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Generalized Statistical Convergence for Uncertain Double Sequences of Fuzzy Numbers Defined by an Orlicz Function

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ABSTRACT: In this study we propose the notion of $\mathcal{I}_2$-statistical convergence for uncertain double sequences of fuzzy numbers defined by an Orlicz function. We investigate several associated convergence types such as convergence in measure, convergence in mean, convergence in distribution and uniformly almost sure convergence. Furthermore we present illustrative examples to clarify the connections and differences among these distinct convergence forms.

Key Words: Uncertain sequence, fuzzy number, Orlicz function, ideal convergence, statistical convergence, double sequence.

Contents

1. Introduction and Preliminaries

Fast originally presented the idea of statistical convergence [8] which was subsequently investigated within the framework of sequence spaces by numerous scholars. One may refer to [2,12,13,14,26,27,41,48] for further details. This concept has developed into a dynamic field of research owing to its extensive applications in diverse mathematical disciplines including mathematical analysis [10,11,15,16], number theory and probability theory. The study of statistical convergence regarding double sequences was independently commenced by Mursaleen and Edely [28] as well as Tripathy [44].

Ideal convergence constitutes a broad generalization covering both classical and statistical convergence. Kostyrko et al. [18] further analyzed the notion of $\mathcal{I}$-convergence from a sequence space perspective and linked it with summability theory. Following this, researchers generalized the concept in various ways such as n-normed spaces and random spaces [42,49,50]. Das et al. [4] introduced $\mathcal{I}$-convergence for double sequences in metric spaces and examined its characteristics.

Zadeh [52] initially proposed the concept of fuzzy numbers and investigated their arithmetic attributes. This fundamental notion has gained widespread application in fields like artificial intelligence, computer science, control engineering, decision theory, management science, medicine, pattern recognition, operations research and robotics. Matloka [30] extended this idea by integrating it into summability theory and sequence spaces whereas Nanda [32] applied it to vector spaces and topology through fuzzy metrics. Moreover, fuzzy numbers hold a crucial position in the analysis of double sequence spaces, sequence spaces with fuzzy mappings, approximation theory and ideal convergence.

Uncertainty theory was first suggested by Liu [25] as a branch of measure theory and has garnered considerable attention due to its versatile applications in probability theory, statistics, fuzzy set theory, measure theory and summability theory. It possesses significant value in both practical and theoretical realms including risk evaluation and uncertain reliability analysis [22], modeling human language via uncertainty logic [23] and developing continuous uncertain measure theory [9].

Apart from theoretical progress, uncertainty theory provides practical utility in areas like uncertain finance [25] and uncertain optimization [24] where modeling uncertainty is essential. Liu [21] established uncertainty theory in 2007 which stimulated subsequent studies on various sequence convergence types in uncertainty spaces. In his pioneering work, Liu [21] described four main convergence forms for real

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uncertain sequences which are convergence in mean, in measure, in distribution and almost surely. Later, You [51] proposed a novel convergence concept termed uniformly almost sure convergence and analyzed its relation to existing types. Chen et al. [3] broadened the theory to complex uncertain variables while Nath and Tripathy [35] investigated the convergence of complex uncertain sequences via Orlicz functions. Datta and Tripathy [6] studied the convergence of double sequences with complex uncertain variables. Furthermore, Tripathy et al. [46] initiated the research on statistical convergence for complex uncertain sequences and later analyzed this notion using Orlicz functions [34]. Relevant studies on this subject appear in [5,7,36,39,40,43,47].

Recently, Baliarsingh et al. [1] presented the idea of statistical convergence for uncertain sequences of fuzzy numbers. Kişi and Choudhury [17] subsequently extended this concept to double sequences. Moreover, Raj et al. [38] investigated lacunary statistical convergence using Orlicz functions.

In recent years, the study of fuzzy numbers and uncertain sequences has gained significant attention due to their pivotal role in modeling imprecise data and complex systems. A comprehensive review of the existing literature indicates that $\mathcal{I}_2$-statistical convergence via Orlicz functions for uncertain fuzzy sequences remains largely unexplored. This gap presents a valuable opportunity to advance the theory of uncertain fuzzy sequences using natural density and Orlicz functions. In this context, our work introduces the concept of $\mathcal{I}_2$-statistical convergence for uncertain double sequences of fuzzy numbers, providing a novel framework to analyze convergence in settings where uncertainty and fuzziness are inherent. By examining related convergence types—such as convergence in measure, in mean, in distribution, and uniformly almost surely—this study not only extends the theoretical foundations of statistical convergence but also bridges gaps between different convergence notions. Illustrative examples are provided to clarify these connections, highlighting the practical and theoretical relevance of the proposed framework for researchers in mathematical analysis and uncertainty theory.

We provide the definition of a fuzzy number below.

Definition 1.1 *A fuzzy set $\mathcal{U}$ is a mapping $\mathcal{U}:\mathbb{R}\to[0,1]$ which is termed a fuzzy number if it satisfies the conditions below.*

1. ***Normality.*** *A point $u\in\mathbb{R}$ exists such that $\mathcal{U}(u)=1$.*
2. ***Fuzzy convexity.*** *For any $m,n\in\mathbb{R}$ and $\lambda\in[0,1]$, the inequality holds*
$$\mathcal{U}(\lambda m+(1-\lambda)n)\geq\min\{\mathcal{U}(m),\mathcal{U}(n)\}.$$
3. ***Upper semicontinuity.*** *The function $\mathcal{U}$ is upper semicontinuous.*
4. ***Compact support.*** *The closure of the set $\mathcal{U}^0=\{u\in\mathbb{R}:\mathcal{U}(u)>0\}$ is compact.*

We denote the collection of all fuzzy numbers defined on the real line by $\mathcal{L}(\mathbb{R})$. Specifically, we can identify each real number $t\in\mathbb{R}$ with a corresponding function $\overline{s}(u)\in\mathcal{L}(\mathbb{R})$ defined as follows

$$\overline{s}(u)=\begin{cases}1, & \text{if } u=t,\\ 0, & \text{if } u\neq t.\end{cases}$$

For any $\alpha\in(0,1]$, the α-level set or α-cut of a fuzzy number $\mathcal{U}$ is

$$[\mathcal{U}]_\alpha=\{\,u\in\mathbb{R}:\mathcal{U}(u)\geq\alpha\,\}.$$

Throughout this work, $\mathcal{L}(\mathbb{R})$ represents the space of all fuzzy numbers on $\mathbb{R}$.

Let $\mathcal{U}$ and $\mathcal{V}$ be two fuzzy numbers. We define their distance by

$$d(\mathcal{U},\mathcal{V})=\sup_{0\leq\alpha\leq 1} d_H\left([\mathcal{U}]_\alpha,[\mathcal{V}]_\alpha\right),$$

where d_H signifies the Hausdorff metric given by

$$d_H\left([\mathcal{U}]_\alpha,[\mathcal{V}]_\alpha\right)=\max\left\{\left([\mathcal{U}]_\alpha^- - [\mathcal{V}]_\alpha^-\right),\ \left([\mathcal{U}]_\alpha^+ - [\mathcal{V}]_\alpha^+\right)\right\},$$

while $[\mathcal{U}]_\alpha^-$ and $[\mathcal{U}]_\alpha^+$ represent the lower and upper endpoints of the α-level set of $\mathcal{U}$ respectively.

The space $\mathcal{L}(\mathbb{R})$ equipped with the metric d forms a complete metric space.

We say a sequence $\{\zeta_m\}$ is statistically convergent to ζ if for every $\varphi > 0$ the set

$$\mathcal{K}(\varphi) = \{ m \in \mathbb{N} : |\zeta_m - \zeta| \geq \varphi \}$$

possesses natural density zero as stated by Fast [8]. This means

$$\delta(\mathcal{K}(\varphi)) = \lim_{p\to\infty} \frac{1}{p} |\{ m \leq p : |\zeta_m - \zeta| \geq \varphi \}| = 0,$$

where $|\cdot|$ indicates the cardinality of the set.

Nuray and Savaş [33] introduced the fuzzy analogue following this primary notion of statistical convergence. A sequence $\{\mathcal{X}_s\}$ of fuzzy numbers is statistically convergent to $\mathcal{X}_0$ if for every $\varphi > 0$

$$\delta(\{ s : d(\mathcal{X}_s, \mathcal{X}_0) \geq \varphi \}) = \lim_{n\to\infty} \frac{1}{n} |\{ s \leq n : d(\mathcal{X}_s, \mathcal{X}_0) \geq \varphi \}| = 0.$$

As mentioned in [19], an Orlicz function is a mapping $\Xi : [0, \infty) \to [0, \infty)$ which is continuous, convex and non-decreasing satisfying $\Xi(0) = 0$, $\Xi(\vartheta) > 0$ for $\vartheta > 0$ and $\Xi(\vartheta) \to \infty$ as $\vartheta \to \infty$. If we replace the convexity condition with the inequality

$$\Xi(\vartheta + \alpha) \leq \Xi(\vartheta) + \Xi(\alpha),$$

we refer to the function as a modulus function. Nakano [31] first introduced this concept which was later extended by Musielak [29] and Tripathy [45] in their analysis of sequence spaces.

It is a well-known result that if Ξ is convex and satisfies $\Xi(0) = 0$ then

$$\Xi(\lambda\vartheta) \leq \lambda\,\Xi(\vartheta) \qquad \text{for all } 0 < \lambda < 1.$$

Similarly, Lindenstrauss and Tzafriri [20] used Orlicz functions to define the Orlicz sequence space as

$$\ell_\Xi = \left\{ w = \{w_k\} \in \omega : \sum_{k=1}^{\infty} \Xi\left(\frac{|w_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\},$$

which becomes a Banach space with the norm

$$\|w\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} \Xi\left(\frac{|w_k|}{\rho}\right) \leq 1 \right\}.$$

The space ℓ_Φ coincides with the classical ℓ_p space for $\Xi(\vartheta) = \vartheta^p$ with $1 \leq p < \infty$. An Orlicz function satisfies the Δ_2-condition if a constant $K > 0$ exists such that $\Xi(\gamma\vartheta) \leq K\gamma\,\Xi(\vartheta)$ for all $\gamma > 1$.

This study introduces several new variations of $\mathcal{I}_2$-statistical convergence for uncertain double sequences of fuzzy numbers defined via an Orlicz function. These variants consist of strong $\mathcal{I}_2$-Cesàro summability, $\mathcal{I}_2$-statistical boundedness, $\mathcal{I}_2$-statistical convergence almost surely, $\mathcal{I}_2$-statistical convergence in mean, $\mathcal{I}_2$-statistical convergence in measure, $\mathcal{I}_2$-statistical convergence in distribution and $\mathcal{I}_2$-statistical uniformly almost sure convergence. Moreover, we establish several crucial results regarding these concepts which are summarized below.

1. Examples 2.1 and 2.2 illustrate the concept of $\mathcal{I}_2$-statistical convergence for uncertain sequences of fuzzy numbers with respect to the Orlicz function Ξ.

2. Strong $\mathcal{I}_2$-Cesàro convergence implies $\mathcal{I}_2$-statistical convergence relative to Ξ. The converse holds if the sequence is bounded as shown in Theorem 2.1.

3. $\mathcal{I}_2$-statistical convergence in mean implies $\mathcal{I}_2$-statistical convergence in measure with respect to natural density and the Orlicz function as seen in Theorem 2.2 although the converse is not true as shown in Example 2.5.

4. $\mathcal{I}_2$-statistical convergence almost surely does not generally imply $\mathcal{I}_2$-statistical convergence in measure relative to an Orlicz function as demonstrated in Example 2.6.

5. Convergence in measure does not necessarily imply convergence almost surely with respect to natural density and Ξ as shown in Example 2.7.

6. $\mathcal{I}_2$-statistical convergence almost surely does not necessarily imply $\mathcal{I}_2$-statistical convergence in mean relative to Ξ as seen in Example 2.8.

7. We establish several significant results in Theorems 2.4, 2.5 and 2.6 concerning almost sure and uniformly almost sure convergence within the framework of $\mathcal{I}_2$-statistical convergence for uncertain sequences of fuzzy numbers relative to an Orlicz function.

We first provide several key definitions and preliminary results before delving into the main contributions which will be essential for developing the subsequent sections of this study.

Definition 1.2 *[21] Let $\mathcal{L}$ be a σ-algebra on a non-empty set $\mathbb{Y}$. We call a set function $\mathcal{M}$ on $\mathcal{L}$ an* uncertain measure *if it complies with the axioms below.*

Axiom 1. Normality. $\mathcal{M}(\mathbb{Y}) = 1$.

Axiom 2. Duality. *For any $\mathcal{G} \in \mathcal{L}$,*

$$\mathcal{M}(\mathcal{G}) + \mathcal{M}(\mathcal{G}^c) = 1.$$

Axiom 3. Subadditivity. *For every countable collection $\{\mathcal{G}_j\}_{j=1}^{\infty} \subset \mathcal{L}$,*

$$\mathcal{M}\left(\bigcup_{j=1}^{\infty} \mathcal{G}_j\right) \leq \sum_{j=1}^{\infty} \mathcal{M}(\mathcal{G}_j).$$

We refer to the triplet $(\mathbb{Y}, \mathcal{L}, \mathcal{M})$ as an uncertainty space *and each $\mathcal{G} \in \mathcal{L}$ as an* event*. A product uncertain measure is defined via the following axiom to obtain the uncertain measure of compound events.*

Axiom 4. Product Axiom. *Let $\{(\mathbb{Y}_k, \mathcal{L}_k, \mathcal{M}_k)\}_{k=1}^{\infty}$ be a sequence of uncertainty spaces. The product uncertain measure $\mathcal{M}$ is the uncertain measure satisfying*

$$\mathcal{M}\left(\prod_{k=1}^{\infty} \mathcal{G}_k\right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\mathcal{G}_k),$$

where each $\mathcal{G}_k$ represents an arbitrarily chosen event from $\mathcal{L}_k$.

Definition 1.3 *[21] An* uncertain variable *ξ is a measurable function defined on an uncertainty space $(\mathbb{Y}, \mathcal{L}, \mathcal{M})$ mapping to the set of real numbers. Specifically, for any Borel set $B \subset \mathbb{R}$ the set*

$$\{\varsigma \in B\} = \{\kappa \in \mathbb{Y} : \varsigma(\kappa) \in B\}$$

constitutes an event in $\mathcal{L}$.

Definition 1.4 *[37] An* uncertain variable *ξ is a measurable function from an uncertainty space $(\mathbb{Y}, \mathcal{L}, \mathcal{M})$ to the set of complex numbers. Specifically, for any Borel set $B \subset \mathbb{C}$ the set*

$$\{\varsigma \in B\} = \{\kappa \in \mathbb{Y} : \varsigma(\kappa) \in B\}$$

constitutes an event in $\mathcal{L}$.

Definition 1.5 *[46] We state that a complex uncertain sequence $\{\varsigma_s\}$ is statistically convergent almost surely to ς if for every $\varphi > 0$ an event $\mathcal{G}$ exists with $\mathcal{M}(\mathcal{G}) = 1$ such that*

$$\lim_{p\to\infty} \frac{1}{p} \left|\{\, s \leq p : \|\varsigma_s(\kappa) - \varsigma(\kappa)\| \geq \varphi \,\}\right| = 0, \qquad \textit{for every } \kappa \in \mathcal{G}.$$

Definition 1.6 *[46] We state that a complex uncertain sequence $\{\varsigma_s\}$ is statistically convergent in measure to ς if for every $\lambda,\ \varphi > 0$ the following holds*

$$\lim_{p\to\infty} \frac{1}{p} \left|\{\, s \le p : \mathcal{M}\left(\|\varsigma_s - \varsigma\| \ge \lambda\right) \ge \varphi\}\right| = 0.$$

Definition 1.7 *[46] We state that a complex uncertain sequence $\{\varsigma_s\}$ is statistically convergent in mean to ς if for every $\varphi > 0$ the following holds*

$$\lim_{p\to\infty} \frac{1}{p} \left|\{\, s \le p : \mathbf{E}\left[\|\varsigma_s - \varsigma\|\right] \ge \varphi\,\}\right| = 0.$$

Definition 1.8 *[46] Let $\Psi, \Psi_1, \Psi_2, \ldots$ represent the complex uncertainty distributions of the complex uncertain variables $\varsigma, \varsigma_1, \varsigma_2, \ldots$ respectively. The complex uncertain sequence $\{\varsigma_s\}$ is statistically convergent in distribution to ς if for every $\varphi > 0$*

$$\lim_{p\to\infty} \frac{1}{p} \left|\{\, s \le p : |\Psi_s(\kappa) - \Psi(\kappa)| \ge \varphi\,\}\right| = 0,$$

for all points κ where Ψ is continuous.

2. Main Results

We detail the primary findings of our study in this section.

Definition 2.1 *Let $\{\mathcal{U}_{\alpha\beta}\}$ be an uncertain double sequence of fuzzy numbers or UDSFN. We term the sequence $\mathcal{I}_2$-statistically convergent to $\mathcal{U}_0$ concerning the Orlicz function Ξ provided that for every $\varphi, \sigma > 0$ an event $\mathcal{A}$ exists such that the following condition holds for every $\varsigma \in \mathcal{A}$*

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \ge \varphi\right\}\right| \ge \sigma\right\} \in \mathcal{I}_2,$$

for some $\rho > 0$.

Definition 2.2 *Let $\{\mathcal{U}_{\alpha\beta}\}$ be an UDSFN. We define the sequence as $\mathcal{I}_2$-statistically bounded concerning the Orlicz function Ξ if an event $\mathcal{A}$ exists such that a real number $r_0 > 0$ exists for every $\varsigma \in \mathcal{A}$ satisfying*

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \overline{0}\right)}{\rho}\right) \ge r_0\right\}\right| \ge \sigma\right\} \in \mathcal{I}_2,$$

for every $\sigma > 0$ and some constant $\rho > 0$.

Definition 2.3 *Let $\{\mathcal{U}_{\alpha\beta}\}$ be an UDSFN. We say the sequence is $\mathcal{I}_2$-statistically convergent almost surely to $\mathcal{U}_0$ concerning the Orlicz function Ξ if an event $\mathcal{A}$ with $\mathcal{M}(\mathcal{A}) = 1$ exists for every $\varphi, \sigma > 0$ such that*

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \ge \varphi\right\}\right| \ge \sigma\right\} \in \mathcal{I}_2,$$

for every $\varsigma \in \mathcal{A}$ and some constant $\rho > 0$.

Definition 2.4 *Let $\{\mathcal{U}_{\alpha\beta}\}$ be an UDSFN. We designate the sequence as strongly $\mathcal{I}_2$-Cesàro summable almost surely to $\mathcal{U}_0$ relative to the Orlicz function Ξ if an event $\mathcal{A}$ with $\mathcal{M}(\mathcal{A}) = 1$ exists for any given positive real $\varphi > 0$ such that*

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\sum_{\alpha,\beta=1}^{f,g} \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \ge \varphi\right\} \in \mathcal{I}_2,$$

for some constant $\rho > 0$.

Definition 2.5 *Let $\{\mathcal{U}_{\alpha\beta}\}$ be an UDSFN. The sequence is $\mathcal{I}_2$-statistically convergent in measure to $\mathcal{U}_0$ concerning the Orlicz function Ξ provided that the following condition holds for every $\tau > 0$ and $\varphi, \sigma > 0$*

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \mathcal{M}\left(\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}, \mathcal{U}_0\right)}{\rho}\right) \ge \varphi\right) \ge \sigma\right\}\right| \ge \tau\right\} \in \mathcal{I}_2,$$

for some constant $\rho > 0$.

Definition 2.6 *Let $\{\mathcal{U}_{\alpha\beta}\}$ be an UDSFN. We call the sequence strongly $\mathcal{I}_2$-Cesàro summable in measure to $\mathcal{U}_0$ relative to the Orlicz function Ξ if the condition below holds for every $\varphi, \sigma > 0$ and some $\rho > 0$*

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\sum_{\alpha,\beta=1}^{f,g} \mathcal{M}\left(\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}, \mathcal{U}_0\right)}{\rho}\right) \ge \varphi\right) \ge \sigma\right\} \in \mathcal{I}_2.$$

Definition 2.7 *Let $\{\mathcal{U}_{\alpha\beta}\}$ be an UDSFN. The sequence is $\mathcal{I}_2$-statistically convergent in mean to $\mathcal{U}_0$ concerning the Orlicz function Ξ if the following condition holds for every $\varphi, \sigma > 0$*

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \mathbf{E}\left[\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}, \mathcal{U}_0\right)}{\rho}\right)\right] \ge \varphi\right\}\right| \ge \sigma\right\} \in \mathcal{I}_2,$$

for some constant $\rho > 0$.

Definition 2.8 *Let Ψ and $\Psi_{\alpha\beta}$ represent the uncertainty distributions of the uncertain variables $\mathcal{U}$ and $\mathcal{U}_{\alpha\beta}$ respectively. An uncertain double sequence $\{\mathcal{U}_{\alpha\beta}\}$ of fuzzy numbers is $\mathcal{I}_2$-statistically convergent in distribution to $\mathcal{U}_0$ relative to the Orlicz function Ξ if the following condition is satisfied for every $\varphi, \sigma > 0$ and some $\rho > 0$*

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \Xi\left(\frac{d\left(\Psi_{\alpha\beta}(z), \Psi(z)\right)}{\rho}\right) \ge \varphi\right\}\right| \ge \sigma\right\} \in \mathcal{I}_2,$$

for all z at which $\Psi(z)$ is continuous.

Definition 2.9 *Let $\{\mathcal{U}_{\alpha\beta}\}$ be an UDSFN. We say the sequence is $\mathcal{I}_2$-statistically convergent uniformly almost surely to $\mathcal{U}_0$ concerning the Orlicz function Ξ if a sequence of events $\{\Omega_{\alpha\beta}\}$ with $\mathcal{M}(\Omega_{\alpha\beta}) \to 0$ exists for every $\varphi, \sigma > 0$ such that the following condition holds for every $\varsigma \in \mathbb{Y} \setminus \{\Omega_{\alpha\beta}\}$*

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \ge \varphi\right\}\right| \ge \sigma\right\} \in \mathcal{I}_2,$$

for some $\rho > 0$.

We provide several examples below to illustrate $\mathcal{I}_2$-statistical convergence for *UDSFN* with respect to the Orlicz function Ξ.

Example 2.1 *Consider an event $\varsigma \in \mathcal{A}$ along with the corresponding uncertain fuzzy number sequence $\{\mathcal{U}_{\alpha\beta}\}$ defined as*

$$\mathcal{U}_{\alpha\beta}(\varsigma) = \begin{cases} \frac{1}{2}, & \text{if } \alpha = j^2,\ \beta = k^2;\ j,k \in \mathbb{N}, \\ \mathcal{U}_0(\varsigma), & \text{otherwise.} \end{cases}$$

Assume that $\Xi(\mathcal{U}) = \mathcal{U}$ and $\rho = 1$. Then we have for every $\varphi, \sigma > 0$ and every $\varsigma \in \mathcal{A}$

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \ge \varphi\right\}\right| \ge \sigma\right\} \in \mathcal{I}_2,$$

Consequently the sequence $\{\mathcal{U}_{\alpha\beta}\}$ is $\mathcal{I}_2$-statistically convergent to $\mathcal{U}_0$ relative to the Orlicz function Ξ.

Example 2.2 *Let $\varsigma \in \mathcal{A}$ be an event and let $\mathcal{U}_0$ be a fixed fuzzy number. Consider the uncertain fuzzy number sequence $\{\mathcal{U}_{\alpha\beta}\}$ defined by*

$$\mathcal{U}_{\alpha\beta}(\varsigma) = \begin{cases} \mathcal{U}_0(\varsigma), & \text{if } \alpha = j^2,\ \beta = k^2;\ j,k \in \mathbb{N}, \\ \dfrac{1}{4}, & \text{if } \alpha = j^3,\ \beta = k^3;\ j,k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that $\Xi(\mathcal{U}) = \mathcal{U}$ and $\rho = 1$. Then we obtain for every $\varphi, \sigma > 0$ and every $\varsigma \in \mathcal{A}$

$$\left\{(f,g) \in \mathbb{N} \times \mathbb{N} : \frac{1}{fg} \left| \left\{ \alpha \leq f, \beta \leq g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \overline{0}\right)}{\rho}\right) \geq \varphi \right\} \right| \geq \sigma \right\} \in \mathcal{I}_2,$$

Therefore the sequence $\{\mathcal{U}_{\alpha\beta}\}$ is $\mathcal{I}_2$-statistically convergent to $\overline{0}$ concerning the Orlicz function Ξ.

Example 2.3 *Consider the uncertainty space $(\Upsilon, \mathcal{L}, \mathcal{M})_{\mathcal{U}}$ to be $\{\varsigma_1, \varsigma_2, ...\}$ with*

$$\mathcal{M}(\Lambda) = \begin{cases} \sup_{\varsigma_{\alpha+\beta} \in \mathcal{A}} \frac{\alpha+\beta}{2(\alpha+\beta)+1}, & \text{when } \sup_{\varsigma_{\alpha+\beta} \in \mathcal{A}} \frac{\alpha+\beta}{2(\alpha+\beta)+1} < \frac{1}{2}; \\ 1 - \sup_{\varsigma_{\alpha+\beta} \in \mathcal{A}^c} \frac{\alpha+\beta}{2(\alpha+\beta)+1}, & \text{when } \sup_{\varsigma_{\alpha+\beta} \in \mathcal{A}^c} \frac{\alpha+\beta}{2(\alpha+\beta)+1} < \frac{1}{2}; \\ 0.5, & \text{otherwise,} \end{cases}$$

and the uncertain variable $\{\mathcal{U}_{\alpha\beta}\}$ described as

$$\mathcal{U}_{\alpha\beta}(\varsigma) = \begin{cases} \alpha+\beta, & \text{when } \varsigma = \varsigma_{\alpha+\beta}; \\ 0, & \text{otherwise,} \end{cases}$$

and $\mathcal{U}_0 \equiv 0$ for all $\alpha, \beta \in \mathbb{N}$. Assume that $\Xi(\mathcal{U}) = \mathcal{U}$ and $\rho = 1$. The sequence is clearly strongly $\mathcal{I}_2$-Cesàro summable and $\mathcal{I}_2$-statistically convergent almost surely to $\mathcal{U}_0$ concerning the Orlicz function Ξ.

Example 2.4 *Suppose that the uncertainty space $(\Upsilon, \mathcal{L}, \mathcal{M})_{\mathcal{U}}$ to be $\{\varsigma_1, \varsigma_2, ...\}$ with*

$$\mathcal{M}(\mathcal{A}) = \begin{cases} \sup_{\varsigma_{\alpha+\beta} \in \mathcal{A}} \frac{1}{\alpha+\beta+1}, & \text{when } \sup_{\varsigma_{\alpha+\beta} \in \mathcal{A}} \frac{1}{\alpha+\beta+1} < \frac{1}{2}; \\ 1 - \sup_{\varsigma_{\alpha+\beta} \in \mathcal{A}^c} \frac{1}{\alpha+\beta+1}, & \text{when } \sup_{\varsigma_{\alpha+\beta} \in \mathcal{A}^c} \frac{1}{\alpha+\beta+1} < \frac{1}{2}; \\ 0.5, & \text{otherwise,} \end{cases}$$

and the uncertain variable $(\mathcal{U}_{\alpha\beta})$ described as

$$\mathcal{U}_{\alpha\beta}(\varsigma) = \begin{cases} \alpha+\beta+1, & \text{when } \varsigma = \varsigma_{\alpha+\beta}; \\ 0, & \text{otherwise,} \end{cases}$$

and $\mathcal{U}_0 = 0$ for all $\alpha, \beta \in \mathbb{N}$. Assume that $\Xi(\mathcal{U}) = \mathcal{U}$ and $\rho = 1$. The sequence is evidently strongly $\mathcal{I}_2$-Cesàro summable and $\mathcal{I}_2$-statistically convergent in measure to $\mathcal{U}_0$ relative to the Orlicz function Ξ.

The subsequent theorem investigates the relationship between strong $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-statistical convergence for UDSFN relative to an Orlicz function.

Theorem 2.1 *Consider an UDSFN $\{\mathcal{U}_{\alpha\beta}\}$ which is strongly $\mathcal{I}_2$-Cesàro summable almost surely to $\mathcal{U}_0$ concerning the Orlicz function Ξ. This implies that it is also $\mathcal{I}_2$-statistically convergent almost surely to $\mathcal{U}_0$ relative to Ξ. The converse implication holds provided the sequence is bounded.*

Proof: Let us assume that $\{\mathcal{U}_{\alpha\beta}\}$ is strongly $\mathcal{I}_2$-Cesàro summable almost surely to $\mathcal{U}_0$ with respect to the Orlicz function Ξ. Consequently an event $\mathcal{A}$ exists with $\mathcal{M}(\mathcal{A}) = 1$ such that

$$\left\{(f,g) \in \mathbb{N} \times \mathbb{N} : \frac{1}{fg} \sum_{\alpha,\beta=1}^{f,g} \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \varphi \right\} \in \mathcal{I}_2, \tag{2.1}$$

holds for every $\varsigma \in \mathcal{A}$ and any preassigned positive real $\varphi > 0$ for some $\rho > 0$. We proceed with the computation

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{f,g} \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \\
&\geq \sum_{\substack{\alpha,\beta=1 \\ d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))\geq\varphi}}^{f,g} \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \varphi . \left|\left\{\alpha \leq f, \beta \leq g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \varphi\right\}\right|.
\end{aligned}$$

which leads to

$$\frac{1}{\varphi f g} \sum_{\alpha,\beta=1}^{f,g} \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \frac{1}{fg}\left|\left\{\alpha \leq f, \beta \leq g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \varphi\right\}\right|.$$

We observe the following inclusion for every $\sigma > 0$

$$\begin{aligned}
&\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{fg}\left|\left\{\alpha \leq f, \beta \leq g : \Xi\left(\tfrac{d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))}{\rho}\right) \geq \varphi\right\}\right| \geq \sigma\right\} \\
&\subseteq \left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{fg} \sum_{\alpha,\beta=1}^{f,g} \Xi\left(\tfrac{d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))}{\rho}\right) \geq \varphi\sigma\right\}.
\end{aligned}$$

The relation (2.1) allows us to conclude that $\{\mathcal{U}_{\alpha\beta}\}$ is $\mathcal{I}_2$-statistically convergent almost surely to $\mathcal{U}_0$ relative to Ξ.

Conversely we assume that the UDSFN $\{\mathcal{U}_{\alpha\beta}\}$ is bounded and $\mathcal{I}_2$-statistically convergent almost surely to $\mathcal{U}_0$ relative to Ξ. An event $\mathcal{A}$ exists such that we can find $r_0 > 0$ for every $\varsigma \in \mathcal{A}$ satisfying

$$\sup_{\alpha,\beta}\left(\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right)\right) \leq r_0 < \infty.$$

We calculate

$$\begin{aligned}
&\frac{1}{fg}\sum_{\alpha,\beta=1}^{f,g} \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \\
&= \frac{1}{fg}\left[\sum_{\substack{\alpha,\beta=1 \\ d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))\geq\frac{\varphi}{2}}}^{f,g} \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) + \sum_{\substack{\alpha,\beta=1 \\ d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))<\frac{\varphi}{2}}}^{f,g} \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right)\right] \\
&\leq \frac{r_0}{fg}\left|\left\{\alpha \leq f, \beta \leq g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \frac{\varphi}{2}\right\}\right| + \frac{\varphi}{2}.
\end{aligned}$$

We define the sets

$$D_1 := \left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\sum_{\alpha,\beta=1}^{f,g} \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \varphi\right\}$$

and

$$D_2 := \left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \leq f, \beta \leq g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \frac{\varphi}{2}\right\}\right| \geq \frac{\varphi}{2r_0}\right\}.$$

If $(f,g) \notin D_2$ then

$$\frac{1}{fg}\left|\left\{\alpha \leq f, \beta \leq g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \frac{\varphi}{2}\right\}\right| < \frac{\varphi}{2r_0}.$$

We can also write

$$\frac{1}{fg}\sum_{\alpha,\beta=1}^{f,g}\Xi\left(\frac{d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))}{\rho}\right) \leq \frac{r_0}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\Xi\left(\frac{d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))}{\rho}\right)\geq\frac{\varphi}{2}\right\}\right|+\frac{\varphi}{2}$$
$$=\frac{\varphi}{2}+\frac{\varphi}{2}=\varphi.$$

Hence $(f,g)\notin D_1$. Consequently we have

$$\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\frac{1}{fg}\sum_{\alpha,\beta=1}^{f,g}\Xi\left(\frac{d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))}{\rho}\right)\geq\varphi\right\}$$
$$\subseteq\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\frac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\Xi\left(\frac{d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))}{\rho}\right)\geq\frac{\varphi}{2}\right\}\right|\geq\frac{\varphi}{2r_0}\right\}\in\mathcal{I}_2.$$

We conclude that $\{\mathcal{U}_{\alpha\beta}\}$ is strongly $\mathcal{I}_2$-Cesàro summable almost surely to $\mathcal{U}_0$ relative to the Orlicz function Ξ. □

The result below shows that $\mathcal{I}_2$-statistical convergence in mean implies $\mathcal{I}_2$-statistical convergence in measure for UDSFN relative to an Orlicz function within this framework.

Theorem 2.2 *Let $\{\mathcal{U}_{\alpha\beta}\}$ be an UDSFN. The sequence is $\mathcal{I}_2$-statistically convergent in measure to $\mathcal{U}_0$ with respect to the Orlicz function Ξ if it is $\mathcal{I}_2$-statistically convergent in mean to the same limit $\mathcal{U}_0$ relative to Ξ.*

Proof: Suppose that $\{\mathcal{U}_{\alpha\beta}\}$ is $\mathcal{I}_2$-statistically convergent in mean to $\mathcal{U}_0$ relative to the Orlicz function Ξ. We have for any $\sigma,\tau>0$

$$\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\frac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\mathbf{E}\left[\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta},\mathcal{U}_0\right)}{\rho}\right)\right]\geq\sigma\right\}\right|\geq\tau\right\}\in\mathcal{I}_2,$$

for some constant $\rho>0$. The following condition holds for every $\varphi,\sigma>0$ and $\tau>0$

$$\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\frac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\mathcal{M}\left(\Xi\left(\frac{d(\mathcal{U}_{\alpha\beta},\mathcal{U}_0)}{\rho}\right)\geq\varphi\right)\geq\sigma\right\}\right|\geq\tau\right\}$$
$$\subseteq\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\frac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\mathbf{E}\left[\Xi\left(\frac{d(\mathcal{U}_{\alpha\beta},\mathcal{U}_0)}{\rho\varphi}\right)\right]\geq\sigma\right\}\right|\geq\tau\right\}\in\mathcal{I}_2.$$

Thus $\{\mathcal{U}_{\alpha\beta}\}$ is $\mathcal{I}_2$-statistically convergent in measure to the same limit $\mathcal{U}_0$ relative to Ξ. □

Remark 2.1 *The converse of Theorem 2.2 does not necessarily hold. We illustrate this fact with the following example.*

Example 2.5 *Let us consider the uncertainty space $(\mathbb{Y},\mathcal{L},\mathcal{M})_{\mathcal{U}}$ generated by the collection $\{\varsigma_1,\varsigma_2,\ldots\}$ where*

$$\mathcal{B}_1(\varsigma)=\sup_{\varsigma_{\alpha+\beta}\in\mathcal{A}}\frac{1}{\alpha+\beta+1}\qquad\text{and}\qquad\mathcal{B}_2(\varsigma)=\sup_{\varsigma_{\alpha+\beta}\in\mathcal{A}^c}\frac{1}{\alpha+\beta+1},$$

and we define the measure $\mathcal{M}$ as

$$\mathcal{M}(\mathcal{A})=\begin{cases}\mathcal{B}_1(\varsigma), & \text{if } \mathcal{B}_1(\varsigma)<0.5,\\ 1-\mathcal{B}_2(\varsigma), & \text{if } \mathcal{B}_2(\varsigma)<0.5,\\ 0.5, & \text{otherwise.}\end{cases}$$

We consider the uncertain variables in this uncertainty space

$$\mathcal{U}_{\alpha\beta}(\varsigma)=(\alpha+\beta+1)\,\mathfrak{D}(\varsigma,\varsigma_{\alpha+\beta})\qquad(\alpha,\beta=1,2,3,\ldots),\qquad\text{and}\qquad\mathcal{U}_0\equiv\overline{0},$$

where $\mathfrak{D}(\varsigma, \varsigma_{\alpha+\beta})$ *represents the Kronecker delta function*

$$\mathfrak{D}(e,f)=\begin{cases}1, & \text{if } e=b,\\ 0, & \text{otherwise.}\end{cases}$$

Assume that $\Xi(\mathcal{U})=\mathcal{U}$ *and* $\rho=1$. *We have for any given positive values* φ, σ *and all* $\alpha,\beta\geq 2$

$$\begin{aligned}&\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\mathcal{M}\left(\Xi\left(\tfrac{d(\mathcal{U}_{\alpha\beta},\mathcal{U}_0)}{\rho}\right)\geq\varphi\right)\geq\sigma\right\}\right|\geq\tau\right\}\\ &=\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\mathcal{M}\left(\Xi\left(\tfrac{\varsigma:d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}_0(\varsigma))}{\rho}\right)\geq\varphi\right)\geq\sigma\right\}\right|\geq\tau\right\}\\ &=\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\left|\{\alpha\leq f,\beta\leq g:\mathcal{M}\{\varsigma=\varsigma_{\alpha+\beta}\}\geq\sigma\}\right|\geq\tau\right\}\\ &=\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\tfrac{1}{\alpha+\beta+1}\geq\sigma\right\}\right|\geq\tau\right\}\in\mathcal{I}_2.\end{aligned}$$

The uncertain sequence $\{\mathcal{U}_{\alpha\beta}\}$ *is therefore* $\mathcal{I}_2$*-statistically convergent in measure to* 0 *relative to* Ξ.

Assume that $\Xi(\mathcal{U})=\mathcal{U}$ *and* $\rho=1$. *Furthermore the uncertain distribution of* $d(\mathcal{U}_{\alpha\beta},\mathcal{U}_0)=d(\mathcal{U}_{\alpha\beta},\overline{0})$ *for each* $\alpha,\beta\geq 2$ *is given by*

$$\Psi_{\alpha\beta}(z)=\begin{cases}1, & \text{if } z\geq\alpha+\beta+1,\\ 1-\dfrac{1}{\alpha+\beta+1}, & \text{if } 0\leq z\leq\alpha+\beta+1,\\ 0, & \text{if } z<0.\end{cases}$$

Additionally

$$\begin{aligned}&\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\mathbf{E}\left[\Xi\left(\tfrac{d(\mathcal{U}_{\alpha\beta},\mathcal{U})}{\rho}\right)-1\right]\geq\sigma\right\}\right|\geq\tau\right\}\\ &=\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\left[\int_0^{\alpha+\beta+1}1-\left(1-\tfrac{1}{\alpha+\beta+1}\right)ds-1\right]\geq\sigma\right\}\right|\geq\tau\right\}\end{aligned}$$

We observe that for each $\alpha,\beta\geq 2$

$$\begin{aligned}&\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\left|\left\{\alpha\leq f,\beta\leq g:\mathbf{E}\left[\Xi\left(\tfrac{d(\mathcal{U}_{\alpha\beta},\mathcal{U})}{\rho}\right)\right]\geq\sigma\right\}\right|\geq\tau\right\}\\ &=\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\left|\{\alpha\leq f,\beta\leq g\}\right|\geq\tau\right\}\in\mathcal{F}(\mathcal{I}_2),\end{aligned}$$

which is impossible. Consequently the sequence $\{\mathcal{U}_{\alpha\beta}\}$ *is not* $\mathcal{I}_2$*-statistically convergent in mean to* 0 *relative to* Ξ *even though it is* $\mathcal{I}_2$*-statistically convergent in measure to* 0.

We illustrate the remark above with the subsequent example.

Example 2.6 *Let us examine the uncertainty space* $(\mathbb{Y},\mathcal{L},\mathcal{M})_{\mathcal{U}}$ *formed by the collection* $\{\varsigma_1,\varsigma_2,\ldots\}$ *where we define the boundary functions as*

$$\mathcal{B}_1(\varsigma)=\sup_{\varsigma_{\alpha+\beta}\in\mathcal{A}}\frac{\alpha+\beta}{2(\alpha+\beta)+1}\qquad\text{and}\qquad\mathcal{B}_2(\xi)=\sup_{\varsigma_{\alpha+\beta}\in\mathcal{A}^c}\frac{\alpha+\beta}{2(\alpha+\beta)+1},$$

and the measure $\mathcal{M}$ *follows the rule*

$$\mathcal{M}(\mathcal{A})=\begin{cases}\mathcal{B}_1(\varsigma), & \text{if } \mathcal{B}_1(\varsigma)<0.5,\\ 1-\mathcal{B}_2(\varsigma), & \text{if } \mathcal{B}_2(\varsigma)<0.5,\\ 0.5, & \text{otherwise.}\end{cases}$$

We define the uncertain variables as follows

$$\mathcal{U}_{\alpha\beta}(\varsigma)=\begin{cases}\alpha+\beta, & \text{if } \varsigma=\varsigma_{\alpha+\beta},\\ 1, & \text{otherwise,}\end{cases}$$

for all $\alpha, \beta = 1, 2, 3, \dots$ *and we set* $\mathcal{U}_0 \equiv \overline{0}$. *We assume that* $\Xi(\mathcal{U}) = \mathcal{U}$ *and* $\rho = 1$.

One can verify that the sequence $\{\mathcal{U}_{\alpha\beta}\}$ $\mathcal{I}_2$*-statistically converges almost surely to* $\mathcal{U}_0$ *concerning* Ξ. *Nevertheless we observe*

$$\begin{aligned}
&\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \mathcal{M}\left(\Xi\left(\tfrac{d(\mathcal{U}_{\alpha\beta},\mathcal{U}_0)}{\rho}\right) \ge \varphi\right) \ge \tfrac{1}{2}\right\}\right| \ge \tau\right\} \\
&= \left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \mathcal{M}\left(\Xi\left(\tfrac{\varsigma:\ d(\mathcal{U}_{\alpha\beta}(\varsigma),\overline{0})}{\rho}\right) \ge \varphi\right) \ge \tfrac{1}{2}\right\}\right| \ge \tau\right\} \\
&= \left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \mathcal{M}\left(\varsigma_{\alpha+\beta}\right) \ge \tfrac{1}{2}\right\}\right| \ge \tau\right\} \in \mathcal{F}\left(\mathcal{I}_2\right).
\end{aligned}$$

The sequence $\{\mathcal{U}_{\alpha\beta}\}$ *is not* $\mathcal{I}_2$*-statistically convergent in measure relative to* Ξ *even though it* $\mathcal{I}_2$*-statistically converges almost surely to* $\mathcal{U}_0$.

Theorem 2.3 *Statistical convergence in measure of a sequence* $\{\mathcal{U}_{\alpha\beta}\}$ *concerning* Φ *does not necessarily imply statistical convergence almost surely relative to* Φ.

We provide the example below to illustrate the previous theorem.

Example 2.7 *Consider the uncertainty space* $(\mathbb{Y}, \mathcal{L}, \mathcal{M})_{\mathcal{U}} = [0,1]$ *using the Borel algebra and the Lebesgue measure. Integers* p_1 *and* p_2 *exist for each integers* α *and* β *such that* $\alpha = 2^{p_1} + Q$ *and* $\beta = 2^{p_2} + Q$ *where* Q *is an integer between* 0 *and* $\min\{2^{p_1}, 2^{p_2}\} - 1$. *We define the uncertain variable for any* $\alpha, \beta \in \mathbb{N}$

$$\mathcal{U}_{\alpha\beta}(\varsigma) = \begin{cases} 1, & \text{if } \dfrac{K}{2^{p_1+p_2}} \le \varsigma \le \dfrac{K+1}{2^{p_1+p_2}}, \\ 0, & \text{otherwise}, \end{cases}$$

and we take $\mathcal{U}_0 \equiv \overline{0}$.

Assume that $\Xi(\mathcal{U}) = \mathcal{U}$ *and* $\rho = 1$. *We obtain for given* $\varphi, \sigma > 0$ *and* $\alpha, \beta \ge 2$

$$\begin{aligned}
&\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \mathcal{M}\left(\Xi\left(\tfrac{\varsigma:\ d(\mathcal{U}_{\alpha\beta}(\varsigma),\mathcal{U}(\varsigma))}{\rho}\right) \ge \varphi\right) \ge \sigma\right\}\right| \ge \tau\right\} \\
&= \left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \mathcal{M}\left(\Xi\left(\tfrac{\varsigma:\ d(\mathcal{U}_{\alpha\beta}(\varsigma),\overline{0})}{\rho}\right) \ge \varphi\right) \ge \sigma\right\}\right| \ge \tau\right\} \\
&= \left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \mathcal{M}\left(\mathcal{U}_{\alpha\beta}\right) \ge \sigma\right\}\right| \ge \tau\right\} \in \mathcal{I}_2.
\end{aligned}$$

The sequence $\{\mathcal{U}_{\alpha\beta}\}$ *is* $\mathcal{I}_2$*-statistically convergent in measure to* $\mathcal{U}_0$ *relative to* Ξ. *Furthermore we have for any* $\varphi, \sigma > 0$

$$\left\{(f,g) \in \mathbb{N}\times\mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \le f, \beta \le g : \mathbf{E}\left[\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta},\mathcal{U}\right)}{\rho}\right)\right] \ge \sigma\right\}\right| \ge \tau\right\} \in \mathcal{I}_2,$$

which demonstrates that $\{\mathcal{U}_{\alpha\beta}\}$ *is also* $\mathcal{I}_2$*-statistically convergent in mean to* $\mathcal{U}_0$ *regarding* Ξ.

However infinitely many closed intervals of the form

$$\left[\frac{K}{2^{p_1+p_2}}, \frac{K+1}{2^{p_1+p_2}}\right]$$

containing ς *exist for any* $\varsigma \in [0,1]$. *Consequently the sequence* $\{\mathcal{U}_{\alpha\beta}\}$ *fails to be* $\mathcal{I}_2$*-statistically convergent almost surely to* $\mathcal{U}_0$ *relative to the Orlicz function* Ξ.

Remark 2.2 *It is crucial to highlight that* $\mathcal{I}_2$*-statistical convergence almost surely of a sequence* $\{\mathcal{U}_{\alpha\beta}\}$ *concerning* Ξ *does not inevitably result in* $\mathcal{I}_2$*-statistical convergence in mean relative to* Ξ.

We present an example below to demonstrate this point.

Example 2.8 *Let us analyze the uncertainty space* $(\mathbb{Y}, \mathcal{L}, \mathcal{M})_{\mathcal{U}}$ *established by the collection* $\{\varsigma_1, \varsigma_2, \ldots\}$ *where the measure follows the rule*

$$\mathcal{M}(\mathcal{A}) = \sum_{\varsigma_{\alpha+\beta} \in \mathcal{A}} \frac{1}{2^{\alpha+\beta}}.$$

We define the complex uncertain variables as

$$\mathcal{U}_{\alpha\beta}(\varsigma) = 2^{\alpha+\beta}\,\mathfrak{D}(\varsigma, \varsigma_{\alpha+\beta}), \qquad \textit{and} \qquad \mathcal{U}_0 \equiv \bar{0}.$$

One can easily verify that the sequence $\{\mathcal{U}_{\alpha\beta}\}$ *is* $\mathcal{I}_2$*-statistically convergent almost surely to* $\mathcal{U}_0$ *relative to* Ξ*. The uncertainty distributions of* $\|\mathcal{U}_{\alpha\beta}\|$ *are*

$$\mathcal{U}_{\alpha\beta}(\varsigma) = \begin{cases} 1, & \textit{if } \varsigma = \varsigma_{\alpha+\beta}, \\ 0, & \textit{otherwise,} \end{cases}$$

and

$$\Psi_{\alpha\beta}(z) = \begin{cases} 1, & \textit{if } z \geq 2^{\alpha+\beta}, \\ 1 - \frac{1}{2^{\alpha+\beta}}, & \textit{if } 0 \leq z < 2^{\alpha+\beta}, \\ 0, & \textit{if } z \leq 0. \end{cases}$$

We subsequently obtain

$$\left\{(f,g) \in \mathbb{N} \times \mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \leq f, \beta \leq g : \mathbf{E}\left[\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}, \bar{0}\right)}{\rho}\right)\right] \geq 1\right\}\right| \geq \tau\right\} \in \mathcal{I}_2.$$

This result indicates that the sequence $\{\mathcal{U}_{\alpha\beta}\}$ *fails to be* $\mathcal{I}_2$*-statistically convergent in mean to* $\mathcal{U}_0$ *relative to* Ξ*.*

We provide several results below regarding almost sure and uniformly almost sure convergence within the context of $\mathcal{I}_2$-statistical convergence for UDSFN relative to an Orlicz function.

Theorem 2.4 *A sequence* $\{\mathcal{U}_{fg}\}$ *displays* $\mathcal{I}_2$*-statistical convergence almost surely to* $\mathcal{U}_0$ *regarding the Orlicz function* Ξ *provided that the condition below holds for any* $\varphi, \sigma, \tau > 0$

$$\Big\{(f,g) \in \mathbb{N} \times \mathbb{N} : \tfrac{1}{fg}\Big|\Big\{\alpha \leq f, \beta \leq g : \mathcal{M}\Big(\Big\{\varsigma : \textstyle\bigcap_{\alpha,\beta=1}^{\infty} \bigcup_{f=\alpha,g=\beta}^{\infty} \Xi\left(\frac{d(\mathcal{U}_{\alpha\beta}, \mathcal{U}_0)}{\rho}\right) \geq \varphi\Big\}\Big) \geq \sigma\Big\}\Big| \geq \tau\Big\} \in \mathcal{I}_2.$$

for some constant $\rho > 0$*.*

Proof: Let us assume that the sequence $\{\mathcal{U}_{fg}\}$ is $\mathcal{I}_2$-statistically convergent almost surely to $\mathcal{U}_0$ relative to an Orlicz function Ξ. An event $\mathcal{A}$ with $\mathcal{M}(\mathcal{A}) = 1$ exists for any $\varphi, \sigma > 0$ such that

$$\left\{(f,g) \in \mathbb{N} \times \mathbb{N} : \frac{1}{fg}\left|\left\{\alpha \leq f, \beta \leq g : \Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right)}{\rho}\right) \geq \varphi\right\}\right| \geq \sigma\right\} \in \mathcal{I}_2,$$

holds for every $\varsigma \in \mathcal{A}$ and some constant $\rho > 0$. Consequently we can find $\alpha, \beta \in \mathbb{N}$ for any $\varphi, \sigma > 0$ such that

$$d\left(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma)\right) < \varphi \quad \text{for all } \alpha \leq f, \beta \leq g \text{ and } \varsigma \in \mathcal{A},$$

which we can equivalently express as

$$\Big\{(f,g) \in \mathbb{N} \times \mathbb{N} : \tfrac{1}{fg}\Big|\Big\{\alpha \leq f, \beta \leq g : \mathcal{M}\Big(\Big\{\varsigma : \textstyle\bigcap_{\alpha,\beta=1}^{\infty} \bigcup_{f=\alpha,g=\beta}^{\infty} \Xi\left(\frac{d(\mathcal{U}_{\alpha\beta}(\varsigma), \mathcal{U}_0(\varsigma))}{\rho}\right) < \varphi\Big\}\Big) \geq 1\Big\}\Big| \geq \sigma\Big\} \in \mathcal{I}_2.$$

We may simply write this as

$$\Big\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\Big|\Big\{\alpha\le f,\beta\le g:\mathcal{M}\Big(\Big\{\bigcap_{\alpha,\beta=1}^{\infty}\bigcup_{f=\alpha,g=\beta}^{\infty}\Xi\Big(\tfrac{d(\mathcal{U}_{\alpha\beta},\mathcal{U}_0)}{\rho}\Big)<\varphi\Big\}\Big)\ge 1\Big\}\Big|\ge\sigma\Big\}\in\mathcal{I}_2.$$

We finally invoke the duality axiom of the uncertain measure to obtain for every $\tau > 0$

$$\Big\{(f,g)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{fg}\Big|\Big\{\alpha\le f,\beta\le g:\mathcal{M}\Big(\Big\{\varsigma:\bigcap_{\alpha,\beta=1}^{\infty}\bigcup_{f=\alpha,g=\beta}^{\infty}\Xi\Big(\tfrac{d(\mathcal{U}_{\alpha\beta},\mathcal{U}_0)}{\rho}\Big)\ge\varphi\Big\}\Big)\ge\tau\Big\}\Big|\ge\sigma\Big\}\in\mathcal{I}_2.$$

□

Theorem 2.5 *A sequence $\{\mathcal{U}_{fg}\}$ exhibits $\mathcal{I}_2$-statistical convergence uniformly almost surely to $\mathcal{U}_0$ relative to the Orlicz function Ξ if the following condition holds for any $\varphi, \sigma, \tau > 0$*

$$\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\frac{1}{fg}\left|\left\{\alpha\le f,\beta\le g:\mathcal{M}\left(\bigcup_{f=\alpha,g=\beta}^{\infty}\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta},\mathcal{U}_0\right)}{\rho}\right)\ge\varphi\right)\ge\sigma\right\}\right|\ge\tau\right\}\in\mathcal{I}_2,$$

for some constant $\rho > 0$.

Proof: Let us assume that the sequence $\{\mathcal{U}_{fg}\}$ is $\mathcal{I}_2$-statistically convergent uniformly almost surely to $\mathcal{U}_0$ with respect to the Orlicz function Ξ. We can find a set $\mathcal{C}$ for a given $\tau > 0$ such that $\mathcal{M}(\mathcal{C}) < \tau$ where the sequence $\{\mathcal{U}_{fg}\}$ exhibits statistically uniform almost sure convergence to $\mathcal{U}_0$ on $\mathbb{Y}\setminus\mathcal{A}$. We know by definition that $\alpha \ge f$ and $\beta \ge g$ exist for any $\varphi > 0$ such that $d(\mathcal{U}_{\alpha\beta},\mathcal{U}_0) < \varphi$ for all $\varsigma \in \mathbb{Y}\setminus\mathcal{A}$. Consequently

$$\bigcup_{f=\alpha,g=\beta}^{\infty}\{d(\mathcal{U}_{fg},\mathcal{U}_0)<\varphi\}\subset\mathcal{C}.$$

The subadditivity axiom leads to

$$\frac{1}{fg}\left|\left\{\alpha\le f,\beta\le g:\mathcal{M}\left(\bigcup_{f=\alpha,g=\beta}^{\infty}\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta},\mathcal{U}_0\right)}{\rho}\right)\ge\varphi\right)\ge\sigma\right\}\right|\le\tau(\mathcal{M}(\mathcal{C}))<\tau.$$

We therefore write

$$\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\frac{1}{fg}\left|\left\{\alpha\le f,\beta\le g:\mathcal{M}\left(\bigcup_{f=\alpha,g=\beta}^{\infty}\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta},\mathcal{U}_0\right)}{\rho}\right)\ge\varphi\right)\ge\sigma\right\}\right|\ge\tau\right\}\in\mathcal{I}_2,$$

for some constant $\rho > 0$. □

Theorem 2.6 *If a sequence $\{\mathcal{U}_{fg}\}$ is $\mathcal{I}_2$-statistically convergent uniformly almost surely to $\mathcal{U}_0$ relative to the Orlicz function Ξ then it is also $\mathcal{I}_2$-statistically convergent almost surely to $\mathcal{U}_0$ concerning Ξ.*

Proof: Suppose that the sequence $\{\mathcal{U}_{fg}\}$ is statistically convergent uniformly almost surely to $\mathcal{U}_0$ relative to the Orlicz function Ξ. Theorem 2.5 states that we have for any $\varphi > 0$ and $\sigma, \tau > 0$

$$\left\{(f,g)\in\mathbb{N}\times\mathbb{N}:\frac{1}{fg}\left|\left\{\alpha\le f,\beta\le g:\mathcal{M}\left(\bigcup_{f=\alpha,g=\beta}^{\infty}\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta},\mathcal{U}_0\right)}{\rho}\right)\ge\varphi\right)\ge\sigma\right\}\right|\ge\tau\right\}\in\mathcal{I}_2,$$

for some constant $\rho > 0$. Furthermore

$$\delta_{\mathcal{I}_2}\left(\mathcal{M}\left(\bigcap_{\alpha,\beta=1}^{\infty}\bigcup_{f=\alpha,g=\beta}^{\infty}\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta},\mathcal{U}_0\right)}{\rho}\right)\geq\varphi\right)\right)\leq\delta_{\mathcal{I}_2}\left(\mathcal{M}\left(\bigcup_{f=\alpha,g=\beta}^{\infty}\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta},\mathcal{U}_0\right)}{\rho}\right)\geq\varphi\right)\right).$$

We subsequently obtain

$$\delta_{\mathcal{I}_2}\left(\mathcal{M}\left(\bigcap_{\alpha,\beta=1}^{\infty}\bigcup_{f=\alpha,g=\beta}^{\infty}\Xi\left(\frac{d\left(\mathcal{U}_{\alpha\beta},\mathcal{U}_0\right)}{\rho}\right)\geq\varphi\right)\right)=0.$$

Theorem 2.4 confirms the required conclusion. □

3. Conclusion and Future Scope

In this study we established the notion of $\mathcal{I}_2$-statistical convergence concerning uncertain double sequences of fuzzy numbers characterized by an Orlicz function. We investigated diverse associated convergence forms including convergence in measure, in mean, in distribution and uniformly almost surely. We elucidated the connections and distinctions among these convergence types through illustrative examples thereby offering a more profound comprehension of their relationships within the context of uncertain fuzzy sequences.

The results presented in this paper add to the continuous advancement of uncertainty theory and its uses in mathematical analysis. Nevertheless numerous paths for future investigation persist. One possible avenue involves extending the notion of $\mathcal{I}_2$-statistical convergence to more intricate structures like higher-dimensional uncertain sequences or sequences comprising multiple variables. Moreover investigating the practical uses of these convergence concepts in areas such as fuzzy optimization, uncertain control systems and risk analysis could result in significant progress. Another promising field for future research is the combination of these convergence concepts with other mathematical instruments like dynamic systems or stochastic processes to expand their applicability and theoretical richness.

References

1. Baliarsingh, P., Nanda, S., Nayak, L., *On statistical convergence of uncertain sequence of fuzzy numbers*. Int. J. Uncertain. Fuzziness Knowledge-based Syst. 30(6), 975-989, (2022).
2. Belen, C., Mohiuddine, S. A., *Generalized weighted statistical convergence and application*. Appl. Math. Comput. 219, 9821-9826, (2013).
3. Chen, X., Ning, Y., Wang, X., *Convergence of complex uncertain sequences*. J. Intell. Fuzzy Syst. 30, 3357-3366, (2016).
4. Das, P., Kostyrko, P., Wilczynski, W., Malik, P., *$\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences*. Math. Slovaca 58(5), 605-620, (2008).
5. Das, B., Tripathy, B. C., Debnath, P., Bhattacharya, B., *Characterization of statistical convergence of complex uncertain double sequence*. Anal. Math. Phys. 10(71), 1-20, (2020).
6. Datta, D., Tripathy, B. C., *Convergence of complex uncertain double sequences*. New Math. Nat. Comput. 16(3), 447-459, (2020).
7. Datta, D., Tripathy, B. C., *Double sequences of complex uncertain variables defined by Orlicz function*. New Math. Nat. Comput. 16(3), 541-550, (2020).
8. Fast, H., *Sur la convergence statistique*. Colloq. Math. 2(3-4), 241-244, (1951).
9. Gao, X., *Some properties of continuous uncertain measure*. Int. J. Uncertain. Fuzziness Knowledge-based Syst. 17, 419-426, (2009).
10. Gürdal, M., *Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra*. Expo. Math. 27, 153-160, (2009).
11. Gürdal, M., Garayev, M. T., Saltan, S., *Some concrete operators and their properties*. Turkish J. Math. 39, 970-989, (2015).
12. Gürdal, M., Yamancı, U., *Statistical convergence of operator theory*. Dyn. Syst. Appl. 24(3), 305-311, (2015).
13. Hazarika, B., Alotaibi, A., Mohiuddine, S. A, *Statistical convergence in measure for double sequences of fuzzy-valued functions*. Soft Comput. 24, 6613-6622, (2020).

14. Kadak, U., Mohiuddine, S. A., *Generalized statistically almost convergence based on the difference operator which includes the (p, q)-Gamma function and related approximation theorems*. Results Math. 73(9), 1-31, (2018).

15. Karaev, M. T., Gürdal, M., Huban, M. B., *Reproducing kernels, Englis algebras and some applications*. Studia Math. 232, 113-141, (2016).

16. Karaev, M., Gürdal, M., Saltan, S., *Some applications of Banach algebra techniques*. Math. Nachr. 284, 1678-1689, (2011).

17. Kişi, Ö., Choudhury, C., *Some observations on statistical convergence of uncertain double sequences of fuzzy numbers*. J. Nonlinear Sci. Appl. 17(1), 19-29, (2024).

18. Kostyrko, P., Šalát, T., Wilczynski, W., *$\mathcal{I}$-convergence*. Real Anal. Exchange 26(2), 669-686, (2000).

19. Krasnoselskii, M. A., *Convex functions and Orlicz spaces*. Vol. 4311, US Atomic Energy Commission, (1960).

20. Lindenstrauss, J., Tzafriri, L., *Orlicz sequence spaces*. In Classical Banach Spaces I, (1971).

21. Liu, B., *Uncertainty Theory*. 2nd edn. Springer, Berlin, (2007).

22. Liu, B., *Some research problems in uncertainty theory*. J. Uncertain Syst. 3, 3-10, (2009).

23. Liu, B., *Uncertainty theory: A branch of mathematics for modelling human uncertainty*. Springer-Verlag, Berlin, (2010).

24. Liu, B., *Why is there a need for uncertainty theory?*. J. Uncertain Syst. 6, 3-10, (2012).

25. Liu, B., *Uncertainty theory*. Springer-Verlag, Berlin, (2016).

26. Mohiuddine, S. A., Şevli, H., Cancan, M., *Statistical convergence in fuzzy 2-normed space*. J. Comput. Anal. Appl. 12(4), 787-798, (2010).

27. Mohiuddine, S. A., Asiri, A., Hazarika, B., *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*. Int. J. Gen. Syst. 48(5), 492-506, (2019).

28. Mursaleen, M., Edely, O. H. H., *Statistical convergence of double sequences*. J. Math. Anal. Appl. 288(1), 223-231, (2003).

29. Musielak, J., *Orlicz spaces and modular spaces*. Springer-Verlag, Berlin Heidelberg, New York, (1983).

30. Matloka, M., *Sequences of fuzzy numbers*. Busefal 28, 28-37, (1986).

31. Nakano, H., *Concave modulars*. J. Math. Soc. Jpn. 5(1), 29-49, (1953).

32. Nanda, S., *On sequences of fuzzy numbers*. Fuzzy Sets Syst. 33, 123-126, (1989).

33. Nuray, F., Savaş, E., *Statistical convergence of sequences of fuzzy real numbers*. Math. Slovaca 45, 269-273, (1995).

34. Nath, P. K., Tripathy, B. C., *Convergent complex uncertain sequences defined by Orlicz function*. Ann. Univ. Craiova, Math. Comput. Sci. Ser. 46(1), 139-149, (2019).

35. Nath, P. K., Tripathy, B. C., *Statistical convergence of complex uncertain sequences defined by Orlicz function*. Proyecciones J. Math. 39(2), 301-315, (2020).

36. Nath, P. K., Tripathy, B. C., *Convergence of complex uncertain double sequences*. New Math. Nat. Comput. 16(03), 447-459, (2020).

37. Peng, Z., *Complex uncertain variable*. Doctoral Dissertation, Tsinghua University, (2012).

38. Raj, K., Gorka, S., Esi, A., *Lacunary statistical convergence of uncertain fuzzy number sequences via Orlicz function*. Math. Found. Comput. 8(6), 998-1011, (2025).

39. Raj, K., Esi, A., Sharma, C., *Orlicz-lacunary bicomplex sequence spaces of difference operators*. Filomat 37(8), 2421-2435, (2023).

40. Saha, S., Esi, A., Roy, S., *Some new classes of multiplier ideal convergent triple sequence spaces of fuzzy numbers defined by Orlicz functions*. Palestine J. Math. 9(1), 174-186, (2020).

41. Savaş, E., Kişi, Ö., Gürdal, M., *On statistical convergence in credibility space*. Numerical Funct. Anal. Optim. 43(8), 987-1008, (2022).

42. Şahiner, A., Gürdal, M., Yiğit, T., *Ideal convergence characterization of the completion of linear n-normed spaces*. Comput. Math. Appl. 61(3), 683-689, (2011).

43. Tanushri, T., Ahmad, A., Esi, A., *A Framework for $\mathcal{I}^*$-statistical convergence of fuzzy numbers*. Axioms 13(1), 639-645, (2024).

44. Tripathy, B. C., *Statistically convergent double sequences*. Tamkang J. Math. 34(3), 231-237, (2003).

45. Tripathy, B. C., Hazarika, B., *Some $\mathcal{I}$-convergent sequence spaces defined by Orlicz functions*. Acta Math. Appl. Sin. (Engl. Ser.) 27(1), 149-154, (2011).

46. Tripathy, B. C., Nath, P. K., *Statistical convergence of complex uncertain sequences*. New Math. Nat. Comput. 13(3), 359-374, (2017).

47. Tripathy, B. C., Datta, D., *Double sequences of complex uncertain variables associated with multiplier sequences*. Rand. Oper. Stoch. Equ. 32(4), 357-362, (2024).

48. Yamancı, U., Gürdal, M., *Statistical convergence and operators on Fock space*. New York J. Math. 22, 199-207, (2016).

49. Yamancı, U., Gürdal, M., *On lacunary ideal convergence in random n-normed space*. J. Math. 2013, Art. ID 868457, (2013).

50. Yamancı, U., Gürdal, M., *$\mathcal{I}$-statistical convergence in 2-normed space*. Arab J. Math. Sci. 20(1), 41-47, (2014).

51. You, C., *On the convergence of uncertain sequences*. Math. Comput. Modell. 49, 482-487, (2009).

52. Zadeh, L. A., *Fuzzy sets*. Inform. Control 8, 338-353, (1965).

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