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A Three-Dimensional Mixed Finite Element Scheme for Nonlinear Quasi-Static Eddy Current Problem

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ABSTRACT: We present the first fully discrete mixed finite element formulation for the stationary three-dimensional p-curl problem. The method relies on Nédélec edge elements and Raviart–Thomas face elements, and yields a discrete scheme that is consistent with the underlying continuous mixed formulation. The core of the analysis is the proof of existence and uniqueness of the discrete solution for $p > 2$. This work provides the first rigorous numerical framework for the mixed finite element approximation of the three-dimensional p-curl problem, and paves the way for future extensions, including non-homogeneous boundary conditions, the fully time-dependent problem, and the implementation of the method on representative test cases.

Key Words: Finite element method, superconductors, Nédélec elements, discrete mixed formulation, eddy current problem.

Contents

1. Introduction

The macroscopic behavior of type-II superconductors subjected to strong magnetic fields is governed by highly nonlinear electromagnetic laws. In the widely used power-law framework, the constitutive relation between the electric field $\mathbf{E}$ and the induced current density $\mathbf{J}$ is given by

$$\mathbf{E} = |\mathbf{J}|^{p-2}\mathbf{J}, \qquad 1 < p < \infty,$$

which, when substituted into the magnetoquasistatic approximation of Maxwell's equations, yields a nonlinear curl–curl equation for the magnetic or electric field. The corresponding differential operator,

$$\nabla \times \left(|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H}\right),$$

is commonly referred to as the p-curl operator and can be viewed as the rotational analogue of the classical p-Laplacian. Let $\Omega \subset \mathbb{R}^3$ be a compact, simply connected domain with a smooth boundary $\partial\Omega$, which is partitioned into two disjoint components,

$$\partial\Omega = \Gamma_{\mathbf{H}} \cup \Gamma_{\mathbf{E}}.$$

We prescribe two tangential vector fields, $\mathbf{g_H}$ on $\Gamma_{\mathbf{H}}$ and $\mathbf{g_E}$ on $\Gamma_{\mathbf{E}}$, and consider a divergence-free source term $\mathbf{F} : \Omega \to \mathbb{R}^3$.
In our previous work [6], we established the existence, uniqueness, and stability of the magnetic field $\mathbf{H}$ and the electric field $\mathbf{E}$ that solve the p-curl problem, formulated as follows:

$$\nabla \times \mathbf{E} = \mathbf{F} \qquad \text{in } \Omega, \tag{1.1a}$$
$$\mathbf{E} - \left|\nabla \times \mathbf{H}\right|^{p-2}\nabla \times \mathbf{H} = 0 \qquad \text{in } \Omega, \tag{1.1b}$$
$$\nabla \cdot \mathbf{H} = 0 \qquad \text{in } \Omega, \tag{1.1c}$$
$$\mathbf{n} \times \mathbf{E} = \mathbf{g_E} \qquad \text{in } \Gamma_{\mathbf{E}}, \tag{1.1d}$$
$$(\mathbf{n} \times \mathbf{H}) \times \mathbf{n} = \mathbf{g_H} \qquad \text{in } \Gamma_{\mathbf{H}}, \tag{1.1e}$$

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where $p \in (1, \infty)$ and $\mathbf{n}$ is the outward normal along the boundary.
From an applied perspective, the p-curl model is essential for the numerical simulation of a wide range of superconducting devices, including high-field magnets, fault-current limiters, superconducting tapes and cables, magnetic shielding systems, and rotating electrical machines. Accurate prediction of AC losses, current penetration, hysteresis cycles, and quench initiation requires numerical methods capable of resolving sharp magnetic fronts while robustly handling strong nonlinearities. Since the electric field is the primary quantity of engineering interest, its accurate approximation imposes stringent requirements on the stability and robustness of any discretization scheme.

At the continuous level, the stationary p-curl problem has been analyzed in [12,13,14] and in the authors' previous work [6], a continuous mixed formulation was studied, yielding existence, uniqueness, and stability results in the natural Banach spaces $W^q(curl, \Omega)$ and $W^p(div^0, \Omega)$, defined by

$$W^q(curl;\Omega) := \left\{ \mathbf{v} \in L^q(\Omega)^3 \;\middle|\; \nabla \times \mathbf{v} \in L^q(\Omega)^3 \right\},$$

and

$$W^p(div^0;\Omega) := \left\{ \mathbf{u} \in L^p(\Omega)^3 \;:\; \langle \mathbf{u}, \nabla\phi \rangle = 0 \quad \forall\, \phi \in W_0^{1,q}(\Omega) \right\},$$

where $q = p/(p-1)$. Although the continuous theory is now relatively mature, extending these results to stable and convergent finite element discretizations remains a significant challenge. Until now, to our knowledge, no fully discrete mixed finite element framework has been available for the three-dimensional p-curl problem. The purpose of the present work is to establish existence and uniqueness for a discrete mixed formulation of the stationary p-curl problem for $p > 2$. Following the approach developed in the authors' previous work on the continuous problem [6], well-posedness of the discrete system is obtained via equivalence to a finite-dimensional strictly convex minimization problem.

2. The continuous mixed formulation

For any $s \in [1, \infty)$, we introduce the standard Sobolev and vector-valued Banach spaces

$$\begin{aligned} W^{1,s}(\Omega) &:= \{ \mathbf{u} \in L^s(\Omega) \;:\; \nabla \mathbf{u} \in L^s(\Omega)^3 \}, \\ W^s(div;\Omega) &:= \{ \mathbf{u} \in L^s(\Omega)^3 \;:\; \nabla\cdot \mathbf{u} \in L^s(\Omega) \}, \\ W^s(curl;\Omega) &:= \{ \mathbf{u} \in L^s(\Omega)^3 \;:\; \nabla\times \mathbf{u} \in L^s(\Omega)^3 \}. \end{aligned}$$

Each space is equipped with its natural norm:

$$\begin{aligned} \|\mathbf{u}\|_{W^{1,s}(\Omega)} &:= \left(\|\mathbf{u}\|^s_{L^s(\Omega)} + \|\nabla \mathbf{u}\|^s_{L^s(\Omega)} \right)^{1/s}, \\ \|\mathbf{u}\|_{W^s(div;\Omega)} &:= \left(\|\mathbf{u}\|^s_{L^s(\Omega)} + \|\nabla\cdot \mathbf{u}\|^s_{L^s(\Omega)} \right)^{1/s}, \\ \|\mathbf{u}\|_{W^s(curl;\Omega)} &:= \left(\|\mathbf{u}\|^s_{L^s(\Omega)} + \|\nabla\times \mathbf{u}\|^s_{L^s(\Omega)} \right)^{1/s}. \end{aligned}$$

We also consider the tangential trace operator

$$\begin{aligned} \gamma_t :\; W^s(curl;\Omega) &\longrightarrow Y_t^{\,s}, \\ \mathbf{u} &\longmapsto \mathbf{n} \times \mathbf{u}, \end{aligned}$$

where Y_t^s denotes the Banach space of tangential traces on $\partial\Omega$.

Let r be the conjugate exponent of s, i.e., $1/s + 1/r = 1$. The weakly divergence-free subspace of $W^s(div^0;\Omega)$ is defined by

$$W^s(div^0;\Omega) := \left\{ \mathbf{u} \in L^s(\Omega)^3 \;:\; \langle \mathbf{u}, \nabla\phi \rangle = 0 \quad \forall\, \phi \in W_0^{1,r}(\Omega) \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 duality pairing and $W_0^{1,r}(\Omega)$ is the subspace of $W^{1,r}(\Omega)$ consisting of functions with vanishing trace on $\partial\Omega$.

Let q be the conjugate exponent, satisfying $p^{-1} + q^{-1} = 1$. In our previous work [6], we developed a mixed variational formulation for the stationary problem (1.1a,1.1e) and proved its well-posedness. More precisely, for any divergence-free source term $\mathbf{F} \in L^q(\Omega)^3$, there exists a unique pair $(\mathbf{E}, \mathbf{H})$ belonging to the spaces

$$X := W^q(curl;\Omega) \cap W^q(div^0;\Omega), \qquad M := W^p(div^0;\Omega),$$

equipped with the natural norms

$$\|\mathbf{u}\|_X := \left(\|\mathbf{u}\|^q_{L^q(\Omega)} + \|\nabla\times\mathbf{u}\|^q_{L^q(\Omega)}\right)^{1/q},$$

$$\|\mathbf{v}\|_M := \|\mathbf{v}\|_{L^p(\Omega)}.$$

such that $(\mathbf{E}, \mathbf{H})$ satisfies the mixed variational system

$$\begin{aligned} \int_\Omega |\mathbf{E}|^{q-2}\,\mathbf{E}\cdot\mathbf{u}\,d\mathbf{x} &= \int_\Omega (\nabla\times\mathbf{u})\cdot\mathbf{H}\,d\mathbf{x} - \int_{\partial\Omega} \mathbf{g_H}\cdot(\mathbf{n}\times\mathbf{u})\,d\mathbf{s}, && \forall \mathbf{u}\in X,\\ \int_\Omega (\nabla\times\mathbf{E})\cdot\mathbf{v}\,d\mathbf{x} &= \int_\Omega \mathbf{F}\cdot\mathbf{v}\,d\mathbf{x}, && \forall \mathbf{v}\in M. \end{aligned} \tag{2.1}$$

Moreover, the solution satisfies the stability estimate

$$\|\mathbf{E}\|_X + \|\mathbf{H}\|_M \;\le\; C\Big(\|\mathbf{F}\|_{L^q(\Omega)} + \|\mathbf{g_H}\|_{Y_t{}^r} + \|\mathbf{F}\|^{\frac{q}{p}}_{L^q(\Omega)}\Big), \tag{2.2}$$

for some constant $C > 0$ independent of the solution $(\mathbf{E}, \mathbf{H})$.

3. Well-posedness of the discrete mixed formulation

In this section, we introduce the discrete mixed finite element formulation for the p-curl problem. We begin with a recapitulation of the construction of Nédélec and Raviart-Thomas elements on in $\mathbb{R}^3$, and we also provide an overview of the interpolation operator suitable for sufficiently smooth vector fields, as previously outlined by Nédélec [8], Monk [7], and Ern [1,2,3]. Let $\mathcal{T}_h$ denote a triangulation of Ω into tetrahedrons. We assume that the triangulation $\mathcal{T}_h$ is regular [7]. To approximate the system described in equation (2.1), we will recall the elements introduced in [7] and [9] studies, which are used to construct the approximation spaces of the continuous functional spaces.

Lemma 3.1 *[9] Let Ω be the union of two domains K_1 and K_2, sharing a common face f with normal vector. A vector field $\mathbf{p}$ belonging to the space $(H^1(K_1))^3 \cup (H^1(K_2))^3$ is in $H(\mathrm{curl},\Omega)$ if and only if tangential trace of $\mathbf{p}$ across the face f is continuous on both sides of f.*

Definition 3.1 *Let $\mathbb{P}_k$ denote the space of polynomials of degree $k \ge 1$, and $\tilde{\mathbb{P}}_k$ the space of homogeneous polynomials of degree k. We define the polynomial space $\mathbb{S}_k$ as follows*

$$\mathbb{S}_k = \{\boldsymbol{q} \in (\tilde{\mathbb{P}}_k)^3 \mid \boldsymbol{q}(\mathbf{x})\cdot\mathbf{x} = 0\,\forall\mathbf{x}\in\mathbb{R}^3\}.$$

We introduce the vector space of Nédélec polynomials in $\mathbb{R}^3$ as the direct sum of $(\mathbb{P}_{k-1})^3$ and $\mathbb{S}_k$

$$\begin{aligned} \mathbb{N}_k &:= (\mathbb{P}_{k-1})^3 \oplus \mathbb{S}_k \\ &:= \{\boldsymbol{p}(\mathbf{x}) + \boldsymbol{q}(\mathbf{x}) : \boldsymbol{p}\in(\mathbb{P}_{k-1})^3,\ \boldsymbol{q}\in\mathbb{S}_k\}. \end{aligned}$$

Let $\boldsymbol{\tau}$ be the unit tangent vector along an edge e, and let $\boldsymbol{\nu}$ be the normal vector to a face f of a tetrahedron $K \in \mathcal{T}_h$. We introduce the set of degrees of freedom $\mathcal{D}_k$ for K, which contains

1. *Degrees of freedom for edges e*

$$d_e(\boldsymbol{p}) = \int_e (\boldsymbol{p}\cdot\boldsymbol{\tau})\,\boldsymbol{q}\,dl, \quad \forall \boldsymbol{q}\in\mathbb{P}_{k-1}(e),$$

2. *Degrees of freedom for faces f*

$$d_f(\boldsymbol{p}) = \int_f (\boldsymbol{p}\cdot\boldsymbol{q})\,ds, \quad \forall \boldsymbol{q}\in\mathbb{P}_{k-2}(f)^2_,$$

3. *Degrees of freedom for tetrahedron K*

$$d_K(\boldsymbol{p}) = \int_K (\boldsymbol{p}\cdot\boldsymbol{q})\,dv, \quad \forall \boldsymbol{q}\in \mathbb{P}_{k-3}(K)^3.$$

Lemma 3.2 *[9] Let Ω be the union of two domains K_1 and K_2, sharing a common face f with normal vector . A vector field $\mathbf{p}$ belonging to the space $(H^1(K_1))^3 \cup (H^1(K_2))^3$ is in $H(\mathrm{div},\Omega)$ if and only if the normal trace of $\mathbf{p}$ across the face f is continuous on both sides of f.*

Definition 3.2 *We introduce the vector space of Raviart-Thomas polynomials in $\mathbb{R}^3$ as follows*

$$\begin{aligned}\mathbb{RT}_k &:= (\mathbb{P}_{k-1})^3 \oplus \mathbf{x}\tilde{\mathbb{P}}_{k-1}\\ &:= \{\boldsymbol{p}(\mathbf{x}) + \boldsymbol{q}(\mathbf{x})\mathbf{x} : \boldsymbol{p}\in(\mathbb{P}_{k-1})^3, \boldsymbol{q}\in\tilde{\mathbb{P}}_{k-1}\}.\end{aligned}$$

We introduce the set of degrees of freedom $\mathcal{R}_k$ which contains

1. *Degrees of freedom for faces f*

$$d_f(\boldsymbol{p}) = \int_f (\boldsymbol{p}\cdot\boldsymbol{\nu})\boldsymbol{q}\,ds, \quad \forall \boldsymbol{q}\in\mathbb{P}_{k-1}(f),$$

2. *Degrees of freedom for tetrahedron K*

$$d_K(\boldsymbol{p}) = \int_K \boldsymbol{p}\cdot\boldsymbol{q}\,dv, \quad \forall \boldsymbol{q}\in(\mathbb{P}_{k-2}(K))^3.$$

The finite elements $(K,\mathcal{D}_k,\mathbb{N}_k)$ and $(K,\mathcal{R}_k,\mathbb{RT}_k)$ are conforming to the spaces $H(\mathrm{curl},\Omega)$ and $H(\mathrm{div},\Omega)$, and are unisolvent for the degrees of freedom $\mathcal{D}_k$ and $\mathcal{R}_k$ respectively. We define the discrete spaces that approximate these functional spaces as follows

$$\begin{aligned}\mathbf{F}^c_{h,k} &:= \left\{\mathbf{u}_h\in H(\mathrm{curl},\Omega)\,\middle|\,\mathbf{u}_h|_K\in\mathbb{N}_k,\,\forall K\in\mathcal{T}_h\right\},\\ \mathbf{F}^d_{h,k} &:= \left\{\mathbf{v}_h\in H(\mathrm{div},\Omega)\,\middle|\,\mathbf{v}_h|_K\in\mathbb{RT}_k,\,\forall K\in\mathcal{T}_h\right\},\end{aligned}$$

and for $p>1$ we define the spaces

$$\begin{aligned}\mathbf{F}^{c,p}_{h,k} &:= \left\{\mathbf{u}_h\in W^p(\mathrm{curl},\Omega)\,\middle|\,\mathbf{u}_h|_K\in\mathbb{N}_k,\,\forall K\in\mathcal{T}_h\right\},\\ \mathbf{F}^{d,p}_{h,k} &:= \left\{\mathbf{v}_h\in W^p(\mathrm{div},\Omega)\,\middle|\,\mathbf{v}_h|_K\in\mathbb{RT}_k,\,\forall K\in\mathcal{T}_h\right\}.\end{aligned}$$

We conclude our discussion of functional spaces by introducing the conforming discrete subspaces $X_h\subset X$ and $M_h\subset M$, in which we will seek the discrete solution.We propose to seek the electric field $\mathbf{E}_h$ in the space X_h which is given by

$$X_h := \left\{\mathbf{u}_h\in W^q(\mathrm{curl},\Omega)\cap W^q(\mathrm{div}^0,\Omega)\,\middle|\,\mathbf{u}_h|_K\in\mathbb{N}_k,\,\forall K\in\mathcal{T}_h\right\},$$

and we seek the magnetic field $\mathbf{H}_h$ in the discrete space

$$M_h := \left\{\mathbf{v}_h\in W^p(\mathrm{div},\Omega)\,\middle|\,\mathrm{div}\,\mathbf{v}_h=0,\,\mathbf{v}_h|_K\in\mathbb{RT}_k,\,\forall K\in\mathcal{T}_h\right\},$$

where $k\geq 1$.

Lemma 3.3 *[3], [9] Let $p\in[1,\infty)$, the interpolation operator $\pi^d : W^{1,p}(\Omega)^3\cap W^p(\mathrm{div},\Omega)\to\mathbf{F}^{d,p}_{h,k}$ is well-defined, and we have*

$$\|\mathbf{v}-\pi^d(\mathbf{v})\|_{L^p(\Omega)^3}\leq Ch|\mathbf{v}|_{W^{1,p}(\Omega)^3} \qquad \forall\mathbf{v}\in W^{1,p}(\Omega)^3. \tag{3.1}$$

Similarly, for r satisfying $rp>2$ for $p\in(1,\infty)$ or $r=2$ if $p=1$, the interpolation operator π^c : $W^{r,p}(\Omega)^3\cap W^p(\mathrm{curl},\Omega)\to\mathbf{F}^{c,p}_{h,k}$, is also well-defined, and if $p\in(2,\infty]$ we have for every $r\in\{1:k+1\}$

$$\|\mathbf{u}-\pi^c(\mathbf{u})\|_{L^p(\Omega)^3}\leq Ch^r|\mathbf{u}|_{W^{r,p}(\Omega)^3} \qquad \forall\mathbf{u}\in W^p(\mathrm{curl},\Omega)\cap W^{r,p}(\Omega)^3. \tag{3.2}$$

On the other hand if $p\in[1,2]$, and $k\geq 1$, the estimate (3.2) holds for every $r\in\{2:k+1\}$.

Let $\mathcal{C} = W^q(\text{curl}, \Omega) \cap W^{r,q}(\Omega)^3$ where $rq > 2$ for $q \in (1, \infty)$ or $r = 2$ if $q = 1$, and $\mathcal{D} = W^p(div^0) \cap W^{1,p}(\Omega)^3$. We define the vector field space $\tilde{\mathcal{C}}$ as follows

$$\tilde{\mathcal{C}} = \{\mathbf{u} \in \mathcal{C} \,|\, \nabla \times \mathbf{u} \in \mathcal{D}\}.$$

For every $\mathbf{u} \in \tilde{\mathcal{C}}$, the following relationship holds (see Lemma 19.6 from [3])

$$\nabla \times (\pi^c(\mathbf{u})) = \pi^d(\nabla \times \mathbf{u}). \tag{3.3}$$

In other words, the following diagram commutes

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{\nabla\times} & \mathcal{D} \\ \downarrow{\scriptstyle \pi^c} & & \downarrow{\scriptstyle \pi^d} \\ \mathbf{F}^{c,p}_{h,k} & \xrightarrow{\nabla\times} & \mathbf{F}^{d,p}_{h,k} \end{array}$$

We are now in a position to describe the discrete mixed formulation. Given $\mathbf{F} \subset L^q(\Omega)^3$ that is divergence free and a triangular mesh $\mathcal{T}_h$, we seek $(\mathbf{H}_h, \mathbf{E}_h) \in M_h \times X_h$ satisfying

$$\begin{aligned} \int_\Omega |\mathbf{E}_h|^{q-2}\mathbf{E}_h \cdot \mathbf{u}_h \, d\mathbf{x} &= \int_\Omega \nabla \times \mathbf{u}_h \cdot \mathbf{H}_h \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g_H} \cdot (\mathbf{n} \times \mathbf{u}_h) \, \forall \mathbf{u}_h \in X_h, \\ \int_\Omega \nabla \times \mathbf{E}_h \cdot \mathbf{v}_h \, d\mathbf{x} &= \int_\Omega \mathbf{F} \cdot \mathbf{v}_h \, d\mathbf{x} \quad \forall \mathbf{v}_h \in M_h. \end{aligned} \tag{3.4}$$

Before stating the main theorems of this section, which establish the existence and uniqueness of the solution for $p > 2$, we first prove an inf–sup conditiin for the discrete spaces. This condition is the key ingredient in the subsequent analysis.

Theorem 3.1 *(Theorem 7, [9]) Let $\mathcal{T}_h$ be a regular triangulation of the domain Ω. We define the spaces*

$$\begin{aligned} \mathbf{F}^{d,0}_{h,k} &:= \Big\{ \mathbf{v}_h \in H(\text{div}, \Omega) \mid \text{div}(\mathbf{v}_h) = 0 \text{ and } \mathbf{v}_h|_K \in \mathbb{RT}_k, \, K \in \mathcal{T}_h \Big\}, \\ F^g_{h,k} &:= \Big\{ \mathbf{w}_h \in H^1(\Omega) \mid \mathbf{w}_h|_K \in \mathbb{P}_k, K \in \mathcal{T}_h \Big\}. \end{aligned}$$

where $k \geq 1$, and For each element $\mathbf{v}_h$ in the space $\mathbf{F}^{d,0}_{h,k}$, there exists a unique element $\mathbf{u}_h$ in the space $\mathbf{F}^c_{h,k}$ such that

$$\nabla \times \mathbf{u}_h = \mathbf{v}_h, \tag{3.5}$$

and it satisfies the weak divergence-free condition

$$\int_\Omega \mathbf{u}_h \cdot \nabla \mathbf{w}_h = 0, \quad \forall \mathbf{w}_h \in F^g_{h,k}.$$

Moreover, there exists a constant $c > 0$ such that

$$\|\mathbf{u}_h\|_{H(\text{curl},\Omega)} \leq c\|\mathbf{v}_h\|_{L^2}. \tag{3.6}$$

Proposition 3.1 *Let Ω be a compact and simply connected domain with a boundary $\partial\Omega_h$ of class C^1. Then, there exists $\beta > 0$ such that*

$$\inf_{\mathbf{v}_h \in M_h} \sup_{\mathbf{u}_h \in X_h} \frac{\int_\Omega (\nabla \times \mathbf{u}_h) \cdot \mathbf{v}_h \, dx}{\|\mathbf{v}_h\|_M \|\mathbf{u}_h\|_X} \geq \beta.$$

Proof: Let $\mathbf{v}_h \in M_h$. Since $p > 2$, we have $\mathbf{v}_h \in M_h \subset \mathbf{F}^{d,0}_{h,k-1}$. From equation (3.5), there exists a unique element in the space $\mathbf{F}^c_{h,k}$ such that $\nabla \times \mathbf{u}_h = \mathbf{v}_h$. Since $q \leq 2$, thus $\mathbf{u}_h \in X_h$. Therefore, we obtain

$$\int_\Omega (\nabla \times \mathbf{u}_h) \cdot \mathbf{v}_h \, dx = \|\mathbf{v}_h\|^2_{L^2}.$$

From inequality (3.6), we have

$$\|\mathbf{u}_h\|_{H(\mathrm{curl})} \leq c\|\mathbf{v}_h\|_{L^2},$$

Using the continuous embeddings $L^2(\Omega) \hookrightarrow L^q(\Omega)$ and $L^p(\Omega) \hookrightarrow L^2(\Omega)$, we deduce that

$$\|\mathbf{v}_h\|_{L^2} \leq c_1\|\mathbf{v}_h\|_{L^p}, \quad \|\mathbf{u}_h\|_X \leq c_2\|\mathbf{u}_h\|_{H(\mathrm{curl})}.$$

Thus, there exists $c_3 > 0$ satisfying

$$\|\mathbf{u}_h\|_X \leq c_3\|\mathbf{v}_h\|_{L^p}.$$

This gives

$$\frac{1}{c_3}\frac{\|\mathbf{v}_h\|_{L^2}}{\|\mathbf{v}_h\|_{L^p}} \leq \frac{\int_\Omega (\nabla \times \mathbf{u}_h) \cdot \mathbf{v}_h \, dx}{\|\mathbf{u}_h\|_X}.$$

Since in finite dimension all norms are equivalent, there exists $c_4 > 0$ such that

$$c_4\|\mathbf{v}_h\|_{L^p} \leq \|\mathbf{v}_h\|_{L^2},$$

thus

$$\frac{c_4^2\|\mathbf{v}_h\|^2_{L^p}}{c_3} \leq \frac{\int_\Omega (\nabla \times \mathbf{u}_h) \cdot \mathbf{v}_h \, dx}{\|\mathbf{u}_h\|_X}.$$

By setting $\beta = \frac{c_4^2}{c_3} > 0$, we obtain

$$\beta \leq \inf_{\mathbf{v}_h \in M_h} \sup_{\mathbf{u}_h \in X_h} \frac{\int_\Omega (\nabla \times \mathbf{u}_h) \cdot \mathbf{v}_h \, dx}{\|\mathbf{v}_h\|_M \|\mathbf{u}_h\|_X}.$$

□

Before stating the main result establishing the existence and uniqueness of the solution of the discrete mixed formulation, we first introduce the following fundamental theorem.

Theorem 3.2 *[10] Let $(Y, \|\cdot\|_Y)$ and $(S, \|\cdot\|_S)$ be two reflexive Banach spaces. Let $(Y^*, \|\cdot\|_{Y^*})$ and $(S^*, \|\cdot\|_{S^*})$ be their corresponding dual spaces. Let $B : Y \to S^*$ be a continuous linear operator and $B^* : S \to Y^*$ the dual operator of B. Let $K_B = \ker(B)$ be the kernel of B, and $\dot{B} : (Y/K_B) \to S^*$ the quotient operator associated with B. The following three properties are equivalent*

(*i*) *There exists $\beta > 0$ such that*

$$\beta \leq \inf_{\mathbf{v} \in S} \sup_{\mathbf{u} \in Y} \frac{\langle B\mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{v}\|_S \|\mathbf{u}\|_Y},$$

(*ii*) *B^* is an isomorphism and we have*

$$\beta\|\mathbf{v}\|_S \leq \|B^*\mathbf{v}\|_{Y^*} \quad \forall \mathbf{v} \in S,$$

(*iii*) *$\dot{B}$ is an isomorphism from (Y/K_B) onto S^* and*

$$\beta\|\dot{\mathbf{u}}\|_{(Y/K_B)} \leq \|\dot{B}\dot{\mathbf{u}}\|_{S^*} \quad \forall \dot{\mathbf{u}} \in (Y/K_B).$$

Theorem 3.3 *We consider the following mixed formulation: find $(\mathbf{E}_h, \mathbf{H}_h) \in X_h \times M_h$ such that*

$$\begin{aligned} \int_\Omega |\mathbf{E}_h|^{q-2}\mathbf{E}_h \cdot \mathbf{u}_h \, d\mathbf{x} &= \int_\Omega \nabla \times \mathbf{u}_h \cdot \mathbf{H}_h \, d\mathbf{x} - \int_{\partial\Omega} g_{\mathbf{H}} \cdot (\mathbf{n} \times \mathbf{u}_h) \, \forall \mathbf{u}_h \in X_h \\ \int_\Omega \nabla \times \mathbf{E}_h \cdot \mathbf{v}_h \, d\mathbf{x} &= \int_\Omega \boldsymbol{F} \cdot \mathbf{v}_h \, d\mathbf{x} \quad \forall \mathbf{v}_h \in M_h. \end{aligned} \tag{3.7}$$

There exists a unique solution to this problem.

Proof: We define the discrete kernel

$$K_B = \{\mathbf{u}_h \in X_h \,|\, \int_\Omega (\nabla \times \mathbf{u}_h) \cdot \mathbf{v}_h \, d\mathbf{x} = 0 \quad \forall \mathbf{v}_h \in M_h\}.$$

By Proposition 3.1 and Theorem 3.2, the operator $\nabla \times : X_h \to (M_h)^*$ is an isomorphism from X_h/K_B onto $(M_h)^*$. Consequently, there exists a unique $\overline{\mathbf{e}}_h \in X_h$ such that

$$\int_\Omega \nabla \times (\mathbf{e}_h + \overline{\mathbf{e}}_h) \cdot \mathbf{h}_h \, d\mathbf{x} = \int_\Omega \mathbf{F} \cdot \mathbf{h}_h \, d\mathbf{x} \quad \forall \mathbf{h}_h \in M_h, \quad \forall \mathbf{e}_h \in K_B, \tag{3.8}$$

Furthermore, the original discrete mixed problem is thus equivalent to finding $\mathbf{e}_h \in K_B$ and $\mathbf{h}_h \in M_h$ satisfying

$$\int_\Omega |\mathbf{e}_h + \overline{\mathbf{e}}_h|^{q-2}(\mathbf{e}_h + \overline{\mathbf{e}}_h) \cdot \mathbf{u}_h \, d\mathbf{x} = \int_\Omega (\nabla \times \mathbf{u}_h) \cdot \mathbf{h}_h \, d\mathbf{x} \quad \forall \mathbf{u}_h - \int_{\partial\Omega} g_{\mathbf{H}} \cdot (\mathbf{n} \times \mathbf{u}_h) \in X_h,$$
$$\int_\Omega \nabla \times (\mathbf{e}_h + \overline{\mathbf{e}}_h) \cdot \mathbf{h}_h \, d\mathbf{x} = \int_\Omega \mathbf{F} \cdot \mathbf{h}_h \, d\mathbf{x} \quad \forall \mathbf{h}_h \in M_h.$$

Since the second equation is always satisfied, it suffices to focus on the first one. Now consider the auxiliary problem: find $\hat{\mathbf{e}}_h \in K_B$ such that

$$\int_\Omega |\hat{\mathbf{e}}_h + \overline{\mathbf{e}}_h|^{q-2}(\hat{\mathbf{e}}_h + \overline{\mathbf{e}}_h) \cdot \mathbf{u}_h \, d\mathbf{x} = 0 \quad \forall \mathbf{u}_h \in K_B. \tag{3.9}$$

As established in [4,5,6], this equation is the first variation of the functional

$$\mathcal{L}(\mathbf{w}_h) = \frac{1}{q} \int_\Omega |\mathbf{w}_h + \overline{\mathbf{e}}_h|^q \, d\mathbf{x}.$$

Thus, problem (3.9) is equivalent to the minimization problem

$$\mathcal{L}(\hat{\mathbf{e}}_h + \overline{\mathbf{e}}_h) = \inf_{\mathbf{u}_h \in K_B} \mathcal{L}(\mathbf{u}_h + \overline{\mathbf{e}}_h).$$

By Theorem 1.1 of [11], this minimization problem has a unique solution if $\mathcal{L}$ is continuous, strictly convex, and coercive on K_B. In our previous work [6], we established that the functional $\mathcal{L}$ is continuous and strictly convex, it thus remains to prove its coercivity on the discrete subspace $K_B \subset X_h$. For any $\mathbf{e}_h \in K_B$,

$$\mathcal{L}(\mathbf{e}_h) = \frac{1}{q} \int_\Omega |\mathbf{e}_h + \overline{\mathbf{e}}_h|^q \, d\mathbf{x} = \frac{1}{q} \|\mathbf{e}_h + \overline{\mathbf{e}}_h\|_{L^q}^q \geq \frac{1}{q} \big(\|\mathbf{e}_h\|_{L^q} - \|\overline{\mathbf{e}}_h\|_{L^q}\big)^q,$$

where the last inequality follows from the reverse triangle inequality. Since K_B is a finite-dimensional subspace, all norms on K_B are equivalent. In particular, there exists a constant $c = c(h) > 0$, depending on the mesh size h and the shape-regularity of the triangulation but independent of $\mathbf{e}_h$, such that

$$\|\mathbf{e}_h\|_{L^q(\Omega)} \geq c \|\mathbf{e}_h\|_X \quad \forall \mathbf{e}_h \in K_B.$$

Combining this with the norm equivalence gives

$$\mathcal{L}(\mathbf{e}_h) \geq \frac{1}{q} \big(c\|\mathbf{e}_h\|_X - \|\overline{\mathbf{e}}_h\|_{L^q(\Omega)}\big)^q$$

Now consider any sequence $\{\mathbf{e}_h^{(n)}\}_{n \geq 1} \subset K_B$ such that $\|\mathbf{e}_h^{(n)}\|_X \to +\infty$ as $n \to \infty$. Hence, $\mathcal{L}$ is coercive on K_B. Therefore, there exists a unique $\hat{\mathbf{e}}_h \in K_B$ solving the minimization problem. On the other hand, for the first equation, by applying the same argument as in our previous work [6], we conclude that there exists a unique $\mathbf{h}_h \in M_h$ such that

$$\int_\Omega |\mathbf{e}_h + \overline{\mathbf{e}}_h|^{q-2}(\mathbf{e}_h + \overline{\mathbf{e}}_h) \cdot \mathbf{u}_h \, d\mathbf{x} = \int_\Omega (\nabla \times \mathbf{u}_h) \cdot \mathbf{h}_h \, d\mathbf{x} \quad \forall \mathbf{u}_h - \int_{\partial\Omega} g_{\mathbf{H}} \cdot (\mathbf{n} \times \mathbf{u}_h) \in X_h \tag{3.10}$$

Therefore, from equations (3.8), (3.9), and (3.10), there exists a unique solution $(\hat{\mathbf{e}}_h + \overline{\mathbf{e}}_h, \mathbf{h}_h) \in X_h \times M_h$ to our discrete mixed formulation.

□

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