Y$.

3. G-Sequential Convergences in Submethods

We define restriction of method G_s associated with a G -method on X to subset Y under the name of G_s -submethod as follows:

Assume that G is a method on set X and $Y \subseteq X$ subset. Then associated G_s -method $G_s: G_s(X) \rightarrow \mathcal{P}X$ provided with property that if whenever $\mathbf{x} = (x_n) \in G_s(X) \cap s(Y)$ then $G(\mathbf{x}) \subseteq Y$, is called *reducible* to G_s -method $G_s^Y: G_s(Y) \rightarrow \mathcal{P}Y$ defined by $G_s^Y(\mathbf{x}) = G_s(\mathbf{x})$ for sequences in $G_s(Y) = G_s(X) \cap s(Y)$. The induced method G_s^Y is said to be *G-sequentially submethod* or in short *G_s -submethod* on Y .

In below we have examples reducible and irreducible G_s -methods.

Example 3.1 Consider G -method on set X defined by $G(\mathbf{x}) = x_1$ for any sequence $\mathbf{x} = (x_n)$. The method is defined for all sequences in X . Hence it is reducible to any non-empty subset of X . All subsets of X are G -closed and therefore G -open. Since G -open subsets are G_s -open, we conclude that all subsets are G_s -open as well.

We now set G_s -method associated with the method G . For any sequence $\mathbf{x} = (x_n)$ if $u \in G_s(\mathbf{x})$, then any G -open neighbourhood of u contains a tail of the sequence. By the fact that all subsets are G -open, singleton set $\{u\}$ is a G -open neighbourhood of u and therefore it contains a tail of the sequence, which means the sequence $\mathbf{x} = (x_n)$ has a tail with constant term u and therefore the method G_s is defined for only the sequences with constant tails. In other word $u \in G_s(\mathbf{x})$ for any sequence $\mathbf{x} = (x_n)$ if and only if it has a tail with fixed term u . Hence the G_s -method is reducible to any non-empty subset under the same rule. As we can see in the example both methods G and G_s are reducible to any non-empty subset.

Example 3.2 Consider constant method on set X such that any sequence in X is mapped to a constant $x_0 \in X$. Accordingly G -closed subsets are those containing the constant x_0 and G -open subsets are those excluding x_0 . Then the method G is reducible to G -closed subsets, because if Y is G -closed, whenever a sequence $\mathbf{x} = (x_n)$ in Y is in $G(X)$, then $G(\mathbf{x}) \in Y$.

We now produce G_s -method associated with the method G . Since only G -open neighbourhood of x_0 is X itself, any sequence $\mathbf{x} = (x_n)$ is G -sequentially convergent to x_0 which means x_0 is G_s -limit of any sequence $\mathbf{x} = (x_n)$, i.e., $x_0 \in G_s(\mathbf{x})$. Further for $u \in X$ with $u \neq x_0$, if u is G_s -limit of a sequence $\mathbf{x} = (x_n)$, then $\{u\}$ includes a tail of (x_n) , since $\{u\}$ is G -open neighbourhood of u and therefore a tail of the sequence has constant term u , which means that if a sequence (x_n) has a tail with constant term u , then G_s -limits of (x_n) are x_0 and u . If (x_n) has no any tail with constant term, then G_s -limit of $\mathbf{x} = (x_n)$ is only x_0 . As a result the G_s -method corresponding to G is defined to be $G_s: G_s(X) \rightarrow \mathcal{P}(X)$ assigning the singleton set $\{x_0\}$ to any sequence without any tail with constant term and assigning $\{x_0, u\}$ to any sequence with a constant tail u . Here it is understood that the domain $G_s(X)$ of G_s -method is the set $s(X)$ of all sequences in X .

If a non-empty subset A is G_s -closed, then $x_0 \in X$; and if $x_0 \in A$ then A contains G_s -hull $[A]^{G_s}$ which means A is G_s -closed iff $x_0 \in A$. Hence G -closed and G_s -closed subsets agree. Here we can easily check that G_s -hull of A is $[A]^{G_s} = A \cup \{x_0\}$.

As we can follow, G_s -method is reducible to any G -closed subset Y , under the same mapping rule $G_s^Y: G_s(Y) \rightarrow Y$.

4. Results

In G -method case of the following result appears in [27].

Theorem 4.1 Let G_s be G -sequentially method associated with G -method on X and $Y \subseteq X$. Then existence of submethod G_s^Y on Y is equivalent to G -sequentially closedness of Y .

Proof: If G_s -method $G_s: G_s(X) \rightarrow \mathcal{P}(X)$ corresponding to the method G on X exists and Y is G -closed subset, then whenever $\mathbf{x} = (x_n)$ is a sequence with the terms of Y in $G_s(X)$, then $G_s(\mathbf{x}) \subseteq Y$ since Y is G -closed and therefore G_s -closed subset. Here we remark that G -open subsets are G_s -open as well. Hence existence of G_s -submethod $G_s^Y: G_s(Y) \rightarrow \mathcal{P}(Y)$ holds.

For opposite side of the proof assume that the G_s -method is reducible to a subset $Y \subseteq X$ and prove that Y is G_s -closed. Let (x_n) be a sequence with terms of Y such that $u \in G(X)$. Since the method G_s is reducible to Y , we have that $u \in Y$. Hence Y is G_s -closed. \square

Example 4.1 *Since according to the G -method in Example 2.1, finite subsets are G_s -closed, by Theorem 4.1, the G_s -method is reducible to finite subsets. However as it is stated in Example 2.1, the G -method is not reducible to finite subsets.*

Theorem 4.2 *For G_s -method related to a G -method on X and G -closed subset $Y \subseteq X$, G_s^Y -closedness of subset $B \subseteq Y$ is equivalent to G -closedness of B in X .*

Proof: G_s is also a method $G_s: G_s(X) \rightarrow \mathcal{P}(X)$, the proof follows from [27, Proposition 1.2.1]. \square

Theorem 4.3 *For G_s -method induced by G -method on X and G -closed subset $Y \subseteq X$, a subset $B \subseteq Y$ is G_s^Y -closed in Y if and only if there exists G_s -closed subset F in X with the property $B = F \cap Y$.*

Proof: The proof is similar to the proof of [27, Theorem 1.2.1] and therefore it is omitted. \square

Theorem 4.4 *If G_s and H_s are sequential methods associated with methods G and H on X and Y respectively, then one has $(G \times H)_s = G_s \times H_s$.*

Proof: By given methods G and H respectively on X and Y , We have product method $G \times H$ on $X \times Y$ defined by $(G \times H)(\mathbf{x}, \mathbf{y}) = (G(\mathbf{x}), H(\mathbf{y}))$ for sequences $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ in domains of G and H respectively. Then we have the sequential methods G_s , H_s and $(G \times H)_s$ respectively on X , Y and $X \times Y$ corresponding to the methods G , H and $G \times H$. For any sequence $(\mathbf{x}, \mathbf{y}) = (x_n, y_n)$ in domain of the method $(G \times H)_s$ we have that the sequences $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ are respectively in domains of the methods G_s and H_s . Hence the proof is completed by the evaluation

$$(G \times H)_s(\mathbf{x}, \mathbf{y}) = (G_s(\mathbf{x}), H_s(\mathbf{y})) = (G_s \times H_s)(\mathbf{x}, \mathbf{y})$$

\square

We know from [28] that, a G -method on X leads to a topology on X denoted by τ^G in which G_s -open subsets are open sets of the topology.

Theorem 4.5 [27, Theorem 3.2.2] *Let G -method be given on set X and $Y \subseteq X$ subset. Let τ^G be the topology created by G_s -open subsets, G_Y submethod on Y reduced by the method G on X and τ^{G_Y} the topology created by the submethod G_Y on Y . Then τ^{G_Y} is the subtopology on Y induced by the topology τ^G .*

In the following theorem we prove the reducibility of product methods.

Theorem 4.6 *Let methods G and H be given on sets X and Y along with subsets $X' \subseteq X$ and $Y' \subseteq Y$. If the method G is reducible to X' and H is reducible to Y' , then the product method $G \times H$ is reducible to $X' \times Y'$.*

Proof: Let the methods G and H be reducible to X' and Y' respectively and $(\mathbf{x}, \mathbf{y}) = (x_n, y_n)$ a sequence in domain of the product method $G \times H$ with terms in $X' \times Y'$. Then by $(G \times H)(\mathbf{x}, \mathbf{y}) = (G(\mathbf{x}), H(\mathbf{y}))$ we have that the sequences \mathbf{x} and \mathbf{y} are respectively in domains of G and H with the terms in X' and Y' . Since the methods G and H are reducible to X' and Y' , we have that $G(\mathbf{x}) \in X'$ and $H(\mathbf{y}) \in Y'$ and therefore $(G \times H)(\mathbf{x}, \mathbf{y}) = (G(\mathbf{x}), H(\mathbf{y})) \in X' \times Y'$. Hence the product method $G \times H$ is reducible to $X \times Y$ as well. \square

5. Conclusion

In the paper, we have considered G -sequentially methods or G_s -methods and study the reducibility of them to subsets. We have tried to relate the G_s -method to G with some counter examples and properties. It would be interesting to investigate more relations between the methods G and G_s such as regularity, compactness, connectedness and reducibility in terms of these methods.

References

1. Connor, J., Grosse-Erdmann, K.-G., *Sequential definitions of continuity for real functions*, Rocky Mountain J. Math., 33 no. 1, 93–121, (2003).
2. Çakallı, H., *On G -continuity*, Comput. Math. Appl., 61, 313–318, (2011).
3. Mucuk, O., Şahan, T., *On G -sequential continuity*, Filomat, 28 no. 6, 1181–1189, (2014).
4. Lin, S., Liu, L., *G -methods, G -sequential spaces and G -continuity in topological spaces*, Topology Appl., 212, 29–48, (2016).
5. Çanak, İ., Dik, M., *New types of continuities*, Abstr. Appl. Anal., Art. ID 258980, 6 pp., (2010).
6. Çakallı, H., *New kinds of continuities*, Comput. Math. Appl., 61 no. 4, 960–965, (2011).
7. Çakallı, H., *Sequential definitions of compactness*, Appl. Math. Lett., 21 no. 6, 594–598, (2008).
8. Liu, L., Zhou, X.G., Liu, F., *G -sequentially compact spaces*, (Chinese) Appl. Math. J. Chinese Univ. Ser. A, 34 no. 4, 473–480, (2019).
9. Mucuk, O., Çakallı H., *G -compactness and local G -compactness of topological groups with operations*, AIP Conference Proceedings 2334, 020007 (2021); <https://doi.org/10.1063/5.004223>.
10. Mucuk, O., Çakallı, H., *On G -compactness of topological groups with operations*, Filomat, 36 no. 20, 7113–7121, (2022).
11. Çakallı, H., *Sequential definitions of connectedness*, Appl. Math. Lett., 25 no. 3, 461–465, (2012).
12. Çakallı, H., Mucuk, O., *On connectedness via a sequential method*, Rev. Un. Mat. Argentina., 54 no. 2, 101–109, (2013).
13. Mucuk, O., Çakallı, H., *G -connectedness for topological groups with operations*, Filomat, 32 no.3, 1079–1089 (2018).
14. Wu, Y., Lin, F., *The G -connected property and G -topological groups*, Filomat, 33 4441–4450, (2019).
15. Liu, L., Ping, Z., *Product methods and G -connectedness*, Acta Math. Hungar., 162 no. 1, 1–13, (2020).
16. Brown, R., Mucuk, O., *Covering groups of non-connected topological groups revisited*, Math. Proc. Cambridge Philos. Soc., 115, 97–110, (1994).
17. Liu, L., *G -kernel-open sets, G -kernel-neighbourhoods and G -kernel-derived sets*, J. Math. Res. Appl., 38 no.3, 276–286, (2018).
18. Behram S., Mucuk, O., *About varieties of G -sequential methods, G -hulls and G -closures*, Proceedings of International Mathematical Sciences, 5 no.2, 81–86, (2023).
19. Mucuk, O. Behram, S., *Some counter examples of G -convergence*, Sixth International Conference of Mathematical Sciences (ICMS 2022), AIP Conf. Proc. 2879, 1–4.
20. Di Maio, Giuseppe, Kočinac, L. D. R., *Statistical convergence in topology*, Topology Appl., 156, no. 1, 28–45, (2008).
21. Chen, J., Zhang, J., *On G -mappings defined by G -methods and G -topological group*, Filomat, 35 no. 7, 2245–2256, (2021).
22. Mucuk, O., *About G -topological groups*, Filomat, 39 no. 21, 7339–7346, (2025).
23. Liu, F., Lin, S., Zhou, X.G., *On G -quotient spaces*, Filomat, 39, 1219–1225 (2025).
24. Açıkgöz, A., Esenbel, F., Mucuk, O., *Neutrosophication-compactness*, Maltepe J. Math., 6, 90–102, (2024)
25. Açıkgöz, A., ÇakallıH., Esenbel, H., Kočinac, L. D. R., *A quest of G -continuity in neutrosophic spaces*, Math. Methods Appl. Sci., 44 no. 9, 7834–7844, (2021).
26. Liu, L., Lin, S., Zhou, X., *On G_s -convergence and G_s -countable compactness*, Topology Appl., 375 , Paper No. 109554, 23 pp., (2025).
27. Behram, S., *G -convergence and G -sequential spaces for G -methods*, Ph.D Thesis. Erciyes University, (2025).
28. Behram, S., Mucuk, O., *G -convergence and G -sequential spaces*, Hacet. J. Math. Stat., 54 no.6, 2326–2334, (2025).

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