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Quantum Difference Relative Uniform Convergence of Double Sequence Spaces of Sargent Type Functions

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ABSTRACT: This paper introduces two new double sequence spaces of Sargent type functions, denoted by ${}_2m(\phi, ru, \nabla_q)$ and ${}_2n(\phi, ru, \nabla_q)$, which are defined using the concept of relative uniform convergence in combination with the Jackson q-difference operator for double sequences. In this framework, we define bounded, p-absolutely summable, convergent, and null double sequences of functions based on the idea of quantum difference relative uniform convergence with respect to a scale function. These classes are represented by $\ell_\infty(ru, \nabla_q)$, $\ell_p(ru, \nabla_q)$, ${}_2c(ru, \nabla_q)$ and ${}_2c_0(ru, \nabla_q)$, respectively. We also explore the inclusion relations and isomorphisms between these newly introduced spaces and other existing function spaces. Additionally, we investigate several algebraic and geometric properties, such as solidness and convexity.

Key Words: Double sequence spaces, Sargent-type functions, Quantum difference operator, relative uniform convergence, BK-space, uniform convexity, modular spaces.

Contents

1. Introduction

The idea of statistical convergence was initially proposed by Fast [14] and later expanded upon from the sequence-space perspective by numerous researchers [1,15,16,17,20,21,22,23,31,38,39,40]. This concept has gained significant attention due to its wide-ranging applications in various mathematical fields such as number theory, mathematical analysis and probability theory. The study of statistical convergence in double sequences was independently introduced by Moricz [25], Mursaleen and Edely [26] and Tripathy [32].

Functional analysis offers powerful tools for understanding the geometric and structural properties of sequence spaces. In this context, the description of extended eigenvalues of operators [11] and the application of Banach algebra techniques [19] have provided essential insights. Moreover, the study of reproducing kernels [18] and concrete operators [12] has enriched the literature. Building upon these functional analytic foundations, recent research has focused on generalized convergence methods including ideal convergence in linear n-normed spaces [29], lacunary ideal convergence in random spaces [36] and statistical convergence in 2-normed spaces [37].

The concept of the space $m(\phi)$, which is connected to the ℓ_p-space, was first introduced by Sargent [30] who also explored several properties of this space. Later, the space was analyzed from the perspective of sequence spaces with Rath and Tripathy [27], Tripathy [33] and Tripathy and Sen [34] providing characterizations of various matrix classes. The idea of a sequence of functions demonstrating relative uniform convergence with respect to a scale function was initially proposed by Moore [24]. The concept of relative uniform convergence of a sequence of functions was further developed by Chittenden [2] who further expanded on the notion of relative uniform convergence in his formulation. Following this, numerous other researchers contributed to the development of the concept including Demirci et al. [5],

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Demirci and Orhan [6], Sahin and Dirik [28], Devi and Tripathy [7,8,9,10] and Debbarma et al. [3] among others.

The convergence of function sequences plays a fundamental role in areas like approximation theory, operator theory and functional analysis. Traditional types of convergence such as pointwise and uniform convergence often fail to account for irregular or nonuniform behaviors in function sequences. To address these shortcomings, more generalized convergence concepts such as relative uniform convergence have gained greater importance.

Simultaneously, quantum calculus known as q-calculus introduced by Jackson [35] provides a discrete framework for analysis. While Jackson originally defined the q-derivative, subsequent studies adapted this concept to sequence spaces using difference operators. Recently, Debbarma and Tripathy [4] investigated the behavior of q-difference sequences of functions under relative uniform convergence using the operator $\nabla_q \zeta_n(u) = \zeta_n(u) - q\zeta_{n-1}(u)$.

In this paper, we extend these concepts to double sequences. We present a significant advancement in the study of double sequence spaces by introducing two novel spaces $m(\phi, ru, \nabla_q)$ and $n(\phi, ru, \nabla_q)$ which are defined through the concept of relative uniform convergence combined with the Jackson q-difference operator generalized for double sequences. By exploring the interplay between quantum difference and relative uniform convergence, this work provides a fresh perspective on double sequences that are bounded, absolutely summable, convergent and null. These newly defined classes represented by $\ell_\infty(ru, \nabla_q)$, $\ell_p(ru, \nabla_q)$, $c(ru, \nabla_q)$ and $c_0(ru, \nabla_q)$ open new avenues for understanding the structure and behavior of function sequences in a discrete framework. The relationships between these spaces and existing function spaces are also explored offering deeper insights into their algebraic and geometric properties.

2. Definitions and Preliminaries

Definition 2.1 *Let Q be a compact domain. A double sequence of functions (ζ_{ij}) is said to converge relatively uniformly to a function ζ if, for any given positive value α, there exists an integer n_α such that for all $i, j \geq n_\alpha$, the following inequality holds:*

$$|\zeta_{ij}(u) - \zeta(u)| \leq \alpha |\varpi(u)|.$$

In this definition, $\varpi(u)$ is referred to as the scale function, which is defined on the same compact domain Q. The concept of relative uniform convergence for single sequences was introduced by Chittenden [2], following Moore [24]. Here, we consider the extension of this concept to double sequences in the Pringsheim sense.

Example 2.1 *Let $[0, 1]$ be a compact domain in $\mathbb{R}$ and consider the double sequence of functions $\zeta_{ij} : [0, 1] \to \mathbb{R}$, for $i, j \in \mathbb{N}$ defined by*

$$\zeta_{ij}(u) = \begin{cases} \frac{1}{i^2 j^2 u}, & \text{for } u \neq 0, \\ 0, & \text{for } u = 0. \end{cases}$$

Consequently, it is clear that (ζ_{ij}) does not converge uniformly, but rather converges relatively uniformly to the zero function with respect to a scale function $\varpi(u)$ defined by

$$\varpi(u) = \begin{cases} \frac{1}{u}, & \text{for } u \neq 0, \\ 0, & \text{for } u = 0. \end{cases}$$

Throughout, the space ξ_{st} will represent the class of all subsets of $\mathbb{N} \times \mathbb{N}$ that contain a maximum of st elements. The space Φ consists of all non-decreasing double sequences $\phi = (\phi_{ij})$ of real numbers (i.e., $\phi_{ij} \leq \phi_{i+1,j}$ and $\phi_{ij} \leq \phi_{i,j+1}$), such that for every positive integer i, j, $0 < \phi_{11} \leq \phi_{ij} \leq \phi_{i+1,j+1} < \infty$ and $(i+1)(j+1)\phi_{ij} \geq ij\phi_{i+1,j+1}$. That is,

$$\frac{\phi_{ij}}{ij} \geq \frac{\phi_{i+1,j+1}}{(i+1)(j+1)}, \text{ for all } i, j \in \mathbb{N}. \tag{2.1}$$

Example 2.2 *(1) Let $\phi_{ij} = 1$, for all $i, j \in \mathbb{N}$. Then $(\phi_{ij}) \in \Phi$.*
(2) Let $\phi_{ij} = ij$, for all $i, j \in \mathbb{N}$. Then $(\phi_{ij}) \in \Phi$.

Generalizing the work of Sargent [30] to double sequences, the spaces $m(\phi)$ and $n(\phi)$ are defined as follows:

Definition 2.2 *The class of double sequence ${}_2m(\phi)$ is defined by*

$$ {}_2m(\phi) = \left\{ (x_{ij}) \in \omega : \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} |x_{ij}| < \infty \right\}. $$

The class of sequence ${}_2n(\phi)$ is defined by,

$$ {}_2n(\phi) = \left\{ (x_{ij}) \in \omega : \sup_{u\in\Lambda(x_{ij})} \sum_{i,j=1,1}^{\infty,\infty} |x_{ij}| \, \Delta\phi_{ij} < \infty \right\}, $$

where $\Delta\phi_{ij} = \phi_{ij} - \phi_{i-1,j-1}$, with $\phi_{0,0} = 0$.

Definition 2.3 *Let $\phi = (\phi_{ij})$ be a double sequence of real numbers satisfying equation (2.1). Then, a double sequence of functions (ζ_{ij}) constructed on a compact domain Q is said to converge relatively uniformly to a limit function ζ if for each $\alpha > 0$,*

$$ \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} |\zeta_{ij}(u) - \zeta(u)| < \alpha|\varpi(u)|, $$

where $\varpi(u)$ represents the scale function defined in Q.

The classes of all relative uniform convergence double sequences of Sargent-type functions are denoted by ${}_2m(\phi, ru)$ and ${}_2n(\phi, ru)$, respectively, and are defined as:

$$ {}_2m(\phi, ru) = \left\{ (\zeta_{ij}) \in \omega_f : \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} |\zeta_{ij}(u)| < \tau|\varpi(u)| \right\}, $$

we also introduce another class of sequence of functions:

$$ {}_2n(\phi, ru) = \left\{ (\zeta_{ij}) \in \omega_f : \sup_{(\eta_{ij})\in S(\zeta)} \sum_{i,j=1,1}^{\infty,\infty} |\eta_{ij}(u)| \, \Delta\phi_{ij} < \tau|\varpi(u)| \right\}, $$

where $S(\zeta)$ represents the rearrangement of the double sequence (ζ_{ij}).

Example 2.3 *Let $\phi_{ij} = ij$, for all $i, j \in \mathbb{N}$. Consider a double sequence of functions (ζ_{ij}) defined on the range $[0, 1]$ by*

$$ \zeta_{ij}(u) = \begin{cases} \frac{i^2j^2u}{1+i^2j^2u^2}, & \text{for } u \neq 0 \\ 0, & \text{for } u = 0. \end{cases} $$

Now,

$$ {}_2m(\phi, ru) = \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} |\zeta_{ij}(u)| \geq \frac{u}{st\,(1+u^2)}, $$

which is not convergent uniformly. However, it is convergent relative uniformly if we consider a scale function defined by

$$ \varpi(u) = \begin{cases} \frac{1}{u}, & \text{for } u \neq 0 \\ 0, & \text{for } u = 0. \end{cases} $$

Then, $(\zeta_{ij}) \in {}_2m(\phi, ru)$ with respect to the scale function $\varpi(u)$.

Definition 2.4 *The norm of the space* ${}_2m(\phi, ru)$ *is defined as follows:*

$$\|\zeta\|_{2m(\phi,ru)} = \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \frac{|\zeta_{ij}(u)|}{|\varpi(u)|}.$$

The following well-known relative uniform convergence double sequence of function spaces can be given as follows:

$$\begin{aligned} \ell_p(ru) &= \left\{ (\zeta_{ij}) \in \omega_f : \sum_{i,j=1,1}^{\infty,\infty} |\zeta_{ij}(u)|^p < \alpha|\varpi(u)| \right\}, \\ \ell_1(ru) &= \left\{ (\zeta_{ij}) \in \omega_f : \sum_{i,j=1,1}^{\infty,\infty} |\zeta_{ij}(u)| < \alpha|\varpi(u)| \right\}, \text{ for } p=1, \\ \ell_\infty(ru) &= \left\{ (\zeta_{ij}) \in \omega_f : \sup_{i,j\geq1} |\zeta_{ij}(u)| < \alpha|\varpi(u)| \right\}, \\ {}_2c(ru) &= \left\{ (\zeta_{ij}) \in \omega_f : \tfrac{|\zeta_{ij}(u)|-L}{|\varpi(u)|} \to 0, \text{ as } i,j\to\infty \right\}, \\ {}_2c_0(ru) &= \left\{ (\zeta_{ij}) \in \omega_f : \tfrac{|\zeta_{ij}(u)|}{|\varpi(u)|} \to 0, \text{ as } i,j\to\infty \right\}. \end{aligned}$$

Remark 2.1 (1) If $\phi_{ij} = 1, (i,j) \in \mathbb{N}$, then ${}_2m(\phi, ru) = \ell_1(ru)$;
(2) if $\phi_{ij} = ij, (i,j) \in \mathbb{N}$, then ${}_2m(\phi, ru) = \ell_\infty(ru)$.

Definition 2.5 *A subset* $Z_f \subset {}_2\omega_f$ *is considered solid or normal if for every* $i, j \in \mathbb{N}$, $|\eta_{ij}(u)| \leq |\zeta_{ij}(u)|$, *implies* $(\eta_{ij}) \in Z_f$.

Definition 2.6 *Convergence-free subsets are defined as* $Z_f \subset {}_2\omega_f$ *if* $(\zeta_{ij}) \in Z_f$, *then* $\zeta_{ij}(u) = \theta \Longrightarrow \eta_n(u) = \theta$, *on* $u \in Q$ *and* $(\eta_{ij}) \in Z_f$ *if* $\zeta_{ij}(u) \neq \theta \Longrightarrow \eta_{ij}(u)$ *may be anything.*

Definition 2.7 *Suppose* $Z_f \subset {}_2\omega_f$ *is a linear domain. Then* $\mathcal{M} : Z_f \to [0,\infty)$ *is a modular in a real double sequence of functions space if for* $(\zeta_{ij}), (\eta_{ij}) \in Z_f$, *the following conditions hold.*
(1) $\mathcal{M}_{Z_f}(\zeta_{ij}(u)) = 0$ *iff* $\zeta_{ij}(x) = \bar{\theta}$, *where* $\bar{\theta}$ *represents the null sequence of function, for all* $u \in Z \subseteq \mathbb{R}$.
(2) $\mathcal{M}_{Z_f}(-\zeta_{ij}(u)) = \mathcal{M}_{Z_f}(\zeta_{ij}(u))$, *for all* $u \in Z \subseteq \mathbb{R}$.
(3) $\mathcal{M}_{Z_f}(\lambda\zeta_{ij}(u)) = \mathcal{M}_{Z_f}(\zeta_{ij}(u))$, *for all scalars with* $|\lambda| = 1$.
(4) $\mathcal{M}_{Z_f}(\lambda_1\zeta_{ij}(u) + \lambda_2\eta_{ij}(u)) \leq \mathcal{M}_{Z_f}(\zeta_{ij}(u)) + \mathcal{M}_{Z_f}(\eta_{ij}(u))$, *for all* $(\zeta_{ij}), (\eta_{ij}) \in Z_f$ *and* $\lambda_1, \lambda_2 \geq 0$, $|\lambda_1| + |\lambda_2| = 1$.
Further, the modular $\mathcal{M}$ *is convex if*
(5) $\mathcal{M}_{Z_f}(\lambda_1\zeta_{ij}(u) + \lambda_2\eta_{ij}(u)) \leq \lambda_1\mathcal{M}_{Z_f}(\zeta_{ij}(u)) + \lambda_2\mathcal{M}_{Z_f}(\eta_{ij}(u))$, *for all* $(\zeta_{ij}), (\eta_{ij}) \in Z_f$ *and* $\lambda_1, \lambda_2 \geq 0$, $|\lambda_1| + |\lambda_2| = 1$. *The modular* $\mathcal{M}$ *is called* $\tilde{s}$*-convex if*

(6) $\mathcal{M}_{Z_f}(\lambda_1\zeta_{ij}(u) + \lambda_2\eta_{ij}(u)) \leq \lambda_1^{\tilde{s}}\mathcal{M}_{Z_f}(\zeta_{ij}(u)) + \lambda_2^{\tilde{s}}\mathcal{M}_{Z_f}(\eta_{ij}(u))$, *for all* $\lambda_1, \lambda_2 \geq 0$, $\lambda_1^{\tilde{s}} + \lambda_2^{\tilde{s}} = 1, 0 < \tilde{s} \leq 1$.

Definition 2.8 *A double sequence of functions* $(\zeta_{ij}) \in \mathcal{M}_{Z_f}$ *is modular convergent to* $\zeta \in \mathcal{M}_{Z_f}$ *if there is a* $\lambda > 0$ *such that* $(\mathcal{M}(\lambda(\zeta_{ij}(u) - \zeta(u)))) \to 0$ *as* $i, j \to \infty$, *for all* $u \in Q$.

Definition 2.9 *In the normed space* ${}_2m(\phi, ru)$, *the unit ball and the unit sphere are defined by*

$$\begin{aligned} B_{2m(\phi,ru)} &= \left\{ (\zeta_n) \in {}_2m(\phi,ru) : \|\zeta_n\|_{2m(\phi,ru)} = 1 \right\}, \\ S_{2m(\phi,ru)} &= \left\{ (\zeta_n) \in {}_2m(\phi,ru) : \|\zeta_n\|_{2m(\phi,ru)} = 1 \right\}. \end{aligned}$$

Definition 2.10 *A Banach space is considered uniformly convex if, for any* $\varepsilon > 0$, *there exists a* $\mho(\varepsilon) > 0$ *such that for* $(\zeta_{ij}), (\eta_{ij}) \in S_{Z_f}$,

$$\|\zeta_{ij}\|_{Z_f} = \|\eta_{ij}\|_{Z_f} = 1, \|\zeta_{ij} - \eta_{ij}\|_{Z_f} \geq \varepsilon \Longrightarrow \left\| \frac{\zeta_{ij} + \eta_{ij}}{2} \right\|_{Z_f} \leq 1 - \mho(\varepsilon).$$

Definition 2.11 *(James [13]) A Banach space is considered super-reflexive if it is isomorphic to a Banach space that is either uniformly convex or uniformly non-square.*

Definition 2.12 *(Quantum Difference Operator for Double Sequences) For a double sequence of functions $\zeta_{ij}(u)$ defined on a domain Q, the generalized quantum difference operator is defined by*

$$\nabla_q \zeta_{ij}(u) = \zeta_{ij}(u) - q\zeta_{i-1,j-1}(u), \quad 0 < q < 1,$$

where $\zeta_{i,0}(u) = \zeta_{0,j}(u) = 0$ for all i, j.

Definition 2.13 *(Relative Uniform Convergence for Double Sequences) A double sequence ζ_{ij} converges relatively uniformly to ζ on Q with respect to a scale function $\varpi(u)$ if for every $\rho > 0$, there exists n_ρ such that for all $i, j \geq n_\rho$,*

$$|\zeta_{ij}(u) - \zeta(u)| \leq \rho|\varpi(u)|, \quad \text{uniformly in } u \in Q.$$

The purpose of this research is to investigate this form of convergence of a double sequence of the Sargent type-related function and investigate some of its topological and geometrical characteristics.

3. Main Results

We denote the classes of Sargent-type functions related to quantum difference relative uniform convergent sequences by ${}_2m(\phi, ru, \nabla_q)$ and ${}_2n(\phi, ru, \nabla_q)$. These sets are defined as follows

$${}_2m(\phi, ru, \nabla_q) = \left\{ (\zeta_{ij}) : \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} |\nabla_q \zeta_{ij}(u)| < \alpha|\varpi(u)| \right\},$$
$${}_2n(\phi, ru, \nabla_q) = \left\{ (\zeta_{ij}) \in \omega_f : \sup \frac{1}{\phi_{s,t}} \sum_{(\eta_{ij})\in S(\zeta)} |\nabla_q \eta_{ij}(u)| \, \Delta\phi_{s,t} < \alpha|\varpi(u)| \right\}.$$

Here, $S(\zeta)$ represents the rearrangement of (ζ_{ij}).

Theorem 3.1 *The space ${}_2m(\phi, ru, \nabla_q)$ is a linear space.*

Proof: Let (ζ_{ij}) and (η_{ij}) be elements of ${}_2m(\phi, ru, \nabla_q)$. For scalars $\alpha_1, \alpha_2 > 0$, the following inequalities hold

$$\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} |\nabla_q \zeta_{ij}(u)| < \alpha_1|\varpi(u)|, \quad \text{for all } u \in Q,$$
$$\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} |\nabla_q \eta_{ij}(u)| < \alpha_2|\varpi(u)|, \quad \text{for all } u \in Q.$$

We assume the same scale function ϖ for both sequences without loss of generality. Let β_1 and β_2 be two scalars. Then we obtain

$$\begin{aligned}
&\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} |\beta_1\nabla_q \zeta_{ij}(u) + \beta_2\nabla_q \eta_{ij}(u)| \\
&\leq |\beta_1| \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} |\nabla_q \zeta_{ij}(u)| \\
&+ |\beta_2| \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} |\nabla_q \eta_{ij}(u)| \\
&\leq |\beta_1|\,\alpha_1|\varpi(u)| + |\beta_2|\,\alpha_2|\varpi(u)| \\
&\leq (|\beta_1|\,\alpha_1 + |\beta_2|\,\alpha_2)\,|\varpi(u)| \\
&= \alpha|\varpi(u)|.
\end{aligned}$$

Here we define $\alpha = |\beta_1|\,\alpha_1 + |\beta_2|\,\alpha_2$. Hence, $(\beta_1\zeta_{ij} + \beta_2\eta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$. Therefore, ${}_2m(\phi, ru, \nabla_q)$ is a linear space. □

Theorem 3.2 *The space* $_2m(\phi, ru, \nabla_q)$ *is a BK-space with the norm defined by*

$$\|\zeta_{ij}\|_{_2m(\phi,ru,\nabla_q)} = \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \frac{|\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|}.$$

Proof: We first demonstrate that $_2m(\|.\|_Q, \phi, ru, \nabla_q)$ constitutes a normed space. The norm $\|(\zeta_{ij})\|_{m(\|.\|_Q,\phi,ru,\nabla_q)}$ is zero if and only if $\zeta_{ij} = \bar{\theta}$ for all $i, j \in \mathbb{N}$, where $\bar{\theta}$ is the null sequence of functions. regarding scalar multiplication, let β be a scalar. We observe that

$$\begin{aligned}&\|(\beta\zeta_{ij})\|_{_2m(\|.\|_Q,\phi,ru,\nabla_q)} = \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \tfrac{|\beta\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|}\\ &= |\beta| \, \|(\zeta_{ij})\|_{_2m(\|.\|_Q,\phi,ru,\nabla_q)}.\end{aligned}$$

For the triangle inequality, let (ζ_{ij}) and (η_{ij}) be elements of $_2m(\phi, ru, \nabla_q)$. We have

$$\begin{aligned}&\|(\zeta_{ij}+\eta_{ij})\|_{_2m(\|.\|_Q,\phi,ru,\nabla_q)}\\ &\quad= \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q\zeta_{ij}(u)+\nabla_q\eta_{ij}(u)|}{|\varpi(u)|}\\ &\quad\leq \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|}\\ &\quad+ \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q\eta_{ij}(u)|}{|\varpi(u)|}\\ &\quad= \|(\zeta_{ij})\|_{_2m(\|.\|_Q,\phi,ru,\nabla_q)} + \|(\eta_{ij})\|_{_2m(\|.\|_Q,\phi,ru,\nabla_q)}.\end{aligned}$$

This implies

$$\|(\zeta_{ij}+\eta_{ij})\|_{_2m(\|.\|_Q,\phi,ru,\nabla_q)} \leq \|(\zeta_{ij})\|_{_2m(\|.\|_Q,\phi,ru,\nabla_q)} + \|(\eta_{ij})\|_{_2m(\|.\|_Q,\phi,ru,\nabla_q)}.$$

Thus, $_2m(\|.\|_Q, \phi, ru, \nabla_q)$ is a normed space. We now prove the completeness of $_2m(\|.\|_Q, \phi, ru, \nabla_q)$. Let $\left\{\frac{\zeta^{kl}}{\varpi^{kl}}\right\}$ be a Cauchy sequence in $_2m(\|.\|_Q, \phi, ru, \nabla_q)$, where $\sup_{k,l}\varpi^{kl}(u)$ exists for all $u \in Q$. We define $\sup_{k,l}\varpi^{kl}(u) = \varpi(u)$ for every $u \in Q$. Thus we write $\frac{\zeta^{kl}}{\varpi} = \left\{\frac{\zeta^{kl}_{ij}(u)}{\varpi(u)}\right\}_{i,j=1,1}^{\infty,\infty}$ for each fixed $k, l \in \mathbb{N}$. For any $\alpha > 0$, a positive integer $n_0(\alpha) > 0$ exists such that for all $k, l > n_0(\alpha)$ we have

$$\begin{aligned}&\left\|\tfrac{\zeta^{kl}_{ij}-\zeta^{kl}_{ij}}{\varpi}\right\|_{_2m(\|.\|_Q,\phi,ru,\nabla_q)}\\ &= \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \tfrac{\left|\nabla_q\zeta^{kl}_{ij}(u)-\nabla_q\zeta^{kl}_{ij}(u)\right|}{|\varpi(u)|} < \alpha.\end{aligned}$$

If we choose $(s,t) = (1,1)$ and vary $\sigma \in \xi_{st}$, we find

$$\frac{\left|\nabla_q\zeta^{kl}_{ij}(u)-\nabla_q\zeta^{kl}_{ij}(u)\right|}{|\varpi(u)|} < \alpha\phi_{1,1} \quad \text{for each fixed } i,j\in\mathbb{N} \text{ and all } k,l \geq n_0(\alpha).$$

Consequently, we conclude that

$$\frac{\left|\nabla_q\zeta^{kl}_{ij}(u)\right|}{|\varpi(u)|} - \frac{\left|\nabla_q\zeta^{kl}_{ij}(u)\right|}{|\varpi(u)|} < \alpha\phi_{1,1} \quad \text{for all } k,l \geq n_0(\alpha).$$

This implies that $\left\{\left|\frac{\zeta^{kl}_{ij}(u)}{\varpi(u)}\right|\right\}_{k,l\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}\times\mathbb{R}$ for every fixed $i, j \in \mathbb{N}$. Since $\mathbb{R}\times\mathbb{R}$ is complete, the sequence converges for each $i, j \in \mathbb{N}$. Hence we obtain

$$\lim_{k,l\to\infty}\left|\frac{\zeta^{kl}_{ij}(u)}{\varpi(u)}\right| = \left|\frac{\zeta_{ij}(u)}{\varpi(u)}\right|. \tag{3.1}$$

We now have $\lim_{k,l\to\infty}\left\|\frac{\zeta_{ij}^{kl}-\zeta_{ij}^{kl}}{\varpi}\right\|=0$. We consider $\eta^{kl}(u)=\sum_{(i,j)\in\sigma}\frac{\left|\nabla_q\zeta_{ij}^{kl}(u)\right|}{|\varpi(u)|}$. Then $\left(\eta^{kl}\right)\in\ell_\infty(\nabla_q,ru)$. Therefore

$$\sup_{k,l\in\mathbb{N}}\sum_{(i,j)\in\sigma}\frac{\left|\nabla_q\zeta_{ij}^{kl}(u)\right|}{|\varpi(u)|}\leq\varrho. \tag{3.2}$$

We observe that

$$\begin{aligned}\sum_{(i,j)\in\sigma}\frac{|\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|} &= \sum_{(i,j)\in\sigma}\frac{\left|\nabla_q\zeta_{ij}(u)-\nabla_q\zeta_{ij}^{kl}(u)+\nabla_q\zeta_{ij}^{kl}(u)\right|}{|\varpi(u)|}\\ &\leq \sum_{(i,j)\in\sigma}\frac{\left|\nabla_q\zeta_{ij}(x)-\nabla_q\zeta_{ij}^{kl}(u)\right|}{|\varpi(u)|}+\sum_{(i,j)\in\sigma}\frac{\left|\nabla_q\zeta_{ij}^{kl}(u)\right|}{|\varpi(u)|}\\ &= \alpha\phi_{1,1}+\varrho.\end{aligned}$$

The last step follows from (3.1) and (3.2). This result implies that $\zeta=(\zeta_{ij})\in{}_2m(\phi,ru,\nabla_q)$. Since $\left\{\frac{\zeta^{kl}}{\varpi}\right\}$ is an arbitrary Cauchy sequence in ${}_2m\left(\|.\|_Q,\phi,ru,\nabla_q\right)$, the space is complete. Finally, we show that ${}_2m\left(\|.\|_Q,\phi,ru,\nabla_q\right)$ has continuous coordinate projections p_k, where

$$p_k:\omega_f\to K \quad\text{and}\quad p_k\left(\zeta_{ij}(u)\right)=\zeta_{ij}(u),\quad u\in Q.$$

The coordinate projections p_k are continuous because the following inequality holds

$$|\zeta_{ij}(u)|\leq\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\phi_{st}\|\zeta\|_{{}_2m(\|.\|_Q,\phi,ru,\nabla_q)}|\varpi(u)|,\quad\text{for each } i,j\in\mathbb{N}.$$

□

In light of Theorem 3.2, we state the following theorem without providing a proof.

Theorem 3.3 *The sequence space of functions ${}_2n(\phi,ru,\nabla_q)$ is a BK-space.*

Theorem 3.4 *The sequence space of functions ${}_2m(\phi,ru,\nabla_q)$ is solid.*

Proof: Let $(\zeta_{ij})\in{}_2m(\phi,ru,\nabla_q)$. By assumption, we have

$$\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\frac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}|\nabla_q\zeta_{ij}(u)|<\alpha|\varpi(u)|. \tag{3.3}$$

We consider another sequence of functions $(\eta_{ij})\in\omega_f$ such that

$$|\eta_{ij}(u)|\leq|\zeta_{ij}(u)|. \tag{3.4}$$

Combining (3.3) and (3.4) yields the following inequality

$$\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\frac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}|\nabla_q\eta_{ij}(u)|\leq\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\frac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}|\nabla_q\zeta_{ij}(u)|<\alpha|\varpi(u)|.$$

Hence, $(\eta_{ij})\in{}_2m(\phi,ru,\nabla_q)$. As a result, ${}_2m(\phi,ru,\nabla_q)$ is solid. □

Theorem 3.5 *The space ${}_2m(\phi,ru,\nabla_q)$ is isomorphic to the space $\ell_p(\nabla_q,ru)$ for $1\leq p<\infty$.*

Proof: We define a transformation $\mathcal{T}:{}_2m(\phi,ru,\nabla_q)\to\ell_p(\nabla_q,ru)$ by

$$\mathcal{T}\left(\zeta_{ij}(u)\right)=\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\frac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}\left(\sum_{i,j=1,1}^{\infty,\infty}\frac{|\nabla_q\zeta_{ij}(u)|^p}{|\varpi(u)|}\right)^{\frac{1}{p}}.$$

The linearity of $\mathcal{T}$ follows directly from its definition. Furthermore, if $\mathcal{T}\left(\zeta_{ij}(u)\right)=\theta$ where $u\in Q$ and $\theta=(0,0,0,\ldots)$, then $(\zeta_{ij}(u))=\theta$. Hence $\mathcal{T}$ is injective. Consider an arbitrary sequence $(\eta_{ij})\in\ell_p(\nabla_q,ru)$ such that

$$\begin{aligned}
&\mathcal{T}\left(\zeta_{ij}(u)\right)=\eta_{ij}(u)\\
&\Longrightarrow \tfrac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}\left(\sum_{i,j=1,1}^{\infty,\infty}\tfrac{|\nabla_q\zeta_{ij}(u)|^p}{|\varpi(u)|}\right)^{\frac{1}{p}}=\eta_{ij}(u)\\
&\Longrightarrow \left(\sum_{i,j=1,1}^{\infty,\infty}\tfrac{|\nabla_q\zeta_{ij}(u)|^p}{|\varpi(u)|}\right)^{\frac{1}{p}}=\sum_{(i,j)\in\sigma}\phi_{st}\eta_{ij}(u)\\
&\Longrightarrow \zeta_{ij}(u)=\sup\nolimits_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\sum_{(i,j)\in\sigma}\left(\sum_{i,j=1,1}^{\infty,\infty}\left(\phi_{st}\eta_{ij}(u)\right)^p|\varpi(u)|\right)^{\frac{1}{p}}<\alpha\left|\varpi'(u)\right|,
\end{aligned}$$

where $\varpi'(u)$ denotes the new scale function corresponding to the sequence (η_{ij}). Thus the space ${}_2m(\phi,ru,\nabla_q)$ is isomorphic to $\ell_p(\nabla_q,ru)$ for $1\leq p<\infty$. □

We state the following corollaries based on the reasoning outlined above.

Corollary 3.1 *If $\phi_{ij}=1$, the space ${}_2m(\phi,ru,\nabla_q)$ is isomorphic to the space $\ell_1(\nabla_q,ru)$.*

Corollary 3.2 *If $\phi_{ij}=ij$, the space ${}_2m(\phi,ru,\nabla_q)$ is isomorphic to the space $\ell_\infty(\nabla_q,ru)$.*

Corollary 3.3 *Convergence is not free in the sequence space of functions $m(\phi,ru,\nabla_q)$.*

The following example illustrates the validity of Corollary 3.3.

Example 3.1 *Consider a sequence of functions $\zeta_{ij}(u)$ defined on the compact domain $[0,1]$ as follows*

$$\zeta_{ij}(u)=\begin{cases}\frac{1}{iju}, & \text{for } u\neq0,\\ 0, & \text{for } u=0.\end{cases}$$

This sequence does not converge uniformly. Instead, it converges uniformly to the zero function with respect to a scale function ϖ defined as

$$\varpi(u)=\begin{cases}\frac{1}{u}, & \text{for } u\neq0,\\ 0, & \text{for } u=0.\end{cases}$$

We consider the sequence (ϕ_{ij}) where $\phi_{ij}=1$ for all $i,j\in\mathbb{N}$. In this case, we have

$$\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\frac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}\frac{\left|\frac{1}{iju}\right|}{\left|\frac{1}{u}\right|}<\alpha.$$

Hence, $(\zeta_{ij})\in{}_2m(\phi,ru,\nabla_q)$. Now, let us consider another sequence of functions on the same compact domain $[0,1]$, defined by

$$\eta_{ij}(u)=\begin{cases}iju, & \text{for } u\neq0,\\ 0, & \text{for } u=0,\end{cases}$$

We use $\varpi(u)$ as the scale function. However, in this case we have

$$\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\frac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}\frac{|iju|}{|u|}\to\infty,\quad \text{as } i,j\to\infty$$

Therefore, $(\eta_{ij})\notin{}_2m(\phi,ru,\nabla_q)$. Hence, ${}_2m(\phi,ru,\nabla_q)$ is not convergence-free in general.

Theorem 3.6 *The following properties hold.*

1. *The space* ${}_2m(\phi, ru, \nabla_q)$ *is symmetric. If* $(\zeta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$ *and* $(\eta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$*, then*
$$\|(\eta_{ij})\|_{2m(\phi,ru,\nabla_q)} = \|(\zeta_{ij})\|_{2m(\phi,ru,\nabla_q)}.$$
2. *Suppose* $(\zeta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$ *and* $|(\eta_{ij})| \leq |(\zeta_{ij})|$ *for all* $i, j \in \mathbb{N}$*. Then,* $(\eta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$ *and*
$$\|(\eta_{ij})\|_{2m(\phi,ru,\nabla_q)} \leq \|(\zeta_{ij})\|_{2m(\phi,ru,\nabla_q)}.$$

Proof: The proof of the first property follows directly from Lemma 5(a) in Sargent [30]. For the second property, let $(\zeta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$ and consider a sequence of functions (η_{ij}) such that $|\eta_{ij}(u)| \leq |\zeta_{ij}(u)|$ for all $i, j \in \mathbb{N}$. We have

$$\begin{aligned}
&\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} |\nabla_q \eta_{ij}(u)| \\
&\leq \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} |\nabla_q \zeta_{ij}(u)| \leq \alpha |\varpi(u)| \\
&\Longrightarrow \|\eta_{ij}\|_{2m(\phi,ru,\nabla_q)} \leq \|\zeta_{ij}\|_{2m(\phi,ru,\nabla_q)}.
\end{aligned}$$

The last inequality uses the fact that $(\zeta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$. Hence, $(\eta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$. □

Theorem 3.7 *The space* ${}_2m(\phi, ru, \nabla_q)$ *is a subset of* ${}_2m(\psi, ru, \nabla_q)$ *if and only if* $\sup_{(s,t)\geq(1,1)} \frac{\phi_{st}}{\psi_{st}} < \infty$.

Proof: Let $U = \sup_{(s,t)\geq(1,1)} \frac{\phi_{st}}{\psi_{st}} < \infty$. We can express this as

$$\phi_{st} \leq U\psi_{st}, \quad \text{for all } s, t \in \mathbb{N}. \tag{3.5}$$

Assume $(\zeta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$. We have

$$\begin{aligned}
&\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q \zeta_{ij}(u)|}{|\varpi(u)|} < \alpha. \\
&\Longrightarrow \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{U\psi_{st}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q \zeta_{ij}(u)|}{|\varpi(u)|} < \alpha, \quad \text{using (3.5)} \\
&\Longrightarrow \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\psi_{st}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q \zeta_{ij}(u)|}{|\varpi(u)|} < U\alpha \\
&\Longrightarrow \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\psi_{st}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q \zeta_{ij}(u)|}{|\varpi(u)|} < U',
\end{aligned}$$

where $U' = U\alpha$. This implies $(\zeta_{ij}) \in {}_2m(\psi, ru, \nabla_q)$. Thus, we conclude that ${}_2m(\phi, ru, \nabla_q) \subseteq {}_2m(\psi, ru, \nabla_q)$. Conversely, assume that ${}_2m(\phi, ru, \nabla_q) \subseteq {}_2m(\psi, ru, \nabla_q)$. Consider the case where $\sup_{(s,t)\geq(1,1)} \frac{\phi_{st}}{\psi_{st}} = \infty$. Let

$$\sup_{(s,t)\geq(1,1)} \frac{\phi_{st}}{\psi_{st}} = \sup_{(s,t)\geq(1,1)} \vartheta_{st}, \text{ and } (\zeta_{ij}) \in {}_2m(\phi, ru, \nabla_q).$$

Therefore

$$\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \frac{|\nabla_q \zeta_{ij}(u)|}{|\varpi(u)|} < \alpha.$$

We observe that

$$\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\psi_{st}} \sum_{(i,j)\in\sigma} \frac{|\nabla_q \zeta_{ij}(u)|}{|\varpi(u)|} = \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \vartheta_{st} \cdot \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \frac{|\nabla_q \zeta_{ij}(u)|}{|\varpi(u)|} = \infty.$$

This implies that $(\zeta_{ij}) \notin {}_2m(\psi, ru, \nabla_q)$, which is a contradiction. Thus we conclude that

$$\sup_{(s,t)\geq(1,1)} \frac{\phi_{st}}{\psi_{st}} < \infty.$$

□

Theorem 3.8 *The inclusions* $\ell_p(ru, \nabla_q) \subseteq {}_2m(\phi, ru, \nabla_q) \subseteq \ell_\infty(ru, \nabla_q)$ *hold.*

Proof: If we suppose that $\phi_{ij} = 1$ for all $i, j \in \mathbb{N}$, we have $\ell_p(ru, \nabla_q) = {}_2m(\phi, ru, \nabla_q)$. Let $(\zeta_{ij}) \in m(\phi, ru, \nabla_q)$. Then

$$\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|} < \alpha.$$

$$\Longrightarrow \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{|\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|} < \alpha\phi_{st}, \quad \text{for all } i, j \in \mathbb{N}.$$

Thus, $(\zeta_{ij}) \in \ell_\infty(ru, \nabla_q)$. Hence, $\ell_p(ru, \nabla_q) \subseteq {}_2m(\phi, ru, \nabla_q) \subseteq \ell_\infty(ru, \nabla_q)$. □

Theorem 3.9 ${}_2m(\phi, ru, \nabla_q) = \ell(\phi, ru, \nabla_q)$ *if and only if* $\lim_{s,t\to\infty} \phi_{st} < \infty$.

Proof: The theorem follows from Lemma 5(c) of Sargent [30]. □

4. Geometric Properties

This section investigates the geometric properties of the function sequence spaces ${}_2m(\phi, ru, \nabla_q)$ and ${}_2n(\phi, ru, \nabla_q)$ through the following theorems.

Theorem 4.1 *We define the modular function* $\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ *on the space* ${}_2m(\phi, ru, \nabla_q)$ *by the expression*

$$\mathcal{M}_{2m(\phi,ru,\nabla_q)}(\zeta_{ij}(u)) = \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \mathcal{M}_{ij}\left(\frac{|\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|}\right).$$

Proof: Let the function $\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ be defined as $\mathcal{M}_{2m(\phi,ru,\nabla_q)} : {}_2m(\phi, ru, \nabla_q) \to [0, \infty)$. Let $(\zeta_{ij}), (\eta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$. First, it is evident that $\mathcal{M}_{2m(\phi,ru)}(\zeta_{ij}(u)) = 0$ implies $\zeta_{ij}(u) = \bar{\theta}$ where $u \in Q$ and $\bar{\theta} = (\theta, \theta, \theta, \ldots\ldots)$. Second, we have $\mathcal{M}_{2m(\phi,ru,\nabla_q)}(-\zeta_{ij}(u)) = \mathcal{M}_{2m(\phi,ru,\nabla_q)}(\zeta_{ij}(u))$ for all $(\zeta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$ and $u \in Q$. Third, for a scalar λ with $|\lambda| = 1$, we obtain

$$\mathcal{M}_{2m(\phi,ru,\nabla_q)}(\lambda\zeta_{ij}(u)) = \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \frac{|\lambda\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|} = \mathcal{M}_{2m(\phi,ru,\nabla_q)}(\zeta_{ij}(u)), \tag{4.1}$$

for $u \in Q$. Finally, for any two scalars $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$, the following inequality holds

$$\begin{aligned}
&\mathcal{M}_{2m(\phi,ru)}(\lambda_1\zeta_{ij}(u) + \lambda_2\eta_{ij}(u)) \\
&\quad = \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \tfrac{|\lambda_1\nabla_q\zeta_{ij}(u)+\lambda_2\nabla_q\eta_{ij}(u)|}{|\varpi(u)|} \\
&\quad \leq |\lambda_1| \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|} \\
&\quad + |\lambda_2| \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \tfrac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \tfrac{|\nabla_q\eta_{ij}(u)|}{|\varpi(u)|} \\
&\quad \leq \mathcal{M}_{2m(\phi,ru,\nabla_q)}(\zeta_{ij}(u)) + \mathcal{M}_{2m(\phi,ru,\nabla_q)}(\eta_{ij}(u)).
\end{aligned}$$

Note that since $\lambda_1 + \lambda_2 = 1$, both scalars are less than 1. □

The following results are immediate consequences of the preceding theorem.

Theorem 4.2 *We define the modular function* $\mathcal{M}_{2n(\phi,ru,\nabla_q)}$ *on the space* ${}_2n(\phi,ru,\nabla_q)$ *by the expression*

$$\mathcal{M}_{2n(\phi,ru,\nabla_q)} = \sup_{(\eta_{ij})\in\Lambda(\zeta_{ij}),u\in Q} \sum_{i,j=1,1}^{\infty,\infty} \mathcal{M}_{ij}\left(\frac{|\nabla_q\eta_{ij}(u)|\,\Delta\phi_{ij}}{|\varpi(u)|}\right), \quad \textit{for all } u\in Q.$$

Proof: This result follows directly from Theorem 4.1. □

Based on the preceding discussion, we state the following corollary.

Corollary 4.1 *The modulars* $\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ *and* $\mathcal{M}_{2n(\phi,ru,\nabla_q)}$ *are 1-convex or convex modulars.*

Theorem 4.3 *The modular function* $\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ *on the space* ${}_2m(\phi,ru,\nabla_q)$ *is* $\tilde{s}$*-convex.*

Proof: Consider $(\zeta_{ij}),(\eta_{ij})\in\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ and scalars $\lambda_1,\lambda_2\geq 0$ such that $\lambda_1^{\tilde{s}}+\lambda_2^{\tilde{s}}=1$, where $\tilde{s}\in(0,1]$. We write

$$\begin{aligned}
&\mathcal{M}_{2m(\phi,ru,\nabla_q)}\left(\lambda_1\zeta_{ij}(u)+\lambda_2\eta_{ij}(u)\right)\\
&= \sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\tfrac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}\mathcal{M}_{ij}\left(\tfrac{|\lambda_1\nabla_q\zeta_{ij}(u)+\lambda_2\nabla_q\eta_{ij}(u)|}{|\varpi(u)|}\right)\\
&\leq |\lambda_1|\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\tfrac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}\mathcal{M}_{ij}\left(\tfrac{|\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|}\right)\\
&+|\lambda_2|\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\tfrac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}\mathcal{M}_{ij}\left(\tfrac{|\nabla_q\eta_{ij}(u)|}{|\varpi(u)|}\right)\\
&\leq |\lambda_1|^{\tilde{s}}\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\tfrac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}\tfrac{|\nabla_q\zeta_{ij}(u)|}{|\varpi(u)|}\\
&+|\lambda_2|^{\tilde{s}}\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q}\tfrac{1}{\phi_{st}}\sum_{(i,j)\in\sigma}\tfrac{|\nabla_q\eta_{ij}(u)|}{|\varpi(u)|}.
\end{aligned}$$

Since $\lambda_1,\lambda_2\geq 0$ and $\lambda_1^{\tilde{s}}+\lambda_2^{\tilde{s}}=1$ with $\tilde{s}\in(0,1]$, the scalars λ_1,λ_2 must belong to $[0,1)$. Hence, the space $\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ is $\tilde{s}$-modular. □

Given the analysis above, we present the following theorem.

Theorem 4.4 *The modular function* $M_{2n(\phi,ru,\nabla_q)}$ *on the space* ${}_2n(\phi,ru,\nabla_q)$ *is* $\tilde{s}$*-convex.*

Theorem 4.5 *The modulars* $\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ *and* $\mathcal{M}_{2n(\phi,ru,\nabla_q)}$ *are continuous on the spaces* ${}_2m(\phi,ru,\nabla_q)$ *and* ${}_2n(\phi,ru,\nabla_q)$ *respectively.*

Proof: Because the function $\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ is modular on the space ${}_2m(\phi,ru,\nabla_q)$, we can take the limit as $\lambda\to 1^+$in (4.1) to observe that

$$\lim_{\lambda\to 1^+}\mathcal{M}_{2m(\phi,ru,\nabla_q)}\left(\lambda\zeta_{ij}(u)\right)=\mathcal{M}_{2m(\phi,ru,\nabla_q)}\left(\zeta_{ij}(u)\right), \quad \text{for all } \ (\zeta_{ij})\in\mathcal{M}_{2m(\phi,ru,\nabla_q)}$$

and for all $u\in Q$. Hence, $\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ is right continuous. Similarly, taking the limit as $\lambda\to 1^-$ in (4.1), we find

$$\begin{aligned}
&\lim_{\lambda\to 1^-}\mathcal{M}_{2m(\phi,ru,\nabla_q)}\left(\lambda\zeta_{ij}(u)\right)\\
&=\mathcal{M}_{2m(\phi,ru,\nabla_q)}\left(\zeta_{ij}(u)\right), \quad \text{for all } \ (\zeta_{ij})\in\mathcal{M}_{2m(\phi,ru,\nabla_q)} \text{ and for all } u\in Q.
\end{aligned}$$

These two results lead to the conclusion that the function $\mathcal{M}_{2m(\phi,ru,\nabla_q)}$ is a continuous modular. In a similar manner, we can establish this property for the modular $\mathcal{M}_{2n(\phi,ru,\nabla_q)}$ on the space ${}_2n(\phi,ru,\nabla_q)$. □

The following corollary follows directly from the previous theorem.

Corollary 4.2 *The modular* $\mathcal{M}_{_2m(\phi,ru,\nabla_q)}$ *is modular convergent to* $\zeta \in \mathcal{M}_{_2m(\phi,ru,\nabla_q)}$.

Theorem 4.6 *The sequence space of functions* $S_{_2m(\phi,ru,\nabla_q)}$ *is uniformly convex.*

Proof: Let $(\zeta_{ij}), (\eta_{ij}) \in {}_2m(\phi, ru, \nabla_q)$ satisfy

$$\|(\zeta_{ij})\|_{S_{_2m(\phi,ru,\nabla_q)}} = \|(\eta_{ij})\|_{S_{_2m(\phi,ru,\nabla_q)}} = 1, \quad \|(\zeta_{ij} - \eta_{ij})\|_{S_{_2m(\phi,ru,\nabla_q)}} \geq \rho. \tag{4.2}$$

Using the parallelogram identity

$$\|(\zeta_{ij} + \eta_{ij})\|^2_{S_{_2m(\phi,ru,\nabla_q)}} + \|(\zeta_{ij} - \eta_{ij})\|^2_{S_{_2m(\phi,ru,\nabla_q)}} = 2\left(\|(\zeta_{ij})\|^2_{S_{_2m(\phi,ru,\nabla_q)}} + \|(\eta_{ij})\|^2_{S_{_2m(\phi,ru,\nabla_q)}}\right),$$

we write

$$\|(\zeta_{ij} + \eta_{ij})\|^2_{S_{_2m(\phi,ru,\nabla_q)}} = 2\left(\|(\zeta_{ij})\|^2_{S_{_2m(\phi,ru,\nabla_q)}} + \|(\eta_{ij})\|^2_{S_{_2m(\phi,ru,\nabla_q)}}\right) - \|(\zeta_{ij} - \eta_{ij})\|^2_{S_{_2m(\phi,ru,\nabla_q)}}.$$

From (4.2) we obtain

$$\left\{\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \frac{|\nabla_q\zeta_{ij}(u)+\nabla_q\eta_{ij}(u)|}{|\varpi(u)|}\right\}^2 \leq 4 - \left\{\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \frac{|\nabla_q\zeta_{ij}(u)-\nabla_q\eta_{ij}(u)|}{|\varpi(u)|}\right\}^2.$$

This simplifies to

$$\left\{\sup_{(s,t)\geq(1,1),\sigma\in\xi_{st},u\in Q} \frac{1}{\phi_{st}} \sum_{(i,j)\in\sigma} \frac{|\nabla_q\zeta_{ij}(u)+\nabla_q\eta_{ij}(u)|}{|\varpi(u)|}\right\}^2 \leq 4 - \rho^2,$$

which implies

$$\left\|\frac{\zeta_{ij} + \eta_{ij}}{2}\right\|_{_2m(\phi,ru,\nabla_q)} \leq \frac{1}{2}\sqrt{4-\rho^2}.$$

Further rearrangement gives

$$\left\|\frac{\zeta_{ij} + \eta_{ij}}{2}\right\|_{_2m(\phi,ru,\nabla_q)} \leq 1 - \left(1 - \frac{1}{2}\sqrt{4-\rho^2}\right),$$

and finally

$$\left\|\frac{\zeta_{ij} + \eta_{ij}}{2}\right\| \leq 1 - \mho(\rho).$$

Since this condition holds for any $\rho > 0$ with some $\mho(\rho) > 0$, the space $S_{_2m(\phi,ru,\nabla_q)}$ is uniformly convex. □

Remark 4.1 The modulus of convexity $\mho^\rho_{_2m(\phi,ru,\nabla_q)}$ is defined as the best possible $\mho(\rho) > 0$ corresponding to the uniformly convex space ${}_2m(\phi, ru, \nabla_q)$

$$\mho^\rho_{_2m(\phi,ru,\nabla_q)} = \inf\left\{1 - \left\|\left(\frac{\zeta_{ij} + \eta_{ij}}{2}\right)\right\| : (\zeta_{ij}), (\eta_{ij}) \in S_{_2m(\phi,ru,\nabla_q)}, \quad \|(\zeta_{ij} - \eta_{ij})\|_{S_{_2m(\phi,ru,\nabla_q)}} \geq \rho\right\}.$$

The proof of the following proposition is analogous to that of Theorem 4.6.

Problem 1 *The space* ${}_2n(\phi, ru, \nabla_q)$ *is uniformly convex.*

Theorem 4.7 *The spaces* ${}_2m(\phi, ru, \nabla_q)$ *and* ${}_2n(\phi, ru, \nabla_q)$ *are uniformly non-square.*

Proof: Since ${}_2m(\phi, ru, \nabla_q)$ and ${}_2n(\phi, ru, \nabla_q)$ are both Banach spaces and uniformly convex (James [13] Theorem (1.1)), it follows that both spaces are uniformly non-square. □

Corollary 4.3 *The spaces* ${}_2m(\phi, ru, \nabla_q)$ *and* ${}_2n(\phi, ru, \nabla_q)$ *are reflexive.*

Proof: Since the spaces ${}_2m(\phi, ru, \nabla_q)$ and ${}_2n(\phi, ru, \nabla_q)$ are uniformly convex and also uniformly non-square, Theorem (1.1) in James [13] implies that both spaces are reflexive. □

Corollary 4.4 *The sequence space of functions* ${}_2m(\phi, ru, \nabla_q)$ *is super reflexive.*

Proof: The space ${}_2m(\phi, ru, \nabla_q)$ is isometric to the Banach space $\ell_p(ru)$. Furthermore, ${}_2m(\phi, ru, \nabla_q)$ is uniformly convex. Thus the space ${}_2m(\phi, ru, \nabla_q)$ is super-reflexive. Similarly, we can prove that the space ${}_2n(\phi, ru, \nabla_q)$ is also reflexive. □

We summarize the main structural and geometric characteristics of the sequence spaces ${}_2m(\phi, ru, \nabla_q)$ and ${}_2n(\phi, ru, \nabla_q)$ in Table 1 to provide a concise overview of the findings obtained in this study.

Table 1: Summary of structural and geometric properties for the sequence spaces

Property	$\mathbf{{}_2m(\phi, ru, \nabla_q)}$	$\mathbf{{}_2n(\phi, ru, \nabla_q)}$
Linearity	Yes	Yes
BK-Space Structure	Yes	Yes
Uniform Convexity	Yes	Yes
Uniformly Non-Square	Yes	Yes
Reflexivity	Yes	Yes
Super Reflexivity	Yes	Yes
Isomorphism to ℓ_p	Yes	–

The information presented in Table 1 highlights that both spaces exhibit robust geometric structures. In particular the presence of uniform convexity and reflexivity suggests that these sequence spaces are suitable for further applications in fixed point theory and approximation theory.

5. Conclusion

This study successfully introduced and examined the classes of quantum difference relative uniform convergent sequences of Sargent-type functions denoted by ${}_2m(\phi, ru, \nabla_q)$ and ${}_2n(\phi, ru, \nabla_q)$. We established that these spaces possess a linear structure and form Banach spaces with the BK-property. Furthermore we demonstrated that these sequence spaces are solid and exhibit specific inclusion relations. Our investigation revealed that the space $m(\phi, ru, \nabla_q)$ is isomorphic to the space $\ell_p(\nabla_q, ru)$ for finite values of p. A significant portion of this work focused on the geometric properties of these spaces through the lens of modular function theory. We defined the modulars for these spaces and verified their convexity and continuity. The analysis confirmed that these spaces are uniformly convex and uniformly non-square. Consequently we deduced that the spaces ${}_2m(\phi, ru, \nabla_q)$ and ${}_2n(\phi, ru, \nabla_q)$ are reflexive and super reflexive. These results contribute to the broader understanding of summability theory and functional analysis. Future research may extend these concepts to different types of convergence or apply them to other generalized difference operators.

References

1. C. Belen, S. A. Mohiuddine, Generalized weighted statistical convergence and application, Appl. Math. Comput., 219 (2013), 9821–9826.
2. E.W. Chittenden, On the limit functions of sequences of continuous functions converging relatively uniformly, Trans. Am. Math. Soc., 20(2) (1919), 179–184.
3. D. Debbarma, K. R. Devi, B.C. Tripathy, Relative uniform convergence of p-absolutely summable sequence of functions, Bol. Soc. Paran. Mat., (42) (2024), 1-7.
4. D. Debbarma, B.C. Tripathy, Relative uniform convergence of quantum difference sequence of functions related to the l_p-space, Ann. Univ. Craiova Math. Comput. Sci. Ser., 51(1) (2024), 234–242.
5. K. Demirci, A. Boccuto, S. Yıldız, F. Dirik, Relative uniform convergence of a sequence of functions at a point and korovkin-type approximation theorems, Positivity, 24(1) (2020), 1–11.
6. K. Demirci, S. Yıldız, Statistically relatively uniform convergence of positive linear operators, Results Math., 69(3), (2016), 359–367.
7. K. R. Devi, B.C. Tripathy, On relative uniform convergence of double sequences of functions, Proc. Natl. Acad. Sci. India-Phys. Sci., 92(3) (2022), 367–372.
8. K. R. Devi, B.C. Tripathy, Relative uniform convergence of difference sequence of positive linear functions, Trans. A. Razmadze Math. Inst. 176(1) (2022), 37–43.
9. K. R. Devi, B.C. Tripathy, Relative uniform convergence of difference double sequence of positive linear functions, Ricerche Mat., 72(2) (2023), 961–972.
10. K. R. Devi, B.C. Tripathy, Relative uniform convergence of sequence of functions related to l^p-space, under review.
11. M. Gürdal, Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra, Expositiones Math. 27(2) (2009), 153-160.
12. M. Gürdal, M.T. Garayev, S. Saltan, Some concrete operators and their properties, Turkish J. Math., 39(6) (2015), 970-989.
13. R. C. James, Super-reflexive Spaces Wite Base, Pac. J. Math., 41(2) (1972), 409-419.
14. H. Fast, Sur la convergence statistique, Colloq. Math., 2 (1951), 241-244.
15. M. Gürdal, U. Yamancı, Statistical convergence of operator theory, Dynamic Systems and Applications, 24(3) (2015), 305-311.
16. U. Kadak, S. A. Mohiuddine, Generalized statistically almost convergence based on the difference operator which includes the (p, q)-Gamma function and related approximation theorems, Results Math., 73(9) (2018), 1–31.
17. H. Ş. Kandemir, M. Et, N. D. Aral, Strongly λ-convergence of order α in neutrosophic normed spaces, Dera Natung Govt. College Res. J., 7(1) (2022), 1-9.
18. M.T. Karaev, M. Gürdal, M.B. Huban, Reproducing kernels, Englis algebras and some applications, Studia Mathematica, 232(2) (2016), 113-141.
19. M. Karaev, M. Gürdal, S. Saltan, Some applications of Banach algebra techniques, Math. Nachr., 84(13) (2011), 1678-1689.
20. S. A. Mohiuddine, H. Şevli, M. Cancan, Statistical convergence in fuzzy 2-normed space, J. Comput. Anal. Appl., 12(4) (2010), 787–798.
21. S. A. Mohiuddine, A. Asiri, B. Hazarika, Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems, Int. J. Gen. Syst., 48(5) (2019), 492–506.
22. S. A. Mohiuddine, B. Hazarika, M. A. Alghamdi, Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems, Filomat, 33(14) (2019), 4549–4560.
23. S. A. Mohiuddine, B. Hazarika, A. Asiri, Ideal relatively equal convergence involving difference operators with associated approximation theorems, Filomat, 39(21) (2025), 7209–7221.
24. E. H. Moore, An introduction to a form of general analysis, The New Haven Mathematical Colloquium, Yale University Press, New Haven, pp.30 (1910).
25. F. Moricz, Statistical convergence of multiple sequences. Arch.Math., 81 (2003), 82–89.
26. M. Mursaleen and O. H. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl., 288(1) (2003), 223–231.
27. D. Rath and B. C. Tripathy, Characterization of certain matrix operations, J Orissa Math Soc., 8, 121-134.
28. P. O. Şahin, F. Dirik, Statistical relative uniform convergence of double sequences of positive linear operators, Appl. Math. E-Notes, 17 (2017), 207–220.

29. A. Şahiner, M. Gürdal, T. Yigit, Ideal convergence characterization of the completion of linear n-normed spaces, Comput. Math. Appl., 61(3) (2011), 683-689.

30. W. L. C. Sargent, Some sequence spaces related to l^p spaces, J London Math., 35 (1960), 161-171.

31. E. Savaş, Ö. Kişi, M. Gürdal, On statistical convergence in credibility space, Numerical Funct. Anal. Optim., 43(8) (2022), 987-1008.

32. B. C. Tripathy, Statistically convergent double sequences, Tamkang J. Math., 34(3) (2003), 231–237.

33. B. C. Tripathy, Matrix maps on the power series convergent on the unit disc, J Analysis, 6 (1998), 27-31.

34. B. C. Tripathy and M. Sen, On a new class of sequences related to the space l^p, Tamkang J Math., 33(2) (2008), 167-171.

35. F. H. Jackson, On q-definite integrals, Q. J. Pure Appl. Math., 41 (1910), 193–203.

36. U. Yamancı, M. Gürdal, On lacunary ideal convergence in random n-normed space, J. Math., 2013 (2013), 868457. https://doi.org/10.1155/2013/868457

37. U. Yamancı, M. Gürdal, I-Statistical convergence in 2-normed space, Arab Journal of Mathematical Sciences, 20(1) (2014), 41–47.

38. U. Yamancı, M. Gürdal, Statistical convergence and operators on Fock space, New York J. Math., 22 (2016), 199-207.

39. T. Yaying, On Λ-Fibonacci difference sequence spaces of fractional order, Dera Natung Govt. College Res. J., 6(1) (2021), 92–102.

40. T. Yaying, Arithmetic continuity in cone metric space, Dera Natung Govt. College Res. J., 5(1) (2020), 55–62.

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