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Solution of Partial Differential Equations via "Double SEE-Aboodh Transform"

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ABSTRACT: In this paper, we introduce a new double integral technique namely "Double SEE-Aboodh Teachnique" to evaluate the solution of the general linear partial differential equations (LPDE's). Many basic functions are used to prove the importance of this double technique. This technique depends on the SEE (Sadik-Emad-Eman) and Aboodh transforms, where these two transforms are combined to obtain the double SEE-Aboodh transform. It turns out that been this technique is used; the steps for solving partial differential equations are easy and simple to obtain the exact solution. Combining two different transformations is aimed at finding a new technique for finding exact solutions to partial differential equations in a way and manner that is easier and simpler than previous techniques. Especially after the emergence of many new life problems, it requires finding simple modern techniques that are easy to use in algebraic calculations, far from difficulty and complexity, and this is what we found in our paper through the examples for which exact solutions were found. Through this double transform we obtain many double transforms, the most important of which are the double Aboodh-Laplace, double Aboodh transforms, and others.

Keywords: Aboodh transform, SEE transform, double SEE transform, double Aboodh transform, double integral transform.

Contents

1. Introduction

Due to the emergence of many modern applications in the various natural sciences, engineering and its branches, we need different, simple techniques to evaluate the exact solutions. One of the simplest of these techniques is the method of integral transformations. Several authors and those interested in developing integral techniques have appeared, especially modifying the Laplace transform, as it is the origin of l transformations, we notice this in many integral techniques and each transform has feature that differs from the other through applications,and the most important of these transforms are Emad-Sara transform [11], AEM transform [3], Sadik and complex Sadik transforms [13], Emad-Falih transform [10], Shehu transform [2], [4], Alzughair transform, Altememe transform [9], KAJ transform, complex EE technique [5], Emad-Issa transform [12] and others [9], [13], [7]-[14], [5]. authors began to develop transforms by combining transforms among themselves to obtain the easiest and simplest way to find the exact solution to partial differential equations (PDE's) and their real life applications, such as heat and wave problems, telegraph and Poisson partial equations, the Laplace equation, and others,for examples Kuffi and other authors they solved PDE's and their physical problems with easy and simple algebraic calculation steps and also what physicists need [1], [6], [8]. It is one of the most important double integral transforms: double SEE, double complex SEE, double Emad-Falih, double complex EE, double SEJI, double Shehu and others. All of these double techniques solved partial differential operator equations,

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but they differ in the way they are used depending on the real-life problem [11], [1]. SEE technique is knowledge as a function of an exponential order in

$$A = \left\{ h(x) \ : \ \exists M > 0, \ \exists L_1, L_2 > 0, |h(x)| < M^{L_j|x|}, \ x \in (-1)^j \times [0,\infty), \ j = 1,2 \right\}.$$

as:

$$\mathcal{S}\{h(x)\} = H(u) = \frac{1}{u^n}\int_0^\infty e^{-ux}h(x)\,dx, \quad n \in \mathbb{Z}, x \geq 0, L_1 \leq u \leq L_2, \Re e(u) > 0.[2].$$

And the Aboodh transform is defined as:

$$\mathcal{A}\{h(t)\} = H(s) = \frac{1}{s}\int_0^\infty e^{-st}h(t)\,dt, \quad t \geq 0, \quad L_1 \leq s \leq L_2, \quad \Re e(s) > 0,$$

see [5].

2. Double See-Aboodh Transform Technique

If $h(x,t)$, where t and x belong to the positive set real numbers is a convergent infinite expansion, so it is double SEE-Aboodh technique is introduced as:

$$\mathcal{SA}\{h(t,x)\} = \frac{1}{u^n s}\int_0^\infty \int_0^\infty e^{-(ux+st)}h(x,t)\,dt\,dx = H(u,s).$$

where u and s are complex parameters, $\Re(u) > 0, \ \Re(s) > 0, \ n \in \mathbb{Z}$.

And the inverse of this double integral transform is:

$$h(x,t) = \mathcal{SA}^{-1}\{H(u,s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} u^n e^{ux}\,du \left(\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} se^{st}H(u,s)\,ds\right).$$

where γ and μ are real constants.

Occasions, such as program code (Section 2.3.7).
Table 1: Relation Between SEE-Aboodh Transform with Other Double Transforms.

n	Name of double Transform	Formula
$n=0$	Double Laplace – Aboodh [6]	$\frac{1}{v}\int_0^\infty\int_0^\infty e^{-(ux+vt)}\,h(x,t)\,dt\,dx$
$n=1$	Double Aboodh [5]	$\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(ux+vt)}\,h(x,t)\,dt\,dx$
$n=-1$	Double Mahgoub – Aboodh	$\frac{u}{v}\int_0^\infty\int_0^\infty e^{-(ux+vt)}\,h(x,t)\,dt\,dx$
$n=2$	Double Emad – Sara – Aboodh	$\frac{1}{u^2v}\int_0^\infty\int_0^\infty e^{-(ux+vt)}\,h(x,t)\,dt\,dx$
$n=3$	Double Gupta – Aboodh	$\frac{1}{u^3v}\int_0^\infty\int_0^\infty e^{-(ux+vt)}\,h(x,t)\,dt\,dx$

Table 1: Different forms of double Aboodh-type transforms

3. Double See-Aboodh Technique for Elementary Functions

1. If $h(x,t) = k$, where k is a constant, $x, t > 0$, then

$$SA\{k\} = \frac{1}{u^n s}\int_0^\infty\int_0^\infty e^{-ux}e^{-st}\,k\,dt\,dx$$
$$= \frac{k}{u^{n+1}s^2}.$$

2. If $h(x,t) = xt,$ then

$$SA\{xt\} = \frac{1}{u^n s}\int_0^\infty\int_0^\infty e^{-ux}e^{-st}\,xt\,dt\,dx$$
$$= \frac{1}{u^{n+2}s^3}.$$

Hence,

$$SA\{xt\} = \frac{1}{u^{n+2}s^3}.$$

Also,

$$SA\{x^2t^2\} = \frac{2!\,2!}{u^{n+3}s^4},$$
$$SA\{x^3t^3\} = \frac{3!\,3!}{u^{n+4}s^5}.$$

In general,

$$SA\{x^m t^m\} = \frac{m!\,m!}{u^{m+n+1}s^{m+2}}, \qquad m = 0,1,2,\dots$$

And,

$$SA\{x^m t^m\} = \frac{\Gamma(m+1)\Gamma(m+1)}{u^{m+n+1}s^{m+2}}, \qquad m > -1,$$

where $\Gamma(\cdot)$ is the Gamma function.

3. If $h(x,t) = e^{\alpha x+\beta t},$ where α, β are constants, then

$$SA\{e^{\alpha x+\beta t}\} = \frac{1}{u^n s}\int_0^\infty\int_0^\infty e^{-(ux+st)}e^{\alpha x+\beta t}\,dt\,dx$$
$$= \frac{1}{u^n s(u-\alpha)(s-\beta)}.$$

4. If $h(x,t) = \cos(\alpha x+\beta t),$ then

$$SA\{\cos(\alpha x+\beta t)\} = \frac{1}{u^n s}\int_0^\infty\int_0^\infty e^{-(ux+st)}\cos(\alpha x+\beta t)\,dt\,dx$$
$$= \frac{us-\alpha\beta}{u^n s\,(s^2+\beta^2)(u^2+\alpha^2)},$$

where α and β are constants.

4. Double See-Aboodh Technique for the Second Partial Derivatives

1. $SA\left\{\frac{\partial^2 h}{\partial x^2}\right\} = u^2 H(u,s) - \frac{u}{u^n}H(0,u) - \frac{1}{u^n}\frac{\partial h(0,s)}{\partial x}.$

Proof:

$$SA\left\{\frac{\partial^2 h}{\partial x^2}\right\} = \frac{1}{u^n s}\int_{x=0}^\infty\int_{t=0}^\infty e^{-(ux+st)}\frac{\partial^2 h}{\partial x^2}\,dt\,dx,$$
$$= \frac{1}{s}\int_{t=0}^\infty e^{-st}\left[\frac{1}{u^n}\int_{x=0}^\infty e^{-ux}\frac{\partial^2 h}{\partial x^2}\,dx\right]dt.$$

Integrate

$$\frac{1}{u^n}\int_{x=0}^{\infty} e^{-ux}\frac{\partial^2 h}{\partial x^2}\,dx \quad \text{by parts:}$$

$$\Rightarrow \left[\frac{1}{u^n}\int_{x=0}^{\infty} e^{-ux}\frac{\partial^2 h}{\partial x^2}\,dx\right] = \frac{1}{u^n}\left[e^{-ux}\frac{\partial h(x,t)}{\partial x}\right]_0^{\infty} + \frac{u}{u^n}\int_{x=0}^{\infty} e^{-ux}\frac{\partial h}{\partial x}\,dx,$$

$$= \frac{1}{u^n}\left[-\frac{\partial h(x,t)}{\partial x}\right]_{x=0} - \frac{u}{u^n}h(0,t) + \frac{u^2}{u^n}\int_{x=0}^{\infty} e^{-ux}h\,dx,$$

$$= u^2H(u,s) - u^{1-n}h(0,t) - \frac{1}{u^n}\frac{\partial h(0,t)}{\partial x}.$$

Then

$$SA\left\{\frac{\partial^2 h}{\partial x^2}\right\} = \frac{1}{s}\int_{t=0}^{\infty} e^{-st}\left[u^2H(u,s) - u^{1-n}h(0,t) - \frac{1}{u^n}\frac{\partial h(0,t)}{\partial x}\right]dt,$$

$$= u^2H(u,s) - u^{1-n}H(0,s) - \frac{1}{u^n}\frac{\partial H(0,s)}{\partial x}.$$

It is possible to prove (2) and (3) using the same approach.

2. $SA\left\{\frac{\partial^2 h}{\partial t^2}\right\} = s^2H(u,s) - H(u,0) - \frac{1}{s}\frac{\partial H(u,0)}{\partial t}$.
3. $SA\left\{\frac{\partial^2 h}{\partial x\partial t}\right\} = su\,H(u,s) - uH(u,0) - sH(0,s) - H(0,0)$.

5. Application of Appling Double See-Aboodh Technique on Partial Differential Equations

The Validity of using double SEE-Aboodh technique to evaluate the solution of partial differential equation is established during a practical problem.

Example 5.1. *Consider the following 2^{nd} order PDE:*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

with

$$U(x,0) = \sin(x), \qquad U_t(x,0) = 2,$$
$$U(0,t) = 2t, \quad \textit{and} \qquad U_x(0,t) = \cos(t).$$

Solution: Applying the double SEE–Aboodh technique to the given equation,

$$s^2H(u,s) - H(u,0) - \frac{1}{s}\frac{\partial}{\partial t}H(u,0) = u^2H(u,s) - \frac{u}{u^n}H(0,s) - \frac{1}{u^n}\frac{\partial}{\partial x}H(0,s).$$

For

$$H(u,0) = \frac{1}{u^n(u^2+1)}, \qquad H_t(u,0) = \frac{2}{u^{n+1}},$$
$$H(0,s) = \frac{2}{s^3}, \quad \textit{and} \qquad H_x(0,s) = \frac{1}{s^2+1}.$$

Then

$$s^2H(u,s) - \frac{1}{u^n(u^2+1)} - \frac{2}{s\,u^{n+1}} = u^2H(u,s) - \frac{2u}{u^ns^3} - \frac{1}{u^n(s^2+1)}.$$

So,

$$(s^2 - u^2)H(u,s) = \frac{1}{u^n(u^2+1)} + \frac{2}{s\,u^{n+1}} - \frac{2u}{u^n s^3} - \frac{1}{u^n(s^2+1)}.$$

$$H(u,s) = \frac{(s^2+1)-(u^2+1)}{u^n(u^2+1)(s^2-u^2)(s^2+1)} + \frac{2s^2-2u^2}{u^{n+1}s^3(s^2-u^2)}.$$

$$H(u,s) = \frac{1}{u^n(u^2+1)(s^2+1)} + \frac{2}{u^{n+1}s^3}.$$

Take the inverse of double SEE-Aboodh technique to the both sides of above algebraic equation; we obtain the solution to the 2^{nd} order PDE

$$u_{xx} = u_{tt},$$

with conditions, which is:

$$u(x,t) = \sin(x)\cos(t) + 2t.$$

Example 5.2. *Solve the partial differential equation*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t},$$

with the initial conditions

$$u(0,t) = t, \qquad u(x,0) = x.$$

Solution: Let

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}.$$

Using the double SEE–Aboodh technique on the given equation, we obtain

$$uH(u,s) - \frac{1}{u^n}H(0,s) = sH(u,s) - \frac{1}{s}H(u,0).$$

$$\Rightarrow uH(u,s) - \frac{1}{u^n}H(s) = sH(u,s) - \frac{1}{s}H(u).$$

$$\Rightarrow uH(u,s) - \frac{1}{u^n}\frac{1}{s^3} = sH(u,s) - \frac{1}{s}\frac{1}{u^{n+2}}.$$

$$\Rightarrow H(u,s)(u-s) = \frac{1}{s^3u^n} - \frac{1}{su^{n+2}}.$$

$$\Rightarrow H(u,s)(u-s) = \frac{1}{u^n s}\left(\frac{1}{s^2} - \frac{1}{u^2}\right).$$

$$\Rightarrow H(u,s)(u-s) = \frac{1}{u^n s}\left(\frac{u^2-s^2}{u^2s^2}\right).$$

$$\Rightarrow H(u,s) = \frac{1}{u^n s}\left(\frac{1}{u^2s^2}\right)(u+s).$$

$$\Rightarrow H(u,s) = \left(\frac{1}{u^{n+1}s^3} + \frac{1}{u^{n+2}s^2}\right).$$

Take the inverse double SEE-Aboodh technique to the both sides above equation, we get the exact solution to the second order partial equation $u_x = u_t$ with given conditions, which is:

$$u(x,t) = x + t.$$

6. Conclusion

A new double integral technique derived from the well-known SEE (Sadik-Emad-Eman) and Aboodh integral techniques is said to be "double SEE-Aboodh technique" has been presented. The outputs of using the suggested integral technique to important basic functions and the partial derivatives has been discussed and showed. The competency of the double SEE-Aboodh technique for finding the solution of PDE's has been confirmed via finding an exact solution of PDE's problem, which opens the door for further usage of the double SEE-Aboodh technique in evaluating the solution for a vast range of PDE's. This double technique is characterized via its generality, as many double transforms are extracted, as we noted in Table 1. Through this work and by using the double technique, we observe the ease and simplicity of finding and calculating solutions to linear partial differential equations. This means that the above technique is not a general technique that includes many duoble transformations, but rather it is characterized by its simplicity and the simplicity of the computational operations as well.

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