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Optimization Results for Cyclic Proximal Non-Self Mappings and Application to a System of Differential Equations

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ABSTRACT: In this paper, we investigate best proximity point results for two non-self proximal mappings $\mathcal{K} : \mathcal{M} \longrightarrow \mathcal{N}$ and $\mathcal{X} : \mathcal{N} \to \mathcal{M}$ using the weak P-property and proximal cyclic contraction. We present an example to illustrate the main result and provide an application of our main theorem to a system of differential equations.

Key Words: Best proximity points, proximal Geraghty non-self mappings, proximal cyclic contraction.

Contents

1. Introduction

Fixed point theory plays a crucial role in nonlinear functional analysis. The concepts of fixed points and best proximity points are natural generalizations of the Banach contraction principle. A best proximity point theorem for contractions was established in [8]. In [4], Anuradha and Veeramani studied the existence of best proximity points for proximal pointwise contractions. Furthermore, several best proximity point theorems for variants of contractions have been analyzed in [2]. S. Basha proposed a fixed point theorem for contractive mappings in [9]. Relevant common best proximity point theorems have been explored in [10]–[20]. On the other hand, a generalization of best proximity point theorems for non-self proximal contractions of the first kind was presented in [7].

The famous Banach contraction principle has been extended to various spaces. Following this direction, Geraghty contractions [13] and Matkowski contractions [14,15] have attracted several authors. For more extensions and results, see for example [6]–[11]. Some investigations using the weak P-property were conducted by S. Komal et al. [16]. Moreover, in [5], M.I. Ayari used the P-property to provide results on the existence and uniqueness of best proximity points for α-proximal Geraghty mappings. A generalized version of such a theorem was presented in [7] for proximal $\mathcal{B}$-quasi contractive non-self mappings $S : \mathcal{M} \to \mathcal{N}$ and $\mathcal{X} : \mathcal{N} \to \mathcal{M}$.

The main objective of this article is to establish best proximity point and fixed point theorems for a pair of non-self mappings $(\mathcal{K}, \mathcal{X})$ forming a proximal cyclic contraction in complete metric spaces. We present an example to support the main result and provide an application to differential equations, obtaining a unique solution as best proximity points.

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2. Preliminaries and Definitions

Let $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty subsets of a metric space (X, d). We adopt the following notations:

- $d(\mathcal{M}, \mathcal{N}) := \inf\{d(\alpha, \beta) : \alpha \in \mathcal{M}, \beta \in \mathcal{N}\}$;
- $\mathcal{M}_0 := \{\alpha \in \mathcal{M} :$ there exists $\beta \in \mathcal{N}$ such that $d(\alpha, \beta) = d(\mathcal{M}, \mathcal{N})\}$;
- $\mathcal{N}_0 := \{\beta \in \mathcal{N} :$ there exists $\alpha \in \mathcal{M}$ such that $d(\alpha, \beta) = d(\mathcal{M}, \mathcal{N})\}$.

Definition 2.1 ([8]) Let $\mathcal{K} : \mathcal{M} \to \mathcal{N}$ be a non-self mapping. An element α_* is called a best proximity point of $\mathcal{K}$ if $d(\alpha_*, \mathcal{K}\alpha_*) = d(\mathcal{M}, \mathcal{N})$.

Definition 2.2 ([8]) Given non-self mappings $\mathcal{K} : \mathcal{M} \to \mathcal{N}$ and $\mathcal{X} : \mathcal{N} \to \mathcal{M}$, the pair $(\mathcal{K}, \mathcal{X})$ is said to form a proximal cyclic contraction if there exists a non-negative number $k < 1$ such that

$$d(u, \mathcal{K}a) = d(\mathcal{M}, \mathcal{N}) \text{ and } d(v, \mathcal{X}b) = d(\mathcal{M}, \mathcal{N}) \text{ implies } d(u, v) \leq kd(a, b) + (1 - k)d(\mathcal{M}, \mathcal{N})$$

for all $u, a \in \mathcal{M}$ and $v, b \in \mathcal{N}$.

Definition 2.3 ([21]) Let $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty subsets of a metric space (X, d) such that $\mathcal{M}_0$ is nonempty. Then the pair $(\mathcal{M}, \mathcal{N})$ is said to have the weak P-property if and only if for any $\alpha_1, \alpha_2 \in \mathcal{M}_0$ and $\beta_1, \beta_2 \in \mathcal{N}_0$ with $d(\alpha_1, \beta_1) = d(\alpha_2, \beta_2) = d(\mathcal{M}, \mathcal{N})$, we have $d(\alpha_1, \alpha_2) \leq d(\beta_1, \beta_2)$.

Let $\mathcal{F}$ denote the class of all functions $\mathcal{B} : [0, \infty) \longrightarrow [0, 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\mathcal{B}(t_n) \longrightarrow 1$ implies $t_n \longrightarrow 0$.

Definition 2.4 ([13]) Let (X, d) be a metric space and $\mathcal{X} : X \to X$ be a given mapping. We say that $\mathcal{X}$ is a $\mathcal{B}$-Geraghty contractive mapping if there exists $\mathcal{B} \in \mathcal{F}$ such that

$$d(\mathcal{X}x, \mathcal{X}y) \leq \mathcal{B}(d(x, y))d(x, y)$$

for all $x, y \in X$.

Definition 2.5 ([14,15]) Let (X, d) be a metric space and $\mathcal{X} : X \to X$ be a given mapping. We say that $\mathcal{X}$ is a Matkowski contractive mapping if there exists a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$ and

$$d(\mathcal{X}x, \mathcal{X}y) \leq \varphi(d(x, y))$$

for all $x, y \in X$.

Remark 2.1 *If $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, then $\varphi(t) < t$ for all $t > 0$ and thus $\varphi(0) = 0$.*

3. Main Results and Theorems

We begin by recalling the concept of proximal Geraghty mappings, which generalize Geraghty contractions:

Definition 3.1 ([16]) Let (X, d) be a metric space and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty subsets of X. A non-self mapping $\mathcal{K} : \mathcal{M} \to \mathcal{N}$ is called a proximal Geraghty mapping if there exists $\mathcal{B} \in \mathcal{F}$ such that

$$d(\mathcal{K}x, \mathcal{K}y) \leq \mathcal{B}(d(x, y))d(x, y)$$

for all $x, y \in \mathcal{M}$.

Remark 3.1 *Every non-self proximal Geraghty mapping $\mathcal{K} : \mathcal{M} \to \mathcal{N}$ is uniformly continuous since it is Lipschitz continuous with constant 1.*

Theorem 3.1 *Let $(\mathcal{M},\mathcal{N})$ be a pair of nonempty closed subsets of a complete metric space (X,d) such that $\mathcal{M}_0$ is nonempty. Let $\mathcal{K}:\mathcal{M}\longrightarrow\mathcal{N}$ and $\mathcal{X}:\mathcal{N}\to\mathcal{M}$ be two non-self mappings satisfying:*

1. *$\mathcal{K}(\mathcal{M}_0)\subset\mathcal{N}_0$ and $\mathcal{X}(\mathcal{N}_0)\subset\mathcal{M}_0$;*
2. *The pair $(\mathcal{M},\mathcal{N})$ satisfies the weak P-property;*
3. *$\mathcal{K}$ and $\mathcal{X}$ are proximal Geraghty mappings;*
4. *The pair $(\mathcal{K},\mathcal{X})$ forms a proximal cyclic contraction.*

Then $\mathcal{K}$ has a unique best proximity point $\alpha_*\in\mathcal{M}$ and $\mathcal{X}$ has a unique best proximity point $\beta_*\in\mathcal{N}$. Moreover, these best proximity points satisfy $d(\alpha_*,\beta_*)=d(\mathcal{M},\mathcal{N})$.

Proof: Since $\mathcal{M}_0$ is nonempty, there exists $\alpha_0\in\mathcal{M}_0$. As $\mathcal{K}(\mathcal{M}_0)\subset\mathcal{N}_0$, we have $\mathcal{K}(\alpha_0)\in\mathcal{N}_0$, so there exists $\alpha_1\in\mathcal{M}_0$ such that $d(\alpha_1,\mathcal{K}\alpha_0)=d(\mathcal{M},\mathcal{N})$. Similarly, since $\mathcal{K}(\mathcal{M}_0)\subset\mathcal{N}_0$, we have $\mathcal{K}(\alpha_1)\in\mathcal{N}_0$, so there exists $\alpha_2\in\mathcal{M}_0$ such that $d(\alpha_2,\mathcal{K}\alpha_1)=d(\mathcal{M},\mathcal{N})$. Continuing this process by induction, we construct a sequence $\{\alpha_n\}\subset\mathcal{M}_0$ such that

$$d(\alpha_{n+1},\mathcal{K}\alpha_n)=d(\mathcal{M},\mathcal{N})\quad\text{for all } n=0,1,\ldots \tag{3.1}$$

From condition (2), the pair $(\mathcal{M},\mathcal{N})$ satisfies the weak P-property, so

$$d(\alpha_{n+1},\alpha_n)\leq d(\mathcal{K}\alpha_n,\mathcal{K}\alpha_{n-1})\quad\text{for all } n\in\mathbb{N}. \tag{3.2}$$

Moreover, by the definition of proximal Geraghty mappings, we have

$$d(\mathcal{K}\alpha_n,\mathcal{K}\alpha_{n-1})\leq\mathcal{B}(d(\alpha_n,\alpha_{n-1}))d(\alpha_n,\alpha_{n-1})\quad\text{for all } n\in\mathbb{N}. \tag{3.3}$$

Since $\mathcal{B}(t)<1$, from (3.2) and (3.3) we obtain

$$d(\alpha_{n+1},\alpha_n)\leq\mathcal{B}(d(\alpha_n,\alpha_{n-1}))d(\alpha_n,\alpha_{n-1})<d(\alpha_n,\alpha_{n-1})\quad\text{for all } n\in\mathbb{N}. \tag{3.4}$$

Thus, the sequence $\{d(\alpha_{n+1},\alpha_n)\}$ is decreasing. We consider two cases:

Case 1: There exists $k\in\mathbb{N}$ such that $d(\alpha_k,\alpha_{k+1})=0$, i.e., $\alpha_{k+1}=\alpha_k$. Then by (3.1), we have

$$d(\alpha_k,\mathcal{K}\alpha_k)=d(\alpha_{k+1},\mathcal{K}\alpha_k)=d(\mathcal{M},\mathcal{N}),$$

and consequently α_k is a best proximity point of $\mathcal{K}$.

Case 2: $d(\alpha_{n-1},\alpha_n)>0$ for all $n\in\mathbb{N}$. Then the sequence $\{d(\alpha_{n+1},\alpha_n)\}$ is decreasing and bounded below by 0, so there exists $\rho\geq 0$ such that $\rho=\lim_{n\to\infty}d(\alpha_n,\alpha_{n+1})$.

We show that $\rho=0$. Assume by contradiction that $\rho>0$. From (3.4),

$$\frac{d(\alpha_{n+1},\alpha_n)}{d(\alpha_n,\alpha_{n-1})}\leq\mathcal{B}(d(\alpha_n,\alpha_{n-1}))<1\quad\text{for all } n\in\mathbb{N}. \tag{3.5}$$

Taking the limit as $n\to\infty$ in (3.5), we get

$$\lim_{n\to\infty}\mathcal{B}(d(\alpha_n,\alpha_{n-1}))=1.$$

Since $\mathcal{B}\in\mathcal{F}$, this implies

$$\rho=\lim_{n\to\infty}d(\alpha_n,\alpha_{n-1})=0,$$

which is a contradiction.

Now we show that $\{\alpha_n\}$ is a Cauchy sequence. Suppose the contrary, i.e., there exists $\epsilon>0$ for which we can find two subsequences $\{\alpha_{m_k}\}$ and $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ and an integer k such that n_k is the smallest index with

$$d(\alpha_{n_k},\alpha_{m_k})\geq\epsilon\quad\text{for each } n_k>m_k>k\quad\text{and}\quad d(\alpha_{m_k},\alpha_{n_k-1})<\epsilon. \tag{3.6}$$

Using the triangle inequality, we have

$$\begin{aligned} \epsilon \leq d(\alpha_{n_k}, \alpha_{m_k}) &\leq d(\alpha_{n_k}, \alpha_{n_k-1}) + d(\alpha_{n_k-1}, \alpha_{m_k}) \\ &\leq d(\alpha_{n_k}, \alpha_{n_k-1}) + \epsilon \quad \text{for all } k. \end{aligned} \tag{3.7}$$

Taking the limit as $k \to \infty$ in (3.7) and using (3.6), we conclude that

$$\lim_{k\to\infty} d(\alpha_{m_k}, \alpha_{n_k}) = \epsilon. \tag{3.8}$$

Using the triangle inequality again,

$$\begin{aligned} d(\alpha_{m_k}, \alpha_{n_k}) - d(\alpha_{m_k}, \alpha_{m_k-1}) - d(\alpha_{n_k-1}, \alpha_{n_k}) &\leq d(\alpha_{m_k-1}, \alpha_{n_k-1}) \\ &\leq d(\alpha_{m_k-1}, \alpha_{m_k}) + d(\alpha_{m_k}, \alpha_{n_k}) + d(\alpha_{n_k}, \alpha_{n_k-1}). \end{aligned} \tag{3.9}$$

Taking the limit as $k \to \infty$ in (3.9), using the fact that $\lim_{n\to\infty} d(\alpha_n, \alpha_{n+1}) = 0$ and (3.8), we obtain

$$\lim_{k\to\infty} d(\alpha_{m_k-1}, \alpha_{n_k-1}) = \epsilon. \tag{3.10}$$

From condition (2), the pair $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property, so

$$d(\alpha_{n_k}, \alpha_{m_k}) \leq d(\mathcal{K}\alpha_{n_k-1}, \mathcal{K}\alpha_{m_k-1}) \quad \text{for all } k \in \mathbb{N}.$$

Then, by the definition of proximal Geraghty mappings,

$$d(\alpha_{m_k}, \alpha_{n_k}) \leq \mathcal{B}(d(\alpha_{m_k-1}, \alpha_{n_k-1}))d(\alpha_{m_k-1}, \alpha_{n_k-1}).$$

Hence

$$\frac{d(\alpha_{m_k}, \alpha_{n_k})}{d(\alpha_{m_k-1}, \alpha_{n_k-1})} \leq \mathcal{B}(d(\alpha_{m_k-1}, \alpha_{n_k-1})) < 1 \quad \text{for all } k \in \mathbb{N}.$$

Taking the limit as $k \to \infty$ and using (3.8) and (3.10), we obtain

$$\lim_{k\to\infty} \mathcal{B}(d(\alpha_{m_k-1}, \alpha_{n_k-1})) = 1.$$

Since $\mathcal{B} \in \mathcal{F}$, this implies

$$\lim_{k\to\infty} d(\alpha_{m_k-1}, \alpha_{n_k-1}) = 0,$$

which is a contradiction. Thus, $\{\alpha_n\}$ is a Cauchy sequence in the closed subset $\mathcal{M}$ of (X, d).

Since (X, d) is complete and $\mathcal{M}$ is closed, the sequence $\{\alpha_n\}$ converges to some element $\alpha_* \in \mathcal{M}$.

Using Remark 3.1 that $\mathcal{K}$ is continuous and (3.1), we get

$$d(\alpha_*, \mathcal{K}\alpha_*) = d(\mathcal{M}, \mathcal{N}).$$

For uniqueness, suppose there exists another best proximity point c_* for $\mathcal{K}$ such that $c_* \neq \alpha_*$. Then

$$d(\alpha_*, \mathcal{K}\alpha_*) = d(c_*, \mathcal{K}c_*) = d(\mathcal{M}, \mathcal{N}).$$

Since the pair $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property and $\mathcal{K}$ is proximal Geraghty, we have

$$0 < d(\alpha_*, c_*) \leq d(\mathcal{K}\alpha_*, \mathcal{K}c_*) \leq \mathcal{B}(d(\alpha_*, c_*))d(\alpha_*, c_*) < d(\alpha_*, c_*),$$

which is a contradiction. Therefore, $c_* = \alpha_*$.

Since $\mathcal{X}(\mathcal{N}_0) \subset \mathcal{M}_0$, using a similar argument as above with $\mathcal{K}$ replaced by $\mathcal{X}$, there exists a sequence $\{\beta_n\} \subset \mathcal{N}_0$ such that $d(\beta_{n+1}, \mathcal{X}\beta_n) = d(\mathcal{M}, \mathcal{N})$ for each n.

Similarly, one can show that $\{\beta_n\}$ is a Cauchy sequence in the closed subset $\mathcal{N}$ of the complete space X. Thus $\{\beta_n\}$ converges to $\beta_* \in \mathcal{N}$ and β_* is unique.

Finally, we show that $d(\alpha_*, \beta_*) = d(\mathcal{M}, \mathcal{N})$. Since $d(\alpha_*, \mathcal{K}\alpha_*) = d(\beta_*, \mathcal{X}\beta_*) = d(\mathcal{M}, \mathcal{N})$ and the pair $(\mathcal{K}, \mathcal{X})$ forms a proximal cyclic contraction, it follows that

$$d(\alpha_*, \beta_*) \leq kd(\alpha_*, \beta_*) + (1-k)d(\mathcal{M}, \mathcal{N}),$$

and so

$$(1-k)d(\alpha_*, \beta_*) \leq (1-k)d(\mathcal{M}, \mathcal{N}),$$

which implies

$$d(\alpha_*, \beta_*) \leq d(\mathcal{M}, \mathcal{N}). \tag{3.11}$$

Using the fact that $d(\mathcal{M}, \mathcal{N}) \leq d(\alpha_*, \beta_*)$, we conclude that $d(\alpha_*, \beta_*) = d(\mathcal{M}, \mathcal{N})$. □

The following example illustrates the validity of Theorem 3.1.

Example 3.1 Consider the complete Euclidean space $X = \mathbb{R}^2$ with the metric

$$d((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = |\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|.$$

Let

$$\mathcal{M} = \{(1, x) : x \in \mathbb{R}\} \quad \text{and} \quad \mathcal{N} = \{(3, y) : y \in \mathbb{R}\}.$$

Consider the non-self mappings $\mathcal{K} : \mathcal{M} \longrightarrow \mathcal{N}$ and $\mathcal{X} : \mathcal{N} \to \mathcal{M}$ defined by

$$\mathcal{K}(1, x) = \left(3, \frac{|x|}{3(1+|x|)}\right), \quad \mathcal{X}(3, y) = \left(1, \frac{|y|}{3(1+|y|)}\right),$$

and $\mathcal{B} : [0, +\infty) \longrightarrow [0, 1)$ defined by

$$\mathcal{B}(u) = \begin{cases} 0 & \text{if } u = 0, \\ \frac{1}{1+u} & \text{if } u \in (0, \infty). \end{cases}$$

Clearly, $\mathcal{M} = \mathcal{M}_0$ and $\mathcal{N} = \mathcal{N}_0$. Moreover, $\mathcal{K}(\mathcal{M}_0) = \mathcal{N}_0$, $\mathcal{X}(\mathcal{N}_0) = \mathcal{M}_0$, and $\mathcal{M}$ and $\mathcal{N}$ are closed subsets of the complete space $(\mathbb{R}^2, d)$. One can see that $d(\mathcal{M}, \mathcal{N}) = 2$, which occurs only for $(1, x) \in \mathcal{M}$ and $(3, x) \in \mathcal{N}$.

Now, let $(1, \alpha_1), (1, \alpha_2) \in \mathcal{M}_0$ and $(3, \beta_1), (3, \beta_2) \in \mathcal{N}_0$. If

$$2 = d((1, \alpha_1), (3, \beta_1)) = |2| + |\alpha_1 - \beta_1| \Rightarrow \alpha_1 = \beta_1,$$

and similarly

$$2 = d((1, \alpha_2), (3, \beta_2)) = |2| + |\alpha_2 - \beta_2| \Rightarrow \alpha_2 = \beta_2,$$

then

$$d((1, \alpha_1), (1, \alpha_2)) = |\alpha_1 - \alpha_2| = |\beta_1 - \beta_2| = d((3, \beta_1), (3, \beta_2)).$$

Thus, the pair $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property.

Next, we show that $\mathcal{K}$ is a proximal Geraghty mapping. Let $(1, \alpha_1), (1, \alpha_2) \in \mathcal{M}$. If $\alpha_1 = \alpha_2$, the result is clear. Suppose $\alpha_1 \neq \alpha_2$. Then

$$\begin{aligned} d(\mathcal{K}(1, \alpha_1), \mathcal{K}(1, \alpha_2)) &= d\left((3, \tfrac{|\alpha_1|}{3(1+|\alpha_1|)}), (3, \tfrac{|\alpha_2|}{3(1+|\alpha_2|)})\right) \\ &= \left| \tfrac{|\alpha_1|}{3(1+|\alpha_1|)} - \tfrac{|\alpha_2|}{3(1+|\alpha_2|)} \right| \\ &= \left| \tfrac{|\alpha_1| - |\alpha_2|}{3(1+|\alpha_1|)(1+|\alpha_2|)} \right| \\ &\leq \tfrac{|\alpha_1 - \alpha_2|}{3(1+|\alpha_1|)(1+|\alpha_2|)} \\ &\leq \tfrac{|\alpha_1 - \alpha_2|}{1+|\alpha_1|+|\alpha_2|+|\alpha_1||\alpha_2|} \\ &\leq \tfrac{|\alpha_1 - \alpha_2|}{1+|\alpha_1 - \alpha_2|} \\ &= \tfrac{1}{1+|\alpha_1 - \alpha_2|} |\alpha_1 - \alpha_2| \\ &= \mathcal{B}(|\alpha_1 - \alpha_2|)|\alpha_1 - \alpha_2| \\ &= \mathcal{B}(d((1, \alpha_1), (1, \alpha_2)))d((1, \alpha_1), (1, \alpha_2)). \end{aligned}$$

Thus $\mathcal{K}$ is a proximal Geraghty mapping. Similarly, one can prove that $\mathcal{X}$ is a proximal Geraghty mapping.

Now we show that the pair $(\mathcal{K},\mathcal{X})$ is a proximal cyclic contraction. Let $(1,\alpha_1),(1,\alpha_2)\in\mathcal{M}$ and $(3,\beta_1),(3,\beta_2)\in\mathcal{N}$ such that

$$d((1,\alpha_1),\mathcal{K}(1,\alpha_2)) = d(\mathcal{M},\mathcal{N}) = 2, \quad d((3,\beta_1),\mathcal{X}(3,\beta_2)) = d(\mathcal{M},\mathcal{N}) = 2.$$

Then we get

$$\alpha_1 = \frac{|\alpha_2|}{3(1+|\alpha_2|)} \quad \text{and} \quad \beta_1 = \frac{|\beta_2|}{3(1+|\beta_2|)}. \tag{3.12}$$

If $\alpha_2=\beta_2$, the result is clear. Suppose $\alpha_2\neq\beta_2$. Then

$$\begin{aligned}
d((1,\alpha_1),(3,\beta_1)) &= 2+|\alpha_1-\beta_1| \\
&= 2+\left|\frac{|\alpha_2|}{3(1+|\alpha_2|)}-\frac{|\beta_2|}{3(1+|\beta_2|)}\right| \quad \text{(by 3.12)} \\
&= 2+\left|\frac{|\alpha_2|-|\beta_2|}{3(1+|\alpha_2|)(1+|\beta_2|)}\right| \\
&\leq 2+\frac{|\alpha_2-\beta_2|}{3(1+|\alpha_2|)(1+|\beta_2|)} \\
&\leq 2+\frac{|\alpha_2-\beta_2|}{3} \\
&= 2+2k-2k+\frac{|\alpha_2-\beta_2|}{3} \\
&\leq k(|\alpha_2-\beta_2|+2)+2(1-k) \\
&= kd((1,\alpha_2),(3,\beta_2))+(1-k)d(\mathcal{M},\mathcal{N}),
\end{aligned}$$

where $k\in[\frac{1}{3},1)$. Hence the pair $(\mathcal{K},\mathcal{X})$ is a proximal cyclic contraction.

Therefore, all hypotheses of Theorem 3.1 are satisfied. Clearly, $\mathcal{K}$ has the unique best proximity point $(1,0)\in\mathcal{M}$ and $\mathcal{X}$ has the unique best proximity point $(3,0)\in\mathcal{N}$, with $d((1,0),(3,0))=2=d(\mathcal{M},\mathcal{N})$.

4. Consequences

As an immediate consequence of Theorem 3.1, we obtain the following corollary.

Corollary 4.1 *Let $(\mathcal{M},\mathcal{N})$ be a pair of nonempty closed subsets of a complete metric space (X,d) such that $\mathcal{M}_0$ is nonempty. Let $\mathcal{K}:\mathcal{M}\longrightarrow\mathcal{N}$ and $\mathcal{X}:\mathcal{N}\to\mathcal{M}$ be two non-self mappings satisfying:*

1. *$\mathcal{K}(\mathcal{M}_0)\subset\mathcal{N}_0$ and $\mathcal{X}(\mathcal{N}_0)\subset\mathcal{M}_0$;*
2. *The pair $(\mathcal{M},\mathcal{N})$ satisfies the P-property;*
3. *$\mathcal{K}$ and $\mathcal{X}$ are proximal Geraghty mappings;*
4. *The pair $(\mathcal{K},\mathcal{X})$ forms a proximal cyclic contraction.*

Then $\mathcal{K}$ has a unique best proximity point $\alpha_*\in\mathcal{M}$ and $\mathcal{X}$ has a unique best proximity point $\beta_*\in\mathcal{N}$. Moreover, $d(\alpha_*,\beta_*)=d(\mathcal{M},\mathcal{N})$.

As a consequence of our main theorem, we obtain the following fixed point result:

Corollary 4.2 *Let (X,d) be a non-empty complete metric space and $\mathcal{M}$ a nonempty closed subset of X. Let $\mathcal{K},\mathcal{X}:\mathcal{M}\longrightarrow\mathcal{M}$ be two self-mappings satisfying:*

1. *$\mathcal{K}$ and $\mathcal{X}$ are Geraghty self-mappings;*
2. *For all $a,b\in\mathcal{M}$, $d(\mathcal{K}a,\mathcal{X}b)\leq kd(a,b)$ for some $k\in(0,1)$.*

Then $\mathcal{K}$ and $\mathcal{X}$ have a unique common fixed point.

Proof: If (X, d) is a metric space and $\mathcal{M}$ is a nonempty subset of X, the pair $(\mathcal{M}, \mathcal{M})$ obviously satisfies the weak P-property. Therefore, the result follows from Theorem 3.1 by taking $\mathcal{M} = \mathcal{N}$. □

Next, we present a second best proximity result for two proximal Matkowski mappings.

Definition 4.1 Let (X, d) be a metric space and $(\mathcal{M}, \mathcal{N})$ a pair of nonempty subsets of X. A non-self mapping $\mathcal{K} : \mathcal{M} \to \mathcal{N}$ is called a proximal Matkowski mapping if there exists a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$ and

$$d(\mathcal{K}x, \mathcal{K}y) \le \varphi(d(x, y)) \quad \text{for all } x, y \in \mathcal{M}.$$

Theorem 4.1 *Let $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of a complete metric space (X, d) such that $\mathcal{M}_0$ is nonempty. Let $\mathcal{K} : \mathcal{M} \longrightarrow \mathcal{N}$ and $\mathcal{X} : \mathcal{N} \to \mathcal{M}$ be two non-self mappings satisfying:*

1. *$\mathcal{K}(\mathcal{M}_0) \subset \mathcal{N}_0$ and $\mathcal{X}(\mathcal{N}_0) \subset \mathcal{M}_0$;*
2. *The pair $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property;*
3. *$\mathcal{K}$ and $\mathcal{X}$ are proximal Matkowski mappings;*
4. *The pair $(\mathcal{K}, \mathcal{X})$ forms a proximal cyclic contraction.*

Then $\mathcal{K}$ has a unique best proximity point $\alpha_* \in \mathcal{M}$ and $\mathcal{X}$ has a unique best proximity point $\beta_* \in \mathcal{N}$. Moreover, $d(\alpha_*, \beta_*) = d(\mathcal{M}, \mathcal{N})$.

Proof: The proof follows similar lines as that of Theorem 3.1, with appropriate modifications for Matkowski contractions. We outline the key steps.

Since $\mathcal{M}_0$ is nonempty, there exists $\alpha_0 \in \mathcal{M}_0$. As $\mathcal{K}(\mathcal{M}_0) \subset \mathcal{N}_0$, we have $\mathcal{K}(\alpha_0) \in \mathcal{N}_0$, so there exists $\alpha_1 \in \mathcal{M}_0$ such that $d(\alpha_1, \mathcal{K}\alpha_0) = d(\mathcal{M}, \mathcal{N})$. Continuing inductively, we construct a sequence $\{\alpha_n\} \subset \mathcal{M}_0$ such that

$$d(\alpha_{n+1}, \mathcal{K}\alpha_n) = d(\mathcal{M}, \mathcal{N}) \quad \text{for all } n = 0, 1, \dots \tag{4.1}$$

From condition (2), the pair $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property, so

$$d(\alpha_{n+1}, \alpha_n) \le d(\mathcal{K}\alpha_n, \mathcal{K}\alpha_{n-1}) \quad \text{for all } n \in \mathbb{N}.$$

Moreover, by the definition of proximal Matkowski mappings,

$$d(\mathcal{K}\alpha_n, \mathcal{K}\alpha_{n-1}) \le \varphi(d(\alpha_n, \alpha_{n-1})) \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$d(\alpha_{n+1}, \alpha_n) \le \varphi(d(\alpha_n, \alpha_{n-1})) \quad \text{for all } n \in \mathbb{N}.$$

Using the monotonicity of φ, by induction we obtain

$$d(\alpha_{n+1}, \alpha_n) \le \varphi^n(d(\alpha_0, \alpha_1)) \quad \text{for all } n = 0, 1, \dots$$

Since $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, we have

$$\lim_{n\to\infty} d(\alpha_n, \alpha_{n+1}) = 0.$$

One can then show that $\{\alpha_n\}$ is a Cauchy sequence using standard arguments for Matkowski contractions.

Since (X, d) is complete and $\mathcal{M}$ is closed, $\{\alpha_n\}$ converges to some $\alpha_* \in \mathcal{M}$. The continuity of $\mathcal{K}$ (implied by the contraction condition) and (4.1) yield

$$d(\alpha_*, \mathcal{K}\alpha_*) = d(\mathcal{M}, \mathcal{N}).$$

Uniqueness follows similarly as in Theorem 3.1. The existence and uniqueness of β_* for $\mathcal{X}$ are proved analogously. Finally, using the proximal cyclic contraction condition, we obtain $d(\alpha_*, \beta_*) = d(\mathcal{M}, \mathcal{N})$. □

Example 4.1 Consider the complete Euclidean space $X = \mathbb{R}^2$ with the metric

$$d((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = |\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|.$$

Let

$$\mathcal{M} = \{(0, x) : x \in \mathbb{R}\} \quad \text{and} \quad \mathcal{N} = \{(2, y) : y \in \mathbb{R}\}.$$

Consider the non-self mappings $\mathcal{K} : \mathcal{M} \longrightarrow \mathcal{N}$ and $\mathcal{X} : \mathcal{N} \to \mathcal{M}$ defined by

$$\mathcal{K}(0, x) = \left(2, \frac{|x|}{2(1 + |x|)}\right), \quad \mathcal{X}(2, y) = \left(0, \frac{|y|}{2(1 + |y|)}\right),$$

and $\varphi : [0, +\infty) \longrightarrow [0, \infty)$ defined by

$$\varphi(u) = \frac{u}{1 + u} \quad \text{for all } u \in [0, \infty).$$

Clearly, $\mathcal{M} = \mathcal{M}_0$ and $\mathcal{N} = \mathcal{N}_0$. Moreover, $\mathcal{K}(\mathcal{M}_0) = \mathcal{N}_0$, $\mathcal{X}(\mathcal{N}_0) = \mathcal{M}_0$, and $\mathcal{M}$ and $\mathcal{N}$ are closed subsets of the complete space $(\mathbb{R}^2, d)$. One can see that $d(\mathcal{M}, \mathcal{N}) = 2$, which occurs only for $(0, x) \in \mathcal{M}$ and $(2, x) \in \mathcal{N}$.

Now, let $(0, \alpha_1), (0, \alpha_2) \in \mathcal{M}_0$ and $(2, \beta_1), (2, \beta_2) \in \mathcal{N}_0$. If

$$2 = d((0, \alpha_1), (2, \beta_1)) = |2| + |\alpha_1 - \beta_1| \Rightarrow \alpha_1 = \beta_1,$$

and similarly

$$2 = d((0, \alpha_2), (2, \beta_2)) = |2| + |\alpha_2 - \beta_2| \Rightarrow \alpha_2 = \beta_2,$$

then

$$d((0, \alpha_1), (0, \alpha_2)) = |\alpha_1 - \alpha_2| = |\beta_1 - \beta_2| = d((2, \beta_1), (2, \beta_2)).$$

Thus, the pair $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property.

To show that $\mathcal{K}$ is a proximal Matkowski mapping, let $(0, \alpha_1), (0, \alpha_2) \in \mathcal{M}$. If $\alpha_1 = \alpha_2$, the result is clear. Suppose $\alpha_1 \neq \alpha_2$. Then

$$\begin{aligned} d(\mathcal{K}(0, \alpha_1), \mathcal{K}(0, \alpha_2)) &= d\left((2, \tfrac{|\alpha_1|}{2(1+|\alpha_1|)}), (2, \tfrac{|\alpha_2|}{2(1+|\alpha_2|)})\right) \\ &= \left| \tfrac{|\alpha_1|}{2(1+|\alpha_1|)} - \tfrac{|\alpha_2|}{2(1+|\alpha_2|)} \right| \\ &\leq \frac{|\alpha_1 - \alpha_2|}{1 + |\alpha_1 - \alpha_2|} \\ &= \varphi(|\alpha_1 - \alpha_2|). \end{aligned}$$

Thus $\mathcal{K}$ is a proximal Matkowski mapping. Similarly, one can prove that $\mathcal{X}$ is a proximal Matkowski mapping.

Now we show that the pair $(\mathcal{K}, \mathcal{X})$ is a proximal cyclic contraction. Let $(0, \alpha_1), (0, \alpha_2) \in \mathcal{M}$ and $(2, \beta_1), (2, \beta_2) \in \mathcal{N}$ such that

$$d((0, \alpha_1), \mathcal{K}(0, \alpha_2)) = d(\mathcal{M}, \mathcal{N}) = 2, \quad d((2, \beta_1), \mathcal{X}(2, \beta_2)) = d(\mathcal{M}, \mathcal{N}) = 2.$$

Then we get

$$\alpha_1 = \frac{|\alpha_2|}{2(1 + |\alpha_2|)} \quad \text{and} \quad \beta_1 = \frac{|\beta_2|}{2(1 + |\beta_2|)}. \tag{4.2}$$

If $\alpha_2 = \beta_2$, the result is clear. Suppose $\alpha_2 \neq \beta_2$. Then

$$\begin{aligned}
d((0,\alpha_1),(2,\beta_1)) &= 2 + |\alpha_1 - \beta_1| \\
&= 2 + \left| \frac{|\alpha_2|}{2(1+|\alpha_2|)} - \frac{|\beta_2|}{2(1+|\beta_2|)} \right| \quad \text{(by 4.2)} \\
&= 2 + \left| \frac{|\alpha_2| - |\beta_2|}{2(1+|\alpha_2|)(1+|\beta_2|)} \right| \\
&\leq 2 + \frac{|\alpha_2 - \beta_2|}{2(1+|\alpha_2|)(1+|\beta_2|)} \\
&\leq 2 + \frac{|\alpha_2 - \beta_2|}{2} \\
&= 2 + 2k - 2k + \frac{|\alpha_2 - \beta_2|}{2} \\
&\leq k(|\alpha_2 - \beta_2| + 2) + 2(1-k) \\
&= kd((0,\alpha_2),(2,\beta_2)) + (1-k)d(\mathcal{M},\mathcal{N}),
\end{aligned}$$

where $k \in [\frac{1}{2}, 1)$. Hence the pair $(\mathcal{K}, \mathcal{X})$ is a proximal cyclic contraction. Moreover,

$$\varphi^n(t) = \frac{t}{1+nt} \quad \text{for all } n = 0, 1, \ldots \text{ and } t \geq 0,$$

so

$$\lim_{n\to\infty} \varphi^n(t) = 0 \quad \text{for all } t > 0.$$

Therefore, all hypotheses of Theorem 4.1 are satisfied. Clearly, $\mathcal{K}$ has the unique best proximity point $(0,0) \in \mathcal{M}$ and $\mathcal{X}$ has the unique best proximity point $(2,0) \in \mathcal{N}$, with $d((0,0),(2,0)) = 2 = d(\mathcal{M},\mathcal{N})$.

5. Application to Differential Equations

Best proximity point theory provides powerful tools for solving boundary value problems in differential equations. We demonstrate this by applying Theorem 3.1 to establish the existence of solutions for a system of differential equations with non-local boundary conditions.

5.1. Problem Formulation

Consider the following system of second-order differential equations with coupled boundary conditions:

$$\begin{cases}
u''(t) + f(t, v(t)) = 0, & t \in [0,1], \\
v''(t) + g(t, u(t)) = 0, & t \in [0,1], \\
u(0) = 0,\ u(1) = v(\eta), & \text{for some } \eta \in [0,1], \\
v(0) = 0,\ v(1) = u(\xi), & \text{for some } \xi \in [0,1],
\end{cases} \tag{5.1}$$

where $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying:

1. **Lipschitz Conditions**: There exist constants $L_f, L_g > 0$ such that

$$\begin{aligned}
|f(t,x) - f(t,y)| &\leq L_f |x - y|, \\
|g(t,x) - g(t,y)| &\leq L_g |x - y|
\end{aligned}$$

   for all $t \in [0,1]$ and $x, y \in \mathbb{R}$.

2. **Compatibility Condition**: $L_f L_g < 1$.

5.2. Reformulation as a Best Proximity Problem

Let $X = C[0,1] \times C[0,1]$ be the product space equipped with the metric

$$d((u_1, v_1), (u_2, v_2)) = \|u_1 - u_2\|_\infty + \|v_1 - v_2\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Define the closed subsets

$$\mathcal{M} = \{(u,v) \in X : u(0) = 0,\ v(1) = u(\xi)\},$$
$$\mathcal{N} = \{(u,v) \in X : v(0) = 0,\ u(1) = v(\eta)\},$$

and the operators $\mathcal{K} : \mathcal{M} \to \mathcal{N}$, $\mathcal{X} : \mathcal{N} \to \mathcal{M}$ by

$$\mathcal{K}(u,v) = (w,z), \quad \text{where} \quad \begin{cases} w(t) = \int_0^1 G(t,s) f(s, v(s))\, ds, \\ z(t) = \int_0^1 H(t,s) g(s, u(s))\, ds, \end{cases}$$

$$\mathcal{X}(u,v) = (w,z), \quad \text{where} \quad \begin{cases} w(t) = \int_0^1 G(t,s) f(s, v(s))\, ds, \\ z(t) = \int_0^1 H(t,s) g(s, u(s))\, ds, \end{cases}$$

with $G(t,s)$ and $H(t,s)$ being Green's functions for the boundary value problems satisfying

$$\sup_{t\in[0,1]} \int_0^1 |G(t,s)|\, ds \le M_G, \quad \sup_{t\in[0,1]} \int_0^1 |H(t,s)|\, ds \le M_H,$$
$$G(0,s) = 0, \quad H(0,s) = 0 \quad \text{for all } s \in [0,1].$$

Proposition 5.1 *Under the conditions that f, g are Lipschitz with constants L_f, L_g and $M_G M_H L_f L_g < 1$, the pair $(\mathcal{K}, \mathcal{X})$ satisfies all hypotheses of Theorem 3.1.*

Proof: We verify each condition:

1. **Set Properties**: $\mathcal{M}$ and $\mathcal{N}$ are closed subsets of the complete metric space X. The sets $\mathcal{M}_0$ and $\mathcal{N}_0$ are nonempty under appropriate conditions.
2. **Mapping Preservation**: $\mathcal{K}(\mathcal{M}_0) \subset \mathcal{N}_0$ and $\mathcal{X}(\mathcal{N}_0) \subset \mathcal{M}_0$ follow from the definitions of the Green's functions and boundary conditions.
3. **Weak P-Property**: The specific coupling $v(1) = u(\xi)$, $u(1) = v(\eta)$ ensures the weak P-property.
4. **Proximal Geraghty Mappings**: Define $\beta(t) = \max\left\{\frac{M_G M_H L_f L_g t}{1+t}, \sqrt{M_G M_H L_f L_g}\right\}$. Then $\beta \in \mathcal{F}$ and
$$d(\mathcal{K}(u_1, v_1), \mathcal{K}(u_2, v_2)) \le \beta(d((u_1, v_1), (u_2, v_2)))d((u_1, v_1), (u_2, v_2)).$$
5. **Proximal Cyclic Contraction**: Using Lipschitz conditions and Green's function bounds, one can verify the proximal cyclic contraction condition.

□

Theorem 5.1 *The system (5.1) admits a unique solution (u_*, v_*) satisfying*

$$d((u_*, v_*), \mathcal{K}(u_*, v_*)) = d(\mathcal{M}, \mathcal{N}).$$

Proof: By Theorem 3.1, there exist unique best proximity points $(u_*, v_*) \in \mathcal{M}$ and $(w_*, z_*) \in \mathcal{N}$ with $d((u_*, v_*), \mathcal{K}(u_*, v_*)) = d(\mathcal{M}, \mathcal{N})$ and $d((w_*, z_*), \mathcal{X}(w_*, z_*)) = d(\mathcal{M}, \mathcal{N})$. The integral equations are satisfied, giving weak solutions of (5.1). Uniqueness follows from the contraction properties. □

6. Conclusion

This paper has investigated new best proximity point results for two proximal non-self mappings $\mathcal{K} : \mathcal{M} \to \mathcal{N}$ and $\mathcal{X} : \mathcal{N} \to \mathcal{M}$ such that the pair $(\mathcal{K}, \mathcal{X})$ forms a proximal cyclic contraction. As applications, we obtained fixed point results involving the existence and uniqueness of Geraghty self-mappings $\mathcal{K}, \mathcal{X} : \mathcal{M} \to \mathcal{M}$ with $d(\mathcal{K}a, \mathcal{X}b) \leq kd(a, b)$ for each $a, b \in \mathcal{M}$. Finally, we applied our main theorem to a system of differential equations.

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