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Assessing the Fuzzy n-Inner Product Spaces and Its Impact on Linear Operators

Md Arman Ahmad, Ayaz Ahmad and Ayhan Esi

ABSTRACT: Fuzzy n-inner product space ($f-n-IPS$) and its concept induced fuzzy n-normed linear space ($f-n-NLS$), which generalizes the classical n-inner product spaces to the fuzzy setting, and investigates what effects these generalized fuzzy structures have on the properties of linear operators. We construct a specific $f-n-IPS$ and obtain some fundamental properties that relate the fuzzy inner product to the fuzzy norm. Then, the paper characterizes fuzzy bounded and fuzzy continuous linear operators $T: (X, N_1) \to (Y, N_2)$ between these fuzzy spaces. One of the surprising findings is that, in this fuzzy context, a fuzzy bounded linear operator is always fuzzy continuous; however, the converse may not be true in general, which represents a critical difference from the classic theory of operators. Nevertheless, we prove that in the case when the space X is finite-dimensional, fuzzy continuity yields fuzzy boundedness. Finally, this study generalizes basic concepts from classical operator theory, such as adjoint, self-adjoint, normal, and unitary operators, to this new fuzzy context. With this work, a solid framework for the study of linear operators in imprecise contexts is laid, which provides the possibility for investigation into areas such as fuzzy differential equations and signal processing under uncertainty. Through a rigorous investigation of operator behavior in these uncertain situations, it represents a significant contribution to the growing field of fuzzy mathematics. We provide a basis for possible applications in fuzzy modeling, uncertain signal processing, and multidimensional decision-making systems.

Keywords: Fuzzy n-inner product spaces, linear operators, fuzzy n-norm, fuzzy bounded operators.

Contents

1. Introduction

1.1. Fuzzy inner product spaces

Katsaras [1] provided the first introduction to the concept of a fuzzy norm that exists within a linear space. After that, a great number of mathematicians, such as Felbin [2], Cheng [3], and Mordeson, as well as Bag and Samanta [4], have presented a variety of definitions of fuzzy normed spaces. There has been a significant amount of study carried out in the field of fuzzy functional analysis in recent circumstances. On the other hand, there has been a relatively small amount of research carried out on confusing interior product spaces. The rigorous definition of fuzzy inner product spaces and the accompanying fuzzy norm functions was initially established by R. Biswas [5] and A. M. El-Abyed & Hassan M. El-Hamouly [6]. Both of these individuals were among the pioneers in this field. Following that, J. K. Kohli and R. Kumar [7] made some changes to the notion that R. Biswas had initially developed. In the years that followed,

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in 1995, they attempted to rethink the concept of the fuzzy inner product. Additionally, the idea of hazy co-inner product space is presented as a result of these investigations.

1.2. Preliminaries

To prepare for the subsequent talk, this section will first go over some fundamental ideas that are required for the discussion.

Definition 1.1 *Let n be a natural integer that exceeds 1, and let X be a real linear space with dimensions that are greater than or equal to n. Let $(\bullet, \bullet|\bullet, \ldots, \bullet)$ represent a real-valued function defined on the quotient space $\frac{X\times\cdots\times X}{n+1}$, which is equivalent to X^{n+1}, and that meets the conditions listed below:*
(a) $(x, x|x_2, \ldots, x_2) \geq 0$,
$(x, x|x_2, \ldots, x_n) = 0$ *if any only if* $x, x_2, \ldots, x_n$ *are linearly dependent,*
(b) $(x, y|x_2, \ldots, x_n) = (y, x|x_2, \ldots, x_n)$,
$(x, y|x_2, \ldots, x_n)$ *is invariant under any permutation of* $x_2, \ldots, x_n$,
(c) The formula $(x, x|x_2, \ldots, x_n) = (x_2, x_2|x, x_3, \ldots, x_n)$,
(d) For each real number $a \in R$, *the equation* $(ax, x|x_2, \ldots, x_n) = a(x, x|x_2, \ldots, x_n)$ *holds true.*
(e) $(x + x, y|x_2, \ldots, x_n) = (x, y|x_2, \ldots, x_n) + (x, y|x_2, \ldots, x_n)$.
Then, the product $(\bullet,\bullet|\bullet, \ldots, \bullet)$ is referred to as an n-inner product space, and the product $(X, (\bullet,\bullet|\bullet, \ldots, \bullet))$ is referred to as an n-inner product on X.

Definition 1.2 *Let n belong to the set of natural numbers N, and let X be a real linear space with dimensions that are the same as or bigger than n. An example of a real-valued function that satisfies the following four qualities is the function $||\bullet, \ldots, \bullet||$ on the set $\frac{X\times\cdots\times X}{n} = X^n$.*

(1) *The norm $||x_1, x_2, \ldots, x_n|| = 0$ if and only if the vectors $x_1, x_2, \ldots, x_n$ are linearly dependent.*

(2) *The norm $||x_1, x_2, \ldots, x_n||$ remains unchanged under any permutation.*

(3) $||x_1, x_2, \ldots, ax_n|| = |a|||x_1, x_2, \ldots, x_n||$ *for any $a \in R$ (real) remains invariant under any permutation.*

(4) *An n-norm on x is denoted by $||x_1, x_2, , x_{n-1}, y + z|| \leq ||x_1, x_2, \ldots, x_{n-1}, y|| + ||x_1, x_2, \ldots, x_{n-1}, z||$, and the pair $(X, ||\bullet, \ldots, \bullet||)$ is denoted by the notation n-normed linear space.*

Remark 1.1 *If the replace the third term in the definition with the following: (3) by, (3) $||x_1, x_2, \ldots, ax_n||$ $= |a|p||x_1, x_2, \ldots, x_n||$, where an is a real number and 0 is less than or equal to p, then the space $(X, ||\bullet, \ldots, \bullet||)$ is referred to as a quasi n-normed linear space.*

Remark 1.2 *An n-norm on X is defined by $||x_1, x_2, \ldots, x_n|| = \sqrt{(x_1, |x_2, \ldots, x_n)}$ if an n-inner product space $(X, (\bullet, \bullet|\bullet, \ldots, \bullet))$ is given. Additionally, the Cauchy-Buniakowski inequality's subsequent extension is also accurate:*

$$|(x, y|x_2, \ldots, x_n)| \leq \sqrt{(x, x|\, x_2, \ldots, x_n)}\sqrt{(y, y|x_2, \ldots, x_n)}.$$

Definition 1.3 *Let's say that X is a vector space that takes place over the real field F. A fuzzy subset N of X raised to the power of R, where R is the set of real numbers, is considered to be a fuzzy n-norm on X if and only if the following conditions are met:*

- *(N1) For all cases $t \in R$ such that $t \leq 0, N\,(x_1, x_2, \ldots, x_n, t) = 0$.*
- *(N2) For all cases $t \in R$ with $t > 0, N(x_1, x_2, \ldots, x_n, t) = 1$ solely in the event that $x_1, x_2,\ldots, x_n$ consists of a linearly dependence.*
- *(N3) $N(x_1, x_2, \ldots, x_n, t)$ is unchanged regardless of the permutation of $x_1, x_2, \ldots, x_n$.*
- *(N4) For all cases $t \in R$ with $t > 0, N(xx_1, x_2, \ldots, cx_n, t) = N\left(x_1, x_2, \ldots, x_n, \frac{t}{|c|}\right)$, if $c \neq 0, c \in F$ (field).*

- *(N5) For all cases* $s, t \in R$, $N(x_1, x_2, \ldots, x_n + x'n, s+t) \geq \min\{N(x_1, x_2, \ldots, x_n, s), N(x_1, x_2, \ldots, x'_n, t)\}$.
- *(N6)* $N(x_1, x_2, \ldots, x_n, t)$ *is a non-decreasing function of* $t \in R$ *and* $\lim_{t\to\infty} N(x_1, x_2, \ldots, x_n, t) = 1$.

  In this case, the space (X, N) *is referred to as a fuzzy n-normed linear space, or* $f - n - NLS$. *for short.*

Remark 1.3 *In Definition 1.3, substituting (N4) with* $(N4)'$ *yields: For any* $t \in R$ *where* $t > 0$ $N(x_1, x_2, \ldots, cx_n, t) = N\left(x_1, x_2, \ldots, x_n, \frac{t}{|c|p}\right)$, *if* $c \neq 0, c \in F$ *(field), and* $0 \leq p < 1$ *consequently,* (X, N) *is referred to as a fuzzy quasi n-normed linear space, abbreviated as* $f - q - n - NLS$.

Theorem 1.1 *Let* (X, N) *denote a functional non-linear system. Assume the requirement that* $(N*) N(x_1, x_2, \ldots, x_n, t) > 0$ *for every* $t > 0$ *necessitates that* $x_1, x_2, \ldots, x_n$ *are linearly dependent. Define* $||x_1, x_2, \ldots, x_n||\,\alpha = \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}$ *where,* $\alpha \in (0, 1)$.

Then $\{||\bullet, \bullet, \ldots, \bullet||_a : \alpha \in (0, 1)\}$ constitutes an ascending family of n-norms on X. It designate these n-norms as $\alpha - n - norms$ on X, equivalent to the fuzzy n-norm on X.

2. Fuzzy n-Inner Product Spaces

This section provides an explanation of the satisfactory idea of fuzzy n-inner product space, which functions as a generalization of Definition 2.1.

Definition 2.1 *A fuzzy subset* $J : X^{n+1} \times R$ *(R-set of real numbers) in a linear space* X *over a field* F *is designated as a fuzzy n-inner product on* X *if and only if the subsequent requirements are fulfilled: Moreover, Definition 2.1 in this section is extended by the concept of a fuzzy n-inner product space.*

(1) *For any* $t \in R$ *where* $t \leq 0$, $J(x, x|x_2, \ldots, x_n, t) = 0$.

(2) *For all cases* $t \in R$ *where* $t > 0$ $J(x, x|x_2, \ldots, x_n, t) = 1$ *equals* 1 *if and only if* $x, x_2, \ldots, x_n$ *are linearly dependent.*

(3) *For all cases* $t > 0$, $J(x, y|x_2, \ldots, x_n, t) = J(y, x|x_2, \ldots, x_n, t)$.

(4) $J(x, y|x_2, \ldots, x_n, t)$ *remains invariant under any permutation of* $x_2, \ldots, x_n$.

(5) *For all cases* $t > 0$, $J(x, x|x_2, \ldots, x_n, t) = J(x_2, x_2|x, x_3, \ldots, x_n, t)$.

(6) *For every* $t > 0$, $J(ax, bx|x_2, \ldots, x_n, t) = J\left(x, x|x_2, \ldots, x_n, \frac{t}{|ab|}\right)$, *where* $a, b \in R$ *(real numbers).*

(7) *For every* $s, t \in R$,

$$J(x + x', y|x_2, \ldots, x_n, t + s) \\ \geq \min\ \{J(x, y|x_2, \ldots, x_n, t), J(x', y|x_2, \ldots, x_n, s)\}.$$

(8) *For all real numbers* s *and* t, *where both* s *and* t *are greater than* 0,

$$J(x, y|x_2, \ldots, x_n, \sqrt{ts}) \\ \geq \min\{J(x, x|x_2, \ldots, x_n, t), J(y, y|x_2, \ldots, x_n, s)\}.$$

(9) *The non-decreasing function* $J(x, y|x_2, \ldots, x_n, t)$ *is a function of* $t \in R$, *and the limit of the function* $\lim_{t\to\infty} J(x, y|x_2, \ldots, x_n, t)$ *is equal to* 1.

In this case, the space (X, J) *is referred to as a fuzzy n-inner product space, or* $f - n - IPS$ *for short.*

Example 2.1 *Let $(X,(\bullet,\bullet|,\bullet\dots,\bullet))$ signify an n-inner product space. Offer a precise and formal definition.*

$$J(x,y|x_2,\dots,x_n,t)=\frac{t}{t+|(x,y|x_2,\dots,x_n)|},\ \textit{where } t>0,\ \textit{and } t\in R,$$
$$(x,y|x_2,\dots,x_n)\in X^{n+1},$$

where t is less than or equal to zero.

Then (X,J) is a $f-n$-IPS.

Proof: Following is a verification of each of the nine requirements for $f-n-IPS$:

(1) For all cases $t\in R$ where $t\le 0$, it follows definition that $J(x,x|x_2,\dots,x_n,t)=0$.

(2) For all cases $t\in R$ where $t>0$, it follows that

$$\begin{aligned}J(x,x|x_2,\dots,x_n,t)&=1\\ &\Leftrightarrow\frac{t}{t+|(x,x|x_2,\dots,x_n)|}=1\Leftrightarrow|(x,x|x_2,\dots,x_n)|=0\\ &\Leftrightarrow(x,x|x_2,\dots,x_n)=0\Leftrightarrow x,x_2,\dots,x_n\end{aligned}$$

are linearly dependent.

(3) For every $t>0$,

$$J(x,y|x_2,\dots,x_n,t)=\frac{t}{t+|(x,y|x_2,\dots,x_n)|}$$
$$=\frac{t}{t+|(y,x|x2,\dots,xn)|}=J\left(y,x|x_2,\dots,x_n,t\right).$$

(4) Since $(x,x|x_2,\dots,x_n)$ is unchanged under any permutation of $x_2,\dots,x_n$ it follows that $J(x,y|x_2,\dots,x_n,t)$ is also invariant under any permutation.

(5) For all $t>0$,

$$\begin{aligned}J(x,x|x_2,\dots,x_n,t)&=\frac{t}{t+|(x,x|x_2,\dots,x_n)|}\\ &=\frac{t}{t+|(x_2,x_2|x,x_3,\dots,x_n)|}\\ &=J(x_2,x_2|x,x_3,\dots,x_n,t).\end{aligned}$$

(6) For all $t>0$,

$$\begin{aligned}J(x,x|x_2,\dots,x_n,\frac{t}{|ab|})&=\frac{\frac{t}{|ab|}}{\frac{t}{|ab|}+|(x,x|x_2,\dots,x_n)|}\\ &=\frac{t}{t+|ab||(x_2,x_2|x,x_3,\dots,x_n)|}\\ &=\frac{t}{t+|(ax,bx|x_2,\dots,x_n)|}\\ &=J(ax,bx|x_2,\dots,x_n,t).\end{aligned}$$

(7) If (i) $s+t<0$ (ii) $s=t=0$ (iii) $s+t>0;s>0,t<0;s<0,t>0$, then the relationship mentioned above is clear. Assuming (iv) $s>0,t>0,s+t>0$ then assuming, without losing generality,

$$J(x,y|,x_2,\dots,x_n,t)$$

$$\leq J(x', y|x_2, \ldots, x_n, s)$$

$$\frac{t}{t + |(x, y|_2, \ldots, x_n)|} \leq \frac{s}{s + |(x', y|_2, \ldots, x_n)|}$$

$$\frac{t + |(x, y|x_2, \ldots, x_n)|}{t} \geq \frac{s + |(x', y|x_2, \ldots, x_n)|}{s}$$

$$\Rightarrow 1 + \frac{|(x, y|x_2, \ldots, x_n)|}{t} \geq 1 + \frac{|(x', y|x_2, \ldots, x_n)|}{s} \Rightarrow \frac{|(x, y|x_2, \ldots, x_n)|}{t} \geq \frac{|(x', y|x_2, \ldots, x_n)|}{s}$$

$$\Rightarrow \frac{s|(x, y|x_2, \ldots, x_n)|}{s} \geq |(x', y||x_2, \ldots, x_n)|$$

$$\Rightarrow |(x, y|x_2, \ldots, x_n)| + \frac{s|(x, y|x_2, \ldots, x_n)|}{t} \geq |(x, y|x_2, \ldots, x_n)| + |(x', y|x_2, \ldots, x_n)|$$

$$\Rightarrow (1 + \frac{s}{t})|(x, y|x_2, \ldots, x_n)| \geq |(x + x', y|x_2, \ldots, x_n)|$$

$$\Rightarrow (\frac{s+t}{t})|(x, y|x_2, \ldots, x_n)| \geq |(x + x', y|x_2, \ldots, x_n)|$$

$$\Rightarrow \frac{|(x, y|x_2, \ldots, x_n)|}{t} \geq \frac{|(x + x', y|x_2, \ldots, x_n)|}{s + t}$$

$$\Rightarrow 1 + \frac{|(x, y|x_2, \ldots, x_n)|}{t} \geq 1 + |(\frac{x + x', y|x_2, \ldots, x_n)|}{s + t}$$

$$\Rightarrow \frac{t + |(x, y|x_2, \ldots, x_n)|}{t} \geq \frac{s + t + |(x + x', y|x_2, \ldots, x_n)|}{s + t}$$

$$\Rightarrow \frac{t}{t + |(x, y|x_2, \ldots, x_n)|} \leq \frac{s + t}{s + t + |(x + x', y|x_2, \ldots, x_n)|}$$

$$\Rightarrow \min\{J(x, y|x_2, \ldots, x_n, t), J(x', y|x_2, \ldots, x_n, t)\}$$
$$\leq J\left(x + x', y|x_2, \ldots, x_n, s + t\right).$$

Without loss of generality assume that

$$J(x, x|x_2, \ldots, x_n, t)$$
$$\leq J(y, y|x_2, \ldots, x_n, s),$$

for all $s, t \in R$ with $s > 0, t > 0$.

$$\Rightarrow \frac{t}{t + |(x, x|x_2, \ldots, x_n)|} \leq \frac{s}{s + |(y, y|x_2, \ldots, x_n)|}$$

$$\Rightarrow \frac{t + |(x, x|x_2, \ldots, x_n)|}{t} \geq \frac{s + |(y, y|x_2, \ldots, x_n)|}{s}$$

$$\Rightarrow 1 + \frac{|(x, x|x_2, \ldots, x_n)|}{t} \geq 1 + \frac{|(y, y|x_2, \ldots, x_n)|}{s}$$

$$\Rightarrow \frac{|(x, x|x_2, \ldots, x_n)|}{t} \geq \frac{|(y, y|x_2, \ldots, x_n)|}{s}$$

$$\Rightarrow \frac{s|(x, x|x_2, \ldots, x_n)|}{t} \geq |(y, y|x_2, \ldots, x_n)|$$

$$\Rightarrow \frac{|(x, x|x_2, \ldots, x_n)|s|(x, x|x_2, \ldots, x_n)|}{t}$$
$$\geq |(x, x|x_2, \ldots, x_n)||(y, y|x_2, \ldots, x_n)|.$$

By Remark 1.1,

$$|(x,x|x_2,\dots,x_n)|^2\,\frac{s}{t} \\ \geq |(x,y|x_2,\dots,x_n)|^2$$

$$\Rightarrow \frac{|(x,x|x_2,\dots,x_n)|^2\frac{s^2}{t} \geq |(x,y|x_2,\dots,x_n)|^2}{t}$$

$$\Rightarrow \frac{|(x,x|x_2,\dots,x_n)|^2}{t^2} \geq \frac{|(x,y|x_2,\dots,x_n)|^2}{st}$$

Taking square root on both sides,

$$\Rightarrow \frac{|(x,x|x_2,\dots,x_n)|}{t} \geq \frac{|(x,y|x_2,\dots,x_n)|}{\sqrt{st}}$$

$$\Rightarrow 1+\frac{|(x,x|x_2,\dots,x_n)|}{t} \geq 1+\frac{|(x,y|x_2,\dots,x_n)|}{\sqrt{st}} \Rightarrow \frac{t+|(x,x|x_2,\dots,x_n)|}{t} \geq \frac{\sqrt{st}+|(x,y|x_2,\dots,x_n)|}{\sqrt{st}}$$

$$\Rightarrow \frac{t}{t+|(x,x|x_2,\dots,x_n)|} \leq \frac{\sqrt{st}}{\sqrt{st}+|(x,y|x_2,\dots,x_n)|}$$

$$\Rightarrow \min\{J(x,x|x_2,\dots,x_n,t), J(y,y|x_2,\dots,x_n,s)\} \leq J(x,y|x_2,\dots,x_n,\sqrt{ts}).$$

(9) For every $t_1,t_2 \in R$ if $t_1 < t_2 \leq 0$ then, by our definition,

$$J(x,y|x_2,\dots,x_n,t_1) = J(x,y|x_2,\dots,x_n,t_2) = 0.$$

Suppose $t_2 > t_1 > 0$ then,

$$\frac{t_2}{t_2+|(x,y|x_2,\dots,x_n)|} - \frac{t_1}{t_1+|(x,y|x_2,\dots,x_n)|} \\ = \frac{|(x,y|x_2,\dots,x_n)|(t_2-t_1)}{(t_2+|(x,y|x_2,\dots,x_n)|)(t_1+|(x,y|x_2,\dots,x_n)|)} \geq 0$$

for all $(x,y|x_2,\dots,x_n) \in X^{n+1}$

$$\Rightarrow \frac{t_2}{(t_2+|(x,y|x_2,\dots,x_n)|} \geq \frac{t_1}{t_1+|(x,y|x_2,\dots,x_n)|} \\ \Rightarrow J(x,y|x_2,\dots,x_n,t_2) \geq J(x,y|x_2,\dots,x_n,t_1).$$

Thus $J(x,y|x_2,\dots,x_n,t)$ is a non-decreasing function. Also,

$$\lim_{t\to\infty} J(x,y|x_2,\dots,x_n,t) = \lim_{t\to\infty}\frac{t}{t+|(x,y|x_2,\dots,x_n)|} \\ = \lim_{t\to\infty}\frac{t}{(1+\frac{1}{t}|(x,y|x_2,\dots,x_n)|)} = 1.$$

Thus (X,J) is an $f-n-IPS$ [8].

□

3. Fuzzy Based Linear Operator: an Overview

3.1. Fuzzy bounded linear operator

This section offers definitions of fuzzy bounded linear operators and fuzzy continuous operators within the framework of fuzzy strong-normed linear spaces and explores their interrelationship.

Definition 3.1 *The linear operator $T : (X, N_1, \varphi, K_{,*1}) \to (Y, N_2, \varphi, K_{*2})$ applies to fuzzy strong $\varphi-$b-normed linear spaces X and Y. For each $\alpha \in (0, 1), \exists M\alpha > 0$ in T, it is fuzzy bounded.*
$N_1(x, \frac{t}{M\alpha}) \geq 1 - \alpha \Rightarrow N_2(Tx, Ks) \geq \alpha, \forall s > t$ *and* $\forall t > 0$.

Proposition 3.1 *$X$and Y represent fuzzy strong φ-b-normed linear spaces, and a linear operator is represented by $T : X \to Y$. Connection (1) is a relation if T is fuzzy-bounded:*

$$\{t > 0 : N_2(Tx, Kt) \geq \alpha\} \leq M_a \wedge \{t > 0 : N_1(x, t) \geq 1 - \alpha\}, \forall x \in X.$$

Proof: First it show that (1) $\Rightarrow$(2).
Let $r > M_a \wedge \{t > 0 : N_1(x, t) \geq 1 - \alpha\}$

$$\Rightarrow \frac{r}{M\alpha} > \wedge\{t > 0 : N_1(x, t) \geq 1 - \alpha\},$$

$$\Rightarrow \exists \frac{r'}{M\alpha} \text{ such that } \frac{r}{M\alpha} > \frac{r}{(M\alpha)}' > \wedge\{t > 0 : N_1(x, t) \geq 1 - \alpha\},$$

$$\Rightarrow \exists \frac{r'}{M\alpha}\text{such that } N1(x, \frac{r'}{M\alpha}) \geq 1 - \alpha,$$

$$\Rightarrow N_2(Tx, Kr) \geq \alpha \text{ (by(1))}$$

$$\Rightarrow \wedge\{t > 0 : N_2(Tx, Kt) \geq \alpha\} \leq r.$$

$$\Rightarrow \wedge\{t > 0 : N_2(Tx, Kt) \geq \alpha\} \leq rM\alpha \wedge \{t > 0 : N_1(x, t) \geq 1 - \alpha\}\forall x \in X.$$

So, (1)$\Rightarrow$(2). Now this proves (2) $\Rightarrow$ (1).
Assume that $N_1(x, \frac{t}{M\alpha}) \geq 1 - \alpha$. So,

$$\wedge\{r > 0 : N_1(x, r) \geq 1 - \alpha\} \leq \frac{t}{M\alpha}$$

$$\Rightarrow M\alpha \wedge \{r > 0 : N_1(x, r) \geq 1 - \alpha\} \leq t$$
$$\Rightarrow \wedge\{r > 0 : N_2(Tx, Kr) \geq \alpha\} \leq t \text{ (by(2))}$$

$$\Rightarrow \text{for any } s > t, \wedge\{r > 0 : N_2(Tx, Kr) \geq \alpha\} < s$$
$$\Rightarrow N_2(Tx, Ks) \geq \alpha.$$

The collection of all linear operators that are specified from a fuzzy strong φ-b-normed linear space $(X, N_1, \varphi, K_{,*1})$ to another linear space $(Y, N_2, \varphi, K_{,*2})$ is indicated by the notation $L(X, Y)$. For fuzzy bounded linear operators, the collection is denoted as $BF(X, Y)$. In order to fulfill the goal of this proposition, let $(X, K, *)$ be a fuzzy strong φ-b-normed linear space. Additionally, assume the underlying t-norm $*$ is continuous at $(1, 1)$. Then, for any $\alpha \in (0, 1), \exists\beta \geq \alpha$ this scenario where

$$\wedge\{s + t > 0 : N(x + y, K(s + t)) \geq \alpha\} \leq \wedge\{s > 0 : N(x, s) \geq \beta\} + \wedge\{t > 0 : N(y, t) \geq \beta\}, \forall x, y \in X.$$

The evidence: Due to the fact that $*$ is continuous at $(1,1)$, it is possible to locate $\beta \in (0,1)$ in such a way that $\beta * \beta$ is greater than or equal to α. Once more, the equation $\beta \geq \beta * \beta \geq \alpha$ is equivalent to $\beta \geq \alpha$. Currently,

$$\begin{aligned}\wedge\{s>0 : N(x,s) \geq \beta\} &+ \wedge\{t>0 : N(y,t) \geq \beta\} \\ &= \wedge\{s+t>0 : N(x,s) \geq \beta, N(y,t) \geq \beta\} \\ &\geq \wedge\{s+t>0 : N(x,s) * N(y,t) \geq \beta * \beta\} \\ &\geq \wedge\{s+t>0 : N(x+y, s+Kt) \geq \alpha\}.\end{aligned}$$

Hence $\forall x, y \in X, \wedge\{s+t>0 : N(x+y, s+Kt) \geq \alpha\} \leq \wedge\{s>0 : N(x,s) \geq \beta\} + \wedge\{t>0 : N(y,t) \geq \beta\}$.

(3) Now, $K \geq 1 \Rightarrow Ks \geq s \Rightarrow Ks + Kt \geq s + Kt \Rightarrow K(s+t) \geq s + Kt$. So,

$$\begin{aligned}&\{s+t>0 : N(x+y, s+Kt) \geq \alpha\} \subset \{s+t>0 : N(x+y, K(s+t)) \geq \alpha\} \\ &\quad \Rightarrow \wedge\{s+t>0 : N(x+y, s+Kt) \geq \alpha\} \geq \wedge\{s+t>0 : N(x+y, K(s+t)) \geq \alpha\}.\end{aligned}$$

From (3) we get,

$$\wedge\{s+t>0 : N(x+y, K(s+t)) \geq \alpha\} \leq \wedge\{s>0 : N(x,s) \geq \beta\} + \wedge\{t>0 : N(y,t) \geq \beta\}, \forall x, y \in X.$$

□

Theorem 3.1 *It affirms that the collection of all linear operators, denoted by $L(X,Y)$, constitutes a subspace of the set of all fuzzy bounded linear operators, denoted by $BF(X,Y)$. Here, fuzzy strongly b-normed linear spaces are denoted by $(X, N_1, \varphi, K,_{*1})$ and $(Y, N_2, \varphi, K,_{*2})$. $*2$ is a continuous function at the point $(1,1)$.*

The evidence. It consider T_1 and T_2 to be elements of the set $BF(X,Y)$. Now, according to Proposition 3.1, for scalars k_1 and k_2 that are not zero,

$$\begin{aligned}\wedge\{s+t>0 : N_2((k_1T_1 + k_2T_2)x, K(s+t)) \geq \alpha\} \leq \wedge\,&\{s>0 : N_2(k_1T_1x, s) \geq \beta\} \\ &+ \wedge\{t>0 : N(k_2T_2x, t) \geq \beta\}, \forall x \in X,\end{aligned}$$

where β is dependent on α and β is greater than or equal to α.

$$\wedge\{s+t>0 : N_2((k_1T_1 + k_2T_2)x, K(s+t)) \geq \alpha\} \leq \varphi(k_1) \wedge \{s>0 : N_2(T_1x, s) \geq \beta\}$$

$$+\varphi(k_2) \wedge \{t>0 : N(T_2x, t) \geq \beta\}, \forall x \in X.$$

Given that T_1 and T_2 are fuzzy bounded, it is possible that $M_1\beta(\alpha)$ and M_2 are also bounded. $\beta(\alpha)$ is greater than zero, in such a way that

$$\wedge\{s>0 : N_2(T_1x, s) \geq \beta\} \leq \frac{M^1}{\beta(\alpha)} \wedge \{s>0 : N_1(x,s) \geq 1-\beta\}, \forall x \in X,$$

and

$$\wedge\{t>0 : N_2(T_2x, t) \geq \beta\} \leq \frac{M^2}{\beta(\alpha)} \wedge \{t>0 : N_1(x,t) \geq 1-\beta\}, \forall x \in X.$$

Let $M\alpha$ be defined as $\varphi(k_1)$.

$$\frac{M^1}{\beta(\alpha)} + \varphi(k_2)\frac{M^2}{\beta(\alpha)}$$

Consequently, from the preceding analysis, this derive

$$\wedge\{s+t>0 : N_2((k_1T_1 + k_2T_2)x, K(s+t)) \geq \alpha\} \leq M\alpha \wedge \{t>0 : N_1(x,t) \geq 1-\beta\}, \forall x \in X.$$

Since $\beta \geq \alpha$, so $1-\alpha \geq 1-\beta$. Thus,

$$\{t>0: N_1(x,t) \geq 1-\alpha\} \subset \{t>0: N_1(x,t) \geq 1-\beta\}$$

$$\Rightarrow \wedge\{t>0: N_1(x,t) \geq 1-\alpha\} \geq \wedge\{t>0: N_1(x,t) \geq 1-\beta\}.$$

Thus, from (5), it derive

$$\wedge\{s+t>0: N_2((k_1T_1+k_2T_2)x, K(s+t)) \geq \alpha\} \leq M\alpha \wedge \{t>0: N1(x,t) \geq 1-\alpha\}, \forall x \in X'$$

$$\Rightarrow k_1T_1+k_2T_2 \in BF(X,Y).$$

Consequently, $BF(X,Y)$ constitutes a subspace of $L(X,Y)$.

Definition 3.2 *A fuzzy continuous operator is stated to be $(X, N_1, \varphi, K,_{*1}) \to (Y, N_2, \varphi, K,_{*2})$ for $x \in X$ if, for any sequence $\{x_n\}$ in X with $x_n \to x$, it is implied that $Tx_n \to Tx$.In other words, the limit as n approaches infinity of $N(x_n - x, t)$ equals 1; furthermore, if t is greater than zero, then the limit as n approaches infinity of $N(Tx_n - Tx, t)$ also equals 1. Consequently, $BF(X,Y)$ is a subspace of $L(X,Y)$ because of this.*

Theorem 3.2 *T is a linear operator from the fuzzy strong φ-b-normed linear space X $(X, N_1, \varphi, K,_{*1})$ to the fuzzy strong φ-b-normed linear space Y $(Y, N_2, \varphi, K,_{*2})$ is fuzzy continuous throughout X if it is fuzzy continuous at $x_0 \in X$.*

The evidence is unambiguous.

Theorem 3.3 *States that T is a linear operator from the fuzzy strong φ-b-normed linear space: $(X, N_1, \varphi, K,_{*1})$ to the fuzzy strong φ-b-normed linear space $(Y, N_2, \varphi, K,_{*2})$. A fuzzy bounded T is fuzzy continuous, but not vice versa.*

Proof: Assume T is fuzzy-bound. Note that $M\alpha$ exists for each $\alpha \in (0,1)$ such that

$$\wedge\{t>0: N_2(Tx, Kt) \geq \alpha\} \leq M\alpha \wedge \{t>0: N_1(x,t) \geq 1-\alpha\}, \forall x \in X.$$

Let $\{x_n\}$ be a sequence in X such that x_n converges to x. Consequently

$$\lim_{n\to\infty} N_1(x_n - x, t) = 1, \forall t > 0.$$

Let Q be a positive quantity. For every α in the interval $(0,1)$, a positive integer $N(\alpha, Q)$ meets certain requirements.

$$\begin{aligned} N_1(x_n - x, &Q\frac{t}{2KM\alpha}) > 1-\alpha, \forall n \geq N(\alpha, \varepsilon) \\ &\Rightarrow \wedge\{t>0: N_1(x_n - x, t) \geq 1-\alpha\} \leq \frac{\varepsilon}{2KM_a}, \forall n \geq N(\alpha,\varepsilon), \forall \alpha \in (0,1) \\ &\Rightarrow M\alpha \wedge \{t>0: N_1(x_n - x, t) \geq 1-\alpha\} \leq \frac{\varepsilon}{2K} < \frac{\varepsilon}{K}, \forall n \geq N(\alpha, Q), \forall \alpha \in (0,1) \\ &\Rightarrow N_2(Tx_n - Tx, Q) \geq \alpha, \forall n \geq N(\alpha,\varepsilon), \forall \alpha \in (0,1) \Rightarrow \lim_{n\to\infty} N_2(Tx_n - Tx, \varepsilon) = 1. \end{aligned}$$

Given that $Q > 0$ is arbitrary, it follows that $\lim_{n\to\infty} N_2(Tx_n - Tx, t) = 1$ for any $t > 0$.

T is consequently fuzzy and continuous on X.

The following example shows that the opposite outcome might not be true.

$$N_1(x,t) = \begin{cases} 1 \quad \text{if } t \geq \|x\| \\ \frac{1}{2} \quad 0 < t < \|x\| \\ 0 \quad \text{if } t \leq 0 \end{cases}$$

$$N_2(x,t) = \begin{cases} 1-\frac{||x||}{t} & \text{if } t \geq ||x|| \\ 0 & \text{if } t < ||x|| \end{cases}$$

N_1 and N_2 are resilient fuzzy $\varphi - b - norms$ on X, where $*$ signifies min, $\varphi(\alpha) = |\alpha|$ for every $\alpha \in R$, and $K = 1$.

For every $x \in X$, define a linear operator $T : (X, N_1) \to (X, N_2)$ such that $T(x) = 2x$ As seen in the above example, T is fuzzy continuous but not fuzzy bounded. □

Theorem 3.4 *States that if X is a finite-dimensional structure, then T is fuzzy-bound. The linear operator $T : X \to Y$ is a linear operator, where $(X, N_1, \varphi, K,_{*1})$ and $(Y, N_2, \varphi, K,_{*2})$ represent two fuzzy strong, φ-b-normed linear spaces.*

Proof: Let dim $X = n$ and $\{x_1, x_2, \ldots, x_n\}$ be a basis of X. Choose $x(\neq \theta) \in X$. Then $x = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n$ for some suitable scalars $\{\beta_1, \beta_2, \ldots, \beta_n\}$. So,

$$\begin{aligned} Tx &= T(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n) \\ &= T(\beta_1 x_1) + T(\beta_2 x_2) + \cdots + T(\beta_n x_n). \end{aligned}$$

Let's equal the sum from i equals 1 to n of the absolute values of β_i. It is evident that s is not equal to zero.

Let us select $\alpha \in (0, 1)$ arbitrarily. At this moment,

$$\begin{aligned} &\wedge\{\frac{t}{n} > 0 : N_2(sT(\beta_1 x_1), \frac{t}{n}) \geq \alpha\} + \wedge\{\frac{t}{n} > 0 : N_2(sT(\beta_2 x_2), \frac{t}{Kn}) \geq \alpha\} + \cdots \\ &\qquad + \wedge\{\frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{kn}) \geq \alpha\} \\ &= \wedge\{\frac{t}{n} + \frac{t}{n} > 0 : N_2(sT(\beta_1 x_1), \frac{t}{n}) \geq \alpha, N_2(sT(\beta_2 x_2), \frac{t}{Kn}) \geq \alpha\} + \cdots \\ &\qquad + \wedge\{\frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{Kn}) \geq \alpha\} \\ &\geq \wedge\{\frac{t}{n} + \frac{t}{n} > 0 : N_2(sT(\beta_1 x_1) + sT(\beta_2 x_2), \frac{t}{n} + K \cdot \frac{t}{Kn}) \geq \alpha * \alpha\} + \cdots \\ &\qquad + \wedge\{\frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{Kn}) \geq \alpha\} \\ &\geq \wedge\{\frac{t}{n} + \cdots + \frac{t}{n} > 0 : N_2(sT(\beta_1 x_1) + \cdots sT(\beta_n x_n), \frac{t}{n} + \cdots + \frac{t}{n}) \geq \alpha * \cdots * \alpha\} \\ &= \wedge\{t > 0 : N_2(T(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n), \frac{t}{\varphi(s))} \geq \alpha * \alpha * \cdots * \alpha\}. \end{aligned}$$

Let us select $\alpha \in (0, 1)$ arbitrarily. At this moment,

$$\begin{aligned} &\wedge\{\frac{t}{n} > 0 : N_2(sT(\beta_1 x_1), t_n) \geq \alpha\} + \cdots + \wedge\{\frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{Kn}) \geq \alpha\} \geq \\ &\wedge\{t > 0 : N_2(T(\beta_1 x_1 + \cdots + \beta_n x_n), t\varphi(s)) \geq \alpha * \cdots * \alpha\}. \end{aligned}$$

Let, $N_a = \max\{V\{t_n > 0 : N_2(sT(\beta_1 x_1), \frac{t}{n}) \geq \alpha\}, \ldots, V\{\frac{t}{n} > 0 : N_2(sT(\beta_n x_n), \frac{t}{Kn}) \geq \alpha\}\}$. From equation (6), it obtains

$$nN_a \geq \wedge\{t > 0 : N_2(Tx, t\varphi(s)) \geq \alpha * \cdots * \alpha\},$$

this suggests

$$\frac{nN\alpha}{\varphi(s)} \geq \wedge\{t > 0 : N_2(Tx, t) \geq \alpha * \cdots * \alpha\}.$$

From Proposition 3.1, $\exists c\alpha > 0$ such that

$$\wedge\{t > 0 : N_1(\beta_1 x_1 + \cdots + \beta_n x_n, Kt) \geq 1 - \alpha\} \geq \frac{c\alpha}{\varphi(s)}.$$

From (7) and (8), it is consequent that,

$$\frac{nN\alpha}{c\alpha} \wedge \{t > 0 : N_1(x, Kt) \geq 1 - \alpha\} \geq \wedge\{t > 0 : N_2(Tx, t) \geq \alpha * \cdots * \alpha\}$$

that is $\wedge\{t > 0 : N_2(Tx, t) \geq \alpha * \cdots * \alpha\} \leq M_\alpha^{'} \wedge \{t > 0 : N_1(x, Kt) \geq 1 - \alpha\}$ where $M'\alpha = \frac{nN\alpha}{c\alpha}$, that is $K \wedge \{\frac{t}{K} > 0 : N_2(Tx, K \cdot \frac{t}{K}) \geq \alpha * \cdots * \alpha\} \leq \frac{M_\alpha'}{K} \wedge \{Kt > 0 : N_1(x, Kt) \geq 1 - \alpha\}$ that is $K \wedge \{t > 0 : N_2(Tx, Kt) \geq \alpha * \cdots * \alpha\} \leq \frac{M_\alpha'}{K} \wedge \{t > 0 : N_1(x, t) \geq 1 - \alpha\}$.

Consequently, the ultimately derived

$$\wedge\{t > 0 : N_2(Tx, Kt) \geq \alpha * \cdots * \alpha\} \leq M\alpha \wedge \{t > 0 : N_1(x, t) \geq 1 - \alpha\},$$

where $M\alpha = \frac{M'\alpha}{K2}$.

Since α is greater than or equal to α multiplied by itself multiple times, it follows from equation (9) that

$$\wedge\{t > 0 : N_2(Tx, Kt) \geq \alpha * \cdots * \alpha\} \leq M_a \wedge \{t > 0 : N_1(x, t) \geq 1 - \alpha * \alpha * \cdots * \alpha\} \forall x \in X$$

or $\wedge\{t > 0 : N_2(Tx, Kt) \geq \beta(\alpha)\} \leq M\alpha \wedge \{t > 0 : N_1(x, t) \geq 1 - \beta(\alpha)\} \forall x \in X$,where $\beta(\alpha) = \alpha * \alpha * \cdots * \alpha$.

Since $\alpha \in (0, 1)$ is arbitrary, T is consequently imprecisely bounded [9]. □

3.2. Adjoint Fuzzy Operator

Definition 3.3 (X, μ) *is the notation used to represent the fuzzy inner product space. Whenever the convergence of the sequence* $\{f_n\}$ *to* f *entails the convergence of the sequence* $\{\{Tf_n\}$ *to* Tf *for all* $\{f_n\}$ *in* $F(x)$*, where* f *is an element of* $F(x)$*, then a linear functional* T *that is defined on* $F(x)$ *is regarded to be continuous.*

There is a positive integer such that n_0 is a member of the set $\mathbb{N}$.

$$\mu(f_n - f, f_n - f, h, t) > 1 - r.$$

This applies for any given $t > 0$ and for any r such that $0 < r < 1$.

For any integer n greater than n_0 and for any h in $F(x)$, the conditions $0 < r < 1$ are satisfied. A specific and positive number is indicated.

There exists an $n_0 \in N$ such that

$$\mu(Tf_n - Tf, Tf_n - Tf, h, t') > 1 - r'.$$

For all $n > n_0$ and $h \in F(x)$, where $0 < t' \leq 1$ and $r' \in (0, 1)$.

Theorem 3.5 *A fuzzy inner product space is denoted by the notation* X. $\inf\{t : \mu(f, g, h, t) \geq \alpha\} < \infty$ *for all* $f, g \in F(X)$ *then* $\inf\{t + s : \mu(f + g, f_1, h, t + s) \geq \alpha\}$

$$\begin{aligned} &= \inf\{t : \mu(f, f_1, h, t) \geq \alpha\} \\ &= \inf\{t + s : \mu(f + g, f_1, h, t + s) \geq \alpha\}. \end{aligned}$$

Proof: Let us consider

$$\begin{aligned} &\inf\{t : (f, f, h, t)\}. \inf\{s : (g, f, h, s)\} = \inf\{t + s : (f, f, h, t)(g, f, h, s)\} \geq \alpha\mu_1 \geq \alpha \\ &\quad \geq \inf\{t + s : \mu(f + g, f_1, h, t + s\} \geq \alpha. \end{aligned}$$

□

Theorem 3.6 *Let* (X, μ) *be a fuzzy inner product space. If* $\inf\{t : \mu(f, g, h, t) \geq \alpha\} < \infty$ *for all* $f, g \in F(X)$*, then* $\inf\{t+s : \mu(f+g, f_1, h, t+s) \geq \alpha\} = \inf\{t : \mu(f, f_1, h, t) \geq \alpha\} \inf\{t : \mu(g, f_1, h, s) \geq \alpha\}$.

Demonstration. Let us examine the following:

$$\begin{aligned}
&\inf\{t:\mu(f,f_1,h,t)\geq\alpha\}+\inf\{s:\mu(g,f_1,h,s)\geq\alpha\}\\
&=\inf\{t+s:(f,f_1,h,t)\geq\alpha,\mu=+\mu_1,(f,f,h,s)\}\\
&=\inf\{t+s:\inf\{(f,f_1,h,t)\geq\alpha,\mu(g,f_1,h,s)\}\geq\alpha\}\}\\
&\geq\inf\{t+s:\mu(f+g,f_1,h,t+s)\geq\alpha\}.
\end{aligned}$$

Theorem 3.7 *Let (X,μ) be a fuzzy inner product space. If $\inf\{t:\mu(f,g,h,t)\geq\alpha\}<\infty$ for all $f,g\in F(X)$, then*

$$\inf\{t+s:\mu(f+g,f_1,h,t+s)\geq\alpha\}=\inf\{t:\mu(f,f_1,h,t)\,\alpha\}+\inf\{s:\mu(g,f_1,h,s)\geq\alpha\}.$$

Demonstration. Let us examine

$$\begin{aligned}
&\inf\{t:\mu(f,f,h,t)\}\inf\{s:\mu(g,f,h,s)\geq\alpha\}\\
&=\inf\{t+s:(f,f_1,h,t)\,\alpha,\mu(f,f,h,s)\}\inf\{ts:\inf\{(f,f,h,t),(g,f,h,s)\}\\
&\geq\alpha\mu_1\geq\alpha=+\mu_1\geq\alpha\mu_1\geq\alpha\}\\
&\geq\inf\{t+s:\mu(f+g,f_1,h,t+s)\geq\alpha\}.
\end{aligned}$$

(1) Conversely, for all $\delta>0$ Assume that,

$$k=\inf\{1-\left[1-\mu\left[f,f_1,h,\inf\{t:u(f,f_1,h,t)\geq\alpha\}-\frac{\delta}{2}\right]\right]$$

$$\left[1-\mu\left[g,f,f_1,h,\inf\{s:\mu(g,f_1,h,s)\geq\alpha\}-\frac{\delta}{2}\right]\right]\}$$

$$\mu[-g,f,f_1,h-\inf\{s:\mu(g,f_1,h,s)\geq\alpha\}+\frac{\delta}{2}$$

$$\begin{aligned}
&\geq 1-\mu(-f,-g,f_1,h-\inf\{t:\mu(f,f_1,h,s)\geq\alpha\}-\inf\{s:\mu(g,f_1,h,s)\geq\alpha\}+\delta)\\
&=\mu(f+g,f_1,h,\inf\{t:\mu(f,f_1,h,t)\geq a\}+\inf\{s:u(g,f_1,h,s)\geq\alpha\}-\delta).
\end{aligned}$$

By the definition of infimum

$$\mu\left[f,f_1,h,\inf\{t:\mu(f,f_1,h,t)\geq\alpha\}-\frac{\delta}{2}\right]<\alpha.$$

Hence, $1-\mu\left[g,f_1,h,\inf\{t:\mu(g,f_1,h,s)\geq\alpha\}-\frac{\delta}{2}\right)\leq 1-\alpha$.
Similarly, $1-\mu\left[g,f_1,h,\inf\{s:\mu(g,f_1,h,s)\geq\alpha\}-\frac{\delta}{2}\right]<1-\alpha$.
Therefore, $\inf\left\{\{1-\mu(f,f_1,h,\inf\{t:\mu(f,f_1,h,t)>\alpha\}))-\frac{\delta}{2}\right]$

$$[1-\mu\left[g,f_1,h,\inf\{s:\mu(g,f_1,h,s)\geq\alpha\}-\frac{\delta}{2}\right]]\}>1-\alpha$$

(i.e.) $1-k>1-a$, which indicates that $k<a$. Consequently

$$\begin{aligned}
&\mu(f+g,f_1,h,\inf\{t:\mu(f,f_1,h,t)\geq\alpha\}+\inf\{s:\mu(g,f_1,h,s)\geq\alpha\}-\delta k<\alpha\\
&\qquad\text{(i.e.)}\ \inf\{t+s:\mu(f+g,f_1,h,t+s\geq\alpha\}\\
&\qquad\geq\inf\{t:\mu(f,f_1,h,t)\geq\alpha\}+\inf\{s:\mu(g,f_1,h,s)\geq\alpha\}.
\end{aligned}$$

Based on (1) and (2)

$$\inf\{t+s:\mu(f+g,f_1,h,t+s\geq\alpha\}=\inf\{t:\mu(f,f_1,h,t)\geq\alpha\}+\inf\{s:\mu(g,f_1,h,t)\geq\alpha\}.$$

Theorem 3.8 *It is possible to assert that there is a singular T, which is a continuous linear functional on $F(X)$, in the case where T is a continuous linear functional and X is a fuzzy Hilbert space. This assertion may be made because:*

In the event that T is a continuous linear functional and X is a fuzzy Hilbert space, then there exists a singular T, a continuous linear functional on $F(X)$, that is distinct from any other such functional

$$\inf\{t : \mu(Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(f, f_1, h, t) \geq \alpha\}$$

For every $f, g, h \in F(X)$.

Proof: Choose $g \in F(X)$, define: $G_g : F(X) \to R$ by $G_g(f) = \inf\{t : \mu(f, f_1, h, t) \geq \alpha\}$ for every $f \in F(X)$ such that,

$$G_g(f + l) = G_g(f) + G_g(l)$$

$$G_g(kf) = kG_g(f)$$

$$\inf\{t : \mu(Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(T^*f, g, h, t) \geq \alpha\}$$

Where T^* is adjoint fuzzy operator of T. □

Theorem 3.9 *Let X be a fuzzy Hilbert space with $\inf\{t : \mu(f, g, h, t) \geq \alpha\}$ and let T be a continuous linear functional, then T is self-adjoint fuzzy operator.*

Proof: Since Fx is set of all fuzzy sets on X a non empty set and $\inf\{t : \mu(f, g, h, t) \geq \alpha\}$ for every $f, g \in F(X)$, then $\inf\{t : \mu(Tf, g, h, t) \geq \alpha\}$ is real for all $f \in F(X)$.

Now

$$\inf\{t : \mu(f, g, h, t) \geq \alpha\} = \inf\{t : \mu(Tf, g, h, t) \geq \alpha\}$$
$$= \inf\{t : \mu(f, Tg, h, t) \geq \alpha\}$$
$$= \inf\{t : T * f, g, h, t) \geq \alpha\}.$$

Therefore,

$$\inf\{t : \mu(Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(Tf, g, h, t) \geq \alpha\}$$

is a self-adjoint fuzzy operator.

T is a self-adjoint fuzzy operator. □

Theorem 3.10 *Let (X, μ) be a fuzzy Hilbert space with $\inf\{t : \mu(f, g, h, t) \geq \alpha\}$ for every $f, g \in F(X)$ and let T^* be the adjoint fuzzy operator of T is a continuous linear functional then*

(i) $\inf\{t : \mu(f, T^{**}g, h, t) \geq \alpha\} = \inf\{t : \mu(f, Tg, h, t) \geq \alpha\}$,

(ii) $\inf\left\{t : \mu\left(f, (kT)^* g, h, t\right) \geq \alpha\right\} = \inf\{t : \mu(f, kT^*g, h, t) \geq \alpha\}$,

(iii) $\inf\left\{t : s : \mu\left(f, (kT + sD)^* g, h, t\right) \geq \alpha\right\} = \inf\{t + s : \mu(f(kT^* + sD^*), g, h, t) \geq \alpha,$

(iv) $\inf\left\{t : \mu\left(f, (TD)^* g, h, t\right) \geq \alpha\right\} = \inf\{t : \mu(f, (D^*T^*) g, h, t) \geq \alpha\}.$

Proof: (i)

$$\inf\left\{t : \mu\left(f, (kT)^* g, h, t\right) \geq \alpha\right\} = \inf\{t : \mu(T^*f, g, h, t) \geq \alpha\}$$
$$= \inf\{t : \mu(f, Tg, h, t) \geq \alpha.$$

(ii)

$$\begin{aligned}\inf\left\{t:\mu\left(f,(kT)^{*}g,h,t\right)\geq\alpha\right\} &= \inf\{t:\mu(kTf,g,h,t)\geq\alpha\\ &= \inf\{t:\mu(kTf,g,h,t)\geq\alpha\\ &= \inf\{t:\mu\left[Tf,g,h,\frac{t}{|k|}\right]\geq\alpha\\ &= \inf\{t:\mu\left[f,T,g,h,\frac{t}{|k|}\right]\geq\alpha=\inf\{t:\mu\left[f,kT,g,h,\frac{t}{|k|}\right]\geq\alpha.\end{aligned}$$

(iii)

$$\inf\{t+s:\mu\left(f,kT,sD\right)^{*}g,h,t)\geq\alpha\}=\inf\{t+s:\mu(f\left(kT+sD\right),g,h,t)\geq\alpha\}$$

$$\begin{aligned}&=\inf\{t+s:\inf\left\{\mu\left(kT,T^{*}g,h,t\right)\geq\alpha,\mu\left(sf,D^{*}g,h,t\right)\geq\alpha\right\}\}\\ &=\inf\left\{t+s:\inf\left\{\mu\left[kf,T^{*}g,h,\frac{t}{|k|}\right]\geq\alpha,\mu\left[sf,D^{*}g,h,\frac{t}{|s|}\right]\geq\alpha\right\}\right\}\end{aligned}$$

$$\begin{aligned}&=\inf\left\{t+s:\inf\left\{\mu\left(f,kT^{*}g,h,t\right)\geq\alpha,\mu\left(f,sD^{*}g,h,t\right)\geq\alpha\right\}\right\}\\ &=\inf\left\{t+s:\inf\left\{\mu\left(f\left(kT^{*}+sD^{*}\right)g,h,t\right)\geq\alpha\right\}\right\}\end{aligned}$$

As in Theorem 3.9, the equality mentioned above is demonstrated.

(iv)

$$\inf\{t:\mu(Df,T^{*}g,h,t)\geq\alpha\}=\inf\left\{t:\mu\left(f,D^{*}T^{*}\right)g,h,t\right)\geq\alpha\ \ [10].$$

□

3.3. Normal Fuzzy Operator

Theorem 3.11 *If N is a self-adjoint fuzzy operator on $F(x)$ and a normal fuzzy operator, then*

$$\inf\{t:(N*N)f,f,h,t\}=\inf\{t:(Nf,Nf,f,h,t\}$$

Proof: In the event where N is a regular fuzzy operator, the following is true:

$$\inf\left\{t:\mu\left(N^{*}Nf,g,h,t\right)\geq\alpha\right\}=\inf\{t:\mu\left(N*N,f,g,h,t^{2}\right)\geq\alpha\}.$$

By changing $g=f$,

The equation transforms to

$$\inf\left\{t:\mu\left(N^{*}Nf,f,h,t\right)\geq\alpha\right\}=\inf\{t:\mu\left(N^{*}N,f,f,h,t^{2}\right)\geq\alpha\}.$$

This simplifies to:

$$\inf\{t:\mu\left(Nf,Nf,h,t^{2}\right)\}.$$

□

Definition 3.4 *The irregular inner product space is denoted by (X,μ). If an operator T satisfies certain criteria, it is classified as a unitary fuzzy operator.*

$$\begin{aligned}&\inf\{t:\mu(T^{*}Tf,f,h,t)\geq\alpha\}\inf\{t:\mu(TT^{*}f,f,h,t)\geq\alpha\}\\ &\quad=\inf\{t:\mu(f,g,h,t)\geq\alpha\}\end{aligned}$$

Theorem 3.12 *The conditions that are listed below are comparable in the event that T is a fuzzy operator on a fuzzy space (H):*

(i) $\inf t$: *Let* T *represent a function, and let* $g, h,$ *and* t *denote variables such that* $f, g, h,$ *and* t *are defined.*

(ii) $\inf\{t : \mu(Tf, Tg, h, t) \geq \alpha\} = \inf\{t : \mu(f, g, h, t) \geq \alpha\}$ *for all* $f, g, h \in F(X)$.

(iii) $\inf\{t : \mu\left(Tf, Tg, h, t^2\right) \geq \alpha\} \inf\{t : \mu(f, g, h, t) \geq \alpha$ *for all* $f \in F(X)$.

Demonstration.
(i) Implies (ii) Provided

$$\begin{aligned} &\inf\{t : \mu(x) \geq \alpha = \{t : \mu(x) \geq \alpha\} \\ &= \inf\{t : \mu(T^*Tf, g, h, t) \geq \alpha\} \\ &= \inf\{t : \mu(f, g, h, t) \geq \alpha\}. \end{aligned}$$

Contemplate $\mu(x) \geq \alpha = \{\mu(x) \geq \alpha\}$

$$\begin{aligned} &\inf\left\{t : \mu\left(Tf, Tg, h, t\right) \geq \alpha\right\} \\ &= \inf\{t : \mu(f, T^*Tg, h, t) \geq \alpha\} \\ &= \inf\{t : \mu(f, g, h, t) \geq \alpha\} \end{aligned}$$

(ii) $\Rightarrow$ (iii) provided

$$\inf\{t : \mu(Tf, Tg, h, t) \geq \alpha\} = \inf\{t : \mu(f, g, h, t) \geq \alpha\}.$$

(4) Through the act of taking $g = f$, (4) transforms into

$$\inf\left\{t^2 : \mu\left(Tf, Tf, h, t^2\right) \geq \alpha\right\} \inf\{t^2 : \left(f, f, h, t^2\right) \geq \alpha\}.$$

(iii) implies (iv). Provided
Let $\mu\left(x\right) \geq \alpha$ be equivalent to $\mu\left(x\right) \geq \alpha$.

$$\begin{aligned} &= \inf\{t^2 : \mu(Tf, Tf, h, t^2)\} \geq \alpha \\ &= \inf\{t^2 : (f, f, h, t^2)\} \geq \alpha. \end{aligned}$$

Contemplate $\mu\left(x\right) \geq \alpha = \{\mu\left(x\right) \geq \alpha\}$

$$\begin{aligned} &\inf\{t : \mu(T^*Tf, g, h, t) \geq \alpha\} = \inf\{t : \mu(Tf, Tg, h, t) \geq \alpha\} \\ &= \inf\{t : \mu(f, g, h, t) \geq \alpha\} \end{aligned}$$ [11,12].

4. Impact of Fuzzy n-Inner Product and Its Impact on Linear Operators

Fuzzy inner product spaces represent one of the most important developments in the development of fuzzy functional analysis due to the fact that the extension of classical inner product structures is able to capture multi-vector interactions under conditions of uncertainty. This generalization will play an influential role in the theoretical understanding and characterization of linear operators acting within fuzzy environments. By introducing fuzziness into the geometry of the space, this study allows one to conduct research into operators about such aspects as degrees of membership, multidimensional dependence, and distortions induced by uncertainty—all beyond the reach of traditional functional analysis.

One of the major impacts lies in the redefinition of boundedness for linear operators. In fuzzy strong φ-b-normed spaces, boundedness depends not merely on the operator's algebraic behavior but on its ability to preserve fuzzy norms at every membership level. Equivalence results sketched in the paper ensure that fuzzy-bounded operators are a well-defined subspace, hence ensuring stability and predictability even when the underlying data or vectors are vague. Such an approach is imperative for building operator systems with good performance for which crisp norms cannot model uncertainty.

It also indicates that the fuzzy bounded operator entails fuzzy continuity, showing how continuity obtains a stratified meaning under fuzzy conditions. It deepens operator theory by showing that continuity cannot alone define operator stability under uncertainty. Such insights are of importance for extending convergence analysis, iterative methods, and approximation theory to fuzzy environments.

It also introduces the concepts of adjoint, self-adjoint, normal, and unitary fuzzy operators, along with their analyses, which are extensions of classical spectral and structural notions. Accordingly, the fuzzy adjoint operator generalizes the Riesz representation theorem and preserves inner product behavior under vagueness. Self-adjoint and normal fuzzy operators maintain structural symmetry in uncertain systems and allow spectral-like decompositions to be performed that draw support from applications ranging from fuzzy quantum models to fuzzy signal processing and uncertain dynamical systems. Unitary fuzzy operators preserve fuzzy norms and, hence, support transformations that preserve uncertainty levels, crucial for stability in iterative algorithms. In all, the realm of fuzzy n-inner product spaces essentially widens the scope of operator theory within a unified analytical framework where uncertainty, multidimensional dependence, and operator behavior rigorously coexist. The impact fortifies both the theoretical coherence of fuzzy mathematics and its relevance for application to real-world uncertain systems.

5. Conclusion

This paper concludes that studying fuzzy n-inner product spaces extends the realm of classical inner product theory significantly to allow mathematical analyses in environments characterized by uncertainty, vagueness, or dependence on several elements. This paper develops a foundation for the analysis of linear operators under imprecise conditions by generalizing the notions of inner products and norms within a fuzzy structure. The introduction of fuzzy bounded and fuzzy continuous operators has shown that operator theory in a fuzzy strong φ-b-normed space is fundamentally different from the classical theory of norms. This differentiation is emphasized by the fact that every fuzzy bounded operator is fuzzy continuous, but the reverse statement is not necessarily true, which shows the deeper structure and sensitivity of fuzzy operator theory. Such an extension of classical notions as adjoint, self-adjoint, normal, and unitary operators to fuzzy n-inner product spaces reveals the possibility of retaining meaningful operator properties even when placed into an uncertain environment. These generalized operators retain structural properties similar to their classical cousins but allow for membership-based variations and hence are suitable for applications involving complex, uncertain, or incomplete data. The main contribution of this paper is to develop rigorous definitions, prove basic theorems, and show how fuzzy n-inner product spaces influence the stability, continuity, and structural behavior of linear operators in fuzzy functional analysis. This work widens up the theoretical landscape and opens routes to further research in the fields of fuzzy differential equations, signal processing under uncertainty, fuzzy quantum models, and other domains where imprecision should be mathematically integrated rather than ignored.

References

1. A.K. Katsaras, *Fuzzy Topological Vector Spaces II.* Fuzzy Sets and Systems 12, 143-154, (1984). https://doi.org/10.1016/0165-0114(84)90034-4
2. C. Felbin, *Finite Dimensional Fuzzy Normed Linear Spaces.* Fuzzy Sets and Systems 48, 239-248, (1992). https://doi.org/10.1016/0165-0114(92)90338-5
3. S.C. Cheng and J.M. Mordeson, *Fuzzy Linear Operators and Fuzzy Normed Linear Spaces.* Bulletin of the Calcutta Mathematical Society 86, 429-436, (1994).
4. T. Bag, S.K. Samanta, *Some fixed point theorems in fuzzy normed linear spaces*, Inform. Sci. 177, 3271-3289, (2007).
5. R. Biswas, *Fuzzy inner product spaces and fuzzy norm functions.* Information Sciences 53, 185-190, (1991). https://doi.org/10.1016/0020-0255(91)90063-Z
6. A.M. El-Abyad and H. M. El-Hamouly, *Fuzzy inner product spaces.* Fuzzy Sets and Systems 44, 309-326, (1991). https://doi.org/10.1016/0165-0114(91)90014-H
7. J.K. Kohli and R. Kumar, *Linear mappings, fuzzy linear spaces, fuzzy inner product spaces and fuzzy co-inner product spaces*, Bull. Calcutta Math. Soc. 87(3), 237-246, (1995).
8. S. Vijayabalaji and N. Thillaigovindan, *Fuzzy n-inner product space.* Bulletin of the Korean Mathematical Society 44(3), 447-459, (2007). https://doi.org/10.4134/BKMS.2007.44.3.447
9. A. Das and T. Bag, *Fuzzy strong φ-b-normed linear space for fuzzy bounded linear operators*, New Mathematics and Natural Computation 19(3), 855-873, (2022). https://doi.org/10.1142/S1793005723500394

10. T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, Annals of Fuzzy Mathematics and Informatics 6(2), 271-283, (2013)

11. C. Diminnie, S. Gahler and A. White, *2-inner product space II1)*, Demonstratio Mathematica 10(1), 169-188, (1977). https://doi.org/10.1515/dema-1977-0115

12. B. Thangaraj and R. Abirami, *Study on operators in 2-fuzzy 2-inner product space*, Advances and Applications in Mathematical Sciences 21(3), 1303-1314, (2022).

Md Arman Ahmad,
Department of Mathematics,
National Institute of Technology,
Patna, Bihar, India.
E-mail address: mda.ph21.ma@nitp.ac.in

and

Ayaz Ahmad,
Department of Mathematics,
National Institute of Technology,
Patna, Bihar, India.
E-mail address: ayaz@nitp.ac.in

and

Ayhan Esi,
Department of Basic Eng.Sci. (Math.Sect.),
Engineering and Natural Sciences Faculty, Malatya Turgut Ozal University,
44040, Malatya, Türkiye.
E-mail address: ayhan.esi@ozal.edu.tr, aesi23@hotmail.com