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Estimate of Hankel Determinant Bounds for the Class of Three-Leaf-Type Bounded Turning Functions Defined with Subordination

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ABSTRACT: The motive of the present investigation is to study a generalized subclass of bounded turning functions associated with three-leaf type function in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. We focus on the computation of sharp upper bounds of the first four coefficients, Fekete-Szegö inequality, second Hankel determinant, Zalcman inequality and third Hankel determinant, for the class defined here. Furthermore, these results have also been studied for two-fold and three-fold symmetric functions. Some earlier known results follow as consequences of the results derived in this paper.

Keywords: Analytic functions, subordination, bounded turning function, coefficient inequality, Hankel determinant, three-leaf type function.

Contents

1. Introduction

Let $\mathcal{A}$ denotes the class of functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. In addition to this, if any function in the class $\mathcal{A}$ is injective, then it is known as univalent function and the class of univalent functions is denoted by $\mathcal{S}$.

Bieberbach's conjecture, the most important and interesting result of 20^{th} century, was established by L. Bieberbach [9] in 1916 and is concerned with the class $\mathcal{S}$. It states that, if $f \in \mathcal{S}$ is a univalent function, then $|a_n| \leq n$, $n = 2, 3, \ldots$. Several researchers used their diversified approaches to prove this challenging result. Finally, L. De-Branges [13] proved this conjecture in 1985. During the course of proving this conjecture, various inequalities involving moduli of coefficients a_n were come into existence which helped in defining certain new subclasses of analytic functions.

The well known classes $\mathcal{S}^*$ of starlike functions and $\mathcal{K}$ of convex functions are respectively defined as:

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$$

and

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

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Reade [34] introduced the class $\mathcal{CS}^*$ of close-to-star functions which is given by

$$\mathcal{CS}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{f(z)}{g(z)}\right) > 0, g \in \mathcal{S}^*, z \in E \right\}.$$

Taking $g(z) = z$ in the above class, MacGregor [26] studied the following subclass of close-to-star functions:

$$\mathcal{R}^{'} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{f(z)}{z}\right) > 0, z \in E \right\}.$$

Further by replacing $f(z)$ to $zf'(z)$ in $\mathcal{R}^{'}$, MacGregor [25] established the class $\mathcal{R}$ of bounded turning functions which is defined as

$$\mathcal{R} = \{ f : f \in \mathcal{A}, Re(f'(z)) > 0, z \in E \}.$$

As a generalization, Murugusundaramoorthy and Magesh [27] studied the class $\mathcal{R}(\alpha)$ defined as

$$\mathcal{R}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right) > 0, 0 \le \alpha \le 1, z \in E \right\}.$$

Clearly $\mathcal{R}(\alpha)$ is the unification of the classes $\mathcal{R}^{'}$ and $\mathcal{R}$ as $\mathcal{R}(0) \equiv \mathcal{R}^{'}$ and $\mathcal{R}(1) \equiv \mathcal{R}$.

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (denoted by $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Moreover, if g is univalent in E, then the subordination leads to $f(0) = g(0)$ and $f(E) \subset g(E)$. Using the concept of subordination, various subclasses of $\mathcal{S}$ were studied by several authors by associating to different super-ordinating functions $\phi(z)$, some of which are mentioned below:
(i) Janowski [15] studied the class $\mathcal{S}^*(A, B)$ for $\phi(z) = \frac{1+Az}{1+Bz}$.
(ii) For $\phi(z) = 1 + sinz$, Cho et al. [12] studied the class $\mathcal{S}^*_{sin}$.
(iii) Taking $\phi(z) = e^z$, Arif et al. [4] studied the class $\mathcal{S}^*_e$.
(iv) Chosing $\phi(z) = 1 + z - \frac{z^3}{3}$, Wani and Swaminathan [46] studied the class $\mathcal{S}_N$.
(v) Sokol and Stankiewicz [43] studied the class $\mathcal{S}^*_L$ associated with $\phi(z) = \sqrt{1+z}$.
(vi) For $\phi(z) = z + \sqrt{1+z^2}$, Raina and Sokol [31] studied the class $\mathcal{S}_C$.
(vii) Considering $\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, Sharma et al. [38] studied the class $\mathcal{S}^*_C$.
(viii) For $\phi(z) = 1 + sinh^{-1}z$, Arora and Kumar [6] studied the class $\mathcal{S}^*_p$.
(ix) For $\phi(z) = \frac{2}{1+e^{-z}}$, Goel and Kumar [14] studied the class $\mathcal{S}^*_{SG}$.
(x) Alotaibi et al. [1] studied the class $\mathcal{S}^*_{Cosh}$ related to $\phi(z) = coshz$.

For $q \ge 1$ and $n \ge 1$, Pommerenke [29] defined the q^{th} Hankel determinant $H_q(n)$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & \dots & \dots & a_{n+2q-2} \end{vmatrix}.$$

For different values of q and n, the Hankel determinant $H_q(n)$ reduces to various coefficient functionals. For $q = 2$ and $n = 1$, it redues to $H_2(1) = a_3 - a_2^2$, which is the well known Fekete-Szegö functional. For $q = 2$ and $n = 2$, $H_q(n)$ takes the form of $H_2(2) = a_2a_4 - a_3^2$, which is known as Hankel determinant of second order and for $q = 3$ and $n = 1$, it agrees with $H_3(1)$, which is the Hankel determinant of third order.

One more very important and useful functional is given by $J_{n,m}(f) = a_na_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$. This functional is known as generalized Zalcman functional and was introduced by Ma [24]. For $n = 2, m = 3$, it reduces to $J_{2,3}(f) = a_2a_3 - a_4$. The upper bound for the functional $J_{2,3}(f)$ was computed by various authors over different subclasses of analytic functions. It plays very important role in establishing the bounds for the third Hankel determinant.

Now a days, the estimation of Hankel determinants for various subclasses of analytic functions is a topic of great interest. The second Hankel determinants have been studied by various researchers for different subclasses of $\mathcal{A}$. Janteng et al. [16] investigated the second Hankel determinant for the classes of starlike functions, convex functions and the class of functions with bounded boundary rotation. Further, Lee et al. [20], Altinkya and Yalcin [3], Bansal [8], Caglar et al. [11], Kanas et al. [17] and Liu et al. [23] studied the second Hankel determinant for some important subclasses of $\mathcal{A}$.

To obtain the sharp upper bounds of third Hankel determinant for different subclasses of $\mathcal{A}$ is quite challenging. Babalola [7] was the first researcher who successfully obtained the upper bound of third Hankel determinant for some fundamental classes such as the classes of starlike and convex functions. Further, Shanmugam et al. [37], Bucur et al. [10], Khan et al. [19], Altinkaya and Yalcin [2], Singh and Singh [39], Singh et al. [40,41,42], Sun et al. [44], Riaz et al. [35], Raza et al. [33], Sunthrayuth et al. [45] and few other researchers have been actively engaged in the study of third Hankel determinant for various subclasses of analytic functions.

Recently, Arif et al. [5], Murugusundaramoorthy et al. [28] and Riaz et al. [36] established the sharp upper bounds of Hankel determinants for various subclasses of analytic functions associated with the function $1+\frac{4}{5}z+\frac{1}{5}z^4$, which is known as three-leaf function. This function maps the unit disc onto the region that resembles a three leaf clover.

Inspired by the above remarkable vital research in this direction, now we introduce a new and generalized subclass of bounded turning functions associated with three-leaf type function and defined with subordination.

Definition 1.1 A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_{3l}(\alpha)$ $(0 \le \alpha \le 1)$ if it satisfies the condition

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z) \prec 1+\frac{4}{5}z+\frac{1}{5}z^4.$$

For $\alpha = 0$, the class $\mathcal{R}_{3l}(\alpha)$ reduces to the class $\mathcal{R}'_{3l}$. Further for $\alpha = 1$, the class $\mathcal{R}_{3l}(\alpha)$ agrees with $\mathcal{R}_{3l}$, the class studied by Arif et al. [5] and Murugusundaramoorthy et al. [28].

The aim of this paper is to establish the sharp upper bounds of third Hankel determinant for the class $\mathcal{R}_{3l}(\alpha)$. Moreover the bounds of $H_3(1)$ are studied for the two-fold and three-fold symmetric functions. Various known results follow as special cases.

2. Preliminary Results

By $\mathcal{P}$, we denote the class of analytic functions p given by

$$p(z) = 1+\sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

In order to prove our main results, we need the following lemmas:

Lemma 2.1 *[18,38] If $p \in \mathcal{P}$, then*

$$|p_k| \le 2, k \in \mathbb{N},$$

$$\left|p_2-\frac{p_1^2}{2}\right| \le 2-\frac{|p_1|^2}{2},$$

$$|p_{i+j}-\mu p_i p_j| \le 2, 0 \le \mu \le 1,$$

and for complex number ρ, we have

$$|p_2-\rho p_1^2| \le 2max\{1, |2\rho-1|\}.$$

Lemma 2.2 *[4] Let $p \in \mathcal{P}$, then*

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2|J| + 2|K-2J| + 2|J-K+L|,$$

where J, K, L are real numbers.
In particular, it is proved in [30] that

$$|p_1^3 - 2p_1p_2 + p_3| \leq 2.$$

Lemma 2.3 *[21,22] If $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + (4-p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z,$$

for $|x| \leq 1$ and $|z| \leq 1$.

Lemma 2.4 *[32] Let m, n, l and r satisfy the inequalities $0 < m < 1$, $0 < r < 1$ and*

$$8r(1-r)\left[(mn-2l)^2 + (m(r+m)-n)^2\right] + m(1-m)(n-2rm)^2 \leq 4m^2(1-m)^2r(1-r).$$

If $p \in \mathcal{P}$, then

$$\left|lp_1^4 + rp_2^2 + 2mp_1p_3 - \frac{3}{2}np_1^2p_2 - p_4\right| \leq 2.$$

3. Bounds for First four Coefficients

Theorem 3.1 *If $f \in \mathcal{R}_{3l}(\alpha)$, then*

$$|a_2| \leq \frac{4}{5(1+\alpha)}, \tag{3.1}$$

$$|a_3| \leq \frac{4}{5(1+2\alpha)}, \tag{3.2}$$

$$|a_4| \leq \frac{4}{5(1+3\alpha)}, \tag{3.3}$$

and

$$|a_5| \leq \frac{4}{5(1+4\alpha)}. \tag{3.4}$$

The results are sharp.

Proof: Using the principle of subordination in Definition 1.1, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + \frac{4}{5}w(z) + \frac{1}{5}(w(z))^4. \tag{3.5}$$

Taking $p(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + ...$, we obtain $w(z) = \frac{p(z)-1}{p(z)+1}$.
For $f \in \mathcal{A}$, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + (1+\alpha)a_2z + (1+2\alpha)a_3z^2 + (1+3\alpha)a_4z^3 + (1+4\alpha)a_5z^4 + ... \tag{3.6}$$

On simplification, it can be easily obtained that
$1 + \frac{4}{5}w(z) + \frac{1}{5}(w(z))^4 = 1 + \frac{2}{5}p_1z + \left(\frac{2}{5}p_2 - \frac{1}{5}p_1^2\right)z^2$

$$+\left(\frac{1}{10}p_1^3 - \frac{2}{5}p_1p_2 + \frac{2}{5}p_3\right)z^3 + \left(-\frac{3}{80}p_1^4 + \frac{3}{10}p_1^2p_2 - \frac{2}{5}p_3p_1 - \frac{1}{5}p_2^2 + \frac{2}{5}p_4\right)z^4 + ... \tag{3.7}$$

Using (3.6) and (3.7) in (3.5), it yields
$1+(1+\alpha)a_2z+(1+2\alpha)a_3z^2+(1+3\alpha)a_4z^3+(1+4\alpha)a_5z^4+...$
$=1+\frac{2}{5}p_1z+\left(\frac{2}{5}p_2-\frac{1}{5}p_1^2\right)z^2+\left(\frac{1}{10}p_1^3-\frac{2}{5}p_1p_2+\frac{2}{5}p_3\right)z^3$

$$+\left(-\frac{3}{80}p_1^4+\frac{3}{10}p_1^2p_2-\frac{2}{5}p_3p_1-\frac{1}{5}p_2^2+\frac{2}{5}p_4\right)z^4+... \tag{3.8}$$

Equating the coefficients of z, z^2, z^3 and z^4 in (3.8), it gives

$$a_2=\frac{2}{5(1+\alpha)}p_1, \tag{3.9}$$

$$a_3=\frac{1}{5(1+2\alpha)}\left[2p_2-p_1^2\right], \tag{3.10}$$

$$a_4=\frac{1}{1+3\alpha}\left[\frac{1}{10}p_1^3-\frac{2}{5}p_1p_2+\frac{2}{5}p_3\right], \tag{3.11}$$

and

$$a_5=\frac{1}{1+4\alpha}\left[-\frac{3}{80}p_1^4-\frac{1}{5}p_2^2-\frac{2}{5}p_3p_1+\frac{3}{10}p_1^2p_2+\frac{2}{5}p_4\right]. \tag{3.12}$$

Using first inequality of Lemma 2.1 in (3.9), the result (3.1) is obvious.
10) can be writtes as,

$$|a_3|=\frac{2}{5(1+2\alpha)}\left|p_2-\frac{1}{2}p_1^2\right|. \tag{3.13}$$

Using fourth inequality of Lemma 2.1 in (3.13), the result (3.2) can be easily obtained.
(3.11) can be expressed as

$$|a_4|=\frac{2}{5(1+3\alpha)}\left|\frac{1}{4}p_1^3-p_1p_2+p_3\right|. \tag{3.14}$$

Using Lemma 2.2 in (3.14), it leads to result (3.3).
Further, using Lemma 2.4 in (3.12), the result (3.4) is obvious. □

Equality in the results (3.1), (3.2), (3.3) and (3.4) is attained for the functions f_1, f_2, f_3 and f_4, respectively given by

$$(1-\alpha)\frac{f_1(z)}{z}+\alpha f_1'(z)=1+\frac{4}{5}z+\frac{1}{5}z^4, \tag{3.15}$$

$$(1-\alpha)\frac{f_2(z)}{z}+\alpha f_2'(z)=1+\frac{4}{5}z^2+\frac{1}{5}z^8, \tag{3.16}$$

$$(1-\alpha)\frac{f_3(z)}{z}+\alpha f_3'(z)=1+\frac{4}{5}z^3+\frac{1}{5}z^{12}, \tag{3.17}$$

$$(1-\alpha)\frac{f_4(z)}{z}+\alpha f_4'(z)=1+\frac{4}{5}z^4+\frac{1}{5}z^{16}. \tag{3.18}$$

On putting $\alpha=0$, Theorem 3.1 yields the following result:

Corollary 3.1 *If $f\in\mathcal{R}_{3l}^{'}$, then*

$$|a_2|\le\frac{4}{5}, |a_3|\le\frac{4}{5}, |a_4|\le\frac{4}{5}, |a_5|\le\frac{4}{5}.$$

For $\alpha=1$, Theorem 3.1 gives the following result due to Murugusundaramoorthy et al. [28]:

Corollary 3.2 *If $f\in\mathcal{R}_{3l}$, then*

$$|a_2|\le\frac{2}{5}, |a_3|\le\frac{4}{15}, |a_4|\le\frac{1}{5}, |a_5|\le\frac{4}{25}.$$

4. Fekete-Szegö Inequality

Theorem 4.1 *If $f \in \mathcal{R}_{3l}(\alpha)$ and μ is any complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{4}{5(1+2\alpha)} max \left\{1, \frac{8(1+2\alpha)|\mu|}{10(1+\alpha)^2}\right\}. \tag{4.1}$$

The bound is sharp.

Proof: Using (3.9) and (3.10), we obtain

$$|a_3 - \mu a_2^2| = \frac{2}{5(1+2\alpha)} \left|p_2 - \frac{5(1+\alpha)^2 + 4(1+2\alpha)\mu}{10(1+\alpha)^2} p_1^2\right|. \tag{4.2}$$

Using fourth inequality of Lemma 2.1, (4.2) leads to

$$|a_3 - \mu a_2^2| \leq \frac{4}{5(1+2\alpha)} max \left\{1, \frac{8(1+2\alpha)|\mu|}{10(1+\alpha)^2}\right\}.$$

Equality in the result (4.1) is attained for the function f_2 defined in (3.16). □

Substituting for $\alpha = 0$, Theorem 4.1 yields the following result:

Corollary 4.1 *If $f \in \mathcal{R}'_{3l}$, then*

$$|a_3 - \mu a_2^2| \leq \frac{4}{5} max \left\{1, \frac{4}{5}|\mu|\right\}.$$

Putting $\alpha = 1$, Theorem 4.1 gives the following result due to Murugusundaramoorthy et al. [28]:

Corollary 4.2 *If $f \in \mathcal{R}_{3l}$, then*

$$|a_3 - \mu a_2^2| \leq \frac{4}{15} max \left\{1, \frac{3}{5}|\mu|\right\}.$$

For $\mu = 1$, Theorem 4.1 yields the following result:

Corollary 4.3 *If $f \in \mathcal{R}_{3l}(\alpha)$, then*

$$|a_3 - a_2^2| \leq \frac{4}{5(1+2\alpha)}.$$

For $\alpha = 0$, Corollary 4.1C coincides with the following result:

Corollary 4.4 *If $f \in \mathcal{R}'_{3l}$, then*

$$|a_3 - a_2^2| \leq \frac{4}{5}.$$

For $\alpha = 1$, Corollary 4.1C agrees with the following result due to Murugusundaramoorthy et al. [28]:

Corollary 4.5 *If $f \in \mathcal{R}_{3l}$, then*

$$|a_3 - a_2^2| \leq \frac{4}{15}.$$

5. Zalcman Inequality

Theorem 5.1 *If $f \in \mathcal{R}_{3l}(\alpha)$, then*

$$|a_2a_3 - a_4| \leq \frac{4}{5(1+3\alpha)}. \tag{5.1}$$

The estimate is sharp.

Proof: Using (3.9), (3.10), (3.11) and after some easy calculations, we obtain
$|a_2a_3 - a_4| = \frac{1}{50(1+\alpha)(1+2\alpha)(1+3\alpha)}$

$$\times \left|(9+27\alpha+10\alpha^2)p_1^3 - (28+84\alpha+40\alpha^2)p_1p_2 + (20+60\alpha+40\alpha^2)p_3\right|. \tag{5.2}$$

Applying Lemma 2.2 in (5.2), the result (5.1) can be easily obtained. □

Equality in (5.1) holds for the function f_3 defined in (3.17).

For $\alpha = 0$, the following result is a consequence of Theorem 5.1.

Corollary 5.1 *If $f \in \mathcal{R}_{3l}^{'}$, then*

$$|a_2a_3 - a_4| \leq \frac{4}{5}.$$

On putting $\alpha = 1$ in Theorem 5.1, we can obtain the following result due to Murugusundaramoorthy et al. [28]:

Corollary 5.2 *If $f \in \mathcal{R}_{3l}$, then*

$$|a_2a_3 - a_4| \leq \frac{1}{5}.$$

6. Second Hankel Determinant

Theorem 6.1 *If $f \in \mathcal{R}_{3l}(\alpha)$, then*

$$|a_2a_4 - a_3^2| \leq \frac{16}{25(1+2\alpha)^2}. \tag{6.1}$$

The result is sharp.

Proof: Using (3.9), (3.10) and (3.11), we have
$|a_2a_4 - a_3^2| = \frac{1}{25(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$

$$\times \left|4(1+2\alpha)^2p_1p_3 - 4\alpha^2p_1^2p_2 + \alpha^2p_1^4 - 4(1+\alpha)(1+3\alpha)p_2^2\right|.$$

Substituting for p_2 and p_3 from Lemma 2.3 and letting $p_1 = p$, we get

$$|a_2a_4 - a_3^2| = \frac{1}{25(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Bigg| - (1+2\alpha)^2p^2(4-p^2)x^2$$

$$-(1+4\alpha+3\alpha^2)(4-p^2)^2x^2 + 2(1+2\alpha)^2p(4-p^2)(1-|x|^2)z\Bigg|.$$

Since $|p| = |p_1| \leq 2$, we may assume that $p \in [0,2]$. Using the triangle inequality and $|z| \leq 1$ with $|x| = t \in [0,1]$, we obtain
$|a_2a_4 - a_3^2| \leq \frac{1}{25(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$

$$\times \Bigg[(1+2\alpha)^2p^2(4-p^2)t^2 + (1+4\alpha+3\alpha^2)(4-p^2)^2t^2$$

$$+2(1+2\alpha)^2p(4-p^2) - 2(1+2\alpha)^2p(4-p^2)t^2\Bigg] = F(p,t).$$

$$\frac{\partial F}{\partial t}=\frac{2(4-p^2)}{25(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Bigg[(1+2\alpha)^2p^2t+(1+4\alpha+3\alpha^2)(4-p^2)t-2(1+2\alpha)^2pt\Bigg].$$

Easily we can verify that $\frac{\partial F}{\partial t}\geq 0$ and so $F(p,t)$ is an increasing function of t.
Therefore, $max\{F(p,t)\}=F(p,1)=\frac{1}{25(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$
$\times\Bigg[(1+2\alpha)^2p^2(4-p^2)+(1+4\alpha+3\alpha^2)(4-p^2)^2\Bigg]=H(p).$
$H'(p)=0$ gives $p=0$. Also $H''(p)<0$ for $p=0$.
Therefore $max\{H(p)\}=H(0)=\dfrac{16}{25(1+2\alpha)^2}$, which proves (6.1). □

The result (6.1) is sharp for the function f_2 defined in (3.16).

Putting $\alpha=0$, Theorem 6.1 gives the following result:

Corollary 6.1 *If $f\in\mathcal{R}_{3l}'$, then*

$$|a_2a_4-a_3^2|\leq\frac{16}{25}.$$

Substituting for $\alpha=1$ in Theorem 6.1, the following result due to Murugusundaramoorthy et al. [28], is obvious:

Corollary 6.2 *If $f\in\mathcal{R}_{3l}$, then*

$$|a_2a_4-a_3^2|\leq\frac{16}{225}.$$

7. Third Order Hankel Determinant $H_3(1)$

On expanding, the third Hankel determinant can be expressed as

$$H_3(1)=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2),$$

and after applying the triangle inequality, it yields

$$|H_3(1)|\leq|a_3||a_2a_4-a_3^2|+|a_4||a_2a_3-a_4|+|a_5||a_3-a_2^2|. \tag{7.1}$$

Theorem 7.1 *If $f\in\mathcal{R}_{3l}(\alpha)$, then*

$$|H_3(1)|\leq\frac{224+2240\alpha+7952\alpha^2+11584\alpha^3+5440\alpha^4}{125(1+2\alpha)^3(1+3\alpha)^2(1+4\alpha)}. \tag{7.2}$$

Estimate is sharp.

Proof: By using (3.3), (3.4), (3.5), (5.1), (6.1) and Corollary 4.1C in (7.1), the result (7.2) can be easily obtained. □

The result (7.2) is sharp for the function f_3 defined in (3.17).

For $\alpha=0$, Theorem 7.1 yields the following result:

Corollary 7.1 *If $f\in\mathcal{R}_{3l}'$, then*

$$|H_3(1)|\leq\frac{224}{125}.$$

For $\alpha=1$, Theorem 7.1 gives the following result due to Murugusundaramoorthy et al. [28]:

Corollary 7.2 *If $f\in\mathcal{R}_{3l}$, then*

$$|H_3(1)|\leq 0.10163.$$

8. Bounds of $|H_3(1)|$ for Two-Fold and Three-Fold Symmetric Functions

A function f is said to be n-fold symmetric function if it satisfies the following condition,

$$f(\xi z) = \xi f(z)$$

where $\xi = e^{\frac{2\pi i}{n}}$ and $z \in E$.
By $S^{(n)}$, we denote the set of all n-fold symmetric functions which belong to the class S.
The n-fold univalent function have the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1}. \tag{8.1}$$

An analytic function f of the form (8.1) belongs to the family $\mathcal{R}_{3l}^{(n)}(\alpha)$ if and only if

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4, p \in \mathcal{P}^{(n)},$$

where

$$\mathcal{P}^{(n)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} p_{nk} z^{nk}, z \in E \right\}. \tag{8.2}$$

Theorem 8.1 *If $f \in \mathcal{R}_{3l}^{(2)}(\alpha)$, then*

$$|H_3(1)| \leq \frac{16}{25(1+2\alpha)(1+4\alpha)}. \tag{8.3}$$

Proof: If $f \in \mathcal{R}_{3l}^{(2)}(\alpha)$, then there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4. \tag{8.4}$$

Using (8.1) and (8.2) for $n = 2$, (8.4) yields

$$a_3 = \frac{2}{5(1+2\alpha)} p_2, \tag{8.5}$$

$$a_5 = \frac{2}{5(1+4\alpha)}\left(p_4 - \frac{1}{2}p_2^2\right). \tag{8.6}$$

Also

$$H_3(1) = a_3 a_5 - a_3^3. \tag{8.7}$$

Using (8.5) and (8.6) in (8.7), it yields

$$H_3(1) = \frac{4}{25(1+2\alpha)(1+4\alpha)} p_2 \left[p_4 - \frac{5(1+2\alpha)^2 + 4(1+4\alpha)}{10(1+2\alpha)^2} p_2^2 \right]. \tag{8.8}$$

Taking modulus and using third inequality of Lemma 2.1 in (8.8), we can easily get the result (8.3). □

Putting $\alpha = 0$, the following result can be easily obtained from Theorem 8.1.

Corollary 8.1 *If $f \in \mathcal{R}_{3l}^{'(2)}$, then*

$$|H_3(1)| \leq \frac{16}{25}.$$

For $\alpha = 1$, Theorem 8.1 agrees with the following result due to Murugusundaramoorthy et al. [28].

Corollary 8.2 *If* $f \in \mathcal{R}_{3l}^{(2)}$, *then*

$$|H_3(1)| \leq \frac{16}{375}.$$

Theorem 8.2 *If* $f \in \mathcal{R}_{3l}^{(3)}(\alpha)$, *then*

$$|H_3(1)| \leq \frac{16}{25(1+3\alpha)^2}. \tag{8.9}$$

The bound is sharp.

Proof: If $f \in \mathcal{R}_{3l}^{(3)}(\alpha)$, so there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4. \tag{8.10}$$

Using (8.1) and (8.2) for $n = 3$ in (8.10), it gives

$$a_4 = \frac{2}{5(1+3\alpha)} p_3. \tag{8.11}$$

Also

$$H_3(1) = -a_4^2. \tag{8.12}$$

Using (8.11) in (8.12), it yields

$$H_3(1) = -\frac{4}{25(1+3\alpha)^2} p_3^2. \tag{8.13}$$

Taking modulus and using first inequality of Lemma 2.1, (8.9) can be easily obtained from (8.13). □

Equality in (8.9) is attained for the function f_3 defined in (3.17).

Putting $\alpha = 0$ in Theorem 8.2, it gives the following result:

Corollary 8.3 *If* $f \in \mathcal{R}_{3l}^{'(3)}$, *then*

$$|H_3(1)| \leq \frac{16}{25}.$$

For $\alpha = 1$, Theorem 8.2 yields the following result due to Murugusundaramoorthy et al. [28].

Corollary 8.4 *If* $f \in \mathcal{R}_{3l}^{(3)}$, *then*

$$|H_3(1)| \leq \frac{1}{25}.$$

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