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Polynomial Stability of Timoshenko System with Singular Local Fractional Damping in the Transverse Displacement

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ABSTRACT: In this paper, we investigate the stabilization of Timoshenko system with singular local fractional damping in the transverse displacement. Using frequency domain approach combined with the multiplier method, we prove that the energy of our system decays polynomially with different rates.

Keywords: Timoshenko system, fractional damping, polynomial stability.

Contents

1. Introduction

In this work, we investigate the indirect stability of a one-dimensional Timoshenko system with local internal fractional damping in the transverse displacement, by considering:

$$\begin{aligned} &\rho_1 u_{tt} - \kappa_1 (u_x + y)_x + a(x)\partial_t^{\alpha,\eta} u = 0, \quad (x,t) \in (0,L) \times (0,\infty), \\ &\rho_2 y_{tt} - \kappa_2 y_{xx} + \kappa_1 (u_x + y) = 0, \quad (x,t) \in (0,L) \times (0,\infty), \end{aligned} \tag{1.1}$$

with following initial conditions

$$\begin{aligned} &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,L), \\ &y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in (0,L). \end{aligned} \tag{1.2}$$

and the following boundary conditions

$$u(0,t) = y(0,t) = u(L,t) = y(L,t) = 0, \quad t \in (0,\infty). \tag{1.3}$$

The coefficients ρ_1, ρ_2, κ_1 and κ_2 are positive real numbers, $\alpha \in (0,1)$, and $\eta \geq 0$. We suppose that there exists $0 < \beta < \gamma < L$ and a positive constant a_0, such that

$$a(x) = \begin{cases} a_0, & x \in (\beta,\gamma), \\ 0, & x \in (0,\beta) \cup (\gamma, L). \end{cases} \tag{1.4}$$

Fractional calculus includes various extensions of the usual definition of derivative from integer to real order, including Riemann-Liouville derivative, the Caputo derivative, the Riesz derivative, the Weyl derivative, cf. [7]. In this paper, we consider the Caputo's fractional derivative $\partial_t^{\alpha,\eta}$ of order $\alpha \in (0,1)$ with respect to time variable t defined by

$$[D^{\alpha,\eta}\omega](t) = \partial_t^{\alpha,\eta}\omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(s)ds. \tag{1.5}$$

The fractional derivatives are non local and involve singular and non-integrable kernels ($t^{-\alpha}$, $0 < \alpha < 1$). The Timoshenko system is usually considered in describing the transverse vibration of a beam and

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ignoring damping effects of any nature. Recently, the author of [1] studied the indirect stability of a one-dimensional Timoshenko system with local internal fractional damping in the rotation angle and established the following polynomial energy decay rates:

$$\begin{cases} t^{-\frac{2}{1-\alpha}} & \text{if } \dfrac{\kappa_1}{\rho_1} = \dfrac{\kappa_2}{\rho_2}, \\ t^{-\frac{2}{3-\alpha}} & \text{if } \dfrac{\kappa_1}{\rho_1} \neq \dfrac{\kappa_2}{\rho_2}. \end{cases}$$

The originality of the current paper lies in considering the case where the damping is applied to the transverse displacement, and we prove different decay rates compared to those in [1]. This paper is organized as follows: In Section 2, we prove the well-posedness of our system by using semigroup approach. In section 3, by using frequency domain approach combined with a specific multiplier method, we prove a polynomial energy decay rate of orders

$$\begin{cases} t^{-\frac{2}{1-\alpha}} & \text{if } \dfrac{\kappa_1}{\rho_1} = \dfrac{\kappa_2}{\rho_2}, \\ t^{-\frac{2}{5-\alpha}} & \text{if } \dfrac{\kappa_1}{\rho_1} \neq \dfrac{\kappa_2}{\rho_2}. \end{cases}$$

2. Augmented model and well-posedness

In this part, using a semigroup approach, we will establish the well-posedness result for system (1.1)-(1.3). First, we need to reformulate our system. For tis aim, we recall the following theorem.

Theorem 2.1 *(See Theorem 2 in [5]). Let* $\alpha \in (0,1)$*, then the relation between the 'input'* V *and the 'output'* O *of the following system*

$$\begin{array}{lcl} \partial_t \omega(x,\xi,t) + (\xi^2+\eta)\omega(x,\xi,t) - V(x,t)|\xi|^{\frac{2\alpha-1}{2}} &=& 0, \quad (x,\xi,t) \in (0,L)\times\mathbb{R}\times(0,\infty), \\ \omega(\xi,0) &=& 0, \quad (x,\xi) \in (0,L)\times\mathbb{R}, \\ O(x,t) - \kappa(\alpha)\displaystyle\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}}\omega(x,\xi,t)d\xi &=& 0, \quad (x,t) \in (0,L)\times(0,\infty), \end{array}$$

is given by

$$O = I^{1-\alpha,\eta}V,$$

where

$$\kappa(\alpha) = \frac{\sin(\alpha\pi)}{\pi} \quad \text{and} \quad [I^{\alpha,\eta}V](x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\eta(t-s)}V(s)ds.$$

In the above theorem, taking the input $V(x,t) = \sqrt{a(x)}u_t(x,t)$, then using (1.5), we get the output O is given by

$$O(t) = \sqrt{a(x)}I^{1-\alpha,\eta}u_t(x,t) = \frac{\sqrt{a(x)}}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\partial_s u(x,s)ds = \sqrt{a(x)}\partial_t^{\alpha,\eta}u(x,t).$$

Therefore, by taking the input $V(x,t) = \sqrt{a(x)}u_t(x,t)$ in the above theorem and using the above equation, we get

$$\begin{array}{lcl} \partial_t \omega(x,\xi,t) + (\xi^2+\eta)\omega(x,\xi,t) - \sqrt{a(x)}u_t(x,t)|\xi|^{\frac{2\alpha-1}{2}} &=& 0, \quad (x,\xi,t) \in (0,L)\times\mathbb{R}\times(0,\infty), \\ \omega(\xi,0) &=& 0, \quad (x,\xi) \in (0,L)\times\mathbb{R}, \\ \sqrt{a(x)}\partial_t^{\alpha,\eta}u(x,t) - \kappa(\alpha)\displaystyle\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}}\omega(x,\xi,t)d\xi &=& 0, \quad (x,t) \in (0,L)\times(0,\infty), \end{array}$$

From the above system, we deduce that system (1.1)-(1.3) can be rewritten as the following augmented model

$$\begin{cases} \rho_1 u_{tt} - \kappa_1 (u_x+y)_x + \sqrt{a(x)}\kappa(\alpha)\displaystyle\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}}\omega(x,\xi,t)d\xi = 0, & (x,t) \in (0,L)\times\mathbb{R}_*^+, \\ \rho_2 y_{tt} - \kappa_2 y_{xx} + \kappa_1(u_x+y) = 0, & (x,t) \in (0,L)\times\mathbb{R}_*^+, \\ \omega_t(x,\xi,t) + \left(|\xi|^2+\eta\right)\omega(x,\xi,t) - \sqrt{a(x)}u_t(x,t)|\xi|^{\frac{2\alpha-1}{2}} = 0, & (x,\xi,t) \in (0,L)\times\mathbb{R}\times\mathbb{R}_*^+, \end{cases} \tag{2.1}$$

with the boundary conditions

$$u(0,t)=u(L,t)=y(0,t)=y(L,t)=0, \tag{2.2}$$

and the following initial conditions

$$(u(x,0),u_t(x,0),y(x,0),y_t(x,0),\omega(x,\xi,0))=(u_0(x),u_1(x),y_0(x),y_1(x),0),\quad x\in(0,L),\quad \xi\in\mathbb{R}. \tag{2.3}$$

The energy of system (2.1)-(2.3) is given by

$$E(t)=\frac{1}{2}\int_0^L\left(\rho_1|u_t|^2+\rho_2|y_t|^2+\kappa_1|u_x+y|^2+\kappa_2|y_x|^2\right)dx+\frac{\kappa(\alpha)}{2}\int_0^L\int_{\mathbb{R}}|\omega(x,\xi,t)|^2d\xi dx.$$

and

$$E'(t)=-\kappa(\alpha)\int_0^L\int_{\mathbb{R}}(\xi^2+\eta)|\omega(x,\xi,t)|^2d\xi dx.$$

Now, we define the following Hilbert energy space $\mathcal{H}$ by

$$\mathcal{H}=\left(H_0^1(0,L)\times L^2(0,L)\right)^2\times W,$$

where $W=L^2\left((0,L)\times\mathbb{R}\right)$. The energy space $\mathcal{H}$ is equipped with the inner product defined by

$$\begin{aligned}\langle U,U_1\rangle_{\mathcal{H}} \;=\; &\rho_1\int_0^L v\bar{v}_1dx+\kappa_1\int_0^L(u_x+y)\left(\overline{(u_1)_x+y_1}\right)dx+\rho_2\int_0^L z\bar{z}_1dx+\\ &\kappa_2\int_0^L y_x(\bar{y_1})_xdx+\kappa(\alpha)\int_0^L\int_{\mathbb{R}}\omega(x,\xi)\overline{\omega_1}(x,\xi)d\xi dx,\end{aligned} \tag{2.4}$$

for all $U=(u,v,y,z,\omega)$ and $U_1=(u_1,v_1,y_1,z_1,\omega_1)$ in $\mathcal{H}$. We define the unbounded linear operator $\mathcal{A}$ in $\mathcal{H}$ by

$$D(\mathcal{A})=\left\{\begin{array}{l} U=(u,v,y,z,\omega)\in\mathcal{H};\;\; v,z\in H_0^1(0,L),\;\; u,y\in H^2(0,L)\cap H_0^1(0,L),\\ -\left(|\xi|^2+\eta\right)\omega(x,\xi)+\sqrt{a(x)}v(x)|\xi|^{\frac{2\alpha-1}{2}}\in W,\quad\text{and}\quad |\xi|\omega(x,\xi)\in W\end{array}\right\},$$

and

$$\mathcal{A}U^\top=\begin{pmatrix} v\\ \dfrac{\kappa_1}{\rho_1}(u_x+y)_x-\sqrt{a(x)}\dfrac{\kappa(\alpha)}{\rho_1}\displaystyle\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}\omega(x,\xi)d\xi\\ z\\ \dfrac{\kappa_2}{\rho_2}y_{xx}-\dfrac{\kappa_1}{\rho_2}(u_x+y)\\ -\left(|\xi|^2+\eta\right)\omega(x,\xi)+\sqrt{a(x)}v(x)|\xi|^{\frac{2\alpha-1}{2}}\end{pmatrix}$$

If $U=(u,u_t,y,y_t,\omega)$ is a regular solution of system (2.1)-(2.3), then we rewrite this system as the following evolution equation

$$U_t=\mathcal{A}U^\top,\quad U(0):=U_0=(u_0,u_1,y_0,y_1,0)\in\mathcal{H}. \tag{2.5}$$

Using a similar technique to the one mentioned in proposition 2.4 (see [1]), we infer that the unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$. Thus, according to Lumer-Phillips theorem (see [6]) the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $e^{t\mathcal{A}}$. Then, the solution of the evolution equation (2.5) admits the following representation

$$U(t)=e^{t\mathcal{A}}U_0,\qquad t\geq 0,$$

which leads to the well-posedness of (2.5). Hence, we have the following result

Theorem 2.2 *Let $U_0\in\mathcal{H}$, then problem (2.5) admits a unique weak solution U satisfies*

$$U(t)\in C^0\left(\mathbb{R}^+,\mathcal{H}\right).$$

Moreover, if $U_0\in D(\mathcal{A})$, then problem (2.5) admits a unique strong solution U satisfies

$$U(t)\in C^1(\mathbb{R}^+,\mathcal{H})\cap C^0(\mathbb{R}^+,D(\mathcal{A})).$$

3. Polynomial Stability in the case $\eta > 0$

In this section, we show the influence of the wave propagation speed on the stability of system (2.1)-(2.3). Our main result in this part is the following theorem.

Theorem 3.1 *The C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is polynomially stable; i.e. there exists constant $C>0$ such that for every $U_0 \in D(\mathcal{A})$, we have*

$$E(t) \leq \frac{C}{t^{\frac{2}{\ell(\alpha)}}} \|U_0\|^2_{D(\mathcal{A})}, \quad t>0,$$

where

$$\ell(\alpha) = \begin{cases} 1-\alpha & \text{if} \quad \dfrac{\kappa_1}{\rho_1} = \dfrac{\kappa_2}{\rho_2}, \\ 5-\alpha & \text{if} \quad \dfrac{\kappa_1}{\rho_1} \neq \dfrac{\kappa_2}{\rho_2} \end{cases}$$

Following a similar technique to the one mentioned in Section 2.2 (see [1]), we infer $i\mathbb{R} \subset \rho(\mathcal{A})$, according to Borichev and Tomilov in [3] (see also [2], [4]), to proof Theorem 3.1, we still need to prove the following condition

$$\limsup_{\lambda \in \mathbb{R},\ |\lambda| \to \infty} \frac{1}{|\lambda|^{\ell}} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad \text{where } \ell = \ell(\alpha). \tag{H}$$

We will prove condition (H) by a contradiction argument. For this purpose, suppose that (H) is false, then there exists $\left\{ \left(\lambda^n, U^n := (u^n, v^n, y^n, z^n, \omega^n(\cdot,\xi))^\top\right)\right\} \subset \mathbb{R}_+^* \times D(\mathcal{A})$ with

$$\lambda^n \to +\infty \quad \text{and} \quad \|U^n\|_{\mathcal{H}} = \|(u^n, v^n, y^n, z^n, \omega^n(\cdot,\xi))\|_{\mathcal{H}} = 1, \tag{3.1}$$

such that

$$(\lambda^n)^{\ell} \left(i\lambda^n I - \mathcal{A}\right) U^n = F^n := \left(f^{1,n}, f^{2,n}, f^{3,n}, f^{4,n}, f^{5,n}(\cdot,\xi)\right)^\top \to +\infty \ \text{ in } \ \mathcal{H}. \tag{3.2}$$

For simplicity, we drop the index n. Equivalently, from (3.2), we have

$$i\lambda u - v = \lambda^{-\ell} f^1 \to 0 \quad \text{in } H_0^1(0,L), \tag{3.3}$$

$$i\lambda v - \frac{\kappa_1}{\rho_1}(u_x + y)_x + \sqrt{a(x)}\frac{\kappa(\alpha)}{\rho_1}\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x,\xi)d\xi = \lambda^{-\ell} f^2 \to 0 \quad \text{in } L^2(0,L), \tag{3.4}$$

$$i\lambda y - z = \lambda^{-\ell} f^3 \to 0 \quad \text{in } H_0^1(0,L), \tag{3.5}$$

$$i\lambda z - \frac{\kappa_2}{\rho_2} y_{xx} + \frac{\kappa_1}{\rho_2}(u_x + y) = \lambda^{-\ell} f_4 \to 0 \quad \text{in } L^2(0,L), \tag{3.6}$$

$$(i\lambda + \xi^2 + \eta)\omega(\xi) - \sqrt{a(x)} v(x) |\xi|^{\frac{2\alpha-1}{2}} = \lambda^{-\ell} f_5(x,\xi) \to 0 \quad \text{in } W. \tag{3.7}$$

Here we will check the condition (H) by finding a contradiction with (3.2) by showing $\|U\|_{\mathcal{H}} = o(1)$. For clarity, we divide the proof into several Lemmas.

Lemma 3.1 *Assume that $\eta > 0$ and $\ell \geq 1-\alpha$. Then, the solution $(u,v,y,z,\omega) \in D(\mathcal{A})$ of system (3.3)-(3.7) satisfies the following asymptotic behavior estimations*

$$\int_0^L \int_{\mathbb{R}} \left(|\xi|^2 + \eta\right) |\omega(x,\xi)|^2 d\xi dx = o\left(\lambda^{-\ell}\right) \quad \text{and} \quad \int_\beta^\gamma |v|^2 dx = o\left(\lambda^{-\ell-\alpha+1}\right). \tag{3.8}$$

Proof: Using a similar technique to the one mentioned in Lemma 2.8 (see [1]), we obtain the desired result. □

Remark 3.1 Using the second estimation in equation (3.8) and equation (3.3), we obtain

$$\int_\beta^\gamma |u|^2 dx = o\left(\lambda^{-\ell-\alpha-1}\right). \tag{3.9}$$

Lemma 3.2 *Let $0<\alpha<1$, $\eta>0$ and $0<\varepsilon<\frac{\beta-\gamma}{6}$. Then, the solution $(u,v,y,z,\omega)\in D(\mathcal{A})$ of system (3.3)-(3.7) satisfies the following asymptotic behavior*

$$\int_{\beta+\varepsilon}^{\gamma-\varepsilon}|u_x|^2dx=o\left(\lambda^{-\frac{\ell+\alpha-1}{2}}\right). \tag{3.10}$$

Proof: First, we fix a cutt-off function $h_1\in C^1([0,L])$ such that $0\leq h_1(x)\leq 1$, for all $x\in[0,L]$ and

$$h_1(x)=\begin{cases}1 & \text{if} \quad x\in[\beta+\varepsilon,\gamma-\varepsilon],\\ 0 & \text{if} \quad x\in[0,\beta]\cup[\gamma,L].\end{cases} \tag{3.11}$$

Multiplying equation (3.4) by $\frac{\rho_1}{\kappa_1}h_1\overline{u}$ integrating over $(0,L)$, taking the real part and using the fact that λu is uniformly bounded in $L^2(0,L)$ and $f_2\to 0$ in $L^2(0,L)$, we get

$$\begin{aligned}&\Re\left(i\lambda\frac{\rho_1}{\kappa_1}\int_0^L h_1 v\overline{u}dx\right)+\Re\left(\int_0^L h_1'(u_x+y)\overline{u}dx\right)+\int_0^L h_1|u_x|^2dx+\Re\left(\int_0^L h_1 y\overline{u_x}dx\right)\\&+\kappa_1\kappa(\alpha)\Re\left(\int_0^L\sqrt{a(x)}\overline{u}\left(\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}\omega(x,\xi)d\xi\right)dx\right)=\frac{o(1)}{\lambda^{\frac{\ell}{2}+1}}.\end{aligned} \tag{3.12}$$

Using Cauchy-Schwarz's inequality, the definition of the function $a(x)$, equation (3.8) and the fact that u is uniformly bounded in $L^2(0,L)$, we get

$$\left|\Re\left(\int_0^L h_1\sqrt{a(x)}\overline{u}\left(\int_{\mathbb{R}}|\xi|^{\alpha-\frac{1}{2}}\omega(x,\xi)d\xi\right)dx\right)\right|\leq C\left(\int_\beta^\gamma|u|^2dx\right)^{\frac{1}{2}}\left(\int_0^L\int_{\mathbb{R}}\left(|\xi|^2+\eta\right)|\omega(x,\xi)|^2d\xi dx\right)^{\frac{1}{2}} \tag{3.13}$$

where $C=\sqrt{a(x)}\,\mathtt{S}(\alpha,\eta)$ and $\mathtt{S}(\alpha,\eta)=\left(\int_{\mathbb{R}}\frac{|\xi|^{2\alpha-1}}{|\xi|^2+\eta}d\xi\right)^{\frac{1}{2}}$. We have

$$\frac{|\xi|^{2\alpha-1}}{|\xi|^2+\eta}\underset{0}{\sim}\frac{|\xi|^{2\alpha-1}}{\eta}\quad\text{and}\quad\frac{|\xi|^{2\alpha-1}}{|\xi|^2+\eta}\underset{\infty}{\sim}\frac{1}{|\xi|^{3-2\alpha}}.$$

Since $0<\alpha<1$ and $\eta>0$, then $\mathtt{S}(\alpha,\eta)$ is well defined. Consequently, we deduce that

$$\left|\Re\left(\int_0^L h_1\sqrt{a(x)}\overline{u}\left(\int_{\mathbb{R}}|\xi|^{\alpha-\frac{1}{2}}\omega(x,\xi)d\xi\right)dx\right)\right|=\frac{o(1)}{\lambda^{\ell+\frac{\alpha+1}{2}}}. \tag{3.14}$$

Now, using the fact that (u_x+y) is uniformly bounded in $L^2(0,L)$, Cauchy-Schwarz's inequality, the definition of the function h_1 and equation (3.9), we get

$$\Re\left(\int_0^L h_1'(u_x+y)\overline{u}dx\right)=\frac{o(1)}{\lambda^{\frac{\ell+\alpha+1}{2}}}. \tag{3.15}$$

Using the fact that λu is uniformly bounded in $L^2(0,L)$, the definition of the function h_1 and equation (3.8), we get

$$\Re\left(i\lambda\frac{\rho_1}{\kappa_1}\int_0^L h_1 v\overline{u}dx\right)=\frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{2}}}. \tag{3.16}$$

- Estimation of $\Re\left(\int_0^L h_1 y\overline{u_x}dx\right)$. Using by parts integration, we obtain

$$\Re\left(\int_0^L h_1 y\overline{u_x}dx\right)=-\Re\left(\int_0^L h_1' y\overline{u}dx\right)-\Re\left(\int_0^L h_1 y_x\overline{u}dx\right). \tag{3.17}$$

Using the fact that λy is uniformly bounded in $L^2(0,L)$, y_x is uniformly bounded in $L^2(0,L)$, equation (3.9) and Cauchy-Schwarz inequality , we get

$$\Re\left(\int_0^L h_1' y\overline{u}dx\right)=\frac{o(1)}{\lambda^{\frac{\ell+\alpha+3}{2}}}\quad\text{and}\quad\Re\left(\int_0^L h_1 y_x\overline{u}dx\right)=\frac{o(1)}{\lambda^{\frac{\ell+\alpha+1}{2}}}.$$

Inserting the above equation in (3.17), we get

$$\Re\left(\int_0^L h_1 y\overline{u_x}dx\right)=\frac{o(1)}{\lambda^{\frac{\ell+\alpha+1}{2}}}. \tag{3.18}$$

Inserting (3.14), (3.15), (3.16) and (3.18) in (3.12) and the definition of the function h_1, we get the desired result. The proof has been completed. □

Lemma 3.3 *Let $0<\alpha<1$, $\eta>0$ and $0<\varepsilon<\frac{\beta-\gamma}{6}$. Then, the solution $(u,v,y,z,\omega)\in D(\mathcal{A})$ of system of system (3.3)-(3.7) satisfies the following asymptotic behavior*

$$\int_{\beta+2\varepsilon}^{\gamma-2\varepsilon}|y_x|^2dx\le\left|\frac{\rho_1}{\kappa_1}-\frac{\rho_2}{\kappa_2}\right|\frac{o(1)}{\lambda^{\frac{\ell+\alpha-5}{2}}}+\frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{2}}}. \tag{3.19}$$

Proof: First, we fix a cutt-off function $h_2\in C^1([0,L])$ such that $0\le h_2(x)\le 1$, for all $x\in[0,L]$ and

$$h_2(x)=\begin{cases}1 & \text{if} \quad x\in[\beta+2\varepsilon,\gamma-2\varepsilon],\\ 0 & \text{if} \quad x\in[0,\beta+\varepsilon]\cup[\gamma-\varepsilon,L].\end{cases} \tag{3.20}$$

Multiplying (3.6) by $-h_2\frac{\rho_2}{\kappa_2}\overline{u_x}$, integrating over $(0,L)$ and using the fact that u_x is uniformly bounded in $L^2(0,L)$ and $f_4\to 0$ in $L^2(0,L)$, we get

$$-\Re\left(i\lambda\frac{\rho_2}{\kappa_2}\int_0^L h_2 z\overline{u_x}dx\right)+\Re\left(\int_0^L h_2 y_{xx}\overline{u_x}dx\right)-\Re\left(\kappa_1\kappa_2\int_0^L h_2\left(u_x+y\right)\overline{u_x}\right)dx=o\left(\lambda^{-\ell}\right).$$

Using the fact that (u_x+y) is uniformly bounded, (3.10) and the definition of h_2 in the above equation , we get

$$-\Re\left(i\lambda\frac{\rho_2}{\kappa_2}\int_0^L h_2 z\overline{u_x}dx\right)+\Re\left(\int_0^L h_2 y_{xx}\overline{u_x}dx\right)=\frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{2}}}. \tag{3.21}$$

From (3.5), we obtain

$$-i\lambda z=\lambda^2 y+i\lambda^{1-\ell}f_3.$$

Inserting the above equation in (3.21), we get

$$\Re\left(\lambda^2\frac{\rho_2}{\kappa_2}\int_0^L h_2 y\overline{u_x}dx\right)+\Re\left(i\lambda\frac{\rho_2}{\kappa_2}\int_0^L h_2\lambda^{-\ell}f_3\overline{u_x}dx\right)+\Re\left(\int_0^L h_2 y_{xx}\overline{u_x}dx\right)=\frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{2}}}. \tag{3.22}$$

Using integration by parts on the second integral in (3.22), we get

$$\Re\left(i\lambda\frac{\rho_2}{\kappa_2}\int_0^L h_2\lambda^{-\ell}f_3\overline{u_x}dx\right)=-\Re\left(i\lambda\frac{\rho_2}{\kappa_2}\int_0^L h_2'\lambda^{-\ell}f_3\overline{u}dx\right)-\Re\left(i\lambda\frac{\rho_2}{\kappa_2}\int_0^L h_2\lambda^{-\ell}\overline{(f_3)_x}\overline{u}dx\right) \tag{3.23}$$

Using the fact that λu is uniformly bounded in $L^2(0,L)$, $f_3\to 0$ in $H_0^1(0,L)$ and the definition of h_2, we get

$$\Re\left(i\lambda\frac{\rho_2}{\kappa_2}\int_0^L h_2\lambda^{-\ell}f_3\overline{u_x}dx\right)=\frac{o(1)}{\lambda^{\ell}}.$$

Inserting the above equation in (3.21), we get

$$\Re\left(\lambda^2\frac{\rho_2}{\kappa_2}\int_0^L h_2 y\overline{u_x}dx\right)+\Re\left(\int_0^L h_2 y_{xx}\overline{u_x}dx\right)=\frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{2}}}. \tag{3.24}$$

Now, multiplying (3.4) by $-h_2\frac{\rho_1}{\kappa_1}\overline{y_x}$, integrating over $(0,L)$ and using the fact that y_x is uniformly bounded in $L^2(0,L)$, $f_2\to 0$ in $L^2(0,L)$ and the first estimation in (3.8) we get

$$\Re\left(-i\lambda\frac{\rho_1}{\kappa_1}\int_0^L h_2 v\overline{y_x}dx\right)+\Re\left(\int_0^L h_2 u_{xx}\overline{y_x}dx\right)+\int_0^L h_2|y_x|^2dx=\frac{o(1)}{\lambda^{\frac{\ell}{2}}}. \tag{3.25}$$

Using equation (3.3), we obtain

$$-i\lambda v=\lambda^2 u+i\lambda^{1-\ell}f_1.$$

Inserting the above equation in (3.25) and using integration by parts, we get

$$\begin{aligned}&\Re\left(\lambda^2\frac{\rho_1}{\kappa_1}\int_0^L h_2 u\overline{y_x}dx\right)-\Re\left(i\frac{\rho_1}{\kappa_1}\int_0^L h_2'\lambda^{1-\ell}f_1\overline{y}dx\right)-\Re\left(i\frac{\rho_1}{\kappa_1}\int_0^L h_2\lambda^{1-\ell}(f_1)_x\overline{y}dx\right)\\&-\Re\left(\int_0^L h_2' u_x\overline{y_x}dx\right)-\Re\left(\int_0^L h_2 u_x\overline{y_{xx}}dx\right)+\int_0^L h_2|y_x|^2dx=\frac{o(1)}{\lambda^{\frac{\ell}{2}}}.\end{aligned} \tag{3.26}$$

Using the fact that $f_1\to 0$ in $H_0^1(0,L)$, λy is uniformly bounded in $L^2(0,L)$, y_x is uniformly bounded in $L^2(0,L)$, equation (3.10) and the definition of the function h_2, we get

$$\begin{cases}\Re\left(i\dfrac{\rho_1}{\kappa_1}\displaystyle\int_0^L h_2'\lambda^{1-\ell}f_1\overline{y}dx\right)=\dfrac{o(1)}{\lambda^{\ell}},\\ \Re\left(i\dfrac{\rho_1}{\kappa_1}\displaystyle\int_0^L h_2\lambda^{1-\ell}(f_1)_x\overline{y}dx\right)=\dfrac{o(1)}{\lambda^{\ell}},\\ \Re\left(\displaystyle\int_0^L h_2' u_x\overline{y_x}dx\right)=\dfrac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{4}}}.\end{cases}$$

Inserting the above estimations in (3.26), we get

$$\Re\left(\lambda^2\frac{\rho_1}{\kappa_1}\int_0^L h_2 u\overline{y_x}dx\right)-\Re\left(\int_0^L h_2 u_x\overline{y_{xx}}dx\right)+\int_0^L h_2|y_x|^2dx=\frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{4}}}. \tag{3.27}$$

Now, using integration by parts on the first term in (3.27), we get

$$-\Re\left(\lambda^2\frac{\rho_1}{\kappa_1}\int_0^L h_2' u\overline{y}dx\right)-\Re\left(\lambda^2\frac{\rho_1}{\kappa_1}\int_0^L h_2 u_x\overline{y}dx\right)-\Re\left(\int_0^L h_2 u_x\overline{y_{xx}}dx\right)+\int_0^L h_2|y_x|^2dx=\frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{4}}}. \tag{3.28}$$

Using the fact that λy is uniformly bounded in $L^2(0,L)$, equation (3.9) and the definition of h_2, we get

$$\Re\left(\lambda^2\frac{\rho_1}{\kappa_1}\int_0^L h_2' u\overline{y}dx\right)=\frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{2}}}.$$

Inserting the above equation in (3.28), we get

$$-\Re\left(\lambda^2\frac{\rho_1}{\kappa_1}\int_0^L h_2 u_x\overline{y}dx\right)-\Re\left(\int_0^L h_2 u_x\overline{y_{xx}}dx\right)+\int_0^L h_2|y_x|^2dx=\frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{4}}}. \tag{3.29}$$

Adding (3.24) and (3.29), we get

$$\int_0^L h_2|y_x|^2dx = \left(\frac{\rho_1}{\kappa_1}-\frac{\rho_2}{\kappa_2}\right)\lambda^2\int_0^L h_2 u_x\overline{y}dx + \frac{o(1)}{\lambda^{\frac{\ell+\alpha-1}{4}}} \tag{3.30}$$

Finally, using the fact that λy is uniformly bounded in $L^2(0,L)$, (3.10) and the definition of h_2, we get the desired result. □

Lemma 3.4 *Let $0<\alpha<1$, $\eta>0$ and $0<\varepsilon<\frac{\beta-\gamma}{6}$. Then, the solution $(u,v,y,z,\omega)\in D(\mathcal{A})$ of system of system (3.3)-(3.7) satisfies the following asymptotic behavior*

$$\int_{\beta+3\varepsilon}^{\gamma-3\varepsilon}|z|^2dx \le \frac{\kappa_2}{\rho_2}\left|\frac{\rho_1}{\kappa_1}-\frac{\rho_2}{\kappa_2}\right|\frac{o(1)}{|\lambda|^{\frac{\ell+\alpha-5}{4}}}+o(1). \tag{3.31}$$

Proof: First, we fix a cut-off function $h_3\in C^1([0,L])$ such that $0\le h_3(x)\le 1$, for all $x\in[0,L]$ and

$$h_3(x)=\begin{cases}1 & \text{if} \quad x\in[\beta+3\varepsilon,\gamma-3\varepsilon],\\ 0 & \text{if} \quad x\in[0,\beta+3\varepsilon]\cup[\gamma-3\varepsilon,L].\end{cases} \tag{3.32}$$

Multiplying (3.6) by $-i\lambda^{-1}h_3\overline{z}$ where h_3 is defined in (3.32), integrating over $(0,L)$, using integration by parts and using the fact that $f_4\to 0$ in $L^2(0,L)$, z is uniformly bounded in $L^2(0,L)$ and (u_x+y) is uniformly bounded in $L^2(0,L)$, we get

$$\int_0^L h_3|z|^2dx = \Re\left(i\lambda^{-1}\frac{\kappa_2}{\rho_2}\int_0^L y_x h_3\overline{z_x}dx\right)+\Re\left(i\lambda^{-1}\frac{\kappa_2}{\rho_2}\int_0^L h_3' y_x\overline{z}dx\right)+o(1). \tag{3.33}$$

From (3.5), we get

$$i\lambda^{-1}\overline{z_x}=\overline{y_x}-i\lambda^{-1}\lambda^{-\ell}\overline{(f_3)_x}.$$

Inserting the above equation in (3.33) and using the fact that y_x is uniformly bounded in $L^2(0,L)$ and $f_3\to 0$ in $H_0^1(0,L)$, we get

$$\int_0^L h_3|z|^2dx = \frac{\kappa_2}{\rho_2}\int_0^L h_3|y_x|^2dx+\Re\left(i\lambda^{-1}\frac{\kappa_2}{\rho_2}\int_0^L h_3' y_x\overline{z}dx\right)+o(1). \tag{3.34}$$

Using the fact that y_x and z are uniformly bounded in $L^2(0,L)$ in (3.34), we get

$$\int_0^L h_3|z|^2dx = \frac{\kappa_2}{\rho_2}\int_0^L h_3|y_x|^2dx+o(1).$$

Now, using the definition of function h_3 and equation (3.19), we get the desired result. □

Lemma 3.5 *Let $0<\alpha<1$ and $\eta>0$ and let $h_4\in C^1([0,L])$ such that $h_4(0)=h_4(L)=0$. Then the solution $(u,v,y,z,\omega)\in D(\mathcal{A})$ of system of system (3.3)-(3.7) satisfies the following asymptotic behavior*

$$\int_0^L h_4'\left(\rho_1|v|^2+\kappa_1|u_x|^2+\rho_2|z|^2+\kappa_2|y_x|^2\right)dx=o(1).$$

Proof: Using a similar technique to the one mentioned in Lemma 2.12 (see [1]), we get the desired result. □

Proof of Theorem 3.1. The proof of Theorem 3.1 is divided into two cases.
Case 1. Suppose that $\frac{\rho_1}{\kappa_1}=\frac{\rho_2}{\kappa_2}$. The proof of the following case is divided into two steps.

Step 1. Taking $\ell = 1-\alpha$ in Lemmas 3.1-3.4, then we obtain

$$\left\{\begin{array}{l}\displaystyle\int_0^L\int_{\mathbb{R}}\left(|\xi|^2+\eta\right)|\omega(x,\xi)|^2d\xi dx = o\left(\lambda^{-1+\alpha}\right), \int_\beta^\gamma |v|^2dx = o\left(1\right), \int_{\beta+\varepsilon}^{\gamma-\varepsilon}|u_x|^2dx = o\left(1\right),\\ \displaystyle\int_{\beta+2\varepsilon}^{\gamma-2\varepsilon}|y_x|^2dx = o(1) \quad \text{and} \quad \int_{\beta+3\varepsilon}^{\gamma-3\varepsilon}|z|^2dx = o(1).\end{array}\right. \tag{3.35}$$

Step 2. Using the result of Lemma 3.5 with $h_4 = x\theta_1 + (x-L)\theta_2$, where θ_1 and θ_2 are cut-off functions in $C^1([0,L])$ defined by

$$0 \le \theta_1 \le 1, \quad \theta_1 = 1 \text{ in } [0, \beta+3\varepsilon] \quad \text{and} \quad \theta_1 = 0 \text{ in } [\gamma-3\varepsilon, L]$$

and

$$0 \le \theta_2 \le 1, \quad \theta_2 = 0 \text{ in } [0, \beta+3\varepsilon] \quad \text{and} \quad \theta_2 = 1 \text{ in } [\gamma-3\varepsilon, L],$$

we obtain

$$\int_0^L \left(x\theta_1' + \theta_1 + (x-L)\theta_2' + \theta_2\right)\left(\rho_1|v|^2 + \kappa_1|u_x|^2 + \rho_2|z|^2 + \kappa_2|y_x|^2\right)dx = o(1).$$

Using the definition of θ_1 and Step 1., we obtain

$$\|U\|_{\mathcal{H}} = o(1).$$

which contradicts (H).

Case 2. Suppose that $\dfrac{\rho_1}{\kappa_1} \neq \dfrac{\rho_2}{\kappa_2}$. The proof of the following case is divided into two steps.

Step 1. Taking $\ell = 5-\alpha$ in Lemmas 3.1-3.4, then we obtain

$$\left\{\begin{array}{l}\displaystyle\int_0^L\int_{\mathbb{R}}\left(|\xi|^2+\eta\right)|\omega(x,\xi)|^2d\xi dx = o\left(\lambda^{-5+\alpha}\right), \int_\beta^\gamma |v|^2dx = o\left(\lambda^{-4}\right), \int_{\beta+\varepsilon}^{\gamma-\varepsilon}|u_x|^2dx = o\left(\lambda^{-2}\right),\\ \displaystyle\int_{\beta+2\varepsilon}^{\gamma-2\varepsilon}|y_x|^2dx = o(1) \quad \text{and} \quad \int_{\beta+3\varepsilon}^{\gamma-3\varepsilon}|z|^2dx = o(1).\end{array}\right. \tag{3.36}$$

Step 2. Similar to **Step 2.** in Case 1, we obtain

$$\|U\|_{\mathcal{H}} = o(1),$$

which contradicts (H).

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