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Exploring Some Fundamental Properties of Operators in Neutrosophic Banach Spaces

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ABSTRACT: In this paper, we present and examine the concepts of linearity, bounded, injectivity, surjectivity and invertibility of operators in neuromorphic Banach space. Furthermore, we establish important relationships among injectivity, surjectivity with invertibility. Moreover, the definition of the Fredholm operator is provided along with an analysis of its relationship with invertibility. Additionally, we extend the study of these properties to the perturbation operator. Moreover, we discuss Lipschitz and contraction mappings, as well as uniqueness results related to fixed points.

Keywords: Neutrosophic Banach space, neutrosophic Fredholm operator, neutrosophically contraction and neutrosophically invertible.

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1. Introduction

The neutrosophic framework is adopted instead of the fuzzy one due to its ability to explicitly capture indeterminacy, which cannot be adequately represented within the classical fuzzy setting. The concept of neutrosophy was first introduced by Florentine Smarandache [1]. Since then, interest in this generalization known as the neutrosophic space has grown, and it has begun to be applied in various fields such as decision-making, artificial intelligence, image processing, data analysis, and mathematical modeling [2, 3,4] . As new generalization of normed space, [5] proposed the neutrosophic normed space. [6] presented chain rule and some algebraic properties of Fréchet differentiation of operators between neutrosophic normed spaces. [7] studied the analysis of sequences in terms of ordinary and Cauchy convergence under certain conditions in neutrosophic normed spaces. They also defined the forms of operators on these spaces and investigated some equivalent relationships among them. [8] proposesed a novel approach to address the analysis of stability in functional equations within neutrosophic normed spaces, offering a comprehensive framework for investigating stability properties in such contexts. [9] defined and discussed the concepts of continuous and bounded of operators in neuromorphic normed space. In this paper, we present and examine a new extended of concepts linearity , boundedness , injectivity, surjectivity and invertibility of operators in neutrosophic Banach spaces. Also, we define the neutrosophic Fredholm operator and examine its relationship with invertibility, which saves as a fundamental entry point to neutrosophic spectral theory. Consequently, we reveal new properties and relationships between the bounded linear operators and perturbation operators as well as their connections with the invertibility in neutrosophic Banach spaces . also, we obtain a generalized version of boundedness and continuity of intuitionistic fuzzy norms, while will play an important role in study neutrosophic analysis. Finally, we presented the new concepts of neutrosophic Lipschitz and contraction function in neutrosophic Banach space.

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2. Basic Concepts

Definition 2.1 *[10]] let $\bigoplus : [0,1]\times[0,1] \to [0,1]$ be a mapping. Then $\bigoplus$is called $t-$norm, if it satisfies the conditions*

1. $\bigoplus$ *is continuous, associative and commutative,*
2. $\hbar \bigoplus 1 = \hbar$,
3. $\hbar \bigoplus k \leq m \bigoplus \ell$ *implies that* $\hbar \leq m$ *and* $k \leq \ell$

,for all $\hbar, k, m, \ell \in [0,1]$

Definition 2.2 *[5] let $\bigodot : [0,1]\times[0,1] \to [0,1]$ be a mapping. Then, $\bigodot$is said to be continuous $t-$conorm, if it satisfies the conditions*

1. $\bigodot$ *is continuous, associative and commutative,*
2. $\hbar \bigodot 0 = \hbar$,
3. $\hbar \bigodot k \leq \hbar \bigodot \ell$ *implies that* $\hbar \leq \hbar$ *and* $k \leq \ell$

,for all $\hbar, k, \hbar, \ell \in [0,1]$

Definition 2.3 *[11]] An neutrosophic set (NS $\check{N}$) over $\mathbb{N}$ is described as follows*

$$\left\{\left\langle \hbar, \check{N}_1(\hbar), \check{N}_2(\hbar), \check{N}_3(\hbar)\right\rangle ; \hbar \in \mathbb{N}\right\}$$

where, $\check{N}_i : \mathbb{N} \to [0,1]\,,\ i = 1,2,3$

Definition 2.4 (12) *A function NS $\check{N}$ defined on a linear space $\mathcal{M}$ over $\mathbb{R}$ is referred to as a neutrosophic normed linear space NS$\check{N}$LS $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$, if it satisfies the following axioms*

1. $0 \leq \check{N}_1(\hbar,\upsilon), \check{N}_2(\hbar,\upsilon), \check{N}_3(\hbar,\upsilon) \leq 1$
2. $0 \leq \check{N}_1(\hbar,\upsilon) + \check{N}_2(\hbar,\upsilon) + \check{N}_3(\hbar,\upsilon) \leq 3$
3. $\check{N}_1(\hbar,\upsilon) = 0,\ \check{N}_2(\hbar,\upsilon) = 1, \check{N}_3(\hbar,\upsilon) = 1,\ \upsilon \leq 0$
4. $\check{N}_1(\hbar,\upsilon) = 1,\ \check{N}_2(\hbar,\upsilon) = 0, \check{N}_3(\hbar,\upsilon) = 0,\ \upsilon > 0$ *if and only if* $\hbar = 0$
5. $\check{N}_1(\alpha\hbar,\upsilon) = \check{N}_1\left(\hbar, \frac{\upsilon}{|\alpha|}\right),\ \check{N}_2(\alpha\hbar,\upsilon) = \check{N}_2\left(\hbar, \frac{\upsilon}{|\alpha|}\right), \check{N}_3(\alpha\hbar,\upsilon) = \check{N}_3\left(\hbar, \frac{\upsilon}{|\alpha|}\right)$, *for each* $\alpha \neq 0$ *and* $\upsilon > 0$
6. $\check{N}_1(\hbar,\upsilon) \bigoplus \check{N}_1(\ell,\delta) \leq \check{N}_1(\hbar+\ell, \upsilon+\delta)$, *for each* $\upsilon,\ \delta \in \mathbb{R}$
7. $\check{N}_1(\hbar,\upsilon)$ *approaches* 1 *as* υ *approaches infinity such that* $\check{N}_1$ *is a non-decreasing and continuous function for* $\upsilon > 0$
8. $\check{N}_2(\hbar,\upsilon) \bigoplus \check{N}_2(\ell,\delta) \geq \check{N}_2(\hbar+\ell, \upsilon+\delta)$, , *for each* $\upsilon,\ \delta \in \mathbb{R}$
9. $\widetilde{N_2}(\hbar,\upsilon)$ *approaches* 0 *as* υ *approaches infinity such that* $\widetilde{N_2}$ *is a non-increasing and continuous function for* $\upsilon > 0$
10. $\check{N}_3(\hbar,\upsilon) \bigoplus \check{N}_3(\ell,\delta) \geq \check{N}_3(\hbar+\ell, \upsilon+\delta)$, , *for each* $\upsilon,\ \delta \in \mathbb{R}$
11. $\check{N}_3(\hbar,\upsilon)$ *approaches* 0 *as* υ *approaches infinity such that* $\check{N}_3$ *is a non-increasing and continuous function for* $\upsilon > 0$

Definition 2.5 *[13] Let $\{\hbar_n\}$ be a sequence in NS$\check{N}$LS $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$, then the sequence $\{\hbar_n\}$ converges to $\hbar \in \mathcal{M}$, if and only if for each $0 < \epsilon < 1,\ \upsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfies*

$$\check{N}_1 (\hbar_n - \hbar, \upsilon) > 1 - \epsilon$$

$$\check{N}_2 (\hbar_n - \hbar, \upsilon) < \epsilon$$

$\check{N}_3 (\hbar_n - \hbar, \upsilon) < \epsilon$ *,for all $n \geq n_0$*

$$\lim_{n\to\infty} \check{N}_1 (\hbar_n - \hbar, \upsilon) = 1$$

$$\lim_{n\to\infty} \check{N}_2 (\hbar_n - \hbar, \upsilon) = 0$$

$$\lim_{n\to\infty} \check{N}_3 (\hbar_n - \hbar, \upsilon) = 0$$

In this case the sequence $\{\hbar_n\}$ is said to be convergent in the space NS$\check{N}$LS $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$.

Definition 2.6 *[13] Let $\{\hbar_n\}$ be a sequence in the neutrosophic normed linear space $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot)$. Then the sequence $\{\hbar_n\}$ is called a Cauchy sequence if and only if for each $0 < \epsilon < 1$, $\upsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\mathcal{N}_1 (\hbar_n - \hbar_k, \upsilon) > 1 - \epsilon, \quad \mathcal{N}_2 (\hbar_n - \hbar_k, \upsilon) < \epsilon, \quad \mathcal{N}_3 (\hbar_n - \hbar_k, \upsilon) < \epsilon,$$

for all $n, k \geq n_0$. Equivalently,

$$\lim_{n\to\infty} \mathcal{N}_1 (\hbar_n - \hbar_k, \upsilon) = 1, \quad \lim_{n\to\infty} \mathcal{N}_2 (\hbar_n - \hbar_k, \upsilon) = 0, \quad \lim_{n\to\infty} \mathcal{N}_3 (\hbar_n - \hbar_k, \upsilon) = 0.$$

In this case, the sequence $\{\hbar_n\}$ is said to be Cauchy in the space $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot)$.

Definition 2.7 *[14] A Neutrosophic Banach Space is a neutrosophic normed linear space $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot)$ that is complete with respect to the neutrosophic norm; that is, every neutrosophic Cauchy sequence in $\mathcal{M}$ converges neutrosophically to an element of $\mathcal{M}$.*

3. Main Results

Definition 3.1 *Let $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ and $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ be two neutrosophic Banach linear spaces over the same field $\mathbb{R}$. A mapping $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ is said to be linear if and only if it satisfies*

$$\Upsilon (\alpha\hbar_1 + \beta\hbar_2) = \alpha\, \Upsilon (\hbar_1) + \beta\, \Upsilon (\hbar_2) \quad , \text{ for all } \hbar_1, \hbar_2 \in \mathcal{M} \textbf{ and } \alpha, \beta \in \mathbb{R}$$

Definition 3.2 *Let $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ and $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ be two neutrosophic Banach linear spaces over the same field $\mathbb{R}$. A mapping $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ is called neutrosophically bounded if and only if there exist $0 \neq € \in \mathbb{R}$, for each $\hbar \in \mathcal{M}$* ***and*** *$\upsilon > 0$*

$$\check{N}_1 (\, \Upsilon (\hbar) , \upsilon) \geq \check{N}_1 (€\hbar, \upsilon)$$

$$\check{N}_2 (\, \Upsilon (\hbar) , \upsilon) \leq \check{N}_2 (€\hbar, \upsilon)$$

$$\check{N}_3 (\, \Upsilon (\hbar) , \upsilon) \leq \check{N}_3 (€\hbar, \upsilon)$$

Definition 3.3 *Let $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ and $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ be two neutrosophic Banach linear spaces over the same field $\mathbb{R}$. A mapping $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ is called neutrosophically continuous at $\hbar_0 \in \mathcal{M}$ if and only if for all $\hbar \in \mathcal{M}$, $\mathbf{0 < \varepsilon < 1}$* ***and*** *$\upsilon > 0$, there exists $0 < \delta < 1$ and $t > 0$, such that*

$$\check{N}_1 (\, \Upsilon (\hbar) - \, \Upsilon (\hbar_0) , \upsilon) > 1 - \varepsilon$$

$$\check{N}_2 (\, \Upsilon (\hbar) - \, \Upsilon (\hbar_0) , \upsilon) < \varepsilon$$

$$\check{N}_3 (\, \Upsilon (\hbar) - \, \Upsilon (\hbar_0) , \upsilon) < \varepsilon$$

whenever,

$$\check{N}_1(\hbar - \hbar_0, \upsilon) > 1 - \delta$$
$$\check{N}_2(\hbar - \hbar_0, \upsilon) < \delta$$
$$\check{N}_3(\hbar - \hbar_0, \upsilon) < \delta$$

Υ is continuous on $\mathcal{M}$ if it is continuous at every point in $\mathcal{M}$.

Theorem 3.1 *Let $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot)$ and $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$ be two neutrosophic Banach linear spaces over the same field $\mathbb{R}$ and $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$ be a neutrosophically linear operator. Then, the following statements are equivalent:*

1. Υ is neutrosophically continuous on $\mathcal{M}$.

2. Υ is neutrosophically continuous at the origin $\hbar = 0$.

3. Υ is neutrosophically bounded.

Proof: **1.** $\Rightarrow$ **2**. *Suppose that Υ is neutrosophically continuous on $\mathcal{M}$, that it must be continous at every point, including $\hbar = 0$.*

2. $\Rightarrow$ **3**. *Suppose that Υ is neutrosophically continuous at the origin $\hbar = 0$, then for every* neutrosophic number $\mathbf{0 < \varepsilon < 1}$ **and** $\upsilon > 0$, ***there exists*** $0 < \delta < 1$ *and* $t > 0$ *such that*

$$\check{N}_1(\Upsilon(\hbar) - \Upsilon(0), \upsilon) > 1 - \varepsilon$$
$$\check{N}_2(\Upsilon(\hbar) - \Upsilon(0), \upsilon) < \varepsilon$$
$$\check{N}_3(\Upsilon(\hbar) - \Upsilon(0), \upsilon) < \varepsilon$$

whenever,

$$\check{N}_1(\hbar, \upsilon) > 1 - \delta$$
$$\check{N}_2(\hbar, \upsilon) < \delta$$
$$\check{N}_3(\hbar, \upsilon) < \delta$$

But, Υ is neutrosophically linear operator this leads

$$\check{N}_1(\Upsilon(\hbar), \upsilon) > 1 - \varepsilon$$
$$\check{N}_2(\Upsilon(\hbar), \upsilon) < \varepsilon$$
$$\check{N}_3(\Upsilon(\hbar), \upsilon) < \varepsilon$$

This gives

$$\check{N}_1\left(\Upsilon\left(\frac{\hbar}{\check{N}_1(\hbar, \upsilon)}\right), \upsilon\right) > 1 - \varepsilon$$
$$\check{N}_2\left(\Upsilon\left(\frac{\hbar}{\check{N}_2(\hbar, \upsilon)}\right), \upsilon\right) < \varepsilon$$
$$\check{N}_3\left(\Upsilon\left(\frac{\hbar}{\check{N}_2(\hbar, \upsilon)}\right), \upsilon\right) < \varepsilon$$

Consequently,

$$\check{N}_1(\Upsilon(\hbar), \upsilon) > \frac{2\varepsilon}{\delta}$$
$$\check{N}_2(\Upsilon(\hbar), \upsilon) < \frac{2\varepsilon}{\delta}$$
$$\check{N}_3(\Upsilon(\hbar), \upsilon) < \frac{2\varepsilon}{\delta}$$

Therefore, there exists $0 \neq € \in \mathbb{R}$, for each $\hbar \in \mathcal{M}$ ***and*** $\upsilon > 0$

$$\check{N}_1(\Upsilon(\hbar), \upsilon) \geq \check{N}_1(€\hbar, \upsilon)$$
$$\check{N}_2(\Upsilon(\hbar), \upsilon) \leq \check{N}_2(€\hbar, \upsilon)$$
$$\check{N}_3(\Upsilon(\hbar), \upsilon) \leq \check{N}_3(€\hbar, \upsilon)$$

Theorem 3.2 *Let $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ be a neutrosophically continuous on a Banach space and let $0 < \beta < 1$. Then, $\beta I + \Upsilon$ is a continuous on $\mathcal{M}$ if and only if Υ is neutrosophically continuous on $\mathcal{M}$.*

Proof: *Since Υ is neutrosophically continuous on $\mathcal{M}$, then for all $\hbar \in \mathcal{M}$, $\mathbf{0 < \varepsilon < 1}$ **and** $\upsilon > 0$, there exists $0 < \delta < 1$ and $t > 0$, such that*

$$\check{N}_1\left(\Upsilon(\hbar) - \Upsilon(\hbar_0), \upsilon\right) > 1 - \varepsilon$$

$$\check{N}_2\left(\Upsilon(\hbar) - \Upsilon(\hbar_0), \upsilon\right) < \varepsilon$$

$$\check{N}_3\left(\Upsilon(\hbar) - \Upsilon(\hbar_0), \upsilon\right) < \varepsilon$$

whenever,

$$\check{N}_1(\hbar - \hbar_0, \upsilon) > 1 - \delta$$

$$\check{N}_2(\hbar - \hbar_0, \upsilon) < \delta$$

$$\check{N}_3(\hbar - \hbar_0, \upsilon) < \delta$$

The above inequalities can be written as

$$\check{N}_1\left((\beta I + \Upsilon)(\hbar) - (\beta I + \Upsilon)(\hbar_0), \upsilon\right) > 1 - \varepsilon$$

$$\check{N}_2\left((\beta I + \Upsilon)(\hbar) - (\beta I + \Upsilon)(\hbar_0), \upsilon\right) < \varepsilon$$

$$\check{N}_3\left((\beta I + \Upsilon)(\hbar) - (\beta I + \Upsilon)(\hbar_0), \upsilon\right) < \varepsilon$$

whenever,

$$\check{N}_1(\hbar - \hbar_0, \upsilon) > 1 - \delta$$

$$\check{N}_2(\hbar - \hbar_0, \upsilon) < \delta$$

$$\check{N}_3(\hbar - \hbar_0, \upsilon) < \delta$$

This means $\beta I + \Upsilon$ is a continuous on $\mathcal{M}$.

Definition 3.4 *Let $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ be a neutrosophically bounded linear operator on two neutrosophic Banach linear spaces $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ and $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$. Then, Υ is neutrosophically invertible with inverse $\Upsilon^{-1} : (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ if Υ^{-1} is neutrosophically bounded linear operator and it satisfies the following*

$$\check{N}_1\left(\Upsilon^{-1}\Upsilon(\hbar), \upsilon\right) = \check{N}_1\left(\Upsilon\Upsilon^{-1}(\hbar), \upsilon\right) = \check{N}_1(\hbar, \upsilon)$$

$$\check{N}_2\left(\Upsilon^{-1}\Upsilon(\hbar), \upsilon\right) = \check{N}_2\left(\Upsilon\Upsilon^{-1}(\hbar), \upsilon\right) = \check{N}_2(\hbar, \upsilon)$$

$$\check{N}_3\left(\Upsilon^{-1}\Upsilon(\hbar), \upsilon\right) = \check{N}_3\left(\Upsilon\Upsilon^{-1}(\hbar), \upsilon\right) = \check{N}_3(\hbar, \upsilon)$$

*, for each $\hbar \in \mathcal{M}$ **and** $\upsilon > 0$.*

Theorem 3.3 *Let $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ be a neutrosophically bounded linear operator on two neutrosophic Banach linear spaces $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ and $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ and let $0 < \beta < 1$. Then, consider $S = \beta I + \Upsilon$. If the neutrosophically bounded linear operator Υ satisfies*

$$\check{N}_1\left(\Upsilon(\hbar), \upsilon\right) > \beta$$

$$\check{N}_2\left(\Upsilon(\hbar), \upsilon\right) < \beta$$

$$\check{N}_3\left(\Upsilon(\hbar), \upsilon\right) < \beta$$

Then S is invertible, and its inverse $S^{-1} : (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ is bounded and linear in the neutrosophic sense. Moreover, the inverse is given by the Neutrosophic Neumann series:

$$S^{-1} = \frac{1}{\beta}\sum_{n=0}^{\infty}\left(-\frac{\Upsilon}{\beta}\right)^n$$

with convergence component-wise on $(\check{N}_1, \check{N}_2, \check{N}_3)$.

Proof:

Boundedness and linearity of S***:*** *Since* Υ *is linear, then*

$S(\alpha_1\hbar_1 + \alpha_2\hbar_2) = (\beta I + \Upsilon)(\alpha_1\hbar_1 + \alpha_2\hbar_2) = \beta I(\alpha_1\hbar_1 + \alpha_2\hbar_2) + \Upsilon(\alpha_1\hbar_1 + \alpha_2\hbar_2)$, *for all* $\hbar_1, \hbar_2 \in \mathcal{M}$

and $\alpha_1\alpha_2 \in \mathbb{R}$

$$= \beta\alpha_1\hbar_1 + \beta\alpha_2\hbar_2 + \alpha_1\Upsilon(\hbar_1) + \alpha_2\Upsilon(\hbar_2)$$
$$= \alpha_1(\beta I + \Upsilon)(\hbar_1) + \alpha_2(\beta I + \Upsilon)(\hbar_2)$$

Since Υ *is bounded, there exists a neutrosophic constant* $0 \neq € \in \mathbb{R}$, *for each* $\hbar \in \mathcal{M}$ ***and*** $\upsilon > 0$

$$\check{N}_1(\Upsilon(\hbar), \upsilon) \geq \check{N}_1(€\hbar, \upsilon)$$
$$\check{N}_2(\Upsilon(\hbar), \upsilon) \leq \check{N}_2(€\hbar, \upsilon)$$
$$\check{N}_3(\Upsilon(\hbar), \upsilon) \leq \check{N}_3(€\hbar, \upsilon)$$

And by hypotheses of this theorem, we have

$$\check{N}_1(\Upsilon(\hbar), \upsilon) > \beta$$
$$\check{N}_2(\Upsilon(\hbar), \upsilon) < \beta$$
$$\check{N}_3(\Upsilon(\hbar), \upsilon) < \beta$$

Therefore, S *is bounded.*

Neutrosophic Neumann series: *Consider the series*

$$\sum_{n=0}^{\infty}\left(-\frac{\Upsilon}{\beta}\right)^n.$$

Convergence component-wise is guaranteed because the norm of each term decreases geometrically for $\check{N}_1, \check{N}_2, \check{N}_3$. *Thus, this series defines a bounded linear operator on* $\mathcal{M}$.

Verification of the inverse: *Define*

$$S^{-1} := \frac{1}{\beta}\sum_{n=0}^{\infty}\left(-\frac{\Upsilon}{\beta}\right)^n.$$

Then

$$S^{-1}S = \frac{1}{\beta}\sum_{n=0}^{\infty}\left(-\frac{\Upsilon}{\beta}\right)^n(\beta I + \Upsilon) = I,$$

$$SS^{-1} = (\beta I + \Upsilon)\frac{1}{\beta}\sum_{n=0}^{\infty}\left(-\frac{\Upsilon}{\beta}\right)^n = I.$$

Boundedness of S^{-1}***:*** *Component-wise, we have*

$$\check{N}_1\left(S^{-1}(\hbar), \upsilon\right) \geq \check{N}_1\left(\frac{1}{\beta}\sum n\ \hbar, \upsilon\right)$$
$$\check{N}_2(\Upsilon(\hbar), \upsilon) \leq \check{N}_2\left(\frac{1}{\beta}\sum n\ \hbar, \upsilon\right)$$
$$\check{N}_3(\Upsilon(\hbar), \upsilon) \leq \check{N}_3\left(\frac{1}{\beta}\sum n\ \hbar, \upsilon\right)$$

Thus, S^{-1} *is bounded component-wise for* $(\check{N}_1, \check{N}_2, \check{N}_3)$.

Hence, $S = \beta I + T$ *is invertible, and* S^{-1} *is linear and bounded in the neutrosophic sense.*

The theorem ensures that the inverse preserves the neutrosophic structure: each of the truth, indeterminacy, and falsity components is controlled, guaranteeing both algebraic invertibility and component-wise boundedness.

Definition 3.5 *[9] A mapping* $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ *is called neutrosophically Lipschitz on* $\mathcal{M}$*, if there exists* $\mathcal{C} > 0$ *satisfies the following*

$$\check{N}_1\left(\Upsilon(\hbar_1) - \Upsilon(\hbar_2), \upsilon\right) \geq \check{N}_1\left(\hbar_1 - \hbar_2, \frac{\upsilon}{\mathcal{C}}\right)$$

$$\check{N}_2\left(\Upsilon(\hbar_1) - \Upsilon(\hbar_2), \upsilon\right) \leq \check{N}_2\left(\hbar_1 - \hbar_2, \frac{\upsilon}{\mathcal{C}}\right)$$

$$\check{N}_3\left(\Upsilon(\hbar_1) - \Upsilon(\hbar_2)\right) \leq \check{N}_3\left(\hbar_1 - \hbar_2, \frac{\upsilon}{\mathcal{C}}\right)$$

, for each $\hbar_1, \hbar_2 \in \mathcal{M}$ ***and*** $\upsilon > 0$*.* Υ *is said to be neutrosophically contraction, if* $\mathcal{C} < 1$*.*

Theorem 3.4 *[9]* ***A*** *neutrosophically operator* Υ *which defined on Banach space* $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ *has a unique fixed point if,* Υ *is neutrosophically contraction.*

Theorem 3.5 *Let* Υ *be a neutrosophically operator which defined on Banach space* $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ *and let* $0 < \beta < 1$*. Then,* $\beta I + \Upsilon$ *has a unique fixed point if,* Υ *is neutrosophically contraction.*

Proof: *Since* Υ *is neutrosophically contraction, then there exists* $\mathcal{C} < 1$ *such that*

$$\check{N}_1\left(\Upsilon(\hbar_1) - \Upsilon(\hbar_2), \upsilon\right) \geq \check{N}_1\left(\hbar_1 - \hbar_2, \frac{\upsilon}{\mathcal{C}}\right)$$

$$\check{N}_2\left(\Upsilon(\hbar_1) - \Upsilon(\hbar_2), \upsilon\right) \leq \check{N}_2\left(\hbar_1 - \hbar_2, \frac{\upsilon}{\mathcal{C}}\right)$$

$$\check{N}_3\left(\Upsilon(\hbar_1) - \Upsilon(\hbar_2)\right) \leq \check{N}_3\left(\hbar_1 - \hbar_2, \frac{\upsilon}{\mathcal{C}}\right)$$

, for each $\hbar_1, \hbar_2 \in \mathcal{M}$ ***and*** $\upsilon > 0$*.*
This gives

$$\check{N}_1\left((\beta I + \Upsilon)(\hbar_1) - (\beta I + \Upsilon)(\hbar_2), \upsilon\right) \geq \check{N}_1\left(\hbar_1 - \hbar_2, \frac{\upsilon}{\mathcal{C}}\right)$$

$$\check{N}_2\left((\beta I + \Upsilon)(\hbar_1) - (\beta I + \Upsilon)(\hbar_2), \upsilon\right) \leq \check{N}_2\left(\hbar_1 - \hbar_2, \frac{\upsilon}{\mathcal{C}}\right)$$

$$\check{N}_3\left((\beta I + \Upsilon)(\hbar_1) - (\beta I + \Upsilon)(\hbar_2), \upsilon\right) \leq \check{N}_3\left(\hbar_1 - \hbar_2, \frac{\upsilon}{\mathcal{C}}\right)$$

Therefore, $\beta I + \Upsilon$ *is a neutrosophically contraction which defined on Banach space* $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$*, using Theorem 4.* $\beta I + \Upsilon$ *has a unique fixed point.*

Definition 3.6 *Let* $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ *be a neutrosophically bounded linear operator on two neutrosophic Banach linear spaces* $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ *and* $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$*. Then, the neutrosophic kernel set of* Υ *is defined as follows:*

$$Ker_{\Upsilon} = \left\{\hbar \in \mathcal{M}; \check{N}_1\left(\Upsilon(\hbar), \upsilon\right) = \mathbf{0}, \check{N}_2\left(\Upsilon(\hbar), \upsilon\right) = \boldsymbol{I}_{\mathbf{0}}^{\mathcal{N}}, \check{N}_3\left(\Upsilon(\hbar), \upsilon\right) = \boldsymbol{F}_{\mathbf{0}}^{\mathcal{N}}\right\}$$

Where, $\boldsymbol{I}_{\mathbf{0}}^{\mathcal{N}}$ ***and*** $\boldsymbol{F}_{\mathbf{0}}^{\mathcal{N}}$ ***are indeterminacy degree and falsity degree of zero element respectively in*** $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ *space.*

Definition 3.7 *Let* $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ ) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$ *be a neutrosophically bounded linear operator on two neutrosophic Banach linear spaces* $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ *and* $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot\ )$*. Then, the neutrosophically bounded linear operator* Υ *is called neutrosophic injective if and only if it satisfies the following condition*

$$Ker_{\Upsilon} = \left\{\hbar \in \mathcal{M}; \check{N}_1(\hbar, \upsilon) = \mathbf{0}, \check{N}_2(\hbar, \upsilon) = \boldsymbol{I}_{\mathbf{0}}^{\mathcal{M}}, \check{N}_3(\hbar, \upsilon) = \boldsymbol{F}_{\mathbf{0}}^{\mathcal{M}}\right\}$$

Where, $\boldsymbol{I}_{\mathbf{0}}^{\mathcal{M}}$ ***and*** $\boldsymbol{F}_{\mathbf{0}}^{\mathcal{M}}$ ***are indeterminacy degree and falsity degree of zero element respectively in*** $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot\ )$ *space.*

Definition 3.8 *The neutrosophically range of neutrosophically bounded linear operator* $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$ *is defined as*

$$R_{\Upsilon} = \{n \in \mathcal{N}; \textbf{there exists } \hbar \in \mathcal{M} \textbf{ such that } \Upsilon(\hbar) = n$$
$$\check{N}_1(\Upsilon(\hbar), \upsilon) = \check{N}_1(n, \upsilon), \check{N}_2(\Upsilon(\hbar), \upsilon) = \check{N}_2(n, \upsilon), \check{N}_3(\Upsilon(\hbar), \upsilon) = \check{N}_3(n, \upsilon) \}$$

Definition 3.9 *Let* $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$ *be a neutrosophically bounded linear operator on two neutrosophic Banach linear spaces* $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot)$ *and* $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$. *Then, the neutrosophically operator* Υ *is said to be neutrosophic surjective if and only if it satisfies the following condition* $R_{\Upsilon} = \mathcal{N}$.

Theorem 3.6 *Let* $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$ *be a neutrosophically bounded linear operator on two neutrosophic Banach linear spaces* $(\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot)$ *and* $(\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$. *Then,* Υ *is neutrosophically invertible if and only if* Υ *satisfies the neutrosophic injective and neutrosophic surjective properties.*

Proof: if Υ *is neutrosophically invertible then there exists neutrosophically bounded linear operator* Υ^{-1} that satisfies the following

$$\check{N}_1\left(\Upsilon^{-1}\Upsilon(\hbar), \upsilon\right) = \check{N}_1\left(\Upsilon\Upsilon^{-1}(\hbar), \upsilon\right) = \check{N}_1(\hbar, \upsilon)$$

$$\check{N}_2\left(\Upsilon^{-1}\Upsilon(\hbar), \upsilon\right) = \check{N}_2\left(\Upsilon\Upsilon^{-1}(\hbar), \upsilon\right) = \check{N}_2(\hbar, \upsilon)$$

$$\check{N}_3\left(\Upsilon^{-1}\Upsilon(\hbar), \upsilon\right) = \check{N}_3\left(\Upsilon\Upsilon^{-1}(\hbar), \upsilon\right) = \check{N}_3(\hbar, \upsilon)$$

, for each $\hbar \in \mathcal{M}$ ***and*** $\upsilon > 0$.

This mean that $Ker_{\Upsilon} = \left\{\hbar \in \mathcal{M}; \check{N}_1(\hbar, \upsilon) = \mathbf{0}, \check{N}_2(\hbar, \upsilon) = \boldsymbol{I}_{\mathbf{0}}^{\mathcal{M}}, \check{N}_3(\hbar, \upsilon) = \boldsymbol{F}_{\mathbf{0}}^{\mathcal{M}}\right\}$

And $R_{\Upsilon} = \mathcal{N}$

Therefore, Υ *satisfies the neutrosophic injective and neutrosophic surjective properties.*

if Υ *satisfies the neutrosophic injective and neutrosophic surjective properties then*

$$Ker_{\Upsilon} = \left\{\hbar \in \mathcal{M}; \check{N}_1(\hbar, \upsilon) = \mathbf{0}, \check{N}_2(\hbar, \upsilon) = \boldsymbol{I}_{\mathbf{0}}^{\mathcal{M}}, \check{N}_3(\hbar, \upsilon) = \boldsymbol{F}_{\mathbf{0}}^{\mathcal{M}}\right\}$$

And $R_{\Upsilon} = \mathcal{N}$

By boundedness and the *neutrosophic bounded inverse theorem, the neutrosophic inverse operator* Υ^{-1} *exists and bounded.*

Hence, Υ *is neutrosophically invertible.*

Definition 3.10 *A neutrosophically bounded linear operator* $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$ *between neutrosophic Banach spaces is said to be a neutrosophic Fredholm operator if its neutrosophic kernel is finite dimensional and its range is closed with finite codimension.*

Remark 3.1 *in neutrosophic version the codimension is defined as*

$$Codim_{\Upsilon} = dim\left(\mathcal{N}/R_{\Upsilon}\right)$$

Definition 3.11 *for neutrosophic Fredholm operator* $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$*, neutrosophic Fredholm index is defined as*

$$Ind_{\Upsilon} = dim\left(Ker_{\Upsilon}\right) - Codim_{\Upsilon}$$

Remark 3.2

1. *Every neutrosophic invertible bounded linear operator is a neutrosophic Fredholm operator, but the converse is not necessarily true.*

2. *The neutrosophic Fredholm index is stable under compact neutrosophic perturbations.*

Theorem 3.7 *A neutrosophic Fredholm bounded linear operator* $\Upsilon : (\mathcal{M}, \mathbb{R}, \bigoplus, \bigodot) \to (\mathcal{N}, \mathbb{R}, \bigoplus, \bigodot)$ *between neutrosophic Banach spaces is invertible if and only if its neutrosophic kernel is trivial and its neutrosophic Fredholm index is zero.*

Proof: the trivial kernel ensures inactivity of Υ *since*

$$Ker_{\Upsilon} = \left\{ \hbar \in \mathcal{M}; \check{N}_1(\hbar, \upsilon) = \mathbf{0}, \check{N}_2(\hbar, \upsilon) = \boldsymbol{I}_{\mathbf{0}}^{\mathcal{M}}, \check{N}_3(\hbar, \upsilon) = \boldsymbol{F}_{\mathbf{0}}^{\mathcal{M}} \right\}$$

Fredholm index equal zero implies that the codimension of the range is zero
Hence, the range is dense and closed, so it equals the whole space $\mathcal{N}$.
Since Υ *is neutrosophic Fredholm, then* R_{Υ} *is closed.*
If $Ker_{\Upsilon} = \left\{ \hbar \in \mathcal{M}; \check{N}_1(\hbar, \upsilon) = \mathbf{0}, \check{N}_2(\hbar, \upsilon) = \boldsymbol{I}_{\mathbf{0}}^{\mathcal{M}}, \check{N}_3(\hbar, \upsilon) = \boldsymbol{F}_{\mathbf{0}}^{\mathcal{M}} \right\}$
and the index is zero, then

$$Codim_{\Upsilon} = 0$$

By the Neutrosophic Bounded Inverse Theorem, Υ^{-1} *exists and is bounded.*

Remark 3.3 *1. Every invertible operator is Fredholm with index zero.*

4. Conclusions

In this paper, we extend the concepts of linearity, boundedness, continuity, injectivity, surjectivity, and invertibility of bounded operators to the framework of neutrosophic Banach spaces. Within this setting, a new class of bounded linear operators is introduced. Several significant relationships among these operators are rigorously analyzed. Furthermore, the necessary and sufficient conditions for the uniqueness of fixed points, derived via neutrosophic contractions, are established. This study provides a comprehensive foundation for further investigations in neutrosophic operator theory and its applications. Finally, this study is considered a fundamental cornerstone for neutrosophic spectral theory.

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