Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 12** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.81484

Unified Isomorphism Theory for r-Neutrosophic G- Submodules [*]

Preema T. S. and Binu R.

ABSTRACT: Neutrosophy is a logical framework that treats uncertainty, truth and falsity as distinct concepts hence better representing ambiguous information. In this paper, we analyze r-neutrosophic G-submodules, where $r \in [0,1]$, that bring about the application of known neutrosophic ideas in a more constrained algebraic context. The concept of r-neutrosophic quotient submodules is introduced as an extension of this and some characteristics are recognized and explained. The paper then focuses on the isomorphism properties of r-neutrosophic G-submodules by analysing and proving a version of the isomorphism theorems modified to this limited setting. The results show that the fundamental characteristics of the general neutrosophic G-submodule isomorphism is preserved in this constrained form. This illustrates that r-neutrosophic G-submodules maintain the essential algebraic structure of the extensive framework.

Keywords: Neutrosophic set, neutrosophic G-submodule, support, r-neutrosophic G-submodule, r-neutrosophic G-submodule isomorphism.

Contents

1. Introduction

The analysis of uncertainty in Mathematics began with fuzzy sets [15] which allowed elements to have degrees of membership rather than fitting into classical categories. Intuitionistic fuzzy sets [8] refined this approach by differentiating membership from non-membership, capturing partial information or hesitation . Neutrosophic sets [13], introduced by Smarandache, extended the idea further by treating indeterminacy, truth and falsity distinctly, enabling incomplete or inconsistent data modeling. Further developing these concepts, neutrosophic algebraic structures—such as modules and submodules—incorporate uncertainty into elements and operations. Their connection with group representations leads to neutrosophic G-submodules, providing a solid framework for studying uncertainty-based symmetry.

Latest research has advanced neutrosophic theory by introducing concepts such as Complex Neutrosophic Soft Groups [11], which integrate soft-set theory and neutrosophic membership functions to extend algebraic frameworks with truth, indeterminacy, and falsity components. In addition, Neutrosophic Soft Hyperalgebras [10] have been developed to generalize classical algebraic operations like union and intersection, providing a more flexible structure for uncertainty-based reasoning. Decision-making frameworks has been strengthened by the developments in neutrosophic theory [9] by introducing models that more successfully model complex and vague information.

Bounding neutrosophic components within r-neutrosophy prevents unregulated variations in membership values and improves numerical stability. In areas where exact control over uncertainty is vital, such as optimization, classification, algebraic structure analysis and reasoning, it provides an adjustable framework.

---

[*] This research received no external funding.
2020 *Mathematics Subject Classification:* 03E72, 16D10, 20C05.
Submitted January 22, 2026. Published April 21, 2026

2. Preliminary Concepts

For a better understanding of the sessions that follow, some of the fundamental concepts and results are discussed here.

Definition 2.1 [3,12] "Let $(G,*)$ be a group. A vector space S over the field F is called a *G-module*, if for every $g \in G$ and $s \in S$; $\exists$ a product (called the action of G on M), $g \cdot s \in S$ that satisfies the following axioms

1. $1 \cdot s = s; \quad \forall s \in S$ (1 being the identity element of G)
2. $(l * m) \cdot s = l \cdot (m \cdot s); \quad \forall s \in S$ and $l, m \in G$
3. $g \cdot (k_1 s_1 + k_2 s_2) = k_1(g \cdot s_1) + k_2(g \cdot s_2); \forall\, k_1, k_2 \in K; g \in G; s_1, s_2 \in S$ ".

Theorem 2.1 [5,6,4] "(Module-Theoretic First Isomorphism Theorem) - Let $\phi : S \to T$ be an onto R module homomorphism then $S/Ker\phi \cong Img\phi$ "

Theorem 2.2 [5,6] "(Module-Theoretic Second Isomorphism Theorem) - Let S be a module and M,N be submodules of S, then
i) $M + N = \{\, m + n | m \in M, n \in N \,\}$ is a submodule of S
ii) $M \cap N$ is a submodule of S
iii)the quotient module, $(M + N)/N \cong M/(M \cap N)$ "

Theorem 2.3 [5,6] "(Module-Theoretic Third Isomorphism Theorem) - Let S be a module and M be a submodules of S, then if N is a submodule of S such that $M \subseteq N \subseteq S$
i)N/M is a submodule of S/M
ii)$(S/M)/(N/M) \cong S/N$ "

Definition 2.2 [13,14] "Let Ω be the universal set and H be a *neutrosophic set* on Ω defined as $H = \{(\tau, \mathcal{V}_H(\omega_\tau), \mathcal{V}_H(\delta_\tau), \mathcal{V}_H(\kappa_\tau) : \tau \in \Omega\}$ where $\mathcal{V}_H(\omega_\tau), \mathcal{V}_H(\delta_\tau), \mathcal{V}_H(\kappa_\tau) : \Omega \to ({}^{-}0, 1^{+})$. The three components $\mathcal{V}_H(\omega_\tau), \mathcal{V}_H(\delta_\tau), \mathcal{V}_H(\kappa_\tau)$ represent membership value, indeterminacy value and non membership value respectively.
If $\mathcal{V}_H(\omega_\tau), \mathcal{V}_H(\delta_\tau), \mathcal{V}_H(\kappa_\tau) : \Omega \to [0,1]$, then H is a Single valued neutrosophic set (SVNS)"

Remark 2.1 *"Neutrosophic set means SVNS in this study."*

Definition 2.3 [13] "The neutrosophic set H is defined to be contained in the neutrosophic set J, symbolized as $H \subseteq J$ if and only if $H(\tau) \leqslant J(\tau)\ \forall \tau \in \Omega$, this means that $\mathcal{V}_H(\omega_\tau) \leqslant \mathcal{V}_J(\omega_\tau), \mathcal{V}_H(\delta_\tau) \leqslant \mathcal{V}_J(\delta_\tau), \mathcal{V}_H(\kappa_\tau) \geqslant \mathcal{V}_J(\kappa_\tau),\ \forall\ \tau \in \Omega$ ".

Definition 2.4 [13] "The sum of 2 neutrosophic sets H and J of a G-module S is a neutrosophic set $H + J$ of S denoted and defined as follows

$$H + J(\tau) = \{\tau, \mathcal{V}_{H+J}(\omega_\tau), \mathcal{V}_{H+J}(\delta_\tau), \mathcal{V}_{H+J}(\kappa_\tau) : \tau \in S\} \text{ where}$$
$$\mathcal{V}_{H+J}(\omega_\tau) = \vee\{\mathcal{V}_H(\omega_u) \wedge \mathcal{V}_J(\omega_s) | \tau = u + s, u, s \in S\}$$
$$\mathcal{V}_{H+J}(\delta_\tau) = \vee\{\mathcal{V}_H(\delta_u) \wedge \mathcal{V}_J(\delta_s) | \tau = u + s, u, s \in S\}$$
$$\mathcal{V}_{H+J}(\kappa_\tau) = \wedge\{\mathcal{V}_H(\kappa_u) \vee \mathcal{V}_J(\kappa_s) | \tau = u + s, u, s \in S\}\text{"}$$

Definition 2.5 [2,7] "Let S be a G- module over a field F and H be a neutrosophic set on S. The neutrosophic set $H = \{(\tau, \mathcal{V}_H(\omega_\tau), \mathcal{V}_H(\delta_\tau), \mathcal{V}_H(\kappa_\tau) : \tau \in S\}$ is said to be a neutrosophic G-submodule if the following conditions are satisfied:

1. $\mathcal{V}_H(\omega_{a\tau+bs}) \geq \mathcal{V}_H(\omega_\tau) \wedge \mathcal{V}_H(\omega_s)$
   $\mathcal{V}_H(\delta_{a\tau+bs}) \geq \mathcal{V}_H(\delta_\tau) \wedge \mathcal{V}_H(\delta_s)$
   $\mathcal{V}_H(\kappa_{a\tau+bs}) \leq \mathcal{V}_H(\kappa_\tau) \vee \mathcal{V}_H(\kappa_s)$
2. $\mathcal{V}_H(\omega_{gs}) \geq \mathcal{V}_H(\omega_s)$, $\mathcal{V}_H(\delta_{gs}) \geq \mathcal{V}_H(\delta_s)$, $\mathcal{V}_H(\kappa_{gs}) \leq \mathcal{V}_H(\kappa_s)$

where $g \in G; \tau, s \in S; a, b \in F$."

Definition 2.6 [2] "The support H^* of a neutrosophic set H on S can be defined as
$$H^* = \{\tau \in S, \mathcal{V}_H(\omega_\tau) > 0, \mathcal{V}_H(\delta_\tau) > 0, \mathcal{V}_H(\kappa_\tau) < 1\}$$ "

Proposition 2.1 [2] "If $H \in \Omega(G_S)$, then the support $H^* \in \Omega(G_S)$, where $\Omega(G_S)$ denote the family of all neutrosophic G-submodules of S. "

3. r-Neutrosophic G-Submodules

This section defines isomorphism of r-neutrosophic G-submodules and provides proof of isomorphism theorems.

Definition 3.1 [1] "Let $H = \{(\tau, \mathcal{V}_H(\omega_\tau), \mathcal{V}_H(\delta_\tau), \mathcal{V}_H(\kappa_\tau) : \tau \in \Omega\}$ be a neutrosophic set. Then for each $r \in [0, 1]$, the r-neutrosophic subset of H is denoted and defined as

$$H_r = \{\, \tau, \mathcal{V}_{H_r}(\omega_\tau), \mathcal{V}_{H_r}(\delta_\tau), \mathcal{V}_{H_r}(\kappa_\tau) : \tau \in \Omega \,\}$$

where

$$\mathcal{V}_{H_r}(\omega_\tau) = \mathcal{V}_H(\omega_\tau) \wedge r, \mathcal{V}_{H_r}(\delta_\tau) = \mathcal{V}_H(\delta_\tau) \wedge r, \mathcal{V}_{H_r}(\kappa_\tau) = \mathcal{V}_H(\kappa_\tau) \vee (1 - r)$$

"

Remark 3.1 *We set $\Omega_r(G_S)$ as the family of all r-neutrosophic G-submodules associated with neutrosophic G-submodules of S over a field F.*

Remark 3.2 *Let $H \in \Omega(G_S)$, then $H_r \in \Omega_r(G_S)$.*

Definition 3.2 *Let $H \in \Omega(G_S)$ and the H_r be r-neutrosophic subset of H. Let M be a submodule of S, then the restriction $H_r|M$ is defined as*
$H_r|M = \{\tau, \mathcal{V}_{H_r|M}(\omega_\tau), \mathcal{V}_{H_r|M}(\delta_\tau), \mathcal{V}_{H_r|M}(\kappa_\tau) : \tau \in S\}$ where

$$\mathcal{V}_{H_r|M}(\omega_\tau) = \mathcal{V}_H(\omega_\tau) \wedge r, \mathcal{V}_{H_r|M}(\delta_\tau) = \mathcal{V}_H(\delta_\tau) \wedge r, \mathcal{V}_{H_r|M}(\kappa_\tau) = \mathcal{V}_H(\kappa_\tau) \vee (1 - r)$$

Definition 3.3 *r -neutrosophic quotient submodule $(H_r)_M$ of S/M, is of the form $(H_r)_M = \{(\tau + M, \mathcal{V}_{(H_r)_M}(\omega_{\tau+M}), \mathcal{V}_{(H_r)_M}(\delta_{\tau+M}), \mathcal{V}_{(H_r)_M}(\kappa_{\tau+M}) : \tau \in S\}$ where*

$$\mathcal{V}_{(H_r)_M}(\omega_{\tau+M}) = \bigvee_{m \in M} \mathcal{V}_H(\omega_{\tau+m}) \wedge r$$

$$\mathcal{V}_{(H_r)_M}(\delta_{\tau+M}) = \bigvee_{m \in M} \mathcal{V}_H(\delta_{\tau+m}) \wedge r$$

$$\mathcal{V}_{(H_r)_M}(\kappa_{\tau+M}) = \bigwedge_{m \in M} \mathcal{V}_H(\kappa_{\tau+m}) \vee (1 - r)$$

Definition 3.4 *S , S^* be two G -modules over F, $H_r \in \Omega_r(G_S)$ and $J_r \in \Omega_r(G_{S^*})$. Let ψ be homomorphism of S onto S^*.*

If $\psi(H_r) \subseteq J_r$, then ψ is called a weak neutrososphic G -submodule homomorphism of H_r to J_r and is denoted as $H_r \sim J_r$

If $\psi(H_r) = J_r$, then ψ is called a neutrososphic G -submodule homomorphism of H_r to J_r and is denoted as $H_r \approx J_r$

Definition 3.5 *Let S and S^* be G -modules over F, $H_r \in \Omega_r(G_S)$ and $J_r \in \Omega_r(G_{S^*})$. Let ψ be isomorphism of S onto S^*.*

If $\psi(H_r) \subseteq J_r$, then ψ is called a weak neutrososphic G -submodule isomorphism of H_r to J_r and is denoted as $H_r \simeq J_r$

If $\psi(H_r) = J_r$, then ψ is called a neutrososphic G -submodule isomorphism of H_r to J_r and is denoted as $H_r \cong J_r$

Theorem 3.1 *First isomorphism theorem*

Let $H_r \approx J_r$ where $H_r \in \Omega_r(G_S)$ and $J_r \in \Omega_r(G_{S^*})$. Then there exist $T_r \in \Omega_r(G_S) \subseteq H_r$ such that $H_r/T_r \cong J_r$.

Proof: Let $\psi : S \to S^*$ be a neutrosophic G -submodule homomorphism of H_r onto J_r . Define $T_r \in \Omega_r(G_S)$ as follows:

$$\mathcal{V}_{T_r}(\omega_\tau) = \begin{cases} \big(\mathcal{V}_H(\omega_\tau) \wedge r)\big), & \text{if } \tau \in Ker\psi, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{V}_{T_r}(\delta_\tau) = \begin{cases} \big(\mathcal{V}_H(\delta_\tau) \wedge r\big), & \text{if } \tau \in Ker\psi, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{V}_{T_r}(\kappa_\tau) = \begin{cases} \big(\mathcal{V}_H(\kappa_\tau) \vee (1-r)\big), & \text{if } \tau \in Ker\psi, \\ 1, & \text{otherwise.} \end{cases}$$

Obviously $T_r \subseteq H_r$.
To show $T_r \in \Omega_r(G_S)$
Case 1: If $\tau, s \notin Ker\psi$ then $(\tau + s) \notin Ker\psi$ or $(\tau + s) \in Ker\psi$
(i)If $(\tau + s) \notin Ker\psi$ then $\mathcal{V}_{T_r}(\delta_{\tau+s}) = \mathcal{V}_{T_r}(\delta_\tau) = \mathcal{V}_{T_r}(\delta_s) = 0$
(ii)If $(\tau + s) \in Ker\psi$ then $\mathcal{V}_{T_r}(\delta_{\tau+s}) = \mathcal{V}_H(\delta_{\tau+s}) \wedge r$, but $\mathcal{V}_{T_r}(\delta_\tau) = \mathcal{V}_{T_r}(\delta_s) = 0$
$\therefore \mathcal{V}_{T_r}(\delta_{\tau+s}) \geq \mathcal{V}_{T_r}(\delta_\tau) \wedge \mathcal{V}_{T_r}(\delta_s)$

Case 2: If $\tau, s \in Ker\psi$ then $(\tau + s) \in Ker\psi$

$$\begin{aligned} \mathcal{V}_{T_r}(\delta_{\tau+s}) &= \mathcal{V}_H(\delta_{\tau+s}) \wedge r \\ &\geq \mathcal{V}_H(\delta_\tau) \wedge \mathcal{V}_H(\delta_s) \wedge r \\ &= (\mathcal{V}_H(\delta_\tau) \wedge r) \wedge (\mathcal{V}_H(\delta_s) \wedge r) \\ &= \mathcal{V}_{T_r}(\delta_\tau) \wedge \mathcal{V}_{T_r}(\delta_s) \end{aligned}$$

Case 3: If $\tau \in Ker\psi$,$s \notin Ker\psi$ then $\tau + s \notin Ker\psi$, then

$\mathcal{V}_{T_r}(\delta_\tau) = \mathcal{V}_H(\delta_\tau) \wedge r$, $\mathcal{V}_{T_r}(\delta_s) = 0$, $\mathcal{V}_{T_r}(\delta_{\tau+s}) = 0$
$\therefore \mathcal{V}_{T_r}(\delta_{\tau+s}) \geq \mathcal{V}_{T_r}(\delta_\tau) \wedge \mathcal{V}_{T_r}(\delta_s)$

Case 4: If $s \in Ker\psi$,$\tau \notin Ker\psi$ then $\tau + s \notin Ker\psi$, follows from 3

Thus in all cases we have $\mathcal{V}_{T_r}(\delta_{\tau+s}) \geq \mathcal{V}_{T_r}(\delta_\tau) \wedge \mathcal{V}_{T_r}(\delta_s)$ and similarly the other inequalities holds.

Now to show$\mathcal{V}_{T_r}(\delta_{g\tau}) \geq \mathcal{V}_{T_r}(\delta_\tau)$
Case 1: Let $g \in G, \tau \in Ker\psi$ then $\mathcal{V}_{T_r}(\delta_{g\tau}) = \mathcal{V}_H(\delta_{g\tau}) \wedge r \geq \mathcal{V}_H(\delta_\tau) \wedge r = \mathcal{V}_{T_r}(\delta_\tau)$

Case 2: Let $g \in G, \tau \notin Ker\psi$ then if
(i)$g\tau \in Ker\psi$ then $\mathcal{V}_{T_r}(\delta_{g\tau}) = \mathcal{V}_H(\delta_{g\tau}) \wedge r$, $\mathcal{V}_{T_r}(\delta_\tau) = 0$ gives $\mathcal{V}_{T_r}(\delta_{g\tau}) \geq \mathcal{V}_{T_r}(\delta_\tau)$
(ii)$g\tau \notin Ker\psi$ then $\mathcal{V}_{T_r}(\delta_{g\tau}) = 0$, $\mathcal{V}_{T_r}(\delta_\tau) = 0$ gives $\mathcal{V}_{T_r}(\delta_{g\tau}) \geq \mathcal{V}_{T_r}(\delta_\tau)$
Similarly other inequalities hold. Hence $T_r \in \Omega_r(G_S)$

Now consider the map $\gamma : H_r^* \to J_r^*$
It is clear that $\psi(H_r^*) = J_r^*$ and $\gamma = \psi/H_r^*$
We have

$$\begin{aligned} T_r^* &= \{\, \tau \in H_r^* \mid \psi(\tau) = 0 \,\} \\ &= \{\, \tau \in H_r^* \mid \gamma(\tau) = 0 \,\} \\ &= Ker\gamma \end{aligned}$$

So there exist a G-module isomorphism $\phi : H_r^*/T_r^* \to J_r^*$ such that $\phi(\tau + T_r^*) = \gamma(\tau) = \psi(\tau),$ for all $\tau \in H_r^*$
Now $\phi_{(H_r|T_r)}(z) = \{(z, \mathcal{V}_{\phi(H_r|T_r)}(\omega_z), \mathcal{V}_{\phi(H_r|T_r)}(\delta_z), \mathcal{V}_{\phi(H_r|T_r)}(\kappa_z)\}$ where

$$\begin{aligned} \mathcal{V}_{\phi_{(H_r|T_r)}}(\kappa_z) &= \wedge\{\, \mathcal{V}_{(H_r|T_r)}(\kappa_{\tau+T_{r^*}}) : \tau \in H_r^*, \phi(\tau + T_r^*) = z \,\} \\ &= \wedge\{\, \wedge(\mathcal{V}_H(\kappa_{\tau+h}) \vee (1-r)) : \tau \in H_r^*, h \in T_r^*, \psi(\tau) = z \,\} \\ &= \wedge\{\, \mathcal{V}_H(\kappa_{\tau+h}) \vee (1-r) : \tau \in H_r^*, h \in T_r^*, \psi(\tau) = z \,\} \\ &= \wedge\{\, \mathcal{V}_H(u) \vee (1-r) : u \in H_r^*, \psi(u) = z, u = \tau + h \in H_r^* \,\} \\ &= \mathcal{V}_{J_r}(\kappa_z) \end{aligned}$$

Similarly we can prove other equalities.

Hence proved $H_r/T_r \cong J_r$ □

Proposition 3.1 *Let $H_r, J_r \in \Omega_r(G_S)$, then $(H \cap J)_r = H_r \cap J_r$*

Proof: Let $\tau \in S, r \in [0,1]$

$$\begin{aligned} \omega_{(H\cap J)_r}(\tau) &= \omega_{(H\cap J)}(\tau) \wedge r \\ &= (\omega_H(\tau) \wedge \omega_J(\tau)) \wedge r \\ &= (\omega_H(\tau) \wedge r) \wedge (\omega_J(\tau) \wedge r) \\ &= \omega_{H_r}(\tau) \wedge \omega_{J_r}(\tau) \\ &= \omega_{(H_r \cap J_r)}(\tau) \end{aligned}$$

Similarly
$\delta_{(H\cap J)_r}(\tau) = \delta_{(H_r\cap J_r)}(\tau)$, $\kappa_{(H\cap J)_r}(\tau) = \kappa_{(H_r\cap J_r)}(\tau)$ □

Proposition 3.2 *Let $H_r, J_r \in \Omega_r(G_S)$, then $(H + J)_r = H_r + J_r$*

Proof: Let $\tau, u, s \in M, r \in [0,1]$

$$\begin{aligned} \omega_{(H+J)_r}(\tau) &= \omega_{(H+J)}(\tau) \wedge r \\ &= \bigvee_{\tau=u+s} (\omega_H(u) \wedge \omega_J(s)) \wedge r \\ &= \bigvee_{\tau=u+s} [(\omega_H(u) \wedge r) \wedge (\omega_J(s) \wedge r)] \\ &= \bigvee_{\tau=u+s} [\omega_{H_r}(u) \wedge \omega_{J_r}(s)] \\ &= \omega_{(H_r+J_r)}(\tau) \end{aligned}$$

Similarly
$\delta_{(H+J)_r}(\tau) = \delta_{(H_r+J_r)}(\tau)$, $\kappa_{(H+J)_r}(\tau) = \kappa_{(H_r+J_r)}(\tau)$ □

Theorem 3.2 *Second isomorphism theorem*

Let H_r , $J_r \in \Omega_r(G_S)$, *then* $J_r/(H_r \cap J_r) \simeq (H_r + J_r)/H_r$

Proof: There exist an isomorphism $\psi : J^*/(H \cap J)^* \simeq (H+J)^*/H^*$ by second isomorphism theorem for modules if H and J are submodules of G-module S and is defined as $\psi(\tau + (H \cap J)^*) = \tau + H^*, \forall \tau \in J^*$
Now

$$\begin{aligned}
\mathcal{V}_{\phi_{(J_r/(H_r\cap J_r))}}(\delta_{\tau+H_r^*}) &= \mathcal{V}_{(J_r/(H_r\cap J_r))}(\delta_{\tau+(H_r\cap J_r)^*}) \\
&= \bigvee_{z\in\tau+(H_r\cap J_r)^*} (\mathcal{V}_J(\delta_z) \wedge r) \\
&= \bigvee_{z\in\tau+(H_r\cap J_r)^*} (\mathcal{V}_{H+J}(\delta_z) \wedge r) \\
&\leq \{\, (\mathcal{V}_{(H+J)}(\delta_z) \wedge r : z = \tau + H_r^* \,\} \\
&= \mathcal{V}_{(H_r+J_r)/H_r)}(\delta_{\tau+H_r^*}), \forall \tau \in J_r^*
\end{aligned}$$

Similarly $\mathcal{V}_{\phi_{(J_r/(H_r\cap J_r))}}(\omega_{\tau+H_r^*}) \leq \mathcal{V}_{(H_r+J_r)/H_r)}(\omega_{\tau+H_r^*}), \forall \tau \in J_r^*$
Now

$$\begin{aligned}
\mathcal{V}_{\phi_{(J_r/(H_r\cap J_r))}}(\kappa_{\tau+H_r^*}) &= \mathcal{V}_{(J_r/(H_r\cap J_r))}(\kappa_{\tau+(H_r\cap J_r)^*}) \\
&= \bigwedge_{z\in\tau+(H_r\cap J_r)^*} (\mathcal{V}_J(\kappa_z) \vee (1-r)) \\
&= \bigwedge_{z\in\tau+(H_r\cap J_r)^*} (\mathcal{V}_{H+J}(\kappa_z) \vee (1-r)) \\
&\geq \{\, (\mathcal{V}_{(H+J)}(\kappa_z) \vee (1-r) : z = \tau + H_{r^*} \,\} \\
&= \mathcal{V}_{(H_r+J_r)/H_r)}(\kappa_{\tau+H_r^*}), \forall \tau \in J_r^*
\end{aligned}$$

Hence $J_r/(H_r \cap J_r) \simeq (H_r + J_r)/H_r$ □

Theorem 3.3 *Third isomorphism theorem*

Let $H_r \subseteq J_r \subseteq T_r$ *then* $(T_r/H_r)/(J_r/H_r) \cong T_r/J_r$ *where* H_r, J_r, $T_r \in \Omega_r(G_S)$

Proof: There exist an isomorphism $\psi : (T_r^*/H_r^*)/(J_r^*/H_r^*) \cong T_r^*/J_r^*$ by third isomorphism theorem for modules if H, J and T are submodules of $G-module$ S which is defined as $\psi(\tau + H^* + (J^*/H^*)) = \tau + J^*, \forall \tau \in T^*$

$$\begin{aligned}
\mathcal{V}_{\phi_{((T_r/H_r)/(J_r/H_r))}}(\delta_{\tau+J_{r^*}}) &= \mathcal{V}_{((T_r/H_r)/(J_r/H_r))}(\delta_{\tau+H_r^*+(J_r^*/H_r^*)}) \\
&= \vee\{\, \mathcal{V}_{(T_r/H_r)}(\delta_{z+H_r^*}) : z \in T_r^*, z + H_r^* \in \tau + H_r^* + (J_r^*/H_r^*) \,\} \\
&= \vee\{\, \vee\mathcal{V}_{T_r}(\delta_h) : z \in T_r^*, h \in z + H_r^*, h + H_r^* \in \tau + H_r^* + (J_r^*/H_r^*) \,\} \\
&= \vee\{\, \mathcal{V}_{T_r}(\delta_h) : h \in T_r^*, \mathcal{V}(\delta_h) \in \tau + J_r^* \,\} \\
&= \vee\{\, \mathcal{V}_{T_r/J_r}(\delta_{\tau+J_r^*}) : \tau \in T_r^* \,\}
\end{aligned}$$

Therefore
$\mathcal{V}_{\phi_{(T_r/H_r)/(J_r/H_r)}}(\delta_{\tau+J_r^*}) = \vee\{\,\mathcal{V}_{T_r/J_r}(\delta_{\tau+J_r^*}) : \tau \in T_r^*\,\}$
Similarly
$\mathcal{V}_{\phi_{(T_r/H_r)/(J_r/H_r)}}(\omega_{\tau+J_r^*}) = \vee\{\,\mathcal{V}_{T_r/J_r}(\omega_{\tau+J_r^*}) : \tau \in T_r^*\,\}$
$\mathcal{V}_{\phi_{(T_r/H_r)/(J_r/H_r)}}(\kappa_{\tau+J_r^*}) = \wedge\{\,\mathcal{V}_{T_r/J_r}(\kappa_{\tau+J_r^*}) : \tau \in T_r^*\,\}$

Hence $(T_r/H_r)/(J_r/H_r) \cong T_r/J_r$ □

4. Application

Application of neutrosophic isomorphism can be done in information systems, data fusion and medical imaging to convert uncertain data into simpler forms but still retaininig foundational structural properties. In constrained environments where restrictions on parameters,values or required conditions reduce the effectiveness of established methods, usefulness of r-neutrosophic G-submodules becomes evident. Relevant information such as underlying relationships, patterns, and varying degrees of indeterminacy is preserved despite these restrictions. This ensures that analytical procedures like segmentation and feature extraction continue to deliver reliable results. Ultimately, r-neutrosophic structures provide a interconnection between neutrosophic methods and classic algebraic approaches, supporting accurate analysis even under strict value or conditional constraints.

5. Comparison of Different Neutrosophic Frameworks

Table 1: Comparitive study

Structure	**T–I–F Range**	**Control Mechanism**	**Homomorphism / Isomorphism**	**Applications**
General Neutrosophic Set	$(-0, 1+)$	Maximum flexibility	Not applicable	Theoretical and philosophical modeling
Single-Valued Neutrosophic (SVN) Set	$[0, 1]$	Precise and computationally efficient	Not applicable	Ranking and decision making
Neutrosophic G-submodule	$[0, 1]$ with module operation compatibility	Controlled by module operations	Standard module homomorphisms and isomorphisms extendable	Modeling algebraic uncertainty
r-Neutrosophic G-submodule	$[0, 1]$ restricted by $r \in [0, 1]$	Controlled by module operations and parameter r	Homomorphism and isomorphism preserved under r-restriction	Restricted algebraic uncertainty and controlled decision making

6. Conclusion

Isomorphisms offer a logical way to understand structural equivalences in r-neutrosophic G-submodules while preserving truth, indeterminacy, and falsity. They support the construction of quotient modules, classification of neutrosophic structures and the exploration of higher-order algebraic systems, strengthening both conceptual understanding and operational modeling.

In future work, I would like to extend this study by developing hybrid mathematical models that combine multiple uncertainty frameworks, such as fuzzy, rough, and neutrosophic systems. Such a unified approaches could support the construction of more reliable isomorphisms capable of delivering results in highly controlled or multi-conditional environments.

Acknowledgments

The authors thank the reviewers for their helpful comments and constructive suggestions.

Authors' Contribution: Both authors contributed equally to the theoretical analysis, and preparation of the manuscript.

Conflict of Interest: The authors declare no conflict of interest.

References

1. R Binu et al. R-neutrosophic subset of g-submodules. 2025.
2. R Binu and Paul Isaac. Some characterizations of neutrosophic g-submodule. *J. Math. Comput. Sci.*, 10(1):27–39, 2019.
3. C. W. Curties and Irving Reiner. *Representation Theory of Finite Group and Associative Algebra*. Interscience Publishers, Inc., USA, 1962.
4. Sejal K Dharia. Integral and non-integral dimension of modules over non-commutative rings. 2009.
5. David Steven Dummit, Richard M Foote, et al. *Abstract algebra*, volume 3. Wiley Hoboken, 2004.
6. David Eisenbud and E Graham Evans Jr. Generating modules efficiently: theorems from algebraic k-theory. *Journal of Algebra*, 27(2):278–305, 1973.
7. Nasruddin Hassan, Vakkas Uluçay, and Mehmet Şahin. Q-neutrosophic soft expert set and its application in decision making. *International Journal of Fuzzy System Applications (IJFSA)*, 7(4):37–61, 2018.
8. Paul Isaac and Pearly P John. On intuitionistic fuzzy submodules of a module. *Int. J. of Mathematical Sciences and Applications*, 1(3):1447–1454, 2011.
9. Abeer MM Jaradat, Yousef Al-Qudah, Abdullah Alsoboh, Faisal Al-Sharqi, and Abdullrahman A Al-Maqbali. A new approach of possibility single-valued neutrosophic set and its application in decision-making environment. *International Journal of Analysis and Applications*, 23:213–213, 2025.
10. Serkan Onar. A note on neutrosophic soft set over hyperalgebras. *Symmetry*, 16(10):1288, 2024.
11. Fatimah Rahoumah, Kai Siong Yow, Nik Mohd Asri Nik Long, and Menshawi Gasim. On properties and operations of complex neutrosophic soft groups. *Journal of Inequalities and Applications*, 2024(1):103, 2024.
12. PK Sharma. Anti fuzzy g-modules. *Advances in Fuzzy Sets and Systems*, 21(4):303, 2016.
13. Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.
14. Florentin Smarandache and Maissam Ahmad Jdid. *Research on the topics of neutrosophic operations research: Volume I*. Infinite Study, 2023.
15. Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

Preema T. S.,
Research Scholar, Rajagiri School of Engineering and Technology and
Faculty, SCMS School of Engineering and Technology,
APJ Abdul Kalam Technological University, Kerala, India
E-mail address: `preemashinoj87@gmail.com`

and

Binu R.,
Rajagiri School of Engineering and Technology,
APJ Abdul Kalam Technological University, Kerala, India
E-mail address: `binur@rajagiritech.edu.in`