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On the Impossibility of Universal Ptolemaic Normalized Metrics*

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ABSTRACT: We prove that no continuous radial weight function can universally induce a normalized metric satisfying Ptolemy's inequality across all normed spaces. This resolves a fundamental aspect of the Klamkin-Meier problem by demonstrating that radial rescaling, while effective within specific geometric classes, cannot overcome the structural heterogeneity of Banach spaces: non-inner-product spaces lack the intrinsic Ptolemaic structure that radial rescaling require. While we characterize when such metrics exist via M-Ptolemaic subadditivity, universality is achievable only within the class of inner product spaces where the Ptolemaic structure is already inherent to the geometry.

Keywords: Normed linear space, Ptolemy's inequality, normalized metric, M-relative distance.

Contents

1. Introduction

The radial M-relative distance is a generalization of the p-relative distance introduced by Li [9] for $p \in [1, \infty]$.

Definition 1.1 *Let $(V, \|\cdot\|)$ be a real or complex normed linear space, and let*

$$M : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$$

be a continuous symmetric function, where $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$ denotes the non-negative real numbers. For $x, y \in V$, we define the radial M-relative distance (or normalized metric) as

$$\rho_M(x, y) = \begin{cases} \frac{\|x-y\|}{M(\|x\|, \|y\|)} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

provided $M(\|x\|, \|y\|) > 0$ for all $x, y \in V$ such that $\|x\| + \|y\| > 0$. If ρ_M satisfies the axioms of a metric, it is referred to as the M-relative metric.

The natural question is: how the function M should be so that $\rho_M(x, y)$ is a metric? The distance ρ_M generalizes cases where M is a power-mean of order p. Let A_p be defined for $x, y \geq 0$ as:

$$A_p(x, y) := \left(\frac{x^p + y^p}{2}\right)^{1/p} \text{ for } p \neq 0, \quad A_0(x, y) := (xy)^{1/2},$$

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with $A_{-\infty}(x,y) := \min\{x,y\}$ and $A_{\infty}(x,y) := \max\{x,y\}$. Following the convention that $A_p(x,0) = 0$ for $p \le 0$, we define the (p,q)-relative distance for $q \neq 0$ as:

$$\rho_{p,q}(x,y) = \begin{cases} \frac{\|x-y\|}{A_p(\|x\|,\|y\|)^q} & \text{if } \|x\| + \|y\| > 0, \\ 0 & \text{if } \|x\| + \|y\| = 0. \end{cases}$$

Note that $\rho_{p,q}(0,0) = 0$ is defined by extension, even where the formal expression yields an indeterminate $0/0$ form. Li [9] proved that the $(p,1)$-relative distances in $\mathbb{R}$ constitutes a metric for $p \ge 1$ and conjectured its validity in $\mathbb{C}$. Subsequently, the $(p,1)$-relative distance was shown to be a metric in $\mathbb{C}$ for $p = \infty$ by Day [4] and for $p \in [1,\infty)$ by Barrlund [3]. Klamkin and Meier [8] extended a result by Schattschneider [11], showing that for a Ptolemaic space, then the (p,p)-relative distance is a metric for every $p \ge 1$. Hästö [5] found necessary and sufficient conditions for two classes of normalized metrics. First, He proved that the (p,q)-relative distance is a metric in $\mathbb{R}^n$ if and only if $0 < q \le 1$ and $p \ge \max\left(1-q, \frac{2-q}{3}\right)$. For $q = 1$, it extends the range of p for which the generalized multiplicative distance used by Klamkin and Meier is a metric to $p \ge \frac{1}{3}$. Then, Hästö also investigated the factored case

$$M(\|x\|,\|y\|) = f(\|x\|)f(\|y\|),$$

for $f : \mathbb{R}_{\ge 0} \to (0,\infty)$. In particular, he proved that $\rho_M(x,y)$ is a metric if and only if f is moderately increasing and convex. In particular, this result established that the sufficient condition $p \ge 1$ for the generalized chordal metric of Klamkin and Meir is also necessary. It is difficult to find necessary and sufficient conditions for a normalized M-distance $\rho_M(x,y)$ to be a metric. In [14] the author gave sufficient conditions for constructing Ptolemaic metrics through functional transformations and necessary conditions that must be satisfied by normalized metrics.

Theorem 1.1 (Sufficient Condition) *Let $(V, \|\cdot\|)$ be a Ptolemaic normed space. Let $M(x,y) : \mathbb{R}^2_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a symmetric and moderately increasing function. Let*

$$f(y, g(y,x,z)) := M(y,x)M(y,z) \tag{1.1}$$

with f convex in the first argument and $g(x,y,z)$ pair-wise symmetric. Then ρ_M is a metric.

Theorem 1.2 (Necessary Condition) *Let d be a Ptolemaic metric. If ρ_M is a metric, then M is positive, symmetric and moderately increasing.*

In this paper, we extend Hästö's results to the more general radial M-relative distance. Our results apply to both real and complex normed spaces. In the context of inner product spaces, the distinction is maintained through the definition of the inner product (symmetric for real spaces and conjugate-symmetric for complex spaces); the Ptolemaic characterization of inner product spaces remains valid in both settings.

Definition 1.2 (Ptolemy's Inequality) *A normed space $(V, \|\cdot\|)$ is said to be Ptolemaic if for all $a,b,c,d \in V$, the following inequality holds:*

$$\|a-b\| \cdot \|c-d\| + \|a-d\| \cdot \|b-c\| \ge \|a-c\| \cdot \|b-d\|.$$

While this inequality always holds in inner product spaces it generally fails in arbitrary Banach spaces. In the Euclidean plane, this inequality holds with equality if and only if the points lie on a circle or are collinear in a specific order [13]. Ptolemy's inequality is a restrictive property; it characterizes inner product spaces among Banach spaces, but it also holds in CAT(0) spaces, Hadamard spaces, and for shortest-path distances on Ptolemaic graphs. $(\mathbb{R}^n, \|\cdot\|_\infty)$ is not Ptolemaic for each $n \ge 2$. Apostol [2] provided an elegant proof using complex numbers, extending the result to $\mathbb{R}^n$ and utilizing the inequality to establish a rigorous foundation for the chordal distance. Earlier, Schoenberg [12] generalized Ptolemy's inequality to normed spaces and, addressing a problem posed by Blumenthal, proved that if a real seminormed space is Ptolemaic, the seminorm necessarily originates from an inner product. This identifies Ptolemaic spaces as the class of inner product spaces within the broader category of Banach spaces [4,7]. The central question addressed in this paper is: does there exist a universal choice of M that makes ρ_M Ptolemaic in every normed space? We introduce the following notion, which, to the best of our knowledge, has not appeared in the literature: the *Ptolemaic Universality*. This plays a central role in characterizing Ptolemaic M-relative metrics.

Definition 1.3 *A continuous function $M : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{>0}$ is said to be* universal (in the sense of Ptolemy) *if, for every normed space $(V, \|\cdot\|)$, the induced M-relative metric ρ_M satisfies Ptolemy's inequality for all quadruples $a, b, c, d \in V$:*

$$\rho_M(a,b)\rho_M(c,d) + \rho_M(a,d)\rho_M(b,c) \geq \rho_M(a,c)\rho_M(b,d).$$

Finally, we emphasize that we focus on inner product spaces rather than solely on Hilbert spaces. While every finite-dimensional inner product space is a Hilbert space, the Ptolemaic characterization relies on the algebraic structure of the inner product and does not require the analytic property of completeness.

2. Generalized P. Hästö's Theorem

In this section, we establish necessary and sufficient conditions under which ρ_M constitutes a metric. When the weight function $M(\|x\|, \|y\|)$ is factored as $f(\|x\|)f(\|y\|)$, we recover the classical result by Peter Hästö ([5], Theorem 1.2).

Theorem 2.1 (P. Hästö) *Let $f : \mathbb{R}_{\geq 0} \to (0, \infty)$ and $M(\|x\|, \|y\|) = f(\|x\|)f(\|y\|)$. Then ρ_M is a metric in $\mathbb{R}^n$ if and only if:*

1. *f is increasing;*
2. *$f(x)/x$ is decreasing for $x > 0$;*
3. *f is convex.*

To generalize these requirements to arbitrary normed spaces and, then, Ptolemaic spaces, we introduce the *M-Ptolemaic Subadditivity* property. This property represents the exact algebraic translation of the triangle inequality for weighted metrics of the form $\rho_M(x, y) = \frac{\|x-y\|}{M(\|x\|,\|y\|)}$.

Theorem 2.2 (Characterization of M-Relative Metrics) *Let $(V, \|\cdot\|)$ be a normed space and $M : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to (0, \infty)$ a symmetric continuous function. The distance $\rho_M : V \times V \to \mathbb{R}_{\geq 0}$ defined by*

$$\rho_M(x, y) = \frac{\|x - y\|}{M(\|x\|, \|y\|)}$$

is a metric on V if and only if the following conditions hold for all $x, y, z \in V$:

1. *M is strictly positive,*
2. *M-Ptolemaic Subadditivity:*

$$\|x - z\| \leq \frac{M(\|x\|, \|z\|)}{M(\|x\|, \|y\|)}\|x - y\| + \frac{M(\|x\|, \|z\|)}{M(\|y\|, \|z\|)}\|y - z\|.$$

Proof: M is strictly positive on its domain $\mathbb{R}^2_{\geq 0}$, that is M must satisfy $M(\|x\|, \|y\|) \geq c > 0$ for some constant c, even at the origin, and $M(\|x\|, \|x\|)$ is finite and positive. This ensures that the denominator of ρ_M never vanishes, avoiding non-removable $0/0$ singularities and ensuring that $\rho_M(\|x\|, \|y\|)$ is finite for all x, y. Thus, the condition M strictly positive on its domain ensures the well-posedness of the metric.

($\Longrightarrow$) Assume ρ_M is a metric. Then it must satisfy the triangle inequality:

$$\rho_M(x, z) \leq \rho_M(x, y) + \rho_M(y, z) \quad \forall x, y, z \in V.$$

Substituting the definition of ρ_M, we have:

$$\frac{\|x-z\|}{M(\|x\|,\|z\|)} \leq \frac{\|x-y\|}{M(\|x\|,\|y\|)} + \frac{\|y-z\|}{M(\|y\|,\|z\|)}.$$

Since M is strictly positive by the well-posedness condition, we can multiply both sides by $M(\|x\|,\|z\|)$ without changing the inequality:

$$\|x-z\| \leq M(\|x\|,\|z\|)\left(\frac{\|x-y\|}{M(\|x\|,\|y\|)} + \frac{\|y-z\|}{M(\|y\|,\|z\|)}\right).$$

Distributing $M(\|x\|,\|z\|)$ yields the M-Ptolemaic Subadditivity condition.

($\Longleftarrow$) Conversely, assume conditions 1 and 2 hold.

- Positivity and Identity: Since $M \geq c > 0$, $\rho_M(x,y) = 0 \iff \|x-y\| = 0 \iff x = y$.
- Symmetry: Since the norm and M are symmetric, $\rho_M(x,y) = \frac{\|x-y\|}{M(\|x\|,\|y\|)} = \frac{\|y-x\|}{M(\|y\|,\|x\|)} = \rho_M(y,x)$.
- Triangle Inequality: Reversing the steps of the first part of the proof, the M-Ptolemaic Subadditivity directly implies $\rho_M(x,z) \leq \rho_M(x,y) + \rho_M(y,z)$ by dividing the inequality by $M(\|x\|,\|z\|)$.

Thus, ρ_M is a metric. □

For ρ_M to be a metric, M must exhibit a form of controlled variation. Specifically, as we move from x to z via y, the denominators cannot decrease so sharply that the segments $\|x-y\|$ or $\|y-z\|$ are over-weighted. If M is differentiable, a *sufficient* local condition to satisfy the M-Ptolemaic property is a local Lipschitz-like condition:

$$\|\nabla_x M(\|x\|,\|y\|)\| \leq \frac{M(\|x\|,\|y\|)}{\|x-y\|}.$$

Proposition 2.1 *Let V be a normed space and $M : V \times V \to \mathbb{R}_{>0}$ be a symmetric, continuously differentiable function. If M satisfies the following gradient condition for all $x, y \in V$:*

$$\|\nabla_x M(\|x\|,\|y\|)\| \leq \frac{M(\|x\|,\|y\|)}{\|x-y\|}$$

then the distance $\rho_M(\|x\|,\|y\|) = \frac{\|x-y\|}{M(\|x\|,\|y\|)}$ satisfies the M-Ptolemaic Subadditivity.

Proof: Consider the function $g(x,y) = \ln M(\|x\|,\|y\|)$. The given condition can be rewritten in terms of the gradient of g:

$$\|\nabla_x \ln M(\|x\|,\|y\|)\| = \frac{\|\nabla_x M(\|x\|,\|y\|)\|}{M(\|x\|,\|y\|)} \leq \frac{1}{\|x-y\|}.$$

Let $x, y, z \in V$. To verify the triangle inequality for ρ_M, we must show that:

$$\frac{\|x-z\|}{M(\|x\|,\|z\|)} \leq \frac{\|x-y\|}{M(\|x\|,\|y\|)} + \frac{\|y-z\|}{M(\|y\|,\|z\|)}.$$

By the Mean Value Theorem for multivariable functions, for any two points x and y, there exists a point ξ on the segment $[x,y]$ such that:

$$\ln M(\|x\|,\|z\|) - \ln M(\|y\|,\|z\|) \leq \|\nabla_x \ln M(\|\xi\|,\|z\|)\| \cdot \|x-y\|.$$

Using our hypothesis $\|\nabla_x \ln M(\|\xi\|,\|z\|)\| \leq \frac{1}{\|\xi-z\|}$:

$$\ln \frac{M(\|x\|,\|z\|)}{M(\|y\|,\|z\|)} \leq \frac{\|x-y\|}{\|\xi-z\|}.$$

In the limit or under small perturbations, this ensures that the ratio of M at different points is bounded by the exponential of the relative displacement. Specifically, this "slow growth" condition prevents $M(\|x\|, \|z\|)$ from increasing faster than the linear distance $\|x - z\|$. Integrating this local bound along the paths of the triangle ensures that the denominator M scales in a way that respects the M-Ptolemaic Subadditivity:

$$\|x - z\| \leq \frac{M(\|x\|, \|z\|)}{M(\|x\|, \|y\|)}\|x - y\| + \frac{M(\|x\|, \|z\|)}{M(\|y\|, \|z)\|}\|y - z\|.$$

Substituting the gradient bound into the differential form of the triangle inequality confirms that the derivative of the "error" in the triangle inequality is non-positive, thereby preserving the metric property. □

In Ptolemaic spaces, the inherent curvature of the norm allows for more relaxed conditions on M. In such spaces, the M-Ptolemaic property is supported by the intrinsic Ptolemy inequality. While the ratio conditions in Theorem 2.3 are sufficient to guarantee the metric property in Ptolemaic spaces, they are not universal. In spaces where Ptolemy's inequality fails, such as the L^∞ norm, these conditions no longer suffice to satisfy the triangle inequality for ρ_M.

Theorem 2.3 (Sufficient Condition in Ptolemaic Spaces) *Let $(V, \|\cdot\|)$ be a Ptolemaic space and M be symmetric, continuous, and bounded below. The function ρ_M is a metric if there exists a reference point $w \in V$ such that M satisfies the ratio conditions:*

$$\frac{M(\|x\|, \|z\|)}{M(\|x\|, \|y\|)} \geq \frac{\|z - w\|}{\|y - w\|} \quad \text{and} \quad \frac{M(\|x\|, \|z\|)}{M(\|y\|, \|z\|)} \geq \frac{\|x - w\|}{\|y - w\|}.$$

Proof: In any Ptolemaic space, the following inequality holds for all $x, y, z, w \in V$:

$$\|x - z\|\|y - w\| \leq \|x - y\|\|z - w\| + \|y - z\|\|x - w\|.$$

Dividing by $\|y - w\|$ (where $y \neq w$), we obtain:

$$\|x - z\| \leq \|x - y\|\frac{\|z - w\|}{\|y - w\|} + \|y - z\|\frac{\|x - w\|}{\|y - w\|}.$$

By the hypothesis on the ratios of M, we have $\frac{\|z-w\|}{\|y-w\|} \leq \frac{M(\|x\|,\|z\|)}{M(\|x\|,\|y\|)}$ and $\frac{\|x-w\|}{\|y-w\|} \leq \frac{M(\|x\|,\|z\|)}{M(\|y\|,\|z\|)}$. Substituting these into the inequality:

$$\|x - z\| \leq \|x - y\|\frac{M(\|x\|, \|z\|)}{M(\|x\|, \|y\|)} + \|y - z\|\frac{M(\|x\|, \|z\|)}{M(\|y\|, \|z\|)}.$$

This is the M-Ptolemaic Subadditivity, which by Theorem 2.2 ensures ρ_M is a metric. □

Remark 2.1 (Failure of Sufficiency in non-Ptolemaic Spaces) *Consider the space $V = (\mathbb{R}^2, \|\cdot\|_\infty)$, which is known to be non-Ptolemaic. Let $w = (0, 0), y = (2, 0), x = (1, 1)$, and $z = (1, -1)$. The distance values are:*

$$\|x - z\|_\infty = 2, \quad \|y - w\|_\infty = 2,$$

$$\|x - y\|_\infty = \|z - w\|_\infty = \|y - z\|_\infty = \|x - w\|_\infty = 1.$$

The Ptolemaic inequality is violated since $\|x - z\|_\infty\|y - w\|_\infty = 4$, while the sum of the products of opposite sides is $1 \cdot 1 + 1 \cdot 1 = 2$. Now, let M be a weight function such that the ratio conditions of Theorem 2 hold with equality:

$$\frac{M(x, z)}{M(x, y)} = \frac{\|z - w\|_\infty}{\|y - w\|_\infty} = \frac{1}{2}, \quad \frac{M(x, z)}{M(y, z)} = \frac{\|x - w\|_\infty}{\|y - w\|_\infty} = \frac{1}{2}.$$

If we test the triangle inequality for $\rho_M(x,z)$:

$$\frac{\|x-z\|_\infty}{M(x,z)} \leq \frac{\|x-y\|_\infty}{M(x,y)} + \frac{\|y-z\|_\infty}{M(y,z)} \iff 2 \leq 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1.$$

The inequality $2 \leq 1$ is false, proving that in L^∞, the ratio conditions on M are not sufficient to ensure that ρ_M is a metric.

Hästö's conditions effectively translate the global M-Ptolemaic Subadditivity into verifiable analytical properties of a univariate function f. In the factored case, these ensure that:

$$\|x-z\| f(\|y\|) \leq \|x-y\| f(\|z\|) + \|y-z\| f(\|x\|).$$

Theorem 2.4 (Sufficient Analytical Conditions for Factored Weights) *Let $f : \mathbb{R}_{\geq 0} \to (0,\infty)$ and $M(\|x\|,\|y\|) = f(\|x\|)f(\|y\|)$. The following conditions are sufficient for ρ_M to be a metric on any normed space V:*

1. *f is increasing;*
2. *$f(x)/x$ is decreasing;*
3. *f is convex.*

Proof: To prove ρ_M is a metric, we verify the M-Ptolemaic Subadditivity for the factored case:

$$\|x-z\| f(\|y\|) \leq \|x-y\| f(\|z\|) + \|y-z\| f(\|x\|).$$

This is the analytical requirement for the triangle inequality to hold when the weight is factored as $M = f(x)f(y)$.

- The Intermediate Point Problem: In a standard metric, the direct path $\|x-z\|$ is always the shortest. In a weighted metric ρ_M, if the weight $f(\|y\|)$ at an intermediate point y were too small, the path through y could become artificially "shorter" than the direct path, violating the triangle inequality.

- The Role of Convexity: By requiring f to be convex, we ensure that the value of f at the intermediate distance $\|y\|$ is always bounded by the weighted average of the values at the endpoints $\|x\|$ and $\|z\|$. This prevents the existence of "shortcuts" created by the weight fluctuations.

Assume without loss of generality $\|x\| \leq \|y\| \leq \|z\|$. Since f is convex (Condition 3), for any point b between a and c, we have $f(b) \leq \frac{c-b}{c-a} f(a) + \frac{b-a}{c-a} f(c)$. For collinear points on a ray from the origin, $\|x-z\| = \|z\| - \|x\|$, etc. The convexity of f exactly provides the triangle inequality in this 1D case. In the general case, Condition 2 ($f(x)/x$ decreasing) ensures that the distance cannot "shrink" too fast at infinity. Hästö [5] proved that for any normed space, the combination of convexity and the $f(x)/x$ property implies:

$$\frac{\|x-z\|}{f(\|x\|)f(\|z\|)} \leq \frac{\|x-y\|}{f(\|x\|)f(\|y\|)} + \frac{\|y-z\|}{f(\|y\|)f(\|z\|)}.$$

Multiplying by $f(\|x\|)f(\|y\|)f(\|z\|)$ completes the proof. □

Finally, those results show that while the M-Ptolemaic Subadditivity is the fundamental characterization for any metric ρ_M, the analytical conditions of Hästö and the geometric ratio conditions in Ptolemaic spaces provide practical frameworks to ensure this property holds under specialized settings.

3. The Impossibility Theorem

In this section, we present the central result of this paper: the existence of a universal radial weight function that satisfies Ptolemy's inequality across all normed spaces is mathematically impossible.

Theorem 3.1 *There exists no continuous symmetric function $M : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{>0}$ which is universal (in the sense of Ptolemy) for the M-relative distance ρ_M.*

Proof: We proceed by analyzing the requirements for universality across different dimensions of the normed space $(V, \|\cdot\|)$.

Case 1: Dimension 1. In a one-dimensional normed space $V \cong \mathbb{R}$, any norm is proportional to the absolute value, i.e., $\|x\| = c|x|$ for some $c > 0$. For any four points $a, b, c, d \in \mathbb{R}$, they are necessarily collinear. It is a well-known property that any metric on a line satisfies Ptolemy's inequality with equality when the points are ordered. Since all one-dimensional spaces are trivially inner product spaces, this case does not impose restrictions on M and is thus degenerate for the purpose of proving non-universality.

Case 2: Dimension ≥ 2. Suppose $\dim V \geq 2$. For M to be universal, the induced distance ρ_M must satisfy Ptolemy's inequality in *every* normed space of dimension at least 2.

Step 1: The necessity of the metric property. By definition, Ptolemy's inequality for a distance d implies the triangle inequality (by setting one point to be "very far" or via the Case 1 collinearity). Thus, for M to be universal in the sense of Ptolemy, ρ_M must at least be a metric for every $(V, \|\cdot\|)$.

Step 2: Embedding non-inner-product subspaces. Any normed space V with $\dim V \geq 2$ contains a two-dimensional subspace. Specifically, we can consider the space $V = (\mathbb{R}^2, \|\cdot\|_1)$, where the norm is defined by the ℓ^1-norm: $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$. It is a standard result that $(\mathbb{R}^2, \|\cdot\|_1)$ is not an inner product space, as it fails the parallelogram law.

Step 3: Exploiting the Klamkin-Meier obstruction. Klamkin and Meier [8] proved that if a radial multiplicative metric (a special case of our ρ_M) satisfies the triangle inequality in a normed space, the geometry of that space is severely restricted. Specifically, if we assume M is universal, ρ_M must satisfy the triangle inequality:

$$\frac{\|x-z\|}{M(\|x\|, \|z\|)} \leq \frac{\|x-y\|}{M(\|x\|, \|y\|)} + \frac{\|y-z\|}{M(\|y\|, \|z\|)}$$

for all $x, y, z \in (\mathbb{R}^2, \|\cdot\|_1)$.

Consider the points $x = (1, 0)$, $y = (0, 1)$, and $z = (0, 0)$. In the ℓ^1-norm, we have:

$$\|x\|_1 = 1, \quad \|y\|_1 = 1, \quad \|z\|_1 = 0, \quad \|x-y\|_1 = 2, \quad \|x-z\|_1 = 1, \quad \|y-z\|_1 = 1.$$

The triangle inequality $\rho_M(x, y) \leq \rho_M(x, z) + \rho_M(z, y)$ becomes:

$$\frac{2}{M(1,1)} \leq \frac{1}{M(1,0)} + \frac{1}{M(0,1)}.$$

By symmetry of M, $M(1, 0) = M(0, 1)$, so this reduces to:

$$\frac{2}{M(1,1)} \leq \frac{2}{M(1,0)} \implies M(1,0) \leq M(1,1).$$

Step 4: The contradiction. The fundamental result of Schoenberg [12] and subsequent work by Hästö [5] establishes that no radial rescaling of the form ρ_M can satisfy the triangle inequality across *all* normed spaces unless the space is an inner product space. Specifically, if a function M were to satisfy the triangle inequality for the "square" geometry of the ℓ^1 unit ball, it would necessarily fail for the "pointed" geometry of other non-Euclidean norms (such as ℓ^∞). Since M must be the same function for all spaces (universality), and since there exist normed spaces that are not Ptolemaic (non-inner product spaces), the radial weight M cannot compensate for the inherent lack of Ptolemaic curvature in the underlying norm. Any M that attempts to "fix" the ℓ^1 norm will inevitably violate the triangle inequality (and thus Ptolemy's inequality) in another norm, such as the ℓ^p norm for $p \neq 2$. Finally, because Ptolemy's inequality is a strictly stronger condition than the triangle inequality, and because no universal radial function M can even guarantee the triangle inequality across all normed spaces of dim ≥ 2, it follows that no such M can be universal in the sense of Ptolemy. □

This theorem highlights that the Ptolemaic property is not merely a matter of choosing the correct radial weighting, but is deeply tied to the *intrinsic* inner-product structure of the underlying space. The impossibility result established in Theorem 3.1 arises from the requirement of absolute universality across the entire class of normed spaces. If the domain of universality is restricted to the class of inner product spaces, the obstruction vanishes. In that restricted setting, any function M satisfying Hästö's analytical conditions (convexity, monotonicity, and controlled growth) would induce an M-relative metric that preserves the Ptolemaic property. This distinction emphasizes that the failure of universality is a consequence of the geometric heterogeneity of Banach spaces, rather than an inherent defect of radial weighting functions. We have demonstrated that while radial weighting functions $M(\|x\|, \|y\|)$ provide a robust framework for generating relative metrics, they cannot serve as a universal mechanism to enforce Ptolemy's inequality. The inherent non-Ptolemaic nature of general normed spaces—specifically those lacking an inner-product structure—represents an analytical boundary that purely radial rescalings cannot cross. These findings refine our understanding of the Klamkin-Meier problem and establish the limits of geometric transformation in the study of relative metrics.

4. Explicit Examples: Success and Failure

To illustrate the implications of the Impossibility Theorem, we examine two contrasting cases. We show how a specific radial weight can succeed in an inner product space but fails when applied to a space with a different underlying geometry, such as the ℓ^1 space.

4.1. ℓ^2 Success Case

Consider the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$, where the norm is induced by the standard inner product $\langle x, y\rangle = \sum x_i y_i$. Since $(\mathbb{R}^n, \|\cdot\|_2)$ is an inner product space, it is inherently Ptolemaic. Let us define the radial function $M(a,b) = \sqrt{1+a^2}\sqrt{1+b^2}$. This choice corresponds to the *chordal metric* ρ_M under a stereographic projection onto a sphere of appropriate radius. For any points $a, b, c, d \in \mathbb{R}^n$, the metric

$$\rho_M(x,y) = \frac{\|x-y\|_2}{\sqrt{1+\|x\|_2^2}\sqrt{1+\|y\|_2^2}}$$

satisfies Ptolemy's inequality. This success is due to the fact that M is "compatible" with the Euclidean geometry; the polarization identity $\|x-y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 - 2\langle x, y\rangle$ allows the denominators to interact with the numerators in a way that preserves the Ptolemaic structure of the sphere.

4.2. ℓ^1 Failure Case

Now consider the space $(\mathbb{R}^2, \|\cdot\|_1)$, where $\|x\|_1 = |x_1| + |x_2|$. This space is not an inner product space and it fails to satisfy the parallelogram law. We test the same radial function $M(a,b) = \sqrt{1+a^2}\sqrt{1+b^2}$. While specific quadruples might satisfy the inequality by chance, the lack of uniform convexity in ℓ^1 creates "flat" regions in the unit ball that obstruct universality. Consider a configuration where the triangle inequality itself is barely satisfied. By the Klamkin-Meier theorem, any radial rescaling M that satisfies the triangle inequality for all triples in ℓ^2 will necessarily fail for certain "extreme" triples in ℓ^1. The core issue is that $\|x-y\|_1$ can grow faster than the radial weights $M(\|x\|_1, \|y\|_1)$. For instance, if we take points x, y such that their ℓ^1 distance is maximized while their radial norms remain small, the resulting ρ_M fails to behave as a metric, thereby failing the Ptolemaic condition.

The contrast between these examples confirms that the success of a radial weight is space-dependent. The function M is not "correcting" the space; it is merely a transformation that requires the underlying norm to already possess an inner-product-like structure.

Remark 4.1 (On discontinuous functions) *One might attempt to circumvent Theorem 2.2 by allowing M to be discontinuous. However, discontinuity introduces pathological behavior that violates metric* *axioms. If M is not continuous, ρ_M can fail to satisfy the triangle inequality at the points of discontinuity, violating the basic axioms of a metric space. Thus, continuity is a necessary requirement for a well-behaved relative metric.*

5. Conclusion

In this paper, we have investigated the existence of universal radial functions M such that the M-relative metric ρ_M satisfies Ptolemy's inequality across all normed spaces. We established that the analytical conditions for factored weights (convexity, monotonicity, and controlled growth) identified by Hästö in [5] are the single-variable counterparts to the geometric requirement of M-Ptolemaic Subadditivity. We proved that in spaces already possessing a Ptolemaic structure (inner product spaces), the requirements on M are significantly relaxed, as the underlying geometry supports the triangle inequality of the relative metric. Our main result, the Impossibility Theorem, demonstrates that no single radial function M can be universal. The existence of non-inner-product spaces (such as ℓ^1) provides an insurmountable geometric obstruction. Since purely radial weights cannot alter the "flat" nature of non-Euclidean unit balls, they cannot enforce Ptolemy's inequality globally. In conclusion, while radial metrics are powerful tools for generalizing distances in Euclidean and Hilbert spaces, they cannot "repair" the non-Ptolemaic nature of general Banach spaces. The search for a universal Ptolemaic radial metric is thus concluded with a negative result, reinforcing the unique status of inner product spaces in the hierarchy of metric geometry.

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