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Convergent G_n-Methods via Nets

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ABSTRACT: As generalisation of limit notion in topological spaces, the idea of G-method is set valued function defined on subset of some sequences in set X. Hence G-methods enable us extending sequential versions of some topological definitions such as sequential continuity, sequential compactness and sequential connectedness. In this paper we consider convergent methods defined for nets on X, rather than sequences and denote such method by G_n. Then we also give some properties and characterizations of properties involving G_n-methods.

Keywords: Nets, G_n-methods, G_n-convergence.

Contents

1. Introduction

Although convergent sequences are useful tools for characterisations of some topological properties, they sometimes fail and therefore we need substitution of convergent nets as well. For instance, continuity of a function implies sequentially continuity, and being Hausdorff of topological space means uniqueness of limit for convergence sequence. However wise-versa versions of the statements do not usually hold. To overcome this problems we experience, convergent nets are used. By the fact of these problems we are motivated studying not only convergent sequences but also convergent nets.

In metric space or more generally Hausdorff space X, uniqueness of limit of convergent sequence defines a function lim with domain set of convergent sequences and with range X. Motivated by this lim function, *G-method* is initially defined in [1] to be a function from a set of sequences in X to X. In framework of G-methods, the following works carried out by some authors are different outcomes of G-methods: The references [2–4] are cited for G-continuity and [5,6] other continuities. Our main references on G-compactness are [7–10] and on G-connectedness are [11–15] (see [16] for non-connected topological groups) as well as for G-open subsets and G-neighbourhoods is [3]. Then G-methods extended to arbitrary sets was considered in sets and G-hulls, G-closures, G-kernels and G-interiors were introduced in [4,17]. Various counterexamples of some G-convergences are taken into account [18] and [19] (see also [20] for statistical convergence). As further references for recent works related G-methods we can cite [14,21,22] regarding G-topological groups, [23] the quotient space of G-method and [24,25] an extensions of some of these work to neutrosophic topological spaces.

The idea that in space X any sequence may has many limits carries one into set valued function lim in which any sequence with limits is mapped to set of limit points. A G-method is redefined in [26] as function from set of some sequences in X to power set $\mathcal{P}(X)$.

The collection of G-open subsets does not constitute a topology but the family of G-sequentially open subsets forms one, called *G-sequential topology* in [27] and [28]. Then the latter paper has been

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recently taken into account in [26] to establish relationships between G-convergence and G-sequentially convergence.

In the paper our concern is to substitute nets for sequences and consider set valued function from set of some nets to power set $\mathcal{P}(X)$ called G_n-*method*; and then to extend and characterise sequential versions of some topological concepts for G_n-methods using nets.

2. Preliminaries

Limits of sequences in topological space X induce set valued function $\lim\colon c(X) \to \mathcal{P}(X)$ when $c(X)$ is set of all sequences with limits. Whenever X is Hausdorff space, the set valued lim function becomes $\lim\colon c(X) \to X$. Hence for set X in [26] any G-method is considered to be set valued function $G\colon G(X) \to \mathcal{P}(X)$ which generalises G-method $G\colon c_G(X) \to X$ defined [1].

Since in sequential part of the paper our tools are nets rather than sequences, we sketch them.

A set $\mathcal{D}$ together with a relation $\leq$ is called directed if the following hold:

(i) $a \leq a$ for all $a \in \mathcal{D}$,

(ii) if $a \leq b$ ve $b \leq c$, then $a \leq c$,

(iii) for any pair $a, b \in \mathcal{D}$, there exists $c \in \mathcal{D}$ such that $a \leq c$ ve $b \leq c$.

Definition of net is stated as follows.

Definition 2.1 *Let X be a set and $\mathcal{D}$ directed set. Any map $\phi\colon \mathcal{D} \to X, \phi(a) = x_a$ is called* net *in X and denoted by $\phi = (x_a)$ or $\phi = (x_a)_{a\in\mathcal{D}}$. A net $\phi = (x_a)$ in topological space X is said to be convergent to $x \in X$ whenever any open neighbourhood of x contains a tail of the net $\phi = (x_a)$.*

□

In the definition of net in particular, if directed set is $\mathbb{N}$ with natural ordering relation, then it turns out the definition of sequence.

For first countable spaces, sequential versions of the following results are well known.

Theorem 2.1 *For topological space X, subset A and $x \in X$, the following hold:*

(a) $x \in A'$ is equivalent to existence of a net with terms in $A \setminus \{x\}$ converging to x

(b) $x \in \overline{A}$ is equivalent to existence of a net with terms in A converging to x.

(c) $x \in A^0$ is equivalent to that any net converging to x has a tail in A.

Theorem 2.2 *A function $f\colon X \to Y$ is continuous if and only if whenever any net $\phi = (x_a)$ converges to $x \in X$, then the image net $f(\phi)$ in Y converges to $f(x)$.*

Theorem 2.3 *A topological space is Hausdorff if and only if the limit of any convergent net is unique.*

3. G_n-Methods

Parallel to sequences for topological space X, limits of nets determine set valued function $\lim\colon c_n(X) \to \mathcal{P}(X)$. Then not only for topological space but also for set X, we call a set valued function $G_n\colon G_n(X) \to \mathcal{P}(X)$ to be G_n-*method*, where $G_n(X)$ is *domain* of the G_n-method. Hence in particular the function $\lim\colon c_n(X) \to \mathcal{P}(X)$ is G_n-method, where $c_n(X)$ is set of nets in X with limits.

For given method $G_n\colon G_n(X) \to \mathcal{P}(X)$ on set X and net $\phi = (x_a)$ in X, a point $u \in G_n(\phi)$ is said to be G_n-*limit point* of ϕ or the net ϕ is G_n-*convergent* to x. A method G_n is called *singleton* whenever $G_n(\phi)$ is singleton set with unique member for any net ϕ in the domain $G_n(X)$.

Whenever X is topological space provided with property that any convergent net ϕ is in the domain $G_n(X)$ and $\lim(\phi) = G_n(\phi)$, then the method G_n is said to be *regular*. Here by convergent net we mean it has limits not necessarily unique. Parallel to regularity, G_n-method defined on set X is phrased as *point-wise* if any constant net $\phi = (x_a)$ with fixed term $x_a = u$ is in the domain $G_n(X)$ and $u \in G_n(\phi)$. That notion is originally is due to [17, p.279]. The method G_n is said to be *preserving G_n-convergences of subnets* if whenever any net ϕ is in $G_n(X)$ and G_n-convergent to u, then any subnet φ is also in $G_n(X)$ and G_n-convergent to u, i.e., $u \in G_n(\phi)$ implies $u \in G_n(\varphi)$.

For regular G_n method and constant net $\phi = (x_a)$ with constant term $x_a = u$, the point $u \in X$ is limit point of ϕ and by $\lim(\phi) = G_n(\phi)$ we have $u \in G_n(\phi)$. Hence for topological space X, regular G_n-method is point-wise but, as we can see in the following example, reverse side fails.

Example 3.1 *Let X be topological space with fixed point $x_0 \in X$ and endowed with G_n-method such that any constant net ϕ with constant term u is mapped to constant set $\{u\}$ and other nets are mapped to $\{x_0\}$. Such defined G_n-method is point-wise but not regular, because for any net ϕ, $G_n(\phi)$ is singleton but limit points of convergent net might be more then one and therefore $\lim(\phi)$ and $G_n(\phi)$ become different. For instance, if topological space is discrete, then any net in X converges to all points.*

□

Let G_n-method be defined on set X and $A \subseteq X$ subset. We say $u \in X$ is in G_n-*hull* of A when $u \in G_n(\phi)$ for some net $\phi = (x_a)$ in A; and define G_n-hull $[A]^{G_n}$ as union of $G_n(\phi)$'s for nets ϕ's in A. A subset $A \subseteq X$ with $[A]^{G_n} \subseteq A$ is said to be G_n-*closed*. If G_n-method is point-wise, then $A \subseteq [A]^{G_n}$. Hence in that case A is G_n-closed if and only if $[A]^{G_n} = A$. Even if for point-wise G_n-method, $[A]^{G_n}$ is not necessarily G_n-closed. Intersection of G_n-closed subsets is also G_n-closed but union of G_n-closed subsets is not necessarily G_n-closed. A subset $U \subseteq X$ with G_n-closed complement $X \setminus U$ is called G_n-*open*. Eventually union of G_n-open subsets is G_n-open but intersection of G_n-open subsets is not. For various examples of G-methods and G-closed subsets, we refer to references [18] and [19].

A subset A is said to be neighbourhood of $a \in A$ and written $a \in A^{0G_n}$ whenever there exists G_n-open subset U with $a \in U \subseteq A$. Note that A^{0G_n} is the largest G_n-open subset of A and A is G_n-open if and only if $A \subseteq A^{0G_n}$.

Remark 3.1 If G_n-method on set X is defined by $G_n(\phi) = \{x_0\}$ for fixed $x_0 \in X$, then for $u \in X$ with $u \neq x_0$, singleton set $A = \{u\}$ is not G_n-closed as $[A]^{G_n} = \{x_0\}$. Hence singleton subset is not necessarily G_n-closed.

Example 3.2 Choose a fixed term x_ϕ in any net ϕ in set X and then define G_n-method assigning the singleton set $\{x_\phi\}$ to any net ϕ in X. If ϕ is net in subset $A \subseteq X$ and $u \in G_n(\phi)$, then we have $u = x_\phi$ and therefore $u \in A$. Hence all subsets of X are G_n-closed and therefore G_n-open.

The following is an example of regular G_n-method.

Example 3.3 Define G_n-method in $\mathbb{R}$ to be

$$G_n(\phi) = \{\lim \frac{x_a + x_{a+1}}{2}\}$$

for some nets in form $\phi\colon [0, \infty) \to \mathbb{R}$, $\phi(a) = (x_a)$. If ϕ is convergent net $\phi\colon [0, \infty) \to \mathbb{R}$, then $G_n(\phi) = \lim(\phi)$ which means that the method G_n is regular and therefore it is point-wise.

The following is an example of neither regular nor point-wise G_n-method.

Example 3.4 For topological space X and constant point $x_0 \in X$, the constant G_n-method $G_n(\phi) = \{x_0\}$ is not regular. If $u \in X$ and $x_0 \neq u$, then for constant net ϕ with fixed term u we have $u \in \lim(\phi)$ but $u \notin G_n(\phi)$ which means $G_n(\phi) \neq \lim(\phi)$. Moreover for any constant net ϕ with fixed term u such that $u \neq x_0$, one has $u \notin G_n(\phi) = \{x_0\}$ and therefore the G_n-method is not point-wise.

4. Results

Theorem 4.1 Let X be set with G_n-method and $\{B_k\colon k \in K\}$ a class of subsets of X. The following are satisfied

(a) $\bigcup_{k\in K} [B_k]^{G_n} \subseteq \left[\bigcup_{k\in K} B_k\right]^{G_n}$

(b) $\left[\bigcap_{k\in K} B_k\right]^{G_n} \subseteq \bigcap_{k\in K} [B_k]^{G_n}$

(c) $(\bigcap_{k\in K} B_k)^{0G_n} \subseteq \bigcap_{k\in K} {B_k}^{0G_n}$

(d) $\bigcup_{k\in J} B_k^{0G_n} \subseteq (\bigcup_{k\in K} B_k)^{0G_n}$

Proof: (a) Any point $x \in \bigcup_{k\in K} [B_k]^{G_n}$ belongs to $[B_{k_0}]^{G_n}$ for some $k_0 \in K$ and therefore there exists a net $\phi = (x_a)$ in B_{k_0} with $x \in G_n(\phi))$. As $B_{k_0} \subseteq \bigcup_{k\in K} B_k$, the terms of ϕ are in $\bigcup_{k\in K} B_k$ which proves that $x \in \left[\bigcup_{k\in K} B_k\right]^{G_n}$.

(b) Choose $x \in \left[\bigcap_{k\in K} B_k\right]^{G_n}$ and a net $\phi = (x_a)$ in $\bigcap_{k\in K} B_k$ with $x \in G_n(\phi)$. As the terms of ϕ are in B_k for any $k \in K$ one has $x \in [B_k]^{G_n}$ and therefore $x \in \bigcap_{k\in K}[B_k]^{G_n}$.

The proofs of (c) and (d) are similar to the G-method case and omitted. □

As a result of Teorem 4.1 the following can be stated.

Corollary 4.1 *(i) Intersection of G_n-closed subsets are G_n-closed.*

(ii) Union of G_n-open subsets is G_n-open as well.

(iii) Union of G_n-closed subsets is not necessarily G_n-closed.

(iv) Intersection of G_n-open subsets is not necessarily G_n-open.

(v) G_n-open sunsets do not form a topology.

However the union of G_n-closed subsets is not necessarily G_n-closed.

In the following example union of G_n-closed subsets are not G_n-closed.

Example 4.1 Consider the method G_n defined in Example 3.3. For any $x \in X$, singleton set $A = \{x\}$ is G_n-closed, because any net ϕ with terms in A is constant and by regularity of the method G_n, the net ϕ is G_n-convergent to $x \in A$. Although the subsets $A = \{x\}$ and $B = \{y\}$ are G_n-closed, $A \cup B = \{x, y\}$ is not. The sequence $(x_n) = (x, y, x, y, \dots)$ which can be thought as net whose terms are in union set $A \cup B$ and by

$$G_n(\phi) = \{\lim \frac{x_n + x_{n+1}}{2}\} = \{\frac{x+y}{2}\}$$

$\frac{x+y}{2}$ is G_n-limit point of $A \cup B$ but not in $A \cup B$.

Theorem 4.2 *For a G_n-method defined on set X and $A, B \subseteq X$ subsets, the following hold:*

(i) A^{0G_n} *is G_n-open*

(ii) $A^{0G_n} \subseteq A$

(iii) *A is G_n-open if and only if $A \subseteq A^{0G_n}$.*

(iv) *A is G_n-open if and only if $A = A^{0G_n}$.*

(v) *$A \subseteq B$ implies $A^{0G_n} \subseteq B^{0G_n}$.*

(vi) $(A \cap B)^{0G_n} \subseteq A^{0G_n} \cap B^{0G_n}$

(vii) $A^{0G_n} \cup B^{0G_n} \subseteq (A \cup B)^{0G_n}$.

Proof: The proofs of the items are similar to G-method case and therefore they are ignored. □

5. Conclusion

In the paper, convergent methods ${G_n}'s$ defined in terms of nets within sets are taken into account, and then some properties and characterizations about these G_n-methods are given. It could be interesting to deal with a variant of G_n-method modified by G-sequential method or G_s-method defined in [28] and developed in [26].

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