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Even-Paired Domination with Modified Forgotten, Hyper Zagreb Indices of Graph

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ABSTRACT: In this study, we define a new parameter for each vertex, referred to as the Even-paired domination degree, and denote it by $d_{coe}(v)$. Building on Even-paired domination degrees, we propose several Even-paired domination indices. We also determine the precise values of the Even-paired domination Forgotten, Refined Forgotten index/ Modified Forgotten invariant, Hyper-Zagreb graph invariant for certain families of well-known graphs. Additionally, we identify bounds, both lower and upper, for these indices across various graph structures.

Keywords: Topological Indices, Even-paired dominating sets, degree, Modified Forgotten, hyper Zagreb indices.

Contents

1. Introduction

In mathematics, theory of graph can be viewed as a new tool to deal, with sciences of all types and is applied in the disciplines of topological graphs [2,3], fuzzy graphs [4,5,6] etc. Mathematical chemistry is the most common application of graph theory in the last decade. Mathematical analysis of chemical phenomena can be done using graph theory tools. Topological indices help describe a graph's structure through numerical values that remain fixed for the entire graph. They're widely applied in building structure-activity relationships (QSAR), where they link chemical structures to biological activity and other traits-often via degree-based indices [7,8] or distance-based ones [9,10,11,13,14]. Note that the Zagreb index stands apart as a distance-based measure, defined as [15,17].

$$M_1(G) = \sum_{v \in V(G)} d(v)^2$$
$$M_2(G) = \sum_{e_1 = uv \in E(G)} d(u)d(v).$$

The index has been overlooked in the references. For detailed insights into the Zagreb index and related topological indices, readers may consult the existing survey literature. In a simple graph, let Δ denote the maximum degree of a vertex and let δ denote the minimum degree. A Pendant-augmented n-wheel graph is obtained from a n-wheel graph with n vertices on its cycle by attaching a pendant edge to each vertex of the n-cycle [18]. Consider two path graphs, say P_m and P_n. Their Cartesian product, particularly when one of the paths is P_2, results in a ladder graph with '$(2n)$' vertices and '$(3n-2)$' edges. A dominating set D in a graph G is a set of vertices such that every vertex $v \in V(G)$ either belongs to D or is adjacent to some vertex $u \in D$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of such a set. A. M. Hanan Ahmed and her co-authors determined the domination number of the path graph P_n. For a general graph G, the domination number is denoted by $\gamma(G)$. However, the concept of co-equal domination is not applicable in this context. In this paper, we introduce a new parameter $\gamma_{coe}(G)$, called the *co-pair domination number* or *even-paired domination number* of a graph G. This

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parameter serves as a novel metric associated with the vertices of the graph. Along with this concept, we define several new co-pair domination indices based on classical competitive graph structures. We derive exact expressions for the co-pair Zagreb indices of important families of graphs. Furthermore, lower and upper bounds for the Zagreb index are established using control chart techniques.

2. Graph's Even-paired Domination Indices

The definition of Equivalent Dominance Forgotten, Modified Forgotten, and hyper Zagreb Indices was presented in this part, along with a formula for calculating the accuracy of these indices for specific lines in the Known class of graphs. Graph's even-paired Domination Indices

Definition 2.1 *[1] The Even-paired domination degree of a vertex $v \in V$ is*

$$d_{coe}(v) = \left|\{D \subseteq V \;:\; D \text{ is a even-paired dominating vertices, } v \in D\}\right|.$$

In other words,
$d_{coe}(v)=\{D : D$ *is a even-paired domination set,* $v \in D \subseteq V\}$

$'\delta_{coe}(G)'$ *and* $'\Delta_{coe}(G)'$ *denote the minimal and maximum even-paired domination degrees of* G*, respectively. where;* $\delta_{coe}(G) = min\{d_{coe}(v) : v \in V\}$, $\Delta_{coe}(G) = max\{d_{coe}(v) : v \in V\}$

Definition 2.2 *Suppose G be a graph, and graph's even-paired domination set be D then the graph's Even-paired Domination Forgotten, Modified Forgotten, and Hyper Zagreb Indices are defined as :*

$$D_{coe}F(G) = \sum_{v \in V} d^3_{coe}(v). \tag{2.1}$$

$$D_{coe}F^*(G) = \sum_{e_1=uv \in E} \left[d^2_{coe}(u) + d^2_{coe}(v)\right] \tag{2.2}$$

$$D_{coe}HM_1(G) = \sum_{e_1=uv \in E} \left[d_{coe}(u) + d_{coe}(v)\right]^2. \tag{2.3}$$

Proposition 2.1 *If S_n is a star graph, then*

1. *If n is even*

$$D_{coe}F(S_n) = \sum_{v \in V}(1^2) = n.$$
$$D_{coe}F^*(S_n) = \sum_{e_1=uv \in E}(1^2 + 1^2) = 2n - 2.$$
$$D_{coe}HM(S_n) = \sum_{e_1=uv \in E}(1+1)^2 = 4n - 4.$$

2. *If n is odd;*

$$D_{coe}F(S_n) = 8n - 7.$$
$$D_{coe}F^*(S_n) = 5(n-1).$$
$$D_{coe}HM(S_n) = 9(n-1).$$

Proof: Let a star graph be isomorphic to a graph G. Assume the collection of n vertices of G is $\{v_1, v_2, ..., v_{n-1}, w\}$, with w being the center vertex.

Case 1: There is only single even-paired dominating set \$ if n is even, say $v_1, v_2, ..., v_{n-1}, w$. Hence $d_{coe}(v) = 1$, $T_{coe}(v) = 1$, so we have only $\forall v \in V$, then by the definition 2.2

$$D_{coe}F(S_n) = \sum_{v \in V}(1^2) = n.$$
$$D_{coe}F^*(S_n) = \sum_{e_1=uv \in E}(1^2 + 1^2) = 2(n-1).$$
$$D_{coe}HM_1(S_n) = \sum_{e_1=uv \in E}(1+1)^2 = 4(n-1).$$

Case 2: If n is odd, we have
$D_{coe1} = \{v_1, v_2, ..., v_{n-1}, w\}$, $D_{coe2} = \{v_1, v_2, ..., v_{n-1}\}$, $T_{coe}(S_n) = 2$. Hence ,

$$d_{coe}(v) = \begin{cases} 1, & \text{if } v = w; \\ 2, & \text{if } otherwise\ . \end{cases}$$

then by definition 2.2, we get

$$D_{coe}F(S_n) = 8n - 7.$$
$$D_{coe}F^*(S_n) = 5(n-1).$$
$$D_{coe}HM_1(S_n) = 9(n-1).$$

□

Proposition 2.2 *Consider a wheeled graph (W_n), then*

1. *Suppose n is even,then*

$$D_{coe}F(W_n) = n.$$
$$D_{coe}F^*(W_n) = 4n - 4.$$
$$D_{coe}HM_1(W_n) = 8n - 8.$$

2. *Suppose n is odd,then*

$$D_{coe}F(W_n) = 8n - 7.$$
$$D_{coe}F^*(W_n) = 5n - 5.$$
$$D_{coe}HM_1(W_n) = 9n - 9.$$

Proof: We can prove this as similar to Proposition 2.1 □

Proposition 2.3 *Suppose G is a K_n graph (Fully connected graph) with $n \geq 3$, then*

1. *when n is even, then*

$$D_{coe}F(G) = n.$$
$$D_{coe}F^*(G) = 2n^2 - 2n.$$
$$D_{coe}HM_1(G) = 4n^2 - 4n.$$

2. *when n is odd, then*

$$D_{coe}F(G) = n2^{3n-3}.$$
$$D_{coe}F^*(G) = (n^2 - n)2^{2n-2}.$$
$$D_{coe}HM_1(G) = (n^2 - n)2^{(n^2-n)}.$$

Proof: If G is a fully connected graph (K_n),with $n \geq 3$, then
Case 1: Let n be even, the only even-paired dominating set of G is the vertex set of K_n. i.e, $D_{coe} = \{v_1, v_2, ..., v_n\}$, Hence $d_{coe}(v) = 1$ for all $v \in V(K_n)$, and

$$D_{coe}F(K_n) = n.$$
$$D_{coe}F^*(K_n) = 2n(n-1).$$
$$D_{coe}HM_1(K_n) = 4n(n-1).$$

Case 2: Let n be odd, then the first row of all vertices is even, so there is $\binom{n}{1}$. A co-double dominant set has only one vertex with the first number $\binom{n}{2}$ co-dominant set with two vertices has the same number of degrees and has $\binom{n}{3}$. Co-pair dominated set includes all vertices but not all of the above-mentioned even-pair dominated sets include all vertices at a single level. The total set of public pair keys is: $\binom{n}{1} + \binom{n}{k} + ... + \binom{n}{n} = 2^n - 1$. If we use the definition of co-pair dominance, we get $d_{coe}(v) = 2^{n-1}$. Using the definition of 2.2, we obtain the dominant index of graph which is even-paired.

$$D_{coe}F(K_n) = \sum_{v \in V} \left(2^{n-1}\right)^3 = n2^{3n-3}.$$
$$D_{coe}F^*(K_n) = \sum_{e_1=uv \in E} \left[(2^{n-1})^2 + (2^{n-1})^2\right] = n(n-1)2^{2(n-1)}.$$
$$D_{coe}HM_1(K_n) = n(n-1)2^{2n-1}$$

□

Proposition 2.4 *Let $K_{m,n}$ represent a complete bipartite graph with vertices of m, n.*

1. *Suppose n is even and m is odd, then*

$$D_{coe}F(K_{m,n}) = (m + 8n)2^{3m-3}.$$
$$D_{coe}F^*(K_{m,n}) = 5mn2^{2m-2}.$$
$$D_{coe}HM_1(K_{m,n}) = 9mn2^{2m-2}.$$

2. *Suppose n is odd and m is even, then*

$$D_{coe}F(K_{m,n}) = (8m + n)2^{3n-3}.$$
$$D_{coe}F^*(K_{m,n}) = 3mn2^{2n-1}.$$
$$D_{coe}HM_1(K_{m,n}) = 9mn2^{2n-2}.$$

3. *suppose m and n both are odd, then*

$$D_{coe}F(K_{m,n}) = (m + n).$$
$$D_{coe}F^*(K_{m,n}) = 2mn.$$
$$D_{coe}HM_1(K_{m,n}) = 4mn.$$

Proof: Assume that the first partition's vertex set is A, and the second partition's vertex set is B. It follows that $A = m$, $B = n$
Case 1: When m is odd and n is even, the degree of each vertex in A becomes even, while every vertex in B attains an odd degree in this case there exists $\binom{m}{1}$ jointly double-dominated sets containing a single one vertex from A, $\binom{m}{2}$ jointly such sets involving two vertices from A, and and so on up to $\binom{m}{m}$. Consequently, the co-pair dominating set consists of all vertices belonging to A. The co-pair dominant set contains all vertices which belongs A. (This does not mean that all vertices of B are in Co-equal

dominant declines). Therefore, all co-double dominant clusters are set to B and $\binom{m}{1}+\binom{m}{2}+...+\binom{m}{m}=2^m$. Using the concept of co-dominance, one obtains the total number as 2^m .

$$d_{coe}(v)=\begin{cases} 2^m, \text{ if } v\in B; \\ 2^{m-1}, \text{ if } v\in A \ . \end{cases}$$

By Definition 2.2, we get

$$D_{coe}F(K_{m,n})=\sum_{v\in A}(2^{m-1})^3+\sum_{v\in B}(2^m)^3=(m+8n)2^{3m-3}.$$

$$\begin{aligned} D_{coe}F^*(K_{m,n}) &= \sum_{e_1=uv\in E(k_{m,n})}\left(2^{2m}+2^{2m-2}\right) \\ &= 2^{2m}(1+\frac{1}{4})mn=5mn2^{2m-2}. \\ D_{coe}HM_1(K_{m,n}) &= \sum_{e_1=uv\in E(k_{m,n})}\left(2^m+2^{m-1}\right)^2 \\ &= \left(2^{2m}+2^{2m}+2^{2m-2}\right)=9mn2^{2m-2} \end{aligned}$$

Case 2: suppose m even and n odd. By considering the same method as in Case 1, we get,

$$d_{coe}(v)=\begin{cases} 2^{n-1}, \text{ if } v\in B; \\ 2^n, \text{ if } v\in A \ . \end{cases}$$

From definition 2.2, we get

$$\begin{aligned} D_{coe}F(K_{m,n}) &= \sum_{v\in A}(2^n)^3+\sum_{v\in B}(2^{n-1})^3. \\ &= m2^{3n}+n2^{3n-3}=(8m+n)2^{3n-3} \\ D_{coe}F^*(K_{m,n}) &= \sum_{e_1=uv\in E(k_{m,n})}\left[(2^n)^n+(2^{n-1})^2\right]=2^{2n-1}(mn). \\ D_{coe}HM_1(K_{m,n}) &= \sum_{e_1=uv\in E(k_{m,n})}\left(2^n+2^{n-1}\right)^2=9mn2^{2n-2} \end{aligned}$$

Case 3: When m and n both are odd. In this case there is only single even-paired dominating set, which is the set of all vertices of $K_{m,n}$, hence $d_{coe}(v)=1$ and by using the Definition 2.2 we get the results. □

$$\begin{aligned} D_{coe}F(K_{m,n}) &= m+n. \\ D_{coe}F^*(K_{m,n}) &= 2mn. \\ D_{coe}HM_1(K_{m,n}) &= 4n. \end{aligned}$$

Definition 2.3 For any $v\in V$, the graph G is referred to be the $k-$ even-paired dominance regular graph iff $d_{coe}(v)=k$.
examples:

1. Single even-paired dominating regular graph is S_n, where n is even.

2. Single even-paired dominating regular graph $K_n, n\geq 3$, with even number of vertices n

3. 2^{n-1} Even-paired dominating regular graphs of $K_n$with n odd.

Remark 2.1 (Observation) *[1] Let G be graph with $D_1, D_2, ..., D_t$ as even-paired domination sets and $(\gamma_{coe}(G)\ )$ of $G = (n, m)$ be the even-paired domination number, then*

$$t\gamma_{coe}(G) \leq \sum_{V \in V} d_{coe}(V) \leq t|V|$$

Remark 2.2 (Observation) *[1] Suppose G is the Pendant-augmented n-wheel graph, then*

$$1 \leq d_{coe}(v) \leq T_{coe}(G).$$

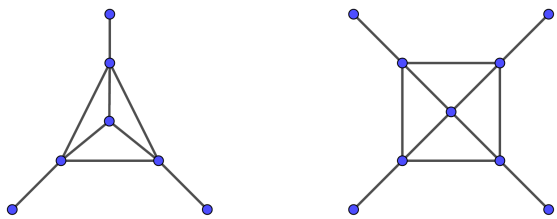

Figure 1: Pendant-augmented n-wheel graph H_4, and H_5.

Theorem 2.1 *Let H_n be Pendant-augmented n-wheel graph with $n \geq 4$. Then*

1. *If n is even*

$$\begin{aligned} D_{coe}F(H_n) =&(9n-1)2^{3n-3}. \\ D_{coe}F^*(H_n) =&(n-1)\left(10 * 2^{2n-4} + 2^{2n-3}\right). \\ D_{coe}HM_1(H_n) =&(n-1)\left(6 * 2^{2n-2} + 2^{2n-2}\right). \end{aligned}$$

2. *If n is odd*

$$\begin{aligned} D_{coe}F(H_n) =&(n-1)\left(2^{3n} + 2^{6n-6} + 3 * 2^{5n-5} + 3 * 2^{4n-4} + 2^{3n-3}\right). \\ D_{coe}F^*(H_n) =&(n-1)\left(2^{2n-1} + 2^{n-2} + 1\right). \\ D_{coe}HM_1(H_n) =&(n-1)\left(7 + (9 * 2^{2n-2}) + 2^{2n} + 2^{2n-2}\right). \end{aligned}$$

Proof: Assume a Pendant-augmented n-wheel graph (H_n) with $n \geq 4$.
Case 1: There are $n-1$ Vertices of even first degree $d(v)$ in this case if n is even. Therefore, $\binom{n-1}{1} + \binom{n-1}{2} + ... + \binom{n-1}{n-1} = 2^{n-1} - 1$ is the number of Even-paired dominant sets generated by all the various substitutions for $(n-1)$ vertices. The $2^{n-1} - 1$ even-paired dominant sets all include vertices of odd initial degree. Furthermore, another even-paired dominant set that only has the vertices of $d(v)$ odd is present. It follows that $T_{coe}(G) = 2^{n-1} - 1 + 1 = 2^{n-1}$. Thus, applying the even-paired domination degree concept, we obtain

$$d_{coe}(v) = \begin{cases} 2^{n-2}, & \text{for even } d(v); \\ 2^{n-1}, & \text{for odd } d(v)\ . \end{cases}$$

Therefore by definition 2.2, we get

$$\begin{aligned} D_{coe}F(H_n) = & \sum_{d(v) is even} (2^{n-2})^3 + \sum_{d(v) is even} (2^{n-1})^3 \\ = & (n-1)2^{3n-6} + n2^{3n-3} = (9n-1)2^{3n-3}. \end{aligned}$$

Let E_1 represent the set of edges joining vertices with odd first degrees to Vertices with even first degrees. Let E_2 represent the set of edges joining vertices with even first degrees to other vertices with even first degrees, therefore

$$\begin{aligned} D_{coe}F^*(H_n) = & \sum_{d(v)=even} \left((2^{n-2})^2 + (2^{n-1})^2 \right) + \sum_{d(v)=even} \left((2^{n-2})^2 + (2^{n-2})^2 \right) \\ = & 5 * 2^{2n-4}(2n-4) + (n-1)2^{2n-3} \\ = & (n-1)\left(10 * 2^{2n-4} + 2^{2n-3} \right) \\ D_{coe}HM_1(H_n) = & \sum_{d(v)=even} (2^{n-2} + 2^{n-1})^2 + \sum_{d(v)=even} (2^{n-2} + 2^{n-2})^2 \\ = & \sum_{d(v)=even} (2^{2n-2} + 2^{2n-2} + 2^{2n-2}) + \sum_{d(v)=even} (2^{2n-2}) \\ = & (n-1)\left(6 * 2^{2n-2} + 2^{2n-2} \right) \end{aligned}$$

Case 2: Here, there exist n vertices of even first degree when n is odd. Therefore, the number of distinct ways to substitute n is equal to the number of even-paired dominant sets. even-paired dominant sets: $\binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n - 1$. It should be noted that every $2^n - 1$ even-paired dominating set contains every pendent vertex.
Another even-paired dominating set with the center vertex and all pendent vertices is also present. Hence $T_{coe}(H_n) = 2^n$.
By the definition of Even-paired dominating degree, we get

$$d_{coe}(v) = \begin{cases} 2^n, & \text{for off } d(v); \\ 2^{n-1}, & \text{excluding the central vertex in } d(v); \\ 2^{n-1} + 1, & \text{for } v \text{ even and } v \text{ the center vertices.} \end{cases}$$

Let B be the set of all vertices on the cycle, and A be the set of all pendant vertices.
By considering the Definition 2.2, we get

$$\begin{aligned} D_{coe}F(H_n) = & \sum_{v \in A} (2^{3n}) + \sum_{v \in B} \Big((2^{3n-3})(2^{n-1} + 1)^3 \\ = & (n-1)2^{3n} + (n-1)\left(2^{6n-6} + 3 * 2^{5n-5} + 3 * 2^{4n-4} + 2^{3n-3} \right) t \end{aligned}$$

Let $E_1 = \{e_1 = uv : d_{coe}(u) = 2^n,$
$d_{coe}(v) = 2^{n-1}\},$
$E_2 = \{e_1 = uv : d_{coe}(u) = 2^{n-1},$
$d_{coe}(v) = 2^{n-1}\},$
$E_3 = \{e_1 = uv : d_{coe}(u) = 2^{n-1},\ \ d_{coe}(v) = 2^{n-1}+1\}$, where; $|E_1| = (n-1),\ |E_2| = (n-1), |E_3| = (n-1)$

Applying the Definition 2.2, we get

$$\begin{aligned}
D_{coe}F^*(H_n) =& \sum_{E_1}\Big(2^{2n}+2^{2n-2}\Big)+\sum_{E_2}\Big(2^{2n-2}+2^{2n-2}\Big)+\sum_{E_3}\Big(2^{2n-2}+2^{2n-2}+2^{n-2}+1\Big).\\
=&(n-1)*\Big(2^{2n}+2^{2n-2}\Big)+(n-1)*\Big(2^{2n-2}+2^{2n-2}\Big)+(n-1)*\Big(2^{2n-2}+2^{2n-2}+2^{n-2}+1\Big)\\
=&(n-1)*\Big(2^{3n}+2^{6n-6}+3*2^{5n-5}+3*2^{4n-4}+2^{3n-3}\Big).\\
D_{coe}HM_1(H_n) =& \sum_{E_1}\Big(2^{n}+2^{n-1}\Big)^2+\sum_{E_2}\Big(2^{n-1}+2^{n-1}\Big)^2+\sum_{E_3}\Big(2^{n-1}+2^{n-1}+1\Big)^2\\
=&\sum_{E_1}\Big(2^{2n}+2^{2n}+2^{2n-2}\Big)+\sum_{E_2}\Big(2^{2n}\Big)\\
=&\sum_{E_3}\Big(2^{2n-2}+2^{2n-1}+2^{n}+1\Big)\\
=&(n-1)2^{2n}(2+\frac{1}{4})+(n-1)2^{2n}+(n-1)\Big(2^{2n-2}+2^{2n-1}+2^{n}+1\Big)\\
=&9(n-1)2^{2n-2}+(n-1)2^{2n}+7(n-1)+(n-1)2^{2n-2}\\
=&(n-1)*\Big(7+(9*2^{2n-2})+2^{2n}+2^{2n-2}\Big).
\end{aligned}$$

□

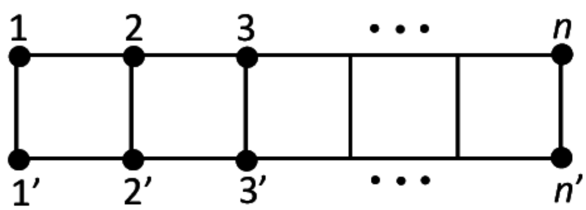

Figure 2: *Ladder graph L_n.*

Proposition 2.5 *Let $P_2 \times P_n$ be Ladder graph.*
Then

$$\begin{aligned}
D_{coe}F(P_2\times P_n) =& 8192n-14336.\\
D_{coe}F^*(P_2\times P_n) =& 1535n-3440.\\
D_{coe}HM_1(P_2\times P_n) =& 1536n-1280.
\end{aligned}$$

Proof: Let $P_2 \times P_n$ be Ladder graph. Then from the Figure 2. the first degree of the vertices sets $\{w_1, w_2.., w_n, w_1^{'}, w_2^{'}, ..w_n^{'}\}$are three, and the first degree of the vertices sets $\{v_1, v_2, v_1^{'}, v_2^{'}\}$ are two. Then the number of even-paired dominating sets are :
$\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}+$ the set of all vertices of first degree3(d(v)=3). Hence $T_{coe}((P_2 \times P_n) = 15+1 = 16$ even-paired dominating sets.
By considering, definition of even-paired dominating degree, we get

$$d_{coe}(v) = \begin{cases} 8, \text{ if } d(v) \text{ } is \text{ } even; \\ 16, \text{ if } d(v) \text{ } is \text{ } odd \text{ }. \end{cases}$$

Therefore by the definition

$$\begin{aligned} D_{coe}F(P_2 \times P_n) &= \sum_{d(v)=even} 8^3 + \sum_{d(v)=odd} 16^3 \\ &= 2048 + 4096(2n-4) \\ &= 2048 + 8192n - 16384 \\ &= 8192n - 14336. \end{aligned}$$

Let $|E_1 = \{e_1 = uv : d(u) \textit{ is even and}$
$d(v) \textit{ is even}\}, |E_1| = 2$.
$E_2 = \{e_1 = uv : d(u) \textit{ is odd and } d(v) \textit{ is even}, |E_2| = 4$.
$E_2 = \{e_1 = uv : d(u) \textit{ is odd and } d(v) \textit{ is odd},$
$|E_3| = 3n - 8$. Then

$$\begin{aligned} D_{coe}F^*(P_2 \times P_n) = &\sum_{e_1=uv\in E} (P_2 \times P_n) \\ = &\sum_{e_1=uv\in E_1(G)} (P_2 \times P_n) + \sum_{e_1=uv\in E_2(G)} (P_2 \times P_n) + \sum_{e_1=uv\in E_3(G)} (P_2 \times P_n) \\ = &\sum_{e_1=uv\in E_1(G)} (64+64) + \sum_{e_1=uv\in E_2(G)} (64+256) + \sum_{e_1=uv\in E_3} 512(3n-8) \\ = &1535n - 3440. \\ D_{coe}HM_1(P_2 \times P_n) = &\sum_{e_1=uv\in E} (P_2 \times P_n) \\ = &\sum_{e_1=uv\in E_1(G)} (P_2 \times P_n) + \sum_{e_1=uv\in E_2(G)} (P_2 \times P_n) + \sum_{e_1=uv\in E_3(G)} (P_2 \times P_n) \\ = &1536n - 1280. \end{aligned}$$

□

Figure 3: *double Star graph.*

Theorem 2.2 *Let $S_{m,n}$ be double Star graph with w_1, w_2 as center vertices of first star and second star respectively.*

1. *If m and n are even,*

$$\begin{aligned} D_{coe}F(S_{m,n}) &= n + m + 2. \\ D_{coe}F^*(S_{m,n}) &= 2(n + m + 1). \\ D_{coe}HM_1(S_{m,n}) &= 2D_{coe}F^*(S_{m,n}). \end{aligned}$$

2. If m and n are odd,

$$\begin{aligned} D_{coe}F(S_{m,n}) =& 16+27(m+n). \\ D_{coe}F^*(S_{m,n}) =& 8+13(m+n). \\ D_{coe}HM_1(S_{m,n}) =& 16+25(m+n). \end{aligned}$$

3. If m is even and n is odd,

$$\begin{aligned} D_{coe}F(S_{m,n}) =& 8(m+n+1)+1. \\ D_{coe}F^*(S_{m,n}) =& 3+8m+5n. \\ D_{coe}HM_1(S_{m,n}) =& 3+16m+9n. \end{aligned}$$

4. If m odd and n is odd,

$$\begin{aligned} D_{coe}F(S_{m,n}) =& 10+9(m+n). \\ D_{coe}F^*(S_{m,n}) =& 3+5m+8n. \\ D_{coe}HM_1(S_{m,n}) =& 3+9m+16n. \end{aligned}$$

Proof: Let $S_{m,n}$ be star graph with $n, m \geq 3$. Then

1. If m and n are even. In this case there is only single even-paired dominating set, $D = V(S_{m,n})$. Hence $d_{coe}(v) = 1$ for all $v \in V(S_{m,n})$. Applying the Definition2.2, we get

$$\begin{aligned} D_{coe}F(S_{m,n}) =& \sum_{v\in V} 1^3 = n+m+2. \\ D_{coe}F^*(S_{m,n}) =& \sum_{e_1=uv\in E} (1+1) = 2(n+m+1). \\ D_{coe}HM_1(S_{m,n}) =& 2(m+n+1) = \sum_{e_1=uv\in E} (1+1)^2 = 2D_{coe}M_2(S_{m,n}). \end{aligned}$$

2. When m and n are both odd, In this case the Even-paired dominating sets are $D_1 = \{all\ pendant\ vertices\ , w_1\}$, $D_2 = \{all\ pendant\ vertices\ w_2\}$, $D_3 = \{all\ pendant\ vertices\ w_1, w_2\}$, $D_4 = \{all\ pendant\ vertices\ \}$Hence $d_{coe}(v) = \begin{cases} 3, \text{ if } d(v)\ is\ pendant\ vertex\ or\ d(v) = 1; \\ 2, \text{ if } d(v) = w_1,\ or\ w_2\ . \end{cases}$

$$D_{coe}F(S_{m,n}) = \sum_{v\in A} 3^3 + \sum_{v\in\{w_1,w_2\}} 2^3 = 16+27(m+n).$$

Let $E_1 = \{e_1 = uv :\ d(u)\ is\ even\ and\ d(v) = 1\}$. Then

$$\begin{aligned} D_{coe}F^*(S_{m,n}) =& 8 + \sum_{e_1=uv\in E_1(S_{m,n})} (3^2+2^2) = 8+13(m+n), \\ D_{coe}HM_1(S_{m,n}) =& 16+25(m+n). \end{aligned}$$

3. If m is even and n is odd. In this case, we have $D_1 = v(S_{m,n})$, $D_2 = \{v(S_{m,n}) - w_2\}$,Hence

$$d_{coe}(v) = \begin{cases} 2, \text{ if } v \neq w_2; \\ 1, \text{ if } v = w_2\ . \end{cases}$$

$$D_{coe}F(S_{m,n}) = \sum_{V-1} 2^3 + 1 = 8(m+n+1).$$

let $E_1 = \{v_i w_1 : i = 1, 2, ..., m\}$,and
$E_2 = u_i w_2 : i = 1, 2, ..., n\}$

$$D_{coe}F^*(S_{m,n}) = 3 + \sum_{e_1=uv\in E_1} (2^2 + 2^2) + \sum_{e_1=uv\in E_2} (1 + 2^2) = 3 + 8m + 5n.$$
$$D_{coe}HM_1(S_{m,n}) = 3 + \sum_{e_1=uv\in E_1} 16 + \sum_{e_1=uv\in E_2} 9 = 3 + 16m + 9n.$$

4. If m odd and n is odd.
In this Case we have $D_1 = v(S_{m,n})$, $D_2 = \{v_{m,n} - w_1\}$, Hence

$$d_{coe}(V) = \begin{cases} 2, \text{ if } v \neq w_1; \\ 1, \text{ if } v = w_1 \ . \end{cases}$$

$$D_{coe}F(S_{m,n}) = \sum_{V(S_{m,n}-1)} 2^3 + 1 = 10 + 9(m + n).$$

Let $E_1 = \{V_i w_1 : i = 1, 2, .., m\}$, then we get

$$D_{coe}F^*(S_{m,n}) = 3 + \sum_{e_1=uv\in E_1} 15 + \sum_{e_1=uv\in E_2} 8 = 3 + 5m + 8n,$$
$$D_{coe}HM_1(S_{m,n}) = 3 + \sum_{e_1=uv\in E_1} 9 + \sum_{e_1=uv\in E_2} 16 = 3 + 9m + 16n.$$

□

3. Some bounds for Modified Forgotten, hyper Zagreb Indices of Graph and Even-paired Domination Forgotten

In the present section, we find some lower and upper bounds on even-paired domination, Zagreb indices of graph, and some graph operations.

Proposition 3.1 *Let $G = (V, E)$. Then*

$$D_{coe}F(G) \leq nT^2_{coe}(G)$$
$$D_{coe}F(G) \leq T^2_{coe}(G)|E|$$
$$D_{coe}M_3(G) \leq 2T_{coe}(G)|E|.$$

The equality holds iff G is one even-paired dominating regular graph.

Proof: Case:1 Let $G = (V, E)$. Then

$$d_{coe}(v) \leq T_{coe}(G)$$
$$d^3_{coe}(v) \leq T^3_{coe}(G)$$
$$\sum_{v\in V} d^3_{coe}(v) \leq \sum_{v\in V} T^3_{coe}(G)$$
$$D_{coe}F(G) \leq T^3_{coe}(G)|V|.$$

Case 2:Let $G=(V,E)$. Then

$$\begin{aligned}
d_{coe}(v) \leq & T_{coe}(G)\\
d^2_{coe}(v) \leq & T^2_{coe}(G)\\
d^2_{coe}(v)+d^2_{coe}(u) \leq & T^2_{coe}(G)+T^2_{coe}(G)\\
\sum_{e_1=uv\in E} d^2_{coe}(v)+d^2_{coe}(u) \leq & \sum_{e_1=uv\in E} T^2_{coe}(G)+\\
T^2_{coe}(G) D_{coe}F^*(G) \leq & 2T^2_{coe}(G)|E|.
\end{aligned}$$

Case 3:Let G be a graph. Then

$$\begin{aligned}
d_{coe}(v) \leq & T_{coe}(G)\\
d_{coe}(v)+d_{coe}(u) \leq & T_{coe}(G)+T_{coe}(G)\\
\Big(d_{coe}(v)+d_{coe}(u)\Big)^2 \leq & \Big(T_{coe}(G)+T_{coe}(G)\Big)^2\\
\sum_{e_1=uv\in E}\Big(d_{coe}(v)+d_{coe}(u)\Big)^2 \leq & \sum_{e_1=uv\in E}\Big(T_{coe}(G)+T_{coe}(G)\Big)^2\\
\sum_{e_1=uv\in E}\Big(d_{coe}(v)+d_{coe}(u)\Big)^2 \leq & \sum_{e_1=uv\in E}\Big(4T^2_{coe}(G)\Big)\\
D_{coe}HM_1(G) \leq & 4T^2_{coe}(G)|E|.
\end{aligned}$$

To prove the equality if G has only one Even-paired domination set, then the only Even-paired domination set is the set of all Vertices of G, so $d_{coe}(v)=1$,and $(1),(2),(3)$ holds.

□

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