



## Graph Associative on Disjoint Open Subsets of Topological Space

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**ABSTRACT:** In this study, a new idea of forming a statement from a discrete topological space is presented. Several properties and limitations of this topological statement are presented. This graph has no isolated vertices. We define the graph of some topological space also collection degree of every vertex and some normal numbers like an independent number, covering number, and chromatic number finally we discussed where  $\Gamma\tau$  planar and nonplanar graph

**Keywords:** Graph theory, discrete topology, independent number, girth, chromatic number, planer graph.

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### 1. Introduction

Graph theory encompasses various directions, including algebraic and topological graph theory. While algebraic graph theory focuses on algebraically motivated graph constructions, the present paper is concerned exclusively with topological graphs and their basic topological properties. Related studies in algebraic graph theory may be found in [14],[15]. There are several studies on graphs that rely on topological structures which can be seen in [1], [2], [6]. The relationship between graph theory, topology, non-discrete topology, joint topology, joint countable topology, and nested topological spaces was reviewed [7],[8],[13]. Converting the topological graph to a special topology by adjacent vertices is studied [4]. A novel topological concept, termed "symmetrical," was introduced for two topologies. It was observed that the structural drawings of the two topologies exhibit symmetry [3]. Studying the topological spaces resulting from graphs, the cases of being an accumulation point and an internal point of a point in these spaces were examined [5]. In this paper we define a new graph associated with a discrete open set of topological space which  $\Gamma\tau$  denotes, this graph consists of the set of vertices that contain all open sets and every two vertices A and B adjacent if  $A \cap B = \emptyset$ , one of the important characteristics this graph has no isolated vertices if there are in the topological space at least two non-empty disjoint open sets then  $\text{girth}(\Gamma\tau) = 3$ , also we calculate some numerical numbers like independent, covering, chromatic number, where the graph define on discrete topology which denoted it, in this case,  $\Gamma d$ , so we find the degree of each vertex of  $\Gamma d$  and discussion the planer and non-planer of it.

### 2. Properties of Topological Graph

In this section, we introduce a novel definition for a discrete topological graph and then proceed to explore several properties of this particular space graph

**Definition [2.1]** Let  $X$  be a non-empty set and  $\tau$  be a topological space on set  $X$  Then the graph is defined on  $\tau$  which  $\Gamma\tau$  denotes,  $\Gamma\tau$  with vertices set as  $V(\Gamma\tau) = \{A; A \in \mathcal{G}\tau\}$  and the edge as  $E(\Gamma\tau) = \{(A, B); A \cap B = \emptyset, \text{ where } A \text{ or } B \neq \emptyset\}$ .

**Definition [2.2]** the shortest cycle of graph  $\Gamma$  is the girth of  $\Gamma$  [9]

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**Definition [2.3]** A complete graph is a graph  $\Gamma$  in which every vertex is connected to the other vertices and denoted by  $K_n$

**Definition [2.4]** A clique in graph  $\Gamma$  It is a subgraph that is complete and the largest of these complete subgraphs is denoted by  $\omega(\Gamma)$ [10].

**Definition [2.5]** Coloring in a graph in which two adjacent vertices each have a different Moreover, the least coloration of the heads is called the chromatic number, represented by the symbol  $\chi(G)$  [11],[12]

**Definition [2.6]** A planer graph is a graph whose edges intersect each other only at the final point.

**proposition[2.7]** the graph  $(\Gamma\tau)$  has no isolated vertices.

**Proof** by (Definition[2.1]), every two vertices  $A, B \in \Gamma\tau$  are adjacent if  $A \cap B = \emptyset$ , so  $\forall A \in V(\Gamma\tau), A \cap \emptyset = \emptyset$  this given that  $\emptyset$  adjacent with all vertices. Therefore graph  $(\Gamma\tau)$  has no isolated vertices

**Example [2.8]** let  $X = \{1, 2, 3, 4, 5\}$  and  $\tau = \{\emptyset, X, \{1\}, \{2\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 5\}, \{1, 2, 5\}\}$  then the  $V(\Gamma\tau) = \{\emptyset, X, \{1\}, \{2\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 5\}, \{1, 2, 5\}\}$  we can see the edge of  $\Gamma\tau$  as follows: **figure[1]**

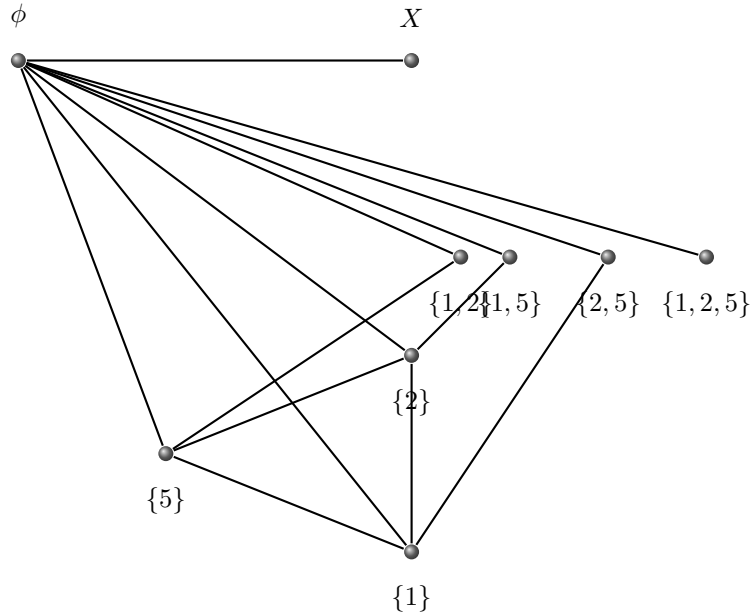


Figure 1:  $\Gamma\tau$  where  $X=5$

**Theorem [2.9]** let  $(X, \tau)$  be a topological space and  $\tau$  has one of followed separations axioms  $(\tau_2, \tau_3, \tau_4)$ , then the girth of  $(\Gamma\tau)$  is 3

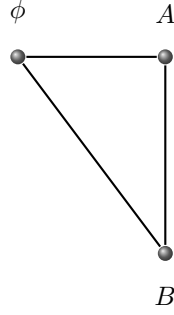
**Proof:** it's clear that  $\emptyset$  they are adjacent to all elements of  $V(\Gamma\tau)$  because  $A \cap \emptyset = \emptyset \forall A \in V(\Gamma\tau)$ , from the definition of separations axioms  $\tau_2, \tau_3$  and  $\tau_4$  we have two disjoint open sets  $A$  and  $B$  they adjacent to  $B$ , also we have everyone  $A$  and  $B$  adjacent with  $\emptyset$  so.

And the proof is complete.

**Theorem [2.10]** let  $X$  be an infinite set and  $\tau$  co-finite topology on  $X$ , then  $(\Gamma\tau)$  is a star graph.

**Proof** let  $A, B \in \Gamma\tau$ , then  $A = X / \{a_1, a_2, a_3, \dots, a_n\}$  and  $B = X / \{b_1, b_2, b_3, \dots, b_n\}$ ,  $A \cap B \neq \emptyset$  This  $A$  is not adjacent to  $B$ . On the other hand, we have  $\emptyset$  Adjacent to all vertices of  $(\Gamma\tau)$ , so this graph with all vertices except  $\emptyset$  have degree one and degree of  $\emptyset$  equal  $n-1$ .

**Theorem [2.11]** let  $X$  be a finite set and  $\tau$  be discrete topology on  $X$ ,  $\Gamma\tau$  is a graph defined on  $\tau$  then


 Figure 2: The girth of graph  $\Gamma\tau$ 

$$\begin{aligned} \deg(\emptyset) &= n \left( 2 + \sum_{i=2}^{n-1} \frac{\prod_{j=1}^{i-1} (n-1)}{i!} \right) \\ \deg(\{a_1\}) &= n + \sum_{i=2}^{n-1} \frac{\prod_{j=1}^i (n-j)}{i!} \\ \deg(\{a_1, a_2, \dots\}) &= (n-1) + \sum_{i=2}^{n-1} \frac{\prod_{j=1}^{i-1} (n-j)}{i!} \end{aligned}$$

In general  $\forall A \in \tau$

$$\deg(A) = (n - (|A| - 1)) + \sum_{i=2}^{n-1} \frac{\prod_{j=1}^{i+(|A|-1)} (n-j)}{i!}$$

**Proof** since for every  $A \in \tau$ , then  $A \cap \emptyset = \emptyset$ , so  $\emptyset$  is adjacent to all vertices of  $\Gamma\tau$ , they  $\deg(\emptyset) = |v(\Gamma\tau)| - 1$  because  $\emptyset$  it is not adjacent to itself. Now we find  $V(\Gamma\tau)$  as follows:

We divided the vertices ( $\Gamma\tau$ ) into several collections such that everyone contains a subset of  $X$  equal numbers, so the number of vertices set can be accounted as following

$$\begin{aligned} & \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} + 1 \text{ where } \emptyset \text{ is } 1. \\ &= n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \dots + n + 1 + 1 \\ &= n \left( 1 + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \dots + 1 + 1 \right) \\ &= n \left( 2 + \sum_{i=2}^{n-1} \frac{\prod_{j=1}^{i-1} (n-j)}{i!} \right) + 1 \end{aligned}$$

$$\text{There for } \deg(\emptyset) = n \left( 2 + \sum_{i=2}^{n-1} \frac{\prod_{j=1}^{i-1} (n-j)}{i!} \right) + 1, \quad i = (2, 3, \dots, n-1)$$

By using some method we can find the degree of all vertices. So the result holds.

**Corollary [2.12]** let  $|X| = n$  and  $\tau$  be discrete topology on  $X$  then  $\chi(\Gamma\tau) = n - 1$

**Proof** since the graph has a complete subgraph containing  $n + 1$  vertices, we can color by  $n + 1$  color. It can also be noted that  $\binom{n}{1}$  of the vertices can be colored in one of these colors because all other vertices are not adjacent to at least one of the vertices of the complete subgraph. So we can use some colors that we mentioned in the offline vertices.

**Corollary [2.13]** let  $|X| = n$  and  $\tau$  be discrete topology on  $X$  then the clique number of  $\Gamma\tau$  is  $\omega(\Gamma\tau) =$

$n + 1$

**Proof:** we can see that all vertices in the collection of a singleton set are adjacent to gather, also these vertices are adjacent to  $\emptyset$ , so we have a complete subgraph of  $\Gamma\tau$  containing  $n+1$  vertices, so  $\omega(\Gamma\tau) = n+1$ .

**Theorem[2.14]** let  $|X| = n$  and  $\tau$  be discrete topology on  $X$ , and  $\Gamma\tau$  is a graph defined on  $\tau$  then

$$\alpha(\Gamma\tau) = (n + 1) + \sum_{i=2}^{n-1} (n - i) + \sum_{k=3}^{n-1} \sum_{j=k}^{n-1} \sum_{j=i}^{n-1} (n - i)$$

When  $k$  is the count of the subset minus one

**Proof:** we divided the set of vertices in this graph into serval families every family contains of subset that has the same number, so we have the following

For the family of singleton subset and from this family we can one vertice as independent because all vertices in this family are adjacent to each other the remaining families of vertices we can choose  $(n - 1)$  of vertices in the family of binary subset because the element has appear in  $n - 1$  subset. The family contains a subset of three elements we can select the vertices as

$(n - 2) + (n - 3) + \dots + (n - (n - 1))$ , so we have  $\sum_{i=2}^{n-1} (n - 1)$  the families which contain subset consist of the manor that three, we can choices the vertices as

$$\sum_{k=3}^{n-1} \sum_{j=k}^{n-1} \sum_{j=i}^{n-1} (n - i)$$

Also, we can take the set  $X$  as an independent element, because  $X$  is adjacent to  $\emptyset$ , thus from above the number of independent sets is

$$\alpha(\Gamma\tau) = 1 + (n + 1) + \sum_{i=2}^{n-1} (n - i) + \sum_{k=3}^{n-1} \sum_{j=k}^{n-1} \sum_{j=i}^{n-1} (n - i) + L$$

When  $K$  is the number of subsets minus one, so

$$\alpha(\Gamma\tau) = (n + 1) + \sum_{i=2}^{n-1} (n - i) + \sum_{k=3}^{n-1} \sum_{j=k}^{n-1} \sum_{j=i}^{n-1} (n - i)$$

**Theorem [2.15]** let  $X$  be a non-empty finite set,  $\tau$  be discrete topology on  $X$  and  $G\tau$  is a graph defined on  $\tau$  then

$$\beta(\Gamma\tau) = \left( n \left( 1 + \sum_{i=2}^{n-1} \frac{\prod_{j=1}^{i-1} (n - j)}{i!} \right) + 2 \right) - n + \sum_{i=2}^{n-1} (n - i) + \sum_{k=3}^{n-1} \sum_{j=k}^{n-1} \sum_{j=i}^{n-1} (n - i)$$

**Proof:** by theorem  $\beta(\Gamma\tau) = |V(\Gamma\tau)| + \alpha(\Gamma\tau)$  we can calculate the number of vertices as follows

$$\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} + 1$$

So

$$V(\Gamma\tau) = \left( n \left( 1 + \sum_{i=2}^{n-1} \frac{\prod_{j=1}^{i-1} (n - j)}{i!} \right) + 2 \right),$$

Also from theorem ((2.14 )) we have

$$\alpha(\Gamma\tau) = (n + 1) + \sum_{i=2}^{n-1} (n - i) + \sum_{k=3}^{n-1} \sum_{j=k}^{n-1} \sum_{j=i}^{n-1} (n - i)$$

Then the proof is complete.

**Theorem[2.16]:** the graph  $(\Gamma\tau)$ , Is anon planer for every  $n \geq 4$  were  $|X| = n$  and  $\tau$  be discrete topology on  $X$ .

**Proof:**if  $n < 4$  the graph  $\Gamma\tau$  is planer .graph show figure  $((3,4,5))$ , if  $n \geq 4$  then we have a subgraph is isomorphic  $Xh$ , which  $\phi$  consists of a singleton set and  $\emptyset$ , so the graph is nonplanar for  $n \geq 4$

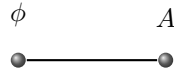


Figure 3: The graph  $\Gamma\tau$  for  $n=1$

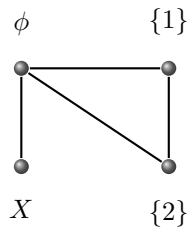


Figure 4: The graph  $\Gamma\tau$  for  $n=2$

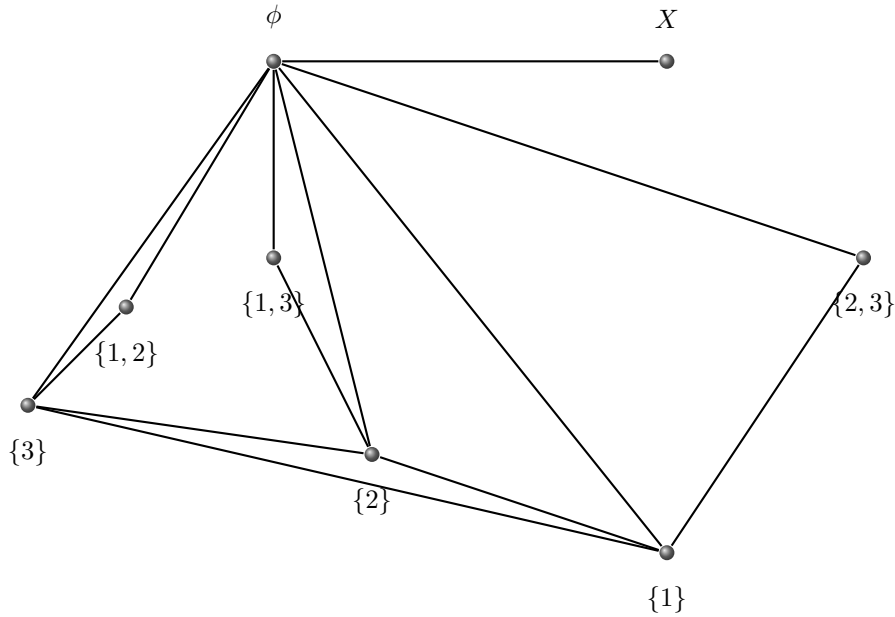


Figure 5:  $\Gamma\tau$  where  $n=3$

### 3. Conclusions

In this study, we were able to describe topological spaces in the form of a graph, In addition, many properties of the graph were studied, such as the chromatic number, the independence number, the clique number, and the Girth number. The degree of vertices was also calculated for the graphs of the discrete

topological space as this plays a crucial role in facilitating the analysis of many properties associated with these graphs.

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