Bol. Soc. Paran. Mat. (3s.) **v. 2026 (44) 4** : 1–7.
©SPM – E-ISSN-2175-1188 ISSN-0037-8712
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.81530

Weakly Berwald Space with (α, β)-Metric

Sheelavathi Ramaswamy, Vasantha Dogehalli Mylarappa

ABSTRACT: Douglas spaces and Landsberg spaces are regarded as generalizations of Berwald spaces. S. Bacso proposed weakly Berwald space as one more general form of it. This paper provides a detailed insight into the verification of the conditions of a weakly Berwald space under which a Finsler space equipped with an special (α, β)-metric of the form $L(\alpha, \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$ reduces to a weakly-Berwald space, here α denotes Riemannian metric and β denotes differential 1-form.

Keywords: Finsler space, Berwald space, weakly-Berwald space, (α, β)-metric.

Contents

1. Introduction

A Berwald space is said to be Finsler space if it satisfies the condition $G^i_{hjk} = 0$ [1]. The notions of Douglas spaces, Landsberg spaces and weakly Berwald spaces are regarded as generalizations of Berwald spaces [2]. S.Bacso and B. Szilagyi [3] proposed the concept of a weakly Berwald space as a one of the general form of Berwald space. The idea of weakly Berwald spaces has been studied by several authors, like Il-Yong Lee and Myung-Han Lee [6], particularly in the context of various (α, β)-metrics ([4], [5], [7], [8]). A Finsler space equipped with an (α, β)-metric is said to be weakly Berwald space if and only if $B^m_m = \frac{\partial B^m}{\partial y^m}$ is a 1-form i.e., B^m_m is a homogeneous polynomial of degree one in y^i.

In this article, we verify the conditions $r_{ij} = 0$ and $s_j = 0$ [3] under which a Finsler space equipped with an (α, β)-metric of the form $L(\alpha, \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta},\ c_1 \neq 0$, becomes a weakly Berwald space.

2. Preliminaries

Let $F^n = (M^n, L)$ be the Finsler space for an $(\alpha,\ \beta)$-metric $L(\alpha,\ \beta)$, where

$$\alpha = (a_{ij}(x)y^i y^j)^{1/2},\ \beta = b_i(x)y^i,$$

is called Riemannian metric and β is a differential 1-form on the M^n. The geodesic spray coefficients G^i for F^n with $L(\alpha,\ \beta)$-metric is defined as,

$$2G^i = \gamma_0{}^i{}_0 + 2B^i,$$

where, $\gamma_j{}^i{}_k$ are the Christoffel symbols of the Levi-Civita connection with α and $\gamma_0{}^i{}_0 = \gamma_j{}^i{}_k\ y^j y^k$. From this, we have the following derived quantities:

$$G\,{}^i_j = \frac{\partial G^i}{\partial y^j} = \gamma_0{}^i{}_j + B^i{}_j,\ G_j{}^i{}_k = \frac{\partial^2 G^i}{\partial y^j \partial y^k} = \gamma_j{}^i{}_k + B_j{}^i{}_k,$$

2020 *Mathematics Subject Classification:* 53B40, 53C60.
Submitted January 24, 2026. Published February 26, 2026

where, the notations $B^i{}_j = \dot{\partial}_j B^i$ and $B_j{}^i{}_k = \dot{\partial}_k B^i{}_j$ are used and $\dot{\partial}_j = \partial/\partial y^j$ denotes the derivative with respect to the directional argument y^j.
We now introduce the information listed below [7]:

$$\begin{aligned}
(a)\quad r_{ij} =& \frac{1}{2}(b_{i:j} + b_{j:i}),\ r^i_j = a^{ih} r_{hj},\ r_j = b_i r^i_j \\
(b)\quad s_{ij} =& \frac{1}{2}(b_{i:j} - b_{j:i}),\ s^i_j = a^{ih} s_{hj},\ s_j = b_i s^i_j \\
(c)\quad b^i =& a^{ih} b_h,\ b^2 = b^i b_i,
\end{aligned}$$

where, the symbol (:) denotes the h-covariant derivative with respect to the Riemannian connection in the space $(M^n,\ \alpha)$ [7].
Now, consider the relation of G^m to F^n with an $(\alpha,\ \beta)$-metric as,

$$2G^m = \gamma_0{}^m{}_0 + 2B^m, \tag{2.1}$$

where

$$B^m = \left(\frac{E^*}{\alpha}\right) y^m + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^m_0 - \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right) C^* \left\{ \left(\frac{y^m}{\alpha}\right) - \left(\frac{\alpha}{\beta}\right) b^m \right\}, \tag{2.2}$$

where, we put [3]

$$\begin{aligned}
E^* =& \left(\frac{\beta L_\beta}{L}\right) C^*, \\
C^* =& \frac{\{\alpha\beta\,(r_{00} L_\alpha - 2\alpha s_0 L_\beta)\}}{\{2\,(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})\}}, \\
\gamma^2 =& b^2\alpha^2 - \beta^2.
\end{aligned} \tag{2.3}$$

We have the following [6]:

$$B^m_m = \frac{1}{2\alpha L(\beta L_\alpha)^2 \Omega^2} \left\{ 2\Omega^2 A C^* + 2\alpha L \Omega^2 B s_0 + \alpha^2 L L_\alpha L_{\alpha\alpha}(C r_{00} + D s_0 + E r_0) \right\}, \tag{2.4}$$

where

$$\begin{aligned}
A &= (1+n) L_\alpha (\beta^3 L_\alpha L_\beta - \alpha\beta^2 L L_{\alpha\alpha}) + \alpha\gamma^2 L\{\alpha(L_{\alpha\alpha})^2 - 2L_\alpha L_{\alpha\alpha} - \alpha L_\alpha L_{\alpha\alpha\alpha}\}, \\
B &= \alpha^2 L L_{\alpha\alpha}, \\
C &= \beta\gamma^2\{-\beta^2 (L_\alpha)^2 + 2b^2\alpha^3 L_\alpha L_{\alpha\alpha} - \alpha^2\gamma^2 (L_{\alpha\alpha})^2 + \alpha^2\gamma^2 L_\alpha L_{\alpha\alpha\alpha}\}, \\
D &= 2\alpha\{\beta^3(\gamma^2 - \beta^2) L_\alpha L_\beta - \alpha^2\beta^2\gamma^2 L_\alpha L_{\alpha\alpha} - 2\alpha\beta\gamma^2(\gamma^2 + 2\beta^2) L_\beta L_{\alpha\alpha} \\
&\quad - \alpha^3\gamma^4 (L_{\alpha\alpha})^2 - \alpha^2\beta\gamma^4 L_\beta L_{\alpha\alpha\alpha}\}, \\
E &= 2\alpha^2\beta^2 L_\alpha \Omega.
\end{aligned} \tag{2.5}$$

Also we use the following statements:

Theorem 2.1 *[3] A Finsler space F^n with an $(\alpha,\ \beta)$-metric is a weakly Berwald space if and only if $G^m_m = \gamma^m_m + B^m_m$ and B^m_m is a homogeneous polynomial of degree one in the directional variables (y^m). The expression for B^m_m is given by the equations (2.4) and (2.5), such that $\Omega \neq 0$.*

Lemma 2.1 *[3] If $\alpha^2 \equiv 0 (mod \beta)$, that is, if the expression $a_{ij} y^i y^j$ contains $\beta = b_i(x) y^i$ as a factor, then the manifold must be two-dimensional, and $b^2 = a^{ij} b_i b_j$ vanishes. In this case, there exists a 1-form $\delta = d_i(x) y^i$ such that $\alpha^2 = \beta\delta$ and the condition $d_i b^i = 2$ holds.*

3. Conditions of a Weakly-Berwald Space for an $(\alpha,\ \beta)$-Metric

Theorem 3.1 *A Finsler space $F^n = (M^n, L)$ for an $(\alpha,\ \beta)$-metric, $L(\alpha,\ \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$ is a weakly-Berwald space iff the conditions $r_{ij} = 0$ and $s_j = 0$ are satisfied.*

Proof: : Consider the metric

$$L(\alpha,\ \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta},\ \textit{where}\ c_1 \neq 0. \tag{3.1}$$

From (3.1), we have

$$L_\alpha = c_1\alpha + \frac{2\alpha}{\beta},\quad L_\beta = c_2 - \frac{\alpha^2}{\beta^2},\quad L_{\alpha\alpha} = \frac{2}{\beta},\quad L_{\alpha\alpha\alpha} = 0. \tag{3.2}$$

Now consider

$$B^m = \left(\frac{E^*}{\alpha}\right) y^m + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^m_{\ 0} - \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right) C^* \left\{\left(\frac{y^m}{\alpha}\right) - \left(\frac{\alpha}{\beta}\right) b^m\right\}, \tag{3.3}$$

where

$$\begin{aligned} C^* &= \frac{\alpha P}{2Q}, \\ E^* &= \frac{\alpha(c_2\beta^2 - \alpha^2)P}{2(c_1\alpha\beta + c_2\beta^2 + \alpha^2)Q}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} P &= \beta r_{00}(c_1\beta + 2\alpha) - 2\alpha s_0(c_2\beta^2 - \alpha^2), \\ Q &= c_1\beta^3 + 2\alpha^3 b^2. \end{aligned} \tag{3.5}$$

Substituting (3.2), (3.4), (3.5), in (3.3), we get

$$\begin{aligned} B^m = &\left(\frac{\beta r_{00}(c_1\beta + 2\alpha) - 2\alpha s_0(c_2\beta^2 - \alpha^2)}{(c_1\beta^3 + 2\alpha^3 b^2)(c_1\beta + 2\alpha)}\right)\left\{\frac{\alpha^3}{\beta} b^m + \frac{c_1c_2\beta^3 - 3c_1\alpha^2\beta - 4\alpha^3}{2(c_1\alpha\beta + c_2\beta^2 + \alpha^2)} y^m\right\} \\ &+ \left(\frac{\alpha(c_2\beta^2 - \alpha^2)}{\beta(c_1\beta + 2\alpha)}\right) s^m_0. \end{aligned} \tag{3.6}$$

Substituting (3.2) in the equations of (2.5), we get

$$\begin{aligned} E &= 2\alpha^2(c_1\beta^3 + 2\alpha^3 b^2)(c_1\beta + 2\alpha), \\ A &= \frac{1}{\beta^3}\Big[\beta^2(n+1)(c_1\beta + 2\alpha)(c_1c_2\beta^3 - 3c_1\alpha^2\beta - 4\alpha^3) - 4\alpha(\alpha + c_1\beta)(b^2\alpha^2 - \beta^2)(c_1\alpha\beta + c_2\beta^2 + \alpha^2)\Big], \\ B &= 2\alpha^2\left\{\frac{c_1\alpha\beta + c_2\beta^2 + \alpha^2}{\beta^2}\right\}, \\ C &= \left(\frac{b^2\alpha^2 - \beta^2}{\beta}\right)\left\{4c_1\alpha\beta(b^2\alpha^2 - \beta^2) + 4b^2\alpha^4 - c_1^2\beta^4\right\}, \\ D &= 2\alpha\Big[(b^2\alpha^2 - 2\beta^2)(c_1\beta + 2\alpha)(c_2\beta^2 - \alpha^2) - 2\alpha^2(b^2\alpha^2 - \beta^2)(c_1\beta + 2\alpha) \\ &\quad - \frac{4\alpha}{\beta^2}\left\{(b^2\alpha^2 + \beta^2)(c_2\beta^2 - \alpha^2) - \alpha^2(b^2\alpha^2 - \beta^2)\right\}\Big]. \end{aligned}$$

Thus from the above equations, (2.4) reduces to

$$
\begin{aligned}
&B_m^m\left\{\alpha^4\beta^6(4c_1^4b^2+16c_1^2c_2b^2+4c_1^2)+\alpha^6\beta^6(32c_1^2b^2+4c_1^2c_2b^4)+\alpha^2\beta^{10}(5c_1^4+4c_1^2c_2)\right.\\
&+\alpha^8\beta^4(20c_1^2b^4+16c_1b^4)+c_1^4c_2\beta^{12}+\alpha\beta^{11}(c_1^5+4c_1^3c_2)+\alpha^7\beta^5(4c_1^3b^4+16c_1c_2b^4+16c_1b^2)\\
&\left.+\alpha^3\beta^9(8c_1^3+4c_1^4b^2)+\alpha^9\beta^3(32c_1b^4)+\alpha^4\beta^7(20c_1^3b^2+16c_1c_2b^2)+16\alpha^{10}\beta^2b^4\right\}\\
&=r_{00}\left\{\alpha\beta^{11}((n+1)c_1^4c_2)+\alpha^3\beta^9(c_1^4(1-3n)+4c_1^2c_2(n+6)+2c_1^3+4c_1c_2)\right.\\
&+\alpha^5\beta^7((n+16)c_1^2-4c_1^4b^2-20c_1^2c_2b^2-2c_1^3b^2-4c_1c_2b^2+4c_1+4n+4c_1^2c_2)+\alpha^7\beta^5(c_1^2b^2(-32n-80))\\
&+\alpha^9\beta^3(-32(n+1)b^2)+\alpha^4\beta^8(-16c_1^3(n+1)+24c_1^3+c_1^3c_2b^2(2n-2)+24c_1c_2+4c_1^2-2c_1^2c_2b^2)\\
&+\alpha^2\beta^{10}(4c_1^3c_2(n+2)+2c_1^2c_2)+\alpha^6\beta^6(c_1^3b^2(-6n-30)-c_1(16n-8)+c_1c_2b^2(8n-16)-6c_1^2b^2)\\
&\left.+\alpha^8\beta^4(c_1b^2(-56n-80))+\alpha^{10}\beta^2(16c_1b^4-16c_2b^4)\right\}\\
&+s_0\left\{\alpha^4\beta^9(c_1^3c_2(2n+8)-8c_1c_2^2-8c_1^4-32c_1^2c_2)+\alpha^6\beta^7(c_1^3(-6n+18)+8c_1^3c_2b^2+c_1^4b^2+16c_1^2c_2^2b^2\right.\\
&-64c_1^2+c_1c_2(16n+96)-112c_2b^2)+\alpha^{10}\beta^3((-50n-170)c_1b^2+48c_1c_2b^2-32c_1)\\
&+\alpha^8\beta^5((40n-20)c_1c_2b^2-(16n-152)c_1+8c_1c_2-16c_1^2b^2+16c_1^2c_2^2b^4-28c_1^3b^2+24c_1^2b^2)\\
&+\alpha^{12}\beta(16c_1b^4+32c_1)+\alpha^{10}\beta^7(16c_1c_2b^4)+\alpha^8\beta^4(-16c_1^3b^2-32c_1c_2b^2)+\alpha^8\beta^2(16c_1c_2b^2)+\alpha^{10}\beta(48c_1b^2)\\
&+\alpha^6\beta^5(-48c_1^2+16c_1c_2)+\alpha^8\beta^3(64c_1+16c_1c_2b^2)+\alpha^4\beta^7(-16c_1^2c_2)+\alpha^{10}\beta(-32c_1b^2)\\
&+\alpha^2\beta^{11}(-2(n+1)c_1^3c_2^2)+\alpha^7\beta^6(c_1^2c_2b^2(16n+18)-(20n-70)c_1^2+12c_1^2c_2b^2-16c_2^2b^2+20c_1^3b^2\\
&+c_1c_2b^2+96c_2-32c_1)+\alpha^5\beta^8(c_1^2c_2^2b^2(-4n+4)+c_1^2c_2(8n+32)+4c_1^2c_2b^2-32c_1c_2-40c_1^3+4c_1^3)\\
&+\alpha^9\beta^4((-12n-56)c_1^2b^2+32(n+1)c_2b^2+32c_2^2b^4+16c_1^2c_2b^4+96)\\
&+\alpha^3\beta^{10}((-4n-8)c_1^2c_2^2-8c_1^3c_2)+\alpha^{11}\beta^2(-32(n+1)b^2+48c_2b^4-16c_1^2b^4+16c_1^2b^2-96b^2)\\
&+\alpha^5\beta^5(8c_1^2c_2)+\alpha^{14}(32b^4)+\alpha^7\beta^2(16c_2b^4)+\alpha^{11}\beta^6(16c_1^2b^4)+\alpha^7\beta^7(8c_1^3+16c_1^2-16c_1^2c_2b^2)\\
&\left.+\alpha^9\beta^2(16c_1^2b^2+32c_2b^2+64)+\alpha^5\beta^6(-16c_1^3-32c_1c_2)+\alpha^7\beta^4(16c_1^2+64c_1)+\alpha^9\beta^3(-48c_1^2b^2)+\alpha^{11}(32b^2)\right\}\\
&+r_0\left\{\alpha^6\beta^8(20c_1^3+16c_1c_2)+\alpha^8\beta^6(8c_1^3b^2+32c_1c_2b^2+16c_1)+\alpha^{10}\beta^4(64c_1b^2)\right.\\
&+\alpha^6\beta^7(8c_1^2c_2b^2)+\alpha^4\beta^{10}(4c_1^3c_2)+\alpha^7\beta^7(32c_1^2)+\alpha^5\beta^9(4c_1^4+16c_1^2c_2)+\alpha^9\beta^5(40c_1^2b^2+32c_2b^2)\\
&\left.+\alpha^{11}\beta^3(32b^2)\right\}.
\end{aligned}
$$

Separating rational and irrational parts, we get

$$
\begin{aligned}
&F_1\beta^2B_m^m+r_{00}\beta^2G_1+\alpha^2s_0H_1+r_0\alpha^4\beta^2I_1=0,\\
&F_2\beta^2B_m^m+r_{00}\beta^2G_2+\alpha^2s_0H_2+r_0\alpha^4\beta^2I_2=0,
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
F_1&=\beta^9(c_1^5+4c_1^3c_2)+\alpha^6\beta^3(4c_1^3b^4+16c_1c_2b^4+16c_1b^2)+\alpha^2\beta^7(8c_1^3+4c_1^4b^2)\\
&\quad+\alpha^8\beta(32c_1b^4)+\alpha^4\beta^5(20c_1^3b^2+16c_1c_2b^2)+16\alpha^9b^4,\\
F_2&=\alpha^4\beta^3(4c_1^4b^2+16c_1^2c_2b^2+4c_1^2+\alpha^6\beta^3(32c_1^2b^2+4c_1^2c_2b^4)\\
&\quad+\alpha^2\beta^7(5c_1^4+4c_1^2c_2)+\alpha^8\beta(20c_1^2b^4+16c_1b^4)+c_1^4c_2\beta^9,\\
G_1&=-\alpha^3\beta^6(-16c_1^3(n+1)+24c_1^3+c_1^3c_2b^2(2n-2)+24c_1c_2+4c_1^2-2c_1^2c_2b^2)\\
&\quad-\alpha\beta^8(4c_1^3c_2(n+2)+2c_1^2c_2)-\alpha^5\beta^4(c_1^3b^2(-6n-30)-c_1(16n-8)\\
&\quad+c_1c_2b^2(8n-16)-6c_1^2b^2)-\alpha^7\beta^2(c_1b^2(-56n-80))-\alpha^9(16c_1b^4-16c_2b^4),\\
G_2&=-\alpha\beta^8((n+1)c_1^4c_2)-\alpha^3\beta^6(c_1^4(1-3n)+4c_1^2c_2(n+6)+2c_1^3+4c_1c_2)\\
&\quad-\alpha^5\beta^4((n+16)c_1^2-4c_1^4b^2-20c_1^2c_2b^2-2c_1^3b^2-4c_1c_2b^2+4c_1+4n+4c_1^2c_2)\\
&\quad-\alpha^7\beta^2(c_1^2b^2(-32n-80))-\alpha^8(-32(n+1)b^2),
\end{aligned}
$$

$$
\begin{aligned}
H_1 =& -\alpha^4\beta^6(c_1^2c_2b^2(16n+18)-(20n-70)c_1^2+12c_1^2c_2b^2-16c_2^2b^2+20c_1^3b^2+c_1c_2b^2+96c_2-32c_1)\\
&-\alpha^2\beta^8(c_1^2c_2^2b^2(-4n+4)+c_1^2c_2(8n+32)+4c_1^2c_2b^2-32c_1c_2-40c_1^3+4c_1^3)\\
&-\alpha^6\beta^4((-12n-56)c_1^2b^2+32(n+1)c_2b^2+32c_2^2b^4+16c_1^2c_2b^4+96)\\
&-\beta^{10}((-4n-8)c_1^2c_2^2-8c_1^3c_2)-\alpha^8\beta^2(-32(n+1)b^2+48c_2b^4-16c_1^2b^4+16c_1^2b^2-96b^2)\\
&-\alpha^2\beta^5(8c_1^2c_2)-\alpha^{11}(32b^4)-\alpha^4\beta^2(16c_2b^4)-\alpha^8\beta^6(16c_1^2b^4)-\alpha^4\beta^7(8c_1^3+16c_1^2-16c_1^2c_2b^2)\\
&-\alpha^6\beta^2(16c_1^2b^2+32c_2b^2+64)-\alpha^2\beta^6(-16c_1^3-32c_1c_2)-\alpha^4\beta^4(16c_1^2+64c_1)\\
&-\alpha^6\beta^3(-48c_1^2b^2)-\alpha^8(32b^2),\\
H_2 =& -\alpha^2\beta^8(c_1^3c_2(2n+8)-8c_1c_2^2-8c_1^4-32c_1^2c_2)-\alpha^4\beta^6(c_1^3(-6n+18)+8c_1^3c_2b^2+c_1^4b^2\\
&+16c_1^2c_2^2b^2-64c_1^2+c_1c_2(16n+96)-112c_2b^2)-\alpha^8\beta^2((-50n-170)c_1b^2+48c_1c_2b^2-32c_1)\\
&-\alpha^6\beta^4((40n-20)c_1c_2b^2(16n-152)c_1+8c_1c_2-16c_1^2b^2+16c_1^2c_2^2b^4-28c_1^3b^2+24c_1^2b^2)\\
&-\alpha^{10}(16c_1b^4+32c_1)-\alpha^8\beta^6(16c_1c_2b^4)-\alpha^6\beta^3(-16c_1^3b^2-32c_1c_2b^2)-\alpha^6\beta(16c_1c_2b^2)-\alpha^8(48c_1b^2)\\
&-\alpha^4\beta^4(-48c_1^2+16c_1c_2)-\alpha^6\beta^2(64c_1+16c_1c_2b^2)-\alpha^2\beta^6(-16c_1^2c_2)\\
&-\alpha^8\beta(-32c_1b^2)-\beta^{10}(-2(n+1)c_1^3c_2^2),\\
I_1 =& -\alpha^2\beta^5(32c_1^2)-\beta^7(4c_1^4+16c_1^2c_2)-\alpha^4\beta^3(40c_1^2b^2+32c_2b^2)+\alpha^4\beta(32b^2),\\
I_2 =& -\alpha^2\beta^5(20c_1^3+16c_1c_2)-\alpha^4\beta^3(8c_1^3b^2+32c_1c_2b^2+16c_1)\\
&-\alpha^6\beta(64c_1b^2)-\alpha^2\beta^4(8c_1^2c_2b^2)-\beta^7(4c_1^3c_2).
\end{aligned}
$$

Now eliminate B^m_m from the above two equations, we get

$$\beta^2 r_{00}R+\alpha^2 s_0 S+\alpha^4\beta^2 r_0 T=0, \tag{3.8}$$

where

$$
\begin{aligned}
R &= G_1F_2-G_2F_1,\\
S &= H_1F_2-H_2F_1,\\
T &= I_1F_2-I_2F_1.
\end{aligned}
$$

Since the only term $16\alpha^{17}b^4$ of Ss_0 does not contains β, then we have $hp(17)V_{17}$ such that

$$\alpha^{17}s_0=\beta V_{17}.$$

We consider two different cases:
Case (i): Suppose if $\alpha^2\not\equiv 0(mod\ \beta)$ and $b^2\neq 0$ then there exits a function $k(x)$ satisfying $V_{17}=k\alpha^{17}$ and hence $s_0=k\beta$.
Then (3.8), reduced to

$$r_{00}R+\frac{\alpha^2}{\beta^2}Sk+\alpha^4 Tr_0=0. \tag{3.9}$$

Then only the term of (3.9) which is free from β is $\alpha^{18}(512b^4)$ such that

$$512b^4=\beta U_{19},$$

which is a contradiction $\therefore$ $k=0$, gives $s_0=0$, $\Rightarrow$ $s_j=0$.
Thus from the expression (3.9), we get

$$r_{00}R+\alpha^4 r_0 T=0. \tag{3.10}$$

In Rr_{00} the term which does not contain α^2 is $\beta^{17}r_{00}((n+1)c_1^4c_2(c_1^5+4c_1^3c_2)+(4c_1^3c_2(n+1)+2c_1^2c_2)c_1^4c_2)$, thus we have $hp(17)V_{17}$ such that $\beta^{17}r_{00}=\alpha^2V_{17}$.
Thus for $\alpha^2\not\equiv 0(mod\ \beta)$ there exists a function $f(x)$ such that

$$r_{00}=\alpha^2 f(x);\quad r_{ij}=a_{ij}f(x). \tag{3.11}$$

Transvecting (3.11) by $b^i y^j$, gives

$$r_0 = \beta f(x); \quad r_j = b_j f(x). \tag{3.12}$$

Using equations(3.11) and (3.12) in (3.10), we get

$$f(x)(R + \alpha^2 \beta T) = 0. \tag{3.13}$$

Let us assume that $f(x) \neq 0$, then the equation (3.13), gives

$$R + \alpha^2 \beta T = 0.$$

The term of $(R + \alpha^2 T)$ which does not contain β is $\alpha^{17}(512(n+1)b^6)$.
Thus there exits $hp(16)V_{16}$ satisfying $\alpha^{17}(512(n+1)b^6) = \beta V_{16}$, implies $V_{16} = 0$, provided $b^2 \neq 0$.
Hence $f(x) = 0$.

Gives $r_{00} = 0$, $r_{ij} = 0$, hence $r_0 = 0$ and $r_j = 0$.
Thus, since $r_{00} = 0$, $r_0 = 0$, $s_0 = 0 \quad \Rightarrow \quad B^m_m = 0$.
Hence, A F^n with the metric $L(\alpha,\ \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$ is a weakly Berwald space.

Conversely, if $r_{00} = 0$, $r_0 = 0$, and $s_0 = 0 \quad \Rightarrow \quad B^m_m = 0$ from (3.7). Thus a Finsler space with the metric $L(\alpha,\ \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$ is a weakly Berwald space.
Since the the Finsler space F^n with the metric $L(\alpha,\ \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$ is a weakly-Berwald Finsler space, then we have $r_{00} = 0$, $r_0 = 0$, and $s_0 = 0$, from the above discussions.
Substitute above in B^m, we get $B^m = 0$.

Case (ii): Suppose if $\alpha^2 \equiv 0(mod\ \beta)$, then $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$ where $\delta = d_i(x)y^i$.
Consider (3.8), substitution the above it reduces to

$$\beta^2 r_{00} R + \beta\delta s_0 S + \beta^4 \delta^2 r_0 T = 0. \tag{3.14}$$

In this equation the term which does not contain δ is $\beta^2 r_{00} R$, hence there is a $hp(1)V_1$ such that $\beta^2 r_{00} = \delta V_1$.
Thus equation (3.8), reduces to

$$\beta^2 r_{00} R + \alpha^2(S s_0 + \alpha^2 \beta^2 r_0 T) = 0,$$

$\Rightarrow \quad S s_0 + \alpha^2\beta^2 r_0 T = 0$ and $r_{00} = 0$.
Then there exists a function $g(x)$ such that $s_0 = \delta g(x) \ \Rightarrow \ s_i = d_i g(x)$.
Transvecting by b^i, we get $s_i b^i = d_i b^i g(x)$ and $d_i b^i = 2$ therefore $g(x) = 0$ hence $s_0 = 0$, then $r_0 = 0$.
Thus, we have $r_{00} = 0$, $r_0 = 0$, and $s_0 = 0$ therefore $B^m_m = 0$. □

References

1. P.L. Antonelli, *Handbook of Finsler Geometry,* kluwer academic publisher, London.
2. P.L. Antonelli, R.S. Ingarden and M. Matsumoto, *The theory of Sprays and Finsler spaces with applications in physics and biology,* Kluwer Acad. Dordrecht, (1993).
3. S. Bacso and B. Szilagyi, *On a weakly Berwald Finsler space of Kropina type,* Math.Pannon. **13**(2002), No. 1, 91-95.
4. S. Bacso and R. Yoshikawa, *Weakly-Berwald spaces,* Publ. Math. Debrecen, **61**(2002),No. 2, 219-231.
5. R. Yoshikawa and K. Okubo, *The conditions for some* (α, β)*-metric spaces to be weakly-Berwald spaces.* Proceedings of 38th symposium on Finsler geometry, (2003), 54-57.
6. Il-Yong Lee and Myung-Han Lee, *On weakly Berwald spaces of special* (α, β)*-metric,* Bull. Korean Math. Soc. **43**(2006), No. 2, 425-441.
7. Gauri Shanker and Deepti Choudhary, *On a new class of weakly spaces with* (α, β)*-metric,* British J. Math. **14**(2016), No. 6, 1-12.
8. V.D. Mylarappa and N.S. Kampalappa, *The study of Berwald connection of a Finsler space with a special* (α, β)*-Metric,* Advances in Mathematics: Scientific Journal, Vol. 9, No. 6(2020), 3221-3228.

Sheelavathi R.,
School of Applied Sciences,
Department of Mathematics,
REVA University, Yelahanka, Bengaluru - 560064, India.
E-mail address: sheeladileep66@gmail.com

and

Vasantha D. M.,
School of Applied Sciences,
Department of Mathematics,
REVA University, Yelahanka, Bengaluru - 560064, India.
E-mail address: dmvasantha@gmail.com