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Analysis of Semi-Discrete Finite Element Approximations for the Fisher-Kolmogorov Model

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ABSTRACT: In this paper,der we investigate the numerical analysis of Fisher-Kolmogorov Model type reaction-diffusion equations on open bounded convex domains $\Omega \subset \mathbb{R}^L(L = 1, 2, 3)$ with Neumann boundary conditions. Tow finite element schemes have introduce which is the semi discrete approximation. The existence and uniqueness of their solutions is demonstrated for semi discrete finite element approximations.The convergence of semi-discrete approximations to the exact solutions is presented.Analysis is done on the error boundaries between semi-discrete and exact solutions, semi-discrete and completely discrete solutions.

Keywords: Semi-discrete approximations, Neumann boundary conditions, the Fisher-Kolmogorov system, error bounds.

Contents

1. Introduction

During the modeling of chemical and biological processes, reaction-diffusion equations are crucial. The Fisher-Kolmogorov Model is a reaction-diffusion equation established by Brussels School [7,4,5] to investigate the behavior of chemical models with non-linear oscillations, this model is widely applied in Chemical reactions and diffusive mass transport, include enzymatic processes, the triple collision production of ozone from atomic oxygen, and the multiple mode coupling in laser and plasma physics. In [19,6], Fisher-Kolmogorov Model has been introduced in the following form.:
Find $\{\zeta\}$ so that

$$\frac{\partial\, \zeta^h}{\partial t} - \nabla(D_\zeta \nabla \zeta^h) = \rho_\zeta \zeta^h (1 - \frac{(\zeta^h)^2}{\theta_\zeta}) \quad on \;\; \Omega \times (0, T), \tag{1.1}$$

$$\frac{\partial \zeta}{\partial \nu} = 0, \qquad \text{on} \quad \partial\Omega, \tag{1.2}$$

$$\zeta(\cdot, 0) = \zeta_0, \qquad \text{in} \quad \Omega. \tag{1.3}$$

Here, Ω denotes a bounded domain in $\mathbb{R}^L(L = 1, 2, 3)$ with Lipschitz boundary $\partial\Omega$, ζ denote dimensionless concentrations , ν represents the exterior unit normal to $\partial\Omega$, ζ_0 denote initial data,where D_ζ is diffusion coefficients; furthermore, ρ_ζ represents the net tumor cell proliferation rate and θ_ζ is the tissue carrying capacity. (i.e., the maximum admissible tumor cell density). In general, these parameters may be defined pointwise and over time (i.e.,$D_\zeta = D_\zeta(x, t), \theta_\zeta = (x, t), \rho_\zeta = (x, t)$) [28]. We assume D_ζ , ρ_ζ and θ_ζ are a non-nagative functions satisfying the following condition:

$$|D_\zeta| > \alpha_1 \;, |\rho_\zeta| < \alpha_2 \;, |\theta_\zeta| < \alpha_3 \tag{1.4}$$

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where $\alpha_1, \alpha_2, \alpha_3$ are posstive constant.
Throughout, we assume the following result:

Owing to the widespread prevalence use of the Fisher-Kolmogorov model and the lack of an exact solution, several studies have employed various numerical methods. Notwithstanding the above, there is a lack within the scope of research on numerical solution of the Fisher-Kolmogorov Model. In [27], For clarification, the second-order finite difference method has been implemented to obtain a numerical solution for the diffusion-free Fisher-Kolmogorov Model. It also introduced a dual-reciprocity boundary element method in [11].For a system of two dimensions, a composite cubic B-spline curve method with a Runge Kutta technique was employed in [17]. It also introduced a differential quadrature method for reaction-diffusion Fisher-Kolmogorov systems in [22], this approach Condenses the system into an ordinary differential equation, which is then solved using the fourth-order Runge Kutta method.. In [14],Galerkin has also developed element-free variable multiscale and discontinuous local Galerkin methods. In [9], It presents a non-lattice radial basis function approach for a two-dimensional Fisher-Kolmogorov system. The Crank-Nicholson ETD method was applied in [20] to solve the Fisher-Kolmogorov Model in one and two dimensions. Furthermore, a boundary knot method has been proposed for solving two-dimensional reaction-diffusion equations in [15]. Additionally, the boundary node method for the purpose of solving two-dimensional response-diffusion differential equations was proposed in [10,25]. Furthermore, many recent studies have focused on analyzing a nonlinear diffusion system using the finite element method , see for example [16,3,29,24,23,8,30,18,1].

In this study, a semi-discrete approximation of finite elements to the diffusion equations of the procallator reaction is carried out. The existence and uniqueness of the approximation of finite semi separate elements is also determined. Then some stability limits are derived. And determines the error associated with the semi-discrete approximation. The next step is to present an approximation of a completely discrete finite element in which some stability limit was found in the solution and a study of its existence and uniqueness was carried out. Moreover, the error associated with the approximation of completely discrete finite elements is given. Moreover, at each time step, an efficient algorithm is proposed to cope with the approximation of completely discrete finite elements. Numerical simulations are performed in one and two dimensions to show the expected behavior of a physical problem.

Section 1.1 the notation employed in this study is advanced. Sections 2 define the semi-discrete approximation for the problem 1.1, and Sections 2.1, 2.2, and 2.3 demonstrate the local existence, global existence, and uniqueness, respectively.. Section 2.4 presents the error bounds of semi-discrete approximation.

1.1. Notation and auxiliary results

We present a weak formulation of the Model (1.1)-(1.3).
find $\zeta(.,t) \in H^1(\Omega)$ so that $\zeta(.,0) = \zeta_0(.)$, $\forall \psi \in H^1(\Omega)$ and for *a.e.* $t \in (0,T)$

$$(\frac{\partial\, \zeta}{\partial t}, \psi) + (D_\zeta \nabla \zeta, \nabla \psi) = (\rho_\zeta \zeta (1 - \frac{(\zeta)^2}{\theta_\zeta}), \psi), \tag{1.5}$$

Let $\mathfrak{D}^h$ be a quasi-uniform partitioning of Ω into disjoint open simplices $\{\mathcal{I}\}$ with $h_{\mathcal{I}} := \text{diam}(\mathcal{I})$ and $h := \max\, h_{\mathcal{I}}$, so that $\overline{\Omega} = \cup_{\mathcal{I} \in \mathfrak{D}^h} \overline{\mathcal{I}}$. The space of basis functions that are piecewise linear $\mathfrak{D}^h$ Can be formulated as follows:

$$S^h := \{\xi \in C(\overline{\Omega}) : \xi|_{\mathcal{I}} \text{ is linear} \, \forall \, \mathcal{I} \in \mathfrak{D}^h\}.$$

Let $\{\phi_j\}_{j=0}^k$ the basis functions of the space should be S^h, which hold for the relation $\phi_j(\zeta_i) = \delta_{ij}$, where $\{\zeta_i\}_{i=0}^k$ is the set of nodes of $\mathfrak{D}^h$. Let $\pi^h : C(\overline{\Omega}) \to S^h$ be the Lagrange interpolation operator such that $\pi^h \upsilon(\zeta_j) = \upsilon(\zeta_j)$, $\forall j = 0, \dots, k$. We also introduce a discrete L^2 inner product on $C(\overline{\Omega})$ via

$$(\omega(\kappa), \upsilon(\kappa))^h := \int_\Omega \pi^h\{\omega(\kappa)\upsilon(\kappa)\} d\kappa = \sum_{j=0}^{k} \widehat{N}_{jj} \omega(\zeta_j) \upsilon(\zeta_j), \tag{1.6}$$

where $\widehat{N}_{jj} = (\phi(\zeta_j), \phi(\zeta_j))^h = (1, \phi(\zeta_j)) > 0$ it is referred to as the concentrated mass matrix.

It can be demonstrated that

$$(\pi^h\omega(\kappa), \upsilon(\kappa))^h := (\omega(\kappa), \upsilon(\kappa))^h \text{ for all } \omega(\kappa), \upsilon(\kappa) \in C(\overline{\Omega}), \tag{1.7}$$

We present $\widehat{N}_{ij} := (\phi_i, \phi_j)^h$, $H_{ij} := (\nabla\phi_i, \nabla\phi_j)$. The inner product (1.6) induces a norm on S^h such that $|\omega|_h^2 := (\pi^h\omega, \omega)^h$, for all $\omega(\kappa) \in S^h$, $|.|_h$ and $\|.\|_0$ are norms that are equivalent to each other, namely,

$$\mathcal{I}_1\|\omega\|_0 \leq |\omega|_h \leq \mathcal{I}_2\|\omega\|_0 \text{ for all } \omega \in S^h, \tag{1.8}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are independent of h. For all $\omega(\kappa) \in S^h$clarify:

$$|\omega^h|_{h,q} := \Big(\int_\Omega \pi^h\{|\omega(\kappa)^h|^q\}d\eta\Big)^{\frac{1}{q}} \equiv \Big(\sum_{i=0}^{k} \widehat{N}_{jj}|\omega(\zeta_i)^h|^q\Big)^{\frac{1}{q}} \text{ if } \ 0 \leq q < \infty, \tag{1.9}$$

and

$$|\omega^h|_{h,\infty} := \max_{0\leq j\leq k} |\omega(\zeta_j)^h|, \qquad \text{if } \ q = \infty. \tag{1.10}$$

The discrete Hölder and Minkowski inequalities can be formulated, $\forall\omega_1, \omega_2 \in C(\overline{\Omega})$, see [21], pp. 271-274, as follows:

$$|\big(\omega_1, \omega_2\big)^h| \leq |\omega_1|_{h,q_1}|\omega_2|_{h,q_2}, \quad \frac{1}{q_1} + \frac{1}{q_2} = 1 \ \ 0 \leq q_1, q_2 \leq \infty, \tag{1.11}$$

$$\big|\omega_1 + \omega_2\big|_{h,q} \leq |\omega_1|_{h,q} + |\omega_2|_{h,q}, \quad 0 \leq q \leq \infty. \tag{1.12}$$

The following estimate is required (see [26], Lemma 15.1):

$$|\big(\omega_1, \omega_2\big) - \big(\omega_1, \omega_2\big)^h| \leq Ch^{k+1}\|\omega_1\|_k\|\omega_2\|_1, \text{ for all } \omega_1, \omega_2 \in S^h \ \ k = 0 \text{ or } 1. \tag{1.13}$$

Also the following estimates are required [13]):

$$\|(I - \pi^h)\omega\|_0 + h|(I - \pi^h)\omega|_1 \leq Ch^2|\omega|_2, \text{ for all } \omega \in H^2(\Omega), \tag{1.14}$$

$$\|(I - \pi^h)\omega\|_{0,1} \leq Ch^2|\omega|_{2,1}, \text{ for } 1 \leq p < \infty. \tag{1.15}$$

Also, it is useful to introduce the following result for $1 \leq q < \infty$. We have that:

$$|\omega|_{2,q} \leq \Big(\int_\Omega \sum_{k,m}^{k} \Big|\frac{\partial^2\omega}{\partial x_k x_m}\Big|^q d\kappa\Big)^{\frac{1}{q}} \leq 2^{\frac{1}{q}}|\omega|_{2,q}, \text{ for all } \omega \in W^{2,q}(\Omega). \tag{1.16}$$

Moreover, the inverse inequalities are also presented here, for all $\omega \in S^h$, Theorem 3.2.6 in [12]), We obtain

$$|\omega|_{1,q,"\tau"} \leq Ch^{-1}_{"\tau"}|\omega|_{0,q,"\tau"}, \quad 1 \leq q \leq \infty, \tag{1.17}$$

$$\|\omega\|_{0,q_2,"\tau"} \leq Ch_{"\tau"}^{-d(\frac{1}{q_1}-\frac{1}{q_2})}\|\omega\|_{0,q_1,"\tau"}, \quad 1 \leq q_1 \leq q_2 \leq \infty, \tag{1.18}$$

$$|\omega|_{1,q_2,"\tau"} \leq Ch_{"\tau"}^{-d(\frac{1}{q_1}-\frac{1}{q_2})}|\omega|_{1,q_1,"\tau"}, \quad 1 \leq q_1 \leq q_2 \leq \infty. \tag{1.19}$$

Let $p^h : L^2 \mapsto S^h$ be the orthogonal projection from L^2 onto S^h, which achieves

$$(p^h\nu, \chi^h) = (\nu, \chi^h) \quad \text{for all} \quad \nu \in L^2, \ \chi^h \in S^h. \tag{1.20}$$

We shall utilize the projection property can be written in the following :

$$\|p^h\nu\|_1 \leq C\|\nu\|_1 \quad \text{for all,} \quad \nu \in H^1. \tag{1.21}$$

Moreover, the Sobolev interpolation theorem will be studied , which is stated in the following form [2]): Let $v \in W^{k,\hbar}(\Omega)$, for $\hbar \in [1,\infty]$, $k \geq 1$, then there are constants C and $\varpi = \frac{d}{k}\left(\frac{1}{\hbar} - \frac{1}{\xi}\right)$ In such a way that the following Gagliardo-Nirenberg inequality valid

$$\|v\|_{0,\xi} \leq C\|v\|_{0,\hbar}^{1-\varpi}\|v\|_{k,\hbar}^{\varpi}, \quad \text{for } \xi \in \begin{cases} [\hbar,\infty] & \text{if } k - \frac{d}{\hbar} > 0, \\ [\hbar,\infty) & \text{if } k - \frac{d}{\hbar} = 0, \\ [\hbar, -\frac{d}{k-d/p}] & \text{if } k - \frac{d}{\hbar} < 0. \end{cases} \tag{1.22}$$

We also require the following Grönwall lemma in differential form: Let $\Gamma(t) \in W^{1,1}(0,T)$ and $\theta_1(t), \theta_2(t), \theta_3(t) \in L^1(0,T)$, where all functions are non-negative. It follows from

$$\frac{d\Gamma(t)}{dt} + \theta_2(t) \leq \theta_3(t)\Gamma(t) + \theta_1(t) \text{ a.e. } t \in [0,T],$$

that

$$\Gamma(T) + \int_0^T \theta_2(t)dt \leq exp(\int_0^T \theta_3(\tau)d\tau)\Gamma(0) + exp(\int_0^T \theta_3(\tau)d\tau)\int_0^T \theta_1(\tau)d\tau. \tag{1.23}$$

We will be frequently need Young's inequality in the form

$$\nu_1\nu_2 \leq \varepsilon^{\frac{\varepsilon_1}{\varepsilon_2}}\frac{\nu_1^{\varepsilon_1}}{\varepsilon_1} + \varepsilon^{-1}\frac{\nu_2^{\varepsilon_2}}{\varepsilon_2}, \qquad \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} = 1, \tag{1.24}$$

that valids for any $\nu_1, \nu_2 \geq 0$, $\varepsilon > 0$ and $\varepsilon_1, \varepsilon_2 > 1$.Another valuable implication of Young's inequality is as follows:

$$\nu_1\nu_2 \geqslant -\varepsilon\frac{\nu_1^2}{2} - \varepsilon^{-1}\frac{\nu_2^2}{2}, \qquad \forall\nu_1, \nu_2 \in \mathbb{R}, \forall\varepsilon > 0. \tag{1.25}$$

$$|\varphi^h(0)|_h^2 \equiv |p^h\varphi_0|_h^2 \leq C\|p^h\varphi_0\|_1^2 \leq C\|\varphi_0\|_1^2 \leq C. \tag{1.26}$$

2. A Semi-Discrete Approximation

We consider a semi-discrete (in time) of the Model (1.1)-(1.3).
Find $\zeta^h(.,t)$, $\frac{\partial\zeta^h}{\partial t} \in S^h(\Omega)$ in a way that $\zeta(.,0) = \zeta_0 \, in \, L^2(\Omega)$, $\frac{\partial\zeta}{\partial v} = 0 \; on \; \partial\Omega$ and the relation is satisfied for a.e $t \in (0,T)$

$$(\frac{\partial\,\zeta^h}{\partial t}, \psi^h)^h + (D_\zeta\nabla\zeta^h, \nabla\psi^h) = (\rho_\zeta\zeta^h(1 - \frac{(\zeta^h)^2}{\theta_\zeta}), \psi^h)^h. \tag{2.1}$$

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^L$ $(L = 1,2,3)$ is an convex , bounded , open domain and $\zeta_0 \in H^1(\Omega)$, then the Model (1.1)-(1.3) possesses a unique semi-discrete solution ζ^h satisfying*

$$\begin{aligned} &\zeta^h(x,t) \in L^{2n+2}(0,T; L^{h,2n+2}(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^{h,4}(\Omega)) \cap L^6(\Omega_T)) \\ &\frac{\partial\zeta^h}{\partial t} \in L^2(\Omega_T). \end{aligned} \tag{2.2}$$

Proof: The proof is organized into three parts, outlined according to the following:

2.1. Local existence

Initially, the parameters ζ^h is fined and expressed according to the following :

$$\zeta^h(.,t) = \sum_{i=0}^{k} C_i(t)\varpi_i(.), \tag{2.3}$$

and set $\psi^h(.,t) = \varpi_j(.)$ for $j = 0,\ldots,k$, where $C_i(t)$ represents a time-dependent constant. The initial condition can be approximated as follows:

$$\zeta_0^h(.) = P^h\;\zeta_0(.) = \sum_{i=0}^{k} C_i(0)\varpi_i(.), \tag{2.4}$$

with

$$\sum_{i=0}^{k} C_i(0)(\varpi_i, \varpi_j) = (\ \zeta_0^h(.), \varpi_j) = (P^h\ \zeta_0(.), \varpi_j), \quad j = 0, \ldots, k, \tag{2.5}$$

Through substitution (2.3) and (2.4) in (2.1) , and having $\psi^h = \varpi_j,\ \ j = 0, \ldots, k$, lead to

$$\sum_{i=0}^{k} \frac{dC_i(t)}{dt}(\varpi_i, \varpi_j)^h + \sum_{i=0}^{k} C_i(t)(D_\zeta \nabla\varpi_i, \nabla\varpi_j) = (\mu_\zeta(\zeta^h), \varpi_j)^h, \tag{2.6}$$

where $\mu_\zeta(\zeta^h) = \rho_\zeta \zeta^h - \rho_\zeta \frac{(\zeta^h)^3}{\theta_\zeta}$, such that

$$(\mu_\zeta(\zeta^h), \varpi_j)^h = (\mu_\zeta(\sum_{i=0}^{k} C_i(t)\varpi_i(.)), \varpi_j)^h$$

$$= \int_\lambda \pi^h(\mu_\zeta(\sum_{i=0}^{k} C_i(t)\varpi_i(.)), \varpi_j)^h dx = \widehat{M}_{jj}\mu_\zeta(C_i). \tag{2.7}$$

Thus from (2.6) and (2.7) we obtain a system of $(k+1)$ ordinary differential equations:

$$\widehat{M}_{ij}\frac{dC_j(t)}{dt} +\ C_j(t)\nabla\widehat{M}_{ij} = \widehat{M}_{jj}\mu_\zeta(C_i), \tag{2.8}$$

where $\widehat{M}_{ij} = (\varpi_i, \varpi_j)$, $\nabla\widehat{M}_{ij} = (D_\zeta\nabla\varpi_i, \nabla\varpi_j)$.

2.2. Global Eistence of a Semi-Discrete Approximation Solution

To establish the global existence of approximate solutions, we will determine bounds for the solutions ζ^h the fact that these bounds do not depend on the function h.

Estimate I:
Setting $\psi^h = \zeta^h$ in (2.1) and noting (1.4), we have

$$\frac{1}{2}\frac{d}{dt}|\zeta^h|_h^2 + \alpha_1||\nabla\zeta^h||_0^2 + \frac{\alpha_2}{\alpha_3}|\zeta^h|_{h,4}^4 \le \alpha_2|\zeta^h|_h^2. \tag{2.9}$$

Multiplying by 2 and using (1.23), we have that

$$|\zeta^h(T)|_h^2 + 2\alpha_1\int_0^T |\zeta^h|_1^2 dt + 2\frac{\alpha_2}{\alpha_3}\int_0^T |\zeta^h|_{h,4}^4 dt \le \exp(CT)(|\zeta^h(0)|_h^2), \tag{2.10}$$

and noting that $\zeta_0 \in H^1(\Omega)$,this lead to

$$|\zeta^h(T)|^2_{L^\infty(0,T;L^{h,2}(\Omega))} + 2\alpha_1|\zeta^h|^2_{L^2(0,T;H^1(\Omega))} + 2\frac{\alpha_2}{\alpha_3}|\zeta^h|^4_{L^{h,4}(\Omega_T)} \le C. \tag{2.11}$$

we obtime to ζ^h is uniformly bounded in $\zeta^h \in L^\infty(0,T;L^{h,2}(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^{h,4}(\Omega_T)$.

Estimate II :
Setting $\psi^h = (\zeta^h)^3$ in (2.1) and noting (1.4) , we get

$$\frac{1}{4}\frac{d}{dt}|\zeta^h|_{h,4}^4 + \alpha_1(\nabla\zeta^h, \nabla(\zeta^h)^3) \le \alpha_2(\zeta^h, (\zeta^h)^3)^h - \frac{\alpha_2}{\alpha_3}((\zeta^h)^3, (\varsigma^h)^3)^h, \tag{2.12}$$

$$\frac{1}{4}\frac{d}{dt}|\zeta^h|_{h,4}^4 + 3\alpha_1(\nabla\zeta^h, (\zeta^h)^2\nabla\zeta^h) \le \alpha_2(\zeta^h, (\zeta^h)^3)^h - \frac{\alpha_2}{\alpha_3}((\zeta^h)^3, (\zeta^h)^3)^h, \tag{2.13}$$

and

$$\frac{1}{4}\frac{d}{dt}|\zeta^h|^4_{h,4} + 3\alpha_1||\zeta^h\nabla\zeta^h||^2_0 \leq \alpha_2|\zeta^h|^4_{h,4} - \frac{\alpha_2}{\alpha_3}|\zeta^h|^6_{h,6}. \tag{2.14}$$

By transferring the negative term from the right-hand side to the left-hand side and using (1.23),and also multiplying by 4, we have that

$$|\zeta^h(T)|^4_{h,4} + 12\alpha_1\int_0^T ||\zeta^h\nabla\varsigma^h||^2_0 dt + 4\frac{\alpha_2}{\alpha_3}\int_0^T |\varsigma^h|^6_{h,6}dt \leq \exp(CT)(|\zeta^h(0)|^4_{h,4}). \tag{2.15}$$

By recalling $\zeta_0 \in H^1(\Omega)\, and\, H^1 \hookrightarrow L^4$ this lead to

$$|\zeta^h(T)|^4_{L^\infty(0,T;L^{h,4}(\Omega))} + 12\alpha_1|\zeta^h\nabla\zeta^h|^2_{L^2(\Omega_T)} + 4\frac{\alpha_2}{\alpha_3}|\zeta^h|^6_{L^6(\Omega_T)} \leq C. \tag{2.16}$$

we arrive to ζ^h is uniformly bounded in $L^\infty(0,T;L^{h,4}(\Omega)) \cap L^6(\Omega_T) and\; \zeta^h\nabla\zeta^h \in L^2(\Omega_T)$.

Estimate III:

Setting $\psi^h = \frac{\partial\varsigma^h}{\partial t}$ in (2.1) and noting (1.4) ,this lead to

$$|\frac{\partial\zeta^h}{\partial t}|^2_h + \alpha_1(\nabla\zeta^h, \nabla(\frac{\partial\zeta^h}{\partial t})) \leq \alpha_2(\zeta^h, \frac{\partial\zeta^h}{\partial t})^h - \frac{\alpha_2}{\alpha_3}((\zeta^h)^3, \frac{\partial\zeta^h}{\partial t})^h \tag{2.17}$$

$$|\frac{\partial\zeta^h}{\partial t}|^2_h + \frac{\alpha_1}{2}\frac{d}{dt}|\nabla\zeta^h|^2_0 \leq \frac{\alpha_2}{2}\frac{d}{dt}|\zeta^h|^2_0 - \frac{\alpha_2}{\alpha_3}((\zeta^h)^3, \frac{\partial\zeta^h}{\partial t})^h \tag{2.18}$$

By using (1.24) and (1.11)

$$\frac{\alpha_2}{2}\frac{d}{dt}|\zeta^h|^2_h - \frac{\alpha_2}{\alpha_3}|\zeta^h|^3_{h,3}|\frac{\partial\zeta^h}{\partial t}|_h \leq \frac{\alpha_2}{2}\frac{d}{dt}|\zeta^h|^2_h + \frac{\alpha_2}{2\alpha_3}|\zeta^h|^6_{h,6} + \frac{1}{2}|\frac{\partial\zeta^h}{\partial t}|^2_h \tag{2.19}$$

substituting (2.19) in (2.18) lead to

$$\frac{1}{2}|\frac{\partial\zeta^h}{\partial t}|^2_h + \frac{\alpha_1}{2}\frac{d}{dt}|\nabla\zeta^h|^2_0 \leq \frac{\alpha_2}{2}\frac{d}{dt}|\zeta^h|^2_h + \frac{\alpha_2}{2\alpha_3}|\zeta^h|^6_{h,6}. \tag{2.20}$$

Multiplying the result by 2 and Integrating over time, we have that

$$\begin{aligned}&|\frac{\partial\zeta^h}{\partial t}|^2_{L^2(0,T;L^{h,2}(\Omega))} + \alpha_1|\nabla\zeta^h(T)|^2 + |\zeta^h(0)|^2_h \\ &\leq \alpha_1|\nabla\zeta^h(0)|^2 + |\zeta^h|^6_{L^6(\Omega T)} + |\zeta^h(T)|^2_h.\end{aligned} \tag{2.21}$$

By recalling $\zeta_0 \in H^1(\Omega)$ and using **Estimate II** , and **Estimate I**, noting that $\frac{\partial\zeta^h}{\partial t}$ is uniformly bounded in $L^2(0,T;L^{h,2}(\Omega)) and\; \zeta^h \in L^\infty(0,T;H^1(\Omega))$.

Estimate IV:

Setting $\psi^h = (\zeta^h)^{2n-1}$ in (1.1) where $1 \leq n < \infty$,and noting (1.4) we have

$$\frac{1}{2n}\frac{d}{dt}|\zeta^h|^{2n}_{h,2n} + \alpha_1||(\zeta^h)^{n-1}\nabla\zeta^h||^2_0 \leq \alpha_2|\zeta^h|^{2n}_{h,2n} - \frac{\alpha_2}{\alpha_3}|\zeta^h|^{2n+2}_{h,2n+2}, \tag{2.22}$$

multiplying by 2n, we have that

$$\frac{d}{dt}|\zeta^h|_{h,2n}^{2n} + 2n\alpha_1||(\zeta^h)^{n-1}\nabla\zeta^h||_0^2 + \frac{2n\alpha_2}{\alpha_3}|\zeta^h|_{h,2n+2}^{2n+2} \leq 2n\alpha_2|\zeta^h|_{h,2n}^{2n}, \tag{2.23}$$

By using (1.23) lead to

$$|\zeta^h(T)|_{h,2n}^{2n} + 2n\alpha_1\int_0^T ||(\zeta^h)^{n-1}\nabla\zeta^h||_0^2 dt + \frac{2n\alpha_2}{\alpha_3}\int_0^T |\zeta^h|_{h,2n+2}^{2n+2}dt \tag{2.24}$$

$$\leq \exp(2n\alpha_2 T)(|\zeta^h(0)|_{h,2n}^{2n}). \tag{2.25}$$

By recalling $\zeta_0 \in H^1(\Omega)\, and\, H^1 \hookrightarrow L^{2n}$, this lead to

$$|\zeta^h(T)|_{h,2n}^{2n} + 2n\alpha_1\int_0^T ||\nabla\zeta^h(\zeta^h)^{n-1}||_0^2 dt + \frac{2n\alpha_2}{\alpha_3}\int_0^T |\zeta^h|_{h,2n+2}^{2n+2}dt \leq C. \tag{2.26}$$

we arrive to ζ^h is uniformly bounded in $L^\infty(0,T;L^{h,2n}(\Omega)) \cap L^{2n+2}(0,T;L^{h,2n+2}(\Omega))$
and $(\zeta^h)^{n-1}\nabla\zeta^h \in L^2(0,T;H^1(\Omega))$.

2.3. Uniqueness

Let ζ_1^h, ζ_2^h are represent solutions of the semi-discrete approximations (1.1). Assignment $\beta^h = \zeta_1^h - \zeta_2^h$, and setting $\psi^h = \beta^h$, in (2.1), this leads to the following formulation:

$$\frac{1}{2}\frac{d}{dt}|\beta^h|_h^2 + \alpha_1||\nabla\beta^h||_0^2 \leq \alpha_2|\beta^h|_h^2 - \frac{\alpha_2}{\alpha_3}((\zeta_1^h)^3 - (\zeta_1^h)^3, \beta^h)^h, \tag{2.27}$$

Now,by using (1.24)and (1.11)

$$\alpha_2|\beta^h|_h^2 - \frac{\alpha_2}{\alpha_3}((\zeta_1^h)^3 - (\zeta_1^h)^3, \beta^h)^h \leq \alpha_2|\beta^h|_h^2 - \frac{\alpha_2}{\alpha_3}((\zeta_1^h)^3 - (\zeta_1^h)^3, \beta^h)^h,$$

$$\leq \alpha_2|\beta^h|_h^2 - \frac{\alpha_2}{\alpha_3}((\zeta_1^h - \zeta_1^h)((\zeta_1^h)^2 + \zeta_1^h\zeta_2^h + (\zeta_2^h)^2), \beta^h)^h,$$

$$\leq |\beta^h|_h^2(\alpha_2 + \frac{\alpha_2}{\alpha_3} \quad |\frac{3}{2}(\zeta_1^h)^2 + \frac{3}{2}(\zeta_2^h)^2|, \tag{2.28}$$

Substituting (2.28) in (2.27) lead to

$$\frac{1}{2}\frac{d}{dt}|\beta^h|_h^2 + \alpha_1||\nabla\beta^h||_0^2 \leq |\beta^h|_h^2(\alpha_2 + \frac{\alpha_2}{\alpha_3} \quad |\frac{3}{2}(\zeta_1^h)^2 + \frac{3}{2}(\zeta_2^h)^2|, \tag{2.29}$$

multiplying by 2 and using (1.23) ,lead to

$$|\beta^h(T)|_h^2 + 2\alpha_1\int_0^T ||\nabla\beta^h||_0^2 dt \leq |\beta^h(0)|_h^2 \exp(\int_0^T (\alpha_2 + \frac{3\alpha_2}{\alpha_3}|(\zeta_1^h)^2 + (\zeta_2^h)^2|)dt). \tag{2.30}$$

By recalling $\zeta \in H^1(\Omega)$ and using **Estimate I**,noting that ζ^h is uniformly bounded in $L^\infty(0,T;L^{h,2}(\Omega))$
and since $L^\infty(0,T;L^{h,2}(\Omega)) \hookrightarrow L^2(\Omega_T)$

$$|\beta^h(T)|_h^2 + 2\alpha_1||\beta^h||^2_{L^2(H^1(\Omega_T))} \leq C|\ \beta^h(0)|_h^2. \tag{2.31}$$

Thus if $\zeta_1^h(0) = \zeta_2^h(0)$ hence we deduce uniqueness for all t ,however,if $\zeta_1^h(0) \neq \zeta_2^h(0)$,then we have continuous dependence in $L^2(\Omega)$. □

2.4. A Semi-Discrete Error

An error estimate between the solutions of equations (2.1) and (1.5) is derived.

Theorem 2.2 *The outcome* $\{\zeta^h\}$ *of (1.1) and (1.3) fulfills the error bound:*

$$\|\zeta - \zeta^h\|_{L^\infty(0,T;L^2(\Omega))} + \|\zeta - \zeta^h\|_{L^2(0,T;H^1(\Omega))} \leq Ch^2. \tag{2.32}$$

Proof: Define $\check{E} = \zeta - \zeta^h$, $\check{E}^A = \zeta - \pi^h\zeta$, $\check{E}^h = \pi^h\zeta - \zeta^h$,
we have that

$$\check{E} = \check{E}^A + \check{E}^h. \tag{2.33}$$

Choosing $\psi = \psi^h = \check{E}^h$ in (2.1) and (1.5), and subtract (2.1) from (1.5), yielding

$$\begin{aligned}&\big(\frac{\partial\zeta}{\partial t}, \check{E}^h\big) - \big(\frac{\partial\zeta^h}{\partial t}, \check{E}^h\big)^h + (D_\zeta\nabla\check{E}^h, \nabla\check{E}^h)\\ &= (\rho_\zeta\zeta - \frac{\rho_\zeta\zeta^2}{\theta_\zeta}, \check{E}^h) - (\rho_\zeta\zeta_h - \frac{\rho_\zeta(\zeta^h)^2}{\theta_\zeta}, \check{E}^h)^h\ .\end{aligned} \tag{2.34}$$

Performing addition and subtraction for each terms $(\frac{\partial\zeta^h}{\partial t}, \check{E}^h)$, and $(\frac{\partial\zeta^h}{\partial t}, \check{E}^h)$ to (2.34) and rearranging leads to

$$\begin{aligned}&\big(\frac{\partial\zeta}{\partial t}, \check{E}^h\big) - \big(\frac{\partial\zeta^h}{\partial t}, \check{E}^h\big)^h - \big(\frac{\partial\zeta^h}{\partial t}, \check{E}^h\big) + \big(\frac{\partial\zeta^h}{\partial t}, \check{E}^h\big) + (D_\zeta\nabla\check{E}, \nabla\check{E}^h)\\ &\qquad = (\rho_\zeta\zeta - \frac{\rho_\zeta\zeta^2}{\theta_\zeta}, \check{E}^h) - (\rho_\zeta\zeta_h - \frac{\rho_\zeta(\zeta^h)^2}{\theta_\zeta}, \check{E}^h)^h\ .\end{aligned} \tag{2.35}$$

By transferring the negative term from the left-hand side to the right-hand side

$$\begin{aligned}&\big(\frac{\partial\check{E}}{\partial t}, \check{E}^h\big) + (D_\zeta\nabla\check{E}, \nabla\check{E}^h)\\ &= \big(\frac{\partial\zeta^h}{\partial t}, \check{E}^h\big)^h - \big(\frac{\partial\zeta^h}{\partial t}, \check{E}^h\big) + (\rho_\zeta\zeta - \frac{\rho_\zeta\zeta^2}{\theta_\zeta}, \check{E}^h) - (\rho_\zeta\zeta_h - \frac{\rho_\zeta(\zeta^h)^2}{\theta_\zeta}, \check{E}^h)^h.\end{aligned} \tag{2.36}$$

By useing (2.33) and noting that (1.4),this lead to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\check{E}\|_0^2 + \alpha_1\|\nabla\check{E}\|_0^2 \leq \big(\frac{\partial\check{E}}{\partial t}, \check{E}^A\big) + \alpha_1(\nabla\check{E}, \nabla\check{E}^A)\\ &+\big(\frac{\partial\zeta^h}{\partial t}, \check{E}^h\big)^h - \big(\frac{\partial\zeta^h}{\partial t}, \check{E}^h\big) + (\rho_\zeta\zeta - \frac{\rho_\zeta\zeta^2}{\theta_\zeta}, \check{E}^h) - (\rho_\zeta\zeta_h - \frac{\rho_\zeta(\zeta^h)^2}{\theta_\zeta}, \check{E}^h)^h.\end{aligned} \tag{2.37}$$

Now , bounded right-hand side of(2.37)
by using (1.11) and (1.15) , to give

$$\Big(\frac{\partial \check{E}}{\partial t},\check{E}^A\Big)\le \Big\|\frac{\partial \check{E}}{\partial t}\Big\|_0\|\check{E}^A\|_0\le Ch^2\Big\|\frac{\partial \check{E}}{\partial t}\Big\|\|\zeta\|_2. \tag{2.38}$$

By using (1.11), (1.15)and (1.24) and choosing $\epsilon=\frac{\alpha_1}{3}$, we obtain that

$$\begin{aligned}(\nabla\check{E},\nabla\check{E}^A)\le\|\nabla\check{E}\|_0\|\nabla\check{E}^A\|_0&\le\|\check{E}\|_1\|\check{E}^A\|_1\\ &\le Ch\|\check{E}\|\|\zeta\|_2\\ \le Ch^2\|\zeta\|_2^2+\frac{\alpha_1}{6}\|\check{E}\|_0^2&+\frac{\alpha_1}{6}|\check{E}|_1^2.\end{aligned} \tag{2.39}$$

By using (1.13)and (1.24)and choosing $\epsilon=\frac{\alpha_1}{3}$, we obtain that

$$\begin{aligned}\Big(\frac{\partial\zeta^h}{\partial t},\check{E}^h\Big)^h-\Big(\frac{\partial\zeta^h}{\partial t},\check{E}^h\Big)&\le Ch\Big\|\frac{\partial\zeta^h}{\partial t}\Big\|_0\|\check{E}^h\|_1\\ &\le Ch\Big\|\frac{\partial\zeta^h}{\partial t}\Big\|_0\|\check{E}-\check{E}^A\|_1\\ &\le Ch^2\Big\|\frac{\partial\zeta^h}{\partial t}\Big\|_0^2+\frac{\alpha_1}{6}\|\check{E}\|_1^2+\frac{\alpha_1}{6}\|\check{E}^A\|_1^2\\ \le Ch^2\Big\|\frac{\partial\zeta^h}{\partial t}\Big\|_0^2+\frac{\alpha_1}{6}\|\check{E}\|_0^2&+\frac{\alpha_1}{6}|\check{E}|_1^2+Ch^2\|\zeta\|_2^2\end{aligned} \tag{2.40}$$

Now,to prove bounded $(\rho_\varsigma\zeta-\frac{\rho_\varsigma\zeta^2}{\theta_\varsigma},\check{E}^h)-(\rho_\varsigma\zeta_h-\frac{\rho_\varsigma(\zeta^h)^2}{\theta_\varsigma},\check{E}^h)^h$
By Performing addition and subtraction for each term $(\rho_\varsigma\zeta-\frac{\rho_\varsigma\zeta^2}{\theta_\varsigma},\check{E}^h)$ and useing (1.15) and rearranging leads to

$$\begin{aligned}&\alpha_2\Big[(\zeta^h-\frac{(\zeta^h)^2}{\alpha_3},\check{E}^h)-(\zeta^h-\frac{(\zeta^h)^2}{\alpha_3},\check{E}^h)^h\Big]+\alpha_2\Big[(\zeta-\frac{\zeta^2}{\alpha_3},\check{E}^h)-(\zeta^h-\frac{(\zeta^h)^2}{\alpha_3},\check{E}^h)\Big]\\ &\le C\Big[(\zeta^h-(\zeta^h)^2,\check{E}^h)-(\zeta^h-(\zeta^h)^2,\check{E}^h)^h\Big]+C\Big[(\zeta-\zeta^2,\check{E}^h)-(\zeta^h-(\zeta^h)^2,\check{E}^h)\Big].\end{aligned} \tag{2.41}$$

Firstly , bounded the first term in(2.41) ,by using (1.15)and (1.16), we obtain that

$$C\Big[(\zeta^h-(\zeta^h)^2,\check{E}^h)-(\zeta^h-(\zeta^h)^2,\check{E}^h)^h\Big]\le Ch^2\sum_{i,j}^{d}\int_\Omega\Big|\frac{\partial^2((\zeta^h-(\zeta^h)^2,\check{E}^h))}{\partial x_i\partial x_j}\Big|dx \tag{2.42}$$

Now, to find partial derivative in (2.42),we have

$$\frac{\partial((\zeta^h-(\zeta^h)^2\check{E}^h)}{\partial x_i}=\Big(\frac{\partial\zeta^h}{\partial x_i}-2\zeta^h\frac{\partial\zeta^h}{\partial x_i}\Big)\check{E}^h+(\zeta^h-(\zeta^h)^2)\frac{\partial\check{E}^h}{\partial x_i}$$

and

$$\begin{aligned}\frac{\partial}{\partial x_j}\Big[\Big(\frac{\partial\zeta^h}{\partial x_i}-2\zeta^h\frac{\partial\zeta^h}{\partial x_i}\Big)\check{E}^h+(\zeta^h-(\zeta^h)^2)\frac{\partial\check{E}^h}{\partial x_i}\Big]&=-2\frac{\partial\zeta^h}{\partial x_i}\frac{\partial\zeta^h}{\partial x_j}\check{E}^h+\Big(\frac{\partial\zeta^h}{\partial x_i}-2\zeta^h\frac{\partial\zeta^h}{\partial x_i}\Big)\frac{\partial\check{E}^h}{\partial x_j}\\ &=-2\frac{\partial\zeta^h}{\partial x_i}\frac{\partial\zeta^h}{\partial x_j}\check{E}^h+(1-2\zeta^h)\frac{\partial\zeta^h}{\partial x_i}\frac{\partial\check{E}^h}{\partial x_j}+(1-2\zeta^h)\frac{\partial\zeta^h}{\partial x_j}\frac{\partial\check{E}^h}{\partial x_i}.\end{aligned} \tag{2.43}$$

Substituting (2.43) in (2.42) lead to

$$C\Big[(\zeta^h-(\zeta^h)^2,\check{E}^h)-(\zeta^h-(\zeta^h)^2,\check{E}^h)^h\Big]$$
$$\leq Ch^2\sum_{i,j}^{d}\int_\Omega\Big|-2\frac{\partial\zeta^h}{\partial x_i}\frac{\partial\zeta^h}{\partial x_j}\check{E}^h+(1-2\zeta^h)\frac{\partial\zeta^h}{\partial x_i}\frac{\partial\check{E}^h}{\partial x_j}+(1-2\zeta^h)\frac{\partial\zeta^h}{\partial x_j}\frac{\partial\check{E}^h}{\partial x_i}\Big|dx$$
$$\leq Ch^2\sum_{i,j}^{d}\Big[\Big|\frac{\partial\zeta^h}{\partial x_i}\Big|\Big|\frac{\partial\zeta^h}{\partial x_j}\Big||\check{E}^h|+(1+2|\zeta^h|)\Big|\frac{\partial\zeta^h}{\partial x_i}\Big|\Big|\frac{\partial\check{E}^h}{\partial x_j}\Big|+(1+2|\zeta^h|)\Big|\frac{\partial\zeta^h}{\partial x_j}\Big|\Big|\frac{\partial\check{E}^h}{\partial x_i}\Big|\Big]$$
$$\leq Ch^2\Big[\|\zeta^h\|_{1,3}^2\ \|\check{E}^h\|_{0,3}+\big(1+\|\zeta^h\|_{0,3}\big)\ \|\zeta^h\|_{1,3}\ \|\check{E}^h\|_{1,3}\Big]$$
$$\leq Ch^2\Big[\|\zeta^h\|_{1,3}^2\ \|\check{E}^h\|_{0,3}+|\zeta^h\|_{1,3}\ \|\check{E}^h\|_{1,3}+\|\zeta^h\|_{0,3}\ \|\zeta^h\|_{1,3}\ \|\check{E}^h\|_{1,3}\Big] \tag{2.44}$$

We use the injections $H^1\hookrightarrow L^3$, (1.19) in the following form $|\omega|_{1,3}\leq Ch^{-d/6}|\omega|_{1,2}$, One can say that ζ^h is bounded in $L^\infty(0,T;H^1(\Omega))$, (1.24), and (1.14), to have that

$$C\Big[(\zeta^h-(\zeta^h)^2,\check{E}^h)-(\zeta^h-(\zeta^h)^2,\check{E}^h)^h\Big]$$
$$\leq Ch^2\Big[Ch^{\frac{-d}{6}}\|\zeta^h\|_1\ \|\check{E}^h\|_1+\|\zeta^h\|_1\,\|\check{E}^h\|_1+Ch^{\frac{-d}{6}}\|\zeta^h\|_1^2\ \ \|\check{E}^h\|_1\Big]$$
$$\leq Ch^{2-\frac{d}{6}}\Big[\|\zeta^h\|_1\ \|\check{E}^h\|_1+\|\zeta^h\|_1^2\,\|\check{E}^h\|_1\Big]\leq Ch^{2-\frac{-d}{6}}\Big[\|\check{E}^h\|_1\Big] \tag{2.45}$$

after using **Estimate III** and (1.24) We have obtained that result.

It follows from the fact that $e^h=e-e^A$, Cauchy Schwarz inequality (1.14),(1.15) and (1.24) and choosing $\epsilon=\frac{\alpha_1}{3}$ that

$$C\Big[(\zeta^h-(\zeta^h)^2,\check{E}^h)-(\zeta^h-(\zeta^h)^2,\check{E}^h)^h\Big]\leq Ch^{4-\frac{d}{3}}+\frac{\alpha_1}{6}\|\check{E}\|_0^2+\frac{\alpha_1}{6}\|\check{E}\|_1^2+Ch^2\|\zeta^h\|_2^2\ . \tag{2.46}$$

Secondly,we prove the bounded of the second term in(2.41)

$$C\Big[(\zeta-\zeta^2,\check{E}^h)-(\zeta^h-(\zeta^h)^2,\check{E}^h)\Big]\leq C\Big((\zeta-\zeta^h)-(\zeta^2-\zeta^h)^2),\check{E}^h\Big)$$
$$\leq C\Big((\zeta-\zeta^h)-(\zeta-\zeta^h)(\zeta+\zeta^h),\check{E}^h\Big) \tag{2.47}$$

since $\check{E}=\zeta-\zeta^h$,we obtain that

$$\leq C\Big(\check{E}-\check{E}(\zeta+\zeta^h),\check{E}^h\Big) \tag{2.48}$$

Using Hölder (1.12), it follows that

$$C\Big(\check{E}-\check{E}(\zeta+\zeta^h),\check{E}^h\Big)\leq C\Big[\|\check{E}\|_0\ \ \|\check{E}^h\|_0+\|\check{E}^h\|_{0,3}\ \ \|\check{E}\|_{0,3}\ \ \|\zeta\|_{0,3}+\|\check{E}^h\|_{0,3}\ \ \|\check{E}\|_{0,3}\ \ \|\zeta^h\|_{0,3}\Big]$$
$$\leq C\|\check{E}\|_0^2+Ch^2\|\zeta\|_2^2. \tag{2.49}$$

Now,put the equations (2.38),(2.39),(2.40),(2.47) and (2.49) in equation (2.37), this lead to

$$\frac{1}{2}\frac{d}{dt}\|\check{E}\|_0^2+\alpha_1\|\nabla\check{E}\|_0^2\le Ch^2\Big\|\frac{\partial\check{E}}{\partial t}\Big\|\|\zeta\|_2+Ch^2\|\zeta\|_2^2+\frac{\alpha_1}{6}\|\check{E}\|_0^2+\frac{\alpha_1}{6}|\check{E}|_1^2+Ch^2\Big\|\frac{\partial\zeta^h}{\partial t}\Big\|_0^2+$$
$$\frac{\alpha_1}{6}\|\check{E}\|_0^2+\frac{\alpha_1}{6}|\check{E}|_1^2+Ch^2\|\zeta\|_2^2+Ch^{4-\frac{d}{3}}+\frac{\alpha_1}{6}\|\check{E}\|_0^2+\frac{\alpha_1}{6}\|\check{E}\|_1^2$$
$$+Ch^2\|\zeta^h\|_2^2+C\|\check{E}\|_0^2+Ch^2\|\zeta\|_2^2. \quad (2.50)$$

$$\frac{1}{2}\frac{d}{dt}\|\check{E}\|_0^2+\frac{\alpha_1}{2}\alpha_1\|\nabla\check{E}\|_0^2\le$$
$$Ch^2\Big\|\frac{\partial\check{E}}{\partial t}\Big\|_0\|\zeta\|_2+Ch^2\|\zeta\|_2^2+Ch^2\Big\|\frac{\partial\zeta^h}{\partial t}\Big\|_0^2+Ch^2\|\zeta\|_2^2+Ch^{4-\frac{d}{3}}. \quad (2.51)$$

$$\frac{1}{2}\frac{d}{dt}\|\check{E}\|_0^2+\frac{\alpha_1}{2}\alpha_1\|\nabla\check{E}\|_0^2\le$$
$$Ch^2\|\zeta\|_0^2+Ch^2\Big[1+\Big\|\frac{\partial\check{E}}{\partial t}\Big\|_0\|\zeta\|_2+\|\zeta\|_2^2+\Big\|\frac{\partial\zeta^h}{\partial t}\Big\|_0^2+\|\zeta\|_2^2\Big] \quad (2.52)$$

We noting $h^{4-\frac{d}{3}}\le h^2$ and $h^{2-\frac{d}{6}}\le h^2$ as $h<1$ and $d<3$

$$\frac{d}{dt}\|\check{E}\|_0^2+\alpha_1\|\nabla\check{E}\|_0^2\le$$
$$Ch^2\|\check{E}\|_0^2+Ch^2\Big[1+\Big\|\frac{\partial\check{E}}{\partial t}\Big\|_0\|\zeta\|_0^2+\|\zeta\|_2^2+\Big\|\frac{\partial\zeta^h}{\partial t}\Big\|_0^2+\|\zeta\|_2^2\Big] \quad (2.53)$$

By Integral the two said on time (T) , this lead to

$$\|\check{E}\|_0^2+\alpha_1\int_0^T\|\nabla\check{E}\|_0^2\,dt\le$$
$$\|\check{E}(0)\|_0^2+Ch^2\int_0^T\Big[1+\Big\|\frac{\partial\check{E}}{\partial t}\Big\|_0\|\zeta\|_0^2+\|\zeta\|_2^2+\Big\|\frac{\partial\zeta^h}{\partial t}\Big\|_0^2+\|\zeta\|_2^2\Big]dt \quad (2.54)$$

Furthermore ,by using**Estimate I** and using **Estimate III**the other terms on the right hand side of equation(2.54) are bounded

$$\|\check{E}\|_{L^\infty(0,T,L^2(\Omega))}+\alpha_1\|\check{E}\|_{L^2(0,T,H^1(\Omega))}\le Ch^2. \quad (2.55)$$

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