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Optimality and Duality for Semi-Infinite Mathematical Programs with Equilibrium Constraints via Tangential Subdifferentials

Ali Ansari Ardali, Maryam Zeinali, Rishabh Pandey and Vinay Singh

ABSTRACT: In this research, we consider a class of semi-infinite mathematical programs with equilibrium constraints (SIMPEC). Then, we introduce new generalized Abadie constraint qualifications and stationary conditions and derive necessary and sufficient optimality conditions for the SIMPEC by aid of the concept of the tangential subdifferentials. Further, we formulate the Mond-Weir and Wolfe type dual models for SIMPEC in a framework of tangential subdifferentials. Furthermore, we establish weak and strong duality results for each of the said models under appropriate convexity assumptions. In addition, we illustrate some results by given example.

Keywords: Equilibrium constraints, duality, optimality conditions, inequalities in optimization, constraint qualifications.

Contents

1. Introduction

In this paper, we consider an extended class of optimization problems that are known in the literature as the semi-infinite mathematical program with equilibrium constraints. It has the form

$$\begin{aligned} (SIMPEC) \qquad & \min \ \Upsilon(z) \\ & s.t. \ \ \pi_t(z) \leq 0 \quad \forall t \in T, \quad \theta(z) = 0, \\ & \qquad \Pi(z) \geq 0, \quad \Theta(z) \geq 0, \quad \Pi(z)^T \Theta(z) = 0, \end{aligned} \tag{1.1}$$

where $\Upsilon : \mathbb{R}^n \to \mathbb{R}$, $\pi_t : \mathbb{R}^n \to \mathbb{R} \ \ \forall t \in T$, $\theta : \mathbb{R}^n \to \mathbb{R}^q$, $\Pi : \mathbb{R}^n \to \mathbb{R}^m$, $\Theta : \mathbb{R}^n \to \mathbb{R}^m$ are given functions. The index set T is an arbitrary compact nonempty set, not necessarily finite.

An optimization problem that involves an infinite number of inequality constraints on a feasible set is referred to as a semi-infinite programming problem (SIP) in the literature. Numerous researchers (see [1,2,3,4,5]) have explored the fundamental theory and applications of SIP. Additionally, studies [6,7,8,9] have examined nonsmooth semi-infinite programming problems using certain generalized gradients. Their approach relaxes the assumptions of differentiability and convexity of the objective functions.

Mathematical programs with equilibrium constraints (MPECs) frequently emerge in in various fields, including natural sciences, engineering, economic modeling, transportation science, game theory, and machine learning. As a result, this topic has garnered significant attention from researchers, particularly in recent years; see, e.g., [10,11,12]. Actually, the class of MPEC is a generalization of the class of bilevel optimization problems. Moreover, since the feasible area of these problems is not necessarily convex, MPECs fall into the category of nonconvex optimization problems and, more importantly, the usual nonlinear programming constraint qualifications (CQ) like the linear independence CQ and the

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Mangasarian-Fromovitz CQ typically are violated at any feasible point (see [13]). The standard nonlinear optimization problem has only one stationary condition that is the KKT-type condition; but since there are several variant approaches to reformulate MPEC, different stationarity concepts appear such as S(Strong)-stationary, M(Mordukhovich)-stationary, A(Alternative)-stationary, C(Clarke)-stationary, W(Weakly)-stationary. Among these stationary concepts, S-stationary is the strongest (which is equivalent to the classical KKT condition) and W-stationary is the weakest. The MPECs have been studied in both smooth and nonsmooth cases by different authors.

In [14], the authors showed that, under an MPEC variant of the strict MFCQ, S-stationarity is a necessary optimality condition for smooth MPEC. For MPECs with smooth data, one can see, e.g., [15,16,17]. Movahedian and Nobakhtian in [18], by the Michel-Penot subdifferential, introduced a nonsmooth version of M-stationarity and proved that it is a necessary condition for MPEC without any smoothness requirements. Later on, in [19], they derived new necessary and sufficient conditions for nonsmooth MPEC based on the Clarke-Rockafellar subdifferential. Ye and Zhang [20] studied the enhanced version of M-stationary conditions for nonsmooth MPEC. Ansari et al. [21] investigated the nonsmooth optimality conditions for MPEC by means of the convexificator concept. Lafhim and Kalmoun in [22] have obtained first-order optimality conditions for MPEC using directional convexificators. They consider a nonsmooth MPEC where its functions are not necessarily locally Lipschitz/continuous. Recently, Pandey et al. [23] introduced a nonsmooth variant constraint qualification and generalized stationary conditions using the tangential subdifferential and developed optimality conditions for MPEC. Most of the previous works have been done on ordinary MPEC, but in this paper, with the idea of the article [21], we study the nonsmooth semi-infinite MPEC.
Duality results are very fruitful and effective in the development of numerical algorithms for solving certain classes of optimization problems. In nonlinear programming, Mond-Weir and Wolfe dual models have been studied by several researchers. As an example, one can see [24,25,27,28,29]. Here, we also introduce Mond-Weir and Wolfe type dual models to the SIMPEC and establish weak and strong duality theorems for both dual models. Our main tool during this study is the notion of tangential subdifferential. The notion of tangential subdifferential was offered and applied to obtain optimality conditions for nonlinear programming by Pshenichnyi in [30]. The tangential subdifferentials include both convex subdifferentials and Gâteaux derivatives. In [31], Martínez-Legaz used this notion to extend the results of [32,33] and weakened the convexity or regularity assumptions on the objective and constraint functions. Recently, several articles applied the tangential subdifferentials for various classes of nonsmooth and nonconvex optimization problems. For instance, in [34], the authors investigated optimality conditions on a nonsmooth optimization problem with just inequality constraints using this notion. Additionally, the papers [35,36] applied the tangential subdifferential to study KKT optimality conditions for the multiobjective semi-infinite programming (MSIP) problem. To the best of our knowledge, no paper has used this notion to examine the optimality conditions and duality for the class of SIMPEC. The tangential subdifferentials also include Clarke and Michel-Penot regular subdifferentials. Hence, the optimality conditions in terms of the tangential subdifferential provide sharper results. Motivated by the above observations, in this paper, we study optimality conditions and Mond-Weir and Wolfe type duality theorems for a large class of nondifferentiable and nonconvex equilibrium optimization problems. To highlight the novelty and contribution of the current study, Table 1 summarizes the key differences and improvements of the present work in comparison with the existing literature on MPECs and semi-infinite programs.

The organization of the next sections of the paper is as follows. Section 2 is assigned to provide the essential notations and initial results that be needed in the rest of the paper. In Section 3, new versions of Abadie constraint qualification are introduced based on the tangential subdifferentials. Also, we define new SIMPEC stationarity conditions which are expressed in the framework of tangential subdifferentials. Moreover, we derive some necessary and sufficient optimality conditions for the local minimizer of SIMPEC using these concepts. Finally, in Section 4, Mond-Weir and Wolfe type dual models are formulated for SIMPEC and relevant theorems are presented.

2. Preliminaries

This section consists some primary definitions and auxiliary results from the nonsmooth analysis that will serve as foundational tools in subsequent discussions; see, e.g., [37] for more comprehensive

Reference	Model description	Smooth/Non-smooth	Assumptions	CQ	Results
Flegel and Kanzow (2003)	MPEC-single objective	Smooth	-	MPEC-LICQ MPEC-MFCQ MPEC-SMFCQ	Fritz John optimality conditions
Flegel and Kanzow (2005)	MPEC-single objective	Smooth	-	Abadie CQ Guignard CQ MPEC-LICQ	Necessary and sufficient conditions
Ye (2005)	MPEC-single objective	Smooth	Pseudoconvex Quasiconvex	MPEC-LICQ NNAMCQ MPEC linear CQ Abadie CQ MPEC Kuhn-Tucker CQ MPEC Zangwill CQ	Necessary and sufficient conditions
Flegel and Kanzow (2005)	MPEC-single objective	Smooth	Locally Lipschitz continuous continuously differentiable	MPEC-ACQ	Necessary optimality conditions
Movahedian and Nobakhtian (2010)	MPEC-single objective	Non-smooth	Pseudoconvex	MF-CQ	Necessary and sufficient conditions
Ansari et al. (2014)	MPEC-single objective	Non-smooth	Locally Lipschitz ∂^*-pseudoconvex ∂^*-quasiconvex	MPEC Abadie CQ MPEC KT CQ MPEC Zangwill CQ	Necessary and sufficient conditions
Pandey and Mishra (2016)	MPEC-single objective	Non-smooth	∂^*-pseudoconvex ∂^*-quasiconvex	GS-ACQ	Duality for MPEC
Pandey and Mishra (2018)	MPEC-single objective and semi-infinite	Non-smooth	Locally Lipschitz ∂^*-convex	GS-ACQ	Optimality conditions and duality for MPEC
Lafhim and Kalmoun (2023)	MPEC-single objective	Non-smooth	∂^*_D-generalized convex	∂^*_D-GS ACQ ∂^*_D-MPEC ACQ	Necessary optimality conditions
Van Su (2023)	MPEC-single objective	Non-smooth	Pseudo-convex	-	Sufficient optimality conditions and duality for MPEC
Pandey et al. (2025)	MPEC-single objective	Non-smooth	Dini-pseudoconvex Dini-quasiconvex	∂_T-ACQ	Necessary and sufficient conditions

Table 1: Literature comparison: Model description Smooth/Nonsmooth, assumptions, CQ, and main results.

exposition. Throughout this paper, $\mathbb{R}^n$ stands for the n-dimensional Euclidean space. For a given subset $\mathcal{H} \subseteq \mathbb{R}^n$, denote by $cone\,\mathcal{H}$, $co\,\mathcal{H}$, $cl\,\mathcal{H}$, the convex cone containing the origin generated by $\mathcal{H}$, the convex hull of $\mathcal{H}$, and the closure of $\mathcal{H}$, respectively. The negative and strictly negative polar cones $\mathcal{H}^-$ and $\mathcal{H}^s$ are defined, respectively, by

$$\mathcal{H}^- := \{u \in \mathbb{R}^n \,|0 \geq \langle z, u\rangle \;\; , \forall z \in \mathcal{H}\},$$
$$\mathcal{H}^s := \{u \in \mathbb{R}^n \,|0 > \langle z, u\rangle \;\; , \forall z \in \mathcal{H} \setminus \{0\}\}.$$

Definition 1. *For $\mathcal{H} \subseteq \mathbb{R}^n$,*

(i) The contingent cone (or the Bouligand tangent cone) of $\mathcal{H}$ at $z \in cl\,\mathcal{H}$ is

$$\mathcal{T}(\mathcal{H}, z) := \{v \in \mathbb{R}^n \mid \;\; \exists v_r \to v, \;\; \exists t_r \downarrow 0, \quad \text{such that} \quad z + t_r v_r \in \mathcal{H}, \;\; \forall r\}.$$

(ii) The cone of feasible directions of $\mathcal{H}$ at z is

$$\mathcal{D}(\mathcal{H}, z) := \{v \in \mathbb{R}^n \mid \;\; \exists \delta > 0 \quad \text{such that} \quad z + tv \in \mathcal{H}, \;\; \forall t \in (0, \delta)\}.$$

(iii) The cone of attainable directions of $\mathcal{H}$ at z is

$$\mathcal{A}(\mathcal{H}, z) := \Big\{ v \in \mathbb{R}^n \mid \;\; \exists \delta > 0 \;\; \text{and} \;\; \alpha : \mathbb{R} \to \mathbb{R}^n \;\; \text{such that} \;\; \alpha(t) \in \mathcal{H}, \;\; \forall t \in (0, \delta),$$
$$\alpha(0) = 0, \;\; \lim_{t \downarrow 0} \frac{\alpha(t) - \alpha(0)}{t} = v \Big\}.$$

Definition 2. *Assume that $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function, $\bar{z} \in dom\psi := \{z \in \mathbb{R}^n \,|\psi(z) < \infty\}$ and $v \in \mathbb{R}^n$. Then, the directional derivative (or Dini derivative) of ψ at $\bar{z}$ in the direction v is defined by*

$$\psi'(\bar{z}; v) := \lim_{t \downarrow 0} \frac{\psi(\bar{z} + tv) - \psi(\bar{z})}{t}.$$

If the directional derivative of ψ exists in all directions v, then it is said that ψ is directionally differentiable at $\bar{z}$.

Definition 3. *[38,30] A function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be tangentially convex at $\bar{z} \in \psi^{-1}(\mathbb{R})$ if for each of $v \in \mathbb{R}^n$, $\psi'(\bar{z}; v)$ exists, is finite and the function $\psi'(\bar{z}; .) : \mathbb{R}^n \to \mathbb{R}$ is a convex function of v.*

Note that, $\psi'(\bar{z}; .)$ will be a sublinear function if ψ is tangentially convex at $\bar{z}$ since $\psi'(\bar{z}; .)$ is positively homogeneous. So, there is a nonempty compact convex set of $\mathbb{R}^n$ such that $\psi'(\bar{z}, .)$ is the support function of that set, by [27, Lemma 2.1].

It is noteworthy that the class of tangentially convex functions is relatively broad. Clearly, that includes Gâteaux differentiable functions and convex functions with open-domain. In addition, an everywhere Gâteaux differentiable function with open-domain is tangentially convex. This class is closed under multiplication by scalars and addition. Thus, the sum of a differentiable function with a convex function gives an example of a tangentially convex function which, in general, is nondifferentiable and nonconvex.

Corresponding to the concept of a tangentially convex function, the following definition of subdifferential is implicitly given in [30].

Definition 4. *Suppose that $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function and $\bar{z} \in dom\psi$. Then, the tangential subdifferential of ψ at $\bar{z}$ is the set*

$$\partial_T \psi(\bar{z}) := \{\xi \in \mathbb{R}^n \,|\langle v, \xi\rangle \leq \psi'(\bar{z}; v), \forall v \in \mathbb{R}^n\}.$$

Obviously, $\partial_T\psi(\bar{z})$ is a nonempty closed convex subset of $\mathbb{R}^n$ if ψ is tangentially convex at $\bar{z}$, and also $\psi'(\bar{z};.)$ will be the support function of $\partial_T\psi(\bar{z})$, that is, for every $v\in\mathbb{R}^n$ one has

$$\psi'(\bar{z};v)=\max_{\xi\in\partial_T\psi(\bar{z})}\langle\xi,v\rangle.$$

If two functions ψ and φ are tangentially convex at a point $\bar{z}$, then the following relationships are established:

$$\partial_T(\lambda\psi)(\bar{z})=\lambda\partial_T\psi(\bar{z}),\quad\forall\lambda\in\mathbb{R}_+,$$
$$\partial_T(\varphi+\psi)(\bar{z})=\partial_T\varphi(\bar{z})+\partial_T\psi(\bar{z}),$$

where $\mathbb{R}_+$ stands for positive real numbers.

Remark 1. *It is noteworthy that every Gâteaux differentiable function $\psi:\mathbb{R}^n\to\mathbb{R}$ at $\bar{z}$ is tangentially convex at $\bar{z}$ and $\partial_T\psi(\bar{z})=\{\nabla\psi(\bar{z})\}$. Also, if ψ be a convex function at $\bar{z}$, then ψ is tangentially convex at $\bar{z}$ and $\partial_T\psi(\bar{z})$ coincides with the classical convex subdifferential $\partial\psi(\bar{z})$ (see [37]). Further, a locally Lipschitz function ψ at $\bar{z}$ which is regular Clarke [39] at $\bar{z}$ is also tangentially convex at $\bar{z}$ and $\partial_T\psi(\bar{z})=\partial_C\psi(\bar{z})$. Moreover, if ψ be a locally Lipschitz function at $\bar{z}$ and regular Michel-Penot [40] at $\bar{z}$, then ψ is tangentially convex at $\bar{z}$ and $\partial_T\psi(\bar{z})=\partial^\diamond\psi(\bar{z})$.*

Now, we illustrate some cases by the following examples.

Example 1. *Assume that $\psi:\mathbb{R}\to\mathbb{R}$ be defined by $\psi(z)=\max\{z^2,3z\}$ and $\bar{z}=0$. Thus, $\psi'(\bar{z};v)=\max\{0,3v\}$ will be a convex function and so $\partial_T\psi(\bar{z})=[0,3]$. Note that ψ is tangentially convex but is not Gâteaux differentiable at $\bar{z}$.*

Example 2. *Consider $\psi:\mathbb{R}^2\to\mathbb{R}$ as the following*

$$\psi(z_1,z_2)=\begin{cases}\frac{z_1^2}{z_1+z_2}+2z_2 & \text{if }\ (z_1,z_2)\neq(0,0),\\ 0 & \text{if }\ (z_1,z_2)=(0,0).\end{cases}$$

For $\bar{z}=(\bar{z}_1,\bar{z}_2)=(0,0)$, $\psi'(\bar{z};v)=2v_2$ and since ψ is Gâteaux differentiable at $\bar{z}$, we get $\partial_T\psi(\bar{z})=\{\nabla\psi(\bar{z})\}=\{(0,2)\}$.

Definition 5. *(see [41]) Let $\psi:\mathbb{R}^n\to\mathbb{R}$, $\mathcal{H}\subseteq\mathbb{R}^n$ be a convex set and $\bar{z}\in\mathcal{H}$. Then, ψ is Dini-convex at $\bar{z}$, if*

$$\psi(z)\geq\psi(\bar{z})+\psi'(\bar{z};z-\bar{z}),\quad\forall z\in\mathcal{H}.$$

Remark 2. *(see [35]) Let $\psi:\mathbb{R}^n\to\mathbb{R}$, $\mathcal{H}\subseteq\mathbb{R}^n$ be a convex set and $\bar{z}\in\mathcal{H}$. Assume that ψ is tangentially convex at $\bar{z}$. If ψ is Dini-convex at $\bar{z}$ and $z\in\mathcal{H}$, then*

$$\psi(z)\geq\psi(\bar{z})+\langle\xi,z-\bar{z}\rangle,\ \forall\xi\in\partial_T\psi(\bar{z}).$$

Now, we also recall the definitions of Dini-pseudoconvexity and Dini-quasiconvexity of a function formulated in terms of tangential subdifferential.

Definition 6. *[23,35,26] Let $\mathcal{H}\subset\mathbb{R}^n$ be a nonempty convex set and $\bar{z}\in\mathcal{H}$ be given. Further, assume that $\psi:\mathcal{H}\to\mathbb{R}$ is a tangentially convex at $\bar{z}$. Then:*

1. *ψ is Dini-pseudoconvex at $\bar{z}$ on $\mathcal{H}$ if the relation*

$$\psi(z)<\psi(\bar{z})\Longrightarrow\big\langle\xi,\ z-\bar{z}\big\rangle<0,\quad\forall\xi\in\partial^T\psi(\bar{z})$$

hold for all $z\in\mathcal{H}$.

2. ψ is Dini-quasiconvex at $\bar{z}$ on $\mathcal{H}$ if the relation

$$\psi(z) \le \psi(\bar{z}) \Longrightarrow \langle \xi,\ z - \bar{z} \rangle \le 0, \quad \forall \xi \in \partial^T \psi(\bar{z})$$

hold for all $z \in \mathcal{H}$.

Lemma 1. *"[42] Let $\{A_t \,|t \in \Gamma\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$ and $K =$ cone $\left(\bigcup_{t\in\Gamma} A_t\right)$. Then, every nonzero vector of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different A_t."*

Lemma 2. *"[1] Assume that P and S are arbitrary index sets, $a_p = a(p) = (a_1(p), \dots, a_n(p))$ maps P onto $\mathbb{R}^n$, and so does a_s. Suppose that the set co$\{a_p, p \in P\}$ + cone$\{a_s, s \in S\}$ is closed. Then, the following statements are equivalent:*

$$I : \begin{cases} \langle a_p, z \rangle < 0, p \in P, P \neq \emptyset \\ \langle a_s, z \rangle \le 0, s \in S \end{cases} \quad \textit{has no solution } z \in \mathbb{R}^n;$$

$$II : 0_n \in co\,\{a_p, p \in P\} + cone\,\{a_s, s \in S\}."$$

3. Constraint Qualifications and Optimality Conditions

In this section, following Ansari et al. [21], we first introduce two generalized form of Abadie constraint qualification for SIMPEC (1.1) in the framework of the concept of tangential subdifferentials. Then, we focus on getting optimality conditions for tangentially convex problems.

Let $\mathcal{F}$ indicates the set of feasible solutions of the problem SIMPEC (1.1). Given a feasible point $\bar{z}$ of SIMPEC (1.1), we define the index sets:

$$\begin{aligned} T_\pi(\bar{z}) &:= T_\pi := \{t \in T \,|\pi_t(\bar{z}) = 0\}, \\ \alpha(\bar{z}) &:= \alpha := \{i \,|\Theta_i(\bar{z}) > 0, \Pi_i(\bar{z}) = 0\}, \\ \beta(\bar{z}) &:= \beta := \{i \,|\Theta_i(\bar{z}) = 0, \Pi_i(\bar{z}) = 0\}, \\ \gamma(\bar{z}) &:= \gamma := \{i \,|\Theta_i(\bar{z}) = 0, \Pi_i(\bar{z}) > 0\}. \end{aligned}$$

Now, assuming that π_{t_i}, θ_i, $-\theta_i$, Π_i, $-\Pi_i$, Θ_i, $-\Theta_i$ are tangentially convex at $\bar{z}$, we present the following notations:

$$\begin{aligned} &\pi = \bigcup_{i=1}^{k} \partial_T \pi_{t_i}(\bar{z}), \quad \theta = \bigcup_{i=1}^{q} \partial_T \theta_i(\bar{z}) \cup \partial_T(-\theta_i)(\bar{z}), \\ &\Pi_\alpha = \bigcup_{i\in\alpha} \partial_T \Pi_i(\bar{z}) \cup \partial_T(-\Pi_i)(\bar{z}), \quad \Pi_\beta = \bigcup_{i\in\beta} \partial_T \Pi_i(\bar{z}), \\ &\Theta_\gamma = \bigcup_{i\in\gamma} \partial_T \Theta_i(\bar{z}) \cup \partial_T(-\Theta_i)(\bar{z}), \quad \Theta_\beta = \bigcup_{i\in\beta} \partial_T \Theta_i(\bar{z}), \\ &(\Pi\Theta)_\beta = \bigcup_{i\in\beta} \partial_T(-\Pi_i)(\bar{z}) \cup \partial_T(-\Theta_i)(\bar{z}), \\ &\Gamma(\bar{z}) := \pi^- \cap \theta^- \cap \Pi_\alpha^- \cap \Theta_\gamma^- \cap (\Pi\Theta)_\beta^-, \\ &\Lambda(\bar{z}) := \pi^- \cap \theta^- \cap \Pi_\alpha^- \cap \Theta_\gamma^- \cap (\Pi\Theta)_\beta^- \cap (\Pi_\beta^- \cup \Theta_\beta^-), \end{aligned}$$

where, $t_1, t_2, \dots, t_k \in T_\pi(\bar{z})$, $k \le n + 1$. Using the above notations, we now focus on defining some new version of Abadie constraint qualification in terms of tangential subdifferential.

Definition 7. *Let $\bar{z}$ be a feasible point of SIMPEC where all functions have tangential subdifferential at $\bar{z}$. Also, suppose that at least one of the dual sets used in each of the definitions of $\Gamma(\bar{z})$ and $\Lambda(\bar{z})$ is nonzero. We say that* *(i) the generalized standard Abadie constraint qualification (SI-SACQ) is satisfied at* $\bar{z}$ *if*

$$\Gamma(\bar{z}) \subseteq \mathcal{T}(\mathcal{F}, \bar{z}).$$

(ii) the generalized SIMPEC Abadie constraint qualification (SI-ACQ) is satisfied at $\bar{z}$ *if*

$$\Lambda(\bar{z}) \subseteq \mathcal{T}(\mathcal{F}, \bar{z}).$$

It is worth mentioning that in differentiable case, two the above constraint qualifications coincide with their corresponding concepts in [17].

since $\Lambda(\bar{z}) \subset \Gamma(\bar{z})$, it follows that:

$$\text{SI-SACQ} \quad \Longrightarrow \quad \text{SI-ACQ}.$$

Now, we present the generalized versions of primal and dual stationarity notions for SIMPEC based on tangential subdifferentials. The notion of primal stationary (Bouligand-stationarity) condition for MPEC was first introduced in [43]. A different form of B-stationary was also recommended in [44]. Now, using the tangential subdifferential, the generalized B-stationarity for SIMPEC (1.1) is defined.

Definition 8. *(GB-stationary Point) We say that a feasible point* $\bar{z}$ *of SIMPEC is a generalized Bouligand stationary point (GB-stationary Point) if for all* $v \in \mathcal{T}(\mathcal{F}, \bar{z})$,

$$\langle \xi, v \rangle \geq 0, \quad \textit{for some} \quad \xi \in \partial_T \Upsilon(\bar{z}).$$

Next, we present dual stationary concepts. All of these concepts are stated in terms of tangential subdifferentials.

Definition 9. *A feasible point* $\bar{z}$ *of SIMPEC, is called*

(a) a generalized weakly stationary point (GW-stationary point) if there exist vectors $\lambda = (\lambda^{\pi}, \lambda^{\theta}, \lambda^{\Pi}, \lambda^{\Theta}) \in \mathbb{R}^{k+q+2m}$, $\mu = (\mu^{\theta}, \mu^{\Pi}, \mu^{\Theta}) \in \mathbb{R}^{q+2m}$ *and indices* $t_1, t_2, \ldots, t_k \in T_{\pi}(\bar{z})$, $k \leq n+1$ *such that the following conditions hold:*

$$\begin{aligned} 0 \in\ & \partial_T \Upsilon(\bar{z}) + \sum_{i=1}^{k} \lambda_i^{\pi} \partial_T \pi_{t_i}(\bar{z}) + \sum_{j=1}^{q} \left[\lambda_j^{\theta} \partial_T \theta_j(\bar{z}) + \mu_j^{\theta} \partial_T(-\theta_j)(\bar{z})\right] \\ & + \sum_{i=1}^{m} \left[\lambda_i^{\Pi} \partial_T(-\Pi_i)(\bar{z}) + \lambda_i^{\Theta} \partial_T(-\Theta_i)(\bar{z})\right] \\ & + \sum_{i=1}^{m} \left[\mu_i^{\Pi} \partial_T \Pi_i(\bar{z}) + \mu_i^{\Theta} \partial_T \Theta_i(\bar{z})\right], \end{aligned} \tag{3.1}$$

$$\lambda_{I_\pi}^{\pi} \geq 0, \quad \lambda_j^{\theta}, \mu_j^{\theta} \geq 0,\ j = 1, 2, \ldots, q, \quad \lambda_i^{\Pi}, \lambda_i^{\Theta}, \mu_i^{\Pi}, \mu_i^{\Theta} \geq 0,\ i = 1, 2, \ldots, m, \tag{3.2}$$

$$\lambda_{\gamma}^{\Pi} = \lambda_{\alpha}^{\Theta} = \mu_{\gamma}^{\Pi} = \mu_{\alpha}^{\Theta} = 0, \tag{3.3}$$

where $I_{\pi} = \{1, 2, \ldots, k\}$.

(b) a generalized alternatively stationary point (GA-stationary point) if $\bar{z}$ *is GW-stationary point and* $\mu_i^{\Pi} = 0 \vee \mu_i^{\Theta} = 0$, $\forall i \in \beta$.

(c) a generalized strong stationary point (GS-stationary point) if $\bar{z}$ *is GW-stationary point and* $\mu_i^{\Pi} = 0 \wedge \mu_i^{\Theta} = 0$, $\forall i \in \beta$.

Instantly, based on the above definitions, the following diagram indicates the relations between the dual stationary concepts:

$$\text{GS-stationary} \implies \text{GA-stationary} \implies \text{GW-stationary}.$$

In the continuation of this section, we investigate the optimality conditions for SIMPEC in the framework of the tangential subdifferentials. The first theorem establishes the necessary and sufficient optimality conditions for a locally optimal point of SIMPEC.

Theorem 1. *Suppose that $\bar{z}$ is a feasible point of SIMPEC and Υ is tangentially convex at $\bar{z}$. Then, the following assertions hold:*

(i) If $\bar{z}$ be a locally optimal point of SIMPEC (1.1) and Υ is Lipschitz near $\bar{z}$, then $\bar{z}$ will be a GB-stationary point.

(ii) If $\bar{z}$ be a GB-stationary point, Υ is Lipschitz near $\bar{z}$ and $\langle \xi, v\rangle > 0$ for all $v \in \mathcal{T}(\mathcal{F}, \bar{z}) \setminus \{0\}$, then $\bar{z}$ will be a locally optimal point.

Proof. (i) Let $v \in \mathcal{T}(\mathcal{F}, \bar{z})$, then

$$\exists t_k \downarrow 0, \quad v_k \to v \quad \text{such that} \quad \bar{z} + t_k v_k \in \mathcal{F}, \quad \forall k.$$

Put $z_k = \bar{z} + t_k v_k$. Since $\bar{z}$ is locally optimal, we get $\Upsilon(z_k) \geq \Upsilon(\bar{z})$ for each k. The locally Lipschitz property of Υ near $\bar{z}$ implies that

$$\left| \frac{\Upsilon(\bar{z} + t_k v_k) - \Upsilon(\bar{z} + t_k v)}{t_k} \right| \leq L \|v_k - v\| \to 0 \quad (\text{when } k \to \infty),$$

where L indicates the Lipschitz constant of Υ near $\bar{z}$. Therefore, we obtain that

$$\begin{aligned} \Upsilon'(\bar{z}; v) &= \lim_{k\to\infty} \frac{\Upsilon(\bar{z} + t_k v) - \Upsilon(\bar{z})}{t_k} \\ &= \lim_{k\to\infty} \frac{\Upsilon(\bar{z} + t_k v) - \Upsilon(\bar{z} + t_k v_k)}{t_k} + \lim_{k\to\infty} \frac{\Upsilon(\bar{z} + t_k v_k) - \Upsilon(\bar{z})}{t_k} \\ &= \lim_{k\to\infty} \frac{\Upsilon(\bar{z} + t_k v_k) - \Upsilon(\bar{z})}{t_k} \\ &= \lim_{k\to\infty} \frac{\Upsilon(z_k) - \Upsilon(\bar{z})}{t_k} \geq 0. \end{aligned}$$

Since $\Upsilon'(\bar{z}; v) = \max_{\xi \in \partial_T \Upsilon(\bar{z})} \langle \xi, v\rangle$, so

$$\langle \xi, v\rangle \geq 0, \quad \text{for some} \quad \xi \in \partial_T \Upsilon(\bar{z}).$$

Therefore, $\bar{z}$ is a GB-stationary point.

(ii) On the contrary, suppose that there exists a sequence of feasible points $z_k \to \bar{z}$ such that $\Upsilon(z_k) < \Upsilon(\bar{z})$ for each k. Taking $v_k := \frac{z_k - \bar{z}}{\|z_k - \bar{z}\|}$, by passing to a subsequence if necessary, it can be assumed that $v_k \to v (\neq 0) \in \mathcal{T}(\mathcal{F}, \bar{z})$. By analogy with the proof of part (i), we get to $\Upsilon'(\bar{z}; v) \leq 0$ which contradicts the assumptions of the part (ii) and the proof is complete.

□

The next theorem is the main result in this section. As will be seen, it is proved that GS-stationary is a necessary condition for local optimality under SI-SACQ.

Theorem 2. *Let $\bar{z}$ be a locally optimal solution of SIMPEC, Υ is locally Lipschitz near $\bar{z}$ and all of the functions are tangentially convex at $\bar{z}$. Also, suppose that SI-SACQ holds at $\bar{z}$ and the cone*

$$D := co\ \partial_T \Upsilon(\bar{z}) + cone\ (\pi \cup \theta \cup \Pi_\alpha \cup \Theta_\gamma \cup (\Pi\Theta)_\beta)$$

is closed. Then, $\bar{z}$ will be a GS-stationary point.

Proof. At first, we claim that $\mathcal{T}(\mathcal{F},\bar{z}) \cap (\partial_T \Upsilon(\bar{z}))^s = \emptyset$. Suppose by contradiction that there exists $v \in \mathcal{T}(\mathcal{F},\bar{z}) \cap (\partial_T \Upsilon(\bar{z}))^s$. It follows from $v \in (\partial_T \Upsilon(\bar{z}))^s$ that

$$\langle \xi, v\rangle < 0, \quad \forall \xi \in \partial_T \Upsilon(\bar{z}). \tag{3.4}$$

According to part (i) of Theorem 1, $\bar{z}$ is a GB-stationary point which contradicts (3.4) and verifies assertion. Since SI-SACQ is satisfied at $\bar{z}$, we get

$$(\partial_T \Upsilon(\bar{z}))^s \cap \Gamma(\bar{z}) = \emptyset.$$

This leads that

$$\begin{cases} \langle \xi, v\rangle < 0, & \forall \xi \in \partial_T \Upsilon(\bar{z}), \\ \langle \eta, v\rangle \le 0, & \forall \eta \in (\pi \cup \theta \cup \Pi_\alpha \cup \Theta_\gamma \cup (\Pi\Theta)_\beta), \end{cases} \tag{3.5}$$

has no solution $v \in \mathbb{R}^n$.Therefore, it follows from (3.5) and Lemma 2 that

$$0 \in co\ \partial_T \Upsilon(\bar{z}) + cone\ (\pi \cup \theta \cup \Pi_\alpha \cup \Theta_\gamma \cup (\Pi\Theta)_\beta).$$

Lemma 1 together with the above inclusion implies that there would be nonnegative multipliers $\lambda_i^\pi, i \in I_\pi = \{1,2,\ldots,k\}$, $\lambda_j^\theta, \mu_j^\theta, j = 1,2,\ldots,q$, $\lambda_i^\Pi, i \in \alpha \cup \beta$, $\lambda_i^\Theta, i \in \gamma \cup \beta$, $\mu_i^\Pi, i \in \alpha$, $\mu_i^\Theta, i \in \gamma$ such that

$$\begin{aligned} 0 \in\ & \partial_T \Upsilon(\bar{z}) + \sum_{i \in I_\pi} \lambda_i^\pi \partial_T \pi_{t_i}(\bar{z}) + \sum_{j=1}^{q} \left[\lambda_j^\theta \partial_T \theta_j(\bar{z}) + \mu_j^\theta \partial_T(-\theta_j)(\bar{z})\right] \\ & + \sum_{i \in \alpha\cup\beta} \lambda_i^\Pi \partial_T(-\Pi_i)(\bar{z}) + \sum_{i \in \gamma\cup\beta} \lambda_i^\Theta \partial_T(-\Theta_i)(\bar{z}) \\ & + \sum_{i\in\alpha} \mu_i^\Pi \partial_T \Pi_i(\bar{z}) + \sum_{i\in\gamma} \mu_i^\Theta \partial_T \Theta_i(\bar{z}). \end{aligned} \tag{3.6}$$

Taking $\lambda_\gamma^\Pi = \lambda_\alpha^\Theta = \mu_{\gamma\cup\beta}^\Pi = \mu_{\alpha\cup\beta}^\Theta = 0$ and using (3.6), we get

$$\begin{aligned} 0 \in\ & \partial_T \Upsilon(\bar{z}) + \sum_{i \in I_\pi} \lambda_i^\pi \partial_T \pi_{t_i}(\bar{z}) + \sum_{j=1}^{q} \left[\lambda_j^\theta \partial_T \theta_j(\bar{z}) + \mu_j^\theta \partial_T(-\theta_j)(\bar{z})\right] \\ & + \sum_{i=1}^{m} \left[\lambda_i^\Pi \partial_T(-\Pi_i)(\bar{z}) + \lambda_i^\Theta \partial_T(-\Theta_i)(\bar{z})\right] \\ & + \sum_{i=1}^{m} \left[\mu_i^\Pi \partial_T \Pi_i(\bar{z}) + \mu_i^\Theta \partial_T \Theta_i(\bar{z})\right], \\ & \lambda_{I_\pi}^\pi \ge 0, \quad \lambda_j^\theta, \mu_j^\theta \ge 0,\ j = 1,2,\ldots,q, \quad \lambda_i^\Pi, \lambda_i^\Theta, \mu_i^\Pi, \mu_i^\Theta \ge 0,\ i = 1,2,\ldots,m, \\ & \lambda_\gamma^\Pi = \mu_\gamma^\Pi = \lambda_\alpha^\Theta = \mu_\alpha^\Theta = 0, \qquad \mu_i^\Pi = \mu_i^\Theta = 0, \quad \forall i \in \beta. \end{aligned}$$

Thus, $\bar{z}$ will be a GS-stationary point and the proof is complete. □

The following corollary is an instant outcome of Theorem 2.

Corollary 1. *Let $\bar{z}$ be a locally optimal solution of SIMPEC and Υ is locally Lipschitz near $\bar{z}$. Suppose that the following conditions hold:*

(i) the function $\pi_t(z)$ is continuous of (z,t) in $\mathbb{R}^n \times T$,

(ii) $\pi_t, t \in T$, are convex functions,

(iii) there exists $\tilde{z} \in \mathbb{R}^n$ such that $\pi_t(\tilde{z}) < 0 \ \ \forall t \in T_\pi(\bar{z})$.

Moreover, assume that Υ *and the other constraint functions are tangentially convex at* $\bar{z}$*. If SI-SACQ holds at* $\bar{z}$*, then* $\bar{z}$ *will be a GS-stationary point.*

Proof. Since T is a compact set, it follows from [1, Theorem 7.9] that $\bigcup_{i=1}^{k} \partial_T \pi_{t_i}(\bar{z})$ is a compact set. Furthermore, since Υ and the other constraint functions are tangentially convex at $\bar{z}$, the cone D in Theorem 2 is closed. □

Now, we give an example to indicate Theorem 2.

Example 3. *Consider the following nonsmooth SIMPEC problem in* $\mathbb{R}^2$*:*

$$\begin{aligned} P_1: \quad \min \quad & \Upsilon(z_1, z_2) = |z_2|^3 + z_1 \\ s.t. \quad & \pi_t(z_1, z_2) = |z_2| - t \le 0, \quad t \in T = [0,1], \\ & \Pi(z_1, z_2) = z_1 \ge 0, \quad \Theta(z_1, z_2) = z_2 \ge 0, \\ & \Pi(z_1, z_2)\Theta(z_1, z_2) = 0. \end{aligned}$$

Obviously, $\bar{z} = (0,0)$ *is the global optimum point for* P_1 *and all of the functions are tangentially convex at* $\bar{z}$*. Also,* Υ *is locally Lipschitz near* $\bar{z}$*. We have*

$$\begin{aligned} &\Upsilon'(\bar{z}; v) = v_1, \\ &\pi_t'(\bar{z}; v) = |v_2|, \\ &(-\Pi)'(\bar{z}; v) = -v_1, \\ &(-\Theta)'(\bar{z}; v) = -v_2. \end{aligned}$$

Thus,

$$\begin{aligned} &\partial_T \Upsilon(\bar{z}) = \{(1,0)\}, \\ &\bigcup_{t \in T_\pi(\bar{z})} \partial_T \pi_t(\bar{z}) = co\{(0,1),(0,-1)\}, \quad T_\pi(\bar{z}) = \{0\}, \\ &\partial_T(-\Pi)(\bar{z}) = \{(-1,0)\}, \quad \partial_T(-\Theta)(\bar{z}) = \{(0,-1)\}. \end{aligned}$$

Therefore, we get

$$\pi^- = \{(v_1, v_2) \mid v_2 = 0\}, \quad (\Pi\Theta)^- = \{(v_1, v_2) \mid v_1 \ge 0, \ v_2 \ge 0\}.$$

Hence, SI-SACQ is satisfied at $\bar{z} = (0,0)$ *and since the cone* D *is closed, it is easy to observe that* $\bar{z}$ *is a GS-stationary point.*

The next result illustrates that under SI-ACQ, GA-stationarity will be a necessary optimality condition for SIMPEC.

Theorem 3. *Let* $\bar{z}$ *be a locally optimal solution of SIMPEC and* Υ *is locally Lipschitz near* $\bar{z}$*. Furthermore, suppse that both* Υ *and all constraint functions are tangentially convex at* $\bar{z}$*. If SI-ACQ holds at* $\bar{z}$*, then* $\bar{z}$ *will be a GA-stationary point.*

Proof. By analogy with the proof of Theorem 2, one can show $\mathcal{T}(\mathcal{F}, \bar{z}) \cap (\partial_T \Upsilon(\bar{z}))^s = \emptyset$. Since SI-ACQ is satisfied at $\bar{z}$, thus

$$(\partial_T \Upsilon(\bar{z}))^s \cap \Lambda(\bar{z}) = \emptyset.$$

On the other hand, since Υ and all constraint functions are tangentially convex at $\bar{z}$, thus

$$co\ \partial_T \Upsilon(\bar{z}) + cone\ (\pi \cup \theta \cup \Pi_\alpha \cup \Theta_\gamma \cup (\Pi\Theta)_\beta \cup \Pi_\beta)$$

is closed. Therefore, the following system

$$\begin{cases} \langle \xi, v\rangle < 0, & \forall \xi \in \partial_T f(\bar{z}), \\ \langle \eta, v\rangle \le 0, & \forall \eta \in \Delta, \end{cases} \tag{3.7}$$

has no solution $v \in \mathbb{R}^n$, where $\Delta = (\pi \cup \theta \cup \Pi_\alpha \cup \Theta_\gamma \cup (\Pi\Theta)_\beta \cup \Pi_\beta), \Delta^- \subset \Lambda(\bar{z})$. Then, it follows from (3.7) and Lemma 2 that

$$0 \in co\ \partial_T \Upsilon(\bar{z}) + cone\ (\pi \cup \theta \cup \Pi_\alpha \cup \Theta_\gamma \cup (\Pi\Theta)_\beta \cup \Pi_\beta).$$

Analogous to the proof of the previous Theorem, Lemma 1 together with the above inclusion implies that there would be nonnegative multipliers $\lambda_i^\pi, i \in I_\pi = \{1, 2, \ldots, k\}$, $\lambda_j^\theta, \mu_j^\theta, j = 1, 2, \ldots, q$, $\lambda_i^\Pi, i \in \alpha \cup \beta$, $\lambda_i^\Theta, i \in \gamma \cup \beta$, $\mu_i^\Pi, i \in \alpha \cup \beta$, $\mu_i^\Theta, i \in \gamma$ such that

$$\begin{aligned} 0 \in\ & \partial_T \Upsilon(\bar{z}) + \sum_{i \in I_\pi} \lambda_i^\pi \partial_T \pi_{t_i}(\bar{z}) + \sum_{j=1}^{q} \left[\lambda_j^\theta \partial_T \theta_j(\bar{z}) + \mu_j^\theta \partial_T(-\theta_j)(\bar{z})\right] \\ & + \sum_{i \in \alpha\cup\beta} \lambda_i^\Pi \partial_T(-\Pi_i)(\bar{z}) + \sum_{i \in \gamma\cup\beta} \lambda_i^\Theta \partial_T(-\Theta_i)(\bar{z}) \\ & + \sum_{i \in \alpha\cup\beta} \mu_i^\Pi \partial_T \Pi_i(\bar{z}) + \sum_{i \in \gamma} \mu_i^\Theta \partial_T \Theta_i(\bar{z}). \end{aligned} \tag{3.8}$$

Taking $\lambda_\gamma^\Pi = \lambda_\alpha^\Theta = \mu_\gamma^\Pi = \mu_{\alpha\cup\beta}^\Theta = 0$ and using (3.8), we get

$$\begin{aligned} 0 \in\ & \partial_T \Upsilon(\bar{z}) + \sum_{i \in I_\pi} \lambda_i^\pi \partial_T \pi_{t_i}(\bar{z}) + \sum_{j=1}^{q} \left[\lambda_j^\theta \partial_T \theta_j(\bar{z}) + \mu_j^\theta \partial_T(-\theta_j)(\bar{z})\right] \\ & + \sum_{i=1}^{m} \left[\lambda_i^\Pi \partial_T(-\Pi_i)(\bar{z}) + \lambda_i^\Theta \partial_T(-\Theta_i)(\bar{z})\right] \\ & + \sum_{i=1}^{m} \left[\mu_i^\Pi \partial_T \Pi_i(\bar{z}) + \mu_i^\Theta \partial_T \Theta_i(\bar{z})\right], \\ & \lambda_{I_\pi}^\pi \ge 0, \quad \lambda_j^\theta, \mu_j^\theta \ge 0,\ j = 1, 2, \ldots, q, \quad \lambda_i^\Pi, \lambda_i^\Theta, \mu_i^\Pi, \mu_i^\Theta \ge 0,\ i = 1, 2, \ldots, m, \\ & \lambda_\gamma^\Pi = \mu_\gamma^\Pi = \lambda_\alpha^\Theta = \mu_\alpha^\Theta = 0, \qquad \mu_i^\Theta = 0, \quad \forall i \in \beta. \end{aligned}$$

Thus, $\bar{z}$ will be a GA-stationary point and the proof is complete. □

The following example illustrates Theorem 3.

Example 4. *Consider the following nonsmooth SIMPEC problem in $\mathbb{R}^2$:*

$$\begin{aligned} P_2: \quad \min\quad & \Upsilon(z_1, z_2) = |z_2| + \sin(z_2^2 - z_1) \\ s.t.\quad & \pi_t(z_1, z_2) = \max\{-z_1, -z_2\} - t \le 0, \quad t \in T = [0, 1], \\ & \Pi(z_1, z_2) = z_1 + z_2 \ge 0, \quad \Theta(z_1, z_2) = z_1 - z_2 \ge 0, \\ & \Pi(z_1, z_2)\Theta(z_1, z_2) = 0. \end{aligned}$$

The graphical representation of objective function $\Upsilon(z_1, z_2)$ is shown in Fig 1. Then, $\mathcal{F} = \{(z_1, z_2)|\ z_1 = z_2,\ z_1 \ge 0\}$ is the feasible set of P_2 and its geometric representation is shown in Fig 2. It can be easy to check that $\bar{z} = (0, 0)$ is a locally optimal solution of P_2 and all of the functions are tangentially convex at $\bar{z}$. Also, Υ is locally Lipschitz at $\bar{z}$. We have

$$\begin{aligned} & \Upsilon'(\bar{z}; v) = |v_2| - v_1, \quad \pi_t'(\bar{z}; v) = \max\{-v_1, -v_2\}, \\ & \Pi'(\bar{z}; v) = v_1 + v_2, \quad \Theta'(\bar{z}; v) = v_1 - v_2, \\ & (-\Pi)'(\bar{z}; v) = -v_1 - v_2, \quad (-\Theta)'(\bar{z}; v) = -v_1 + v_2. \end{aligned}$$

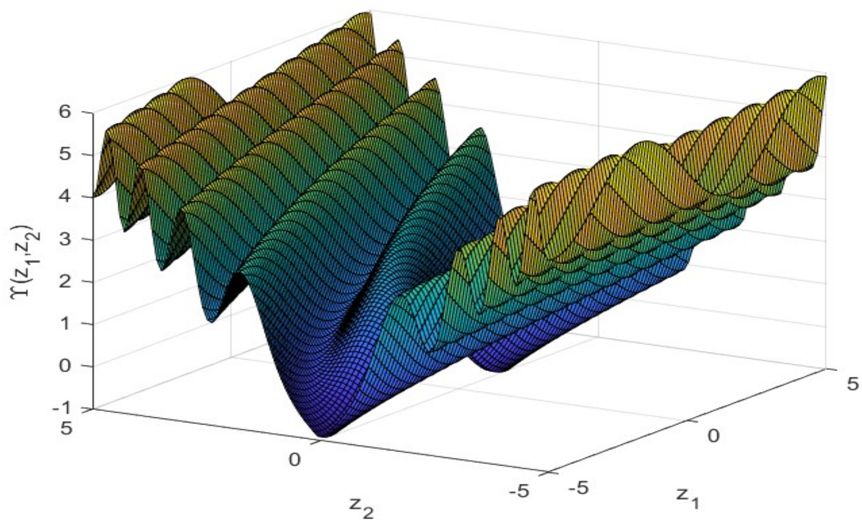

Figure 1: Graph of the objective function $\Upsilon(z_1, z_2)$ considered in Example 4.

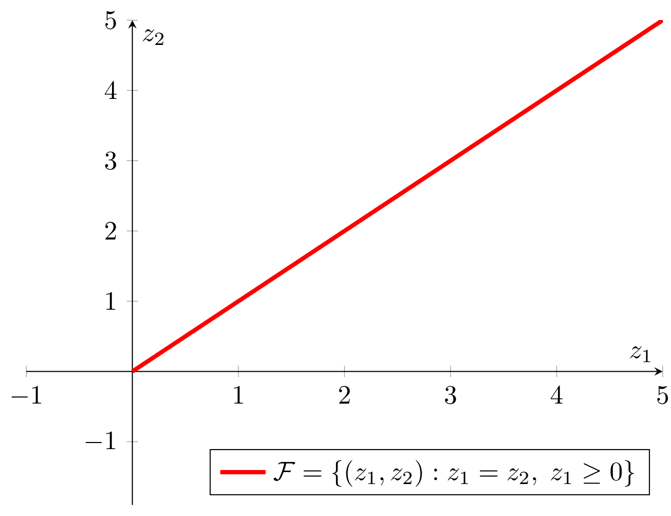

Figure 2: The feasible set $\mathcal{F}$ of P_2 considered in Example 4.

Thus,

$$\partial_T \Upsilon(\bar{z}) = \{-1\} \times [-1,1],$$
$$\bigcup_{t\in T_\pi(\bar{z})} \partial_T \pi_t(\bar{z}) = co\{(-1,0),(0,-1)\}, \quad T_\pi(\bar{z}) = \{0\},$$
$$\partial_T \Pi(\bar{z}) = \{(1,1)\}, \quad \partial_T \Theta(\bar{z}) = \{(1,-1)\}$$
$$\partial_T(-\Pi)(\bar{z}) = \{(-1,-1)\}, \quad \partial_T(-\Theta)(\bar{z}) = \{(-1,1)\}.$$

Hence, we get

$$\pi^- = \{(v_1,v_2) \mid v_1 \geq 0,\ v_2 \geq 0\}, \quad (\Pi\Theta)^- = \{(v_1,v_2) \mid v_1+v_2 \geq 0,\ v_1-v_2 \geq 0\},$$
$$\Pi^- = \{(v_1,v_2) \mid v_1+v_2 \leq 0\}, \quad \Theta^- = \{(v_1,v_2) \mid v_1-v_2 \leq 0\}.$$

Therefore,

$$\Lambda(\bar{z}) = \{(0,0)\} \subset \mathcal{T}(\mathcal{F},\bar{z}) = \{(d_1,d_2) \mid d_1 = d_2,\ d_1 \geq 0\},$$

that is, SI-ACQ is satisfied at $\bar{z}$. For example, by taking $\lambda^\pi = \lambda^\Pi = \mu^\Pi = 0, \lambda^\Theta = 1, \mu^\Theta = 2$, it would be fairly straightforward to see that $\bar{z}$ is a GA-stationary point.

Finally, we prove that under appropriate generalized convexity assumptions, GA-stationarity also would be a global or local sufficient optimality condition for SIMPEC.

Theorem 4. *Let $\bar{z}$ be a GA-stationary point of SIMPEC. We define the following index sets:*

$$\alpha_\mu^+ := \{i \in \alpha \mid \mu_i^\Pi > 0\},$$
$$\gamma_\mu^+ := \{i \in \gamma \mid \mu_i^\Theta > 0\},$$
$$\beta_\mu^\Pi := \{i \in \beta \mid \mu_i^\Theta = 0, \quad \mu_i^\Pi > 0\},$$
$$\beta_\mu^\Theta := \{i \in \beta \mid \mu_i^\Theta > 0, \quad \mu_i^\Pi = 0\}.$$

Thereupon, supposing that Υ be Dini-pseudoconvex at $\bar{z}$, $\pi_t (t \in T_\pi)$, $\pm\theta_j (j = 1,2,\ldots,q)$, $-\Pi_i (i \in \alpha \cup \beta)$, $-\Theta_i (i \in \gamma \cup \beta)$ are Dini-quasiconvex at $\bar{z}$ and all functions are tangentially convex at $\bar{z}$. Then, the following expressions are valid:

- *(i) If $\beta_\mu^\Pi \cup \beta_\mu^\Theta \cup \alpha_\mu^+ \cup \gamma_\mu^+ = \emptyset$, then $\bar{z}$ will be a global optimal solution of SIMPEC.*
- *(ii) If Π_i for $i \in \alpha_\mu^+ \cup \beta_\mu^\Pi$ and Θ_i for $i \in \gamma_\mu^+ \cup \beta_\mu^\Theta$ are continuous and Dini-quasiconvex at $\bar{z}$ and further $\beta_\mu^\Pi \cup \beta_\mu^\Theta = \emptyset$, or if $\bar{z}$ is a interior point relative to the set $\mathcal{F} \cap \{\Pi_i(z) = 0, \quad \Theta_i(z) = 0, \quad i \in \beta_\mu^\Pi \cup \beta_\mu^\Theta\}$, then $\bar{z}$ will be a locally optimal solution of SIMPEC.*

Proof. (i) Let $\bar{z}$ be an arbitrary feasible point of SIMPEC. Then, for each $t_i \in T_\pi(\bar{z})$, we have

$$\pi_{t_i}(z) \leq 0 = \pi_{t_i}(\bar{z}).$$

By the Dini-quasiconvexity of $\pi_{t_i}(t_i \in T_\pi(\bar{z}))$ at $\bar{z}$, it follows that,

$$\langle \xi_i^\pi, z-\bar{z}\rangle \leq 0, \quad \forall \xi_i^\pi \in \partial_T \pi_{t_i}(\bar{z}), \quad \forall t_i \in T_\pi(\bar{z}). \tag{3.9}$$

Similarly, we can get

$$\langle \eta_j, z-\bar{z}\rangle \leq 0, \quad \forall \eta_j \in \partial_T \theta_j(\bar{z}), \quad \forall j = 1,2,\ldots,q, \tag{3.10}$$
$$\langle v_j, z-\bar{z}\rangle \leq 0, \quad \forall v_j \in \partial_T(-\theta)_j(\bar{z}), \quad \forall j = 1,2,\ldots,q, \tag{3.11}$$
$$\langle \xi_i^\Pi, z-\bar{z}\rangle \leq 0, \quad \forall \xi_i^\Pi \in \partial_T(-\Pi_i)(\bar{z}), \quad \forall i \in \alpha \cup \beta, \tag{3.12}$$
$$\langle \xi_i^\Theta, z-\bar{z}\rangle \leq 0, \quad \forall \xi_i^\Theta \in \partial_T(-\Theta_i)(\bar{z}), \quad \forall i \in \gamma \cup \beta. \tag{3.13}$$

Now if $\beta_\mu^{\Pi} \cup \beta_\mu^{\Theta} \cup \alpha_\mu^+ \cup \gamma_\mu^+ = \emptyset$, then multiplying (3.9)-(3.13) by $\lambda_i^\pi \geq 0\,(t_i \in T_\pi(\bar{z}))$, $\lambda_j^\theta \geq 0\,(j = 1, 2, \dots, q)$, $\mu_j^\theta \geq 0\,(j = 1, 2, \dots, q)$, $\lambda_i^{\Pi} \geq 0\,(i \in \alpha \cup \beta)$, $\lambda_i^{\Theta} \geq 0\,(i \in \gamma \cup \beta)$, respectively and by adding them together, we get

$$\left\langle \sum_{i=1}^{k} \lambda_i^\pi \xi_i^\pi + \sum_{j=1}^{q} \left[\lambda_j^\theta \eta_j + \mu_j^\theta v_j\right] + \sum_{i=1}^{m} \left[\lambda_i^{\Pi} \xi_i^{\Pi} + \lambda_i^{\Theta} \xi_i^{\Theta}\right], z - \bar{z} \right\rangle \leq 0,$$

for every $\xi_i^\pi \in \partial_T \pi_{t_i}(\bar{z})$, $\eta_j \in \partial_T \theta_j(\bar{z})$, $v_j \in \partial_T(-\theta_j)(\bar{z})$, $\xi_i^{\Pi} \in \partial_T(-\Pi_i)(\bar{z})$, $\xi_i^{\Theta} \in \partial_T(-\Theta_i)(\bar{z})$. Therefore, as regards $\bar{z}$ is a GA-stationary point, we can select $\theta \in \partial_T \Upsilon(\bar{z})$ such that $\langle \theta, z - \bar{z} \rangle \geq 0$. The Dini-pseudoconvexity Υ at $\bar{z}$ implies that $\Upsilon(z) \geq \Upsilon(\bar{z})$ for every feasible point z. Thus, $\bar{z}$ will be a global optimal solution of SIMPEC.

(ii) At first, we consider the case when $\beta_\mu^{\Pi} \cup \beta_\mu^{\Theta} = \emptyset$. For any $i \in \alpha$, since $\Theta_i(\bar{z}) > 0$, the continuity of Θ_i at $\bar{z}$ implies that $\Theta_i(z) > 0$ for every feasible point z sufficiently close to $\bar{z}$ and so, due to the complementarity condition, we have $\Pi_i(z) = 0$ for such z. Thus,

$$\Pi_i(z) = \Pi_i(\bar{z}), \quad \forall i \in \alpha,$$

for z sufficiently close to $\bar{z}$. According to the Dini-quasiconvexity of $\Pi_i (i \in \alpha_\mu^+)$ at $\bar{z}$, for z sufficiently close to $\bar{z}$ it follows that,

$$\langle \tau_i, z - \bar{z} \rangle \leq 0, \quad \forall \tau_i \in \partial_T \Pi_i(\bar{z}), \quad \forall i \in \alpha_\mu^+. \tag{3.14}$$

Similarly, for z sufficiently close to $\bar{z}$, we can get

$$\langle \delta_i, z - \bar{z} \rangle \leq 0, \quad \forall \delta_i \in \partial_T \Theta_i(\bar{z}), \quad \forall i \in \gamma_\mu^+. \tag{3.15}$$

Analogous to the proof of part (i), one can find some $\theta \in \partial_T \Upsilon(\bar{z})$ for z sufficiently close to $\bar{z}$ such that $\langle \theta, z - \bar{z} \rangle \geq 0$. By the Dini-pseudoconvexity of Υ at $\bar{z}$, it is deduced that $\Upsilon(z) \geq \Upsilon(\bar{z})$ for every feasible point z sufficiently close to $\bar{z}$. That is, $\bar{z}$ will be a locally optimal solution of SIMPEC when $\beta_\mu^{\Pi} \cup \beta_\mu^{\Theta} = \emptyset$.

Next, let's assume that $\bar{z}$ is a interior-point relative to the set $\mathcal{F} \cap \{\Pi_i(z) = 0,\ \Theta_i(z) = 0,\ i \in \beta_\mu^{\Pi} \cup \beta_\mu^{\Theta}\}$. This implies that for any feasible point z sufficiently close to $\bar{z}$,

$$\Pi_i(z) = \Theta_i(z) = 0, \quad \forall i \in \beta_\mu^{\Pi} \cup \beta_\mu^{\Theta}.$$

By the Dini-quasiconvexity of $\Pi_i (i \in \beta_\mu^{\Pi})$ and $\Theta_i (i \in \beta_\mu^{\Theta})$, the following conclusions are obtained,

$$\langle \tau_i, z - \bar{z} \rangle \leq 0, \quad \forall \tau_i \in \partial_T \Pi_i(\bar{z}), \quad \forall i \in \beta_\mu^{\Pi}, \tag{3.16}$$

$$\langle \delta_i, z - \bar{z} \rangle \leq 0, \quad \forall \delta_i \in \partial_T \Theta_i(\bar{z}), \quad \forall i \in \beta_\mu^{\Theta}. \tag{3.17}$$

Multiplying (3.9)-(3.17) by $\lambda_i^\pi \geq 0\,(t_i \in T_\pi(\bar{z}))$, $\lambda_j^\theta \geq 0\,(j = 1, 2, \dots, q)$, $\mu_j^\theta \geq 0\,(j = 1, 2, \dots, q)$, $\lambda_i^{\Pi} \geq 0\,(i \in \alpha \cup \beta)$, $\lambda_i^{\Theta} \geq 0\,(i \in \gamma \cup \beta)$, $\mu_i^{\Pi} > 0\,(i \in \alpha_\mu^+)$, $\mu_i^{\Theta} > 0\,(i \in \gamma_\mu^+)$, $\mu_i^{\Pi} > 0\,(i \in \beta_\mu^{\Pi})$, $\mu_i^{\Theta} > 0\,(i \in \beta_\mu^{\Theta})$ respectively and by adding them together, we get

$$\left\langle \sum_{i=1}^{k} \lambda_i^\pi \xi_i^\pi + \sum_{j=1}^{q} \left[\lambda_j^\theta \eta_j + \mu_j^\theta v_j\right] + \sum_{i=1}^{m} \left[\lambda_i^{\Pi} \xi_i^{\Pi} + \lambda_i^{\Theta} \xi_i^{\Theta} + \mu_i^{\Pi} \tau_i + \mu_i^{\Theta} \delta_i\right], z - \bar{z} \right\rangle \leq 0,$$

for z sufficiently close to $\bar{z}$ and every $\xi_i^\pi \in \partial_T \pi_{t_i}(\bar{z})$, $\eta_j \in \partial_T \theta_j(\bar{z})$, $v_j \in \partial_T(-\theta_j)(\bar{z})$, $\xi_i^{\Pi} \in \partial_T(-\Pi_i)(\bar{z})$, $\xi_i^{\Theta} \in \partial_T(-\Theta_i)(\bar{z})$, $\eta_i \in \partial_T \Pi_i(\bar{z})$, $\delta_i \in \partial_T \Theta_i(\bar{z})$. Therefore, as regards $\bar{z}$ is GA-stationary, we can select $\theta \in \partial_T \Upsilon(\bar{z})$ such that for z sufficiently close to $\bar{z}$, $\langle \theta, z - \bar{z} \rangle \geq 0$. Now, using the Dini-pseudoconvexity of Υ at $\bar{z}$, it follows that $\bar{z}$ is a locally optimal solution for SIMPEC. Therefore, the proof of the Theorem is complete.

□

4. The Wolfe and Mond-Weir Type Duality

The Wolfe and Mond-Weir type dual models are formulated for SIMPEC in terms of tangential subdifferentials in this section. Furthermore, the results related to weak and strong duality are studied using the generalized Dini-convexity. Suppose that $T_\pi, I_\pi, \alpha, \beta, \gamma, \beta_\mu^{\Pi}, \beta_\mu^{\Theta}, \alpha_\mu^+, \alpha_\mu^+$ be defined as before.

Primarily, we formulate the Wolfe type dual model of SIMPEC based on the GS-stationarity as follows:

$$\text{WD-SIMPEC}: \quad \max_{u,\lambda,\mu} \ \Upsilon(u) + \sum_{i=1}^{k} \lambda_i^\pi \pi_{t_i}(u) + \sum_{j=1}^{q} \tilde{\lambda}_j^\theta \theta_j(u) - \sum_{i=1}^{m} \Big[\tilde{\lambda}_i^{\Pi} \Pi_i(u) + \tilde{\lambda}_i^{\Theta} \Theta_i(u)\Big]$$

subject to:

$$\begin{aligned} 0 \in\ & \partial_T \Upsilon(u) + \sum_{i=1}^{k} \lambda_i^\pi \partial_T \pi_{t_i}(u) + \sum_{j=1}^{q} \big[\lambda_j^\theta \partial_T \theta_j(u) + \mu_j^\theta \partial_T(-\theta_j)(u)\big] \\ & + \sum_{i=1}^{m} \big[\lambda_i^{\Pi} \partial_T(-\Pi_i)(u) + \lambda_i^{\Theta} \partial_T(-\Theta_i)(u)\big] + \sum_{i=1}^{m} \big[\mu_i^{\Pi} \partial_T \Pi_i(u) + \mu_i^{\Theta} \partial_T \Theta_i(u)\big], \qquad (4.1) \\ & \lambda_{I_\pi}^\pi \geq 0, \quad \lambda_j^\theta, \mu_j^\theta \geq 0,\ j = 1,2,\ldots,q, \quad \lambda_i^{\Pi}, \lambda_i^{\Theta}, \mu_i^{\Pi}, \mu_i^{\Theta} \geq 0,\ i = 1,2,\ldots,m, \\ & \lambda_\gamma^{\Pi} = \mu_\gamma^{\Pi} = \lambda_\alpha^{\Theta} = \mu_\alpha^{\Theta} = 0, \qquad \mu_i^{\Pi} = 0, \quad \mu_i^{\Theta} = 0, \quad \forall i \in \beta, \end{aligned}$$

where $u \in \mathbb{R}^n$, $\tilde{\lambda}_j^\theta = \lambda_j^\theta - \mu_j^\theta$, $\tilde{\lambda}_i^{\Pi} = \lambda_i^{\Pi} - \mu_i^{\Pi}$, $\tilde{\lambda}_i^{\Theta} = \lambda_i^{\Theta} - \mu_i^{\Theta}$, $\lambda = (\lambda^\pi, \lambda^\theta, \lambda^{\Pi}, \lambda^{\Theta}) \in \mathbb{R}^{k+q+2m}$, $\mu = (\mu^\theta, \mu^{\Pi}, \mu^{\Theta}) \in \mathbb{R}^{q+2m}$, $t_1, t_2, \ldots, t_k \in T_\pi(\bar{z})$, $k \leq n+1$.

The following result represents the weak duality relation between the primal problem SIMPEC and the dual problem WD-SIMPEC.

Theorem 5. *(Weak Duality) Let z is feasible for SIMPEC and (u, λ, μ) is feasible for WD-SIMPEC. Assume that all functions are tangentially convex at u and further, Υ, $\pi_t(t \in T)$, $\pm\theta_j(j = 1,2,\ldots,q)$, $-\Pi_i(i \in \alpha \cup \beta)$, $-\Theta_i(i \in \gamma \cup \beta)$ are Dini-convex functions at u. If $\beta_\mu^{\Pi} \cup \beta_\mu^{\Theta} \cup \alpha_\mu^+ \cup \gamma_\mu^+ = \emptyset$, then*

$$\Upsilon(z) \geq \Upsilon(u) + \sum_{i=1}^{k} \lambda_i^\pi \pi_{t_i}(u) + \sum_{j=1}^{q} \tilde{\lambda}_j^\theta \theta_j(u) - \sum_{i=1}^{m} \Big[\tilde{\lambda}_i^{\Pi} \Pi_i(u) + \tilde{\lambda}_i^{\Theta} \Theta_i(u)\Big].$$

Proof. Let z be any feasible point of SIMPEC. That is, $z \in \mathcal{F}$. By the Dini-convexity of Υ at u,

$$\Upsilon(z) - \Upsilon(u) \geq \langle \xi, z - u \rangle, \quad \forall \xi \in \partial_T \Upsilon(u). \qquad (4.2)$$

Similarly, we can get

$$\begin{aligned} & \pi_{t_i}(z) - \pi_{t_i}(u) \geq \langle \xi_i^\pi, z - u \rangle, \quad \forall \xi_i^\pi \in \partial_T \pi_{t_i}(u), \quad \forall t_i \in T_\pi(z), && (4.3) \\ & \theta_j(z) - \theta_j(u) \geq \langle \eta_j, z - u \rangle, \quad \forall \eta_j \in \partial_T \theta_j(u), \quad j = 1,2,\ldots,q, && (4.4) \\ & -\theta_j(z) + \theta_j(u) \geq \langle v_j, z - u \rangle, \quad \forall v_j \in \partial_T(-\theta_j)(u), \quad j = 1,2,\ldots,q, && (4.5) \\ & -\Pi_i(z) + \Pi_i(u) \geq \langle \xi_i^{\Pi}, z - u \rangle, \quad \forall \xi_i^{\Pi} \in \partial_T(-\Pi_i)(u), \quad \forall i \in \alpha \cup \beta, && (4.6) \\ & -\Theta_i(z) + \Theta_i(u) \geq \langle \xi_i^{\Theta}, z - u \rangle, \quad \forall \xi_i^{\Theta} \in \partial_T(-\Theta_i)(u), \quad \forall i \in \gamma \cup \beta. && (4.7) \end{aligned}$$

If $\beta_\mu^{\Pi} \cup \beta_\mu^{\Theta} \cup \alpha_\mu^+ \cup \gamma_\mu^+ = \emptyset$, then

$$\sum_{i=1}^{m} \big[\mu_i^{\Pi} \Pi_i(u) + \mu_i^{\Theta} \Theta_i(u)\big] = 0. \qquad (4.8)$$

Therefore, multiplying (4.3)-(4.7) by $\lambda_i^\pi \geq 0\,(t_i \in T_\pi)$, $\lambda_j^\theta \geq 0\,(j = 1,2,\ldots,q)$, $\mu_j^\theta \geq 0\,(j = 1,2,\ldots,q)$,

$\lambda_i^{\Pi} \geq 0\,(i \in \alpha \cup \beta)$, $\lambda_i^{\Theta} \geq 0\,(i \in \gamma \cup \beta)$, respectively and by adding them together and to (4.2), we get

$$\begin{aligned}
&\Upsilon(z) - \Upsilon(u) \\
&\quad + \sum_{i=1}^{k} \lambda_i^{\pi} \pi_{t_i}(z) - \sum_{i=1}^{k} \lambda_i^{\pi} \pi_{t_i}(u) + \sum_{j=1}^{q} \lambda_j^{\theta} \theta_j(z) - \sum_{j=1}^{q} \lambda_j^{\theta} \theta_j(u) - \sum_{j=1}^{q} \mu_j^{\theta} \theta_j(z) \\
&\quad + \sum_{j=1}^{q} \mu_j^{\theta} \theta_j(u) - \sum_{i=1}^{m} \lambda_i^{\Pi} \Pi_i(z) + \sum_{i=1}^{m} \lambda_i^{\Pi} \Pi_i(u) - \sum_{i=1}^{m} \lambda_i^{\Theta} \Theta_i(z) + \sum_{i=1}^{m} \lambda_i^{\Theta} \Theta_i(u) \\
&\quad \geq \left\langle \xi + \sum_{i=1}^{k} \lambda_i^{\pi} \xi_i^{\pi} + \sum_{j=1}^{q} \left[\lambda_j^{\theta} \eta_j + \mu_j^{\theta} v_j\right] + \sum_{i=1}^{m} \left[\lambda_i^{\Pi} \xi_i^{\Pi} + \lambda_i^{\Theta} \xi_i^{\Theta}\right], z - u \right\rangle,
\end{aligned}$$

for every $\xi \in \partial_T \Upsilon(u)$, $\xi_i^{\pi} \in \partial_T \pi_{t_i}(u)$, $\eta_j \in \partial_T \theta_j(u)$, $v_j \in \partial_T(-\theta_j)(u)$, $\xi_i^{\Pi} \in \partial_T(-\Pi_i)(u)$, $\xi_i^{\Theta} \in \partial_T(-\Theta_i)(u)$.

From (4.1), there exist $\bar{\xi} \in \partial_T \Upsilon(u)$, $\bar{\xi}_i^{\pi} \in \partial_T \pi_{t_i}(u)$, $\bar{\eta}_j \in \partial_T \theta_j(u)$, $\bar{v}_j \in \partial_T(-\theta_j)(u)$, $\bar{\xi}_i^{\Pi} \in \partial_T(-\Pi_i)(u)$, $\bar{\xi}_i^{\Theta} \in \partial_T(-\Theta_i)(u)$ such that,

$$\bar{\xi} + \sum_{i=1}^{k} \lambda_i^{\pi} \bar{\xi}_i^{\pi} + \sum_{j=1}^{q} \left[\lambda_j^{\theta} \bar{\eta}_j + \mu_j^{\theta} \bar{v}_j\right] + \sum_{i=1}^{m} \left[\lambda_i^{\Pi} \bar{\xi}_i^{\Pi} + \lambda_i^{\Theta} \bar{\xi}_i^{\Theta}\right] = 0.$$

Thus,

$$\begin{aligned}
&\Upsilon(z) - \Upsilon(u) \\
&\quad + \sum_{i=1}^{k} \lambda_i^{\pi} \pi_{t_i}(z) - \sum_{i=1}^{k} \lambda_i^{\pi} \pi_{t_i}(u) + \sum_{j=1}^{q} \lambda_j^{\theta} \theta_j(z) - \sum_{j=1}^{q} \lambda_j^{\theta} \theta_j(u) - \sum_{j=1}^{q} \mu_j^{\theta} \theta_j(z) \\
&\quad + \sum_{j=1}^{q} \mu_j^{\theta} \theta_j(u) - \sum_{i=1}^{m} \lambda_i^{\Pi} \Pi_i(z) + \sum_{i=1}^{m} \lambda_i^{\Pi} \Pi_i(u) - \sum_{i=1}^{m} \lambda_i^{\Theta} \Theta_i(z) + \sum_{i=1}^{m} \lambda_i^{\Theta} \Theta_i(u) \geq 0.
\end{aligned}$$

Now, by using the feasibility of z for SIMPEC, it follows that,

$$\begin{aligned}
&\Upsilon(z) - \Upsilon(u) \\
&\quad - \sum_{i=1}^{k} \lambda_i^{\pi} \pi_{t_i}(u) - \sum_{j=1}^{q} \lambda_j^{\theta} \theta_j(u) + \sum_{j=1}^{q} \mu_j^{\theta} \theta_j(u) + \sum_{i=1}^{m} \lambda_i^{\Pi} \Pi_i(u) + \sum_{i=1}^{m} \lambda_i^{\Theta} \Theta_i(u) \geq 0.
\end{aligned} \tag{4.9}$$

Therefore, from (4.8), (4.9) and feasibility of (u, λ, μ) for WD-SIMPEC, we get

$$\Upsilon(z) \geq \Upsilon(u) + \sum_{i=1}^{k} \lambda_i^{\pi} \pi_{t_i}(u) + \sum_{j=1}^{q} \tilde{\lambda}_j^{\theta} \theta_j(u) - \sum_{i=1}^{m} \left[\tilde{\lambda}_i^{\Pi} \Pi_i(u) + \tilde{\lambda}_i^{\Theta} \Theta_i(u)\right],$$

where $\tilde{\lambda}_j^{\theta} = \lambda_j^{\theta} - \mu_j^{\theta}$, $\tilde{\lambda}_i^{\Pi} = \lambda_i^{\Pi} - \mu_i^{\Pi}$, $\tilde{\lambda}_i^{\Theta} = \lambda_i^{\Theta} - \mu_i^{\Theta}$. This completes the proof. □

The next Theorem expresses a strong duality relation between the primal problem SIMPEC and the dual problem WD-SIMPEC.

Theorem 6. *(Strong Duality) Let $\bar{z}$ be a locally optimal solution of SIMPEC and Υ is locally Lipschitz near $\bar{z}$. Suppose that all functions are tangentially convex at $\bar{z}$, SI-SACQ holds at $\bar{z}$, and D is closed. Then, there exists $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^{k+q+2m} \times \mathbb{R}^{q+2m}$ such that $(\bar{z}, \bar{\lambda}, \bar{\mu})$ is feasible for WD-SIMPEC and respective objective values are equal. Also, if the assumptions of Theorem 5 (Weak Duality) are satisfied at any feasible solution (u, λ, μ) of WD-SIMPEC , then $(\bar{z}, \bar{\lambda}, \bar{\mu})$ is an optimal solution for WD-SIMPEC.*

Proof. According to Theorem 2, there exist vectors $\bar{\lambda} = (\bar{\lambda}^{\pi}, \bar{\lambda}^{\theta}, \bar{\lambda}^{\Pi}, \bar{\lambda}^{\Theta}) \in \mathbb{R}^{k+q+2m}$, $\bar{\mu} = (\bar{\mu}^{\theta}, \bar{\mu}^{\Pi}, \bar{\mu}^{\Theta}) \in \mathbb{R}^{q+2m}$ and indices $t_1, t_2, \ldots, t_k \in T_{\pi}(\bar{z})$, $k \leq n+1$, such that the GS-stationarity conditions are satisfied at $\bar{z}$. Hence, $(\bar{z}, \bar{\lambda}, \bar{\mu})$ is feasible for WD-SIMPEC.

Consequently, from the feasibility conditions $\bar{z}$ for SIMPEC and $(\bar{z}, \bar{\lambda}, \bar{\mu})$ for WD-SIMPEC, we have

$$\Upsilon(\bar{z}) = \Upsilon(\bar{z}) + \sum_{i=1}^{k} \bar{\lambda}_i^{\pi} \pi_{t_i}(\bar{z}) + \sum_{j=1}^{q} \bar{\bar{\lambda}}_j^{\theta} \theta_j(\bar{z}) - \sum_{i=1}^{m} \left[\bar{\bar{\lambda}}_i^{\Pi} \Pi_i(\bar{z}) + \bar{\bar{\lambda}}_i^{\Theta} \Theta_i(\bar{z}) \right], \tag{4.10}$$

where $\bar{\bar{\lambda}}_j^{\theta} = \bar{\lambda}_j^{\theta} - \bar{\mu}_j^{\theta}$, $\bar{\bar{\lambda}}_i^{\Pi} = \bar{\lambda}_i^{\Pi} - \bar{\mu}_i^{\Pi}$, $\bar{\bar{\lambda}}_i^{\Theta} = \bar{\lambda}_i^{\Theta} - \bar{\mu}_i^{\Theta}$.

Moreover, since the assumptions of Theorem 5 are satisfied at any feasible solution (u, λ, μ) of WD-SIMPEC, we have

$$\Upsilon(\bar{z}) \geq \Upsilon(u) + \sum_{i=1}^{k} \lambda_i^{\pi} \pi_{t_i}(u) + \sum_{j=1}^{q} \tilde{\lambda}_j^{\theta} \theta_j(u) - \sum_{i=1}^{m} \left[\tilde{\lambda}_i^{\Pi} \Pi_i(u) + \tilde{\lambda}_i^{\Theta} \Theta_i(u) \right], \tag{4.11}$$

where $\tilde{\lambda}_j^{\theta} = \lambda_j^{\theta} - \mu_j^{\theta}$, $\tilde{\lambda}_i^{\Pi} = \lambda_i^{\Pi} - \mu_i^{\Pi}$, $\tilde{\lambda}_i^{\Theta} = \lambda_i^{\Theta} - \mu_i^{\Theta}$.

Using (4.10) and (4.11), $(\bar{z}, \bar{\lambda}, \bar{\mu})$ is an optimal solution for WD-SIMPEC and the proof is complete. □

We now give an example to illustrate the strong duality Theorem 6.

Example 5. *Consider the following SIMPEC problem in* $\mathbb{R}^2$*:*

$$\begin{aligned} P_3: \quad & \min \quad \Upsilon(z_1, z_2) = z_1 + |z_2| \\ & s.t. \quad \pi_t(z_1, z_2) = -z_1 - t \leq 0, \quad t \in T = [0, 1], \\ & \qquad \Pi(z_1, z_2) = z_1 - z_2 \geq 0, \quad \Theta(z_1, z_2) = z_2 \geq 0, \\ & \qquad \Pi(z_1, z_2)\Theta(z_1, z_2) = 0. \end{aligned}$$

The graphical representation of objective function $\Upsilon(z_1, z_2)$ *is shown in Fig 4. Then,* $\mathcal{F} = \{(z_1, z_2) |\ z_1 \geq 0, z_1 = z_2 \quad or \quad z_1 \geq 0, z_2 = 0\}$ *is the feasible set of the* P_3*, its geometric representation is shown in Fig 3 and for* $u \in \mathbb{R}^2$,

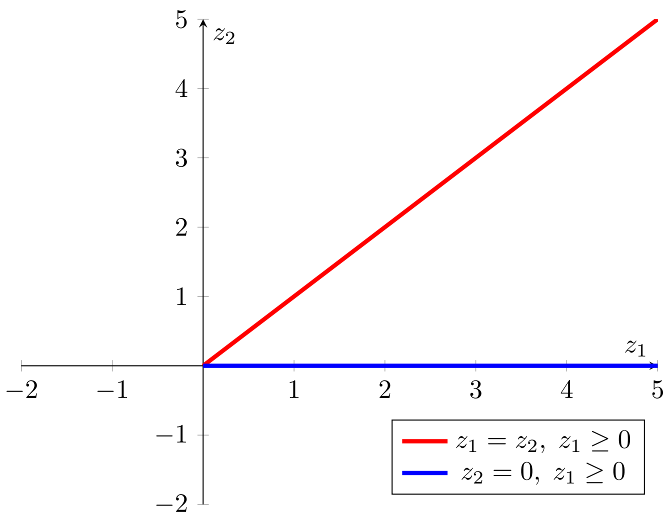

Figure 3: The feasible set $\mathcal{F}$ of P_3 considered in Example 5.

$$\begin{aligned} &\partial_T \Upsilon(u) = \{1\} \times [-1, 1], \quad \partial_T \pi_t(u) = \{(-1, 0)\}, \ \forall t \in T, \\ &\partial_T(-\Pi)(u) = \{(-1, 1)\}, \quad \partial_T(-\Theta)(u) = \{(0, -1)\}, \\ &\partial_T \Pi(u) = \{(1, -1)\}, \quad \partial_T \Theta(u) = \{(0, 1)\}. \end{aligned}$$

Therefore, the Wolfe type dual model of P_3 *is :*

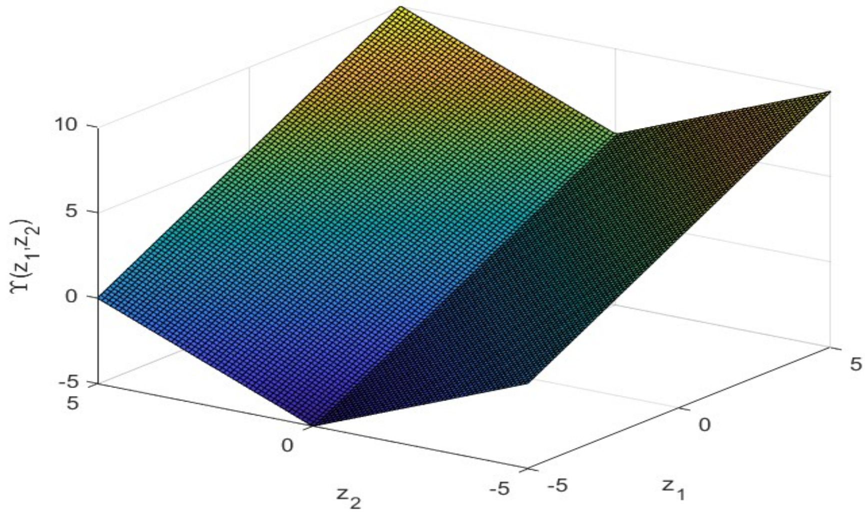

Figure 4: Graph of the objective function $\Upsilon(z_1, z_2)$ considered in Example 5.

$$\begin{aligned} D_3: \quad & \max_{u,\lambda,\mu} \; u_1 + |u_2| + \sum_{t\in T} \lambda_t^{\pi}(-u_1 - t) - \tilde{\lambda}^{\Pi} u_1 - \tilde{\lambda}^{\Theta} u_2 \\ & s.t. \quad (0,0) \in \{1\} \times [-1,1] + \sum_{t\in T} \lambda_t^{\pi}(-1,0) + \lambda^{\Pi}(-1,1) + \lambda^{\Theta}(0,-1) \\ & \qquad\qquad + \mu^{\Pi}(1,-1) + \mu^{\Theta}(0,1), \\ & \lambda_t^{\pi} \geq 0, (t \in T), \quad \lambda^{\Pi} \geq 0, \quad \lambda^{\Theta} \geq 0, \quad \mu^{\Pi} \geq 0, \quad \mu^{\Theta} \geq 0, \end{aligned}$$

where $\tilde{\lambda}^{\Pi} = \lambda^{\Pi} - \mu^{\Pi}$, $\tilde{\lambda}^{\Theta} = \lambda^{\Theta} - \mu^{\Theta}$.

We take $\bar{z} = (0,0)$ *which is optimal solution of primal problem. Then,*

$$\begin{aligned} & \partial_T \Upsilon(\bar{z}) = \{1\} \times [-1,1], \qquad \bigcup_{t\in T_\pi(\bar{z})} \partial_T \pi_t(\bar{z}) = \{(-1,0)\}, \; T_\pi(\bar{z}) = \{0\}, \\ & \partial_T(-\Pi)(\bar{z}) = \{(-1,1)\}, \quad \partial_T(-\Theta)(\bar{z}) = \{(0,-1)\}. \end{aligned}$$

Hence, D *is closed and we have*

$$\pi^- = \{(v_1,v_2)|\; v_1 \geq 0\}, \quad (\Pi\Theta)^- = \{(v_1,v_2)|\; v_1 - v_2 \geq 0, \; v_2 \geq 0\},$$

that is, SI-SACQ holds. As an example, by taking $\lambda^{\pi} = \mu^{\Pi} = \mu^{\Theta} = 0, \lambda^{\Pi} = \lambda^{\Theta} = 1$, *it is clear that Theorem 6 holds between* P_3 *and* D_3.

Now, let us formulate the Mond-Weir dual model for SIMPEC (1.1) and investigate the duality theorems related to it:

$$\text{MWD-SIMPEC}: \qquad \max_{u,\lambda,\mu} \Upsilon(u)$$

subject to:

$$\begin{aligned}
0 \in\ & \partial_T\Upsilon(u) + \sum_{i=1}^{k}\lambda_i^{\pi}\partial_T\pi_{t_i}(u) + \sum_{j=1}^{q}\left[\lambda_j^{\theta}\partial_T\theta_j(u) + \mu_j^{\theta}\partial_T(-\theta_j)(u)\right] \\
& + \sum_{i=1}^{m}\left[\lambda_i^{\Pi}\partial_T(-\Pi_i)(u) + \lambda_i^{\Theta}\partial_T(-\Theta_i)(u)\right] \\
& + \sum_{i=1}^{m}\left[\mu_i^{\Pi}\partial_T\Pi_i(u) + \mu_i^{\Theta}\partial_T\Theta_i(u)\right], \qquad (4.12)\\
& \sum_{i=1}^{k}\lambda_i^{\pi}\pi_{t_i}(u) \geq 0, \quad \sum_{j=1}^{q}\lambda_j^{\theta}\theta_j(u) = 0, \quad \sum_{j=1}^{q}\mu_j^{\theta}\theta_j(u) = 0, \\
& \sum_{i\in\alpha\cup\beta}\lambda_i^{\Pi}\Pi_i(u) \leq 0, \quad \sum_{i\in\beta\cup\gamma}\lambda_i^{\Theta}\Theta_i(u) \leq 0, \\
& \lambda_{I_\pi}^{\pi} \geq 0, \quad \lambda_j^{\theta}, \mu_j^{\theta} \geq 0,\ j = 1,2,\ldots,q, \quad \lambda_i^{\Pi}, \lambda_i^{\Theta}, \mu_i^{\Pi}, \mu_i^{\Theta} \geq 0,\ i = 1,2,\ldots,m, \\
& \lambda_\gamma^{\Pi} = \mu_\gamma^{\Pi} = \lambda_\alpha^{\Theta} = \mu_\alpha^{\Theta} = 0, \qquad \mu_i^{\Pi} = 0, \quad \mu_i^{\Theta} = 0, \quad \forall i \in \beta,
\end{aligned}$$

where $u \in \mathbb{R}^n$, $\lambda = (\lambda^{\pi}, \lambda^{\theta}, \lambda^{\Pi}, \lambda^{\Theta}) \in \mathbb{R}^{k+q+2m}$, $\mu = (\mu^{\theta}, \mu^{\Pi}, \mu^{\Theta}) \in \mathbb{R}^{q+2m}$, $t_1, t_2, \ldots, t_k \in T_\pi(\bar{z})$, $k \leq n+1$.

Theorem 7. *(Weak Duality) Suppose that z is feasible for SIMPEC and (u, λ, μ) is feasible for MWD-SIMPEC. Assume that all functions are tangentially convex at u and further, Υ, $\pi_t(t \in T)$, $\pm\theta_j(j = 1,2,\ldots,q)$, $-\Pi_i(i \in \alpha\cup\beta)$, $-\Theta_i(i \in \gamma\cup\beta)$ are Dini-convex functions at u. If $\beta_\mu^{\Pi} \cup \beta_\mu^{\Theta} \cup \alpha_\mu^{+} \cup \gamma_\mu^{+} = \emptyset$, then*

$$\Upsilon(z) \geq \Upsilon(u).$$

Proof. Let $z \in \mathcal{F}$. By analogy with the proof of Theorem 5, since Υ is Dini-convex at u,

$$\Upsilon(z) - \Upsilon(u) \geq \langle \xi, z-u\rangle, \quad \forall \xi \in \partial_T\Upsilon(u). \qquad (4.13)$$

Similarly, one can get

$$\begin{aligned}
& \pi_{t_i}(z) - \pi_{t_i}(u) \geq \langle \xi_i^{\pi}, z-u\rangle, \quad \forall \xi_i^{\pi} \in \partial_T\pi_{t_i}(u), \quad \forall t_i \in T_\pi(z), && (4.14)\\
& \theta_j(z) - \theta_j(u) \geq \langle \eta_j, z-u\rangle, \quad \forall \eta_j \in \partial_T\theta_j(u), \quad j = 1,2,\ldots,q, && (4.15)\\
& -\theta_j(z) + \theta_j(u) \geq \langle v_j, z-u\rangle, \quad \forall v_j \in \partial_T(-\theta_j)(u), \quad j = 1,2,\ldots,q, && (4.16)\\
& -\Pi_i(z) + \Pi_i(u) \geq \langle \xi_i^{\Pi}, z-u\rangle, \quad \forall \xi_i^{\Pi} \in \partial_T(-\Pi_i)(u), \quad \forall i \in \alpha\cup\beta, && (4.17)\\
& -\Theta_i(z) + \Theta_i(u) \geq \langle \xi_i^{\Theta}, z-u\rangle, \quad \forall \xi_i^{\Theta} \in \partial_T(-\Theta_i)(u), \quad \forall i \in \gamma\cup\beta. && (4.18)
\end{aligned}$$

If $\beta_\mu^{\Pi} \cup \beta_\mu^{\Theta} \cup \alpha_\mu^{+} \cup \gamma_\mu^{+} = \emptyset$, multiplying (4.14)-(4.18) by $\lambda_i^{\pi} \geq 0\,(t_i \in T_\pi)$, $\lambda_j^{\theta} \geq 0\,(j = 1,2,\ldots,q)$, $\mu_j^{\theta} \geq 0\,(j = 1,2,\ldots,q)$, $\lambda_i^{\Pi} \geq 0\,(i \in \alpha\cup\beta)$, $\lambda_i^{\Theta} \geq 0\,(i \in \gamma\cup\beta)$, respectively and by adding them together and to (4.13), we get

$$\begin{aligned}
& \Upsilon(z) - \Upsilon(u) \\
& \quad + \sum_{i=1}^{k}\lambda_i^{\pi}\pi_{t_i}(z) - \sum_{i=1}^{k}\lambda_i^{\pi}\pi_{t_i}(u) + \sum_{j=1}^{q}\lambda_j^{\theta}\theta_j(z) - \sum_{j=1}^{q}\lambda_j^{\theta}\theta_j(u) - \sum_{j=1}^{q}\mu_j^{\theta}\theta_j(z) \\
& \quad + \sum_{j=1}^{q}\mu_j^{\theta}\theta_j(u) - \sum_{i=1}^{m}\lambda_i^{\Pi}\Pi_i(z) + \sum_{i=1}^{m}\lambda_i^{\Pi}\Pi_i(u) - \sum_{i=1}^{m}\lambda_i^{\Theta}\Theta_i(z) + \sum_{i=1}^{m}\lambda_i^{\Theta}\Theta_i(u) \\
& \quad \geq \left\langle \xi + \sum_{i=1}^{k}\lambda_i^{\pi}\xi_i^{\pi} + \sum_{j=1}^{q}\left[\lambda_j^{\theta}\eta_j + \mu_j^{\theta}v_j\right] + \sum_{i=1}^{m}\left[\lambda_i^{\Pi}\xi_i^{\Pi} + \lambda_i^{\Theta}\xi_i^{\Theta}\right], z-u \right\rangle,
\end{aligned}$$

for every $\xi \in \partial_T \Upsilon(u)$, $\xi_i^\pi \in \partial_T \pi_{t_i}(u)$, $\eta_j \in \partial_T \theta_j(u)$, $v_j \in \partial_T(-\theta_j)(u)$, $\xi_i^\Pi \in \partial_T(-\Pi_i)(u)$, $\xi_i^\Theta \in \partial_T(-\Theta_i)(u)$.

From (4.12), there exist $\bar{\xi} \in \partial_T \Upsilon(u)$, $\bar{\xi}_i^\pi \in \partial_T \pi_{t_i}(u)$, $\bar{\eta}_j \in \partial_T \theta_j(u)$, $\bar{v}_j \in \partial_T(-\theta_j)(u)$, $\bar{\xi}_i^\Pi \in \partial_T(-\Pi_i)(u)$, $\bar{\xi}_i^\Theta \in \partial_T(-\Theta_i)(u)$ such that,

$$\bar{\xi} + \sum_{i=1}^{k} \lambda_i^\pi \bar{\xi}_i^\pi + \sum_{j=1}^{q} \left[\lambda_j^\theta \bar{\eta}_j + \mu_j^\theta \bar{v}_j\right] + \sum_{i=1}^{m} \left[\lambda_i^\Pi \bar{\xi}_i^\Pi + \lambda_i^\Theta \bar{\xi}_i^\Theta\right] = 0.$$

Thus,

$$\begin{aligned}
&\Upsilon(z) - \Upsilon(u)\\
&\quad + \sum_{i=1}^{k} \lambda_i^\pi \pi_{t_i}(z) - \sum_{i=1}^{k} \lambda_i^\pi \pi_{t_i}(u) + \sum_{j=1}^{q} \lambda_j^\theta \theta_j(z) - \sum_{j=1}^{q} \lambda_j^\theta \theta_j(u) - \sum_{j=1}^{q} \mu_j^\theta \theta_j(z)\\
&\quad + \sum_{j=1}^{q} \mu_j^\theta \theta_j(u) - \sum_{i=1}^{m} \lambda_i^\Pi \Pi_i(z) + \sum_{i=1}^{m} \lambda_i^\Pi \Pi_i(u) - \sum_{i=1}^{m} \lambda_i^\Theta \Theta_i(z) + \sum_{i=1}^{m} \lambda_i^\Theta \Theta_i(u) \ge 0.
\end{aligned}$$

Therefore, using feasibility of z for SIMPEC and (u, λ, μ) for MWD-SIMPEC, it follows that,

$$\Upsilon(z) \ge \Upsilon(u),$$

and the proof is complete. □

The following example indicates the weak duality Theorem 7.

Example 6. *Consider the following nonsmooth SIMPEC problem in $\mathbb{R}^2$:*

$$\begin{aligned}
P_4: \quad & \min \quad \Upsilon(z_1, z_2) = z_1 + |z_2|\\
& s.t. \quad \pi_t(z_1, z_2) = -z_2 - t \le 0, \quad t \in T = [0,1],\\
& \qquad \Pi(z_1, z_2) = z_1 \ge 0, \quad \Theta(z_1, z_2) = z_1 - |z_2| \ge 0,\\
& \qquad \Pi(z_1, z_2)\Theta(z_1, z_2) = 0.
\end{aligned}$$

Then, the feasible set of the P_4 is $\mathcal{F} = \{(z_1, z_2)|\ z_1 = z_2,\ z_1 \ge 0\}$. If we take $\bar{z} = (0,0)$ which is optimal solution of primal problem, then, the index sets $\alpha(0,0)$ and $\gamma(0,0)$ are empty sets, but $\beta(0,0)$ is nonempty.

Hence, the Mond-Weir type dual model of P_4 described as follows:

$$\begin{aligned}
D_4: \quad & \max_{u,\lambda,\mu} \ |u_2| + u_1\\
& s.t. \quad (0,0) \in \partial_T \Upsilon(u) + \lambda^\pi \partial_T \pi(u) + \lambda^\Pi \partial_T(-\Pi)(u) + \lambda^\Theta \partial_T(-\Theta)(u), && (4.19)\\
& \lambda^\pi(-u_2 - t) \ge 0, \quad t \in T_\pi(\bar{z}) = \{0\}, && (4.20)\\
& \lambda^G u_1 \le 0, && (4.21)\\
& \lambda^\Theta(u_1 - |u_2|) \le 0, && (4.22)\\
& \lambda^\pi \ge 0,\ \lambda^\Pi \ge 0,\ \lambda^\Theta \ge 0. && (4.23)
\end{aligned}$$

From (4.19)-(4.23), we get

$$\Upsilon(u_1, u_2) \le 0 = \Upsilon(z_1, z_2),$$

where (u_1, u_2) is in the feasible region of D_4. Thus, the Theorem 7 holds between P_4 and D_4.

Theorem 8. *(Strong Duality) Let $\bar{z}$ be a locally optimal solution of SIMPEC and Υ is locally Lipschitz near $\bar{z}$. Suppose that all functions are tangentially convex at $\bar{z}$, SI-SACQ holds at $\bar{z}$ and D is closed. Then, there exists $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^{k+q+2m} \times \mathbb{R}^{q+2m}$ such that $(\bar{z}, \bar{\lambda}, \bar{\mu})$ is feasible for MWD-SIMPEC and respective objective values are equal. Also, if the assumptions of Theorem 7 (Weak Duality) are satisfied at any feasible solution (u, λ, μ) of MWD-SIMPEC, then $(\bar{z}, \bar{\lambda}, \bar{\mu})$ is an optimal solution for MWD-SIMPEC.*

Proof. According to Theorem 2, there exist vectors $\bar{\lambda} = (\bar{\lambda}^{\pi}, \bar{\lambda}^{\theta}, \bar{\lambda}^{\Pi}, \bar{\lambda}^{\Theta}) \in \mathbb{R}^{k+q+2m}$, $\bar{\mu} = (\bar{\mu}^{\theta}, \bar{\mu}^{\Pi}, \bar{\mu}^{\Theta}) \in \mathbb{R}^{q+2m}$ and indices $t_1, t_2, \ldots, t_k \in T_{\pi}(\bar{z})$, $k \leq n+1$, such that the GS-stationarity conditions are satisfided at $\bar{z}$. On the other hand, since $\bar{z}$ is an optimal solution of SIMPEC,

$$\sum_{i=1}^{k} \bar{\lambda}_i^{g} g_{t_i}(\bar{z}) = 0, \quad \sum_{j=1}^{q} \bar{\lambda}_j^{\theta} \theta_j(\bar{z}) = 0, \quad \sum_{i=1}^{m} \bar{\lambda}_i^{G} G_i(\bar{z}) = 0, \quad \sum_{i=1}^{m} \bar{\lambda}_i^{H} H_i(\bar{z}) = 0.$$

Hence, $(\bar{z}, \bar{\lambda}, \bar{\mu})$ is a feasible point for MWD-SIMPEC. Obviously, their objective values at $\bar{z}$ are equal.

In addition to what is mentioned above, since the assumptions of Theorem 7 are satisfied at any feasible solution (u, λ, μ) of MWD-SIMPEC, we have

$$\Upsilon(\bar{z}) \geq \Upsilon(u).$$

That is, $(\bar{z}, \bar{\lambda}, \bar{\mu})$ is an optimal solution for MWD-SIMPEC and the proof is complete. □

5. Conclusion

In this paper, we applied the tangential subdifferential concept and studied the necessary and sufficient optimality conditions for semi-infinite mathematical programs with equilibrium constraint (SIMPECs) in the framework of this concept. Also, we introduced the Wolfe and Mond-Weir type duals for SIMPEC. We investigated weak and strong duality theorems relating to the SIMPEC under Dini-convexity assumptions. All our results are based on the S-stationarity or A-stationarity concepts. Since M-stationarity is the second strong stationarity concept after S-stationarity, analyzing optimality conditions for SIMPEC based on the M-stationarity is highly recommended for future studies.

Author Contributions The authors contributed equality in establishing the results of the manuscript. The authors read and approved the final manuscript.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

References

1. Goberna, M.A., López, M.A.: *Linear Semi-Infinite Optimization.* Chichester, Wiley (1998).

2. Goberna, M.A., López, M.A.: *Semi-infinite programming: Recent advances.* Dordrecht, Kluwer (2001).

3. Hettich, R., Kortanek, K.O.: *Semi-infinite programming: Theory, methods and applications.* SIAM Rev. 35, 380–429 (1993)

4. López, M.A., Still, G., *Semi-infinite programming.* European J.Oper.Res. 180, 491–518 (2007)

5. Reemtsen, R., Rückmann, J.J., *Semi-infinite programming. In: Nonconvex Optimization and Its Applications.* Vol. 25. Boston, MA: Kluwer Academic (1998).

6. Kanzi, N., Nobakhtian, S., *Optimality conditions for nonsmooth semi-infinite programming.* Optimization. 53, 717–727 (2008)

7. Kanzi, N., Nobakhtian, S., *Nonsmooth semi-infinite programming problems with mixed constraints.* J. Math. Anal. Appl. 351, 170–181 (2008).

8. Kanzi, N., *Necessary optimality conditions for nonsmooth semi-infinite programming problems.* J. Glob. Optim. 49, 713–725 (2011).

9. Guu, S., Singh, Y., Mishra, S.K., *On strong KKT type sufficient optimality conditions for multiobjective semi-infinite programming problems with vanishing constraints.* J. Inequal. Appl. 282 (2017).

10. Luo, Z.Q., Pang, J.S., Ralph, D. *Mathematical Programs with Equilibrium Constraints.* Cambridge University Press, Cambridge (1996).

11. Moore, G., Bergeron, C., Bennett, K.P., *Model selection for primal SVM.* Mach. Learn. 85, 175–208 (2011).

12. Outrata, J.V., Koćvara, M., Zowe, J., *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints. Nonconvex Optimization and Its Applications.* vol. 28. Kluwer Academic Publishers, Dordrechet (1998).

13. Ye, J.J., Zhu, D.L., Zhu, Q.J., *Exact penalization and necessary optimality conditions for generalized bilevel programming problems.* SIAM J. Optim. 7, 481–507 (1997).

14. Flegel, M.L., Kanzow, C., *A Fritz John approach to first order optimality conditions for mathematical programs with equilibrium constraints.* Optimization. 52, 277–286 (2003).

15. Flegel, M.L., Kanzow, C., *On the Guignard constraint qualification for mathematical programs with equilibrium constraints.* Optimization. 54, 517–534 (2005).

16. Flegel, M.L., Kanzow, C., *On M-stationary points for mathematical programs with equilibrium constraints.* J. Math. Anal. Appl. 310, 286–302 (2005).

17. Ye, J.J., *Necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints.* J. Math. Anal. Appl. 307, 350–369 (2005).

18. Movahedian, N., Nobakhtian, S., *Nondifferentiable multiplier rules for optimization problems with equilibrium constraints.* J. Convex Anal. 16, 187–210 (2009).

19. Movahedian, N., Nobakhtian, S., *Necessary and sufficient conditions for nonsmooth mathematical programs with equilibrium constraints.* Nonlinear Anal. 72, 2694–2705 (2010).

20. Ye, J.J., Zhang, J., *Enhanced Karush-Kuhn-Tucker conditions for mathematical programs with equilibrium constraints.* J. Optim. Theory Appl. 163, 777–794 (2014).

21. Ansari, A.A., Movahedian, N., Nobakhtian, S., *Optimality conditions for nonsmooth mathematical programs with equilibrium constraints using convexificators.* Optimization. 65, 67–85 (2014).

22. Lafhim, L., Kalmoun, E. M., *Optimality conditions for mathematical programs with equilibrium constraints using directional convexificators.* Optimization. 72(5), 1363–1383 (2023).

23. Pandey, R., Pandey, Y., Singh, V.: *Optimality conditions for mathematical programming problem with equilibrium constraints in terms of tangential subdifferentiable.* Commun. Comb. Optim. (2025).

24. Pandey, Y., Miahra, S.K.: *Duality for Nonsmooth Optimization Problems with Equilibrium Constraints Using Convexificators.* J. Optim. Theory Appl. 171, 694–707 (2016).

25. Pandey, Y., Mishra, S.K., *Optimality conditions and duality for semi-infinite mathematical programming problems with equilibrium constraints using convexificators.* Ann. Oper. Res. 269(1-2), 549–564 (2018).

26. Antczak, T., Pandey, R. and Singh, V., *Optimality Conditions for Nondifferentiable Interval-Valued Bilevel Optimization Problems via Tangential Subdifferentials.* Bull. Malays. Math. Sci. Soc. 49, 2 (2026). https://doi.org/10.1007/s40840-025-02000-6

27. Tung, L.T., *Karush-Kuhn-Tucker Optimality Conditions and Duality for Multiobjective Semi-Infinite Programming Via Tangential Subdifferentials.* Numer. Funct. Anal. Optim. 41(6), 659–684 (2019).

28. Giorgi, G., Jiménez, B., Novo, V., *Sufficient optimality conditions and duality in nonsmooth multiobjective optimization problems under generalized convexity.* Generalized Convexity and Related Topics. 538, 265–278 (2007).

29. Su, T. V., *Optimality and duality for nonsmooth mathematical programming problems with equilibrium constraints.* J. Global Optim. 85(3), 663–685 (2023).

30. Pshenichnyi, B.N., *Necessary Conditions for an Extremum.* New York, NY: Marcel Dekker Inc (1971).

31. Martínez-Legaz, J.E., *Optimality conditions for pseudoconvex minimization over convex sets defined by tangentially convex constraints.* Optim. Lett. 9(5), 1017–1023 (2015).

32. Dutta, J., Lalitha, C.S., *Optimality conditions in convex optimization revisited.* Optim. Lett. 7(2), 221–229 (2013).

33. Lasserre, J.B.: *On representations of the feasible set in convex optimization.* Optim. Lett. 4(1), 1–5 (2010).

34. Mashkoorzadeh, F., Movahedian, N., Nobakhtian, S., *Optimality Conditions for Nonconvex Constrained Optimization Problems.* Numer. Funct. Anal. Optim. 40(16), 1918–1938 (2019).

35. Tung, L.T., *Strong Karush-Kuhn-Tucker optimality conditions for multiobjective semi-infinite programming via tangential subdifferential.* Rairo-Oper. Res. 52(4-5), 1019–1041 (2018).

36. Tung, L.T., *Karush-Kuhn-Tucker optimality conditions for nonsmooth multiobjective semidefinite and semi-infinite programming.* J. Appl. Numer. Optim. 1(1), 63–75 (2019).

37. Borwein, J.M., Lewis, A.S., *Convex Analysis and Nonlinear Optimization: Theory and Examples.* New York: Springer-Verlag (2010).

38. Lemaréchal, M., *An introduction to the theory of nonsmooth optimization.* Optimization. 17(6), 827–858 (1986).

39. Clarke, F.H., *Optimization and Nonsmooth Analysis.* New York, NY: Wiley Interscience (1983).

40. Michel, P., Penot, J.P., *A generalized derivative for calm and stable functions.* Diff. Integ. Eq. 5, 433–454 (1992).

41. Giorgi, G., Jiménez, B., Novo, V., *On constraint qualification in directionally differentiable multiobjective optimization problems.* RAIRO Oper.Res. 38, 255–274 (2004).

42. Rockafellar, R.T., *Convex Analysis*. In: Princeton Mathematical Series, Vol. 28. Princeton, NJ: Princeton University Press (1970).

43. Luo, Z.Q., Pang, J.S., Ralph, D., Wu, S.Q., *Exact penalization and stationary conditions of mathematical programs with equilibrium constraints*. Math. Program. 76, 19–76 (1996).

44. Scheel, H., Scholtes, S., *Mathematical programs with complementarity constraints: stationarity, optimality and sensitivity.* Math. Oper. Res. 25, 1–22 (2000).

Ali Ansari Ardali,
Department of Applied Mathematics, Faculty of Mathematical Sciences,
Shahrekord University, Shahrekord, Iran.
ORCID ID: https://orcid.org/0000-0001-9881-8545
E-mail address: ali.ansari9286@gmail.com, ali.ansariardali@sku.ac.ir

and

Maryam Zeinali,
Department of Mathematics, Faculty of Sciences,
University of Qom, Qom, Iran.
E-mail address: m.zeinali84@gmail.com

and

Rishabh Pandey,
Department of Mathematics, National Institute of Technology, Mizoram,
Chaltlang, Aizawl, 796012, Mizoram, India.
ORCID ID: https://orcid.org/0009-0005-8289-8583
E-mail address: pandeyrishabh512@gmail.com

and

Vinay Singh,
Department of Mathematics, National Institute of Technology, Mizoram,
Chaltlang, Aizawl, 796012, Mizoram, India.
ORCID ID: https://orcid.org/0000-0003-3269-8776
E-mail address: vinay.math@nitmz.ac.in

Algorithm 1 An algorithm for finding GA-stationary point of the problem SIMPEC

Step 1. Provide Problem Data
start by supplying the input data for the given SIMPEC problem:

(a) Input $\Upsilon, \pi_t, t \in T, \theta_i, i \in \{1, 2, \ldots, q\}, \Pi, i \in \{1, 2, \ldots, m\}, \Theta_i, i \in \{1, 2, \ldots, m\}$.

Step 2. Identify the Feasible Set

(a) Construct the feasible region as follows:

$$\mathcal{F} := \{\bar{z} \in \mathbb{R}^n \,|\pi_t(\bar{z}) \leq 0, \quad \forall t \in T, \quad \theta(\bar{z}) = 0, \quad \Pi(\bar{z}) \geq 0, \quad \Theta(\bar{z}) \geq 0, \quad \Pi(\bar{z})\Theta(\bar{z}) = 0\}.$$

Step 3. Select a Feasible Point

(a) If the feasible set $\mathcal{F}$ is empty, terminate the algorithm.

(b) Otherwise, choose any point $\bar{z} \in \mathcal{F}$, and update the feasible set by removing this point: $\mathcal{F} = \mathcal{F} \backslash \{\bar{z}\}$.

Step 4. Check tangential convexity of the functions

(a) If all functions are tangentially convex at $\bar{z}$, proceed to step5.

(b) If any functions fails this condition, return to step3.

Step 5. Verify tangential subdifferentiability

(a) Compute the tangential subdifferential of each function at $\bar{z}$.

Step 6. Check the SI-ACQ condition

(a) If the generalized SIMPEC Abadie constraint qualification (SI-ACQ) is satisfied at $\bar{z}$, proceed to the next step.

(b) If not, return to step3.

Step 7. Test the generalized allternatively stationary Conditions
Determine if there exist multipliers $\lambda_i^{\pi}, i \in I_{\pi} = \{1, 2, \ldots, k\}$, $\lambda_j^{\theta}, \mu_j^{\theta}, j = 1, 2, \ldots, q$, $\lambda_i^{\Pi}, \lambda_i^{\Theta}, \mu_i^{\Pi}, \mu_i^{\Theta}, i = 1, 2, \ldots, m$ such that condition (3.1) and (3.2) is satisfied. Also, $\mu_i^{\Pi} = 0 \vee \mu_i^{\Theta} = 0, \forall i \in \beta$.

(a) If such multipliers can be found, then $\bar{z}$ is GA-stationary point of SIMPEC.

(b) If not, return to step 3.