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A Generalized (α, β) Normalization Framework for Super-Additive Games with Applications to the Rank-Shapley Value

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ABSTRACT: Normalization in super-additive cooperative games is a process that transforms a game into an equivalent representation while preserving its core strategic and structural properties. This paper introduces a generalized normalization framework that unifies and extends existing methodologies by defining a flexible family of normalization schemes. The proposed approach involves specifying and estimating the boundary (transformation) parameters to ensure that all coalition values are mapped within predefined bounds. This transformation enables the adjustment of coalition values through either compression or expansion, depending on the original distribution and the desired normalization limits. The framework is designed to be broadly applicable across all super-additive games and maintains essential game-theoretic properties, including transferable utility invariance, player symmetry, and the inessential nature of the original game. These preserved characteristics are fundamental to ensuring that the normalized game retains the strategic equivalence of the original formulation. By offering a unified structure, the framework facilitates consistent analysis and comparison across diverse cooperative scenarios. A practical application is presented to demonstrate the effectiveness of the normalization procedure, illustrating how coalition values can be systematically adjusted to fit within specified constraints. This example underscores the utility of the framework in real-world cooperative settings, particularly in contexts involving resource allocation and strategic decision-making. The generalized normalization scheme proposed in this study provides valuable insights and tools for researchers and practitioners seeking to model, analyze, and optimize cooperative interactions in a mathematically rigorous and operationally meaningful way.

Keywords: Cooperative game, game normalization, weighted Shapley value, TU-Invariant property, strategic equivalence.

Contents

1. Introduction

Transferable Utility (TU) games represent a fundamental class of cooperative games characterized by the assumption that utility is perfectly divisible and valued equally by all participants [1,2]. This assumption facilitates the analysis of coalition formation and resource allocation, making TU games a valuable tool in economics and game theory. However, practical challenges often arise when working with super-additive games, particularly when the coalition values are either constrained by restrictive modeling assumptions or become too large to handle effectively.

In such situations, the technique of normalization offers a solution by transforming the game into an equivalent but more tractable form. Normalization, also referred to as standardization, involves a

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transformation of the game's characteristic function while preserving its essential properties. The aim is to produce a strategically equivalent game in which the coalition values lie within a pre-specified, typically non-negative, boundary [3]. This is conceptually similar to transforming the imputation space, allowing for simplification and comparison across games. The foundational works by Mares [4] provide an in-depth exploration of the properties of normalized coalition games.

Formally, let v and ω represent the characteristic functions of two TU games defined on the same player set, N. The game (N, v) can be transformed into an equivalent game (N, ω) by applying a transformation operator that adjusts the coalition values to lie within a desired boundary. This transformation often simplifies computational processes [5] and facilitates comparison between different games by standardizing their scales [6].

Several specific forms of normalization have been proposed in the literature:

- $(0, 1)$-normalized game: A TU game is $(0, 1)$-normalized if the characteristic function of the transformed game lies between zero and the difference between the worth of the grand coalition and the sum of the stand-alone values, consistent with the egalitarian framework of [7]. This idea was used to characterize structural properties of assignment games in [8].
- (0,1)-normalized game: This form constrains the values to lie within the unit interval $[0, 1]$, as discussed by [9].

While useful, these conventional forms do not accommodate scenarios in which one may wish to normalize coalition values beyond or below these standard boundaries. This limitation motivates the need for a more general and flexible normalization framework.

In this study, we introduce a *generalized family of normalization schemes* for TU games, defined by arbitrarily chosen non-negative boundary parameters α and β, without losing equivalence, thereby providing a unified framework rather than fragmented ad-hoc schemes. This approach enables the generation of multiple, strategically equivalent games by perturbing these parameters, while maintaining the property of transferable utility invariance. The proposed framework enables TU games to be compressed or expanded within any desirable boundary, providing greater adaptability for both theoretical analysis and practical applications.

The remainder of this paper is arranged as follows: Section 2 presents basic definitions and notation, introduces the generalized form of normalization induced by boundary parameters, and explores the transferable utility invariant property. Section 3 provides an illustration to demonstrate the applicability of this generalized family.

2. Materials and Methods

In this section, basic definitions and notation of terms are presented. Also, the general framework for super-additive game normalization is presented. The proposed framework allows for the shrinkage (expansion) of coalition values to lie within a pre-specified boundary.

2.1. Basic Definitions and Notation

A transferable utility (TU) game on a fixed number of players N is defined by a characteristic function v. Thus, a TU game is a pair, (N, v). Without loss of generality, a TU game can be represented as v. Let $\Omega = 2^n$ be the set of all coalitions. A subset θ of N $(\theta \in 2^n)$ is a particular coalition whose size is denoted as $|\theta|$. For every coalition, $\theta \in 2^n$, $v_\theta : 2^n \to \mathbb{R}$ is a function that assigns a real value (worth) to each coalition.

By convention, $v_\phi = 0$ where ϕ is an empty coalition, v_i is the stand-alone worth of player i, and v_N is the worth of the grand coalition. We assume that the characteristic function is super-additive. That is, there exist at least two disjoint coalitions, $S, T \subset N$, such that $v_{S \cup T} \geq v_S + v_T$.

Similarly, let (N, ω) be another game defined on the same player set, N. The game, (N, v) can be transformed to its equivalence (N, ω) such that the coalition values ω are restricted to lie within a pre-specified boundary. This is done with a function given as,

$$\omega_\theta = \sum_{i \in \theta} \tau_i + \sigma v_\theta \quad \forall \theta \in 2^n \tag{2.1}$$

where τ_i $(i = 1, 2, \ldots, n)$ and σ are transformation (normalization) parameters.

2.2. The Generalized Form of Normalization TU Games

Let α and β be arbitrary non-negative integers forming the lower and upper boundaries, respectively. For any such positive integers, we want to establish a generalization for the normalization of super-additive games using the following theorem:

Theorem 1: Every essential game, (N, v) has an (α, β)-normalized (standardized) equivalence (N, ω) such that for any coalition, θ, $\omega_\theta \in [\alpha, \beta], \beta > \alpha$.

Proof: Let the original (essential) game be defined by a characteristics function v and let the (α, β)-normalized game be defined by a characteristics function ω. Our interest is to find the transformation (normalization) parameters $\sigma, \tau_i (i \in N)$ that satisfy $\omega_\theta \in [\alpha, \beta]$, and also make v to be strategically equivalent to ω.

Let

$$\sigma v_i + \tau_i = \alpha \tag{2.2}$$

be a linear function of stand-alone worth, v_i, for $i = 1, 2, \ldots, n$. Also, let

$$\sigma v_N + \sum_{i \in N} \tau_i = \beta \tag{2.3}$$

be a linear function of the grand coalition, v_N. Taking the sum of both sides of equation (2) and evaluating accordingly implies that,

$$\sum_{i=1}^{n} \tau_i = n\alpha - \sigma \sum_{i=1}^{n} v_i \tag{2.4}$$

Substitute for $\sum_{i=1}^{n} \tau_i$ into equation (3) and solve for σ. Hence,

$$\sigma v_N + n\alpha - \sigma \sum_{i=1}^{n} v_i = \beta \tag{2.5}$$

Therefore, $\sigma = \frac{\beta - n\alpha}{v_N - \sum_{i=1}^{n} v_i}$; $\quad v_N \neq \sum_{i=1}^{n} v_i$.

To find τ_i, substitute for σ into equation (2) and solve for τ_i.

$$\left(\frac{\beta - n\alpha}{v_N - \sum_{i=1}^{n} v_i} \right) v_i + \tau_i = \alpha \tag{2.6}$$

$$\tau_i = \alpha - \left(\frac{\beta - n\alpha}{v_N - \sum_{i=1}^{n} v_i} \right) v_i \tag{2.7}$$

Therefore, $\sigma = \frac{\beta - n\alpha}{v_N - \sum_{i=1}^{n} v_i}$ and $\tau_i = \alpha - \left(\frac{\beta - n\alpha}{v_N - \sum_{i=1}^{n} v_i} \right) v_i$ are the $n + 1$ constants that induce (α, β)-normalization on any given super-additive game.

Lemma 1. Let (N, v) be an essential (original) game, and (N, ω) the corresponding (α, β)-normalized game. The Harsanyi dividend of the (α, β)-normalized game is a scaled ratio of the Harsanyi dividend of the original game to the grand benefit due to cooperation. That is

$$H_\omega(\theta) = \frac{(\beta - n\alpha) H_v(\theta)}{v_N - \sum_{i=1}^{n} v_i} \tag{2.8}$$

Proof: It is trivial that for all $\theta = \{i\}$, $H_\omega(\theta) = \omega_i = \alpha$. This follows from the fact that the Harsanyi dividend of a singleton coalition in every super-additive game is equal to the player's stand-alone value. However, for all $\theta \neq \{i\}$, the proof is as follows:

Recall that $H_v(\theta) = \sum_{T \subseteq \theta} (-1)^{|\theta| - t} v_T$, where t is the cardinality of T (that is, the number of players in the subcoalition T). It follows that,

$$H_\omega(\theta) = \sum_{T \subseteq \theta} (-1)^{|\theta| - t} \left(\sum_{i \in T} \tau_i + \sigma v_T \right) \tag{2.9}$$

$$= \sum_{T\subseteq\theta}(-1)^{|\theta|-t}\sum_{i\in T}\tau_i + \sigma\sum_{T\subseteq\theta}(-1)^{|\theta|-t}v_T \tag{2.10}$$

$$= \sum_{T\subseteq\theta}(-1)^{|\theta|-t}\left(\alpha - \frac{(\beta-n\alpha)\sum_{i\in T}v_i}{v_N-\sum_{i=1}^n v_i}\right) + \frac{\beta-n\alpha}{v_N-\sum_{i=1}^n v_i}\sum_{T\subseteq\theta}(-1)^{|\theta|-t}v_T \tag{2.11}$$

$$= \sum_{T\subseteq\theta}(-1)^{|\theta|-t}(t\alpha - \sigma\sum_{i\in T}v_i) + \frac{\beta-n\alpha}{v_N-\sum_{i=1}^n v_i}\sum_{T\subseteq\theta}(-1)^{|\theta|-t}v_T \tag{2.12}$$

But $\sum_{T\subseteq\theta}(-1)^{|\theta|-t}(t\alpha-\sigma\sum_{i\in T}v_i)=0$ for all $|\theta|\geq 2$. Therefore,

$$H_\omega(\theta) = \frac{\beta-n\alpha}{v_N-\sum_{i=1}^n v_i}\sum_{T\subseteq\theta}(-1)^{|\theta|-t}v_T = \frac{(\beta-n\alpha)H_v(\theta)}{v_N-\sum_{i=1}^n v_i} \quad \text{for all } \theta\neq\{i\}. \tag{2.13}$$

Remark: If a given game is inessential, its (α,β)-normalized equivalence is also inessential. Similarly, if α and β are chosen such that $\beta-n\alpha\neq 0$, for any given number of players $n\geq 2, \alpha,\beta\in\mathbb{R}^+; \beta>\alpha$, the Harsanyi dividend of any coalition $\theta:\theta\neq\{i\}$ in (α,β)-normalized game is $(\beta-n\alpha)$ of the Harsanyi dividend of the same coalition in $(0,1)$-normalized game. Furthermore, the dividend of any coalition $\theta:|\theta|\geq 2$ in a $(0,1)$-normalized game is the ratio of the dividend in the main game to the grand coalition benefit.

2.3. Transferable Utility Invariant Property of The Generalized Form of Normalization

Normalizing a super-additive game induces a transformation of the player's imputation space. Thus, the following relation holds:

Let $\varphi_i(v)$ be a weighted Shapley value function (see [10], [11], [12]) for player i in the original game, (N,v). This has been extended to apply to machine learning (fair credit assignment), as recently shown in Fast weighted Shapley value ([13]). Also, let $\varphi_i(\omega)$ be the value function for the (α,β)-normalized game,

$$\varphi_i(v) = \begin{cases} \frac{\varphi_i(\omega)-\tau_i}{\sigma} & \text{if } v_i=v_j \text{ for all } i\neq j \\ \frac{\varphi_i(\omega)-\tau_i}{\sigma} + q_i(v) & \text{otherwise} \end{cases} \tag{2.14}$$

where $q_i(v)=\sum_{i\in\theta;\theta\in 2^n}\frac{\theta w_i-\pi_\theta}{\theta\pi_\theta}H_v(\theta)$ is a correction factor that accounts for heterogeneous stand-alone worths across players, w_i is the (sharing) weight of player i, $\pi_\theta=\sum_{i\in\theta}w_i$, and $H_v(\theta)=\sum_{T\subseteq\theta}(-1)^{|\theta|-t}v_T$ is the dividend accrued to coalition, θ ([14]).

Now, we consider the Rank-Shapley value (see [10], [15], [16]) and show that the Rank-Shapley value of an (α,β)-normalized game is equivalent to the Rank-Shapley value of the original game through the relation in equation (4).

The first part of the proof is based on the premise that the weights (ranks) of the players in the original game are retained in the (α,β)-normalized game. This holds only if $v_i=v_j$ for all $i\neq j$. Recall that if $v_i=v_j$ for all $i\neq j, \varphi_i(v)=v_i+\sum_{i\in\theta;i\neq\theta;\theta\in 2^n}\frac{1}{|\theta|}H_v(\theta)$. Similarly, $\varphi_i(\omega)=\omega_i+\sum_{i\in\theta;i\neq\theta;\theta\in 2^n}\frac{1}{|\theta|}H_\omega(\theta)$ since $\omega_i=\omega_j$ for all $i\neq j$.

From equation (4),

$$\varphi_i(v) = \frac{v_N-\sum_{i=1}^n v_i}{\beta-n\alpha}\left(\omega_i + \sum_{i\in\theta;i\neq\theta;\theta\in 2^n}\frac{1}{|\theta|}H_\omega(\theta)\right) + \frac{v_N-\sum_{i=1}^n v_i}{\beta-n\alpha}(v_i-\alpha) \tag{2.15}$$

But, $H_\omega(\theta)=\frac{(\beta-n\alpha)H_v(\theta)}{v_N-\sum_{i=1}^n v_i}$ for all $\theta\neq\{i\}$ (see Lemma 1).

$$\varphi_i(v) = \frac{v_N-\sum_{i=1}^n v_i}{\beta-n\alpha}\omega_i + \sum_{i\in\theta;i\neq\theta;\theta\in 2^n}\frac{\beta-n\alpha}{|\theta|}\frac{H_v(\theta)}{v_N-\sum_{i=1}^n v_i} + \frac{v_N-\sum_{i=1}^n v_i}{\beta-n\alpha}(v_i-\alpha) \tag{2.16}$$

Recall that in (α, β)-normalized game, $\omega_i = \alpha$ for all i. Therefore,

$$\varphi_i(v) = \frac{v_N - \sum_{i=1}^n v_i}{\beta - n\alpha} \sum_{i \in \theta; i \neq \theta; \theta \in 2^n} \frac{\beta - n\alpha}{v_N - \sum_{i=1}^n v_i} \frac{H_v(\theta)}{|\theta|} + \frac{v_N - \sum_{i=1}^n v_i}{\beta - n\alpha} v_i \tag{2.17}$$

$$= v_i + \sum_{i \in \theta; i \neq \theta; \theta \in 2^n} \frac{1}{|\theta|} H_v(\theta) \tag{2.18}$$

$$= \sum_{i \in \theta; \theta \in 2^n} \frac{1}{|\theta|} H_v(\theta) \tag{2.19}$$

The second part of the proof is a case where $v_i \neq v_j$ for at least one player. In this case, re-weighting of the players is prompted in the normalized game. So, we introduce a correction factor $q_i(v)$ to account for heterogeneous stand-alone worth across players. Hence,

$$\varphi_i(v) = \frac{\varphi_i(\omega) - \tau_i}{\sigma} + q_i(v) \tag{2.20}$$

$$= \sum_{i \in \theta; \theta \in 2^n} \frac{1}{|\theta|} H_v(\theta) + \sum_{i \in \theta; \theta \in 2^n} \frac{\theta w_i - \pi_\theta}{\theta \pi_\theta} H_v(\theta) \tag{2.21}$$

$$= \sum_{i \in \theta; \theta \in 2^n} \frac{w_i}{\pi_\theta} H_v(\theta) \tag{2.22}$$

Since equation (4) holds, it follows that,

$$\varphi_i(\omega) = \begin{cases} \sigma \varphi_i(v) + \tau_i & \text{if } v_i = v_j \text{ for all } i \neq j \\ \sigma(\varphi_i(v) - q_i(v)) + \tau_i & \text{otherwise} \end{cases} \tag{2.23}$$

The introduction of a correction factor in equation (4) handles the case when players have different stand-alone worth. This extends classical normalization theory by showing that normalization does not just rescale coalition values, it also maintains fairness in redistributions when heterogeneity exists. This result is non-trivial and significant because transformations of games often risk changing the solution concept.

3. Examples

To demonstrate the practical applicability of the proposed normalization framework established in Section 2, we present two different examples that illustrate how coalition values in super-additive games can be systematically compressed or expanded to fit within a pre-specified boundary.

Example 1. Here, we consider the normalization of a four-player game with different coalition structures as contained in [17]. The example has four different cases of coalition structure:

Case 1. Symmetric two-player and three-player coalitions

$$v_1 = v_2 = v_3 = v_4 = 0$$
$$v_{1,2} = v_{1,3} = v_{1,4} = v_{2,3} = v_{2,4} = v_{3,4} = c_1$$
$$v_{1,2,3} = v_{1,2,4} = v_{1,3,4} = v_{2,3,4} = c_2$$
$$v_{1,2,3,4} = c$$

In case 1 above, $\tau_i = \alpha$ for all i and $\sigma = \frac{\beta - 4\alpha}{c}$. Using equation (1), we generate the following strategic equivalent game, $(4, \omega)$:

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = \alpha$$
$$\omega_{1,2} = \omega_{1,3} = \omega_{1,4} = \omega_{2,3} = \omega_{2,4} = \omega_{3,4} = 2\alpha + \frac{c_1}{c}(\beta - 4\alpha)$$
$$\omega_{1,2,3} = \omega_{1,2,4} = \omega_{1,3,4} = \omega_{2,3,4} = 3\alpha + \frac{c_2}{c}(\beta - 4\alpha)$$
$$\omega_{1,2,3,4} = \beta$$

The solution of the new game is,

$$\varphi_i(\omega) = \sigma\varphi_i(v) + \tau_i = \alpha + \frac{\beta - 4\alpha}{c}\varphi_i(v) \tag{3.1}$$

Where $\varphi_i(v) = \frac{2.5c}{10}$ is the Rank-Shapley value of the players in the original game (case 1) for all i (symmetry). Hence, $\varphi_i(\omega) = \frac{2.5\beta}{10}$ for all i (symmetry).

Case 2. Symmetric two-player and non-symmetric three-player coalitions

$$\begin{gathered} v_1 = v_2 = v_3 = v_4 = 0 \\ v_{1,2} = v_{1,3} = v_{1,4} = v_{2,3} = v_{2,4} = v_{3,4} = c_1 \\ v_{1,2,3} = v_{1,2,4} = c_2, \quad v_{1,3,4} = v_{2,3,4} = c_3 \\ v_{1,2,3,4} = c \end{gathered}$$

In case 2 above, $\tau_i = \alpha$ for all i and $\sigma = \frac{\beta - 4\alpha}{c}$. Using equation (1), we generate the following strategic equivalent game, $(4, \omega)$:

$$\begin{gathered} \omega_1 = \omega_2 = \omega_3 = \omega_4 = \alpha \\ \omega_{1,2} = \omega_{1,3} = \omega_{1,4} = \omega_{2,3} = \omega_{2,4} = \omega_{3,4} = 2\alpha + \frac{c_1}{c}(\beta - 4\alpha) \\ \omega_{1,2,3} = \omega_{1,2,4} = 3\alpha + \frac{c_2}{c}(\beta - 4\alpha), \quad \omega_{1,3,4} = \omega_{2,3,4} = 3\alpha + \frac{c_3}{c}(\beta - 4\alpha) \\ \omega_{1,2,3,4} = \beta \end{gathered}$$

The solution of the new game is,

$$\varphi_i(\omega) = \sigma\varphi_i(v) + \tau_i = \alpha + \frac{\beta - 4\alpha}{c}\varphi_i(v) \tag{3.2}$$

Where $\varphi_1(v) = \varphi_2(v) = \frac{3c + 2c_2 - 2c_3}{12}$ and $\varphi_3(v) = \varphi_4(v) = \frac{3c - 2c_2 + 2c_3}{12}$ are the Rank-Shapley value of the players in the original game (case 2). Hence, $\varphi_1(\omega) = \varphi_2(\omega) = \alpha + \frac{(\beta - 4\alpha)(3c + 2c_2 - 2c_3)}{12c}$ and $\varphi_3(\omega) = \varphi_4(\omega) = \alpha + \frac{(\beta - 4\alpha)(3c - 2c_2 + 2c_3)}{12c}$.

Case 3. Non-symmetric two-player and symmetric three-player coalitions

$$\begin{gathered} v_1 = v_2 = v_3 = v_4 = 0 \\ v_{1,2} = v_{1,3} = v_{1,4} = c_1, \quad v_{2,3} = v_{2,4} = v_{3,4} = c_2 \\ v_{1,2,3} = v_{1,2,4} = v_{1,3,4} = v_{2,3,4} = c_3 \\ v_{1,2,3,4} = c \end{gathered}$$

In case 3 above, $\tau_i = \alpha$ for all i and $\sigma = \frac{\beta - 4\alpha}{c}$. Using equation (1), we generate the following strategic equivalent game, $(4, \omega)$:

$$\begin{gathered} \omega_1 = \omega_2 = \omega_3 = \omega_4 = \alpha \\ \omega_{1,2} = \omega_{1,3} = \omega_{1,4} = 2\alpha + \frac{c_1}{c}(\beta - 4\alpha) \\ \omega_{2,3} = \omega_{2,4} = \omega_{3,4} = 2\alpha + \frac{c_2}{c}(\beta - 4\alpha) \\ \omega_{1,2,3} = \omega_{1,2,4} = \omega_{1,3,4} = \omega_{2,3,4} = 3\alpha + \frac{c_3}{c}(\beta - 4\alpha) \\ \omega_{1,2,3,4} = \beta \end{gathered}$$

The solution of the new game is,

$$\varphi_i(\omega) = \sigma\varphi_i(v) + \tau_i = \alpha + \frac{\beta - 4\alpha}{c}\varphi_i(v) \tag{3.3}$$

Where $\varphi_1(v) = \frac{c+c_1-c_2}{4}$ and $\varphi_2(v) = \varphi_3(v) = \varphi_4(v) = \frac{3c-c_1+c_2}{12}$ are the Rank-Shapley value of the players in the original game (case 3). Hence, $\varphi_1(\omega) = \alpha + \frac{(\beta-4\alpha)(c+c_1-c_2)}{4c}$ and $\varphi_2(\omega) = \varphi_3(\omega) = \varphi_4(\omega) = \alpha + \frac{(\beta-4\alpha)(3c-c_1+c_2)}{12c}$.

Case 4. Non-symmetric two-player and three-player coalitions

$$v_1 = v_2 = v_3 = v_4 = 0$$
$$v_{1,2} = v_{1,3} = v_{1,4} = c_1, \quad v_{2,3} = v_{2,4} = v_{3,4} = c_2$$
$$v_{1,2,3} = v_{1,2,4} = c_3, \quad v_{1,3,4} = v_{2,3,4} = c_4$$
$$v_{1,2,3,4} = c$$

In case 4 above, $\tau_i = \alpha$ for all i and $\sigma = \frac{\beta-4\alpha}{c}$. Using equation (1), we generate the following strategic equivalent game, $(4, \omega)$:

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = \alpha$$
$$\omega_{1,2} = \omega_{1,3} = \omega_{1,4} = 2\alpha + \frac{c_1}{c}(\beta - 4\alpha)$$
$$\omega_{2,3} = \omega_{2,4} = \omega_{3,4} = 2\alpha + \frac{c_2}{c}(\beta - 4\alpha)$$
$$\omega_{1,2,3} = \omega_{1,2,4} = 3\alpha + \frac{c_3}{c}(\beta - 4\alpha)$$
$$\omega_{1,3,4} = \omega_{2,3,4} = 3\alpha + \frac{c_4}{c}(\beta - 4\alpha)$$
$$\omega_{1,2,3,4} = \beta$$

The solution of the new game is,

$$\varphi_i(\omega) = \sigma\varphi_i(v) + \tau_i = \alpha + \frac{\beta - 4\alpha}{c}\varphi_i(v) \tag{3.4}$$

Where $\varphi_1(v) = \frac{3c+3c_1-3c_2+2c_3-2c_4}{12}$, $\varphi_2(v) = \frac{3c-c_1+c_2+2c_3-2c_4}{12}$, $\varphi_3(v) = \frac{3c-c_1+c_2-2c_3+2c_4}{12}$, and $\varphi_4(v) = \frac{3c-c_1+c_2-2c_3+2c_4}{12}$ are the Rank-Shapley value of the players in the original game (case 4). Hence, $\varphi_1(\omega) = \alpha + \frac{(\beta-4\alpha)(3c+3c_1-3c_2+2c_3-2c_4)}{12c}$, $\varphi_2(\omega) = \alpha + \frac{(\beta-4\alpha)(3c-c_1+c_2+2c_3-2c_4)}{12c}$, $\varphi_3(\omega) = \alpha + \frac{(\beta-4\alpha)(3c-c_1+c_2-2c_3+2c_4)}{12c}$ and $\varphi_4(\omega) = \alpha + \frac{(\beta-4\alpha)(3c-c_1+c_2-2c_3+2c_4)}{12c}$.

Example 2. Here, we consider a classic non-symmetric three-player game drawn from [5].

$$v_1 = 1, v_2 = 1/4, v_3 = -1, v_{1,2} = 3, v_{1,3} = 1, v_{2,3} = 1$$
$$v_{1,2,3} = 4$$

The above game can be transformed to lie within any pre-specified bound, $[\alpha, \beta]$, thereby generating a new game ω that is strategically equivalent to v. For instance, let $\alpha = 1$ and $\beta = 2$ be arbitrarily chosen. We aim to transform the above game to lie within the interval $[1, 2]$. The $n + 1$ parameters that will generate the $(1, 2)$-normalization on the above game are estimated as follows:

$$\sigma = \frac{2 - 3(1)}{4 - (1 + 1/4 - 1)} = \frac{-1}{15/4} = -\frac{4}{15} \tag{3.5}$$

$$\tau_1 = 1 - \left(-\frac{4}{15}\right)(1) = \frac{19}{15}, \quad \tau_2 = 1 - \left(-\frac{4}{15}\right)(1/4) = \frac{16}{15}, \quad \tau_3 = 1 - \left(-\frac{4}{15}\right)(-1) = \frac{11}{15}$$

Using equation (1), the normalized game is generated as follows:

$$\omega_1 = 1, \omega_2 = 1, \omega_3 = 1$$
$$\omega_{1,2} = \frac{23}{15}, \omega_{1,3} = \frac{26}{15}, \omega_{2,3} = \frac{23}{15}, \omega_{1,2,3} = 2$$

Table 1: The original game

Coalition	v_θ	$H_v(\theta)$	π_θ	$\varphi_i(v)$	$q_i(v)$
1	1	1	3	291/120	3/10
2	1/4	1/4	2	28/15	7/60
3	-1	-1	1	-7/24	-5/12
1, 2	3	7/4	5		
1, 3	1	1	4		
2, 3	1	7/4	3		
1, 2, 3	4	-3/4	6		

To justify the transferable utility invariant property of the framework on the example above, we make use of the second part of equation (4) since $v_i \neq v_j$ for all $i \neq j$. The Harsanyi dividend, coalition (players') rank, Rank-Shapley value, and correction factor for non-equality of players' stand-alone worth in the original game are presented in columns 3, 4, 5, and 6, respectively, in Table 1.

From the second part of equation (4), the payoff for the players in the $(1, 2)$-normalized game is $\varphi_i(\omega) = \sigma\varphi_i(v) + \tau_i - \sigma q_i(v)$. Therefore,

$$\varphi_1(\omega) = \left(\frac{291}{120}\right)\left(-\frac{4}{15}\right) + \frac{19}{15} - \left(\frac{3}{10}\right)\left(-\frac{4}{15}\right) = \frac{21}{30}$$

$$\varphi_2(\omega) = \left(\frac{28}{15}\right)\left(-\frac{4}{15}\right) + \frac{16}{15} - \left(\frac{7}{60}\right)\left(-\frac{4}{15}\right) = \frac{3}{5}$$

$$\varphi_3(\omega) = \left(-\frac{7}{24}\right)\left(-\frac{4}{15}\right) + \frac{11}{15} - \left(-\frac{5}{12}\right)\left(-\frac{4}{15}\right) = \frac{21}{30}$$

These generated payoffs correspond to the Rank-Shapley value of the transformed game, ω. By this, the transferable utility invariant property is satisfied.

4. Discussion of Results

The transformation framework in this study enforces equality among all players in terms of individual worth, while maintaining symmetry, efficiency, and invariant property. In Section 2, the transformation equations were systematically derived and solved to estimate the parameters that govern the normalization process. It is important to note that the parameters α and β are constrained to non-negative values, implying that transformations involving negative boundaries are not feasible or meaningful within this framework.

The results demonstrate that the proposed normalization method encompasses and generalizes existing normalization techniques. For instance, setting $\alpha = 0$ and $\beta = 1$ reduces the model to the well-known $(0, 1)$-normalization. On the other hand, choosing α and β such that $\beta - n\alpha = 0$, yields a special case of an inessential game defined as $\omega_\theta = |\theta|\alpha$, which is independent of the original game's data and represents a structurally simplified form.

A key contribution of this framework is the preservation of symmetry, efficiency, and strategic equivalence between the original and normalized games. This is formalized in equation (4), which establishes a link between their respective solution concepts. This feature aligns with the transferable utility invariant (TU-invariant) property discussed in [18], ensuring that the essence of the value function remains unchanged under normalization.

Furthermore, the framework introduces a correction factor that accounts for disparities in individual player worth in the original game, (N, v). This ensures a fairer and more balanced redistribution under the normalized scheme. Corollary 1 highlights the relationship between the dividend allocations in the transformed game and those in the original game, giving a deeper insight into how value is distributed after normalization.

Finally, the practical applicability of the proposed framework was demonstrated using two examples. The examples illustrate how the normalization scheme compresses coalition values to fall within a predefined boundary, $[\alpha, \beta]$, without altering the strategic structure of the game, such as symmetry, efficiency, and strategic equivalence. This makes the normalized game more tractable and computationally efficient for further analysis.

5. Conclusion

This study proposes a generalized framework for transforming (normalizing) super-additive games such that coalition values fall within arbitrarily defined positive boundaries. It introduces a family of normalizations that subsume earlier versions of boundary transformation (normalization) as special cases. The study makes both theoretical and practical contributions to the literature on super-additive games.

On the theoretical side, we develop a generalized α, β-normalization framework that unifies and extends existing normalization schemes, while preserving symmetry, efficiency, and invariance properties. We further introduce a correction mechanism that accounts for heterogeneous stand-alone worths across players and show how normalization proportionally preserves Harsanyi dividends.

On the practical side, the framework offers flexibility to normalize coalition values into any prescribed boundaries, thereby enhancing comparability across games, improving computational tractability, and enabling applications where coalition payoffs must align with real-world constraints such as budget caps or policy limits. Instead of being forced into the narrow $0, 1$ interval, practitioners can normalize games into any desired boundary α, β (e.g., budgetary constraints, policy caps, resource limits). This is useful for applied problems in economics, political science, and multi-agent systems where coalition payoffs must meet external restrictions. For example, in resource allocation or strategic decision-making, organizations can choose α and β according to policy rules (e.g., minimum guaranteed payoff or maximum allowable coalition surplus). This makes the framework not just a mathematical convenience but a policy-relevant tool.

However, since the current formulation restricts α and β to non-negative values, it does not accommodate transformations involving negative boundaries. This limitation highlights an avenue for future research: the development of an even more robust normalization scheme capable of handling broader boundary conditions, including the negative domain. Also, it can be extended to study how the framework preserves (affects) other standard structural properties of games.

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