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# On Some Developments in Positively Bipolar Soft Groups

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ABSTRACT: In this work, we define the concepts of positively bipolar left and right cosets and examine their fundamental properties and structures. Additionally, we define quotient bipolar soft groups. In addition, we also define positively maximal normal bipolar soft groups and the notions of positively simple bipolar soft groups. Moreover, we establish the definitions of positively solvable bipolar soft groups, and we prove several important results.

Keywords: Bipolar soft sets, positively bipolar soft groups, positively bipolar soft cosets, quotient bipolar soft groups, positively maximal normal bipolar soft groups, positively simple bipolar soft groups and positively solvable bipolar soft groups.

## Contents

## 1. Introduction

Soft set theory, first introduced by Molodtsov [18] in 1999, provides a mathematical framework for handling uncertainty and imprecision. Its foundation is based on fuzzy set theory, introduced by Zadeh [24] in 1965. Subsequent works [1,2,5,7,15,20,21,22] developed various operations and established fundamental structural properties of soft sets, enabling researchers to analyze and manipulate soft information systematically. Bipolar soft sets, introduced by Shabir and Naz [23], extended soft set theory by representing both positive and negative information simultaneously. This concept, building on the combination of soft sets with bipolar fuzzy sets as initially discussed by Zhang [25,26] and further developed by Aslam et al. [4], provides a versatile framework for algebraic analysis. Studies by Naz and Shabir [19] explored fuzzy BS-sets and their algebraic properties, while Karaaslan and Karataş [12] refined BS-sets through an extended treatment of the "not set" of parameters. Additional structural investigations of BS-sets were reported in [9,10], and various operations for bipolar soft sets were formalized in [3,6,8,14,17]. A major advance was the introduction of bipolar soft groups by Karaaslan et al. [11], linking bipolar soft theory with classical group structures. The concept of normal bipolar soft subgroups further deepened the algebraic framework [13]. MohammedAmin et al. [16] introduced positively bipolar soft groups, establishing foundational properties and providing a basis for further structural analysis.

In this study, we further develop the concept of $BS^+$-groups, established by MohammedAmin et al. [16], and contribute to the advancement of positively bipolar soft group theory. In this paper, we develop and describe various new notions related to the framework of positively bipolar soft groups, including positively bipolar soft left and right cosets, and provide relevant examples to clarify this concept. We further investigate various properties of positively bipolar soft left and right cosets and present the corresponding results. In addition, we introduce the quotient bipolar soft group and present suitable

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examples to enhance understanding. Besides these, we define a positively maximal normal bipolar soft group and construct some structures related to it. Furthermore, we define the positively simple bipolar soft group and support this concept with examples. Additionally, we study positively solvable bipolar soft groups, examining their properties through various examples.

## 2. Preliminaries

We present the foundational algebraic structures of bipolar soft sets (BS-sets) and positively bipolar soft groups ( $BS^+$-groups) as formulated in [16], which serve as the basis for subsequent constructions.

**Definition 2.1** *[16] Assume that $A \subseteq \mathcal{G}$ and $\chi : A \to \mathcal{G}$ is an injective mapping with the parameter set. The extended parameter set of $A$ is thus $A \cup \chi(A)$ and is denoted by $\varepsilon_A$. The extended parameter set of $A$ will be represented by $\varepsilon$ if $A = \mathcal{G}$.*

**Definition 2.2** *[16] The parameter set $\mathcal{G}$, and the universal set $\mathcal{R}$ are considered. $A \subseteq \mathcal{G}$ and $\varepsilon_A = A \cup \chi(A)$, where $\chi\colon A \to \mathcal{G}$ is an injective mapping (or equivalently, $\chi\colon A \to \chi(A) \subseteq \mathcal{G}$ is a bijective mapping). The triple ( $\mathbb{F}, \mathbb{G}, A$ ) is said to be a BS-set over $\mathcal{R}$ if $\mathbb{F} : A \to P(\mathcal{R})$ and $\mathbb{G} : \chi(A) \to P(\mathcal{R})$ are mappings such that $\mathbb{F}(\mathcal{C}) \cap \mathbb{G}(\chi(\mathcal{C})) = \emptyset$ for every $\mathcal{C} \in A$. Also, a BS-set may be represented as follows: $\psi_A$ such that $\psi_A : A \to P(\mathcal{R}) \times P(\mathcal{R})$ is a mapping in which $\psi_A(\mathcal{C}) = \left(\psi_A^+(\mathcal{C}), \psi_A^-(\mathcal{C})\right)$, where $\psi_A^+(C) = \mathbb{F}(C)$ and $\psi_A^-(C) = \mathbb{G}(\chi(C))$ for each $C \in A$. That is denoted by $(\psi_A^+, \psi_A^-, A)$.*

**Remark 2.1** *[16] Throughout this paper:*

1. *We shall state the image of parameters $C \in A$ by $\psi_A(C) = (\psi_A^+(C), \psi_A^-(C))$ and so $\psi_A = (\psi_A^+, \psi_A^-, A) = \{(\psi_A^+(C), \psi_A^-(C), C) : C \in A\}$.*
2. *For any subset $A$ of $\mathcal{G}$, we shall deal with the BS-set $\psi_A$ over $R$ as:*

$$\psi_A = \{(\psi_A^+(C), \psi_A^-(C), C) : C \in A\} \cup \{(\emptyset, R, C), C \in \mathcal{G} \setminus A\}.$$

3. *Our BS-sets do not contain $(\emptyset, \emptyset, C)$ for all $C \in \mathcal{G}$.*

**Definition 2.3** *[16] Suppose that $\psi_A$ and $\psi_B$ are BS-sets over $\mathcal{R}$. Then,*

1. *If $\psi_A^+(\mathcal{C}) = \emptyset$ and $\psi_A^-(\mathcal{C}) = \mathcal{R}$ for all $\mathcal{C} \in A, \psi_A$ is called a null BS-set, denoted by $\psi_\emptyset = (\widehat{\emptyset}, \widehat{\mathcal{R}}, \mathcal{G})$.*
2. *If $\psi_{\mathcal{G}}^+(\mathcal{C}) = \mathcal{R}$ and $\psi_{\mathcal{G}}^-(\mathcal{C}) = \emptyset$ for all $\mathcal{C} \in \mathcal{G}, \psi_{\mathcal{G}}$ is said to be an absolute BS-set, symbolized by $\mathcal{R}_b = (\widehat{\mathcal{R}}, \widehat{\emptyset}, \mathcal{G})$.*
3. *If $\psi_A^+(\mathcal{C}) = \mathcal{R}$ and $\psi_A^-(\mathcal{C}) = \emptyset$ for all $\mathcal{C} \in A, \psi_A$ is said to be a relative absolute BS-set.*
4. *If $\psi_A^+(\mathcal{C}) \subseteq \psi_B^+(\mathcal{C})$ and $\psi_A^-(\mathcal{C}) \supseteq \psi_B^-(\mathcal{C})$ for all $\mathcal{C} \in \mathcal{G}, \psi_A$ is called a BS-subset of $\psi_B$, represented by $\psi_A \widehat{\subseteq} \psi_B$.*
5. *If $\psi_A \widehat{\subseteq} \psi_B$ and $\psi_B \widehat{\subseteq} \psi_A, \psi_A = \psi_B$.*
6. *The BS-union of $\psi_A$ and $\psi_B$, denoted by $\psi_{A \cup B} = \psi_A \widehat{\cup} \psi_B$, is a BS-set over $\mathcal{R}$ and defined by $\psi_A^+ \widetilde{\cup} \psi_B^+ : A \cup B \to P(\mathcal{R})$ such that $\left(\psi_A^+ \widetilde{\cup} \psi_B^+\right)(\mathcal{C}) = \psi_A^+(\mathcal{C}) \cup \psi_B^+(\mathcal{C})$ and $\psi_A^- \widetilde{\cap} \psi_B^- : A \cup B \to P(\mathcal{R})$ such that $\left(\psi_A^- \widetilde{\cap} \psi_B^-\right)(\mathcal{C}) = \psi_A^-(\mathcal{C}) \cap \psi_B^-(\mathcal{C})$ for all $\mathcal{C} \in \mathcal{G}$.*
7. *The BS-intersection of $\psi_A$ and $\psi_B$, denoted by $\psi_{A \cap B} = \psi_A \widehat{\cap} \psi_B$, is a BS-set over $\mathcal{R}$ and defined by $\psi_A^+ \widetilde{\cap} \psi_B^+ : A \cap B \to P(\mathcal{R})$ such that $\left(\psi_A^+ \widetilde{\cap} \psi_B^+\right)(\mathcal{C}) = \psi_A^+(\mathcal{C}) \cap \psi_B^+(\mathcal{C})$ and $\psi_A^- \widetilde{\cup} \psi_B^- : A \cap B \to P(\mathcal{R})$. such that $\left(\psi_A^- \widetilde{\cup} \psi_B^-\right)(\mathcal{C}) = \psi_A^-(\mathcal{C}) \cup \psi_B^-(\mathcal{C})$ for all $\mathcal{C} \in \mathcal{G}$.*
8. *The BS-complement of $\psi_A$, represented by $\psi_A^{\hat{c}}$ and defined by $\psi_A^{\hat{c}} : A \to P(\mathcal{R}) \times P(\mathcal{R})$ such that $\psi_A^{\hat{c}}(\mathcal{C}) = \left\{(\psi_A^+\mathcal{C}), \psi_A^-(\mathcal{C}), \mathcal{C}) : \mathcal{C} \in \mathcal{G}\right\}$.*

**Remark 2.2** *[16]*

1. *If* $A = \emptyset$*, then* $\psi_A = \psi_\emptyset = (\widehat{\emptyset}, \widehat{R}, A) = (\widehat{\emptyset}, \widehat{R}, \mathcal{G})$.
2. $\psi_A = \psi_B$ *if and only if:*
   - $\psi_A^+(C) = \emptyset$ *and* $\psi_A^-(C) = R$ *for each* $C \in A \setminus B$*, and*
   - $\psi_B^+(C) = \emptyset$ *and* $\psi_B^-(C) = R$ *for each* $C \in B \setminus A$.
3. *For any BS-sets* $\psi_A, \psi_B$*, if* $A = B = \emptyset$*, then* $\psi_A \cap \psi_B = \psi_{A \cap B} = \psi_\emptyset$.

**Definition 2.4** *[16] For any BS-set* $\psi_A$*, the set* $\operatorname{Supp}^+ \psi_A = \left\{\xi \in A : \psi_A^+(\xi) \neq \emptyset\right\}$ *is said to be the positively support of the BS-set* $\psi_A$.

Positive non-null refers to a BS-set $\psi_A$ with a non-empty positively support, while the positive null BS-set is a BS-set with an empty positively support.

**Definition 2.5** *[16] A BS-set* $\psi_A = \left(\psi_A^+, \psi_A^-, A\right)$ *over a group* $\mathcal{R}$ *is called a* $BS^+$*-group over* $\mathcal{R}$*, if* $\psi_A^+(\xi)$ *is a subgroup of* $\mathcal{R}$ *for all* $\xi \in \operatorname{Supp}^+ \psi_A$.

**Definition 2.6** *[16] Let* $\psi_A$ *be a* $BS^+$*-group over* $\mathcal{R}$*. A BS-set* $\psi_B$ *is said to be a* $BS^+$*-subgroup of* $\psi_A$*, represented by* $\psi_B \widehat{<} \psi_A$*. If* $\psi_B^+(\xi)$ *is a subgroup of* $\psi_A^+(\xi)$ *for all* $\xi \in \operatorname{Supp}^+ \psi_B \subseteq \operatorname{Supp}^+ \psi_A$.

**Remark 2.3** *[16] Every* $\mathrm{BS}^+$*-subgroup* $\psi_B$ *of a* $\mathrm{BS}^+$*-group* $\psi_A$ *over* $R$ *is a* $\mathrm{BS}^+$*-group over* $R$.

**Definition 2.7** *[16] A* $BS^+$*-subgroup* $\psi_B$ *of a* $BS^+$*-group* $\psi_A$ *is called a* $NBS^+$*-subgroup of* $\psi_A$*, denoted by* $\psi_B \widehat{\triangleleft} \psi_A$*. If* $\psi_B^+(\xi)$ *is a normal subgroup of* $\psi_A^+(\xi)$ $\big($ *simply,* $\psi_B^+(\xi) \triangleleft \psi_A^+(\xi)\big)$ *for all* $\xi \in \operatorname{Supp}^+ \psi_B \subseteq \operatorname{Supp}^+ \psi_A$.

**Definition 2.8** *[16] A* $BS^+$*-group* $\psi_A$ *over* $\mathcal{R}$ *is called an abelian* $BS^+$*-group over* $\mathcal{R}$ *if for each* $\xi \in \operatorname{Supp}^+ \psi_A$*, the set* $\psi_A^+(\xi)$ *is an abelian subgroup of* $\mathcal{R}$*. That is, for all* $\alpha, \beta \in \psi_A^+(\xi)$*, we have* $\alpha\beta = \beta\alpha$*, for every* $\xi \in \operatorname{Supp}^+ \psi_A$.

**Definition 2.9** *[16] Consider that* $\psi_A$ *is a* $BS^+$*-group over* $\mathcal{R}$ *and* $\psi_B$ *is a* $BS^+$*- subgroup of* $\psi_A$*. We say that* $\psi_B$ *is an abelian* $BS^+$*-subgroup of* $\psi_A$ *if* $\psi_B^+(\xi)$ *is an abelian subgroup of* $\psi_A^+(\xi)$ *for all* $\xi \in \operatorname{Supp}^+ \psi_B \subseteq \operatorname{Supp}^+ \psi_A$.

## 3. The BS-Cosets and $BS^+$-Cosets

In this section, we introduce $BS^+$-left and $BS^+$-right cosets of $\omega_B$ in $\psi_A$. We also present several related remarks and illustrate these concepts with suitable examples.

**Definition 3.1** Let $\omega_B$ be a $BS^+$-subgroup of a $BS^+$-group $\psi_A$ over $\mathcal{R}$, and let $r_B^+ : \operatorname{Supp}^+ \omega_B \to \psi_A^+$ be a function given by $r_B^+(\xi) = r_\xi$, where $r_\xi \in \psi_A^+(\xi)$ and $\xi \in \operatorname{Supp}^+ \omega_B \subseteq \operatorname{Supp}^+ \psi_A$. The BS-left (resp., right) coset of $\omega_B$ in $\psi_A$ is a BS-set $r_B^+\omega_B = \{(r_B^+(\xi)\omega_B^+(\xi), r_B^+(\xi)\omega_B^-(\xi), \xi) : \xi \in \operatorname{Supp}^+ \omega_B\}$ (resp., $\omega_B r_B^+ = \{(\omega_B^+(\xi)r_B^+(\xi), \omega_B^-(\xi)r_B^+(\xi), \xi) : \xi \in \operatorname{Supp}^+ \omega_B\}$ are called the BS-left (resp., BS-right) coset of $\omega_B$ in $\psi_A$, also $r_B^+\omega_B^+ = \{r_B^+(\xi)\omega_B^+(\xi) : \xi \in \operatorname{Supp}^+ \omega_B\}$ (resp., $\omega_B^+ r_B^+ = \{\omega_B^+(\xi)r_B^+(\xi) : \xi \in \operatorname{Supp}^+ \omega_B\}$ are called the $BS^+$-left (resp., $BS^+$-right) coset of $\omega_B$ in $\psi_A$.

**Remark 3.1** *It is easy to see that* $r_B^+\omega_B$ *and* $\omega_B r_B^+$*are BS-sets over* $\mathcal{R}$.

**Example 3.1** *Let the symmetric group* $R = S_3$ *and* $A = \{C_1, C_2\} = B$ *be the set of parameters. Consider the following* $\mathrm{BS}^+$*-group over* $R$.

$$\begin{aligned} &\psi_A^+(C_1) = S_3, && \psi_A^-(C_1) = \emptyset \\ &\psi_A^+(C_2) = \{(1), (13)\}, && \psi_A^-(C_2) = \emptyset \end{aligned}$$

*And,*

$$\omega^+_B(C_1) = \{(1),(12)\}, \qquad \omega^-_B(C_1) = \emptyset$$
$$\omega^+_B(C_2) = \{(1),(13)\}, \qquad \omega^-_B(C_2) = \{(23)\}$$

*Clearly,* $\omega_B$ *is a* $\mathrm{BS}^+$*-subgroup of* $\mathrm{BS}^+$*-group* $\psi_A$ *and* $\mathrm{Supp}^+\psi_A = A = B = \mathrm{Supp}^+\omega_B$. *For all* $r^+_B \in \psi^+_A$*, we have the following table:*

| | $r_1{}^+_B$ | $r_2{}^+_B$ | $r_3{}^+_B$ | $r_4{}^+_B$ | $r_5{}^+_B$ | $r_6{}^+_B$ | $r_7{}^+_B$ | $r_8{}^+_B$ | $r_9{}^+_B$ | $r_{10}{}^+_B$ | $r_{11}{}^+_B$ | $r_{12}{}^+_B$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_1$ | (1) | (1) | (12) | (12) | (13) | (13) | (23) | (23) | (123) | (123) | (132) | (132) |
| $C_2$ | (1) | (13) | (1) | (13) | (1) | (13) | (1) | (13) | (1) | (13) | (1) | (13) |

$$\begin{aligned} r_1{}^+_B\omega_B &= \{((1)\omega^+_B(C_1),(1)\omega^-_B(C_1),C_1),((1)\omega^+_B(C_2),(1)\omega^-_B(C_2),C_2)\}\\ &= \{(\omega^+_B(C_1),\omega^-_B(C_1),C_1),(\omega^+_B(C_2),\omega^-_B(C_2),C_2)\}\\ &= \{(\{(1),(12)\},\emptyset,C_1),(\{(1),(13)\},\{(23)\},C_2)\} = \omega_B. \end{aligned}$$

$$\begin{aligned} r_2{}^+_B\omega_B &= \{((1)\omega^+_B(C_1),(1)\omega^-_B(C_1),C_1),((13)\omega^+_B(C_2),(13)\omega^-_B(C_2),C_2)\}\\ &= \{(\omega^+_B(C_1),\omega^-_B(C_1),C_1),((13)\{(1),(13)\},(13)\{(23)\},C_2)\}\\ &= \{(\{(1),(12)\},\emptyset,C_1),(\{(1),(13)\},\{(132)\},C_2)\}. \end{aligned}$$

$$\begin{aligned} r_3{}^+_B\omega_B &= \{((12)\omega^+_B(C_1),(12)\omega^-_B(C_1),C_1),((1)\omega^+_B(C_2),(1)\omega^-_B(C_2),C_2)\}\\ &= \{((12)\{(1),(12)\},\emptyset,C_1),(\{(1),(13)\},\{(23)\},C_2)\}\\ &= \{(\{(1),(12)\},\emptyset,C_1),(\{(1),(13)\},\{(23)\},C_2)\} = \omega_B. \end{aligned}$$

$$\begin{aligned} r_4{}^+_B\omega_B &= \{((12)\omega^+_B(C_1),(12)\omega^-_B(C_1),C_1),((13)\omega^+_B(C_2),(13)\omega^-_B(C_2),C_2)\}\\ &= \{((12)\{(1),(12)\},\emptyset,C_1),((13)\{(1),(13)\},(13)\{(23)\},C_2)\}\\ &= \{(\{(1),(12)\},\emptyset,C_1),(\{(1),(13)\},\{(132)\},C_2)\}. \end{aligned}$$

$$\begin{aligned} r_5{}^+_B\omega_B &= \{((13)\omega^+_B(C_1),(13)\omega^-_B(C_1),C_1),((1)\omega^+_B(C_2),(1)\omega^-_B(C_2),C_2)\}\\ &= \{((13)\{(1),(12)\},\emptyset,C_1),((1)\{(1),(13)\},(1)\{(23)\},C_2)\}\\ &= \{(\{(13),(123)\},\emptyset,C_1),(\{(1),(13)\},\{(23)\},C_2)\}. \end{aligned}$$

$$\begin{aligned} r_6{}^+_B\omega_B &= \{((13)\omega^+_B(C_1),(13)\omega^-_B(C_1),C_1),((13)\omega^+_B(C_2),(13)\omega^-_B(C_2),C_2)\}\\ &= \{((13)\{(1),(12)\},\emptyset,C_1),((13)\{(1),(13)\},(13)\{(23)\},C_2)\}\\ &= \{(\{(13),(123)\},\emptyset,C_1),(\{(1),(13)\},\{(132)\},C_2)\}. \end{aligned}$$

$$\begin{aligned} r_7{}^+_B\omega_B &= \{((23)\omega^+_B(C_1),(23)\omega^-_B(C_1),C_1),((1)\omega^+_B(C_2),(1)\omega^-_B(C_2),C_2)\}\\ &= \{((23)\{(1),(12)\},\emptyset,C_1),((1)\{(1),(13)\},(1)\{(23)\},C_2)\}\\ &= \{(\{(23),(132)\},\emptyset,C_1),(\{(1),(13)\},\{(23)\},C_2)\}. \end{aligned}$$

$$\begin{aligned} r_8{}_B^+\omega_B &= \{((23)\omega_B^+(C_1),(23)\omega_B^-(C_1),C_1),((13)\omega_B^+(C_2),(13)\omega_B^-(C_2),C_2)\}\\ &= \{((23)\{(1),(12)\},\emptyset,C_1),((13)\{(1),(13)\},(13)\{(23)\},C_2)\}\\ &= \{(\{(23),(132)\},\emptyset,C_1),(\{(1),(13)\},\{(132)\},C_2)\}. \end{aligned}$$

$$\begin{aligned} r_9{}_B^+\omega_B &= \{((123)\omega_B^+(C_1),(123)\omega_B^-(C_1),C_1),((1)\omega_B^+(C_2),(1)\omega_B^-(C_2),C_2)\}\\ &= \{((123)\{(1),(12)\},\emptyset,C_1),((1)\{(1),(13)\},(1)\{(23)\},C_2)\}\\ &= \{(\{(123),(13)\},\emptyset,C_1),(\{(1),(13)\},\{(23)\},C_2)\}. \end{aligned}$$

$$\begin{aligned} r_{10}{}_B^+\omega_B &= \{((123)\omega_B^+(C_1),(123)\omega_B^-(C_1),C_1),((13)\omega_B^+(C_2),(13)\omega_B^-(C_2),C_2)\}\\ &= \{((123)\{(1),(12)\},\emptyset,C_1),((13)\{(1),(13)\},(13)\{(23)\},C_2)\}\\ &= \{(\{(123),(13)\},\emptyset,C_1),(\{(1),(13)\},\{(132)\},C_2)\}. \end{aligned}$$

$$\begin{aligned} r_{11}{}_B^+\omega_B &= \{((132)\omega_B^+(C_1),(132)\omega_B^-(C_1),C_1),((1)\omega_B^+(C_2),(1)\omega_B^-(C_2),C_2)\}\\ &= \{((132)\{(1),(12)\},\emptyset,C_1),((1)\{(1),(13)\},(1)\{(23)\},C_2)\}\\ &= \{(\{(132),(23)\},\emptyset,C_1),(\{(1),(13)\},\{(23)\},C_2)\}. \end{aligned}$$

$$\begin{aligned} r_{12}{}_B^+\omega_B &= \{((132)\omega_B^+(C_1),(132)\omega_B^-(C_1),C_1),((13)\omega_B^+(C_2),(13)\omega_B^-(C_2),C_2)\}\\ &= \{((132)\{(1),(12)\},\emptyset,C_1),((13)\{(1),(13)\},(13)\{(23)\},C_2)\}\\ &= \{(\{(132),(23)\},\emptyset,C_1),(\{(1),(13)\},\{(132)\},C_2)\}. \end{aligned}$$

*We have:* $r_1{}_B^+\omega_B = r_3{}_B^+\omega_B$, $r_2{}_B^+\omega_B = r_4{}_B^+\omega_B$, $r_5{}_B^+\omega_B = r_9{}_B^+\omega_B$, $r_6{}_B^+\omega_B = r_{10}{}_B^+\omega_B$, $r_7{}_B^+\omega_B = r_{11}{}_B^+\omega_B$ *and* $r_8{}_B^+\omega_B = r_{12}{}_B^+\omega_B$.

*Then,* $r_1{}_B^+\omega_B$, $r_2{}_B^+\omega_B$, $r_5{}_B^+\omega_B$, $r_6{}_B^+\omega_B$, $r_7{}_B^+\omega_B$ *and* $r_8{}_B^+\omega_B$ *are BS-left cosets of* $\omega_B$ *in* $\psi_A$.

*Also,*

$$\begin{aligned} r_1{}_B^+\omega_B^+ &= \{r_1{}_B^+(C_1)\omega_B^+(C_1), r_1{}_B^+(C_2)\omega_B^+(C_2)\}\\ &= \{\{(1),(12)\},\{(1),(13)\}\}\\ &= r_2{}_B^+\omega_B^+ = r_3{}_B^+\omega_B^+ = r_4{}_B^+\omega_B^+,\\ r_5{}_B^+\omega_B^+ &= r_6{}_B^+\omega_B^+ = r_9{}_B^+\omega_B^+ = r_{10}{}_B^+\omega_B^+ = \{\{(13),(123)\},\{(1),(13)\}\},\\ r_7{}_B^+\omega_B^+ &= r_8{}_B^+\omega_B^+ = r_{11}{}_B^+\omega_B^+ = r_{12}{}_B^+\omega_B^+ = \{\{(23),(132)\},\{(1),(13)\}\}. \end{aligned}$$

*Thus,* $r_1{}_B^+\omega_B^+$, $r_5{}_B^+\omega_B^+$ *and* $r_7{}_B^+\omega_B^+$ *are* $\mathrm{BS}^+$*-left cosets of* $\omega_B$ *in* $\psi_A$.

*Now, find the BS-right cosets of* $\omega_B$ *in* $\psi_A$.

$$\begin{aligned} \omega_B r_1{}_B^+ &= \{(\omega_B^+(C_1)(1),\omega_B^-(C_1)(1),C_1),(\omega_B^+(C_2)(1),\omega_B^-(C_2)(1),C_2)\}\\ &= \{(\omega_B^+(C_1),\omega_B^-(C_1),C_1),(\omega_B^+(C_2),\omega_B^-(C_2),C_2)\}\\ &= \{(\{(1),(12)\},\emptyset,C_1),(\{(1),(13)\},\{(23)\},C_2)\} = \omega_B. \end{aligned}$$

$$\begin{aligned} \omega_B r_2{}_B^+ &= \{(\omega_B^+(C_1)(1),\omega_B^-(C_1)(1),C_1),(\omega_B^+(C_2)(13),\omega_B^-(C_2)(13),C_2)\}\\ &= \{(\omega_B^+(C_1),\omega_B^-(C_1),C_1),(\{(1),(13)\}(13),\{(23)\}(13),C_2)\}\\ &= \{(\{(1),(12)\},\emptyset,C_1),(\{(1),(13)\},\{(123)\},C_2)\}. \end{aligned}$$

$$\begin{aligned}
\omega_B r3^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(12), \omega^{-}_{B}(C_1)(12), C_1), (\omega^{+}_{B}(C_2)(1), \omega^{-}_{B}(C_2)(1), C_2)\}\\
&= \{(\{(1),(12)\}(12), \emptyset, C_1), (\omega^{+}_{B}(C_2), \omega^{-}_{B}(C_2), C_2)\}\\
&= \{(\{(1),(12)\}, \emptyset, C_1), (\{(1),(13)\}, \{(23)\}, C_2)\} = \omega_B.
\end{aligned}$$

$$\begin{aligned}
\omega_B r4^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(12), \omega^{-}_{B}(C_1)(12), C_1), (\omega^{+}_{B}(C_2)(13), \omega^{-}_{B}(C_2)(13), C_2)\}\\
&= \{(\{(1),(12)\}(12), \omega^{-}_{B}(C_1), C_1), (\{(1),(13)\}(13), \{(23)\}(13), C_2)\}\\
&= \{(\{(1),(12)\}, \emptyset, C_1), (\{(1),(13)\}, \{(123)\}, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
\omega_B r5^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(13), \omega^{-}_{B}(C_1)(13), C_1), (\omega^{+}_{B}(C_2)(1), \omega^{-}_{B}(C_2)(1), C_2)\}\\
&= \{(\{(1),(12)\}(13), \emptyset, C_1), (\omega^{+}_{B}(C_2), \omega^{-}_{B}(C_2), C_2)\}\\
&= \{(\{(13),(132)\}, \emptyset, C_1), (\{(1),(13)\}, \{(23)\}, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
\omega_B r6^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(13), \omega^{-}_{B}(C_1)(13), C_1), (\omega^{+}_{B}(C_2)(13), \omega^{-}_{B}(C_2)(13), C_2)\}\\
&= \{(\{(1),(12)\}(13), \omega^{-}_{B}(C_1), C_1), (\{(1),(13)\}(13), \{(23)\}(13), C_2)\}\\
&= \{(\{(13),(132)\}, \emptyset, C_1), (\{(1),(13)\}, \{(123)\}, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
\omega_B r7^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(23), \omega^{-}_{B}(C_1)(23), C_1), (\omega^{+}_{B}(C_2)(1), \omega^{-}_{B}(C_2)(1), C_2)\}\\
&= \{(\{(1),(12)\}(23), \emptyset, C_1), (\omega^{+}_{B}(C_2), \omega^{-}_{B}(C_2), C_2)\}\\
&= \{(\{(23),(123)\}, \emptyset, C_1), (\{(1),(13)\}, \{(23)\}, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
\omega_B r8^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(23), \omega^{-}_{B}(C_1)(23), C_1), (\omega^{+}_{B}(C_2)(13), \omega^{-}_{B}(C_2)(13), C_2)\}\\
&= \{(\{(1),(12)\}(23), \omega^{-}_{B}(C_1), C_1), (\{(1),(13)\}(13), \{(23)\}(13), C_2)\}\\
&= \{(\{(23),(123)\}, \emptyset, C_1), (\{(1),(13)\}, \{(123)\}, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
\omega_B r9^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(123), \omega^{-}_{B}(C_1)(123), C_1), (\omega^{+}_{B}(C_2)(1), \omega^{-}_{B}(C_2)(1), C_2)\}\\
&= \{(\{(1),(12)\}(123), \emptyset, C_1), (\omega^{+}_{B}(C_2), \omega^{-}_{B}(C_2), C_2)\}\\
&= \{(\{(123),(23)\}, \emptyset, C_1), (\{(1),(13)\}, \{(23)\}, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
\omega_B r10^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(123), \omega^{-}_{B}(C_1)(123), C_1), (\omega^{+}_{B}(C_2)(13), \omega^{-}_{B}(C_2)(13), C_2)\}\\
&= \{(\{(1),(12)\}(123), \omega^{-}_{B}(C_1), C_1), (\{(1),(13)\}(13), \{(23)\}(13), C_2)\}\\
&= \{(\{(123),(23)\}, \emptyset, C_1), (\{(1),(13)\}, \{(123)\}, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
\omega_B r11^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(132), \omega^{-}_{B}(C_1)(132), C_1), (\omega^{+}_{B}(C_2)(1), \omega^{-}_{B}(C_2)(1), C_2)\}\\
&= \{(\{(1),(12)\}(132), \emptyset, C_1), (\omega^{+}_{B}(C_2), \omega^{-}_{B}(C_2), C_2)\}\\
&= \{(\{(132),(13)\}, \emptyset, C_1), (\{(1),(13)\}, \{(23)\}, C_2)\}.
\end{aligned}$$

$$\begin{aligned}\omega_B r_{12}{}^{+}_{B} &= \{(\omega^{+}_{B}(C_1)(132), \omega^{-}_{B}(C_1)(132), C_1), (\omega^{+}_{B}(C_2)(13), \omega^{-}_{B}(C_2)(13), C_2)\}\\ &= \{(\{(1),(12)\}(132), \omega^{-}_{B}(C_1), C_1), (\{(1),(13)\}(13), \{(23)\}(13), C_2)\}\\ &= \{(\{(132),(13)\}, \emptyset, C_1), (\{(1),(13)\}, \{(123)\}, C_2)\}.\end{aligned}$$

*Implies that:* $\omega_B r_1{}^{+}_{B} = \omega_B r_3{}^{+}_{B}$, $\omega_B r_2{}^{+}_{B} = \omega_B r_4{}^{+}_{B}$, $\omega_B r_5{}^{+}_{B} = \omega_B r_{11}{}^{+}_{B}$, $\omega_B r_6{}^{+}_{B} = \omega_B r_{12}{}^{+}_{B}$, $\omega_B r_7{}^{+}_{B} = \omega_B r_9{}^{+}_{B}$, $\omega_B r_8{}^{+}_{B} = \omega_B r_{10}{}^{+}_{B}$.

*Then,* $\omega_B r_1{}^{+}_{B}$, $\omega_B r_2{}^{+}_{B}$, $\omega_B r_5{}^{+}_{B}$, $\omega_B r_6{}^{+}_{B}$, $\omega_B r_7{}^{+}_{B}$ *and* $\omega_B r_8{}^{+}_{B}$ *are BS-right cosets of* $\omega_B$ *in* $\psi_A$.

*Also,*

$$\begin{aligned}\omega^{+}_{B} r_1{}^{+}_{B} &= \{\omega^{+}_{B}(C_1) r_1{}^{+}_{B}(C_1), \omega^{+}_{B}(C_2) r_1{}^{+}_{B}(C_2)\}\\ &= \{\{(1),(12)\},\{(1),(13)\}\}\\ &= \omega^{+}_{B} r_2{}^{+}_{B} = \omega^{+}_{B} r_3{}^{+}_{B} = \omega^{+}_{B} r_4{}^{+}_{B},\\ \omega^{+}_{B} r_5{}^{+}_{B} &= \omega^{+}_{B} r_6{}^{+}_{B} = \omega^{+}_{B} r_{11}{}^{+}_{B} = \omega^{+}_{B} r_{12}{}^{+}_{B} = \{\{(13),(132)\},\{(1),(13)\}\},\\ \omega^{+}_{B} r_7{}^{+}_{B} &= \omega^{+}_{B} r_8{}^{+}_{B} = \omega^{+}_{B} r_9{}^{+}_{B} = \omega^{+}_{B} r_{10}{}^{+}_{B} = \{\{(23),(123)\},\{(1),(13)\}\}.\end{aligned}$$

*Thus,* $\omega^{+}_{B} r^{+}_{1B}$, $\omega^{+}_{B} r^{+}_{5B}$ *and* $\omega^{+}_{B} r^{+}_{7B}$ *are* $\mathrm{BS}^{+}$*-right cosets of* $\omega_B$ *in* $\psi_A$.

**Remark 3.2** *Let* $\psi_A$ *be a* $BS^{+}$*-group over a group* $\mathcal{R}$ *and* $\omega_B$ *be a* $BS^{+}$*-subgroup of* $\psi_A$*. Then, a BS-left coset of* $\omega_B$ *in* $\psi_A$ *over a group* $\mathcal{R}$ *is not equal to a BS-right coset of* $\omega_B$ *in general, but if the group* $\mathcal{R}$ *is abelian or* $\psi_A$ *is an abelian* $BS^{+}$*-group over* $\mathcal{R}$*, then they are equal.As, we illustrated them by the following example:*

**Example 3.2** *See in Example(3.1),* $r_5{}^{+}_{B}\omega^{+}_{B} = \{\{(13),(123)\},\{(1),(13)\}\}$ $\neq \{\{(13),(132)\},\{(1),(13)\}\} = \omega^{+}_{B} r_5{}^{+}_{B}$*.Hence,* $r_5{}^{+}_{B}\omega_B = \big\{\big(\{(13),(123)\},\emptyset,C_1\big),\big(\{(1),(13)\},\{(23)\},C_2\big)\big\}$ $\neq \big\{\big(\{(13),(132)\},\emptyset,C_1\big),\big(\{(1),(13)\},\{(23)\},C_2\big)\big\} = \omega_B r_5{}^{+}_{B}$.

**Example 3.3** *Let* $BS^{+}$*-group* $A_3 = \{(1),(123),(132)\}$, $A = \{\mathcal{C}_1,\mathcal{C}_2\}$, $B = \{\mathcal{C}_1\}$ *and the* $BS^{+}$*-group* $\psi_A = \{(A_3,\emptyset,C_1),((1),\emptyset,C_2)\}$*. Clearly,* $\psi_A$ *is an abelian* $BS^{+}$*-group over* $\mathcal{R}$*. Let* $\omega_B = \{(A_3,\emptyset,C_1)\}$ *be a* $\mathrm{BS}^{+}$*-subgroup of* $\mathrm{BS}^{+}$*-group* $\psi_A$*. For all For all* $r^{+}_{B} \in \psi^{+}_{A}$*, we have the following table*

| | $r_1{}^{+}_{B}$ | $r_2{}^{+}_{B}$ | $r_3{}^{+}_{B}$ |
|---|---|---|---|
| $\mathcal{C}_1$ | (1) | (123) | (132) |

$$\begin{aligned}r_1{}^{+}_{B}\omega_B &= \big\{\big((1)\omega^{+}_{B}(C_1), (1)\omega^{-}_{B}(C_1), C_1\big)\big\}\\ &= \big\{\big(\omega^{+}_{B}(C_1), \omega^{-}_{B}(C_1), C_1\big)\big\}\\ &= \{(A_3,\emptyset,C_1)\} = \omega_B.\end{aligned}$$

$$\begin{aligned}r_2{}^{+}_{B}\omega_B &= \big\{\big((123)\omega^{+}_{B}(C_1), (123)\omega^{-}_{B}(C_1), C_1\big)\big\}\\ &= \{((123)A_3, (123)\emptyset, C_1)\}\\ &= \{(A_3,\emptyset,C_1)\} = \omega_B.\end{aligned}$$

$$\begin{aligned}r_3{}^{+}_{B}\omega_B &= \big\{\big((132)\omega^{+}_{B}(C_1), (132)\omega^{-}_{B}(C_1), C_1\big)\big\}\\ &= \{((132)A_3, (132)\emptyset, C_1)\}\\ &= \{(A_3,\emptyset,C_1)\} = \omega_B.\end{aligned}$$

$\omega_B$ *is the only BS-left coset of* $\omega_B$ *in* $\psi_A$*. Also,* $A_3$ *is only* $BS^{+}$-

*left coset of* $\omega_B$ *in* $\psi_A$. *Also,*

$$\begin{aligned}
\omega_B {r_1}^+_B &= \left\{\left(\omega_B^+\left(\mathcal{C}_1\right)(1), \omega_B^-\left(\mathcal{C}_1\right)(1), \mathcal{C}_1\right)\right\} \\
&= \left\{\left(\omega_B^+\left(\mathcal{C}_1\right), \omega_B^-\left(\mathcal{C}_1\right), \mathcal{C}_1\right)\right\} \\
&= \{(A_3, \emptyset, \mathcal{C}_1)\} = \omega_B \\
\omega_B {r_2}^+_B &= \left\{\left(\omega_B^+\left(\mathcal{C}_1\right)(123), \omega_B^-\left(\mathcal{C}_1\right)(123), \mathcal{C}_1\right)\right\} \\
&= \{(A_3(123), \emptyset(123), \mathcal{C}_1)\} \\
&= \{(A_3, \emptyset, \mathcal{C}_1)\} = \omega_B \\
\omega_B {r_3}^+_B &= \left\{\left(\omega_B^+\left(\mathcal{C}_1\right)(132), \omega_B^-\left(\mathcal{C}_1\right)(132), \mathcal{C}_1\right)\right\} \\
&= \{(A_3(132), \emptyset(132), \mathcal{C}_1)\} \\
&= \{(A_3, \emptyset, \mathcal{C}_1)\} = \omega_B
\end{aligned}$$

*So,* $\omega_B$ *is the only BS-right coset of* $\omega_B$ *in* $\psi_A$. *Also, the only BS-left coset of* $\omega_B$ *in* $\psi_A$. *While,* $A_3$ *is only* $BS^+$*-left and* $BS^+$*-right coset of* $\omega_B$ *in* $\psi_A$.

**Remark 3.3** *A BS-left (resp., BS-right) coset of* $\omega_B$ *in* $\psi_A$ *over a group* $\mathcal{R}$ *need not be a* $BS^+$*-subgroup of* $\psi_A$.

**Example 3.4** *Consider* ${r_{11}}^+_B\omega_B$ *the BS-left coset of* $\omega_B$ *in* $\psi_A$ *in Example (3.1).* ${r_{11}}^+_B\omega_B$ *is not a* $BS^+$*-subgroup of* $\psi_A$*, since* ${r_{11}}^+_B(\mathcal{C}_1)\omega_B^+(\mathcal{C}_1) = \{(132), (23)\}$ *is not a subgroup of* $\psi_A^+(\mathcal{C}_1)$*, while* $\omega_B {r_5}^+_B$ *is the BS-right coset of* $\omega_B$ *in* $\psi_A$*, but* $\omega_B {r_5}^+_B$ *is not a* $BS^+$*-subgroup of* $\psi_A$*, since* $\omega_B^+(\mathcal{C}_1){r_5}^+_B(\mathcal{C}_1) = \{(13), (132)\}$ *is not a subgroup of* $\psi_A^+(\mathcal{C}_1)$.

We recall the following result

**Theorem 3.1** *Let* $R$ *be a group and* $H$ *is a subgroup of* $R$*. Then* $H \lhd R$ *if and only if* $rH = Hr$ *for all* $r \in R$.

**Theorem 3.2** *A* $BS^+$*-subgroup* $\omega_B$ *of a* $BS^+$*-group* $\psi_A$ *over* $\mathcal{R}$ *is an* $NBS^+$*-subgroup of* $\psi_A$ *if and only if the* $BS^+$*-left coset and* $BS^+$*-right coset of* $\omega_B$ *in* $\psi_A$ *are the same.*

**Proof:** Let $\omega_B$ be a $NBS^+$-subgroup of $\psi_A$ $(\omega_B \widehat{\lhd} \psi_A)$. Then, by Definition(2.7), we have that means $\omega_B^+(\xi) \lhd \psi_A^+(\xi)$ for all $\xi \in \operatorname{Supp}^+ \omega_B$. This means that, for each $\xi \in \operatorname{Supp}^+ \omega_B$, we have $r\omega_B^+(\xi) = \omega_B^+(\xi)r$ for each $r \in \psi_A^+(\xi)$ by Theorem(3.1). For each $\xi \in \operatorname{Supp}^+ \omega_B$ and let $r_\xi$ be an element in $\psi_A^+(\xi)$ and put $r_B^+(\xi) = r_\xi$.

Now,

$$\begin{aligned}
r_B^+\omega_B^+ &= \left\{r_B^+(\xi)\omega_B^+(\xi) : \xi \in \operatorname{Supp}^+ \omega_B\right\} \\
&= \left\{r_\xi\omega_B^+(\xi) : \xi \in \operatorname{Supp}^+ \omega_B\right\} \\
&= \left\{\omega_B^+(\xi)r_\xi : \xi \in \operatorname{Supp}^+ \omega_B\right\} \\
&= \left\{\omega_B^+(\xi)r_B^+(\xi) : \xi \in \operatorname{Supp}^+ \omega_B\right\} = \omega_B^+ r_B^+.
\end{aligned}$$

Hence, the $BS^+$-left coset and $BS^+$- right coset of $\omega_B$ in $\psi_A$ are the same.

**Conversely**, suppose that the $BS^+$-left coset and $BS^+$- right coset of $\omega_B$ in $\psi_A$ are the same. This means that, $r_B^+\omega_B^+ = \omega_B^+ r_B^+$ for each $r_B^+$. Let $\xi$ be any fixed point in $\operatorname{Supp}^+ \omega_B$ and $r \in \psi_A^+(\xi)$. Let $r_B^+$: $\operatorname{Supp}^+ \omega_B \to \psi_A^+$ be a function given as $r_B^+(\xi) = r$ and $r_B^+(\xi') = e_{\mathcal{R}}$ for each $\xi' \in \operatorname{Supp}^+ \omega_B$ and $\xi \neq \xi'$. Then, $r_B^+\omega_B^+ = \omega_B^+ r_B^+$. That is, $\left\{r\omega_B^+(\xi)\right\} \cup \left\{\omega_B^+(\xi') : \xi' \in \operatorname{Supp}^+ \omega_B \setminus \{\xi\}\right\} = \left\{\omega_B^+(\xi)r\right\} \cup \left\{\omega_B^+(\xi') : \xi' \in \operatorname{Supp}^+ \omega_B \setminus \{\xi\}\right\}$.

Thus, $\omega_B^+(\xi)r = r\omega_B^+(\xi)$. Hence, $\omega_B^+(\xi) \lhd \psi_A^+(\xi)$. Since $\xi$ is a fixed but arbitrary point of $\operatorname{Supp}^+ \omega_B$, by Definition(2.7), we get $\omega_B \widehat{\lhd} \psi_A$. □

## 4. Quotient BS-Groups and Maximal Normal $BS^+$-groups

In this section, we present the concepts of the quotient BS-groups and the maximal $NBS^+$subgroups, as well as the notion of a simple $BS^+$-group, and establish several significant results related to their structure and properties.

**Remark 4.1** *Let $\omega_B$ be an $NBS^+$-subgroup of a $BS^+$-group $\psi_A$ over a group $\mathcal{R}$.*

1. *The quotient $\psi_A/\omega_B = \Upsilon_B = \left\{r_B^+\omega_B : \text{ for all } r_B^+\right\}$*

$$= \left\{\left(r_B^+(\xi)\omega_B^+(\xi), r_B^+(\xi)\omega_B^-(\xi), \xi\right) : \xi \in \operatorname{Supp}^+\omega_B, r_B^+(\xi) \in \psi_A^+(\xi)\right\}$$

   *is a BS-set over $\mathcal{R}$.*

2. *For each $\xi \in \operatorname{Supp}^+\omega_B$,*

$$\Upsilon_B^+(\xi) = \psi_A^+(\xi)/\omega_B^+(\xi) = \left\{r\omega_B^+(\xi) \mid r \in \psi_A^+(\xi)\right\}$$

   *is a quotient group.*

3. *We give the operation $*$ on $\Upsilon_B = \psi_A/\omega_B$ as follows: For any BS-left cosets $r_B^+\omega_B, t_B^+\omega_B \in \psi_A/\omega_B$,*

$$(r_B^+\omega_B) * (t_B^+\omega_B) = (r_B^+t_B^+)\omega_B,$$

   *where $r_B^+t_B^+(\xi) = r_B^+(\xi)t_B^+(\xi) \in \psi_A^+(\xi)$ for each $\xi \in \operatorname{Supp}^+\omega_B$.*

4. *The map $e_B^+ : \operatorname{Supp}^+\omega_B \to \psi_A^+$ is defined by*

$$e_B^+(\xi) = e_R \quad \text{for each } \xi \in \operatorname{Supp}^+\omega_B,$$

   *where $e_R$ is the identity element of $R$, which is also the identity element of each $\psi_A^+(\xi)$ with $\xi \in \operatorname{Supp}^+\omega_B$.*

5. *For each $r_B^+ \in \psi_A^+$, we define the inverse ${r_B^+}^{-1} : \operatorname{Supp}^+\omega_B \to \psi_A^+$ by*

$${r_B^+}^{-1}(\xi) = \left(r_B^+(\xi)\right)^{-1} \quad \text{for each } \xi \in \operatorname{Supp}^+\omega_B.$$

**Lemma 4.1** *Let $\omega_B$ be an $\mathrm{NBS}^+$-subgroup of a $\mathrm{BS}^+$-group over a group $R$. Then, the operation $*$ which is defined on $\psi_A/\omega_B$ in part (3) of Remark (4.1) is well-defined. Furthermore, $\omega_B$ is the identity element of $\psi_A/\omega_B$.*

**Proof:** Let $r_B^+\omega_B = \mathfrak{s}_B^+\omega_B$ and $t_B^+\omega_B = u_B^+\omega_B$ in $\psi_A/\omega_B$. Then,

$$\begin{aligned}(r_B^+t_B^+)\omega_B &= (r_B^+\omega_B) * (t_B^+\omega_B) \text{ (by part (3) of Remark (4.1))}\\ &= (s_B^+\omega_B) * (u_B^+\omega_B) \text{ (it is given)}\\ &= (s_B^+u_B^+)\omega_B \text{ (by part (3) of Remark (4.1)).}\end{aligned}$$

So, the operation on $\psi_A/\omega_B$ is well-defined.

Furthermore,

$$\begin{aligned}(r_B^+\omega_B) * (\omega_B) &= (r_B^+\omega_B) * (e_B^+\omega_B) \text{ (since } e_B^+\omega_B = \omega_B).\\ &= \left(r_B^+e_B^+\right)\omega_B \text{ (by part (3) of Remark (4.1))}\\ &= r_B^+\omega_B \left(\text{ since } r_B^+e_B^+(\xi) = r_B^+(\xi)e_{\mathcal{R}} = r_B^+(\xi) \text{ for all } \xi \in \operatorname{Supp}^+\omega_B\right).\end{aligned}$$

Similarly, $\omega_B(r_B^+\omega_B) = r_B^+\omega_B$. Thus, $\omega_B$ is the identity element of $\psi_A/\omega_B$. $\square$

**Theorem 4.1** *Let $\omega_B$ be an $NBS^+$-subgroup of a $BS^+$-group $\psi_A$ over $\mathcal{R}$. Then, $(\psi_A/\omega_B, *)$ form a group.*

**Proof:** Clearly, from Lemma(4.1), the operation $*$ is closed, and $\omega_B$ is the identity of $\psi_A/\omega_B$. (That is, $\omega_B^+(\xi)$ represents the identity element of $\psi_A^+(\xi)/\omega_B^+(\xi)$ for each $\xi \in \operatorname{Supp}^+ \omega_B$ ).

$$\begin{aligned} r_B^+\omega_B * \left(t_B^+\omega_B * \mathfrak{s}_B^+\omega_B\right) &= (r_B^+\omega_B) * ((t_B^+ s_B^+)\omega_B) (\text{by part(3) of Remark(4.1)}) \\ &= \left(r_B^+ \left(t_B^+\mathfrak{s}_B^+\right)\right)\omega_B (\text{by part(3) of Remark(4.1)}) \\ &= \left(\left(r_B^+ t_B^+\right)\mathfrak{s}_B^+\right)\omega_B \left(\text{ since } \psi_A^+(\xi) \text{ is a group and } r_B^+(\xi), t_B^+(\xi) \text{ and } \mathfrak{s}_B^+(\xi)\right. \end{aligned}$$

are subsets of $\psi_A^+(\xi)$ for each $\xi \in \operatorname{Supp}^+ \omega_B$ )

$$\begin{aligned} &= \left(\left(r_B^+ t_B^+\right)\omega_B\right) * (\mathfrak{s}_B^+\omega_B) (\text{by part(3) of Remark(4.1)}) \\ &= \left(r_B^+\omega_B * t_B^+\omega_B\right) * (\mathfrak{s}_B^+\omega_B) (\text{by part(3) of Remark(4.1)}) \end{aligned}$$

Let $r_B^+\omega_B \in \psi_A/\omega_B$. Then by part(5) of Remark(4.1), $r_B^{+^{-1}}$ exists such that $r_B^{+^{-1}}(\xi) = \left(r_B^+(\xi)\right)^{-1} \in \psi_A^+(\xi)$ for each $\xi \in \operatorname{Supp}^+ \omega_B$, so $r_B^{+^{-1}}\omega_B \in \psi_A/\omega_B$. Now,

$$\begin{aligned} (r_B^+\,\omega_B) * (r_B^{+^{-1}}\,\omega_B) &= (\,r_B^+\, r_B^{+^{-1}}\,)\,\omega_B \quad (\text{by part(3) of Remark(4.1)}) \\ &= e_B^+\,\omega_B \quad (\text{by part(4) of Remark(4.1)}), \text{since for all } \xi \in \operatorname{Supp}^+ \omega_B, \\ &\quad (r_B^+ r_B^{+^{-1}})(\xi) = r_B^+(\xi)\, r_B^{+^{-1}}(\xi) = e_B^+(\xi)) \\ &= \omega_B. \end{aligned}$$

Similarly, we can get $\left(r_B^{+^{-1}}\omega_B\right) * \left(r_B^+\omega_B\right) = \omega_B$. This means that each member $r_B^+\omega_B$ of $\psi_A/\omega_B$ has the inverse which is $r_B^{+^{-1}}\omega_B$. This implies that $\left(r_B^+\omega_B\right)^{-1} = r_B^{+^{-1}}\omega_B$. Thus, $(\psi_A/\omega_B, *)$ form a group. □

**Definition 4.1** Let $\omega_B$ be an $N$BS$^+$-subgroup of a BS$^+$-group $\psi_A$ over a group $R$. Then $(\psi_A/\omega_B, *)$ is called a quotient BS-group (or simply, a BS-group).

**Remark 4.2** *For simply of writing, we will write $r_B^+\omega_B\, t_B^+\omega_B$ in place of $(r_B^+\omega_B) * (t_B^+\omega_B)$.*

**Example 4.1** *Let $A = \{C_1, C_2\} = B$ and group $R = \{e, a, b, c\}$ given as in the following table.*

| $*$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

*$\psi_A^+$ and $\psi_A^-$ given as follows*

$\psi_A^+(C_1) = \{e, a\}$, $\psi_A^-(C_1) = \{b\}$
$\psi_A^+(C_2) = \{e\}$, $\psi_A^-(C_2) = \emptyset$

*And*

$\omega_B^+(C_1) = \{e, a\}$, $\omega_B^-(C_1) = \{b, c\}$
$\omega_B^+(C_2) = \{e\}$, $\omega_B^-(C_2) = \{b\}$

*Clearly, $\omega_B$ is an $NBS^+$-subgroup of $BS^+$-group $\psi_A$ and $\operatorname{Supp}^+ \psi_A = A = \operatorname{Supp}^+ \omega_B = B$. For all $r_B^+ \in \psi_A^+$. We have the following table*

|  | ${r_1}_B^+$ | ${r_2}_B^+$ |
|---|---|---|
| $\mathcal{C}_1$ | $e$ | $a$ |
| $\mathcal{C}_2$ | $e$ | $e$ |

$$\begin{aligned}
{r_1}_B^+\omega_B &= \left\{\left(e\omega_B^+(C_1), e\omega_B^-(C_1), C_1\right), \left(e\omega_B^+(C_2), e\omega_B^-(C_2), C_2\right)\right\} \\
&= \left\{\left(\omega_B^+(C_1), \omega_B^-(C_1), C_1\right), \left(\omega_B^+(C_2), \omega_B^-(C_2), C_2\right)\right\} \\
&= \left\{\left(\{e, a\}, \{b, c\}, C_1\right), \left(\{e\}, \{b\}, C_2\right)\right\} = \omega_B.
\end{aligned}$$

$$\begin{aligned}
{r_2}_B^+\omega_B &= \left\{\left(a\omega_B^+(C_1), a\omega_B^-(C_1), C_1\right), \left(e\omega_B^+(C_2), e\omega_B^-(C_2), C_2\right)\right\} \\
&= \left\{\left(a\{e, a\}, a\{b, c\}, C_1\right), \left(\omega_B^+(C_2), \omega_B^-(C_2), C_2\right)\right\} \\
&= \left\{\left(\{e, a\}, \{b, c\}, C_1\right), \left(\{e\}, \{b\}, C_2\right)\right\} = \omega_B.
\end{aligned}$$

*Clearly,* $\left({r_1}_B^+\omega_B\right)\left({r_2}_B^+\omega_B\right) = {r_2}_B^+\omega_B = \left({r_2}_B^+\omega_B\right)\left({r_1}_B^+\omega_B\right)$ *and* $\left({r_1}_B^+\omega_B\right)\left({r_1}_B^+\omega_B\right) = e_B^+\omega_B = \omega_B$*, also* $\left({r_2}_B^+\omega_B\right)\left({r_2}_B^+\omega_B\right) = \left({r_2}_B^+{r_2}_B^+\right)\omega_B = e_B^+\omega_B = \omega_B$.

*Thus,* $(\psi_A/\omega_B, *)$ *form a BS-group.*

**Remark 4.3** *Let* $\omega_B$ *be an* $NBS^+$*-subgroup of a* $BS^+$*-group* $\psi_A$ *over* $\mathcal{R}$*. Then,* $\psi_A^+(\xi)/\omega_B^+(\xi)$ *is a group for all* $\xi \in \operatorname{Supp}^+ \omega_B = \operatorname{Supp}^+ \psi_A/\omega_B$*.*

We recall the following result

**Theorem 4.2** *Let* $R$ *be a group and* $N, K$ *be two subgroups of* $R$*. If* $N \lhd R$*,* $K \lhd R$*, and* $N \subseteq K$*, then* $N \lhd K$*.*

**Theorem 4.3** *Let* $\omega_B$ *be an* $NBS^+$*-subgroup of a* $BS^+$*-group* $\psi_A$ *over* $\mathcal{R}$ *and* $D = \operatorname{Supp}^+\omega_B$*. Then:*

1. *If* $\Upsilon_K$ *is an* $NBS^+$*-subgroup of a* $BS^+$*-group* $\psi_A$ *containing* $\omega_B$*, then the quotient BS-group* $\Upsilon_K/\omega_B$ *is a normal BS-subgroup of the BS-group* $\psi_A/\omega_B$*.*

2. *If* $\Upsilon_K/\omega_B$ *is a normal BS-subgroup of* $\psi_A/\omega_B$*, then* $\Upsilon_K$ *containing* $\omega_B$ *and* $\Upsilon_D = \left\{\left(\Upsilon_K^+(\xi), \Upsilon_K^-(\xi), \xi\right) : \xi \in D\right\}$ *is an* $NBS^+$*-subgroup of* $\psi_A$ *containing* $\omega_B$*.*

**Proof:** 1. Since $\omega_B \mathrel{\widehat{\lhd}} \psi_A, \omega_B \mathrel{\widehat{\subseteq}} \Upsilon_K$ and $\Upsilon_K \mathrel{\widehat{\lhd}} \psi_A$. This means that $\omega_B^+(\xi) \lhd \psi_A^+(\xi)$ and $\omega_B^+(\xi) \subseteq \Upsilon_K^+(\xi)$ for each $\xi \in \operatorname{Supp}^+ \omega_B \subseteq \operatorname{Supp}^+ \Upsilon_K \subseteq \operatorname{Supp}^+ \psi_A$ and $\Upsilon_K^+(\xi) \lhd \psi_A^+(\xi)$ for each $\xi \in \operatorname{Supp}^+ \Upsilon_K$. Then, by Theorem (4.2), so $\omega_B^+(\xi) \lhd \Upsilon_K^+(\xi)$ for each $\xi \in \operatorname{Supp}^+ \omega_B$ and by Definition (2.7), we have $\omega_B \mathrel{\widehat{\lhd}} \Upsilon_K$. Then, by Definition (4.1), we have $\Upsilon_K/\omega_B = \mathcal{L}_B$ is a BS-group. Since $\Upsilon_K \mathrel{\widehat{\subseteq}} \psi_A$, then $\mathcal{L}_B = \Upsilon_K/\omega_B \subseteq \psi_A/\omega_B$. This means that $\mathcal{L}_B = \Upsilon_K/\omega_B$ is a BS-subgroup of $\psi_A/\omega_B$. Let $r_B^+\omega_B \in \psi_A/\omega_B$ and $t_B^+\omega_B \in \Upsilon_K/\omega_B$, where $r_B^+ \in \psi_A$ and $t_B^+ \in \Upsilon_K$. Now, $\left(r_B^+\omega_B\right)\left(t_B^+\omega_B\right)\left({r_B^+}^{-1}\omega_B\right) = \left(r_B^+t_B^+{r_B^+}^{-1}\right)\omega_B \mathrel{\widehat{\in}} \Upsilon_K/\omega_B$ (by Remark(4.1), ${r_B^+}^{-1}$ exists such that ${r_B^+}^{-1}(\xi) = \left(r_B^+(\xi)\right)^{-1} \in \psi_A^+(\xi)$ for each $\xi \in \operatorname{Supp}^+ \omega_B$ and since $\Upsilon_K^+(\xi) \lhd \psi_A^+(\xi)$ for each $\xi \in \operatorname{Supp}^+ \Upsilon_K$, we have $\left(r_B^+t_B^+{r_B^+}^{-1}\right)(\xi) = r_B^+(\xi)t_B^+(\xi){r_B^+}^{-1}(\xi) = r_B^+(\xi)t_B^+(\xi)\left(r_B^+(\xi)\right)^{-1} \in \Upsilon_K^+(\xi)$ and $r_B^+(\xi)t_B^+(\xi)\left(r_B^+(\xi)\right)^{-1}\omega_B^+(\xi) \in \Upsilon_K^+(\xi)/\omega_B^+(\xi)\Big)$. Thus, $\mathcal{L}_B = \Upsilon_K/\omega_B$ is a normal BS-subgroup of $\psi_A/\omega_B$.

2. Let $\Upsilon_K/\omega_B$ be a normal BS-subgroup of $\psi_A/\omega_B$. Clearly, $\omega_B \mathrel{\widehat{\subseteq}} \Upsilon_K$ and also $\omega_B \mathrel{\widehat{\subseteq}} \Upsilon_D$. So, we only have to show $\Upsilon_D \mathrel{\widehat{\lhd}} \psi_A$. To this end, let $\xi$ be any fixed but arbitrary point of $D = \operatorname{Supp}^+ \omega_B$. Let $r \in \psi_A^+(\xi)$ and $\mathfrak{s} \in \Upsilon_D^+(\xi)$. Then, $r\omega_B^+(\xi) \in \psi_A^+(\xi)/\omega_B^+(\xi)$ and $\mathfrak{s}\omega_B^+(\xi) \in \Upsilon_K^+(\xi)/\omega_B^+(\xi)$. Define $r_B^+ : D \to \psi_A^+$and $t_B^+ : D \to \Upsilon_K^+$as $r_B^+(\xi) = r$ and $t_B^+(\xi) = \mathfrak{s}$ and $r_B^+(e) = e_{\mathcal{R}} = t_B^+(e)$ for all $e \in D/\{\xi\}$, implies that $r_B^+\omega_B \mathrel{\widehat{\in}} \psi_A/\omega_B$ and $t_B^+\omega_B \mathrel{\widehat{\in}} \Upsilon_K/\omega_B$. Since $\Upsilon_K/\omega_B \lhd \psi_A/\omega_B$, so $\left(r_B^+\omega_B\right)\left(t_B^+\omega_B\right) = \left(t_B^+\omega_B\right)\left(r_B^+\omega_B\right)$. That is, $\left(r_B^+t_B^+\right)\omega_B = \left(t_B^+r_B^+\right)\omega_B$. This implies that

$(rs)\omega_B^+(\xi) = (sr)\omega_B^+(\xi)$, then $rsr^{-1} \in s\omega_B^+(\xi) \in \Upsilon_K^+(\xi)/\omega_B^+(\xi)$.
Hence, $rsr^{-1} \in \Upsilon_K^+(\xi)$, implies that $\Upsilon_D^+(\xi) = \Upsilon_K^+(\xi)$ is a normal subgroup in $\psi_A^+(\xi)$. Thus, $\Upsilon_K$ is an $NBS^+$-subgroup of $\psi_A$. □

**Theorem 4.4** *Let $\psi_A$ be a $BS^+$-group over $\mathcal{R}$, and $\omega_B$ be a $BS^+$-subgroup of $\psi_A$. Then, $r_B^+\omega_B = \omega_B r_B^+$ for all $r_B^+ \in \psi_A^+$ if and only if $r_B^+\omega_B r_B^{+^{-1}} = \omega_B$ for all $r_B^+ \in \psi_A^+$.*

**Proof:** Let $r_B^+\omega_B = \omega_B r_B^+$, for all $r_B^+ \in \psi_A^+$. Then

$$\begin{aligned} r_B^+\,\omega_B r_B^{+^{-1}} &= \omega_B r_B^+\, r_B^{+^{-1}} \quad \text{(by hypothesis)} \\ &= \omega_B e_B^+ = \omega_B \quad \text{(by part (4) of Remark (4.1), since for all } \xi \in \operatorname{Supp}^+\omega_B, \text{ we have} \\ &\quad (r_B^+ r_B^{+^{-1}})(\xi) = r_B^+(\xi)\, r_B^{+^{-1}}(\xi) = e_{\mathcal{R}}). \end{aligned}$$

**Conversely**, suppose that $r_B^+\omega_B r_B^{+^{-1}} = \omega_B$ for all $r_B^+ \in \psi_A^+$. Then, we have

$$\begin{aligned} r_B^+\,\omega_B &= r_B^+\,\omega_B\, e_B^+ \qquad (\text{since } \omega_B e_B^+ = \omega_B) \\ &= r_B^+\,\omega_B r_B^{+^{-1}}\, r_B^+ = \omega_B r_B^+ \qquad (\text{since } r_B^+\omega_B r_B^{+^{-1}} = \omega_B \\ &\quad \text{and by Remark (4.1), since for all } \xi \in \operatorname{Supp}^+\omega_B, \text{ we have} \\ &\quad (r_B^+ r_B^{+^{-1}})(\xi) = r_B^+(\xi)\, r_B^{+^{-1}}(\xi) = e_{\mathcal{R}}). \end{aligned}$$

□

**Definition 4.2** An $NBS^+$-subgroup $\omega_B$ of $BS^+$-group $\psi_A$ is called a maximal $NBS^+$-subgroup if there is no proper $NBS^+$-subgroup $\Upsilon_C$ of $\psi_A$ that contains $\omega_B$ properly with $\operatorname{Supp}^+\omega_B = \operatorname{Supp}^+\Upsilon_C$.

**Example 4.2** *Let $\mathcal{R} = S_3$ be the symmetric group $A = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ and $B = \{\mathcal{C}_1, \mathcal{C}_2\}$ be two sets of parameters. Consider the following BS-set over $\mathcal{R}$.*

$$\begin{aligned} \psi_A^+(C_1) &= \{(1), (12)\}, & \psi_A^-(C_1) &= \emptyset \\ \psi_A^+(C_2) &= \{(1), (13)\}, & \psi_A^-(C_2) &= \emptyset \\ \psi_A^+(C_3) &= \{(1), (23)\}, & \psi_A^-(C_3) &= \emptyset \\ \omega_B^+(C_1) &= \{(1), (12)\}, & \omega_B^-(C_1) &= \emptyset \\ \omega_B^+(C_2) &= \{(1), (13)\}, & \omega_B^-(C_2) &= \emptyset \end{aligned}$$

*Now, $\operatorname{Supp}^+\psi_A = A$ and $\operatorname{Supp}^+\omega_B = B$. Clearly, $\psi_A$ is a $BS^+$-group of $\mathcal{R}$ and $\omega_B$ is a $NBS^+$ subgroup of the $BS^+$-group $\psi_A$. Therefore, $\omega_B$ is not properly contained in any other $NBS^+$-subgroup of $\psi_A$ with the same support. Thus, $\omega_B$ is a maximal $NBS^+$-subgroup of $\psi_A$.*

**Lemma 4.2** *An $NBS^+$-subgroup $\omega_B$ of $BS^+$-group $\psi_A$ is a maximal $NBS^+$-subgroup of $\psi_A$ if and only if $\omega_B^+(\xi)$ is a maximal normal subgroup for all $\xi \in \operatorname{Supp}^+\omega_B$.*

**Proof:** Let $\omega_B$ be a maximal $NBS^+$-subgroup of $\psi_A$. If possible, suppose that there is $\xi_0 \in \operatorname{Supp}^+\omega_B = D$ such that $\omega_B^+(\xi_0)$ is not a maximal normal subgroup in $\psi_A^+(\xi_0)$. This means that there exists a normal subgroup $\mathcal{K}$ of $\psi_A^+(\xi_0)$ such that $\mathcal{K}$ is properly contained in $\psi_A^+(\xi_0)$ and properly contains $\omega_B^+(\xi_0)$. Now, consider the $NBS^+$-subgroup $\Upsilon_B$ of $\psi_A$ such that $\Upsilon_B^+(\xi) = \begin{cases} \omega_B^+(\xi) & \xi \neq \xi_0 \\ \mathcal{K} & \xi = \xi_0 \end{cases}$ and $\Upsilon_B^-(\xi) = \begin{cases} \omega_B^-(\xi) & \xi \neq \xi_0 \\ \omega_B^-(\xi)/\mathcal{K} & \xi = \xi_0 \end{cases}$

Clearly, $\Upsilon_B$ is an $NBS^{+}$-subgroup of $\psi_A, \Upsilon_B \neq \psi_A$ and $\omega_B \neq \Upsilon_B$, and $\text{Supp}^{+}\,\omega_B = \text{Supp}^{+}\,\Upsilon_C$. Thus, $\omega_B$ is not a maximal $NBS^{+}$-subgroup of $\psi_A$. This is impossible. Thus, $\omega_B^{+}(\xi)$ is a maximal normal subgroup of $\psi_A^{+}(\xi)$ for all $\xi \in D$.

Conversely, let $\omega_B$ be an $NBS^{+}$-subgroup of $\psi_A$ in which $\omega_B^{+}(\xi)$ is a maximal normal subgroup of $\psi_A^{+}(\xi)$ for all $\xi \in D = \text{Supp}\,^{+}\omega_B$. To show $\omega_B$ is a maximal $NBS^{+}$-subgroup of $\psi_A$. On the contrary, we suppose that it's not true. That is, there is an $NBS^{+}$-subgroup $\Upsilon_B$ of $\psi_A$ such that $\omega_B \neq \Upsilon_B \neq \psi_A$ and $\omega_B \widehat{\subseteq} \Upsilon_B \widehat{\subseteq} \psi_A$. This means that there exists $\xi_0 \in D$ such that $\omega_B^{+}(\xi_0) \subsetneq \Upsilon_B^{+}(\xi_0) \subsetneq \psi_A^{+}(\xi_0)$ with $\Upsilon_B^{+}(\xi_0) \lhd \psi_A^{+}(\xi_0)$ which is a contradiction. Thus, $\omega_B$ is a maximal $NBS^{+}$-subgroup of $\psi_A$. □

**Definition 4.3** Let $\psi_A$ be a $BS^{+}$-group over $\mathcal{R}$. Then,

1. $\psi_A$ is said to be an identity $BS^{+}$-group over $\mathcal{R}$ if $\psi_A^{+}(\xi) = \{e_{\mathcal{R}}\}$ and $\psi_A^{-}(\xi) = \{\mathcal{R}/\{e_{\mathcal{R}}\}\}$ for all $\xi \in \text{Supp}^{+}\,\psi_A$, where $e_{\mathcal{R}}$ is the identity element of $\mathcal{R}$. Identity $BS^{+}$-group is represented by $I_{\psi_A}$.
2. $\psi_A$ is said to be a relative absolute BS$^{+}$-group over $R$ if $\psi_A^{+}(\xi) = R$, for all $\xi \in \text{Supp}^{+}\,\psi_A$ and it will be denoted by $R_A$.
3. If $R$ is any group and $\mathcal{G}$ is a set of parameters, then the absolute BS-set $R_b = (R, \emptyset, \mathcal{G})$ is said to be an absolute BS$^{+}$-group over $R$.

**Remark 4.4** *1. Since the absolute BS-set $R_b = \{(R, \emptyset, C) : C \in \mathcal{G}\}$ is a* BS$^{+}$*-group over $R$. Then, for any group $R$ and any set of parameters $A \subseteq \mathcal{G}$ and any* BS$^{+}$*-group $\psi_A$ over $R$ can be treated as a* BS$^{+}$*-subgroup of $R_b$.*

*2. Each* BS$^{+}$*-subgroup of a* BS$^{+}$*-group $\psi_A$ over $R$ is a* BS$^{+}$*-group over $R$.*

**Definition 4.4** A $BS^{+}$-group $\psi_A$ over $\mathcal{R}$ is called a strong simple $BS^{+}$-group over $\mathcal{R}$ if and only if it has no $NBS^{+}$-subgroup except the $NBS^{+}$-groups $e_D = \left\{(e_{\mathcal{R}}, \mathcal{R}/\{e_{\mathcal{R}}\}, \xi) : \xi \in D \subseteq \text{Supp}^{+}\,\psi_A\right\}$ and $\psi_A$ itself.

**Definition 4.5** A $BS^{+}$-group $\psi_A$ over $\mathcal{R}$ is said to be a simple $BS^{+}$-group over $\mathcal{R}$, if $\omega_B = I_{\psi_A}$ or $\omega_B = \psi_A$ for all $NBS^{+}$-subgroup $\omega_B$ of $\psi_A$ with $\text{Supp}^{+}\,\omega_B = \text{Supp}^{+}\,\psi_A$.

**Example 4.3** *Take the group $R = \left\{e, m, n, n^2, mn, mn^2\right\}$ which is generated by two $m$ and $n$ which satisfy the relation* $\text{ord}(m) = 2, \text{ord}(n) = 3, \text{ord}(mn) = 2, mn = n^2m$ *and* $nm = mn^2$*. Consider the set of parameters $A = \{\alpha, \beta, \gamma\}$ such that $\psi_A^{+}(\alpha) = \{e, m\}, \psi_A^{+}(\beta) = \left\{e, n, n^2\right\}, \psi_A^{+}(\gamma) = \{e, mn\}$ and $\psi_A^{-}(\alpha) = \left\{n, n^2\right\}, \psi_A^{-}(\beta) = \{m\}, \psi_A^{-}(\gamma) = \{m, n\}$. Then, $\psi_A$ is a $BS^{+}$-group over $R$. The* NBS$^{+}$*-subgroup of $\psi_A$ are only $\psi_A$ and $I_{\psi_A} = \{(e, R/\{e\}, \alpha), (e, R/\{e\}, \beta), (e, R/\{e\}, \gamma)\}$. Thus, $\psi_A$ is a simple* BS$^{+}$*-group over $R$.*

**Theorem 4.5** *Let $\psi_A$ be a $BS^{+}$-group over $\mathcal{R}$. An $NBS^{+}$-subgroup $\omega_B$ of $\psi_A$ is a maximal $NBS^{+}$subgroup of $\psi_A$ if and only if the BS-group $\psi_A/\omega_B$ is a simple BS-group.*

**Proof:** Let $\omega_B$ be a maximal $NBS^{+}$-subgroup of $\psi_A$. On the contrary, we suppose that $\psi_A/\omega_B$ is not simple. Then, there is a normal BS-subgroup $\Upsilon_K/\omega_B$ in which $\Upsilon_K/\omega_B \neq \{\omega_B\}$ and $\Upsilon_K/\omega_B \neq \psi_A/\omega_B$. Then, $\omega_B \widehat{\subsetneq} \Upsilon_K \widehat{\subsetneq} \psi_A$ and $\Upsilon_K$ is an $NBS^{+}$-subgroup of $\psi_A$. This implies that, $\omega_B$ is not a maximal $NBS^{+}$-subgroup of $\psi_A$.

Conversely, let $\psi_A/\omega_B$ be a simple BS-group . If possible, suppose that $\omega_B$ is not a maximal $NBS^{+}$-subgroup of $\psi_A$. This implies that there is an $NBS^{+}$-subgroup $\Upsilon_B$ of $\psi_A$ with $\text{Supp}^{+}\,\omega_B = \text{Supp}^{+}\,\Upsilon_B$ and $\omega_B \widehat{\subseteq} \Upsilon_B \widehat{\subseteq} \psi_A$ with $\omega_B \neq \Upsilon_B$ and $\Upsilon_B \neq \psi_A$. Then, by Theorem (4.3), we get $\Upsilon_A/\omega_B$ is a normal BS-subgroup of $\psi_A/\omega_B$ with $\Upsilon_A/\omega_B \neq \{\omega_B\}$ and $\Upsilon_A/\omega_B \neq \psi_A/\omega_B$. Which means $\psi_A/\omega_B$ is not a simple BS-group, which is impossible. So, $\omega_B$ must be a maximal $NBS^{+}{}_{-}$ subgroup of $\psi_A$. □

## 5. Solvable $BS^{+}$-Group

In this section, we introduce the concept of a solvable $BS^{+}$-group and demonstrate its properties through representative examples.

**Definition 5.1** A $BS^{+}$-group $\psi_A$ over $\mathcal{R}$ is called a solvable $BS^{+}$-group over $\mathcal{R}$, if there exists a finite sequence of $BS^{+}$-subgroups
$\psi_A = \psi_A^n \mathrel{\widehat{>}} \ldots \mathrel{\widehat{>}} \psi_A^1 \mathrel{\widehat{>}} \psi_A^0 = \{I_{\psi_A}\}$ such that:

1. Each $\psi_A^{j-1}$ is an $NBS^{+}$-subgroup of $\psi_A^{j}$.
2. The BS-groups $\psi_A^{j}/\psi_A^{j-1}$ are abilian BS-groups.

**Example 5.1** *Let the alternating group $R = A_3$, and $A = \{C_1, C_2\}$ be the set of parameters. Consider the absolute* BS$^{+}$*-group $\psi_A = R_b$. Then,*

$$\begin{aligned}
\psi_A^{+}(C_1) &= A_3, & \psi_A^{-}(C_1) &= \emptyset\\
\psi_A^{+}(C_2) &= A_3, & \psi_A^{-}(C_2) &= \emptyset\\
\omega_B^{+}(C_1) &= A_3, & \omega_B^{-}(C_1) &= \emptyset\\
\omega_B^{+}(C_2) &= \{(1)\}, & \omega_B^{-}(C_2) &= \emptyset\\
\Upsilon_B^{+}(C_1) &= \{(1)\}, & \Upsilon_B^{-}(C_1) &= \emptyset\\
\Upsilon_B^{+}(C_2) &= \{(1)\}, & \Upsilon_B^{-}(C_2) &= \emptyset
\end{aligned}$$

*are* BS$^{+}$*-subgroups of $\psi_A$ such that $\psi_A \mathrel{\widehat{>}} \omega_A \mathrel{\widehat{>}} \Upsilon_A = I_{\psi_A}$ and* $\operatorname{Supp}^{+} \psi_A = A = \operatorname{Supp}^{+} \omega_A = \operatorname{Supp}^{+} \Upsilon_A$.

*Then, $\Upsilon_A^{+}(C_i) \lhd \omega_A^{+}(C_i)$ for all $C_i \in \operatorname{Supp}^{+} \Upsilon_A$ and $i = 1, 2$ and $\omega_A^{+}(C_i) \lhd \psi_A^{+}(C_i)$ for all $C_i \in \operatorname{Supp}^{+} \omega_A$ and $i = 1, 2$, implies that $\Upsilon_A \mathrel{\widehat{\lhd}} \omega_A$ and $\omega_A \mathrel{\widehat{\lhd}} \psi_A$.*

*Also, find*

$$\omega_A/\Upsilon_A = \left\{t_A^{+}\Upsilon_A : \text{ for all } t_A^{+} \in \omega_A^{+}\right\} = \left\{\left(t_A^{+}(C)\Upsilon_A^{+}(C), t_A^{+}(C)\Upsilon_A^{-}(C), C\right) : C \in \operatorname{Supp}^{+} \Upsilon_A, t_A^{+} \in \omega_A^{+}\right\}.$$

*For all $t_A^{+} \in \omega_A^{+}$, we have the following table:*

| | ${t_1}_A^{+}$ | ${t_2}_A^{+}$ | ${t_3}_A^{+}$ |
|---|---|---|---|
| $C_1$ | (1) | (123) | (132) |
| $C_2$ | (1) | (1) | (1) |

$$\begin{aligned}
{t_1}_A^{+}\Upsilon_A &= \left\{\left((1)\Upsilon_A^{+}(C_1), (1)\Upsilon_A^{-}(C_1), C_1\right), \left((1)\Upsilon_A^{+}(C_2), (1)\Upsilon_A^{-}(C_2), C_2\right)\right\}\\
&= \left\{\left(\Upsilon_A^{+}(C_1), \Upsilon_A^{-}(C_1), C_1\right), \left(\Upsilon_A^{+}(C_2), \Upsilon_A^{-}(C_2), C_2\right)\right\}\\
&= \{(\{(1)\}, \emptyset, C_1), (\{(1)\}, \emptyset, C_2)\} = \Upsilon_A.
\end{aligned}$$

$$\begin{aligned}
{t_2}_A^{+}\Upsilon_A &= \left\{\left((123)\Upsilon_A^{+}(C_1), (123)\Upsilon_A^{-}(C_1), C_1\right), \left((1)\Upsilon_A^{+}(C_2), (1)\Upsilon_A^{-}(C_2), C_2\right)\right\}\\
&= \{((123)\{(1)\}, \emptyset, C_1), ((1)\{(1)\}, \emptyset, C_2)\}\\
&= \{(\{(123)\}, \emptyset, C_1), (\{(1)\}, \emptyset, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
{t_3}_A^{+}\Upsilon_A &= \left\{\left((132)\Upsilon_A^{+}(C_1), (132)\Upsilon_A^{-}(C_1), C_1\right), \left((1)\Upsilon_A^{+}(C_2), (1)\Upsilon_A^{-}(C_2), C_2\right)\right\}\\
&= \{((132)\{(1)\}, \emptyset, C_1), ((1)\{(1)\}, \emptyset, C_2)\}\\
&= \{(\{(132)\}, \emptyset, C_1), (\{(1)\}, \emptyset, C_2)\}.
\end{aligned}$$

*Thus,* $\omega_A/\Upsilon_A = \left\{\Upsilon_A, t_2{}^{+}_{A}\Upsilon_A, t_3{}^{+}_{A}\Upsilon_A\right\}$*. Since* $\Upsilon_A \widehat{\lhd} \omega_A$ *by Definition (4.1), we get* $\omega_A/\Upsilon_A$ *is a BS-group over R. Clearly,*

$$\begin{aligned}
\left(t_1{}^{+}_{A}\Upsilon_A\right)\left(t_2{}^{+}_{A}\Upsilon_A\right) &= t_2{}^{+}_{A}\Upsilon_A = \left(t_2{}^{+}_{A}\Upsilon_A\right)\left(t_1{}^{+}_{A}\Upsilon_A\right)\\
\left(t_1{}^{+}_{A}\Upsilon_A\right)\left(t_3{}^{+}_{A}\Upsilon_A\right) &= t_3{}^{+}_{A}\Upsilon_A = \left(t_3{}^{+}_{A}\Upsilon_A\right)\left(t_1{}^{+}_{A}\Upsilon_A\right)\\
\left(t_2{}^{+}_{A}\Upsilon_A\right)\left(t_3{}^{+}_{A}\Upsilon_A\right) &= t_1{}^{+}_{A}\Upsilon_A = \left(t_3{}^{+}_{A}\Upsilon_A\right)\left(t_2{}^{+}_{A}\Upsilon_A\right).
\end{aligned}$$

*Thus,* $\omega_A/\Upsilon_A$ *is an abelian BS-group over R.*

*Now, find*

$$\psi_A/\omega_A = \left\{r^{+}_{A}\omega_A : \text{ for all } r^{+}_{A} \in \psi^{+}_{A}\right\} = \left\{\left(r^{+}_{A}(C)\omega^{+}_{A}(C), r^{+}_{A}(C)\omega^{-}_{A}(C), C\right) : C \in \operatorname{Supp}^{+}\omega_A, r^{+}_{A} \in \psi^{+}_{A}\right\}.$$

*For all* $r^{+}_{A} \in \psi^{+}_{A}$*, we have the following table:*

| | $r_1{}^{+}_{A}$ | $r_2{}^{+}_{A}$ | $r_3{}^{+}_{A}$ | $r_4{}^{+}_{A}$ | $r_5{}^{+}_{A}$ | $r_6{}^{+}_{A}$ | $r_7{}^{+}_{A}$ | $r_8{}^{+}_{A}$ | $r_9{}^{+}_{A}$ |
|---|---|---|---|---|---|---|---|---|---|
| $C_1$ | (1) | (1) | (1) | (123) | (123) | (123) | (132) | (132) | (132) |
| $C_2$ | (1) | (123) | (132) | (1) | (123) | (132) | (1) | (123) | (132) |

$$\begin{aligned}
r_1{}^{+}_{A}\omega_A &= \left\{\left((1)\omega^{+}_{A}(C_1), (1)\omega^{-}_{A}(C_1), C_1\right), \left((1)\omega^{+}_{A}(C_2), (1)\omega^{-}_{A}(C_2), C_2\right)\right\}\\
&= \{((1)\{(1), (123), (132)\}, \emptyset, C_1), (\{(1)\}, \emptyset, C_2)\}\\
&= \{(\{(1), (123), (132)\}, \emptyset, C_1), (\{(1)\}, \emptyset, C_2)\} = \omega_A.
\end{aligned}$$

$$\begin{aligned}
r_2{}^{+}_{A}\omega_A &= \left\{\left((1)\omega^{+}_{A}(C_1), (1)\omega^{-}_{A}(C_1), C_1\right), \left((123)\omega^{+}_{A}(C_2), (123)\omega^{-}_{A}(C_2), C_2\right)\right\}\\
&= \{((1)\{(1), (123), (132)\}, \emptyset, C_1), ((123)\{(1)\}, \emptyset, C_2)\}\\
&= \{(\{(1), (123), (132)\}, \emptyset, C_1), (\{(123)\}, \emptyset, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
r_3{}^{+}_{A}\omega_A &= \left\{\left(C_1, (1)\omega^{+}_{A}(C_1), (1)\omega^{-}_{A}(C_1)\right), \left((132)\omega^{+}_{A}(C_2), (132)\omega^{-}_{A}(C_2), C_2\right)\right\}\\
&= \{(C_1, (1)\{(1), (123), (132)\}, \emptyset), ((132)\{(1)\}, \emptyset, C_2)\}\\
&= \{(C_1, \{(1), (123), (132)\}, \emptyset), (\{(132)\}, \emptyset, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
r_4{}^{+}_{A}\omega_A &= \left\{\left((123)\omega^{+}_{A}(C_1), (123)\omega^{-}_{A}(C_1), C_1\right), \left((1)\omega^{+}_{A}(C_2), (1)\omega^{-}_{A}(C_2), C_2\right)\right\}\\
&= \{((123)\{(1), (123), (132)\}, \emptyset, C_1), (\{(1)\}, \emptyset, C_2)\}\\
&= \{(\{(1), (123), (132)\}, \emptyset, C_1), (\{(1)\}, \emptyset, C_2)\} = \omega_A.
\end{aligned}$$

$$\begin{aligned}
r_5{}^{+}_{A}\omega_A &= \left\{\left((123)\omega^{+}_{A}(C_1), (123)\omega^{-}_{A}(C_1), C_1\right), \left((123)\omega^{+}_{A}(C_2), (123)\omega^{-}_{A}(C_2), C_2\right)\right\}\\
&= \{((123)\{(1), (123), (132)\}, \emptyset, C_1), ((123)\{(1)\}, \emptyset, C_2)\}\\
&= \{(\{(1), (123), (132)\}, \emptyset, C_1), (\{(123)\}, \emptyset, C_2)\}.
\end{aligned}$$

$$\begin{aligned}
r_6{}^{+}_{A}\omega_A &= \left\{\left((123)\omega^{+}_{A}(C_1), (123)\omega^{-}_{A}(C_1), C_1\right), \left((132)\omega^{+}_{A}(C_2), (132)\omega^{-}_{A}(C_2), C_2\right)\right\}\\
&= \{((123)\{(1), (123), (132)\}, \emptyset, C_1), ((132)\{(1)\}, \emptyset, C_2)\}\\
&= \{(\{(1), (123), (132)\}, \emptyset, C_1), (\{(132)\}, \emptyset, C_2)\}.
\end{aligned}$$

$$\begin{aligned} r_7{}_A^+\omega_A &= \left\{\left((132)\omega_A^+(C_1),(132)\omega_A^-(C_1),C_1\right),\left((1)\omega_A^+(C_2),(1)\omega_A^-(C_2),C_2\right)\right\} \\ &= \{((132)\{(1),(123),(132)\},\emptyset,C_1),(\{(1)\},\emptyset,C_2)\} \\ &= \{(\{(1),(123),(132)\},\emptyset,C_1),(\{(1)\},\emptyset,C_2)\} = \omega_A. \end{aligned}$$

$$\begin{aligned} r_8{}_A^+\omega_A &= \left\{\left((132)\omega_A^+(C_1),(132)\omega_A^-(C_1),C_1\right),\left((123)\omega_A^+(C_2),(123)\omega_A^-(C_2),C_2\right)\right\} \\ &= \{((132)\{(1),(123),(132)\},\emptyset,C_1),((123)\{(1)\},\emptyset,C_2)\} \\ &= \{(\{(1),(123),(132)\},\emptyset,C_1),(\{(123)\},\emptyset,C_2)\}. \end{aligned}$$

$$\begin{aligned} r_9{}_A^+\omega_A &= \left\{\left((132)\omega_A^+(C_1),(132)\omega_A^-(C_1),C_1\right),\left((132)\omega_A^+(C_2),(132)\omega_A^-(C_2),C_2\right)\right\} \\ &= \{((132)\{(1),(123),(132)\},\emptyset,C_1),((132)\{(1)\},\emptyset,C_2)\} \\ &= \{(\{(1),(123),(132)\},\emptyset,C_1),(\{(132)\},\emptyset,C_2)\}. \end{aligned}$$

*Clearly,* $r_1{}_A^+\omega_A = r_4{}_A^+\omega_A = r_7{}_A^+\omega_A = \omega_A$, $r_2{}_A^+\omega_A = r_5{}_A^+\omega_A = r_8{}_A^+\omega_A$ *and* $r_3{}_A^+\omega_A = r_6{}_A^+\omega_A = r_9{}_A^+\omega_A$. *This implies that,* $\psi_A/\omega_A = \left\{r_1{}_A^+\omega_A, r_2{}_A^+\omega_A, r_3{}_A^+\omega_A\right\}$. *Since* $\omega_A \widehat{\lhd} \psi_A$ *by Definition (4.1), we get* $\psi_A/\omega_A$ *is a BS-group over* $R$.

*Clearly,*

$$\begin{aligned} \left(r_1{}_A^+\omega_A\right)\left(r_2{}_A^+\omega_A\right) &= r_2{}_A^+\omega_A = \left(r_2{}_A^+\omega_A\right)\left(r_1{}_A^+\omega_A\right), \\ \left(r_1{}_A^+\omega_A\right)\left(r_3{}_A^+\omega_A\right) &= r_3{}_A^+\omega_A = \left(r_3{}_A^+\omega_A\right)\left(r_1{}_A^+\omega_A\right), \\ \left(r_2{}_A^+\omega_A\right)\left(r_3{}_A^+\omega_A\right) &= r_1{}_A^+\omega_A = \left(r_3{}_A^+\omega_A\right)\left(r_2{}_A^+\omega_A\right). \end{aligned}$$

*Thus,* $\psi_A/\omega_A$ *is an abelian BS-group over* $R$. *Hence,* $\psi_A$ *is a solvable* $\mathrm{BS}^+$*-group over* $R$.

**Theorem 5.1** *Any abelian* $BS^+$*-group over* $\mathcal{R}$ *is solvable* $BS^+$*-group over* $\mathcal{R}$.

**Proof:** Let $\psi_A$ be an abelian $BS^+$-group over $\mathcal{R}$ and consider a finite sequence of $BS^+$-subgroups $\psi_A \widehat{>} \{I_{\psi_A}\} = \left\{(\xi, e_{\mathcal{R}}, \mathcal{R}/\{e_{\mathcal{R}}\}) : \xi \in \operatorname{Supp}^+ \psi_A\right\}$. Clearly, $I_{\psi_A}$ is an $NBS^+$-subgroup of $\psi_A$ and for all $r_A^+, t_A^+ \in \psi_A^+$, we have

$$r_A^+ t_A^+ I_{\psi_A} = \{(r_A^+(\xi)t_A^+(\xi)\{e_R\},\ r_A^+(\xi)t_A^+(\xi)\{R/\{e_R\}\},\ \xi) : \xi \in \operatorname{Supp}^+ \psi_A\}.$$

Since $\psi_A$ is abelian, $r_A^+(\xi)t_A^+(\xi) = t_A^+(\xi)r_A^+(\xi)$ for all $\xi$. Hence,

$$r_A^+ t_A^+ I_{\psi_A} = \{(t_A^+(\xi)r_A^+(\xi)\{e_R\},\ t_A^+(\xi)r_A^+(\xi)R/\{e_R\},\ \xi) : \xi \in \operatorname{Supp}^+ \psi_A\} = t_A^+ r_A^+ I_{\psi_A}$$

. Thus, $\psi_A/I_{\psi_A}$ is an abelian BS-group. Thus, $\psi_A$ is a solvable $BS^+$-group. □

**Theorem 5.2** *Any* $BS^+$*-subgroup of a solvable* $BS^+$*-group* $\psi_A$ *over* $\mathcal{R}$ *is a solvable* $BS^+$*-group.*

**Proof:** Let $\omega_B$ be any $BS^+$-subgroup of a solvable $BS^+$-group $\psi_A$ over $\mathcal{R}$, and let $\psi_A = \psi_A^n \widehat{>} \ldots \widehat{>} \psi_A^1 \widehat{>} \psi_A^0 = \{I_{\psi_A}\}$ be the finitie sequence of $\psi_A$ such that $\psi_A^{j-1}$ is an $NBS^+$-subgroup $\psi_A^j$ and each $\psi_A^j/\psi_A^{j-1}$ is an abelian BS-group. Since $\omega_B$ is a $BS^+$subgroup of $\psi_A$. Let $\omega_B^j = \omega_B \widehat{\cap} \psi_A^j$ for all $j = 0, 1, 2, \ldots, n$. Since for all $\xi \in \operatorname{Supp}^+ \omega_B$, we have $\omega_B^{j+}(\xi) = \omega_B^+(\xi) \cap \psi_A^{j+}(\xi)$, since $\omega_B^+(\xi)$ is a subgroup of $\psi_A^+(\xi)$ and $\psi_A^{j+}(\xi) \lhd \psi_A^+(\xi)$, so $\omega_B^+(\xi) \cap \psi_A^{j+}(\xi)$ is a subgroup of $\psi_A^+(\xi)$ and since $\omega_B^{j+}(\xi) = \omega_B^+(\xi) \cap \psi_A^{j+}(\xi) \subseteq \omega_B^+(\xi) \cap \psi_A^{j+1+}(\xi) = \omega_B^{j+1+}(\xi)$, since $\psi_A^{j+}(\xi) \lhd \psi_A^{j+1+}(\xi)$ for all $\xi \in \operatorname{Supp}{}^+\psi_B$ so $\omega_B^{j+}(\xi) \lhd \omega_B^{j+1+}(\xi)$ for all $\xi \in \operatorname{Supp}^+ \omega_B$. This means that, $\omega_B^j$ is an $NBS^+$-subgroup of $\omega_B^{j+1}$. Also, $\omega_B^j/\omega_B^{j-1} \widehat{\subseteq} \psi_A^j/\psi_A^{j-1}$ and since $\psi_A^j/\psi_A^{j-1}$ is an abelian BS-group and $\omega_B^j/\omega_B^{j-1}$ is a BS-subgroup of $\omega_B^j/\omega_B^{j-1}$, we get $\omega_B^j/\omega_B^{j-1}$ is an abelian BS-group. Thus, $\omega_B$ is a solvable $BS^+$-group over $\mathcal{R}$. □

| $*$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

**Definition 5.2** Let $\omega_B$ be a $BS^+$-subgroup of a $BS^+$-group $\psi_A$ over $\mathcal{R}$. Then, the index of $\omega_B$ in $\psi_A$, denoted by $[\psi_A : \omega_B]$, is the number of the distinct BS -left cosets of $\omega_B$ in $\psi_A$, and the positively index of $\omega_B$ in $\psi_A$, by $[\psi_A :^+ \omega_B]$, is the number of the distinct $BS^+$-left cosets of $\omega_B$ in $\psi_A$.

**Example 5.2** *Let $A = \{C_1, C_2\}$ and group $R = \{e, a, b, c\}$ given as in the following table:*

*with $\psi_A^+$ and $\psi_A^-$ given as follows:*

$$\psi_A^+(C_1) = \{e, a\}, \qquad \psi_A^-(C_1) = \{b\}$$
$$\psi_A^+(C_2) = \{e\}, \qquad \psi_A^-(C_2) = \emptyset$$

*And*

$$\omega_B^+(C_1) = \{e, a\}, \qquad \omega_B^-(C_1) = \{b\}$$
$$\omega_B^+(C_2) = \{e\}, \qquad \omega_B^-(C_2) = \emptyset$$

*Clearly, $\omega_B$ is a $\mathrm{BS}^+$-subgroup of $\mathrm{BS}^+$-group $\psi_A$ and $\mathrm{Supp}^+ \psi_A = A = B = \mathrm{Supp}^+ \omega_B$. For all $r_B^+ \in \psi_A^+$, we present the table:*

| | $r1_B^+$ | $r2_B^+$ |
|---|---|---|
| $\mathcal{C}_1$ | $e$ | $a$ |
| $\mathcal{C}_2$ | $e$ | $e$ |

$$\begin{aligned} r1_B^+\omega_B &= \{(e\omega_B^+(C_1), e\omega_B^-(C_1), C_1), (e\omega_B^+(C_2), e\omega_B^-(C_2), C_2)\} \\ &= \{(\omega_B^+(C_1), \omega_B^-(C_1), C_1), (\omega_B^+(C_2), \omega_B^-(C_2), C_2)\} \\ &= \{(\{e, a\}, \{b\}, C_1), (\{e\}, \emptyset, C_2)\} = \omega_B. \end{aligned}$$

$$\begin{aligned} r2_B^+\omega_B &= \{(a\omega_B^+(C_1), a\omega_B^-(C_1), C_1), (e\omega_B^+(C_2), e\omega_B^-(C_2), C_2)\} \\ &= \{(a\{e, a\}, a\{b\}, C_1), (\omega_B^+(C_2), \omega_B^-(C_2), C_2)\} \\ &= \{(\{e, a\}, \{c\}, C_1), (\{e\}, \emptyset, C_2)\}. \end{aligned}$$

*$r1_B^+\omega_B$, $r2_B^+\omega_B$ are distinct BS-left cosets of $\omega_B$ in $\psi_A$. Thus, $[\psi_A : \omega_B] = 2$.*
*On the other hand,*

$$\begin{aligned} r1_B^+\omega_B^+ &= \{r1_B^+(C_1)\omega_B^+(C_1), r1_B^+(C_2)\omega_B^+(C_2)\} = \{\{e, a\}, \{e\}\}, \\ r2_B^+\omega_B^+ &= \{r2_A^+(C_1)\omega_B^+(C_1), r2_B^+(C_2)\omega_B^+(C_2)\} = \{\{e, a\}, \{e\}\}. \end{aligned}$$

*This implies that $r1_B^+\omega_B^+ = r2_B^+\omega_B^+$ is the only $\mathrm{BS}^+$-left coset of $\omega_B$ in $\psi_A$. Thus, $[\psi_A :^+ \omega_B] = 1$.*

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