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Statistical Convergence for Uncertain Triple Sequences of Fuzzy Numbers

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ABSTRACT: This paper introduces the notion of statistical convergence for uncertain triple sequences of fuzzy numbers. We examine several associated types of convergence, including convergence in measure, in mean, in almost surely, and uniformly almost surely. Moreover, illustrative examples were provided to clarify the relationships and distinctions among these different types of convergence.

Keywords: Uncertainty space, fuzzy numbers, uncertain triple sequence, statistical convergence.

Contents

1. Introduction and Preliminaries

As a significant generalization of the classical notion of convergence for sequences of real numbers, the concept of statistical convergence for single sequences was independently introduced by Fast [8], Steinhaus [29], and Schoenberg [30], marking a pivotal advancement in the field. Since its introduction, this fundamental idea has stimulated extensive research and has evolved in numerous directions.

Zadeh [36] pioneered the concept of fuzzy numbers and examined their arithmetic operations. Since then, this notion has found extensive applications in diverse areas such as artificial intelligence, computer science, medicine, control engineering, decision theory, management science, operations research, pattern recognition, and robotics. Matloka [18] further extended this idea to summability theory and sequence spaces, while Nanda [19] utilized it in vector spaces and topology through fuzzy metrics.

The study of sequences of fuzzy numbers has also been developed in various directions, including the works of Altinok [1] and Çanak et al. [4]. Moreover, fuzzy numbers play an important role in the investigation of double sequence spaces, sequence spaces involving fuzzy mappings, approximation theory, and ideal convergence, among other areas.

We now present the definition of a fuzzy number.

Definition 1.1 *A fuzzy set $\mathcal{U}$ is a mapping $\mathcal{U} : \mathbb{R} \to [0,1]$ and is called a* fuzzy number *if the following conditions are satisfied:*

1. *There exists a point $u \in \mathbb{R}$ such that $\mathcal{U}(u) = 1$.*
2. *For any $m, n \in \mathbb{R}$ and $\lambda \in [0,1]$,*
$$\mathcal{U}(\lambda m + (1-\lambda)n) \geq \min\{\mathcal{U}(m), \mathcal{U}(n)\}.$$
3. *The function $\mathcal{U}$ is upper semicontinuous.*
4. *The closure of the set $\mathcal{U}^0 = \{u \in \mathbb{R} : \mathcal{U}(u) > 0\}$ is compact.*

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The collection of all fuzzy numbers defined on the real line will be denoted by $\mathcal{L}(\mathbb{R})$. In particular, each real number $t \in \mathbb{R}$ can be identified with a corresponding function $\overline{s}(u) \in \mathcal{L}(\mathbb{R})$ defined by

$$\overline{s}(u) = \begin{cases} 1, & \text{if } u = t, \\ 0, & \text{if } u \neq t. \end{cases}$$

For any $\alpha \in (0, 1]$, the α-level set (or α-cut) of a fuzzy number $\mathcal{U}$ is given by

$$[\mathcal{U}]_\alpha = \{\, u \in \mathbb{R} : \mathcal{U}(u) \geq \alpha \,\}.$$

Throughout this paper, the notation $\mathcal{L}(\mathbb{R})$ refers to the space of all fuzzy numbers on $\mathbb{R}$.

Let $\mathcal{U}$ and $\mathcal{V}$ be two fuzzy numbers. Their distance is defined by

$$d(\mathcal{U}, \mathcal{V}) = \sup_{0 \leq \alpha \leq 1} d_H\big([\mathcal{U}]_\alpha, [\mathcal{V}]_\alpha\big),$$

where d_H denotes the Hausdorff metric, given by

$$d_H\big([\mathcal{U}]_\alpha, [\mathcal{V}]_\alpha\big) = \max\left\{\big|[\mathcal{U}]^-_\alpha - [\mathcal{V}]^-_\alpha\big|,\ \big|[\mathcal{U}]^+_\alpha - [\mathcal{V}]^+_\alpha\big|\right\},$$

and $[\mathcal{U}]^-_\alpha$ and $[\mathcal{U}]^+_\alpha$ denote, respectively, the lower and upper endpoints of the α-level set of $\mathcal{U}$.

Equipped with the metric d, the space $\mathcal{L}(\mathbb{R})$ forms a complete metric space.

Following this primary notion of statistical convergence, the fuzzy analogue for triple sequences was introduced by Kumar et al. [21]. A triple sequence $\{\mathcal{Y}_{uvw}\}$ of fuzzy numbers is designated as statistically convergent to some fuzzy number $\mathcal{Y}_0$ provided that, for every $\varphi > 0$,

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left|\left\{u \leq p,\ v \leq q,\ w \leq r : d(\mathcal{Y}_{uvw}, \mathcal{Y}_0) \geq \varphi\right\}\right| = 0.$$

1.1. Research Gaps and Underlying Motivation

Uncertainty theory, originally introduced by Liu [17] within the framework of measure theory, has gained considerable attention due to its broad applicability across diverse fields such as probability theory, statistics, fuzzy set theory, measure theory, and summability theory. Its relevance extends to numerous practical and theoretical contexts, including risk assessment and uncertain reliability analysis [14], the modeling of human language through uncertainty logic [15], and the development of continuous uncertain measure theory [9].

In addition to these theoretical contributions, uncertainty theory has been effectively applied in various practical domains, notably uncertain finance [17], uncertain optimization [16], and several other areas where modeling and managing uncertainty play a central role. The convergence of sequences [26,32] plays a pivotal role in the fundamental theory of functional analysis, and in particular, in the study of sequence spaces. The theory of uncertainty was introduced by Liu [13] in 2007, after which numerous researchers explored various notions of convergence of sequences within the uncertainty space. In his foundational work, Liu [13] established four fundamental types of convergence for real uncertain sequences, namely convergence in mean, in measure, in distribution, and almost surely. Later, You [35] proposed a new concept of convergence, termed convergence with respect to uniformly almost surely, and investigated its interrelations with the existing convergence concepts. Chen et al. [3] further extended the theory to the setting of complex uncertain variables. Nath and Tripathy [23] examined convergence of complex uncertain sequences through the perspective of Orlicz functions. Tripathy and Dowari [34] introduced the notions of Nörlund and Riesz means for complex uncertain sequences and established several related results. Datta and Tripathy [5] studied convergence of double sequences involving complex uncertain variables. In addition, Tripathy et al. [33] initiated the study of statistical convergence for complex uncertain sequences and later examined statistical convergence via Orlicz functions [22]. Different types of convergence for complex uncertain sequences have been developed by several authors, including Roy et al. [27] and, Saini and Raj [31]. The concept of convergence for uncertain triple sequences has been extensively investigated by numerous researchers from various perspectives, including Das et al. [6], Nath et al. [20], Demirci and Gürdal [7], Huban and Gürdal [10], Kişi and Gürdal [11], among others.

More recently, Baliarsingh et al. [2] introduced the concept of statistical convergence for uncertain sequences of fuzzy numbers. This notion was subsequently extended to the double sequence setting by Kişi and Choudhury [12]. Furthermore, Nayak et al. [24] generalized the concept by employing deferred density. Later, Raj et al. [28] investigated lacunary statistical convergence via Orlicz functions.

To date, however, the study of statistical convergence and its generalizations for uncertain fuzzy triple sequences remains largely underdeveloped. A careful review of the existing literature indicates that the concept of statistical convergence for uncertain triple sequences of fuzzy numbers has not yet been explored. This gap highlights a promising avenue for further research in the context of uncertain triple sequences of fuzzy numbers using natural density.

1.2. Key Contributions

This study introduces several new variants of statistical convergence for uncertain triple sequences of fuzzy numbers. These include strong convergence, statistical boundedness, convergence almost surely, convergence in mean, convergence in measure, convergence in almost surely, and uniformly almost sure convergence. Furthermore, we establish several significant results associated with these notions, summarized as follows:

1. Examples 2.1 and 2.2 demonstrate the concept of statistical convergence for uncertain triple sequences of fuzzy numbers.

2. Strong convergence implies statistical convergence. The converse holds whenever the sequence is bounded (Theorem 2.1).

3. Convergence in mean implies convergence in measure in the context of statistical convergence (Theorem 2.2); however, the converse does not hold (Example 2.3).

4. Statistical convergence almost surely does not, in general, imply statistical convergence in measure (Example 2.4).

5. Convergence in measure does not necessarily imply convergence almost surely (Example 2.5).

6. Convergence almost surely does not necessarily imply convergence in mean (Example 2.6).

7. Some significant results (Theorems 2.3, 2.4, and 2.5) are established concerning almost sure and uniformly almost sure convergence within the framework of statistical convergence for uncertain triple sequences of fuzzy numbers.

Before proceeding to the main contributions, we present several fundamental definitions and preliminary results that will play a crucial role in the subsequent sections of this study.

Definition 1.2 *[13] Let $\mathscr{L}$ be a σ-algebra on a non-empty set $\mathcal{X}$. A set function $\mathcal{M}$ on $\mathscr{L}$ is called an* uncertain measure *if it satisfies the following axioms:*

Axiom 1 (Normality). $\mathcal{M}(\mathcal{X}) = 1$.

Axiom 2 (Duality). *For any $\mathcal{G} \in \mathscr{L}$,*

$$\mathcal{M}(\mathcal{G}) + \mathcal{M}(\mathcal{G}^c) = 1.$$

Axiom 3 (Subadditivity). *For every countable collection $\{\mathcal{G}_j\}_{j=1}^{\infty} \subset \mathscr{L}$,*

$$\mathcal{M}\left(\bigcup_{j=1}^{\infty} \mathcal{G}_j\right) \leq \sum_{j=1}^{\infty} \mathcal{M}(\mathcal{G}_j).$$

The triplet $(\mathcal{X}, \mathscr{L}, \mathcal{M})$ is called an uncertainty space*, and each $\mathcal{G} \in \mathscr{L}$ is referred to as an* event*. To obtain the uncertain measure of compound events, a product uncertain measure is defined via the following axiom:*

Axiom 4 (Product Axiom). *Let $\{(\mathcal{X}_k, \mathscr{L}_k, \mathcal{M}_k)\}_{k=1}^{\infty}$ be a sequence of uncertainty spaces. The product uncertain measure $\mathcal{M}$ is the uncertain measure satisfying*

$$\mathcal{M}\left(\prod_{k=1}^{\infty} \mathcal{G}_k\right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\mathcal{G}_k),$$

where each $\mathcal{G}_k$ is an arbitrarily chosen event from $\mathscr{L}_k$.

Definition 1.3 *[13] An* uncertain variable *ξ is a measurable function from an uncertainty space $(\mathcal{X}, \mathscr{L}, \mathcal{M})$ to the set of real numbers. That is, for every Borel set $\mathcal{B} \subset \mathbb{R}$, the set*

$$\{\xi \in \mathcal{B}\} = \{\kappa \in \mathcal{X} : \xi(\kappa) \in \mathcal{B}\}$$

is an event in $\mathscr{L}$.

Definition 1.4 *[25] An* uncertain variable *ξ is a measurable function from an uncertainty space $(\mathcal{X}, \mathscr{L}, \mathcal{M})$ to the set of complex numbers. That is, for every Borel set $\mathcal{B} \subset \mathbb{C}$, the set*

$$\{\xi \in \mathcal{B}\} = \{\kappa \in \mathcal{X} : \xi(\kappa) \in \mathcal{B}\}$$

is an event in $\mathscr{L}$.

Definition 1.5 *[33] A complex uncertain sequence $\{\xi_s\}$ is said to be* statistically convergent almost surely *to ξ if for every $\varphi > 0$ there exists an event $\mathcal{G}$ with $\mathcal{M}(\mathcal{G}) = 1$ such that*

$$\lim_{p\to\infty} \frac{1}{p}\Big|\{\, s \le p : \|\xi_s(\kappa) - \xi(\kappa)\| \ge \varphi \,\}\Big| = 0, \qquad \text{for every } \kappa \in \mathcal{G}.$$

Definition 1.6 *[33] A complex uncertain sequence $\{\xi_s\}$ is said to be* statistically convergent in measure *to ξ if for every $\lambda,\ \varphi > 0$, the following holds:*

$$\lim_{p\to\infty} \frac{1}{p}\Big|\{\, s \le p : \mathcal{M}\big(\|\xi_s - \xi\| \ge \lambda\big) \ge \varphi \,\}\Big| = 0.$$

Definition 1.7 *[33] A complex uncertain sequence $\{\xi_s\}$ is said to be* statistically convergent in mean *to ξ if for every $\varphi > 0$, the following holds:*

$$\lim_{p\to\infty} \frac{1}{p}\Big|\{\, s \le p : \mathbf{E}\big(\|\xi_s - \xi\|\big) \ge \varphi \,\}\Big| = 0.$$

Definition 1.8 *[33] Let $\Phi, \Phi_1, \Phi_2, \ldots$ be the complex uncertainty distributions of the complex uncertain variables $\xi, \xi_1, \xi_2, \ldots$, respectively. The complex uncertain sequence $\{\xi_s\}$ is said to be* statistically convergent in distribution *to ξ if for every $\varphi > 0$,*

$$\lim_{p\to\infty} \frac{1}{p}\big|\{\, s \le p : |\Phi_s(\kappa) - \Phi(\kappa)| \ge \varphi \,\}\big| = 0,$$

for all points κ at which Φ is continuous.

Definition 1.9 ([25]) *Let $\xi = \zeta + i\vartheta$ be a complex uncertain variable. If the expected values $\mathbf{E}(\zeta)$ and $\mathbf{E}(\vartheta)$ exist, then the expected value of ξ is defined as*

$$\mathbf{E}(\xi) = \mathbf{E}(\zeta) + i\,\mathbf{E}(\vartheta).$$

2. Main Results

In this section, we present our main results.

Definition 2.1 *Let $\{\mathcal{Y}_{uvw}\}$ be an uncertain triple sequence of fuzzy numbers. We say that the sequence is statistically convergent to $\mathcal{Y}_0$ if, for every $\varphi > 0$, there exists an event $\mathcal{K}$ such that, for every $\zeta \in \mathcal{K}$, the condition*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \le p,\ v \le q,\ w \le r : d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \ge \varphi \Big\} \right| = 0$$

is satisfied.

Definition 2.2 *Let $\{\mathcal{Y}_{uvw}\}$ be an uncertain triple sequence of fuzzy numbers. The sequence is said to be statistically bounded if, for every $\zeta \in \mathcal{K}$, there exists a real number $\mathcal{N}$ such that*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \le p,\ v \le q,\ w \le r : d\big(\mathcal{Y}_{uvw}(\zeta),\, \tilde{0}\big) \ge \mathcal{N} \Big\} \right| = 0.$$

Definition 2.3 *Let $\{\mathcal{Y}_{uvw}\}$ be an uncertain triple sequence of fuzzy numbers. We say that the sequence is statistically convergent almost surely to $\mathcal{Y}_0$ if, for every $\varphi > 0$, there exists an event $\mathcal{K}$ with $\mathcal{M}(\mathcal{K}) = 1$ such that, for every $\zeta \in \mathcal{K}$,*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \le p,\ v \le q,\ w \le r : d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \ge \varphi \Big\} \right| = 0.$$

Definition 2.4 *Let $\{\mathcal{Y}_{uvw}\}$ be an uncertain triple sequence of fuzzy numbers. We say that the sequence is statistically convergent in measure to $\mathcal{Y}_0$ if, for every $\varphi, \lambda > 0$, the condition*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \le p,\ v \le q,\ w \le r : \mathcal{M}\Big(\{\zeta \in \mathcal{K} : d\big(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)\big) \ge \varphi\} \Big) \ge \lambda \Big\} \right| = 0.$$

Definition 2.5 *Let $\{\mathcal{Y}_{uvw}\}$ be an uncertain triple sequence of fuzzy numbers. We say that the sequence is statistically convergent in mean to $\mathcal{Y}_0$ if, for every $\varphi > 0$, the condition*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \le p,\ v \le q,\ w \le r : \mathbf{E}[d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta))] \ge \varphi \Big\} \right| = 0,$$

for every $\zeta \in \mathcal{K}$.

Definition 2.6 *Let Φ_0 and Φ_{uvw} denote the uncertainty distributions of the uncertain variables $\mathcal{Y}_0$ and $\mathcal{Y}_{uvw}$, respectively. We say that the sequence $\{\mathcal{Y}_{uvw}\}$ is statistically convergent in distribution to $\mathcal{Y}_0$ if, for every $\varphi > 0$, the condition*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \le p,\ v \le q,\ w \le r : d\big(\Phi_{uvw}(g),\, \Phi_0(g)\big) \ge \varphi \Big\} \right| = 0$$

is satisfied for all points g at which $\Phi_0(g)$ is continuous.

Definition 2.7 *Let $\{\mathcal{Y}_{uvw}\}$ be an uncertain triple sequence of fuzzy numbers. We say that the sequence is statistically convergent uniformly almost surely to $\mathcal{Y}_0$ if, for every $\varphi > 0$, there exists a sequence $\{E_u\}$ of events with $\mathcal{M}(E_u) \to 0$ such that*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \le p,\ v \le q,\ w \le r : d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \ge \varphi \Big\} \right| = 0,$$

for every $\zeta \in \mathcal{X} \setminus E_u$.

Definition 2.8 *Let $\{\mathcal{Y}_{uvw}\}$ be an uncertain triple sequence of fuzzy numbers. We say that the sequence is strongly convergent to $\mathcal{Y}_0$ if, for each $\zeta \in \mathcal{K}$,*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \sum_{u=1}^{p}\sum_{v=1}^{q}\sum_{w=1}^{r} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) = 0.$$

We now present a collection of examples that illustrate the concept of statistical convergence for uncertain triple sequences of fuzzy numbers.

Example 2.1 Consider an event $\zeta \in \mathcal{K}$ and the corresponding uncertain fuzzy triple sequence $\{\mathcal{Y}_{uvw}\}$ defined by

$$\mathcal{Y}_{uvw}(\zeta) = \begin{cases} \frac{1}{2}, & \text{if } \;\; u = a^2,\; v = b^2,\; w = c^2, \\ \mathcal{Y}_0(\zeta), & \text{otherwise.} \end{cases}$$

Then, for every $\varphi > 0$ and each $\zeta \in \mathcal{K}$, we have

$$\begin{aligned} &\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \le p,\; v \le q,\; w \le r : d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \ge \varphi \right\} \right| \\ &\le \lim_{p,q,r\to\infty} \frac{\sqrt{pqr}}{(p+1)(q+1)(r+1)} \\ &\le \lim_{p,q,r\to\infty} \frac{\sqrt{pqr}}{pqr} = 0. \end{aligned}$$

Hence, the sequence $\{\mathcal{Y}_{uvw}\}$ is statistically convergent to $\mathcal{Y}_0$.

Example 2.2 Consider an event $\zeta \in \mathcal{K}$ and the corresponding uncertain fuzzy triple sequence $\{\mathcal{Y}_{uvw}\}$ defined by

$$\mathcal{Y}_{uvw}(\zeta) = \begin{cases} \mathcal{Y}_0(\zeta), & \text{if } \;\; u = a^2,\; v = b^2,\; w = c^2,\; (a,b,c \ne 1), \\ \frac{1}{4}, & \text{if } \;\; u = a^3,\; v = b^3,\; w = c^3, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for every $\varphi > 0$ and each $\zeta \in \mathcal{K}$, we obtain

$$\begin{aligned} &\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \le p,\; v \le q,\; w \le r : d\big(\mathcal{Y}_{uvw}(\zeta),\, \tilde{0}\big) \ge \varphi \right\} \right| \\ &\le \lim_{p,q,r\to\infty} \frac{\sqrt{pqr} + \sqrt[3]{pqr}}{(p+1)(q+1)(r+1)} \\ &\le \lim_{p,q,r\to\infty} \frac{\sqrt{pqr} + \sqrt[3]{pqr}}{pqr} = 0. \end{aligned}$$

Therefore, the sequence $\{\mathcal{Y}_{uvw}\}$ is statistically convergent to 0.

In the following theorem, we explore the interplay between strong convergence and statistical convergence for uncertain triple sequences of fuzzy numbers.

Theorem 2.1 *Let $\{\mathcal{Y}_{uvw}\}$ be an uncertain triple sequence of fuzzy numbers that is strongly convergent to $\mathcal{Y}_0$. Then it is also statistically convergent to $\mathcal{Y}_0$. Conversely, the implication holds whenever the sequence is bounded.*

Proof: Suppose that the sequence $\{\mathcal{Y}_{uvw}\}$ is strongly convergent to $\mathcal{Y}_0$. Then there exists an event $\mathcal{K}$ such that, for every $\zeta \in \mathcal{K}$,

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \sum_{u=1}^{p}\sum_{v=1}^{q}\sum_{w=1}^{r} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) = 0. \tag{2.1}$$

For a fixed $\varphi > 0$, define

$$\mathcal{C}_{pqr}(\varphi) = \Big\{ u \le p,\ v \le q,\ w \le r : d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \ge \varphi \Big\}.$$

Then

$$\begin{aligned}
&\frac{1}{(p+1)(q+1)(r+1)} \sum_{u=1}^{p}\sum_{v=1}^{q}\sum_{w=1}^{r} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \\
&= \frac{1}{(p+1)(q+1)(r+1)} \left[\sum_{(u,v,w)\in\mathcal{C}_{pqr}(\varphi)} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) + \sum_{(u,v,w)\notin\mathcal{C}_{pqr}(\varphi)} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \right] \\
&\ge \frac{1}{(p+1)(q+1)(r+1)} \sum_{(u,v,w)\in\mathcal{C}_{pqr}(\varphi)} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \\
&\ge \frac{1}{(p+1)(q+1)(r+1)} \Big|\mathcal{C}_{pqr}(\varphi)\Big|.
\end{aligned}$$

Letting $p, q, r \to \infty$ and using (2.1), we obtain

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \big|\mathcal{C}_{pqr}(\varphi)\big| = 0,$$

showing that $\{\mathcal{Y}_{uvw}\}$ is statistically convergent to $\mathcal{Y}_0$.

Conversely, assume that the sequence $\{\mathcal{Y}_{uvw}\}$ is bounded and statistically convergent to $\mathcal{Y}_0$. Then there exists an event $\mathcal{K}$ such that, for every $\zeta \in \mathcal{K}$, there is a constant $\mathcal{N}_0 > 0$ satisfying

$$\sup_{u,v,w} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \le \mathcal{N}_0 < \infty.$$

Also,

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \big|\mathcal{C}_{pqr}(\varphi)\big| = 0. \tag{2.2}$$

For each $\zeta \in \mathcal{K}$, we compute

$$\begin{aligned}
&\frac{1}{(p+1)(q+1)(r+1)} \sum_{u=1}^{p}\sum_{v=1}^{q}\sum_{w=1}^{r} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \\
&= \frac{1}{(p+1)(q+1)(r+1)} \left[\sum_{(u,v,w)\notin\mathcal{C}_{pqr}(\varphi)} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) + \sum_{(u,v,w)\in\mathcal{C}_{pqr}(\varphi)} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \right] \\
&\le \frac{1}{(p+1)(q+1)(r+1)} \left[\varphi\mathcal{N}_0 \sum_{(u,v,w)\notin\mathcal{C}_{pqr}(\varphi)} 1 + \sum_{(u,v,w)\in\mathcal{C}_{pqr}(\varphi)} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \right].
\end{aligned}$$

Letting $p, q, r \to \infty$ and using (2.2), the right-hand side tends to 0, which proves that $\{\mathcal{Y}_{uvw}\}$ is strongly convergent to $\mathcal{Y}_0$. □

The following result shows that, within the framework of statistical convergence, convergence in mean necessarily implies convergence in measure for uncertain triple sequences of fuzzy numbers.

Theorem 2.2 *Let $\{\mathcal{Y}_{uvw}\}$ be an uncertain triple sequence of fuzzy numbers. If the sequence is statistically convergent in mean to $\mathcal{Y}_0$, then it is also statistically convergent in measure to the same limit $\mathcal{Y}_0$.*

Proof: Suppose that the sequence $\{\mathcal{Y}_{uvw}\}$ is statistically convergent in mean to $\mathcal{Y}_0$. Then, for every $\varphi > 0$, the following condition holds:

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \le p,\ v \le q,\ w \le r : \mathbf{E}[d(\mathcal{Y}_{uvw}(\zeta),\ \mathcal{Y}_0(\zeta))] \ge \varphi \right\} \right| = 0, \tag{2.3}$$

for every $\zeta \in \mathcal{K}$.

Applying Markov's inequality, for any $\lambda > 0$ and $\varphi > 0$, we obtain

$$\begin{aligned} &\Big\{ u \le p,\ v \le q,\ w \le r : \mathcal{M}\Big(\{ \zeta \in \mathcal{K} : d\big(\mathcal{Y}_{uvw}(\zeta),\ \mathcal{Y}_0(\zeta)\big) \ge \varphi \} \Big) \ge \lambda \Big\} \\ &\subseteq \left\{ u \le p,\ v \le q,\ w \le r : \left(\frac{\mathbf{E}[d(\mathcal{Y}_{uvw}(\zeta),\ \mathcal{Y}_0(\zeta))]}{\varphi} \right) \ge \lambda \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \le p,\ v \le q,\ w \le r : \mathcal{M}\Big(\{ \zeta \in \mathcal{K} : d(\mathcal{Y}_{uvw}(\zeta),\ \mathcal{Y}_0(\zeta)) \ge \varphi \} \Big) \ge \lambda \Big\} \right| \\ &\le \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \le p,\ v \le q,\ w \le r : \left(\frac{\mathbf{E}[d(\mathcal{Y}_{uvw}(\zeta),\ \mathcal{Y}_0(\zeta))]}{\varphi} \right) \ge \lambda \right\} \right|. \end{aligned}$$

Now, invoking (2.3), we conclude that the sequence $\{\mathcal{Y}_{uvw}\}$ is statistically convergent in measure to $\mathcal{Y}_0$. □

Remark 2.1 The converse of Theorem 2.2 does not necessarily hold. This fact is illustrated by the following example.

Example 2.3 Consider the uncertainty space $(\mathcal{X}, \mathcal{L}, \mathcal{M})_{\mathcal{Y}} = \{\zeta_1, \zeta_2, \ldots\}$, where

$$\mathcal{C}_1(\zeta) = \sup_{\zeta_{u+v+w} \in \mathcal{K}} \frac{1}{u+v+w+1}, \qquad \mathcal{C}_2(\zeta) = \sup_{\zeta_{u+v+w} \in \mathcal{K}^c} \frac{1}{u+v+w+1},$$

and the measure $\mathcal{M}$ is defined by

$$\mathcal{M}(\mathcal{K}) = \begin{cases} \mathcal{C}_1(\zeta), & \text{if } \mathcal{C}_1(\zeta) < \frac{1}{2}, \\ 1 - \mathcal{C}_2(\zeta), & \text{if } \mathcal{C}_2(\zeta) < \frac{1}{2}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

On this space, consider the uncertain triple sequence defined by

$$\mathcal{Y}_{uvw}(\zeta) = (u+v+w+1)\,\mathfrak{D}(\zeta, \zeta_{u+v+w}), \qquad (u, v, w = 1, 2, \ldots),$$

and let $\mathcal{Y}_0 \equiv \tilde{0}$. Here, $\mathfrak{D}(\zeta, \zeta_{u+v+w})$ denotes the Kronecker delta function:

$$\mathfrak{D}(a, b) = \begin{cases} 1, & a = b, \\ 0, & \text{otherwise.} \end{cases}$$

For given $\varphi, \lambda > 0$ and $u, v, w \geq 2$, we obtain

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \leq p,\ v \leq q,\ w \leq r : \mathcal{M}\Big(\{ \zeta \in \mathcal{K} : d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \geq \varphi \} \Big) \geq \lambda \right\} \right|$$
$$= 0.$$

Thus, the uncertain triple sequence $\{\mathcal{Y}_{uvw}\}$ is statistically convergent in measure to $\tilde{0}$.

Next, the uncertainty distribution of $d(\mathcal{Y}_{uvw}, \tilde{0})$ for each $u, v, w \geq 2$ is given by

$$\Phi_{uvw}(g) = \begin{cases} 1, & g \geq u+v+w, \\ 1 - \dfrac{1}{u+v+w}, & 0 \leq g < u+v+w, \\ 0, & \text{otherwise.} \end{cases}$$

For each $u, v, w \geq 2$, we compute

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \leq p,\ v \leq q,\ w \leq r : \mathbf{E}[d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) - 1] \right\} \right|$$
$$= \lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left[\int_0^{u+v+w+1} \big[1 - \Phi_{uvw}(g)\big]\, dg - 1 \right]$$
$$= \lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left[\int_0^{u+v+w+1} \left(\frac{1}{u+v+w} \right) dg - 1 \right] = 0.$$

Hence, although the sequence is statistically convergent in measure to $\tilde{0}$, it is not statistically convergent in mean to $\tilde{0}$.

Remark 2.2 It is worth noting that an uncertain triple sequence $\{\mathcal{Y}_{uvw}\}$ of fuzzy numbers that is statistically convergent almost surely does not necessarily imply statistical convergence in measure.

The subtlety highlighted above is further illustrated through the following example.

Example 2.4 Consider the uncertainty space $(\mathcal{X}, \mathcal{L}, \mathcal{M})_{\mathcal{Y}} = \{\zeta_1, \zeta_2, \ldots\}$, where

$$\mathcal{C}_1(\zeta) = \sup_{\zeta_{u+v+w} \in \mathcal{K}} \frac{u+v+w}{2(u+v+w)+1}, \qquad \mathcal{C}_2(\zeta) = \sup_{\zeta_{u+v+w} \in \mathcal{K}^c} \frac{u+v+w}{2(u+v+w)+1},$$

and the measure $\mathcal{M}$ is defined by

$$\mathcal{M}(\mathcal{K}) = \begin{cases} \mathcal{C}_1(\zeta), & \text{if } \mathcal{C}_1(\zeta) < \dfrac{1}{2}, \\ 1 - \mathcal{C}_2(\zeta), & \text{if } \mathcal{C}_2(\zeta) < \dfrac{1}{2}, \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases}$$

Define the uncertain variables by

$$\mathcal{Y}_{uvw}(\zeta) = \begin{cases} u+v+w, & \text{if } \zeta = \zeta_{u+v+w}, \\ 1, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mathcal{Y}_0 \equiv \tilde{0}.$$

It is straightforward to verify that the sequence $\{\mathcal{Y}_{uvw}\}$ is statistically convergent almost surely to $\mathcal{Y}_0$. However, note that

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \leq p,\ v \leq q,\ w \leq r : \mathcal{M}\Big(\{ \zeta \in \mathcal{K} : d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \geq \varphi \} \Big) \geq \tfrac{1}{2} \right\} \right|$$
$$= \lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \leq p,\ v \leq q,\ w \leq r : \mathcal{M}(\zeta_{u+v+w}) \geq \tfrac{1}{2} \right\} \right| = \frac{1}{2}.$$

Thus, the sequence fails to be statistically convergent in measure to $\mathcal{Y}_0$, even though it is statistically convergent almost surely.

Remark 2.3 It is worth noting that an uncertain triple sequence $\{\mathcal{Y}_{uvw}\}$ of fuzzy numbers that is statistically convergent in measure does not necessarily imply statistical convergence almost surely.

An illustrative example supporting the above observation is presented below.

Example 2.5 Let $(\mathcal{X}, \mathscr{L}, \mathcal{M})_{\mathcal{Y}}$ to be $[0,1]$ the uncertainty space endowed with the Borel algebra and the Lebesgue measure. Choose integers $a, b, c \in \mathbb{N}$ and define

$$u = 2^a + K, \quad v = 2^b + K, \quad w = 2^c + K,$$

where K is an integer satisfying

$$0 \leq K \leq \min\{2^a, 2^b, 2^c\} - 1.$$

For each $u, v, w \in \mathbb{N}$, define the uncertain variable $\mathcal{Y}_{uvw}$ by

$$\mathcal{Y}_{uvw}(\zeta) = \begin{cases} 1, & \text{if } \dfrac{K}{2^{a+b+c}} \leq \zeta \leq \dfrac{K+1}{2^{a+b+c}}, \\ 0, & \text{otherwise}, \end{cases}$$

and let $\mathcal{Y}_0 \equiv \tilde{0}$.

For arbitrary $\varphi, \lambda > 0$ and $u, v, w \geq 2$, we obtain

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \leq p,\ v \leq q,\ w \leq r : \mathcal{M}\Big(\{\zeta \in \mathcal{K} : d(\mathcal{Y}_{uvw}(\zeta), \mathcal{Y}_0(\zeta)) \geq \varphi\}\Big) \geq \lambda \right\} \right|$$

$$= \lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \leq p,\ v \leq q,\ w \leq r : \mathcal{M}\big(\mathcal{Y}_{uvw}\big) \geq \lambda \right\} \right| = \frac{1}{2^{a+b+c}},$$

which tends to 0 as $a, b, c \to \infty$. Hence, the sequence $\{\mathcal{Y}_{uvw}\}$ is statistically convergent in measure to $\mathcal{Y}_0$.

Furthermore, for any fixed $\varphi > 0$, it follows that

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u \leq p,\ v \leq q,\ w \leq r : \mathbf{E}[d(\mathcal{Y}_{uvw}(\zeta), \mathcal{Y}_0(\zeta))] \geq \varphi \right\} \right| = 0.$$

This shows that $\{\mathcal{Y}_{uvw}\}$ is also statistically convergent in mean to $\mathcal{Y}_0$.

However, for any $\zeta \in [0,1]$, there exist infinitely many intervals of the form

$$\left[\frac{K}{2^{a+b+c}}, \frac{K+1}{2^{a+b+c}} \right]$$

that contains ζ. Consequently, the sequence $\{\mathcal{Y}_{uvw}\}$ does not exhibit statistically convergent almost surely to $\mathcal{Y}_0$.

Remark 2.4 It is worth emphasizing that statistical almost sure convergence of an uncertain triple sequence $\{\mathcal{Y}_{uvw}\}$ of fuzzy numbers to $\mathcal{Y}_0$ does not, in general, guarantee statistical convergence in mean to the same limit $\mathcal{Y}_0$.

The following example illustrates and substantiates the above observation.

Example 2.6 Consider the uncertainty space $(\mathcal{X}, \mathscr{L}, \mathcal{M})_{\mathcal{Y}} = \{\zeta_1, \zeta_2, \ldots\}$, where the uncertain measure is defined by

$$\mathcal{M}(\mathcal{K}) = \sum_{\zeta_{u+v+w} \in \mathcal{K}} \frac{1}{2^{\,u+v+w}}.$$

Define an uncertain triple sequence $\{\mathcal{Y}_{uvw}\}$ by

$$\mathcal{Y}_{uvw}(\zeta) = 2^{\,u+v+w}\, \mathfrak{D}\big(\zeta, \zeta_{u+v+w}\big),$$

and let $\mathcal{Y}_0 \equiv \tilde{0}$ denote the zero fuzzy number. It can be easily verified that the sequence $\{\mathcal{Y}_{uvw}\}$ is statistically convergent almost surely to $\mathcal{Y}_0$.

Observe that the uncertainty distribution of $\|\mathcal{Y}_{uvw}\|$ is given by

$$\mathcal{Y}_{uvw}(\zeta) = \begin{cases} 1, & \text{if } \zeta = \zeta_{u+v+w}, \\ 0, & \text{otherwise,} \end{cases}$$

and the corresponding distribution function takes the form

$$\Phi_{uvw}(g) = \begin{cases} 1, & \text{if } g \geq 2^{\,u+v+w}, \\ 1 - \dfrac{1}{2^{\,u+v+w}}, & \text{if } 0 \leq g < 2^{\,u+v+w}, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \leq p,\ v \leq q,\ w \leq r : \mathbf{E}[d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0)] \geq 1 \Big\} \right| = 0.$$

Hence, the uncertain triple sequence $\{\mathcal{Y}_{uvw}\}$ fails to be statistically convergent in mean to $\mathcal{Y}_0$.

In the sequel, we establish several results concerning almost sure and uniformly almost sure convergence in the setting of statistical convergence for uncertain triple sequences of fuzzy numbers.

Theorem 2.3 *An uncertain triple sequence $\{\mathcal{Y}_{pqr}\}$ is statistically convergent almost surely to $\mathcal{Y}_0$ if, for every $\varphi > 0$ and $\lambda > 0$, the following condition holds:*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ {\scriptstyle u\leq p,\, v\leq q,\, w\leq r:} \mathcal{M}\left(\bigcap_{u=v=w=1}^{\infty} \bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} \{d(\mathcal{Y}_{uvw}, \mathcal{Y}_0) \geq \varphi\} \right) \geq \lambda \right\} \right| = 0.$$

Proof: Assume that the uncertain triple sequence $\{\mathcal{Y}_{pqr}\}$ of fuzzy numbers is statistically convergent almost surely to $\mathcal{Y}_0$. Then, for every $\varphi > 0$, there exists an event $\mathcal{K}$ with $\mathcal{M}(\mathcal{K}) = 1$ such that, for each $\zeta \in \mathcal{K}$,

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \Big\{ u \leq p,\ v \leq q,\ w \leq r : d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) \geq \varphi \Big\} \right| = 0.$$

This implies that, for any $\varphi > 0$ and $\zeta \in \mathcal{K}$, there are $u, v, w \in \mathbb{N}$ such that for all $u \leq p,\ v \leq q,\ w \leq r$,

$$d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) < \varphi.$$

Equivalently,

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ {\scriptstyle u\leq p,\, v\leq q,\, w\leq r:} \mathcal{M}\left(\left\{ \zeta : \bigcap_{u=v=w=1}^{\infty} \bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw}(\zeta),\, \mathcal{Y}_0(\zeta)) < \varphi \right\} \right) \geq 1 \right\} \right| = 0.$$

Consequently, we may write

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ {\scriptstyle u\leq p,\, v\leq q,\, w\leq r:} \mathcal{M}\left(\bigcap_{u=v=w=1}^{\infty} \bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) < \varphi \right) \geq 1 \right\} \right| = 0.$$

Finally, by invoking the duality axiom of the uncertain measure, it follows that for every $\lambda > 0$,

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u\leq p,\, v\leq q,\, w\leq r : \mathcal{M}\left(\bigcap_{u=v=w=1}^{\infty} \bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) \geq \varphi \right) \geq \lambda \right\} \right| = 0.$$

□

Theorem 2.4 *An uncertain triple sequence $\{\mathcal{Y}_{pqr}\}$ of fuzzy numbers is statistically convergent uniformly almost surely to $\mathcal{Y}_0$ if and only if, for any $\varphi > 0$ and $\lambda > 0$, the following condition is satisfied:*

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u\leq p,\, v\leq q,\, w\leq r : \mathcal{M}\left(\bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) \geq \varphi \right) \geq \lambda \right\} \right| = 0.$$

Proof: Assume that the uncertain triple sequence $\{\mathcal{Y}_{pqr}\}$ is statistically convergent uniformly almost surely to $\mathcal{Y}_0$. Then, for a given $\lambda > 0$, there exists a measurable set $\mathcal{C}$ such that $\mathcal{M}(\mathcal{C}) < \lambda$ and the sequence converges statistically uniformly almost surely to $\mathcal{Y}_0$ on $\mathcal{X} \setminus \mathcal{C}$. By definition, for every $\varphi > 0$, there exist indices $u \leq p$, $v \leq q$, and $w \leq r$ such that

$$d(\mathcal{Y}_{pqr},\, \mathcal{Y}_0) < \varphi \quad \text{for all } \zeta \in \mathcal{X} \setminus \mathcal{C}.$$

Consequently,

$$\bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} \{d(\mathcal{Y}_{pqr},\, \mathcal{Y}_0) < \varphi\} \subset \mathcal{C}.$$

Using the subadditivity axiom of the uncertain measure, we obtain

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u\leq p,\, v\leq q,\, w\leq r : \mathcal{M}\left(\bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) \geq \varphi \right) \right\} \right| \leq \lambda\, \mathcal{M}(\mathcal{C}) < \lambda.$$

This establishes the first part.

Conversely, let $\varphi, \lambda > 0$ be such that

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u\leq p,\, v\leq q,\, w\leq r : \mathcal{M}\left(\bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) \geq \varphi \right) \geq \lambda \right\} \right| = 0.$$

For each $m \in \mathbb{N}$, there exist indices m_u, m_v, m_w such that

$$\delta\left((u,v,w) \in \mathbb{N}^3 : \mathcal{M}\left(\bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) \geq \frac{1}{m} \right) \right) < \frac{\lambda}{2^m}.$$

Define the set

$$\mathcal{C} = \bigcup_{m=1}^{\infty} \bigcup_{p=m_u}^{\infty} \bigcup_{q=m_v}^{\infty} \bigcup_{r=m_w}^{\infty} \left\{ d(\mathcal{Y}_{pqr},\, \mathcal{Y}_0) \geq \frac{1}{m} \right\}.$$

By subadditivity,

$$\mathcal{M}(\mathcal{C}) \leq \sum_{m=1}^{\infty} \mathcal{M}\left(\bigcup_{p=m_u}^{\infty} \bigcup_{q=m_v}^{\infty} \bigcup_{r=m_w}^{\infty} d(\mathcal{Y}_{pqr},\, \mathcal{Y}_0) \geq \frac{1}{m} \right) \leq \sum_{m=1}^{\infty} \frac{\lambda}{2^m} = \lambda.$$

Hence, for all $m \in \mathbb{N}$ and for $p > m_u$, $q > m_v$, and $r > m_w$, we have

$$\sup_{\zeta \in \mathcal{X}\setminus\mathcal{C}} d(\mathcal{Y}_{pqr},\, \mathcal{Y}_0) < \frac{1}{m}.$$

Therefore, $\{\mathcal{Y}_{pqr}\}$ is statistically convergent uniformly almost surely to $\mathcal{Y}_0$, which completes the proof. □

Theorem 2.5 *Let $\{\mathcal{Y}_{pqr}\}$ be an uncertain triple sequence of fuzzy numbers that is statistically convergent uniformly almost surely to $\mathcal{Y}_0$. Then $\{\mathcal{Y}_{pqr}\}$ is also statistically convergent almost surely to $\mathcal{Y}_0$.*

Proof: Assume that the uncertain triple sequence $\{\mathcal{Y}_{pqr}\}$ is statistically convergent uniformly almost surely to $\mathcal{Y}_0$. Then, by Theorem 2.4, for every $\varphi > 0$ and $\lambda > 0$, we have

$$\lim_{p,q,r\to\infty} \frac{1}{(p+1)(q+1)(r+1)} \left| \left\{ u\le p,\, v\le q,\, w\le r : \mathcal{M}\left(\bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) \ge \varphi \right) \ge \lambda \right\} \right| = 0.$$

Moreover, for every choice of indices (u, v, w), the uncertain measure yields

$$\mathcal{M}\left(\bigcap_{u=v=w=1}^{\infty} \bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) \ge \varphi \right) \le \mathcal{M}\left(\bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) \ge \varphi \right).$$

Taking asymptotic densities on both sides and passing to the limit, we obtain

$$\delta\left(\mathcal{M}\left(\bigcap_{u=v=w=1}^{\infty} \bigcup_{p=u}^{\infty} \bigcup_{q=v}^{\infty} \bigcup_{r=w}^{\infty} d(\mathcal{Y}_{uvw},\, \mathcal{Y}_0) \ge \varphi \right) \right) = 0.$$

Therefore, by Theorem 2.3, the sequence $\{\mathcal{Y}_{pqr}\}$ is statistically convergent almost surely to $\mathcal{Y}_0$. □

3. Conclusion and Future Scope

In this work, we explored several new variants of statistical convergence for uncertain triple sequences of fuzzy numbers including strong convergence, boundedness, convergence almost surely, convergence in mean, convergence in measure, and uniformly almost sure convergence. By examining these notions collectively, we developed a comprehensive understanding of how different types of convergence interact in uncertain and fuzzy environments.

Our results establish a clear hierarchy among these convergence types. Strong convergence is shown to guarantee statistical convergence, while convergence in mean ensures convergence in measure under uncertain triple sequences of fuzzy numbers. Additional findings reveal that statistical convergence almost surely does not necessarily imply convergence in measure, and conversely, convergence in measure does not guarantee almost sure convergence. Similarly, convergence almost surely does not necessarily imply convergence in mean. The study also provides deeper insights into almost sure and uniformly almost sure convergence in the statistical setting.

This study can be extended by examining multiple sequences of fuzzy numbers and by developing the works based on deferred density for triple sequences.

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