



## Generating Functions for Binary Products of Gaussian Numbers with Special Bivariate Polynomials at Consecutive and Nonconsecutive Indices

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**ABSTRACT:** In this research, we prove a new theorem by applying the symmetrizing operator  $\delta_{a_1 a_2}^{k+l-s+1}$ . Using this theorem, we introduce a novel family of generating functions for products involving Gaussian Pell-Padovan and Gaussian Perrin numbers combined with certain  $(p, q)$ -numbers, as well as special bivariate polynomials with consecutive and nonconsecutive indices. These generating functions enable the derivation of new identities and relationships among the sequences under consideration.

**Keywords:** Generating functions, Gaussian Pell Padovan numbers, Gaussian Perrin numbers,  $(p, q)$ -Fibonacci numbers,  $(p, q)$ -Pell numbers,  $(p, q)$ -Pell Lucas numbers, bivariate polynomials.

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### 1. Introduction

Generating functions are among the most useful tools in discrete mathematics and mathematical analysis. They enable the determination of the terms of linear and recurrent sequences with constant coefficients, making them an effective method for solving a wide range of mathematical problems. This approach also shows how these terms can be derived directly, without the need to compute the preceding ones. This study focuses on constructing new generating functions for the products of various types of numbers and polynomials, based on distinct consecutive and non-consecutive indices. Several previous works have addressed related problems, and the most notable contributions are summarized below.

In 2018, in [6], A. Boussayoud and co-authors introduced new symmetric endomorphism operators using suitable infinite product series. Their main results showed that these operators can be effectively used to derive generating functions for  $k$ -Jacobsthal numbers and Tchebychev polynomials of the first and second kind.

Later, in 2019, in [8], A. Boussayoud and co-authors introduced a new operator to investigate the symmetric properties of Fibonacci numbers and Chebychev polynomials of the first and second kind. By means of this operator, they derived new generating functions for these sequences.

Subsequently, in 2021, in [2], the authors introduced a new family of generating functions for the products of bivariate complex Fibonacci polynomials with various Gaussian numbers, including the Gaussian Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas, Pell, and Pell-Lucas numbers.

Mor recently, in 2025, in [16], the authors constructed new generating functions for the products of various  $(p, q)$ -numbers at consecutive and nonconsecutive indices. They also developed generating

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functions for the products of these  $(p, q)$ -numbers with Mersenne numbers, establishing several new symmetric relations and identities.

In the present work, we focus on sequences of bivariate polynomials defined by second-order recurrence relations. Such polynomials have attracted considerable attention in the literature. One of the earliest studies in this direction was made by M. Catalani in 2004 (see [9]), who introduced the bivariate Fibonacci and Lucas polynomials defined by

$$\begin{cases} F_n(x, y) = xF_{n-1}(x, y) + yF_{n-2}(x, y), & n \geq 2, \\ F_0(x, y) = 0, F_1(x, y) = 1. \end{cases}$$

and

$$\begin{cases} L_n(x, y) = xL_{n-1}(x, y) + yL_{n-2}(x, y), & n \geq 2, \\ L_0(x, y) = 2; L_1(x, y) = x. \end{cases}$$

We also consider the bivariate Jacobsthal and Pell polynomials (see [17,20]), defined, respectively, by

$$\begin{cases} J_n(x, y) = xyJ_{n-1}(x, y) + 2yJ_{n-2}(x, y), & n \geq 2, \\ J_0(x, y) = 0, J_1(x, y) = 1. \end{cases}$$

and

$$\begin{cases} P_n(x, y) = 2xyP_{n-1}(x, y) + yP_{n-2}(x, y), & n \geq 2, \\ P_0(x, y) = 0, P_1(x, y) = 1. \end{cases}$$

Furthermore, we investigate second-order sequence numbers such as the  $(p, q)$ -Fibonacci,  $(p, q)$ -Lucas and  $(p, q)$ -Pell Lucas (see [18]), given by

$$\begin{cases} F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}, & n \geq 2, \\ F_{p,q,0} = 0, F_{p,q,1} = 1, \end{cases}$$

$$\begin{cases} L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2}, & n \geq 2, \\ L_{p,q,0} = 2, L_{p,q,1} = p, \end{cases}$$

and

$$\begin{cases} Q_{p,q,n} = 2pQ_{p,q,n-1} + qQ_{p,q,n-2}, & n \geq 2, \\ Q_{p,q,0} = 2, Q_{p,q,1} = 2p, \end{cases}$$

Finally, we examined third-order sequence numbers, including the Gaussian Pell-Padovan numbers and the Gaussian Perrin numbers (see [19,11]), which are described, respectively, by

$$\begin{cases} GR_n = 2GR_{n-2} + GR_{n-3}, & n \geq 3, \\ GR_0 = 1 - i, GR_1 = 1 + i, GR_2 = 1 + i, \end{cases}$$

and

$$\begin{cases} Gr_n = Gr_{n-2} + Gr_{n-3}, & n \geq 3, \\ Gr_0 = -1 + 3i, Gr_1 = 3, Gr_2 = 2i. \end{cases}$$

This paper begins with an introduction outlining the motivation and contextual background of the study. It then presents the essential definitions and preliminary results that underpin our work. Subsequently, we state and rigorously prove a generalized theorem that forms the foundation for all subsequent developments. We then construct novel generating functions for the products of Gaussian numbers with  $(p, q)$ -numbers and with bivariate polynomials, addressing both consecutive and nonconsecutive indices. The paper concludes with a synthesis of the main results and a discussion of potential directions for future research.

## 2. Notation and some properties

In this section, we introduce a symmetric function and investigate its fundamental properties. We also present relevant definitions and concepts from the existing literature, which will serve as essential tools in the subsequent sections.

**Definition 2.1** [1] *Given two alphabets  $A$  and  $E$ , we define  $S_n(A - E)$  as*

$$\sum_{n=0}^{\infty} S_n(A - E)t^n = \frac{\prod_{e \in E} (1 - et)}{\prod_{a \in A} (1 - at)}, \quad (2.1)$$

with  $S_n(A - E) = 0$  in case  $n < 0$ .

**Corollary 2.1** *Setting  $A = \{0\}$  in relationship (2.1) yields*

$$\sum_{n=0}^{\infty} S_n(-E)t^n = \prod_{e \in E} (1 - et). \quad (2.2)$$

Thus, we obtain

$$\sum_{n=0}^{\infty} S_n(A - E)t^n = \left( \sum_{n=0}^{\infty} S_n(A)t^n \right) \times \left( \sum_{n=0}^{\infty} S_n(-E)t^n \right).$$

**Remark 2.1** *If  $A = E$ , then*

$$\sum_{n=0}^{\infty} S_n(A)t^n = \frac{1}{\sum_{n=0}^{\infty} S_n(-A)t^n}.$$

**Definition 2.2** [12] *Assuming that  $A = \{a_1, a_2\}$  is an alphabet and  $n \in \mathbb{N}$ , we generate the  $n^{\text{th}}$  symmetric function  $S_n(a_1 + a_2)$  by*

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$\begin{aligned} S_0(A) &= S_0(a_1 + a_2) = 1, \\ S_1(A) &= S_1(a_1 + a_2) = a_1 + a_2, \\ S_2(A) &= S_2(a_1 + a_2) = a_1^2 + a_1a_2 + a_2^2, \\ &\vdots \end{aligned}$$

**Definition 2.3** [5] *We define the symmetrizing operator  $\delta_{a_1 a_2}^k$  by*

$$\delta_{a_1 a_2}^k f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \quad \forall k \in \mathbb{N}_0.$$

**Proposition 2.1** *Let  $n \in \mathbb{N}$ . The symmetric functions of the Gaussian Pell Padovan, Gaussian Perrin,  $(p, q)$ -Fibonacci, bivariate Fibonacci and Lucas polynomials, bivariate Jacobsthal and Jacobsthal Lucas, bivariate Pell polynomials are respectively given by*

$$\begin{aligned}
GR_n &= (1-i)S_n(E) + (1+i)S_{n-1}(E) + (-1+3i)S_{n-2}(E), \\
Gr_n &= (-1+3i)S_n(E) + 3S_{n-1}(E) + (1-i)S_{n-2}(E) \quad (\text{see [14]}), \\
F_{p,q,n} &= S_{n-1}(a_1 + [-a_2]), \\
L_{p,q,n} &= 2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2]), \quad (\text{see [15]}), \\
Q_{p,q,n} &= 2S_n(a_1 + [-a_2]) - 2pS_{n-1}(a_1 + [-a_2])P_n(x, y) = S_{n-1}(a_1 + [-a_2]) \quad (\text{see [15,13]}), \\
F_n(x, y) &= S_{n-1}(a_1 + [-a_2]), \\
L_n(x, y) &= 2S_n(a_1 + [-a_2]) - xS_{n-1}(a_1 + [-a_2]) \quad (\text{see [13]}). \\
J_n(x, y) &= S_{n-1}(a_1 + [-a_2]) \quad (\text{see [13]}).
\end{aligned}$$

### 3. The Generalized Theorem and Its Proof

In this section, we present the generalized theorem that constitutes the main theoretical result of this study. A detailed proof is provided to establish its validity, thereby providing the framework for the construction of novel generating functions and the analysis of their associated properties in the subsequent sections.

**Theorem 3.1** Consider the two alphabets  $E = \{e_1, e_2, e_3\}$  and  $A = \{a_1, a_2\}$ . Then we have

$$\frac{\sum_{n=0}^{\infty} S_n(E) S_{n+k+l-s}(A) z^n + S_{k+l-s}(A) + S_1(-E)a_1a_2S_{k+l-s-1}(A)z + S_2(-E)a_1^2a_2^2S_{k+l-s-2}(A)z^2 + S_3(-E)a_1^3a_2^3S_{k+l-s-3}(A)z^3 - a_1^{k+l-s+1}a_2^{k+l-s+1}z^{k+l-s+2} \sum_{n=0}^{\infty} S_{n+k+l-s+2}(-E)S_n(A)z^n}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right) \left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}, \quad (3.1)$$

for all  $n, k \in \mathbb{N}_0$  and  $k, l, s \in \{0, 1, 2, 3\}$

**Proof:** By using the operation  $\delta_{a_1 a_2}^{k+l-s+1}$  to the series  $f(a_1 z) = \sum_{n=0}^{\infty} S_n(E) a_1^n z^n$ , we get

$$\begin{aligned}
\delta_{a_1 a_2}^{k+l-s+1} f(a_1 z) &= \frac{a_1^{k+l-s+1} \left(\sum_{n=0}^{\infty} S_n(E) a_1^n z^n\right) - a_2^{k+l-s+1} \left(\sum_{n=0}^{\infty} S_n(E) a_2^n z^n\right)}{a_1 - a_2} \\
&= \sum_{n=0}^{\infty} S_n(E) \frac{a_1^{n+k+l-s+1} - a_2^{n+k+l-s+1}}{a_1 - a_2} z^n \\
&= \sum_{n=0}^{\infty} S_n(E) S_{n+k+l-s}(A) z^n.
\end{aligned}$$

Alternatively, by using the operation  $\delta_{a_1 a_2}^{k+l-s+1}$  to the series  $f(a_1 z) = \frac{1}{\sum_{n=0}^{\infty} S_n(-E) a_1^n z^n}$ , we get

$$\begin{aligned}
\delta_{a_1 a_2}^{k+l-s+1} f(a_1 z) &= \frac{\frac{a_1^{k+l-s+1}}{\sum_{n=0}^{\infty} S_n(-E) a_1^n z^n} - \frac{a_2^{k+l-s+1}}{\sum_{n=0}^{\infty} S_n(-E) a_2^n z^n}}{a_1 - a_2} \\
&= \frac{a_1^{k+l-s+1} \sum_{n=0}^{\infty} S_n(-E) a_2^n z^n - a_2^{k+l-s+1} \sum_{n=0}^{\infty} S_n(-E) a_1^n z^n}{(a_1 - a_2) \left(\sum_{n=0}^{\infty} S_n(-E) a_1^n z^n\right) \left(\sum_{n=0}^{\infty} S_n(-E) a_2^n z^n\right)}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{\sum_{n=0}^{\infty} S_n(-E) \frac{a_1^{k+l-s+1} a_2^n - a_2^{k+l-s+1} a_1^n}{a_1 - a_2}}{\left( \sum_{n=0}^{\infty} S_n(-E) a_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-E) a_2^n z^n \right)} \\
 &= \frac{\sum_{n=0}^{\infty} S_n(-E) a_1^n a_2^n \frac{a_1^{k+l-s+1-n} - a_2^{k+l-s+1-n}}{a_1 - a_2} z^n}{\left( \sum_{n=0}^{\infty} S_n(-E) a_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-E) a_2^n z^n \right)}.
 \end{aligned}$$

Equivalently

$$\begin{aligned}
 \delta_{a_1 a_2}^{k+l-s+1} f(a_1 z) &= \frac{\sum_{n=0}^{\infty} S_n(-E) a_1^n a_2^n S_{k+l-s-n}(A) z^n}{\left( \sum_{n=0}^{\infty} S_n(-E) a_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-E) a_2^n z^n \right)} \\
 &= \frac{\sum_{n=0}^{k+l-s} S_n(-E) a_1^n a_2^n S_{k+l-s-n}(A) z^n + \sum_{n=k+l-s+2}^{\infty} S_n(-E) a_1^n a_2^n S_{k+l-s-n}(A) z^n}{\left( \sum_{n=0}^{\infty} S_n(-E) a_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-E) a_2^n z^n \right)} \\
 &\quad S_{k+l-s}(A) + S_1(-E) a_1 a_2 S_{k+l-s-1}(A) z \\
 &\quad + S_2(-E) a_1^2 a_2^2 S_{k+l-s-2}(A) z^2 + S_3(-E) a_1^3 a_2^3 S_{k+l-s-3}(A) z^3 \\
 &\quad + \sum_{n=k+l-s+2}^{\infty} S_n(-E) a_1^n a_2^n \left( \frac{a_1^{k+l-s-n+1} - a_2^{k+l-s-n+1}}{a_1 - a_2} \right) z^n \\
 &= \frac{\left( \sum_{n=0}^{\infty} S_n(-E) a_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-E) a_2^n z^n \right)}{S_{k+l-s}(A) + S_1(-E) a_1 a_2 S_{k+l-s-1}(A) z + S_2(-E) a_1^2 a_2^2 S_{k+l-s-2}(A) z^2} \\
 &\quad S_3(-E) a_1^3 a_2^3 S_{k+l-s-3}(A) z^3 + \sum_{n=k+l-s+2}^{\infty} S_n(-E) \frac{a_1^{k+l-s+1} a_2^n - a_2^{k+l-s+1} a_1^n}{a_1 - a_2} z^n \\
 &= \frac{\left( \sum_{n=0}^{\infty} S_n(-E) a_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-E) a_2^n z^n \right)}{S_{k+l-s}(A) + S_1(-E) a_1 a_2 S_{k+l-s-1}(A) z + S_2(-E) a_1^2 a_2^2 S_{k+l-s-2}(A) z^2} \\
 &\quad S_3(-E) a_1^3 a_2^3 S_{k+l-s-3}(A) z^3 \\
 &\quad - \sum_{n=k+l-s+2}^{\infty} S_n(-E) a_1^{k+l-s+1} a_2^{k+l-s+1} \frac{a_1^{n-k-l+s+1} - a_2^{n-k-l+s+1}}{a_1 - a_2} z^n \\
 &= \frac{\left( \sum_{n=0}^{\infty} S_n(-E) a_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-E) a_2^n z^n \right)}{S_{k+l-s}(A) + S_1(-E) a_1 a_2 S_{k+l-s-1}(A) z + S_2(-E) a_1^2 a_2^2 S_{k+l-s-2}(A) z^2} \\
 &\quad S_3(-E) a_1^3 a_2^3 S_{k+l-s-3}(A) z^3 - \sum_{n=k+l-s+2}^{\infty} S_n(-E) a_1^{k+l-s+1} a_2^{k+l-s+1} S_{n-k-l+s-2}(A) z^n \\
 &= \frac{\left( \sum_{n=0}^{\infty} S_n(-E) a_1^n z^n \right) \left( \sum_{n=0}^{\infty} S_n(-E) a_2^n z^n \right)}{S_{k+l-s}(A) + S_1(-E) a_1 a_2 S_{k+l-s-1}(A) z + S_2(-E) a_1^2 a_2^2 S_{k+l-s-2}(A) z^2}
 \end{aligned}$$

which, also, gives

$$\begin{aligned} & S_{k+l-s}(A) + S_1(-E)a_1a_2S_{k+l-s-1}(A)z \\ & + S_2(-E)a_1^2a_2^2S_{k+l-s-2}(A)z^2 + S_3(-E)a_1^3a_2^3S_{k+l-s-3}(A)z^3 \\ \delta_{a_1a_2}^{k+l-s+1} f(a_1z) &= \frac{-a_1^{k+l-s+1}a_2^{k+l-s+1}z^{k+l-s+2} \sum_{n=0}^{\infty} S_{n+k+l-s+2}(-E)S_n(A)z^n}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right) \left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \end{aligned}$$

Thus, the desired outcome is obtained.  $\square$

- For  $E = \{e_1, e_2, e_3\}$  and  $A = \{a_1, a_2\}$  and  $k, l, s \in \{0, 1, 2, 3\}$  in theorem 1, we deduce the following lemmas.

**Lemma 3.1** *Assuming that  $E = \{e_1, e_2, e_3\}$  and  $A = \{a_1, a_2\}$  are two alphabets, we have*

$$\sum_{n=0}^{\infty} S_n(E)S_n(A)z^n = \frac{1 - a_1a_2S_2(-E)z^2 - a_1a_2(a_1 + a_2)S_3(-E)z^3}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right) \left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \quad (3.2)$$

**Lemma 3.2** *Suppose that  $E = \{e_1, e_2, e_3\}$  and  $A = \{a_1, a_2\}$  are two alphabets. Then we have*

$$\sum_{n=0}^{\infty} S_n(E)S_{n+1}(A)z^n = \frac{(a_1 + a_2) + a_1a_2S_1(-E)z - a_1^2a_2^2S_3(-E)z^3}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right) \left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \quad (3.3)$$

**Lemma 3.3** *Let  $E = \{e_1, e_2, e_3\}$  and  $A = \{a_1, a_2\}$  be two alphabets. Then we have*

$$\sum_{n=0}^{\infty} S_n(E)S_{n+2}(A)z^n = \frac{(a_1 + a_2)^2 - a_1a_2 + a_1a_2(a_1 + a_2)S_1(-E)z + a_1^2a_2^2S_2(-E)z^2}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right) \left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \quad (3.4)$$

**Corollary 3.1** *The relationship (3.4) allows us to obtain*

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_n(A)z^n = \frac{[(a_1 + a_2)^2 - a_1a_2]z^2 + a_1a_2(a_1 + a_2)S_1(-E)z^3 + a_1^2a_2^2S_2(-E)z^4}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right) \left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \quad (3.5)$$

**Corollary 3.2** *The relationship (3.4) yields*

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n+1}(A)z^n = \frac{[(a_1 + a_2)^2 - a_1a_2]z + a_1a_2(a_1 + a_2)S_1(-E)z^2 + a_1^2a_2^2S_2(-E)z^3}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right) \left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \quad (3.6)$$

**Lemma 3.4** *Assuming that  $E = \{e_1, e_2, e_3\}$  and  $A = \{a_1, a_2\}$  are two alphabets, we have*

$$\begin{aligned} & (a_1 + a_2)[(a_1 + a_2)^2 - 2a_1a_2] + a_1a_2[(a_1 + a_2)^2 - a_1a_2] \\ \sum_{n=0}^{\infty} S_n(E)S_{n+3}(A)z^n &= \frac{S_1(-E)z + (a_1 + a_2)a_1^2a_2^2S_2(-E)z^2 + a_1^3a_2^3S_3(-E)z^3}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right) \left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \end{aligned} \quad (3.7)$$

**Corollary 3.3** *From the relationship (3.7), we get*

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_{n+1}(A)z^n = \frac{(a_1 + a_2)[(a_1 + a_2)^2 - 2a_1a_2]z^2 + a_1a_2[(a_1 + a_2)^2 - a_1a_2]S_1(-E)z^3 + (a_1 + a_2)a_1^2a_2^2S_2(-E)z^4 + a_1^3a_2^3S_3(-E)z^5}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \quad (3.8)$$

**Corollary 3.4** *Using the relationship (3.8), we obtain*

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n+2}(A)z^n = \frac{(a_1 + a_2)[(a_1 + a_2)^2 - 2a_1a_2]z + a_1a_2[(a_1 + a_2)^2 - a_1a_2]S_1(-E)z^2 + (a_1 + a_2)a_1^2a_2^2S_2(-E)z^3 + a_1^3a_2^3S_3(-E)z^4}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \quad (3.9)$$

**Lemma 3.5** *Suppose that  $E = \{e_1, e_2, e_3\}$  and  $A = \{a_1, a_2\}$  are two alphabets. Then we have*

$$\begin{aligned} & [(a_1 + a_2)^4 - 3a_1a_2(a_1 + a_2)^2 + a_1^2a_2^2] \\ & + S_1(-E)a_1a_2[(a_1 + a_2)^3 - 2a_1a_2(a_1 + a_2)]z \\ \sum_{n=0}^{\infty} S_n(E)S_{n+4}(A)z^n & = \frac{+S_2(-E)a_1^2a_2^2[(a_1 + a_2)^2 - a_1a_2]z^2 + a_1^3a_2^3(a_1 + a_2)S_3(-E)z^3}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \end{aligned} \quad (3.10)$$

**Corollary 3.5** *It follows from(3.10) that*

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_{n+2}(A)z^n = \frac{[(a_1 + a_2)^4 - 3a_1a_2(a_1 + a_2)^2 + a_1^2a_2^2]z^2 + S_1(-E)a_1a_2[(a_1 + a_2)^3 - 2a_1a_2(a_1 + a_2)]z^3 + S_2(-E)a_1^2a_2^2[(a_1 + a_2)^2 - a_1a_2]z^4 + a_1^3a_2^3(a_1 + a_2)S_3(-E)z^5}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-E)a_2^n z^n\right)}. \quad (3.11)$$

- For the case where  $A = \{a_1, -a_2\}$  and  $E = \{e_1, e_2, e_3\}$ , by replacing  $a_2$  with  $(-a_2)$  in equations (3.2), (3.3), (3.4), (3.5), (3.6), (3.8), and (3.11), we obtain

$$\sum_{n=0}^{\infty} S_n(E)S_n(a_1 + [-a_2])z^n = \frac{1 + a_1a_2S_2(-E)z^2 + a_1a_2(a_1 - a_2)S_3(-E)z^3}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}, \quad (3.12)$$

$$\sum_{n=0}^{\infty} S_n(E)S_{n+1}(a_1 + [-a_2])z^n = \frac{(a_1 - a_2) - a_1a_2S_1(-E)z - a_1^2a_2^2S_3(-E)z^3}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}, \quad (3.13)$$

$$\sum_{n=0}^{\infty} S_n(E)S_{n+2}(a_1 + [-a_2])z^n = \frac{(a_1 - a_2)^2 + a_1a_2 - a_1a_2(a_1 - a_2)S_1(-E)z + a_1^2a_2^2S_2(-E)z^2}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}, \quad (3.14)$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(a_1 + [-a_2])z^n = \frac{(a_1 - a_2)z - a_1a_2S_1(-E)z^2 - a_1^2a_2^2S_3(-E)z^4}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}, \quad (3.15)$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n+1}(a_1 + [-a_2])z^n = \frac{[(a_1 - a_2)^2 + a_1a_2]z - a_1a_2(a_1 - a_2)S_1(-E)z^2 + a_1^2a_2^2S_2(-E)z^3}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}, \quad (3.16)$$

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_{n+1}(a_1 + [-a_2])z^n = \frac{(a_1 - a_2)[(a_1 - a_2)^2 + 2a_1a_2]z^2 - a_1a_2[(a_1 - a_2)^2 + a_1a_2]S_1(-E)z^3 + (a_1 - a_2)a_1^2a_2^2S_2(-E)z^4 - a_1^3a_2^3S_3(-E)z^5}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}, \quad (3.17)$$

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1 + [-a_2])z^n = \frac{[(a_1 - a_2)^2 + a_1a_2]z^2 - a_1a_2(a_1 - a_2)S_1(-E)z^3 + a_1^2a_2^2S_2(-E)z^4}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}, \quad (3.18)$$

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_{n+2}(a_1 + [-a_2])z^n = \frac{\begin{aligned} & [(a_1 - a_2)^4 + 3a_1a_2(a_1 - a_2)^2 + a_1^2a_2^2]z^2 \\ & - S_1(-E)a_1a_2[(a_1 - a_2)^3 + 2a_1a_2(a_1 - a_2)]z^3 \\ & + S_2(-E)a_1^2a_2^2[(a_1 - a_2)^2 + a_1a_2]z^4 - a_1^3a_2^3(a_1 - a_2)S_3(-E)z^5 \end{aligned}}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}, \quad (3.19)$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n+2}(a_1 + [-a_2])z^n = \frac{(a_1 - a_2)[(a_1 - a_2)^2 + 2a_1a_2]z - a_1a_2[(a_1 - a_2)^2 + a_1a_2]S_1(-E)z^2 + (a_1 - a_2)a_1^2a_2^2S_2(-E)z^3 - a_1^3a_2^3S_3(-E)z^4}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}. \quad (3.20)$$

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_{n+2}(a_1 + [-a_2])z^n = \frac{\begin{aligned} & [(a_1 - a_2)^4 + 3a_1a_2(a_1 - a_2)^2 + a_1^2a_2^2]z^2 \\ & - S_1(-E)a_1a_2[(a_1 - a_2)^3 + 2a_1a_2(a_1 - a_2)]z^3 \\ & + S_2(-E)a_1^2a_2^2[(a_1 - a_2)^2 + a_1a_2]z^4 - a_1^3a_2^3(a_1 - a_2)S_3(-E)z^5 \end{aligned}}{\prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z)}, \quad (3.21)$$

where

$$\begin{aligned} \prod_{e \in E} (1 - ea_1z) \prod_{e \in E} (1 + ea_2z) &= 1 + (a_1 - a_2)S_1(-E)z + [(a_1 - a_2)^2S_2(-E) \\ &\quad - a_1a_2(S_1^2(-E) - 2S_2(-E))]z^2 \\ &\quad + [(a_1 - a_2)^3S_3(-E) - a_1a_2(a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E))]z^3 \\ &\quad - [a_1a_2(a_1 - a_2)^2S_3(-E)S_1(-E) - a_1^2a_2^2(S_2^2(-E) - 2S_3(-E)S_1(-E))]z^4 \\ &\quad + a_1^2a_2^2(a_1 - a_2)S_3(-E)S_2(-E)z^5 - a_1^3a_2^3S_3^2(-E)z^6. \end{aligned}$$

#### 4. Novel generating functions for products of Gaussian numbers with $(p, q)$ -numbers at consecutive and nonconsecutive index terms

In this section, we develop new generating functions for the products of Gaussian numbers and  $(p, q)$ -numbers, considering both consecutive and nonconsecutive indices terms. These constructions generalize existing results and provide a rigorous framework for establishing new identities and investigating their inherent properties.

**Proposition 4.1** *For  $n \in \mathbb{N}$ , we present a newly proposed generating function for the product of Gaussian Pell-Padovan numbers with  $(p, q)$ -Fibonacci numbers as follows*

$$\sum_{n=0}^{\infty} GR_n F_{p,q,n+1} t^n = \frac{(1-i) + (1+i)pt + [(-p^2 - 3q) + i(3p^2 + 5q)]t^2 + (-1+i)pqt^3 + (3-5i)q^2t^4}{1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6}. \quad (4.1)$$

**Proof:** By citing [14] and [15], we have

$$\begin{aligned} GR_n &= (1-i)S_n(E) + (1+i)S_{n-1}(E) + (-1+3i)S_{n-2}(E), \\ F_{p,q,n+1} &= S_n(a_1 + [-a_2]). \end{aligned}$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} GR_n F_{p,q,n+1} t^n &= \sum_{n=0}^{\infty} [(1-i)S_n(E) + (1+i)S_{n-1}(E) + (-1+3i)S_{n-2}(E)] S_n(a_1 + [-a_2]) t^n \\ &= (1-i) \sum_{n=0}^{\infty} S_n(E) S_n(a_1 + [-a_2]) t^n + (1+i) \sum_{n=0}^{\infty} S_{n-1}(E) S_n(a_1 + [-a_2]) t^n + \\ &\quad (-1+3i) \sum_{n=0}^{\infty} S_{n-2}(E) S_n(a_1 + [-a_2]) t^n, \end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} GR_n F_{p,q,n+1} t^n = \frac{(1-i) + (1+i)pt + [(-p^2 - 3q) + i(3p^2 + 5q)]t^2 + (-1+i)pqt^3 + (3-5i)q^2t^4}{1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6}.$$

This complete the proof.  $\square$

**Proposition 4.2** *Let  $n \in \mathbb{N}$ . The new generating function corresponding to the product of Gaussian Pell Padovan numbers with  $(p, q)$ -Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} GR_n F_{p,q,n+2} t^n = \frac{(1-i)p + (1+i)(p^2 + q)t + (-1+3i)p(p^2 + 2q)t^2 - (1+3i)q^2t^3 + (2-6i)pq^2t^4 + (-1+3i)q^3t^5}{1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6}. \quad (4.2)$$

**Proof:** According to [15], we have

$$F_{p,q,n+2} = pF_{p,q,n+1} + qF_{p,q,n}.$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} GR_n F_{p,q,n+2} t^n &= \sum_{n=0}^{\infty} GR_n \cdot (pF_{p,q,n+1} + qF_{p,q,n}) t^n \\ &= p \sum_{n=0}^{\infty} GR_n \cdot F_{p,q,n+1} t^n + q \sum_{n=0}^{\infty} GR_n \cdot F_{p,q,n} t^n, \end{aligned}$$

On the other hand, we have (see [10] )

$$\sum_{n=0}^{\infty} GR_n F_{p,q,n} t^n = \frac{(1+i)t + p(1+i)t^2 + [(p^2 - q) + i(-p^2 - 3q)]t^3 - pq(1+i)t^4 + q^2(-1+3i)t^5}{1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6}.$$

After performing simple calculations, we get the result.  $\square$

**Proposition 4.3** *For every natural number  $n$ , the new generating function corresponding to the product of Gaussian Perrin numbers with  $(p, q)$ -Fibonacci numbers is introduced as follows*

$$\sum_{n=0}^{\infty} Gr_n F_{p,q,n+1} t^n = \frac{(-1+3i) + 3pt + [(p^2 + 2q) + i(-p^2 - 4q)]t^2 + (1-3i)pqt^3 + (2+i)q^2t^4}{1 - (p^2 + 2q)t^2 - (p^3 + 3pq)t^3 + q^2t^4 + pq^2t^5 - q^3t^6}. \quad (4.3)$$

**Proof:** Using [14] and [15], we have

$$\begin{aligned} Gr_n &= (-1 + 3i)S_n(E) + 3S_{n-1}(E) + (1 - i)S_{n-2}(E), \\ F_{p,q,n+1} &= S_n(a_1 + [-a_2]). \end{aligned}$$

The proof follows the same steps as in Proposition 4.1.  $\square$

**Proposition 4.4** *Let us consider  $n \in \mathbb{N}$ . The new generating function of the product of Gaussian Pell Padovan numbers with  $(p, q)$ -Fibonacci numbers is defined by*

$$\sum_{n=0}^{\infty} Gr_n F_{p,q,n+2} t^n = \frac{(-1 + 3i)p + 3(p^2 + q)t + (1 - i)p(p^2 + 2q)t^2 + [-4q^2 + 3iq^2]t^3 + (-1 + i)pq^2t^4 + (1 - i)q^3t^5}{1 - (p^2 + 2q)t^2 - (p^3 + 3pq)t^3 + q^2t^4 + pq^2t^5 - q^3t^6}, \quad (4.4)$$

**Proof:** By citing [15], we have

$$F_{p,q,n+2} = pF_{p,q,n+1} + qF_{p,q,n}.$$

Then, (see [10])

$$\sum_{n=0}^{\infty} Gr_n F_{p,q,n} t^n = \frac{3t + 2pit^2 + [(-p^2 - 4q) + (3p^2 + 3q)i]t^3 - 3pqt^4 + q^2(1 - i)t^5}{1 - (p^2 + 2q)t^2 - (p^3 + 3pq)t^3 + q^2t^4 + pq^2t^5 - q^3t^6}.$$

Using the same proof as in Proposition 4.2, we obtain the result.  $\square$

**Corollary 4.1** *By setting  $(p = k, q = 1)$ , we obtain new generating functions for product of Gaussian numbers with  $k$ -Fibonacci numbers.*

$$\sum_{n=0}^{\infty} GR_n F_{k,n+1} t^n = \frac{(1 - i) + (1 + i)kt + [(-k^2 - 3) + i(3k^2 + 5)]t^2 + (-1 + i)kt^3 + (3 - 5i)t^4}{1 - (2k^2 + 4)t^2 - (k^3 + 3k)t^3 + 4t^4 + 2kt^5 - t^6}. \quad (4.5)$$

$$\sum_{n=0}^{\infty} GR_n F_{k,n+2} t^n = \frac{(1 - i)k + (1 + i)(k^2 + 1)t + (-1 + 3i)k(k^2 + 2)t^2 - (1 + 3i)t^3 + (2 - 6i)kt^4 + (-1 + 3i)t^5}{1 - (2k^2 + 4)t^2 - (k^3 + 3k)t^3 + 4t^4 + 2kt^5 - t^6}. \quad (4.6)$$

$$\sum_{n=0}^{\infty} Gr_n F_{k,n+1} t^n = \frac{(-1 + 3i) + 3kt + [(k^2 + 2) + i(-k^2 - 4)]t^2 + (1 - 3i)kt^3 + (2 + i)t^4}{1 - (k^2 + 2)t^2 - (k^3 + 3k)t^3 + t^4 + kt^5 - t^6}. \quad (4.7)$$

$$\sum_{n=0}^{\infty} Gr_n F_{k,n+2} t^n = \frac{(-1 + 3i)k + 3(k^2 + 1)t + (1 - i)k(k^2 + 2)t^2 + [-4 + 3i]t^3 + (-1 + i)kt^4 + (1 - i)t^5}{1 - (k^2 + 2)t^2 - (k^3 + 3k)t^3 + t^4 + kt^5 - t^6}. \quad (4.8)$$

**Remark 4.1** *By putting  $k = 1$  we obtain new generating functions for the products of Gaussian numbers with Fibonacci numbers as shown in Table 1.*

Table 1: New generating functions of the products of Gaussian numbers with Fibonacci numbers

Coefficient of $z^n$	Generating function
$GR_n F_{n+1}$	$\frac{(1-i)+(1+i)t+[-4+8i]t^2+(-1+i)t^3+(3-5i)t^4}{1-6t^2-4t^3+4t^4+2t^5-t^6}$
$GR_n F_{n+2}$	$\frac{(1-i)+2(1+i)t+3(-1+3i)t^2-(1+3i)t^3+(2-6i)t^4+(-1+3i)t^5}{1-6t^2-4t^3+4t^4+2t^5-t^6}$
$Gr_n F_{n+1}$	$\frac{(-1+3i)+3t+(3-5i)t^2+(1-3i)t^3+(2+i)t^4}{1-3t^2-4t^3+t^4+t^5-t^6}$
$Gr_n F_{n+2}$	$\frac{(-1+3i)+6t+3(1-i)t^2+[-4+3i]t^3+(-1+i)t^4+(1-i)t^5}{1-3t^2-4t^3+t^4+t^5-t^6}$

**Theorem 4.1** For  $n \in \mathbb{N}$ , we define the newly proposed generating function for the product of Gaussian Pell-Padovan numbers with  $(p, q)$ -Lucas numbers as follows

$$\sum_{n=0}^{\infty} GR_n L_{p,q,n+1} t^n = \frac{(1-i)p + (1+i)(p^2 + 2q)t + p[(-p^2 - q) + i(3p^2 + 7q)]t^2 - q[(-p^2 + 2q) + i(p^2 + 6q)]t^3 + (1-7i)pq^2t^4 + (-2+6i)q^3t^5}{1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6}. \quad (4.9)$$

**Proof:** Using [15], we have

$$L_{p,q,n+1} = 2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2]).$$

The same proof as in Proposition 4.1, applies □

**Proposition 4.5** Suppose that  $n \in \mathbb{N}$ . The new generating function for the product of Gaussian Pell Padovan numbers with  $(p, q)$ -Lucas numbers is introduced as follows

$$\sum_{n=0}^{\infty} GR_n L_{p,q,n+2} t^n = \frac{(1-i)(p^2 + 2q) + (1+i)(p^2 + 3q)pt + [(-p^4 - 4p^2q - 6q^2) + i(3p^4 + 12p^2q + 10q^2)]t^2 - (3+i)pq^2t^3 - q^2[(-2p^2 - 6q) + i(6p^2 + 10q)]t^4 + (-1+3i)pq^3t^5}{1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6}. \quad (4.10)$$

**Proof:** According to [15] and [10], we have

$$L_{p,q,n+2} = pL_{p,q,n+1} + qL_{p,q,n}.$$

$$\sum_{n=0}^{\infty} GR_n L_{p,q,n} t^n = \frac{2(1-i) + p(1+i)t + [(-3p^2 - 6p) + (5p^2 + 10q)i]t^2 + [(-p^3 - pq + (p^3 + 5pq)i)t^3 + [(p^2q + 6q^2) + (p^2q - 10q^2)i]t^4 - pq^2(-1+3i)t^5}{1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6}.$$

After performing simple calculations, we get the result (the same proof as Proposition 4.2) . □

**Proposition 4.6** For  $n \in \mathbb{N}$ , we introduce the new generating function for the product of Gaussian Perrin numbers with  $(p, q)$ -Lucas numbers as follows

$$\sum_{n=0}^{\infty} Gr_n L_{p,q,n+1} t^n = \frac{(-1+3i)p + (3p^2 + 6q)t - p[(-p^2 - 2q) + ip^2]t^2 + q[(-p^2 - 8q) + i(3p^2 + 6q)]t^3 + (-4+i)pq^2t^4 + (2-2i)q^3t^5}{1 - (p^2 + 2q)t^2 - (p^3 + 3pq)t^3 + q^2t^4 + pq^2t^5 - q^3t^6}. \quad (4.11)$$

**Proof:** The same proof as in Proposition 4.1 applies. □

**Proposition 4.7** Let us consider  $n \in \mathbb{N}$ . The new generating function for the product of Gaussian Pell Padovan numbers with  $(p, q)$ -Lucas numbers is presented as follows

$$\sum_{n=0}^{\infty} Gr_n L_{p,q,n+2} t^n = \frac{[(-p^2 - 2q) + i(3p^2 + 6q)] + (3p^3 + 9pq)t + [(p^4 + 4p^2q + 4q^2) + i(-8q^2 - 4p^2q - p^4)]t^2 + pq^2(-2-3i)t^3 + q^2[(4q - p^2) + i(p^2 + 2q)]t^4 + (1-i)pq^3t^5}{1 - (p^2 + 2q)t^2 - (p^3 + 3pq)t^3 + q^2t^4 + pq^2t^5 - q^3t^6}, \quad (4.12)$$

**Proof:** We have

$$\sum_{n=0}^{\infty} Gr_n L_{p,q,n} t^n = \frac{-2 + 6i + 3pt + [(2-4i)(p^2 + 2q)]t^2 + [(p^3 + 6pq) + (-3p^3 - 9pq)i]t^3 + [(4q^2 + 3p^2q) + 2q^2i]t^4 - pq^2(1-i)t^5}{1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6}, \quad (\text{See [10]}).$$

which follows by the same proof as in Proposition 4.2. □

**Corollary 4.2** *Setting ( $p = k$  and  $q = 1$ ) yields new generating functions for the product of Gaussian numbers and  $k$ -Lucas numbers.*

$$\sum_{n=0}^{\infty} GR_n L_{k,n+1} t^n = \frac{(1-i)k + (1+i)(k^2+2)t + k[(-k^2-1) + i(3k^2+7)]t^2 - [(-k^2+2) + i(k^2+6)]t^3 + (1-7i)kt^4 + (-2+6i)t^5}{1 - (2k^2+4)t^2 - (k^3+3k)t^3 + 4t^4 + 2kt^5 - t^6}. \quad (4.13)$$

$$\sum_{n=0}^{\infty} GR_n L_{k,n+2} t^n = \frac{(1-i)(k^2+2) + (1+i)(k^2+3)kt + [(-k^4-4k^2-6) + i(3k^4+12k^2+10)]t^2 - (3+i)kt^3 - [(-2k^2-6) + i(6k^2+10)]t^4 + (-1+3i)kt^5}{1 - (2k^2+4)t^2 - (k^3+3k)t^3 + 4t^4 + 2kt^5 - t^6}. \quad (4.14)$$

$$\sum_{n=0}^{\infty} Gr_n L_{k,n+1} t^n = \frac{(-1+3i)k + (3k^2+6)t - k[(-k^2-2) + ik^2]t^2 + [(-k^2-8) + i(3k^2+6)]t^3 + (-4+i)kt^4 + (2-2i)t^5}{1 - (k^2+2)t^2 - (k^3+3k)t^3 + t^4 + kt^5 - t^6}. \quad (4.15)$$

$$\sum_{n=0}^{\infty} Gr_n L_{k,n+2} t^n = \frac{[(-k^2-2) + i(3k^2+6)] + (3k^3+9k)t + [(k^4+4k^2+4) + i(-8-4k^2-k^4)]t^2 + k(-2-3i)t^3 + [(4-k^2) + i(k^2+2)]t^4 + (1-i)kt^5}{1 - (k^2+2)t^2 - (k^3+3k)t^3 + t^4 + kt^5 - t^6}. \quad (4.16)$$

**Remark 4.2** *By taking  $k = 1$ , we obtain new generating functions for the products of Gaussian numbers with Lucas numbers as show in Table 2.*

Table 2: New generating functions for the products of Gaussians numbers with Lucas numbers

Coefficient of $z^n$	Generating function
$GR_n L_{n+1}$	$\frac{(1-i)+3(1+i)t+(-2+i10i)t^2-(1+7i)t^3+(1-7i)t^4+(-2+6i)t^5}{1-6t^2-4t^3+4t^4+2t^5-t^6}$
$GR_n L_{n+2}$	$\frac{3(1-i)+4(1+i)t+(-11+25i)t^2-(3+i)t^3-(-8+16i)t^4+(-1+3i)t^5}{1-6t^2-4t^3+4t^4+2t^5-t^6}$
$Gr_n L_{n+1}$	$\frac{(-1+3i)+9t-(-3+i)t^2+(-9+9i)t^3+(-4+i)t^4+(2-2i)t^5}{1-3t^2-4t^3+t^4+t^5-t^6}$
$Gr_n L_{n+2}$	$\frac{-3+9i+12t+(9+3i)t^2+(-2-3i)t^3+(3+3i)t^4+(1-i)t^5}{1-3t^2-4t^3+t^4+t^5-t^6}$

**Proposition 4.8** *Let  $n \in \mathbb{N}$ . The newly proposed generating function for the product of Gaussian Pell-Padovan numbers with  $(p, q)$ -Pell Lucas numbers is given by*

$$\sum_{n=0}^{\infty} GR_n Q_{p,q,n+1} t^n = \frac{2(1-i)p + 2(1+i)(2p^2+q)t + p[(-8p^2-2q) + i(24p^2+14q)]t^2 + q[(4p^2+2q) + i(-4p^2-6q)]t^3 + (2-14i)pq^2t^4 + (-2+6i)q^3t^5}{1 - (p^2+4q)t^2 - (8p^3+6pq)t^3 + 4q^2t^4 + 4pq^2t^5 - q^3t^6}. \quad (4.17)$$

**Proof:** Using [15], we have

$$Q_{p,q,n+1} = 2S_n(a_1 + [-a_2] - 2pS_{n-1}(a_1 + [-a_2])).$$

which follows by the same proof as in Proposition 4.2.  $\square$

**Proposition 4.9** *Given  $n \in \mathbb{N}$ , the new generating function corresponding to the product of Gaussian Pell Padovan numbers with  $(p, q)$ -Pell Lucas numbers is introduced as follows*

$$\sum_{n=0}^{\infty} GR_n Q_{p,q,n+2} t^n = \frac{2(1-i)(2p^2+q) + 2(1+i)(4p^2+3q)pt + [(-16p^4-16p^2q-6q^2) + i(48p^4+48p^2q+10q^2)]t^2 - (6+2i)pq^2t^3 + [(8p^2q^2+6q^3) + i(-24p^2q^2-10q^3)]t^4 + (-2+6i)pq^3t^5}{1 - (p^2+4q)t^2 - (8p^3+6pq)t^3 + 4q^2t^4 + 4pq^2t^5 - q^3t^6}. \quad (4.18)$$

**Proof:** From [15] and [10], we get

$$Q_{p,q,n+2} = 2pL_{p,q,n+1} + qL_{p,q,n}.$$

$$\sum_{n=0}^{\infty} GR_n Q_{p,q,n} t^n = \frac{2(1-i) + 2p(1+i)t + [(-12p^2 - 6q) + i(20p^2 + 10q)]t^2 + [(-8p^3 - 2pq) + i(8p^3 + 10pq)]t^3 + [(4p^2q + 6q^2) + i(4p^2q - 10q^2)]t^4 - 2pq^2(-1 + 3i)t^5}{1 - (2p^2 + 4q)t^2 - (p^3 + 3pq)t^3 + 4q^2t^4 + 2pq^2t^5 - q^3t^6}.$$

After performing simple calculations, we obtain the result (the same proof as in Proposition 4.2 applies).  $\square$

**Proposition 4.10** *Given  $n \in \mathbb{N}$ , we define the new generating function corresponding to the product of Gaussian Perrin numbers with  $(p, q)$ -Pell Lucas numbers as follows*

$$\sum_{n=0}^{\infty} Gr_n Q_{p,q,n+1} t^n = \frac{(-2 + 6i)p + [(12p^2 + 6q) - 12p^2i]t + p[(8p^2 + 4q) - 8ipq]t^2 + [(-4p^2q - 8q^2) + i(12p^2q + 6q^2)]t^3 - (8 + 4i)pq^2t^4 + (2 - 2i)q^3t^5}{1 - (4p^2 + 2q)t^2 - (8p^3 + 6pq)t^3 + q^2t^4 + 2pq^2t^5 - q^3t^6}. \quad (4.19)$$

**Proof:** which follows by the same proof as in Proposition 4.1.  $\square$

**Proposition 4.11** *Let us consider  $n \in \mathbb{N}$ . The new generating function for the product of Gaussian Pell Padovan numbers with  $(p, q)$ -Pell Lucas numbers is given by*

$$\sum_{n=0}^{\infty} Gr_n Q_{p,q,n+2} t^n = \frac{[(-4p^2 - 2q) + i(12p^2 + 6q)] + [(24p^3 + 18pq) + i(-24p^3 - 6pq)]t + [(16p^4 + 16p^2q + 4q^2) + i(-16p^4 - 16p^2q - 8q^2)]t^2 - 4pq^2t^3 + q^2[(4q - 4p^2) + i(4p^2 + 2q)]t^4 + 2(1 - i)pq^3t^5}{1 - (4p^2 + 2q)t^2 - (8p^3 + 6pq)t^3 + q^2t^4 + 2pq^2t^5 - q^3t^6}, \quad (4.20)$$

**Proof:** We have

$$\sum_{n=0}^{\infty} Gr_n Q_{p,q,n} t^n = \frac{(-2 + 6i) + (6p - 6pi)t + [(8p^2 + 4q) + i(-16p^2 - 8q)]t^2 + [(8p^3 + 12pq) + i(-24p^3 - 12pq)]t^3 + [(12p^2q + 4q^2) + i(2q^2 + 12p^2q)]t^4 - 2pq^2(1 - i)t^5}{1 - (4p^2 + 2q)t^2 - (8p^3 + 6pq)t^3 + q^2t^4 + 2pq^2t^5 - q^3t^6}, \quad (\text{See [10]}).$$

The proof proceeds in the same manner as in Proposition 4.1.  $\square$

**Corollary 4.3** *From  $(p = k \text{ and } q = 1)$ , we obtain new generating functions for the product of Gaussian numbers with  $k$ -Pell Lucas numbers*

$$\sum_{n=0}^{\infty} GR_n Q_{k,n+1} t^n = \frac{2(1-i)k + 2(1+i)(2k^2 + 1)t + k[(-8k^2 - 2) + i(24k^2 + 14)]t^2 + [(4k^2 + 2) + i(-4k^2 - 6)]t^3 + (2 - 14i)kt^4 + (-2 + 6i)t^5}{1 - (k^2 + 4)t^2 - (8k^3 + 6k)t^3 + 4t^4 + 4kt^5 - t^6}. \quad (4.21)$$

$$\sum_{n=0}^{\infty} GR_n Q_{k,n+2} t^n = \frac{2(1-i)(2k^2 + 1) + 2(1+i)(4k^2 + 3)kt + [(-16k^4 - 16k^2 - 6) + i(48k^4 + 48k^2 + 10)]t^2 - (6 + 2i)kt^3 + [(8k^2 + 6) + i(-24k^2 - 10)]t^4 + (-2 + 6i)kt^5}{1 - (k^2 + 4)t^2 - (8k^3 + 6k)t^3 + 4t^4 + 4kt^5 - t^6}. \quad (4.22)$$

$$\sum_{n=0}^{\infty} Gr_n Q_{k,n+1} t^n = \frac{(-2 + 6i)k + [(12k^2 + 6) - 12k^2i]t + k[(8k^2 + 4) - 8ik]t^2 + [(-4k^2 - 8) + i(12k^2 + 6)]t^3 - (8 + 4i)kt^4 + (2 - 2i)t^5}{1 - (4k^2 + 2)t^2 - (8k^3 + 6k)t^3 + t^4 + 2kt^5 - t^6}. \quad (4.23)$$

$$\sum_{n=0}^{\infty} Gr_n Q_{k,n+2} t^n = \frac{[(-4k^2 - 2) + i(12k^2 + 6)] + [(24k^3 + 18k) + i(-24k^3 - 6k)]t + [(16k^4 + 16k^2 + 4) + i(-16k^4 - 16k^2 - 8)]t^2 - 4kt^3 + [(4 - 4k^2) + i(4k^2 + 2)]t^4 + 2(1 - i)kt^5}{1 - (4k^2 + 2)t^2 - (8k^3 + 6k)t^3 + t^4 + 2kt^5 - t^6}. \quad (4.24)$$

**Remark 4.3** For  $k = 1$ , we obtain new generating functions for the products of Gaussian numbers with Pell Lucas numbers, as presented in Table 3.

Table 3: New generating functions of the products of Gaussians numbers with Pell Lucas numbers

Coefficient of $z^n$	Generating function
$GR_n Q_{n+1}$	$\frac{2(1-i)+6(1+i)t+(-10+38i)t^2+(6-10i)t^3+(2-14i)t^4+(-2+6i)t^5}{1-5t^2-14t^3+4t^4+4t^5-t^6}$
$GR_n Q_{n+2}$	$\frac{6(1-i)+14(1+i)t+(-38+106i)t^2-(6+2i)t^3+(14-34i)t^4+(-2+6i)t^5}{1-5t^2-14t^3+4t^4+4t^5-t^6}$
$Gr_n Q_{n+1}$	$\frac{(-2+6i)+(18-12i)t+(12-8i)t^2+(-12+18i)t^3-(8+4i)t^4+(2-2i)t^5}{1-6t^2-14t^3+t^4+2t^5-t^6}$
$Gr_n Q_{n+2}$	$\frac{(-6+18i)+(42-30i)t+(32-40i)t^2-4t^3+6it^4+2(1-i)t^5}{1-6t^2-14t^3+t^4+2t^5-t^6}$

### 5. Novel generating functions of the products of Gaussian numbers with bivariate polynomials at consecutive and nonconsecutive indices

In this section, we propose and investigate novel generating functions associated with products of Gaussian numbers and bivariate polynomials at both consecutive and nonconsecutive indices. These generating functions are then used to establish new identities and relationships among the sequences under consideration.

**Proposition 5.1** For  $n \in \mathbb{N}$ , the new generating function for the product of Gaussian Pell Padovan numbers with bivariate Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty} GR_n F_{n+1}(x, y) t^n = \frac{(1 - i) + (1 + i)xt + [(-x^2 - 3y) + i(3x^2 + 5y)]t^2 + (-1 + i)xyt^3 + (3 - 5i)y^2t^4}{1 - (2x^2 + 4y)t^2 - (x^3 + 3xy)t^3 + 4y^2t^4 + 2xy^2t^5 - y^3t^6}. \quad (5.1)$$

**Proof:** According to [13], we have

$$F_{n+1}(x, y) = S_n(a_1 + [-a_2]).$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} GR_n F_{n+1}(x, y) t^n &= \sum_{n=0}^{\infty} [(1 - i)S_n(E) + (1 + i)S_{n-1}(E) + (-1 + 3i)S_{n-2}(E)] S_n(a_1 + [-a_2]) t^n \\ &= (1 - i) \sum_{n=0}^{\infty} S_n(E) S_n(a_1 + [-a_2]) t^n + (1 + i) \sum_{n=0}^{\infty} S_{n-1}(E) S_n(a_1 + [-a_2]) t^n \\ &\quad + (-1 + 3i) \sum_{n=0}^{\infty} S_{n-2}(E) S_n(a_1 + [-a_2]) t^n, \end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} GR_n F_{p,q,n+1} t^n = \frac{(1 - i) + (1 + i)xt + [(-x^2 - 3y) + i(3x^2 + 5y)]t^2 + (-1 + i)xyt^3 + (3 - 5i)y^2t^4}{1 - (2x^2 + 4y)t^2 - (x^3 + 3xy)t^3 + 4y^2t^4 + 2xy^2t^5 - y^3t^6}.$$

□

**Proposition 5.2** *Let us consider  $n \in \mathbb{N}$ . The new generating function for the product of Gaussian Pell Padovan numbers with bivariate Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} GR_n F_{n+2}(x, y) t^n = \frac{(1-i)x + (1+i)(x^2 + y)t + (-1+3i)x(x^2 + 2y)t^2 - (1+3i)y^2 t^3 + (2-6i)xy^2 t^4 + (-1+3i)y^3 t^5}{1 - (2x^2 + 4y)t^2 - (x^3 + 3xy)t^3 + 4y^2 t^4 + 2xy^2 t^5 - y^3 t^6}. \quad (5.2)$$

**Proof:** By citing [13], we have

$$F_{n+2}(x, y) = xF_{n+1}(x, y) + yF_n(x, y).$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} GR_n F_{n+2}(x, y) t^n &= \sum_{n=0}^{\infty} GR_n \cdot (xF_{n+1}(x, y) + yF_n(x, y)) t^n \\ &= x \sum_{n=0}^{\infty} GR_n \cdot F_{n+1}(x, y) t^n + y \sum_{n=0}^{\infty} GR_n \cdot F_n(x, y) t^n, \end{aligned}$$

Moreover, we have ([10])

$$\sum_{n=0}^{\infty} GR_n F_n(x, y) t^n = \frac{(1+i)t + x(1+i)t^2 + [(x^2 - y) + i(-x^2 - 3y)]t^3 - xy(1+i)t^4 + y^2(-1+3i)t^5}{1 - (2x^2 + 4y)t^2 - (x^3 + 3xy)t^3 + 4y^2 t^4 + 2xy^2 t^5 - y^3 t^6}.$$

After a brief calculation, the result is obtained.  $\square$

**Proposition 5.3** *Let  $n \in \mathbb{N}$ , we define the new generating function for the product of Gaussian Perrin numbers with bivariate Fibonacci polynomials as follows*

$$\sum_{n=0}^{\infty} Gr_n F_{n+1}(x, y) t^n = \frac{(-1+3i) + 3xt + [(x^2 + 2y) + i(-x^2 - 4y)]t^2 + (1-3i)xyt^3 + (2+i)y^2 t^4}{1 - (x^2 + 2y)t^2 - (x^3 + 3xy)t^3 + y^2 t^4 + xy^2 t^5 - y^3 t^6}. \quad (5.3)$$

**Proof:** The proof follows the same steps as in Proposition 4.1.  $\square$

**Theorem 5.1** *Let us consider  $n \in \mathbb{N}$ . The newly proposed generating function for the product of Gaussian Pell Padovan numbers with bivariate Fibonacci polynomials is*

$$\sum_{n=0}^{\infty} Gr_n F_{n+2}(x, y) t^n = \frac{(-1+3i)x + 3(x^2 + y)t + (1-i)x(x^2 + 2y)t^2 + [-4y^2 + 3iy^2]t^3 + (-1+i)xy^2 t^4 + (1-i)y^3 t^5}{1 - (x^2 + 2y)t^2 - (x^3 + 3xy)t^3 + y^2 t^4 + xy^2 t^5 - y^3 t^6}; \quad (5.4)$$

**Proof:** Using [13], we have

$$F_{n+2}(x, y) = xF_{n+1}(x, y) + yF_n(x, y).$$

Then ([10])

$$\sum_{n=0}^{\infty} Gr_n F_n(x, y) t^n = \frac{3t + 2xit^2 + [(-x^2 - 4y) + (3x^2 + 3y)i]t^3 - 3xyt^4 + y^2(1-i)t^5}{1 - (x^2 + 2y)t^2 - (x^3 + 3xy)t^3 + y^2 t^4 + xy^2 t^5 - y^3 t^6}.$$

After a simple calculation, we have the result.  $\square$

**Proposition 5.4** For  $n \in \mathbb{N}$ , The newly proposed generating function for the product of Gaussian Pell Padovan numbers with bivariate Pell polynomials is given by

$$\sum_{n=0}^{\infty} GR_n P_{n+1}(x, y)t^n = \frac{(1-i) + (1+i)2xyt + [(-4x^2y^2 - 3y) + i(12x^2y^2 + 5y)]t^2 + (-1+i)2xy^2t^3 + (3-5i)y^2t^4}{1 - (8x^2y^2 + 4y)t^2 - (8x^3y^3 + 6xy^2)t^3 + 4y^2t^4 + 4xy^3t^5 - y^3t^6}. \quad (5.5)$$

**Proof:** By citing [13], we have

$$P_{n+1}(x, y) = S_n(a_1 + [-a_2]).$$

The proof follows the same steps as in Proposition 4.1.  $\square$

**Proposition 5.5** Assuming that  $n \in \mathbb{N}$ , we present a new generating function for the product of Gaussian Pell Padovan numbers with bivariate Pell polynomials as

$$\sum_{n=0}^{\infty} GR_n P_{n+2}(x, y)t^n = \frac{(1-i)2xy + (1+i)(4x^2y^2 + y)t + (-1+3i)2xy(4x^2y^2 + 2y)t^2 - (1+3i)y^2t^3 + (2-6i)2xy^3t^4 + (-1+3i)y^3t^5}{1 - (8x^2y^2 + 4y)t^2 - (8x^3y^3 + 6xy^2)t^3 + 4y^2t^4 + 4xy^3t^5 - y^3t^6}. \quad (5.6)$$

**Proof:** According to [13], we have

$$P_{n+2}(x, y) = 2xyF_{n+1}(x, y) + yF_n(x, y).$$

In addition, we have ([10])

$$\sum_{n=0}^{\infty} GR_n P_n(x, y)t^n = \frac{(1+i)t + 2xy(1+i)t^2 + [(4x^2y^2 - y) + i(-4x^2y^2 - 3y)]t^3 - 2xy^2(1+i)t^4 + y^2(-1+3i)t^5}{1 - (8x^2y^2 + 4y)t^2 - (8x^3y^3 + 6xy^2)t^3 + 4y^2t^4 + 4xy^3t^5 - y^3t^6}.$$

After a simple calculation, we have the result.  $\square$

**Proposition 5.6** Suppose that  $n \in \mathbb{N}$ , then the newly proposed generating function for the product of Gaussian Perrin numbers with bivariate Pell polynomials is given by

$$\sum_{n=0}^{\infty} Gr_n P_{n+1}(x, y)t^n = \frac{(-1+3i) + 6xyt + [(4x^2y^2 + 2y) + i(-4x^2y^2 - 4y)]t^2 + (1-3i)2xy^3t^3 + (2+i)y^2t^4}{1 - (4x^2y^2 + 2y)t^2 - (8x^3y^3 + 6xy^2)t^3 + y^2t^4 + 2xy^3t^5 - y^3t^6}. \quad (5.7)$$

**Proof:** The proof proceeds in the same manner as in Proposition 4.1.  $\square$

**Proposition 5.7** Assume that  $n \in \mathbb{N}$ . The new generating function for the product of Gaussian Pell Padovan numbers with bivariate Pell polynomials is given by

$$\sum_{n=0}^{\infty} Gr_n P_{n+2}(x, y)t^n = \frac{(-1+3i)2xy + 3(4x^2y^2 + y)t + (1-i)2xy(4x^2y^2 + 2y)t^2 + [-4y^2 + 3iy^2]t^3 + (-1+i)2xy^3t^4 + (1-i)y^3t^5}{1 - (4x^2y^2 + 2y)t^2 - (8x^3y^3 + 6xy^2)t^3 + y^2t^4 + 2xy^3t^5 - y^3t^6}. \quad (5.8)$$

**Proof:** From [13], we have

$$P_{n+2}(x, y) = 2xyF_{n+1}(x, y) + yF_n(x, y).$$

Note that

$$\sum_{n=0}^{\infty} Gr_n P_n(x, y)t^n = \frac{3t + 4xyit^2 + [(-4x^2y^2 - 4y) + (12x^2y^2 + 3y)i]t^3 - 6xy^2t^4 + y^2(1-i)t^5}{1 - (4x^2y^2 + 2y)t^2 - (8x^3y^3 + 6xy^2)t^3 + y^2t^4 + 2xy^3t^5 - y^3t^6}. \quad (\text{see [10]})$$

A simple calculation yields the result.  $\square$

**Proposition 5.8** Assume that  $n \in \mathbb{N}$ , we define the newly proposed generating function for the product of Gaussian Pell Padovan numbers with Jacobsthal bivariate polynomials, given by

$$\sum_{n=0}^{\infty} GR_n J_{n+1}(x, y) t^n = \frac{(1-i) + (1+i)xyt + [(-x^2y^2 - 6y) + i(3x^2y^2 + 10y)]t^2 + 2(-1+i)xy^2t^3 + 4(3-5i)y^2t^4}{1 - (2x^2y^2 + 8y)t^2 - (x^3y^3 + 6xy^2)t^3 + 16y^2t^4 + 8xy^3t^5 - 8y^3t^6}. \quad (5.9)$$

**Proof:** By citing [13], we have

$$J_{n+1}(x, y) = S_n(a_1 + [-a_2]).$$

The proof proceeds in the same manner as in Proposition 4.1.  $\square$

**Proposition 5.9** Let  $n \in \mathbb{N}$ . The new generating function for the product of the product of Gaussian Pell Padovan numbers with Jacobsthal bivariate polynomials is presented by

$$\sum_{n=0}^{\infty} GR_n J_{n+2}(x, y) t^n = \frac{(1-i)xy + (1+i)(x^2y^2 + 2y)t + (-1+3i)xy(x^2y^2 + 4y)t^2 - 4(1+3i)y^2t^3 + 4(2-6i)xy^3t^4 + 8(-1+3i)y^3t^5}{1 - (2x^2y^2 + 8y)t^2 - (x^3y^3 + 6xy^2)t^3 + 16y^2t^4 + 8xy^3t^5 - 8y^3t^6}. \quad (5.10)$$

**Proof:** Using [13], we have

$$J_{n+2}(x, y) = xyJ_{n+1}(x, y) + 2yJ_n(x, y).$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} GR_n J_{n+2}(x, y) t^n &= \sum_{n=0}^{\infty} GR_n \cdot (xyJ_{n+1}(x, y) + 2yJ_n(x, y)) t^n \\ &= xy \sum_{n=0}^{\infty} GR_n \cdot J_{n+1}(x, y) t^n + 2y \sum_{n=0}^{\infty} GR_n \cdot J_n(x, y) t^n, \end{aligned}$$

Similarly, we have ([10])

$$\sum_{n=0}^{\infty} GR_n J_n(x, y) t^n = \frac{(1+i)t + xy(1+i)t^2 + [(x^2y^2 - 2y) + i(-x^2y^2 - 6y)]t^3 - 2xy^2(1+i)t^4 + 4y^2(-1+3i)t^5}{1 - (2x^2y^2 + 8y)t^2 - (x^3y^3 + 6xy^2)t^3 + 16y^2t^4 + 8xy^3t^5 - 8y^3t^6}.$$

After straightforward computations, we obtain the result.  $\square$

**Proposition 5.10** For  $n \in \mathbb{N}$ , the new generating function for the product of Gaussian Perrin numbers with Jacobsthal bivariate polynomials given by

$$\sum_{n=0}^{\infty} Gr_n J_{n+1}(x, y) t^n = \frac{(-1+3i) + 3xyt + [(x^2y^2 + 4y) + i(-x^2y^2 - 8y)]t^2 + 2(1-3i)xy^2t^3 + 4(2+i)y^2t^4}{1 - (x^2y^2 + 4y)t^2 - (x^3y^3 + 6xy^2)t^3 + 4y^2t^4 + 4y^3xt^5 - 8y^3t^6}. \quad (5.11)$$

**Proof:** By the same argument as in Proposition 4.1, the result follows.  $\square$

**Proposition 5.11** Let  $n \in \mathbb{N}$ . The novel generating function for the product of Gaussian Pell Padovan numbers with Jacobsthal bivariate polynomials is

$$\sum_{n=0}^{\infty} Gr_n J_{n+2}(x, y) t^n = \frac{(-1+3i)p + 3(p^2 + q)t + (1-i)p(p^2 + 2q)t^2 + [-4q^2 + 3iq^2]t^3 + (-1+i)pq^2t^4 + (1-i)q^3t^5}{1 - (x^2y^2 + 4y)t^2 - (x^3y^3 + 6xy^2)t^3 + 4y^2t^4 + 4y^3xt^5 - 8y^3t^6}. \quad (5.12)$$

**Proof:** We have ([10])

$$\sum_{n=0}^{\infty} Gr_n J_n(x, y) t^n = \frac{3t + 2xyit^2 + [(-x^2y^2 - 8y) + (3x^2y^2 + 6y)i]t^3 - 6xy^2t^4 + 4y^2(1-i)t^5}{1 - (x^2y^2 + 4y)t^2 - (x^3y^3 + 6xy^2)t^3 + 4y^2t^4 + 4y^3xt^5 - 8y^3t^6}.$$

After a simple calculation, the result follows.  $\square$

## 6. Conclusion

his study introduces a new theorem that enables the construction of generating functions for various Gaussian numbers and  $(p, q)$ -numbers in combination with bivariate polynomials. The results extend previous work and provide a unified framework for the study of generalized sequences and their analytical properties. Future research may examine potential applications of these generating functions in areas such as complex analysis and differential equations, as well as their connections with other classes of special polynomials and complex numbers.

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