Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 10** : 1–26.
ISSN-0037-8712
doi:10.5269/bspm.81564

A Conformable Fractional Weerakoon-Fernando Method for Solving Nonlinear Equations and its Generalization for Nonlinear Systems

Toshan Kumar Shriwas and Jai Prakash Jaiswal*

ABSTRACT: This study introduces a new conformable Weerakoon-Fernando method designed to solve nonlinear equations, the classical Weerakoon-Fernando method is a special case of the conformable Weerakoon-Fernando method. In addition, the proposed method has been extended to its multi-dimensional version, enabling it to effectively solve systems of nonlinear equations. Moreover, we present a comprehensive analysis of convergence for the proposed methods. Through numerical comparisons, we highlight the substantial improvements in both convergence rate and accuracy offered by these methods. Furthermore, the convergence planes generated by the proposed methods demonstrate strong stability and a significantly higher convergence percentage.

Keywords: Nonlinear equations, fractional Newton-type method, conformable derivative, Weerakoon-Fernando method, convergence analysis, stability.

Contents

1 Introduction **1**

2 Development of Method and Convergence Analysis **5**

3 Numerical Results and Comparisons **11**

4 Numerical Stability of the Proposed Schemes **20**

5 Conclusion **24**

1. Introduction

Fractional calculus serves as a broad generalization of classical calculus while preserving many of its essential characteristics. A wide range of real-world problems can be effectively modeled using tools from fractional calculus, owing to its greater flexibility compared to classical calculus; see, for example, [2,5,17,20,22,26] and the references therein. Solving nonlinear equations of the form $f(y) = 0$ remains a significant challenge in modern science and engineering. To tackle these problems, numerous iterative methods have been developed, which provide effective approaches for approximating solutions. These techniques are highly significant in numerical analysis due to their broad applicability across various areas of pure and applied sciences. Recent studies have introduced fractional iterative methods for solving nonlinear equations, employing Riemann–Liouville, Caputo, and conformable derivatives (one can see [3,4,6,7,8,9,10,21]). The focus of these methods is to enhance the efficiency and accuracy of finding solutions to nonlinear problems by incorporating fractional calculus and fractal techniques into the traditional iterative method framework. While methods based on Riemann–Liouville and Caputo derivatives do not generally preserve the convergence order of their classical counterparts, conformable derivatives offer lower computational complexity by avoiding the evaluation of special functions and retain the theoretical convergence order of the classical iterative method in practice. Furthermore, sometimes it presents some numerical advantages versus the classical version. Specifically, for the conformable versions of the iterative schemes presented in [7,8,9,10,21], we observed several interesting behaviours: solutions can be obtained in situations where the classical method fails (with order of derivative $\alpha \neq 1$); in some cases, the conformable methods requires fewer iterations and attains a slightly higher computational order of convergence; different roots can be identified by choosing distinct fractional index α while using the same

* Corresponding author.
2020 *Mathematics Subject Classification:* 65H04, 65H05, 65H10.
Submitted January 26, 2026. Published April 17, 2026

initial guess; and complex roots can be found even when starting from real initial estimates.

First, we will go over some basic ideas associated with the conformable derivative. The conformable derivative of a function $f : [a, \infty) \longrightarrow \mathbb{R}$ of order α, for any $a \in \mathbb{R}$, $a < y$ and $0 < \alpha \leq 1$ is defined as [1,16]

$$(T_\alpha^a f)(y) = \lim_{\epsilon \longrightarrow 0} \frac{f(y + \epsilon(y-a)^{1-\alpha}) - f(y)}{\epsilon}, \tag{1.1}$$

where, T_α^a represents the conformable fractional derivative operator. If the above limit exists then f is called α-differentiable. Additionally, if f is α-differentiable, then $(T_\alpha^a f)(y) = (y-a)^{1-\alpha} f'(y)$. If for any $b \in \mathbb{R}$, f is α-differentiable in (a, b), then $(T_\alpha^a f)(a) = \lim_{y \longrightarrow a^+} (T_\alpha^a f)(y)$.

The idea of using conformable fractional-order derivatives offers a novel approach to the development of iterative methods. Additionally, its computation does not involve any special functions, making it computationally more efficient. By applying the concept of conformable fractional derivatives Candelario et al. [7] recently introduced a fractional Newton-type iterative method for locating the roots of the nonlinear equation $f(y) = 0$. For any $0 < \alpha \leq 1$, $a \in \mathbb{R}$ the method is given as:

$$y_{n+1} = a + \left((y_n - a)^\alpha - \alpha \frac{f(y_n)}{(T_\alpha^a f)(y_n)} \right)^{\frac{1}{\alpha}}, \qquad n = 0, 1, 2, ... \tag{1.2}$$

After this Candelario et al. [10] proposed three one-point conformable Newton-type methods, along with a general procedure for constructing the conformable form of any iterative method. This approach was then applied to develop the conformable counterparts of four classical multipoint schemes (Traub-type, Chun–Kim-type, Ostrowski-type, Chun-type). Authors developed the following conformable Traub-type procedure (TeCO) for solving the equation $f(y) = 0$,

$$y_{n+1} = a + \left((z_n - a)^\alpha - \alpha \frac{f(z_n)}{(T_\alpha^a f)(y_n)} \right)^{\frac{1}{\alpha}}, \qquad n = 0, 1, 2, ..., \tag{1.3}$$

$$where, \quad z_n = a + \left((y_n - a)^\alpha - \alpha \frac{f(y_n)}{(T_\alpha^a f)(y_n)} \right)^{\frac{1}{\alpha}}, \qquad n = 0, 1, 2, ...$$

And also developed conformable Chun–Kim-type scheme (CKeCO) as:

$$y_{n+1} = a + \left((y_n - a)^\alpha - \frac{\alpha}{2} \left[3 - \frac{(T_\alpha^a f)(z_n)}{(T_\alpha^a f)(y_n)} \right] \frac{f(y_n)}{(T_\alpha^a f)(y_n)} \right)^{\frac{1}{\alpha}}, \qquad n = 0, 1, 2, ..., \tag{1.4}$$

$$where, \quad z_n = a + \left((y_n - a)^\alpha - \alpha \frac{f(y_n)}{(T_\alpha^a f)(y_n)} \right)^{\frac{1}{\alpha}}, \qquad n = 0, 1, 2, ...,$$

where $\alpha \in (0, 1]$, $a \in \mathbb{R}$ and f is α-differentiable. Observe that when $\alpha = 1$, the formulation in equation (1.3) specifically simplifies to the classical Traub method and the formulation in equation (1.4) specifically simplifies to the classical Chun–Kim method. In recent developments, the fractional Newton's iterative method (1.2) has been successfully generalized [9] to handle nonlinear systems of equations, demonstrating promising performance in various numerical experiments. The structure of this method is given as follows:

$$y_{n+1} = a + \left[(y_n - a)^{\odot \alpha} - \alpha [F_a^{\alpha(1)}(y_n)]^{-1} F(y_n) \right]^{\odot \frac{1}{\alpha}}, \qquad n = 0, 1, 2, ..., \tag{1.5}$$

where $a \in \mathbb{R}^m$, $\alpha \in (0, 1]$ and $\odot$ being the Hadamard power [13,14].

After this in 2024, Wang and Xu [24] developed the conformable vector Traub's method (CVTM) for solving nonlinear system as:

$$z_n = a + \Big((y_n - a)^{\odot\alpha} - \alpha(F_a^{\alpha(1)}(y_n))^{-1}F(y_n)\Big)^{\odot\frac{1}{\alpha}},$$
$$y_{n+1} = a + \Big((z_n - a)^{\odot\alpha} - \alpha(F_a^{\alpha(1)}(y_n))^{-1}F(z_n)\Big)^{\odot\frac{1}{\alpha}}, \qquad n = 0, 1, 2, ..., \tag{1.6}$$

where $a \in \mathbb{R}^m$, $0 < \alpha \leq 1$ and $\odot$ being the Hadamard power. To compute the root $\bar{y} \in \mathbb{R}^m$ of the nonlinear system $F(y) = 0$ using method (1.5) and (1.6), we consider the coordinate representation of the function $F(y) = (f_1, ..., f_m)$, where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is assumed to be sufficiently Fréchet-differentiable within the open domain D.
Lastly, some derivative-free iterative methods were formulated, including Steffensen-type and Secant-type procedures that employ approximations of conformable derivatives in [8], and a Traub–Steffensen-type method presented in both scalar and vector forms in [21]. These approaches successfully maintain their theoretical convergence order in practical implementations and demonstrate strong numerical efficiency, often delivering improvements over their classical counterparts in various numerical aspects. The conformable derivative offers the simplest and most natural approach to defining fractional derivatives. Indeed, the definition (1.1) plays a key role in the construction of novel fractional iterative scheme. To facilitate the formulation of the new schemes, it is essential to consider the following result for a deeper understanding of convergence behaviour:

Theorem 1.1 *(Theorem 4.1, [1]). Suppose $f(y)$ is an infinitely α-differentiable function, for some $\alpha \in (0, 1]$ at a neighborhood of a point y_0. Then the fractional power series expansion of $f(y)$ is:*

$$f(y) = \sum_{n=0}^{\infty} \frac{(T_\alpha^{y_0} f)^{(n)}(y_0)(y - y_0)^{n\alpha}}{\alpha^n n!}, \qquad y_0 < y < y_0 + R^{1/\alpha}, R > 0, \tag{1.7}$$

where $(T_\alpha^{y_0} f)^{(n)}(y_0)$ is the n times application of fractional derivative.

In addition to the Theorem 1.1 result, the following power series expansion better serves our purposes.

Theorem 1.2 *(Theorem 4.1, [23]). Suppose $f : [a, \infty) \longrightarrow \mathbb{R}$ is an infinitely $\alpha-differentiable$ function, for some $0 < \alpha \leq 1$ at a neighborhood of $a_1 \in (a, \infty)$. Then the fractional power series expansion of $f(y)$ is:*

$$f(y) = f(a_1) + \frac{(T_\alpha^a f)(a_1)\delta_1}{\alpha} + \frac{(T_\alpha^a f)^{(2)}(a_1)\delta_2}{2!\alpha^2} + \frac{(T_\alpha^a f)^{(3)}(a_1)\delta_3}{3!\alpha^3} + ..., \tag{1.8}$$

where, $\delta_1 = H^\alpha - L^\alpha$, $\delta_2 = H^{2\alpha} - L^{2\alpha} - 2L^\alpha\delta_1$, $\delta_3 = H^{3\alpha} - L^{3\alpha} - 3L^\alpha\delta_2 - 3L^{2\alpha}\delta_1$,...,and $H = y - a$, $L = a_1 - a$. One can easily verify that $\delta_2 = \delta_1^2$, $\delta_3 = \delta_1^3$, etc. Hence, the expansion (1.8) can be rewritten as:

$$f(y) = f(a_1) + \frac{(T_\alpha^a f)(a_1)\delta_1}{\alpha} + \frac{(T_\alpha^a f)^{(2)}(a_1)\delta_1^2}{2!\alpha^2} + \frac{(T_\alpha^a f)^{(3)}(a_1)\delta_1^3}{3!\alpha^2} + ... \tag{1.9}$$

To further enhance the formulation and examination of techniques for solving nonlinear system of equations, some concepts corresponding to the fractional iterative approach for solving nonlinear systems are introduced.

Definition 1.1 *[9] Let f be a function with m variables $y_1, ..., y_m$, then the conformable partial derivative of f of order $\alpha \in (0, 1]$ in $y_i \in (a, \infty)$ is defined as:*

$$\frac{\partial_a^\alpha}{\partial y_i^\alpha} f(y_1, ..., y_m) = \lim_{\epsilon \longrightarrow 0} \frac{f(y_1, ..., y_i + \epsilon(y_i - a)^{1-\alpha}, ..., y_m) - f(y_1, ..., y_m)}{\epsilon}. \tag{1.10}$$

When $y_i = a$, then $\frac{\partial_a^\alpha}{\partial y_i^\alpha} f(y_1, ..., a, ..., y_m) = \lim_{y_i \longrightarrow a^+} \frac{\partial_a^\alpha}{\partial y_i^\alpha} f(y_1, ..., y_i, ..., y_m)$.

Remark 1.1 *The connection between the classical and conformable partial derivatives can be directly derived from equation (1.10) as follows:*

$$\frac{\partial_a^\alpha}{\partial y_i^\alpha} f(y_1, ..., y_m) = (y_i - a)^{1-\alpha} \frac{\partial}{\partial y_i} f(y_1, ..., y_m). \tag{1.11}$$

Remark 1.2 *By Definition 1.1 and Remark 1.1 we can find*

$$F_a^{\alpha(1)}(y) = (y-a)^{\odot(1-\alpha)} F'(y), \tag{1.12}$$

$$F_a^{\alpha(1)}(a) = \lim_{y \longrightarrow a^+} (y-a)^{\odot(1-\alpha)} F'(y), \tag{1.13}$$

where $F'(y)$ is the classical Jacobian matrix.

Additionally, the p^{th} conformable derivative of F at $u \in \mathbb{R}^m$ is the p-linear function defined as follows:

Definition 1.2 *[9] Let $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be sufficiently α-differentiable in D. The p^{th} derivative of F at $u \in \mathbb{R}^m$ is the $\alpha(p)$-linear function $F_a^{\alpha(p)}(u) : \mathbb{R}^m \times ... \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ in which the following remains valid for any $v_1, ..., v_p \in \mathbb{R}^m$*

1. $F_a^{\alpha(p)}(u)(v_1, ..., v_p) \in \mathbb{R}^m$.
2. $F_a^{\alpha(p)}(u)(v_1, ..., v_{p-1}, \cdot) \in \mathcal{L}(\mathbb{R}^m)$, *where $\mathcal{L}(\mathbb{R}^m)$ denotes the space of linear mappings of $\mathbb{R}^m \to \mathbb{R}^m$.*
3. $F_a^{\alpha(p)}(u)(v_{\sigma_1}, ..., v_{\sigma_p}) = F_a^{\alpha(p)}(u)(v_1, ..., v_p)$ *for any permutation $\sigma \in \{1, ..., p\}$.*

Based on these properties, we can adopt the following notations:

1. $F_a^{\alpha(p)}(u)(v_1, ..., v_p) = F_a^{\alpha(p)}(u)v_1 \cdots v_p$.
2. $F_a^{\alpha(p)}(u)v^{p-1} F_a^{\alpha(q)}(u)v^q = F_a^{\alpha(p)}(u) F_a^{\alpha(q)}(u) v^{p+q-1}$.

We now define the conformable Jacobian matrix for the variables $y_1, y_2, ..., y_m$, where y_i lies in the interval (a_i, ∞), $i = 1, 2, ..., m$, with $y = (y_1, y_2, ..., y_m)$ and $a = (a_1, a_2, ..., a_m)$.

Definition 1.3 *[9] Assume that $f_1, f_2, ..., f_m$ are coordinate functions of a vector valued function $F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ defined in terms of the variables $y_1, y_2, ..., y_m$ such that their partial derivatives exist and are continuous, where $y_i > a_i$, $i = 1$ to m, then the conformable Jacobian matrix is:*

$$F_a^{\alpha(1)}(y) = \begin{pmatrix} \frac{\partial_{a_1}^\alpha f_1}{\partial y_1^\alpha} & \frac{\partial_{a_2}^\alpha f_1}{\partial y_2^\alpha} & \cdots & \frac{\partial_{a_m}^\alpha f_1}{\partial y_m^\alpha} \\ \frac{\partial_{a_1}^\alpha f_2}{\partial y_1^\alpha} & \frac{\partial_{a_2}^\alpha f_2}{\partial y_2^\alpha} & \cdots & \frac{\partial_{a_m}^\alpha f_2}{\partial y_m^\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial_{a_1}^\alpha f_m}{\partial y_1^\alpha} & \frac{\partial_{a_2}^\alpha f_m}{\partial y_2^\alpha} & \cdots & \frac{\partial_{a_m}^\alpha f_m}{\partial y_m^\alpha} \end{pmatrix},$$

or

$$F_a^{\alpha(1)}(y) = \begin{pmatrix} (y_1-a_1)^{1-\alpha} \frac{\partial f_1}{\partial y_1} & (y_2-a_2)^{1-\alpha} \frac{\partial f_1}{\partial y_2} & \cdots & (y_m-a_m)^{1-\alpha} \frac{\partial f_1}{\partial y_m} \\ (y_1-a_1)^{1-\alpha} \frac{\partial f_2}{\partial y_1} & (y_2-a_2)^{1-\alpha} \frac{\partial f_2}{\partial y_2} & \cdots & (y_m-a_m)^{1-\alpha} \frac{\partial f_2}{\partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ (y_1-a_1)^{1-\alpha} \frac{\partial f_m}{\partial y_1} & (y_2-a_2)^{1-\alpha} \frac{\partial f_m}{\partial y_2} & \cdots & (y_m-a_m)^{1-\alpha} \frac{\partial f_m}{\partial y_m} \end{pmatrix}. \tag{1.14}$$

Next, the conformable Taylor series for a vector-valued function is given as:

Theorem 1.3 *[9] Suppose $F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is an infinitely α-differentiable function, for some $0 < \alpha \leq 1$ at a neighborhood of $b_j \in (a_j, \infty)$, $\forall j = 1, ..., m$, where $a = (a_1, ..., a_m) \in \mathbb{R}^m$, $b = (b_1, ..., b_m) \in \mathbb{R}^m$. Then the fractional power series expansion of F is:*

$$F(t) = F(b) + \frac{F_a^{\alpha(1)}(b)}{\alpha}\Delta + \frac{F_a^{\alpha(2)}(b)}{2!\alpha^2}\Delta^2 + \frac{F_a^{\alpha(3)}(b)}{3!\alpha^3}\Delta^3 + ..., \tag{1.15}$$

where, $\Delta = H^{\odot\alpha} - L^{\odot\alpha}$, $H = t - a$, $L = b - a$ and $\odot$ be the Hadamard power.

Theorem 1.4 *[9] If $y_1, y_2 \in \mathbb{R}^m$, $r \in \mathbb{R}$, then the Newton's Binomial theorem for vector valued and fractional power is given by*

$$(y_1 + y_2)^{\odot r} = \sum_{n=0}^{\infty} \binom{r}{n} y_1^{\odot(r-n)} \odot y_2^{\odot n}, \qquad n \in \{0\} \cup \mathbb{N}, \tag{1.16}$$

where

$$\binom{r}{n} = \frac{\Gamma(r+1)}{n!\Gamma(r-n+1)}, \qquad n \in \{0\} \cup \mathbb{N}.$$

The remainder of the manuscript is structured as: Section 2 introduces and examines a new fractional iterative method. Furthermore, its extension to the multi-dimensional case is presented for solving nonlinear system of equations. In Section 3, multiple numerical tests are carried out to support the theoretical conclusions and error graphs are presented. Section 4 examines how the iterative method's performance is influenced by initial guess values through the visualization of convergence planes. Ultimately, in Section 5 the most relevant outcomes of the work are discussed.

2. Development of Method and Convergence Analysis

To derive a new fractional iterative method, we consider the second-order approximation of the fractional power series expansion (1.9), evaluated at the solution $\bar{y}$, as follows:

$$f(y) \approx f(\bar{y}) + \frac{(T_\alpha^a f)(\bar{y})\delta_1}{\alpha} + \frac{(T_\alpha^a f)^{(2)}(\bar{y})\delta_1^2}{2!\alpha^2}. \tag{2.1}$$

And hence

$$(T_\alpha^a f)(y) \approx (T_\alpha^a f)(\bar{y}) + \frac{(T_\alpha^a f)^{(2)}(\bar{y})\delta_1}{\alpha} + \frac{(T_\alpha^a f)^{(3)}(\bar{y})\delta_1^2}{2!\alpha^2}. \tag{2.2}$$

Taking up to order two, we get

$$(T_\alpha^a f)(y) \approx (T_\alpha^a f)(\bar{y}) + \frac{(T_\alpha^a f)^{(2)}(\bar{y})\delta_1}{\alpha}, \tag{2.3}$$

which gives

$$\frac{(T_\alpha^a f)^{(2)}(\bar{y})\delta_1}{\alpha} \approx (T_\alpha^a f)(y) - (T_\alpha^a f)(\bar{y}). \tag{2.4}$$

Replacing (2.4) in (2.1), one can find

$$f(y) \approx f(\bar{y}) + \frac{(T_\alpha^a f)(\bar{y})\delta_1}{\alpha} + \frac{1}{2!\alpha}[(T_\alpha^a f)(y) - (T_\alpha^a f)(\bar{y})]\delta_1, \tag{2.5}$$

or

$$f(y) \approx f(\bar{y}) + \frac{1}{2\alpha}[(T_\alpha^a f)(y) + (T_\alpha^a f)(\bar{y})]\delta_1. \tag{2.6}$$

But $f(\bar{y}) = 0$ and $\delta_1 = H^\alpha - L^\alpha$, being $H = y - a$, $L = \bar{y} - a$, so expression (2.6) can be reformulated as

$$f(y) \approx \frac{1}{2\alpha}[(T_\alpha^a f)(y) + (T_\alpha^a f)(\bar{y})]((y-a)^\alpha - (\bar{y}-a)^\alpha). \tag{2.7}$$

Hence, from $(\bar{y} - a)^\alpha$ the value of $\bar{y}$ can be extracted as

$$\bar{y} \approx a + \left((y-a)^\alpha - \alpha \frac{2f(y)}{(T_\alpha^a f)(\bar{y}) + (T_\alpha^a f)(y)}\right)^{\frac{1}{\alpha}}. \tag{2.8}$$

Regarding the iterates y_n and y_{n+1} as approximations to the solution $\bar{y}$, we derive

$$y_{n+1} = a + \left((y_n-a)^\alpha - \alpha \frac{2f(y_n)}{(T_\alpha^a f)(y_{n+1}) + (T_\alpha^a f)(y_n)}\right)^{\frac{1}{\alpha}}. \tag{2.9}$$

Clearly, this forms an implicit scheme since calculating the $(n+1)^{th}$ approximation requires knowing the derivative of the function at that same step. To address this challenge, we can substitute fractional Newton-type formula (1.2) to evaluate the $(n+1)^{th}$ term on the right-hand side. Thus the resulting new scheme is

$$y_{n+1} = a + \left((y_n - a)^\alpha - \alpha \frac{2f(y_n)}{(T^a_\alpha f)(y^*_{n+1}) + (T^a_\alpha f)(y_n)} \right)^{\frac{1}{\alpha}}, \qquad n = 0, 1, 2, ..., \tag{2.10}$$

where $y^*_{n+1} = a + \left((y_n - a)^\alpha - \alpha \frac{f(y_n)}{(T^a_\alpha f)(y_n)} \right)^{\frac{1}{\alpha}}$.

It is important to observe that when $\alpha = 1$, the proposed method simplifies to the well-known **Weerakoon-Fernando** method [25], which is expressed as follows:

$$y_{n+1} = y_n - \frac{2f(y_n)}{f'(y^*_{n+1}) + f'(y_n)}, \quad n = 0, 1, 2, ..., \quad where \;\; y^*_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \tag{2.11}$$

So, let us call the newly proposed method (2.10) as conformable Weerakoon-Fernando method (CWFM). A proof demonstrating the convergence order of method (2.10) is presented below:

Theorem 2.1 *Assume that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function in the interval I enclosing the zero $\bar{y}$ of $f(y)$. Suppose $(T^a_\alpha f)(y)$ be the conformable fractional derivative of $f(y)$ starting from $a \in \mathbb{R}$, with order α, for any $0 < \alpha \leq 1$. Also suppose that $(T^a_\alpha f)(y) \neq 0$, at $\bar{y}$ and is continuous. Let f is α-differentiable sufficient number of times in I. Furthermore, let y_0 is an initial approximation, closed enough to $\bar{y}$ and $\{y_n\}_{\geqslant 1}$ denotes the sequence generated by the iterative scheme (2.10), such that $a < y_n$, for all n. Then the local convergence order of the conformable fractional iterative scheme (2.10) is at least 3 and the error equation is*

$$e_{n+1} = \alpha^2 \left(C_2^2 + \frac{C_3}{2} \right) (\bar{y} - a)^{2\alpha - 2} e_n^3 + O(e_n^4),$$

being $C_k = \frac{1}{k!\alpha^{k-1}} \frac{(T^a_\alpha f)^{(k)}(\bar{y})}{(T^a_\alpha f)(\bar{y})}$, for $k = 2, 3, 4, ...$

Proof: The Taylor series expansion (1.9) of $f(y)$ at y_n around $\bar{y}$ is given by

$$f(y_n) = \frac{(T^a_\alpha f)(\bar{y})}{\alpha} \left[\delta_1 + C_2\delta_1^2 + C_3\delta_1^3 + C_4\delta_1^4 + ... \right],$$

being $C_k = \frac{1}{k!\alpha^{k-1}} \frac{(T^a_\alpha f)^{(k)}(\bar{y})}{(T^a_\alpha f)(\bar{y})}$, for $k = 2, 3, 4, ...$ Using $\delta_1 = H^\alpha - L^\alpha$, $H = y_n - a = e_n + \bar{y} - a$, $L = \bar{y} - a$, we obtain

$$\begin{aligned} f(y_n) = &\frac{(T^a_\alpha f)(\bar{y})}{\alpha} \left[((e_n + \bar{y} - a)^\alpha - (\bar{y} - a)^\alpha) + C_2((e_n + \bar{y} - a)^\alpha - (\bar{y} - a)^\alpha)^2 \right. \\ &\left. + C_3((e_n + \bar{y} - a)^\alpha - (\bar{y} - a)^\alpha)^3 + C_4((e_n + \bar{y} - a)^\alpha - (\bar{y} - a)^\alpha)^4 \right] + O(e_n^5). \end{aligned}$$

Now, applying the Newton's Binomial theorem [7] for fractional power, given by

$$(y_1 + y_2)^r = \sum_{n=0}^{\infty} \binom{r}{n} y_1^{r-n} y_2^n, \qquad n \in \{0\} \cup \mathbb{N},$$

where

$$\binom{r}{n} = \frac{\Gamma(r+1)}{n!\Gamma(r-n+1)}, \qquad n \in \{0\} \cup \mathbb{N},$$

we obtain

$$\begin{aligned} f(y_n) = &\frac{(T^a_\alpha f)(\bar{y})}{\alpha} \left[\alpha(\bar{y} - a)^{\alpha-1} e_n + \left(\frac{\alpha}{2}(\alpha - 1)(\bar{y} - a)^{\alpha-2} + \alpha^2(\bar{y} - a)^{2\alpha-2} C_2 \right) e_n^2 \right. \\ &\left. + \left(\frac{\alpha}{6}(\alpha - 1)(\alpha - 2)(\bar{y} - a)^{\alpha-3} + \alpha^2(\alpha - 1)(\bar{y} - a)^{2\alpha-3} C_2 + \alpha^3(\bar{y} - a)^{3\alpha-3} C_3 \right) e_n^3 \right] + O(e_n^4), \end{aligned} \tag{2.12}$$

and again using (1.9), we get the conformable fraction derivative of $f(y_n)$ as

$$\begin{aligned}(T^a_\alpha f)(y_n) =&\frac{(T^a_\alpha f)(\bar{y})}{\alpha}\big[\alpha + 2\alpha^2(\bar{y}-a)^{\alpha-1}C_2 e_n \\ &+ \big(\alpha^2(\alpha-1)(\bar{y}-a)^{\alpha-2}C_2 + 3\alpha^3(\bar{y}-a)^{2\alpha-2}C_3\big)\, e_n^2 + O(e_n^3).\end{aligned} \tag{2.13}$$

Using (2.12) and (2.13), we can write

$$\frac{f(y_n)}{(T^a_\alpha f)(y_n)} = \frac{1}{\alpha}\big[\alpha(\bar{y}-a)^{\alpha-1}e_n + \Big(\frac{\alpha}{2}(\alpha-1)(\bar{y}-a)^{\alpha-2} - \alpha^2 C_2(\bar{y}-a)^{2\alpha-2}\Big)\, e_n^2\big] + O(e_n^3),$$

and then

$$(y_n-a)^\alpha - \alpha\frac{f(y_n)}{(T^a_\alpha f)(y_n)} = (\bar{y}-a)^\alpha + C_2\alpha^2(\bar{y}-a)^{2\alpha-2}e_n^2 + O(e_n^3).$$

Again, utilizing the generalized Binomial theorem for fractional powers, we get

$$\left((y_n-a)^\alpha - \alpha\frac{f(y_n)}{(T^a_\alpha f)(y_n)}\right)^{\frac{1}{\alpha}} = \bar{y} - a + C_2\alpha(\bar{y}-a)^{\alpha-1}e_n^2 + O(e_n^3). \tag{2.14}$$

Now

$$y^*_{n+1} = a + \left((y_n-a)^\alpha - \alpha\frac{f(y_n)}{(T^a_\alpha f)(y_n)}\right)^{\frac{1}{\alpha}}. \tag{2.15}$$

Using $y^*_{n+1} = e^*_{n+1} + \bar{y}$ and (2.14) in above equation, we obtain

$$e^*_{n+1} = C_2\alpha(\bar{y}-a)^{\alpha-1}e_n^2 + O(e_n^3). \tag{2.16}$$

From (2.13), one can write

$$\begin{aligned}(T^a_\alpha f)(y^*_{n+1}) =&\frac{(T^a_\alpha f)(\bar{y})}{\alpha}\Bigg[\alpha + 2\alpha^2(\bar{y}-a)^{\alpha-1}C_2 e^*_{n+1} + \Big(\alpha^2(\alpha-1)(\bar{y}-a)^{\alpha-2}C_2 \\ &+ 3\alpha^3(\bar{y}-a)^{2\alpha-2}C_3\Big)(e^*_{n+1})^2\Bigg] + O((e^*_{n+1})^3).\end{aligned} \tag{2.17}$$

From (2.16) and (2.17), it can be written as

$$(T^a_\alpha f)(y^*_{n+1}) = (T^a_\alpha f)(\bar{y})\big[1 + 2\alpha^2(\bar{y}-a)^{2\alpha-2}C_2^2 e_n^2 + O(e_n^3). \tag{2.18}$$

And so

$$\begin{aligned}(T^a_\alpha f)(y^*_{n+1}) + (T^a_\alpha f)(y_n) =&2(T^a_\alpha f)(\bar{y})\Bigg[1 + \alpha(\bar{y}-a)^{\alpha-1}C_2 e_n + \frac{1}{2}\Big(\alpha(\alpha-1)(\bar{y}-a)^{\alpha-2}C_2 \\ &+ \alpha^2(2C_2^2 + 3C_3)(\bar{y}-a)^{2\alpha-2}\Big)e_n^2\Bigg] + O(e_n^3).\end{aligned} \tag{2.19}$$

From (2.12) and (2.19), we obtain

$$\begin{aligned}\frac{2f(y_n)}{(T^a_\alpha f)(y^*_{n+1}) + (T^a_\alpha f)(y_n)} =&\frac{1}{\alpha}\Bigg[\alpha(\bar{y}-a)^{\alpha-1}e_n + \frac{1}{2}\alpha(\alpha-1)(\bar{y}-a)^{\alpha-2}e_n^2 \\ &+ \left(\frac{1}{6}\alpha(\alpha-1)(\alpha-2)(\bar{y}-a)^{\alpha-3} - \alpha^3\left(C_2^2 + \frac{C_3}{2}\right)(\bar{y}-a)^{3\alpha-3}\right)e_n^3\Bigg] + O(e_n^4),\end{aligned}$$

and after that

$$(y_n-a)^\alpha - \alpha\frac{2f(y_n)}{(T^a_\alpha f)(y^*_{n+1}) + (T^a_\alpha f)(y_n)} = (\bar{y}-a)^\alpha + \alpha^3\left(C_2^2 + \frac{C_3}{2}\right)(\bar{y}-a)^{3\alpha-3}e_n^3 + O(e_n^4).$$

Again, applying the generalized Binomial theorem for fractional powers, we get

$$\left((y_n-a)^\alpha - \alpha\frac{2f(y_n)}{(T^a_\alpha f)(y^*_{n+1}) + (T^a_\alpha f)(y_n)}\right)^{\frac{1}{\alpha}} = \bar{y} - a + \alpha^2\left(C_2^2 + \frac{C_3}{2}\right)(\bar{y}-a)^{2\alpha-2}e_n^3 + O(e_n^4),$$

and using $y_{n+1} = e_{n+1} + \bar{y}$ and above equation, from (2.10) we obtain

$$e_{n+1} + \bar{y} = a + \bar{y} - a + \alpha^2 \left(C_2^2 + \frac{C_3}{2} \right) (\bar{y} - a)^{2\alpha - 2} e_n^3 + O(e_n^4).$$

Hence, the error equation is

$$e_{n+1} = \alpha^2 \left(C_2^2 + \frac{C_3}{2} \right) (\bar{y} - a)^{2\alpha - 2} e_n^3 + O(e_n^4).$$

The proof is completed. □

Generalized case:

To effectively apply the fractional iterative technique (2.10) to a multi-dimensional function $F : \mathbb{R}^m \to \mathbb{R}^m$, it must first be suitably reformulated into its generalized form. To construct multi-dimensional fractional iterative method consider the second-order approximation of the Taylor series expansion (1.15), around the solution $\bar{y}$, as

$$F(y) \approx F(\bar{y}) + \frac{F_a^{\alpha(1)}(\bar{y})}{\alpha} \Delta + \frac{F_a^{\alpha(2)}(\bar{y})}{2!\alpha^2} \Delta^2. \tag{2.20}$$

And hence

$$F_a^{\alpha(1)}(y) \approx F_a^{\alpha(1)}(\bar{y}) + \frac{F_a^{\alpha(2)}(\bar{y})}{\alpha} \Delta + \frac{F_a^{\alpha(3)}(\bar{y})}{2!\alpha^2} \Delta^2. \tag{2.21}$$

Taking up to order two, we can write

$$F_a^{\alpha(1)}(y) \approx F_a^{\alpha(1)}(\bar{y}) + \frac{F_a^{\alpha(2)}(\bar{y})}{\alpha} \Delta, \tag{2.22}$$

which gives

$$\frac{F_a^{\alpha(2)}(\bar{y})}{\alpha} \Delta \approx F_a^{\alpha(1)}(y) - F_a^{\alpha(1)}(\bar{y}). \tag{2.23}$$

Replacing (2.23) in (2.20), we find

$$F(y) \approx F(\bar{y}) + \frac{F_a^{\alpha(1)}(\bar{y})}{\alpha} \Delta + \frac{1}{2\alpha} (F_a^{\alpha(1)}(y) - F_a^{\alpha(1)}(\bar{y})) \Delta, \tag{2.24}$$

or

$$F(y) \approx F(\bar{y}) + \frac{1}{2\alpha} (F_a^{\alpha(1)}(\bar{y}) + F_a^{\alpha(1)}(y)) \Delta. \tag{2.25}$$

But $F(\bar{y}) = 0$ and $\Delta = H^{\odot\alpha} - L^{\odot\alpha}$, being $H = y - a$, $L = \bar{y} - a$, so (2.25) can be rewritten as

$$F(y) \approx \frac{1}{2\alpha} (F_a^{\alpha(1)}(\bar{y}) + F_a^{\alpha(1)}(y))((y - a)^{\odot\alpha} - (\bar{y} - a)^{\odot\alpha}). \tag{2.26}$$

Multiplying by $2\alpha(F_a^{\alpha(1)}(\bar{y}) + F_a^{\alpha(1)}(y))^{-1}$ from left of both sides of (2.26),

$$2\alpha(F_a^{\alpha(1)}(\bar{y}) + F_a^{\alpha(1)}(y))^{-1} F(y) \approx ((y - a)^{\odot\alpha} - (\bar{y} - a)^{\odot\alpha}), \tag{2.27}$$

which implies

$$\bar{y} \approx a + \left[(y - a)^{\odot\alpha} - 2\alpha(F_a^{\alpha(1)}(\bar{y}) + F_a^{\alpha(1)}(y))^{-1} F(y) \right]^{\odot\frac{1}{\alpha}}. \tag{2.28}$$

Regarding the iterates y_n and y_{n+1} as approximations to the solution $\bar{y}$, we derive

$$y_{n+1} = a + \left[(y_n - a)^{\odot\alpha} - 2\alpha(F_a^{\alpha(1)}(y_{n+1}) + F_a^{\alpha(1)}(y_n))^{-1} F(y_n) \right]^{\odot\frac{1}{\alpha}}. \tag{2.29}$$

But, the above formula is implicit since calculating the $(n + 1)^{th}$ approximation requires knowing the derivative of the function at that same step. To overcome this problem, we can substitute fractional

Newton-type method (1.5) to evaluate the $(n+1)^{th}$ term on the right-hand side. Thus the resulting conformable vector Weerakoon-Fernando method is

$$y_{n+1} = a + \left[(y_n - a)^{\odot\alpha} - 2\alpha(F_a^{\alpha(1)}(y_{n+1}^*) + F_a^{\alpha(1)}(y_n))^{-1}F(y_n)\right]^{\odot\frac{1}{\alpha}}, \quad n = 0, 1, 2, ..., \tag{2.30}$$

where $y_{n+1}^* = a + \left[(y_n - a)^{\odot\alpha} - \alpha[F_a^{\alpha(1)}(y_n)]^{-1}F(y_n)\right]^{\odot\frac{1}{\alpha}}$.

For $\alpha = 1$, method (2.30) simplifies to the generalized Weerakoon-Fernando method (for systems of nonlinear equations) (see [11], [19]), given by

$$y_{n+1} = y_n - 2[F'(y_{n+1}^*) + F'(y_n)]^{-1}F(y_n), \qquad n = 0, 1, 2..., \tag{2.31}$$

where $y_{n+1}^* = y_n - [F'(y_n)]^{-1}F(y_n)$ and $F'(y_n)$ is the classical Jacobian matrix.

Theorem 2.2 *Assume that $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function in an open convex set $D \subseteq \mathbb{R}^m$ enclosing a zero $\bar{y} \in \mathbb{R}^m$ of $F(y)$. Suppose $F_a^{\alpha(1)}(y)$ be the conformable Jacobian matrix of $F(y)$ starting from $a \in \mathbb{R}^m$, with order α, for any $0 < \alpha \leq 1$. Also suppose that $F_a^{\alpha(1)}(y)$ is continuous and non-singular at $\bar{y}$. Let F is α-differentiable sufficient number of times in D. Furthermore, let $y_0 \in \mathbb{R}^m$ is an initial approximation, closed enough to $\bar{y}$ and $\{y_n\}_{\geqslant 1}$ denotes the sequence generated by the iterative scheme (2.30), such that $a < y_n$, for all n. Then the local convergence order of the conformable fractional iterative scheme (2.30) is at least 3 and the error equation is*

$$e_{n+1} = \alpha^2 \left(A_2^2 + \frac{A_3}{2}\right)(\bar{y} - a)^{\odot(2\alpha-2)}e_n^3 + O(e_n^4),$$

being $A_k = \frac{1}{k!\alpha^{k-1}}[F_a^{\alpha(1)}(\bar{y})]^{-1}F_a^{\alpha(k)}(\bar{y}), \quad$ for $k = 2, 3, 4, ...$

Proof: Applying Theorem 1.3 and using the relation $y_n = e_n + \bar{y}$, the conformable vector Taylor expansion of $F(y)$ at y_n around $\bar{y}$ can be expressed as:

$$F(y_n) = \frac{F_a^{\alpha(1)}(\bar{y})}{\alpha}[\Delta + A_2\Delta^2 + A_3\Delta^3] + O(e_n^4), \tag{2.32}$$

being $A_k = \frac{1}{k!\alpha^{k-1}}[F_a^{\alpha(1)}(\bar{y})]^{-1}F_a^{\alpha(k)}(\bar{y}), \quad$ for $k = 2, 3, 4, ...$ Using $\Delta = H^{\odot\alpha} - L^{\odot\alpha}$, $H = y_n - a$, $L = \bar{y} - a$, where $\odot$ be the Hadamard Power, we can write

$$\begin{aligned} F(y_n) =& \frac{F_a^{\alpha(1)}(\bar{y})}{\alpha}\Big[((e_n + \bar{y} - a)^{\odot\alpha} - (\bar{y} - a)^{\odot\alpha}) + A_2((e_n + \bar{y} - a)^{\odot\alpha} - (\bar{y} - a)^{\odot\alpha})^2 \\ &+ A_3((e_n + \bar{y} - a)^{\odot\alpha} - (\bar{y} - a)^{\odot\alpha})^3\Big] + O(e_n^4). \end{aligned} \tag{2.33}$$

Using Theorem 1.4, we can obtain the following form

$$\begin{aligned} F(y_n) =& \frac{F_a^{\alpha(1)}(\bar{y})}{\alpha}\Big[\alpha(\bar{y} - a)^{\odot(\alpha-1)}e_n + \left(\frac{1}{2}\alpha(\alpha-1)(\bar{y} - a)^{\odot(\alpha-2)} + A_2\alpha^2(\bar{y} - a)^{\odot(2\alpha-2)}\right)e_n^2 \\ &+ \left(\frac{1}{6}\alpha(\alpha-1)(\alpha-2)(\bar{y} - a)^{\odot(\alpha-3)} + A_2\alpha^2(\alpha-1)(\bar{y} - a)^{\odot(2\alpha-3)} + A_3\alpha^3(\bar{y} - a)^{\odot(3\alpha-3)}\right)e_n^3\Big] + O(e_n^4). \end{aligned} \tag{2.34}$$

Now the conformable Jacobian matrix $F_a^{\alpha(1)}(y_n)$ of $F(y_n)$ can be written as

$$\begin{aligned} F_a^{\alpha(1)}(y_n) =& \frac{F_a^{\alpha(1)}(\bar{y})}{\alpha}\Big[\alpha I + 2A_2\alpha^2(\bar{y} - a)^{\odot(\alpha-1)}e_n \\ &+ \left(A_2\alpha^2(\alpha-1)(\bar{y} - a)^{\odot(\alpha-2)} + 3A_3\alpha^3(\bar{y} - a)^{\odot(2\alpha-2)}\right)e_n^2\Big] + O(e_n^3). \end{aligned} \tag{2.35}$$

Using (2.34) and (2.35), we can obtain

$$[F_a^{\alpha(1)}(y_n)]^{-1}F(y_n) = \frac{1}{\alpha}\left[\alpha(\bar{y}-a)^{\odot(\alpha-1)}e_n + \left(\frac{1}{2}\alpha(\alpha-1)(\bar{y}-a)^{\odot(\alpha-2)} - A_2\alpha^2(\bar{y}-a)^{\odot(2\alpha-2)}\right)e_n^2\right] + O(e_n^3). \quad (2.36)$$

Again using Theorem 1.4, we get

$$(y_n-a)^{\odot\alpha} - \alpha[F_a^{\alpha(1)}(y_n)]^{-1}F(y_n) = (\bar{y}-a)^{\odot\alpha} + A_2\alpha^2(\bar{y}-a)^{\odot(2\alpha-2)}e_n^2 + O(e_n^3). \quad (2.37)$$

And then

$$\left[(y_n-a)^{\odot\alpha} - \alpha[F_a^{\alpha(1)}(y_n)]^{-1}F(y_n)\right]^{\odot\frac{1}{\alpha}} = \bar{y} - a + A_2\alpha(\bar{y}-a)^{\odot(\alpha-1)}e_n^2 + O(e_n^3). \quad (2.38)$$

Now

$$y_{n+1}^* = a + \left[(y_n-a)^{\odot\alpha} - \alpha[F_a^{\alpha(1)}(y_n)]^{-1}F(y_n)\right]^{\odot\frac{1}{\alpha}}, \quad (2.39)$$

using $y_{n+1}^* = e_{n+1}^* + \bar{y}$ and (2.38) in above equation, we get

$$e_{n+1}^* = A_2\alpha(\bar{y}-a)^{\odot(\alpha-1)}e_n^2 + O(e_n^3). \quad (2.40)$$

From (2.35), we can write

$$F_a^{\alpha(1)}(y_{n+1}^*) = \frac{F_a^{\alpha(1)}(\bar{y})}{\alpha}\left[\alpha I + 2A_2\alpha^2(\bar{y}-a)^{\odot(\alpha-1)}e_{n+1}^* + \left(A_2\alpha^2(\alpha-1)(\bar{y}-a)^{\odot(\alpha-2)} + 3A_3\alpha^3(\bar{y}-a)^{\odot(2\alpha-2)}\right)(e_{n+1}^*)^2\right] + O((e_{n+1}^*)^3). \quad (2.41)$$

Utilizing (2.40) in (2.41), we obtain

$$F_a^{\alpha(1)}(y_{n+1}^*) = F_a^{\alpha(1)}(\bar{y})\left[I + 2A_2^2\alpha^2(\bar{y}-a)^{\odot(2\alpha-2)}e_n^2\right] + O(e_n^3). \quad (2.42)$$

So that

$$F_a^{\alpha(1)}(y_{n+1}^*) + F_a^{\alpha(1)}(y_n) = 2F_a^{\alpha(1)}(\bar{y})\left[I + A_2\alpha(\bar{y}-a)^{\odot(\alpha-1)}e_n + \frac{1}{2}\left(A_2\alpha(\alpha-1)(\bar{y}-a)^{\odot(\alpha-2)} + (2A_2^2+3A_3)\alpha^2(\bar{y}-a)^{\odot(2\alpha-2)}\right)e_n^2\right] + O(e_n^3). \quad (2.43)$$

From (2.34) and (2.43), one can find

$$2\alpha[F_a^{\alpha(1)}(y_{n+1}^*) + F_a^{\alpha(1)}(y_n)]^{-1}F(y_n) = \alpha(\bar{y}-a)^{\odot(\alpha-1)}e_n + \frac{1}{2}\alpha(\alpha-1)(\bar{y}-a)^{\odot(\alpha-2)}e_n^2 + \left(\frac{1}{6}\alpha(\alpha-1)(\alpha-2)(\bar{y}-a)^{\odot(\alpha-3)} - \alpha^3(A_2^2 + \frac{A_3}{2})(\bar{y}-a)^{\odot(3\alpha-3)}\right)e_n^3 + O(e_n^4). \quad (2.44)$$

Again employing Theorem (1.4), we have

$$(y_n-a)^{\odot\alpha} - 2\alpha[F_a^{\alpha(1)}(y_{n+1}^*) + F_a^{\alpha(1)}(y_n)]^{-1}F(y_n) = (\bar{y}-a)^{\odot\alpha} + \alpha^3\left(A_2^2 + \frac{A_3}{2}\right)(\bar{y}-a)^{\odot(3\alpha-3)}e_n^3 + O(e_n^4). \quad (2.45)$$

And hence

$$\left[(y_n-a)^{\odot\alpha} - 2\alpha[F_a^{\alpha(1)}(y_{n+1}^*) + F_a^{\alpha(1)}(y_n)]^{-1}F(y_n)\right]^{\odot\frac{1}{\alpha}} = \bar{y} - a + \alpha^2\left(A_2^2 + \frac{A_3}{2}\right)(\bar{y}-a)^{\odot(2\alpha-2)}e_n^3 + O(e_n^4). \quad (2.46)$$

Now using $y_{n+1} = e_{n+1} + \bar{y}$ and above equation, in (2.30), we can see that

$$e_{n+1} = \alpha^2\left(A_2^2 + \frac{A_3}{2}\right)(\bar{y}-a)^{\odot(2\alpha-2)}e_n^3 + O(e_n^4). \quad (2.47)$$

The proof is completed. □

3. Numerical Results and Comparisons

This section analyzes the numerical performance of the proposed fractional iterative method (2.10) and its multidimensional form (2.30) by evaluating their accuracy and rate of convergence. For this evaluation, various test problems have been chosen, including single-variable nonlinear equations (where $m = 1$) and multi-variable systems (where $m > 1$), to thoroughly assess the method's performance across different types of equations. Here we compare the proposed conformable Weerakoon-Fernando method (CWFM) mentioned in (2.10) with the existing conformable Traub-type method (TeCO) mentioned in (1.3) and conformable Chun-Kim-type method (CKeCO) mentioned in (1.4). It is crucial to highlight that each test (for single-variable nonlinear equations) include a comparison of method (2.10) with the Weerakoon-Fernando method (2.11) when $\alpha = 1$. Similarly proposed multi-dimensional method (2.30) is compared with conformable vector Traub's method (CVTM) mentioned in (1.6) and also each test (for multi-variable system) include a comparison of method (2.30) with the method (2.31) method when $\alpha = 1$. The entire set of calculations was carried out using the MATHEMATICA 14.1 software application with multiple precision arithmetic, the stopping criterion is $\parallel f(y_{n+1}) \parallel < 10^{-8}$ or $\parallel y_{n+1} - y_n \parallel < 10^{-8}$, with a limit of 500 iterations. To ensure that $a < y_n$, for all n, we use $a = -10$ when $m = 1$ and $a = (a_1, ..., a_m) = (-10, ..., -10)$ when $m > 1$. We also use the Approximated Computational Order of Convergence (ACOC) (see [12]) denoted by ρ,

$$\rho \approx \frac{\ln(\parallel y_{n+1} - y_n \parallel / \parallel y_n - y_{n-1} \parallel)}{\ln(\parallel y_n - y_{n-1} \parallel / \parallel y_{n-1} - y_{n-2} \parallel)}, \qquad n = 2, 3, 4, ..., \tag{3.1}$$

to confirm that the theoretical order of convergence is attained in practical computations. So as to evaluate and compare the methods performance the order α is varied within the interval $(0, 1]$, with a step size of 0.1. Considering the above aspects, the performance for each chosen problem will be evaluated with respect to (i) required number of iterations to reach convergence (Itr), (ii) error between successive iterations ($\parallel y_{n+1} - y_n \parallel$), (iii) function value at the last iteration ($\parallel f(y_{n+1}) \parallel$), and (iv) ACOC (ρ) . Also a comparison based on the error $\parallel y_{n+1} - y_n \parallel$ across iterations is graphically represented in Figures 1 to 8.

Problem 1. (Blood Rheology Model [15]) Blood rheology is the study of the properties of blood flow and its behavior within the circulatory system. Modeling blood rheology is essential for gaining insights into different physiological and pathological conditions associated with blood flow. Blood rheology is a medical discipline that examines the physical behavior and flow characteristics of blood. Numerical iterative methods are frequently employed to solve the mathematical equations that describe blood rheology. As blood is a non-Newtonian fluid, it is classified as a Caisson fluid. Based on this concept, the flow within a tube behaves like a plug with minimum deformation, and a velocity gradient forms near the wall. We consider the following nonlinear equation

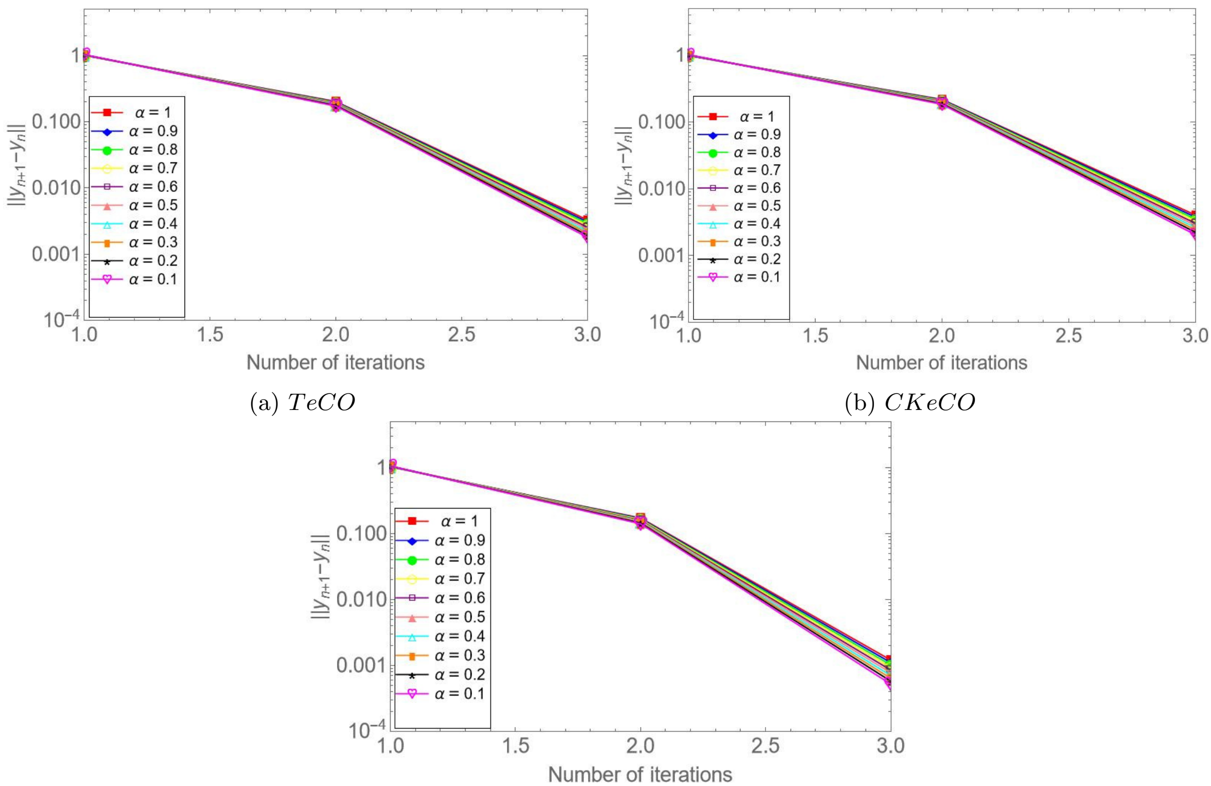
$$f_1(y) = \frac{1}{441}y^8 - \frac{8}{63}y^5 - 0.05714285714y^4 + \frac{16}{9}y^2 - 3.624489796y + 0.36, \tag{3.2}$$

to analyze the plug flow of Caisson fluid flow. Here, y represents the plug flow of Caisson fluid flow. One of the solutions to the $f_1(y)$ is $\bar{y_1} = 0.1046986515...$. To solve $f_1(y) = 0$, we start with an initial approximation $y_0 = -1.1$. Table 1 displays the calculated results of various methods.

Table 1: Numerical result of $TeCO$, $CKeCO$ and $CWFM$ for $f_1(y)$

	$TeCO$					$CKeCO$					$CWFM$				
α	$\bar{y}$	$\parallel f(y_{n+1}) \parallel$	$\parallel y_{n+1} - y_n \parallel$	Itr	ρ	$\bar{y}$	$\parallel f(y_{n+1}) \parallel$	$\parallel y_{n+1} - y_n \parallel$	Itr	ρ	$\bar{y}$	$\parallel f(y_{n+1}) \parallel$	$\parallel y_{n+1} - y_n \parallel$	Itr	ρ
1	$\bar{y_1}$	1.29×10^{-17}	2.08×10^{-8}	4	2.91	$\bar{y_1}$	4.26×10^{-17}	3.90×10^{-8}	4	2.89	$\bar{y_1}$	1.73×10^{-9}	1.21×10^{-3}	3	2.78
0.9	$\bar{y_1}$	1.29×10^{-15}	1.70×10^{-8}	4	2.91	$\bar{y_1}$	7.06×10^{-15}	3.11×10^{-8}	4	2.90	$\bar{y_1}$	1.31×10^{-9}	1.11×10^{-3}	3	2.78
0.8	$\bar{y_1}$	1.29×10^{-15}	1.38×10^{-8}	4	2.91	$\bar{y_1}$	7.06×10^{-15}	2.47×10^{-8}	4	2.90	$\bar{y_1}$	9.85×10^{-10}	1.02×10^{-3}	3	2.78
0.7	$\bar{y_1}$	4.48×10^{-15}	1.12×10^{-8}	4	2.91	$\bar{y_1}$	4.48×10^{-15}	1.95×10^{-8}	4	2.90	$\bar{y_1}$	7.35×10^{-10}	9.33×10^{-4}	3	2.78
0.6	$\bar{y_1}$	7.06×10^{-15}	9.05×10^{-9}	4	2.91	$\bar{y_1}$	7.06×10^{-15}	1.53×10^{-8}	4	2.90	$\bar{y_1}$	5.45×10^{-10}	8.53×10^{-4}	3	2.78
0.5	$\bar{y_1}$	4.48×10^{-15}	7.27×10^{-9}	4	2.92	$\bar{y_1}$	4.48×10^{-15}	1.20×10^{-8}	4	2.91	$\bar{y_1}$	4.01×10^{-10}	7.77×10^{-4}	3	2.78
0.4	$\bar{y_1}$	4.48×10^{-15}	5.82×10^{-9}	4	2.92	$\bar{y_1}$	4.48×10^{-15}	9.27×10^{-9}	4	2.91	$\bar{y_1}$	2.92×10^{-10}	7.06×10^{-4}	3	2.78
0.3	$\bar{y_1}$	1.61×10^{-14}	4.63×10^{-9}	4	2.92	$\bar{y_1}$	1.28×10^{-14}	7.15×10^{-9}	4	2.91	$\bar{y_1}$	2.11×10^{-10}	6.41×10^{-4}	3	2.78
0.2	$\bar{y_1}$	1.02×10^{-14}	3.67×10^{-9}	4	2.92	$\bar{y_1}$	1.02×10^{-14}	5.47×10^{-9}	4	2.91	$\bar{y_1}$	1.52×10^{-10}	5.79×10^{-4}	3	2.78
0.1	$\bar{y_1}$	1.86×10^{-14}	2.90×10^{-9}	4	2.92	$\bar{y_1}$	1.86×10^{-14}	4.16×10^{-9}	4	2.91	$\bar{y_1}$	1.08×10^{-10}	5.22×10^{-4}	3	2.78

Table 1 shows that the $CWFM$ requires fewer iterations than both $TeCO$ and $CKeCO$ for the same value of α. For proposed method Approximated Computational Order of Convergence ρ is approximately equal to 3, which is equal to theoretical result. We can also see in Table 1 that the error $\| y_{n+1} - y_n \|$ of conformable Weerakoon-Fernando method ($\alpha < 1$) is less than conformable Weerakoon-Fernando method ($\alpha = 1$). Conformable Weerakoon-Fernando method is the classical Weerakoon-Fernando method when $\alpha = 1$. In Figure 1, no erratic behavior is observed, as the errors consistently decrease with each iteration for proposed method $CWFM$ and also error $\| y_{n+1} - y_n \|$ in $CWFM$ decreases more swiftly compared to both $TeCO$ and $CKeCO$ for same α value, as well as compared to classical Weerakoon-Fernando method.



(a) $TeCO$ (b) $CKeCO$

(c) $CWFM$

Figure 1: Error plots of $f_1(y)$ across all α values provided in Table 1

Problem 2. Location of Maximum Energy Distribution [15]
Consider Planck's radiation law

$$\gamma = \frac{8\pi ch\lambda^{-5}}{e^{\frac{ch}{\lambda kT}} - 1},$$

where γ is energy density, λ is the wavelength radiation, T is the absolute temperature, k is Boltzmann's constant, h is the Planck's constant, and c is the speed of light. To determine the wavelength λ and maximize the energy density, we first evaluate

$$\frac{d\gamma}{d\lambda} = \frac{8\pi ch\lambda^{-5}}{e^{\frac{ch}{\lambda kT}} - 1}\left(-5 + \frac{(\frac{ch}{\lambda kT})e^{\frac{ch}{\lambda kT}}}{e^{\frac{ch}{\lambda kT}} - 1}\right).$$

As $\lambda \to 0$ and $\lambda \to \infty$, the terms inside the parentheses on the left side become zero, even though the energy density shows minimum in both scenarios. We obtain the desire maximum when the parenthetical terms become zero. Which gives,

$$\frac{(\frac{ch}{\lambda_{max}kT})e^{\frac{ch}{\lambda_{max}kT}}}{e^{\frac{ch}{\lambda_{max}kT}}-1} = 5,$$

where λ_{max} denotes the wavelength to maximize the energy density. For $y = \frac{ch}{\lambda_{max}kT}$, the above equation can be rewritten as

$$1 - \frac{y}{5} = e^{-y}.$$

Next, we define the following non-linear equation,

$$f_2(y) = e^{-y} - 1 + \frac{y}{5}.$$

The problem involves solving the above nonlinear equation, which yields two roots: $\bar{y_1} = 0$ and $\bar{y_2} \approx 4.965114232$. Tables 2 shows the numerical performance of various methods for $f_2(y)$ with initial approximation $y_0 = 2$.

Table 2: Numerical result of $TeCO$, $CKeCO$ and $CWFM$ for $f_2(y)$

	$TeCO$					$CKeCO$					$CWFM$				
α	$\bar{y}$	$\lVert f(y_{n+1}) \rVert$	$\lVert y_{n+1} - y_n \rVert$	Itr	ρ	$\bar{y}$	$\lVert f(y_{n+1}) \rVert$	$\lVert y_{n+1} - y_n \rVert$	Itr	ρ	$\bar{y}$	$\lVert f(y_{n+1}) \rVert$	$\lVert y_{n+1} - y_n \rVert$	Itr	ρ
1	$\bar{y_1}$	7.96×10^{-16}	1.08×10^{-5}	7	2.89	$\bar{y_1}$	9.14×10^{-18}	2.35×10^{-6}	206	2.90	$\bar{y_2}$	2.03×10^{-14}	3.39×10^{-4}	3	3.98
0.9	$\bar{y_1}$	4.07×10^{-12}	1.88×10^{-4}	7	2.85	$\bar{y_2}$	9.19×10^{-13}	1.32×10^{-3}	5	1.16	$\bar{y_2}$	4.23×10^{-15}	2.04×10^{-4}	3	3.88
0.8	$\bar{y_1}$	1.39×10^{-9}	1.32×10^{-3}	7	2.84	$\bar{y_1}$	3.65×10^{-12}	1.75×10^{-4}	6	3.72	$\bar{y_2}$	4.59×10^{-16}	1.04×10^{-4}	3	3.76
0.7	$\bar{y_1}$	0	9.74×10^{-8}	8	2.94	$\bar{y_1}$	1.24×10^{-11}	2.64×10^{-4}	3	1.65	$\bar{y_2}$	2.26×10^{-16}	4.08×10^{-5}	3	3.62
0.6	$\bar{y_1}$	0	1.62×10^{-6}	8	2.91	$\bar{y_1}$	8.24×10^{-10}	1.08×10^{-3}	4	2.84	$\bar{y_2}$	1.16×10^{-16}	9.56×10^{-6}	3	3.44
0.5	$\bar{y_1}$	1.42×10^{-15}	1.16×10^{-5}	8	2.89	$\bar{y_1}$	6.25×10^{-14}	4.58×10^{-5}	5	2.87	$\bar{y_2}$	4.59×10^{-16}	4.86×10^{-7}	3	3.15
0.4	$\bar{y_1}$	5.97×10^{-14}	4.70×10^{-5}	8	2.87	$\bar{y_1}$	0	7.90×10^{-8}	6	2.94	$\bar{y_2}$	1.16×10^{-16}	1.31×10^{-7}	3	3.12
0.3	$\bar{y_1}$	1.11×10^{-12}	1.26×10^{-4}	8	2.86	$\bar{y_1}$	3.60×10^{-13}	8.33×10^{-5}	6	2.86	$\bar{y_2}$	4.59×10^{-16}	4.55×10^{-6}	3	3.79
0.2	$\bar{y_1}$	8.63×10^{-12}	2.51×10^{-4}	8	2.85	$\bar{y_1}$	$2.84{\times}10^{-15}$	3.37×10^{-8}	7	2.94	$\bar{y_2}$	4.59×10^{-16}	1.43×10^{-5}	3	4.60
0.1	$\bar{y_1}$	3.50×10^{-11}	4.02×10^{-4}	8	2.84	$\bar{y_1}$	9.95×10^{-15}	1.80×10^{-5}	7	2.88	$\bar{y_2}$	1.26×10^{-15}	3.70×10^{-6}	3	6.73

We can observe from Table 2, that the $CWFM$ takes less iterations than both $TeCO$ and $CKeCO$ for the same α. We can also observe from Table 2 that the error $\lVert y_{n+1} - y_n \rVert$ of conformable Weerakoon-Fernando method (when $\alpha < 1$) is less than conformable Weerakoon-Fernando method (when $\alpha = 1$). We can also see in Figure 2 that the error drop is better in proposed method than existing methods and the errors reduces steadily with each iteration for proposed method.

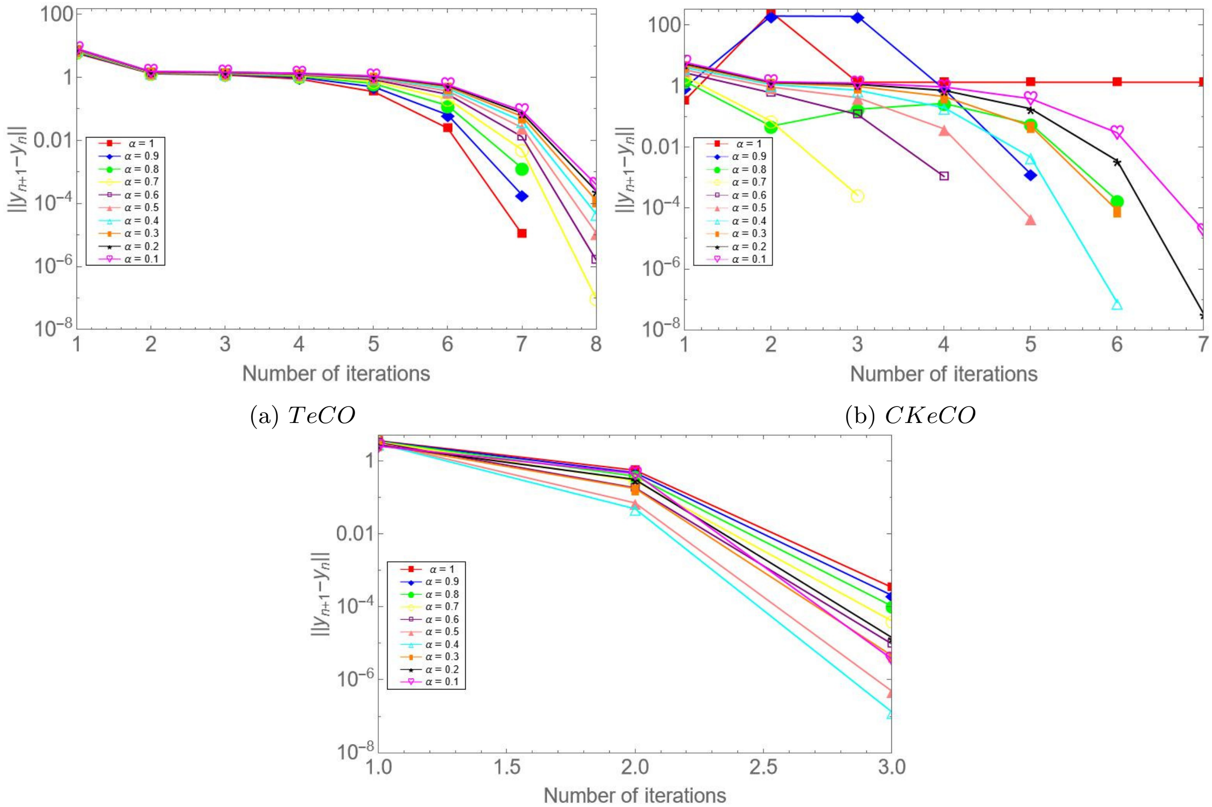
Problem 3. Vertical Stress [15]
Boussinesq's formula determines the vertical stress (s) at a particular point beneath the edge of a rectangular strip footing resting on an elastic medium, resulting from a uniformly distributed pressure q. The formula is given as:

$$\sigma_s = \frac{q}{\pi}y + Cos(y)Sin(y). \tag{3.3}$$

To calculate the value of y at which the vertical stress (s) equals 25 percentage of the applied footing pressure q, it is first necessary to determine the corresponding value of s. This involves identifying the depth at which the stress becomes one-fourth of q, which requires solving the following equation:

$$f_3(y) = \frac{y + Cos(y)Sin(y)}{\pi} - \frac{1}{4}. \tag{3.4}$$

The exact solution of the equation (3.4) is $\bar{y_1} = 0.415856....$ Tables 3 shows the numerical performance of various methods for $f_3(y)$ with initial approximation $y_0 = 1.1$.



(a) $TeCO$ (b) $CKeCO$

(c) $CWFM$

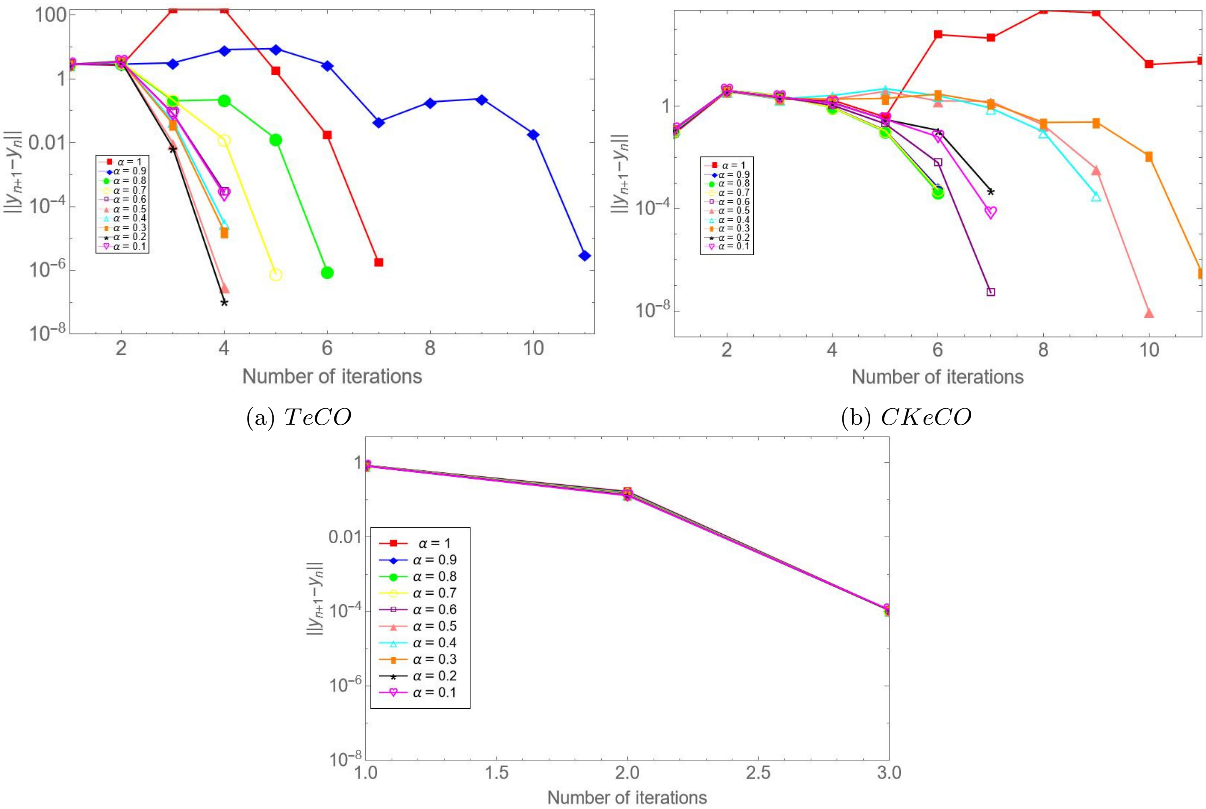
Figure 2: Error plots of $f_2(y)$ across all α values provided in Table 2

Table 3: Numerical result of $TeCO$, $CKeCO$ and $CWFM$ for $f_3(y)$

	$TeCO$					$CKeCo$					$CWFM$				
α	$\bar{y}$	$\parallel f(y_{n+1}) \parallel$	$\parallel y_{n+1} - y_n \parallel$	Itr	ρ	$\bar{y}$	$\parallel f(y_{n+1}) \parallel$	$\parallel y_{n+1} - y_n \parallel$	Itr	ρ	$\bar{y}$	$\parallel f(y_{n+1}) \parallel$	$\parallel y_{n+1} - y_n \parallel$	Itr	ρ
1	$\bar{y_1}$	1.01×10^{-18}	1.69×10^{-6}	7	1.99	$\bar{y_1}$	5.61×10^{-11}	7.43×10^{-4}	29	5.64	$\bar{y_1}$	4.34×10^{-14}	1.10×10^{-4}	3	4.53
0.9	$\bar{y_1}$	1.81×10^{-15}	3.18×10^{-6}	11	3.50	$\bar{y_1}$	3.82×10^{-11}	6.63×10^{-4}	6	2.49	$\bar{y_1}$	3.78×10^{-14}	1.08×10^{-4}	3	4.44
0.8	$\bar{y_1}$	8.63×10^{-16}	9.16×10^{-7}	6	3.40	$\bar{y_1}$	9.54×10^{-12}	4.24×10^{-4}	6	2.56	$\bar{y_1}$	3.21×10^{-14}	1.07×10^{-4}	3	4.35
0.7	$\bar{y_1}$	8.36×10^{-17}	7.78×10^{-7}	5	3.39	$\bar{y_1}$	1.93×10^{-11}	5.44×10^{-4}	6	2.55	$\bar{y_1}$	2.74×10^{-14}	1.07×10^{-4}	3	4.26
0.6	$\bar{y_1}$	4.52×10^{-12}	2.88×10^{-4}	4	1.52	$\bar{y_1}$	8.36×10^{-17}	5.28×10^{-8}	7	3.34	$\bar{y_1}$	2.36×10^{-14}	1.07×10^{-4}	3	4.18
0.5	$\bar{y_1}$	8.63×10^{-16}	3.10×10^{-7}	4	1.76	$\bar{y_1}$	8.63×10^{-16}	9.54×10^{-9}	10	3.27	$\bar{y_1}$	1.88×10^{-14}	1.07×10^{-4}	3	4.11
0.4	$\bar{y_1}$	4.82×10^{-15}	3.02×10^{-5}	4	1.71	$\bar{y_1}$	4.30×10^{-12}	3.46×10^{-4}	9	2.65	$\bar{y_1}$	1.41×10^{-14}	1.08×10^{-4}	3	4.04
0.3	$\bar{y_1}$	8.36×10^{-17}	1.66×10^{-5}	4	1.72	$\bar{y_1}$	8.36×10^{-17}	3.22×10^{-11}	11	3.48	$\bar{y_1}$	1.22×10^{-14}	1.09×10^{-4}	3	3.98
0.2	$\bar{y_1}$	1.03×10^{-15}	1.12×10^{-7}	4	1.78	$\bar{y_1}$	1.32×10^{-11}	5.21×10^{-4}	7	5.52	$\bar{y_1}$	6.54×10^{-15}	1.10×10^{-4}	3	3.92
0.1	$\bar{y_1}$	2.07×10^{-12}	2.30×10^{-4}	4	1.56	$\bar{y_1}$	2.93×10^{-14}	6.54×10^{-4}	7	4.37	$\bar{y_1}$	8.36×10^{-17}	1.12×10^{-4}	3	3.86

It can be observed in Table 3, that for the same value of α the $CWFM$ takes less iterations than both $TeCO$ and $CKeCO$. We can also notice that the error $\parallel y_{n+1} - y_n \parallel$ of conformable Weerakoon-Fernando method (for $0.3 \leqslant \alpha \leqslant 0.9$) is slightly less than conformable Weerakoon-Fernando method (for $\alpha = 1$). Figure 3 shows the absence of erratic trends for $CWFM$ due to the uniform decrease in errors ($\parallel y_{n+1} - y_n \parallel$) per iteration and compared to $TeCO$ and $CKeCO$ a more rapid drop in the error is found.

Problem 4. Consider the non-linear equation $f_4(y) = iy^{1.8} - y^{0.9} - 16$, with solutions $\bar{y_1} \approx 2.90807 - 4.24908i$, $\bar{y_2} \approx -3.85126 + 1.74602i$. Table 4 shows the numerical performance of the methods for $f_4(y)$ with initial approximation $y_0 = 0.5$. We can see in Table 4, that the $CWFM$ needs fewer iterations than both $TeCO$ and $CKeCO$ for the same α (except at $\alpha = 0.6$). For proposed method ρ is approximately equal to the theoretical convergence order. We can also see in Table 4 that the error $\parallel y_{n+1} - y_n \parallel$ of conformable Weerakoon-Fernando method (for $\alpha \leqslant 0.4$) is less than conformable Weerakoon-Fernando



(a) $TeCO$ (b) $CKeCO$

(c) $CWFM$

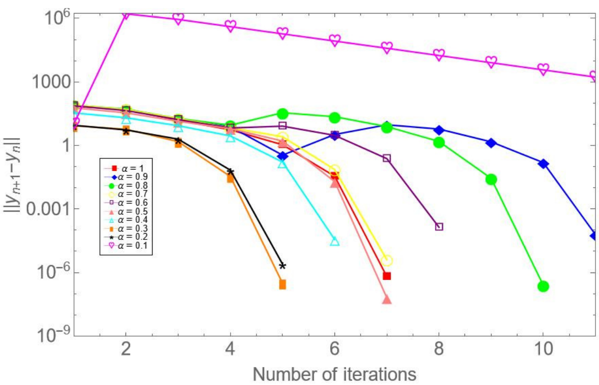
Figure 3: Error plots of $f_3(y)$ across all α values provided in Table 3

method (for $\alpha = 1$). Figure 4 reveals that in the proposed method, error $\| y_{n+1} - y_n \|$ decreases more rapidly than in the earlier proposed $TeCO$ and $CKeCO$ for same α value (except at $\alpha = 0.6$).

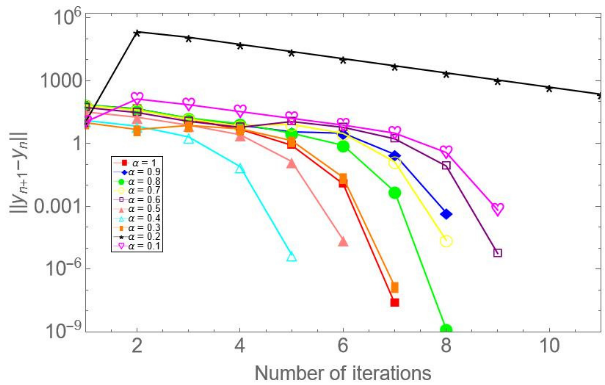
Table 4: Numerical result of $TeCO$, $CKeCO$ and $CWFM$ for $f_4(y)$

	$TeCo$					$CKeCo$					$CWFM$				
α	$\bar{y}$	$\| f(y_{n+1}) \|$	$\| y_{n+1} - y_n \|$	Itr	ρ	$\bar{y}$	$\| f(y_{n+1}) \|$	$\| y_{n+1} - y_n \|$	Itr	ρ	$\bar{y}$	$\| f(y_{n+1}) \|$	$\| y_{n+1} - y_n \|$	Itr	ρ
1	$\bar{y_1}$	4.44×10^{-15}	6.43×10^{-7}	7	3.18	$\bar{y_1}$	3.58×10^{-15}	2.34×10^{-8}	7	3.09	$\bar{y_1}$	3.58×10^{-15}	3.02×10^{-7}	5	2.91
0.9	$\bar{y_1}$	2.26×10^{-14}	6.19×10^{-5}	11	3.46	$\bar{y_1}$	1.14×10^{-11}	4.94×10^{-4}	8	2.76	$\bar{y_1}$	4.72×10^{-15}	6.39×10^{-5}	5	2.84
0.8	$\bar{y_2}$	8.75×10^{-15}	2.49×10^{-7}	10	2.90	$\bar{y_2}$	2.84×10^{-14}	1.35×10^{-9}	8	2.97	$\bar{y_1}$	2.07×10^{-14}	2.89×10^{-8}	6	3.02
0.7	$\bar{y_2}$	1.28×10^{-14}	4.02×10^{-6}	7	2.75	$\bar{y_2}$	8.75×10^{-15}	2.37×10^{-5}	8	2.76	$\bar{y_1}$	8.67×10^{-13}	2.47×10^{-4}	7	3.12
0.6	$\bar{y_2}$	1.57×10^{-13}	1.37×10^{-4}	8	2.95	$\bar{y_2}$	1.57×10^{-14}	5.55×10^{-6}	9	3.22	$\bar{y_1}$	3.58×10^{-15}	1.87×10^{-9}	11	2.97
0.5	$\bar{y_2}$	7.23×10^{-15}	6.41×10^{-8}	7	2.86	$\bar{y_2}$	3.23×10^{-14}	2.49×10^{-5}	6	2.94	$\bar{y_2}$	8.75×10^{-15}	6.93×10^{-7}	5	3.07
0.4	$\bar{y_2}$	8.74×10^{-15}	3.59×10^{-5}	6	2.97	$\bar{y_2}$	8.75×10^{-15}	4.86×10^{-6}	5	2.97	$\bar{y_2}$	1.28×10^{-14}	1.14×10^{-7}	5	3.03
0.3	$\bar{y_2}$	7.23×10^{-15}	3.10×10^{-7}	5	3.07	$\bar{y_2}$	1.57×10^{-14}	1.51×10^{-7}	7	2.96	$\bar{y_2}$	8.75×10^{-15}	3.89×10^{-8}	5	3.00
0.2	$\bar{y_2}$	1.57×10^{-14}	2.47×10^{-6}	5	3.03	$\bar{y_2}$	1.57×10^{-14}	5.19×10^{-9}	19	3.00	$\bar{y_2}$	4.23×10^{-14}	2.98×10^{-8}	5	2.99
0.1	$\bar{y_1}$	2.03×10^{-13}	4.53×10^{-9}	30	2.98	$\bar{y_2}$	2.30×10^{-11}	6.61×10^{-4}	9	2.97	$\bar{y_2}$	1.57×10^{-14}	4.99×10^{-8}	5	2.99

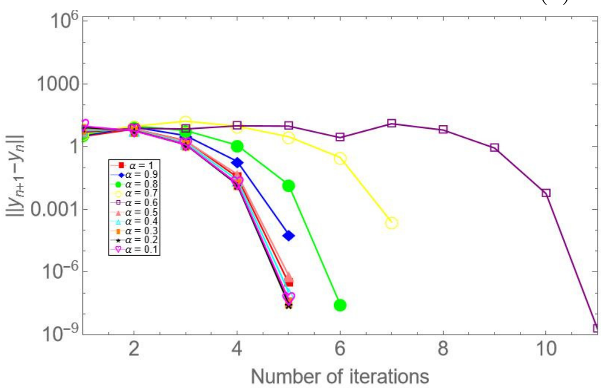
Problem 5. Consider the non-linear equation $f_5(y) = y^4 + 6y^2 + 25$, having four complex roots $\bar{y_1} = -1 - 2i$, $\bar{y_2} = -1 + 2i$, $\bar{y_3} = 1 - 2i$, $\bar{y_4} = 1 + 2i$. Clearly $f_5(y)$ is a polynomial of degree four with real coefficient and having only complex roots, so the classical Weerakoon-Fernando method (2.11) is not able to find the solution of $f_5(y)$ with real initial approximation. Table 5 present the numerical results of the $TeCO$, $CKeCO$ and $CWFM$ methods for $f_5(y)$ with initial approximation $y_0 = -1$. From Table 5, we observe that the $CWFM$ takes fewer iteration than $TeCO$ and $CKeCO$ for same value of α (except at $\alpha = 0.3$). From the table it is clear that the classical method (2.11) (when $\alpha = 1$) fails to converge to the roots. It can also be observed from the Figure 5 that the proposed method exhibits a faster reduction in error compared to the existing methods, with the error decreasing consistently at each iteration.



(a) $TeCO$



(b) $CKeCO$



(c) $CWFM$

Figure 4: Error plots of $f_4(y)$ across all α values provided in Table 4

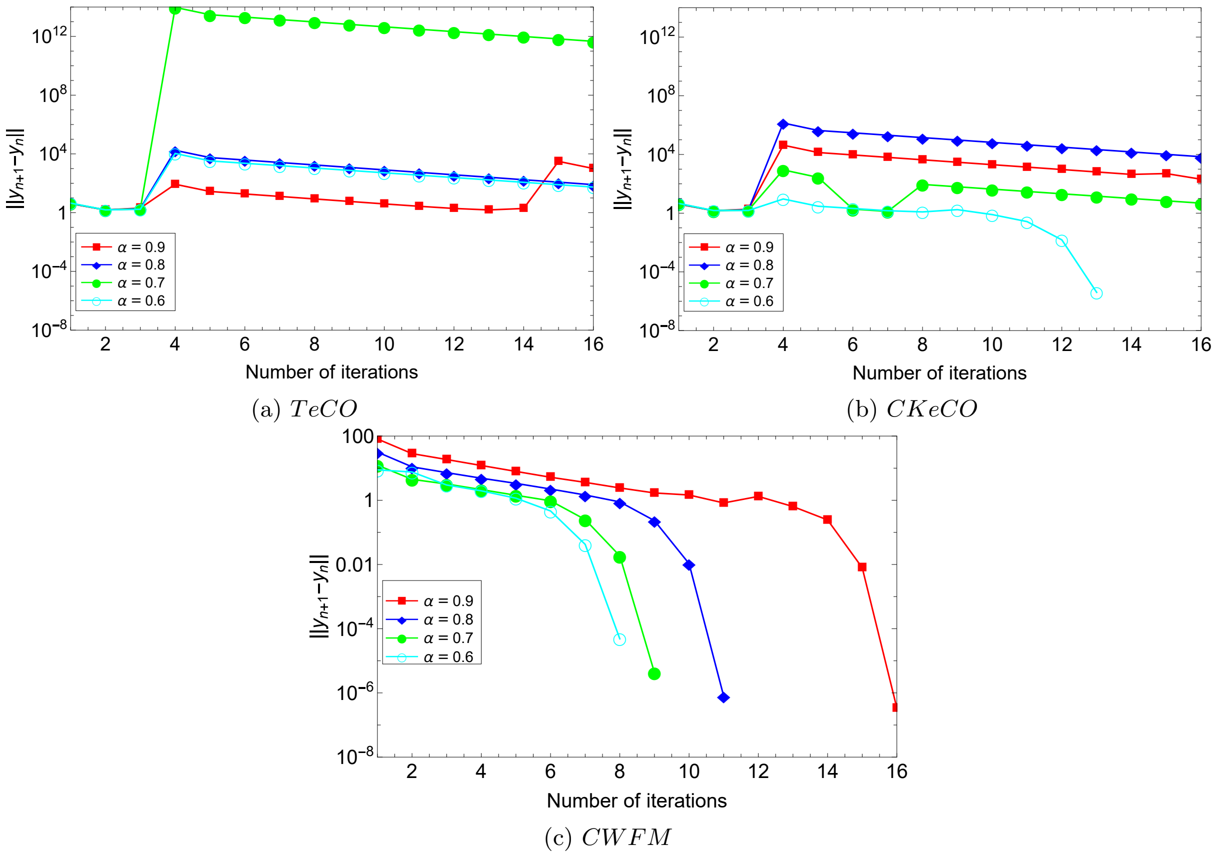
Table 5: Numerical result of $TeCO$, $CKeCO$ and $CWFM$ for $f_5(y)$

	$TeCo$					$CKeCo$					$CWFM$				
α	$\bar{y}$	$\| f(y_{n+1}) \|$	$\| y_{n+1}-y_n \|$	Itr	ρ	$\bar{y}$	$\| f(y_{n+1}) \|$	$\| y_{n+1}-y_n \|$	Itr	ρ	$\bar{y}$	$\| f(y_{n+1}) \|$	$\| y_{n+1}-y_n \|$	Itr	ρ
1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
0.9	$\bar{y_1}$	7.94×10^{-15}	5.50×10^{-6}	61	2.84	$\bar{y_1}$	1.97×10^{-13}	5.77×10^{-9}	57	3.00	$\bar{y_1}$	1.59×10^{-14}	3.39×10^{-7}	16	2.94
0.8	$\bar{y_1}$	1.97×10^{-13}	4.28×10^{-10}	72	3.01	$\bar{y_2}$	1.64×10^{-11}	7.20×10^{-5}	60	2.90	$\bar{y_1}$	1.93×10^{-13}	7.87×10^{-7}	11	3.08
0.7	$\bar{y_2}$	8.81×10^{-12}	6.00×10^{-5}	102	2.74	$\bar{y_1}$	7.94×10^{-15}	9.61×10^{-7}	24	3.18	$\bar{y_1}$	7.94×10^{-15}	4.32×10^{-6}	9	3.21
0.6	$\bar{y_3}$	6.55×10^{-14}	2.98×10^{-7}	30	2.96	$\bar{y_1}$	0	4.03×10^{-6}	13	2.87	$\bar{y_2}$	2.63×10^{-12}	4.84×10^{-5}	8	2.83
0.5	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
0.4	–	–	–	–	–	$\bar{y_2}$	2.57×10^{-13}	2.02×10^{-9}	90	2.95	$\bar{y_2}$	0	3.12×10^{-10}	43	3.01
0.3	–	–	–	–	–	$\bar{y_1}$	1.59×10^{-14}	4.22×10^{-8}	44	2.93	–	–	–	–	–
0.2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
0.1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–

Problem 6. Next, we consider the first vector valued function as $F_1(y_1,y_2)=(y_1^2+y_2^2-1, y_1^2-y_2^2-\frac{1}{2})^T$ having real roots $\bar{y_1}=(\sqrt{3}/2,1/2)^T\approx(0.8660,0.5)^T$, $\bar{y_2}=(-\sqrt{3}/2,1/2)^T\approx(-0.8660,0.5)^T$, $\bar{y_3}=(\sqrt{3}/2,-1/2)^T\approx(0.8660,-0.5)^T$ and $\bar{y_4}=(-\sqrt{3}/2,-1/2)^T\approx(-0.8660,-0.5)^T$. The conformable Jacobian matrix of $F_1(y_1,y_2)$ is

$$F_a^{\alpha(1)}(y_1,y_2)=\begin{pmatrix}(y_1-a_1)^{1-\alpha}(2y_1) & (y_2-a_2)^{1-\alpha}(2y_2)\\(y_1-a_1)^{1-\alpha}(2y_1) & (y_2-a_2)^{1-\alpha}(-2y_2)\end{pmatrix},$$

being $a=(a_1,a_2)=(-10,-10)$. Table 6 displays the numerical results of conformable vector Traub method ($CVTM$) and conformable vector Weerakoon-fernando method ($CVWFM$) for $F_1(y_1,y_2)$ with initial approximation $y_0=(1,-1.5)^T$. It is clear from Table 6 that for same value of α, $CVWFM$ needs fewer iteration than $CVTM$ and also conformable vector Weerakoon-Fernando method ($CVWFM$) (



(a) $TeCO$ (b) $CKeCO$

(c) $CWFM$

Figure 5: Error plots of $f_5(y)$ at $\alpha = 0.9, 0.8, 0.7, 0.6$ provided in Table 5

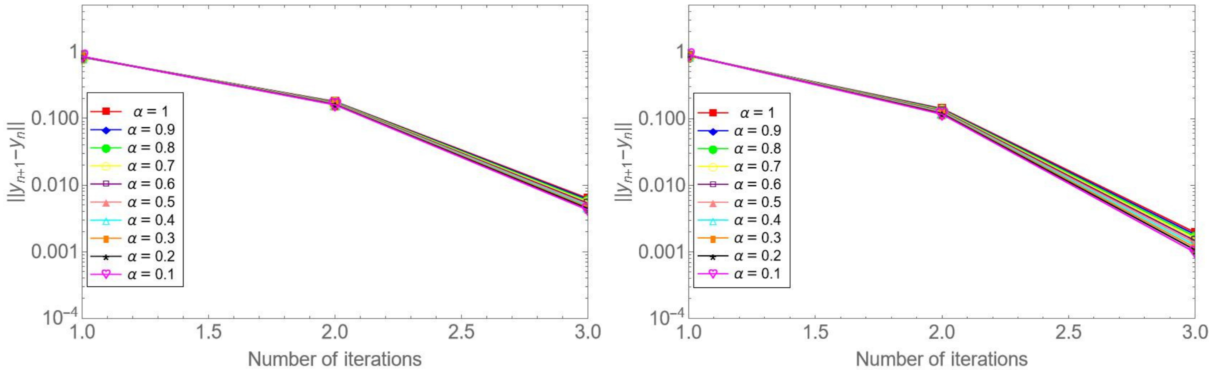
for $\alpha < 1$) require fewer iteration than conformable vector Weerakoon-Fernando method (for $\alpha = 1$). Conformable vector Weerakoon-Fernando method is the classical method (2.31), when $\alpha = 1$.

Table 6: Numerical result of $CVTM$ and $CVWFM$ for $F_1(y_1, y_2)$

	$CVTM$					$CVWFM$				
α	$\bar{y}$	$\Vert F(y_{n+1}) \Vert$	$\Vert y_{n+1} - y_n \Vert$	Itr	ρ	$\bar{y}$	$\Vert F(y_{n+1}) \Vert$	$\Vert y_{n+1} - y_n \Vert$	Itr	ρ
1	$\bar{y_3}$	1.57×10^{-15}	4.88×10^{-7}	4	2.83	$\bar{y_3}$	1.57×10^{-15}	7.45×10^{-9}	4	2.92
0.9	$\bar{y_3}$	6.29×10^{-15}	4.26×10^{-7}	4	2.83	$\bar{y_3}$	8.44×10^{-9}	1.83×10^{-3}	3	2.35
0.8	$\bar{y_3}$	5.73×10^{-15}	3.72×10^{-7}	4	2.83	$\bar{y_3}$	6.73×10^{-9}	1.70×10^{-3}	3	2.35
0.7	$\bar{y_3}$	7.65×10^{-15}	3.24×10^{-7}	4	2.84	$\bar{y_3}$	5.33×10^{-9}	1.58×10^{-3}	3	2.35
0.6	$\bar{y_3}$	2.82×10^{-15}	2.81×10^{-7}	4	2.84	$\bar{y_3}$	4.20×10^{-9}	1.47×10^{-3}	3	2.34
0.5	$\bar{y_3}$	3.78×10^{-15}	2.44×10^{-7}	4	2.84	$\bar{y_3}$	3.28×10^{-9}	1.36×10^{-3}	3	2.34
0.4	$\bar{y_3}$	7.65×10^{-15}	2.11×10^{-7}	4	2.85	$\bar{y_3}$	2.55×10^{-9}	1.26×10^{-3}	3	2.34
0.3	$\bar{y_3}$	2.83×10^{-14}	1.82×10^{-7}	4	2.85	$\bar{y_3}$	1.97×10^{-9}	1.16×10^{-3}	3	2.34
0.2	$\bar{y_3}$	5.26×10^{-15}	1.56×10^{-7}	4	2.85	$\bar{y_3}$	1.51×10^{-9}	1.07×10^{-3}	3	2.34
0.1	$\bar{y_3}$	2.13×10^{-14}	1.34×10^{-7}	4	2.85	$\bar{y_3}$	1.15×10^{-9}	9.81×10^{-4}	3	2.33

In Figure 6, it is evident that for same α, the error $\Vert y_{n+1} - y_n \Vert$ in $CVWFM$ decreases more quickly compared to the earlier proposed $CVTM$, and also compared to the classical method (2.31).

Problem 7. The second vector valued function is $F_2(y_1, y_2, y_3) = (\cos y_2 - \sin y_1, y_3^{y_1} - \frac{1}{y_2}, \exp y_1 - y_3^2)^T$ with solution $\bar{y_1} \approx (0.909569, 0.661227, 1.575834)^T$. The conformable Jacobian matrix of $F_2(y_1, y_2, y_3)$ is



(a) $CVTM$ (b) $CVWFM$

Figure 6: Error plots of F_1 across all α values provided in Table 6

$$F_a^{\alpha(1)}(y_1,y_2,y_3)=\begin{pmatrix}(y_1-a_1)^{1-\alpha}(-\cos y_1) & (y_2-a_2)^{1-\alpha}(-\sin y_2) & 0\\ (y_1-a_1)^{1-\alpha}(y_3^{y_1}\ln y_3) & (y_2-a_2)^{1-\alpha}(\frac{1}{y_2^2}) & (y_3-a_3)^{1-\alpha}(y_3^{y_1}\frac{y_1}{y_3})\\ (y_1-a_1)^{1-\alpha}(\exp y_1) & 0 & (y_3-a_3)^{1-\alpha}(-2y_3)\end{pmatrix},$$

being $a=(a_1,a_2,a_3)=(-10,-10,-10)$. Table 7 displays the numerical performance of $CVTM$ and $CVWFM$ for $F_2(y_1,y_2,y_3)$ with initial approximation $y_0=(0.5,0.1,1)^T$. From Table 7 it can be seen that $CVWFM$ performed better than $CVTM$ for same α value. It can also be seen that the error $\| y_{n+1}-y_n \|$ in $CVWFM$ (for $\alpha<1$) is less than $CVWFM$ (for $\alpha=1$). Figure 7 demonstrates a more accelerated reduction in the error $\| y_{n+1}-y_n \|$ in proposed method when compared to the performance of the $CVTM$.

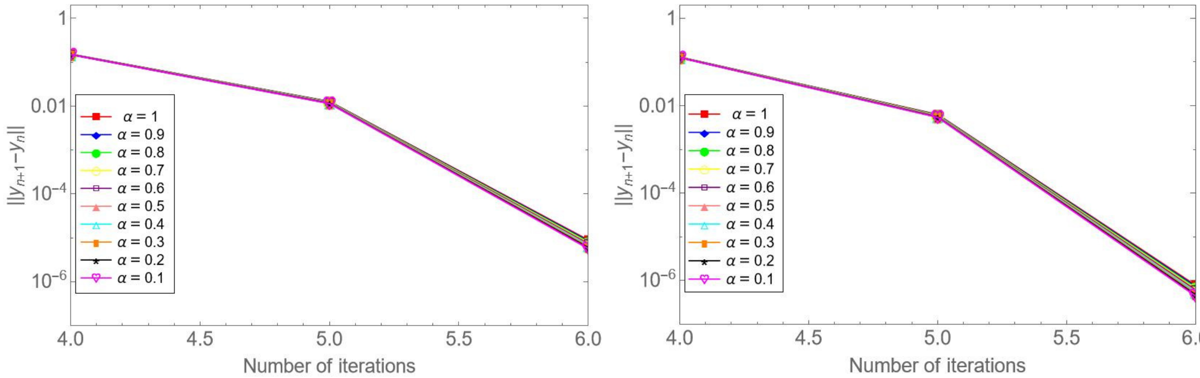
Table 7: Numerical result of $CVTM$ and $CVWFM$ for $F_2(y_1,y_2,y_3)$

	$CVTM$					$CVWFM$				
α	$\bar{y}$	$\| F(y_{n+1}) \|$	$\| y_{n+1}-y_n \|$	Itr	ρ	$\bar{y}$	$\| F(y_{n+1}) \|$	$\| y_{n+1}-y_n \|$	Itr	ρ
1	$\bar{y_1}$	3.39×10^{-15}	8.86×10^{-6}	6	2.93	$\bar{y_1}$	3.58×10^{-15}	7.79×10^{-7}	6	2.98
0.9	$\bar{y_1}$	8.63×10^{-15}	8.44×10^{-6}	6	2.93	$\bar{y_1}$	1.29×10^{-14}	7.73×10^{-7}	6	2.98
0.8	$\bar{y_1}$	4.91×10^{-15}	8.04×10^{-6}	6	2.94	$\bar{y_1}$	4.93×10^{-15}	6.90×10^{-7}	6	2.98
0.7	$\bar{y_1}$	9.07×10^{-15}	7.66×10^{-6}	6	2.94	$\bar{y_1}$	3.58×10^{-15}	6.49×10^{-7}	6	2.98
0.6	$\bar{y_1}$	6.00×10^{-15}	7.29×10^{-6}	6	2.94	$\bar{y_1}$	4.00×10^{-15}	6.10×10^{-7}	6	2.98
0.5	$\bar{y_1}$	7.63×10^{-15}	6.94×10^{-6}	6	2.95	$\bar{y_1}$	7.63×10^{-15}	5.73×10^{-7}	6	2.98
0.4	$\bar{y_1}$	8.54×10^{-15}	6.60×10^{-6}	6	2.95	$\bar{y_1}$	9.07×10^{-15}	5.38×10^{-7}	6	2.98
0.3	$\bar{y_1}$	1.17×10^{-14}	6.27×10^{-6}	6	2.95	$\bar{y_1}$	1.17×10^{-14}	5.05×10^{-7}	6	2.99
0.2	$\bar{y_1}$	7.31×10^{-15}	5.96×10^{-6}	6	2.96	$\bar{y_1}$	7.31×10^{-15}	4.74×10^{-7}	6	2.99
0.1	$\bar{y_1}$	1.55×10^{-14}	5.66×10^{-6}	6	2.96	$\bar{y_1}$	1.55×10^{-14}	4.44×10^{-7}	6	2.99

Problem 8. The third vector valued function is $F_3(y)=(g_1(y),...,g_{15}(y))^T$, where $y=(y_1,...,y_{15})$, $g_i:\mathbb{R}^m\to\mathbb{R}$, $i=1,2,...,15$ such that

$$\begin{cases} g_i(y)=y_iy_{i+1}-1, & i=1,2,...,14\\ g_{15}(y)=y_{15}y_1-1, \end{cases}$$

having real roots $\bar{y_1}=(-1,-1,...,-1)^T$ and $\bar{y_2}=(1,1,...,1)^T$. The conformable Jacobian matrix of



(a) $CVTM$ (b) $CVWFM$

Figure 7: Error plots of F_2 across all α values provided in Table 7

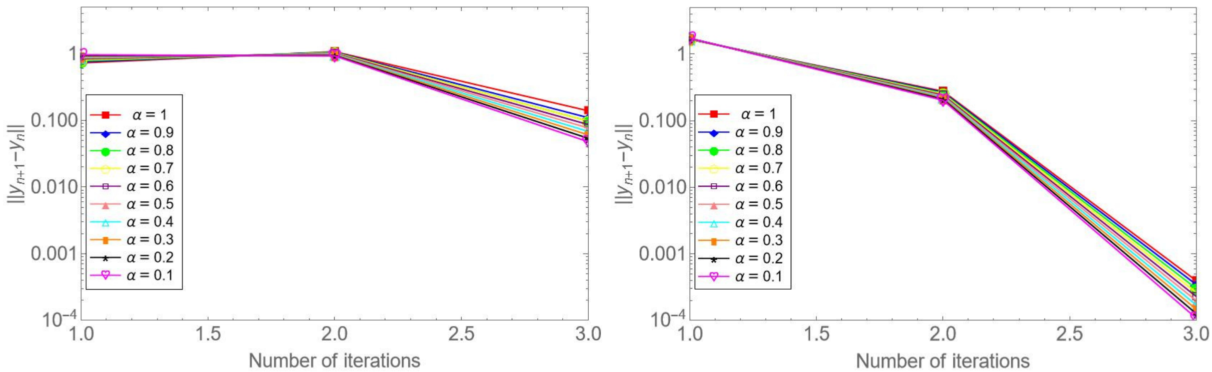
$F_3(y)$ is

$$F_a^{\alpha(1)}(y) = \begin{pmatrix} \xi_{1,1} & \xi_{1,2} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \xi_{2,2} & \xi_{2,3} & 0 & \cdots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & \cdots & \cdots & 0 & \xi_{14,14} & \xi_{14,15} \\ \xi_{15,1} & 0 & \cdots & \cdots & 0 & 0 & \xi_{15,15} \end{pmatrix},$$

where $\xi_{1,1} = (y_1 - a_1)^{1-\alpha}(y_2)$, $\xi_{1,2} = (y_2 - a_2)^{1-\alpha}(y_1)$, $\xi_{2,2} = (y_2 - a_2)^{1-\alpha}(y_3)$, $\xi_{2,3} = (y_3 - a_3)^{1-\alpha}(y_2)$, $\xi_{14,14} = (y_{14} - a_{14})^{1-\alpha}(y_{15})$, $\xi_{14,15} = (y_{15} - a_{15})^{1-\alpha}(y_{14})$, $\xi_{15,1} = (y_1 - a_1)^{1-\alpha}(y_{15})$, $\xi_{15,15} = (y_{15} - a_{15})^{1-\alpha}(y_1)$, being $a = (a_1, ..., a_{15}) = (-10, ..., -10)$. Table 8 shows the numerical results of $CVTM$ and $CVWFM$ for $F_3(y)$ with initial approximation $y_0 = (-0.5, -05, ..., -0.5)^T$. From Table 8 it is clear that $CVWFM$ takes lower iteration as compared to $CVTM$ for same α value. It is also clear that the error $\| y_{n+1} - y_n \|$ of $CVWFM$ (for $\alpha < 1$) is less than $CVWFM$ (for $\alpha = 1$). Figure 8 clearly shows that the error declines at a faster rate in $CVWFM$ than that observed with the $CVTM$.

Table 8: Numerical result of $CVTM$ and $CVWFM$ for $F_3(y)$

	$CVTM$					$CVWFM$				
α	$\bar{y}$	$\| F(y_{n+1}) \|$	$\| y_{n+1} - y_n \|$	Itr	ρ	$\bar{y}$	$\| F(y_{n+1}) \|$	$\| y_{n+1} - y_n \|$	Itr	ρ
1	$\bar{y_1}$	6.88×10^{-14}	9.77×10^{-5}	4	3.55	$\bar{y_1}$	2.04×10^{-12}	3.93×10^{-4}	3	3.65
0.9	$\bar{y_1}$	4.13×10^{-14}	6.62×10^{-5}	4	3.51	$\bar{y_1}$	1.32×10^{-12}	3.44×10^{-4}	3	3.64
0.8	$\bar{y_1}$	2.75×10^{-14}	4.49×10^{-5}	4	3.46	$\bar{y_1}$	8.53×10^{-13}	3.00×10^{-4}	3	3.69
0.7	$\bar{y_1}$	0	3.05×10^{-5}	4	3.42	$\bar{y_1}$	5.23×10^{-13}	2.61×10^{-4}	3	3.60
0.6	$\bar{y_1}$	0	2.07×10^{-5}	4	3.39	$\bar{y_1}$	3.30×10^{-13}	2.27×10^{-4}	3	3.58
0.5	$\bar{y_1}$	0	1.41×10^{-5}	4	3.36	$\bar{y_1}$	2.24×10^{-13}	1.97×10^{-4}	3	3.57
0.4	$\bar{y_1}$	1.38×10^{-14}	9.58×10^{-6}	4	3.33	$\bar{y_1}$	1.51×10^{-13}	1.70×10^{-4}	3	3.55
0.3	$\bar{y_1}$	0	6.52×10^{-6}	4	3.30	$\bar{y_1}$	8.26×10^{-14}	1.46×10^{-4}	3	3.54
0.2	$\bar{y_1}$	1.38×10^{-14}	4.44×10^{-6}	4	3.28	$\bar{y_1}$	2.75×10^{-14}	1.26×10^{-4}	3	3.53
0.1	$\bar{y_1}$	4.13×10^{-14}	3.02×10^{-6}	4	3.26	$\bar{y_1}$	4.13×10^{-14}	1.08×10^{-4}	3	3.51



(a) $CVTM$ (b) $CVWFM$

Figure 8: Error plots of F_3 across all α values provided in Table 8

4. Numerical Stability of the Proposed Schemes

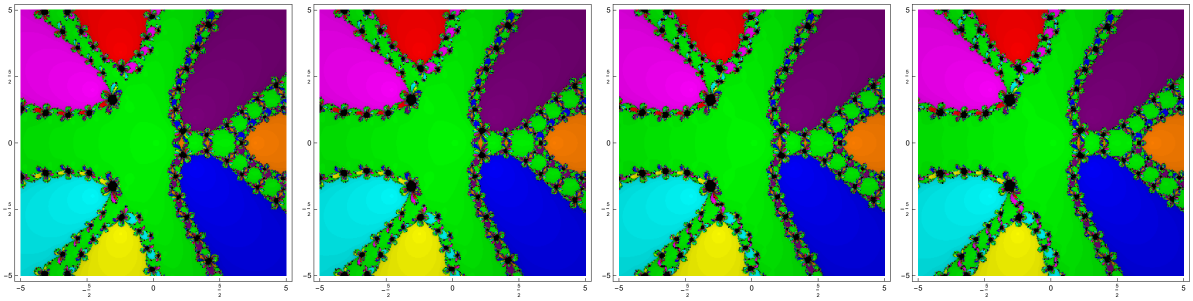
In order to thoroughly assess the stability of the suggested methods, we investigate the influence of the initial approximation on the convergence process, by utilizing the concept of convergence planes [18]. To generate the convergence planes, we select a rectangular region $D \subseteq \mathbb{C}$, and divide it uniformly into a 400×400 grid of cells or mesh points and associate a color with each mesh point z_0 based on the solution at which the respective iterative process initiated from z_0 converges (the color becomes lighter as the number of iterations required is decreases), and we assign a black color to the point if the method fails to converge. We have used stopping criteria 10^{-3} and maximum number of iterations 25 for each method. To generate and compare convergence planes we take the functions f_1, f_2, f_3 and F_1 discussed in the previous section and we take the value of α as 1, 0.75, 0.5 and 0.25. We conducted a comparison of proposed scheme $CWFM$ with known schemes $TeCO$ and $CKeCO$ in the complex domain. Similarly the proposed scheme $CVWFM$ is compared with known scheme $CVTM$. All data and visualizations were obtained using MATHEMATICA 14.1.

We consider the domain $D = [-5, 5]\times[-5, 5]$ for $f_1(z)$, which has eight distinct roots: $-2.27869-1.98748i$, $-2.27869+1.98748i$, $-1.23877-3.40852i$, $-1.23877+3.40852i$, 0.104699, $1.55392-0.940415i$, $1.55392+0.940415i$, 3.82239 contained in D and these roots are represented using the colors cyan, magenta, yellow, red, green, blue, purple, orange respectively. The corresponding fractal basins for $f_1(z) = 0$ are shown in Figure 9 to 11. We can observe from these figures that for same value of α, $CWFM$ has greater convergence percentage compared to $TeCO$ and $CKeCO$ methods. The convergence % of the proposed method is almost 97% for $f_1(z)$.

Figure 12 to 14 presents the convergence planes for the equation $f_2(z) = 0$, which has two roots 0 and 4.96511. We take the domain $D = [-2.5, 2.5]\times[-2.5, 2.5]$. These roots are visually distinguished using the colors yellow and blue, respectively. Figure 12 to 14 shows that $CWFM$ produce simple and wide regions of convergence and has greater convergence percentage as compared to $TeCO$ and $CKeCO$ methods for same α value. The convergence % of the proposed method $CWFM$ is almost 98% for $f_2(z)$.

Figure 15 to 17 shows the convergence planes for the equation $f_3(z) = 0$ having root 0.415856... contained in $D = [-2, 2] \times [-2, 2]$ and this root is presented by yellow color. It can be observed in these Figures that $CWFM$ generates simple fractal patterns and broad convergence regions as compared to $TeCO$ and $CKeCO$ for same α. The convergence % of the proposed method is almost 52% for $f_3(z)$.

Figure 18 and 19 displays the convergence plane for $F_1(y_1, y_2)$, which has four real solutions (0.8660, -0.5), (-0.8660, 0.5), (-0.8660, -0.5) and (0.8660, 0.5) enclosing in the domain $D = [-2.5, 2.5]\times[-2.5, 2.5]$. To distinguish the solutions visually, colors blue, yellow, red and green are applied respectively. Figure 18 to 19 shows that $CVWFM$ maintains structural simplicity in fractals and wide regions of convergence as compared to $CVTM$ for same α value. The convergence % of the proposed method $CVWFM$ is 99.50% for $F_1(y_1, y_2)$. We observe that the proposed schemes exhibit superior and greater stability performance as compare to existing schemes.



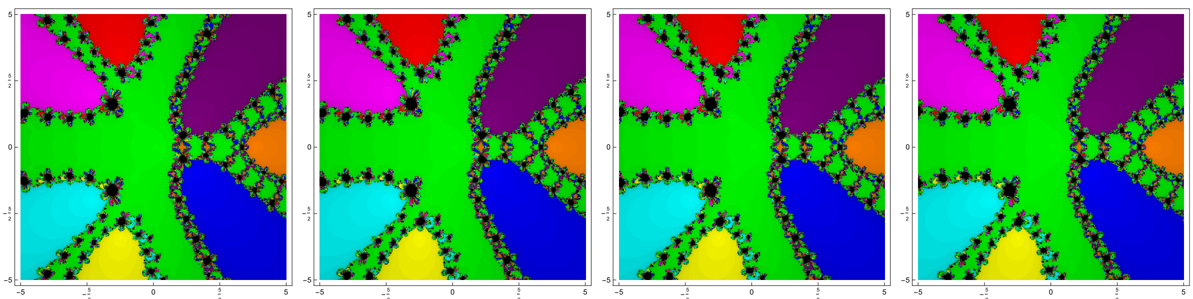
(a) $\alpha = 1$ (96.61% convergence)

(b) $\alpha = 0.75$ (96.39% convergence)

(c) $\alpha = 0.5$ (96.09% convergence)

(d) $\alpha = 0.25$ (95.99% convergence)

Figure 9: Convergence planes for $f_1(z)$ by using $TeCO$



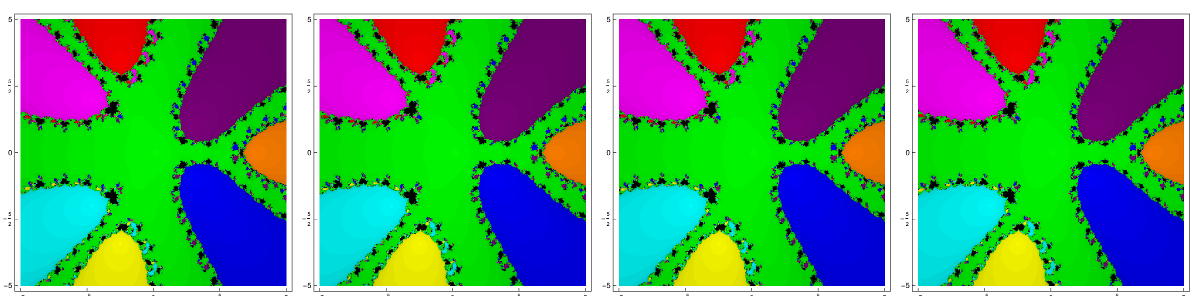
(a) $\alpha = 1$ (95.12% convergence)

(b) $\alpha = 0.75$ (94.92% convergence)

(c) $\alpha = 0.5$ (94.82% convergence)

(d) $\alpha = 0.25$ (94.70% convergence)

Figure 10: Convergence planes for $f_1(z)$ by using $CKeCO$



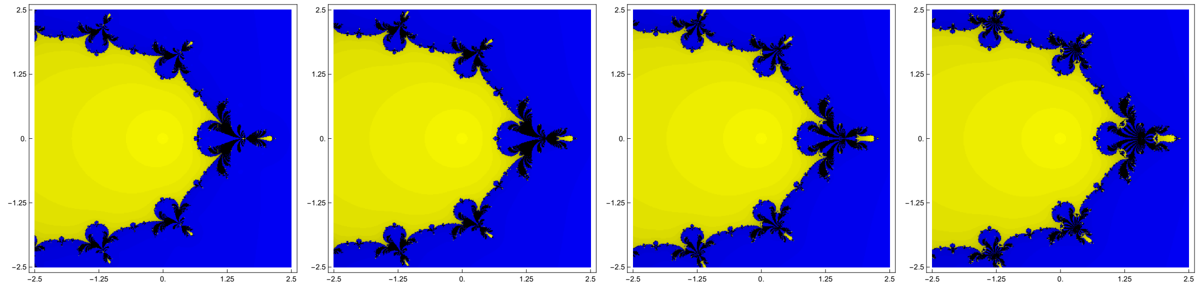
(a) $\alpha = 1$ (97.00% convergence)

(b) $\alpha = 0.75$ (96.99% convergence)

(c) $\alpha = 0.5$ (97.09% convergence)

(d) $\alpha = 0.25$ (97.19% convergence)

Figure 11: Convergence planes for $f_1(z)$ by using $CWFM$



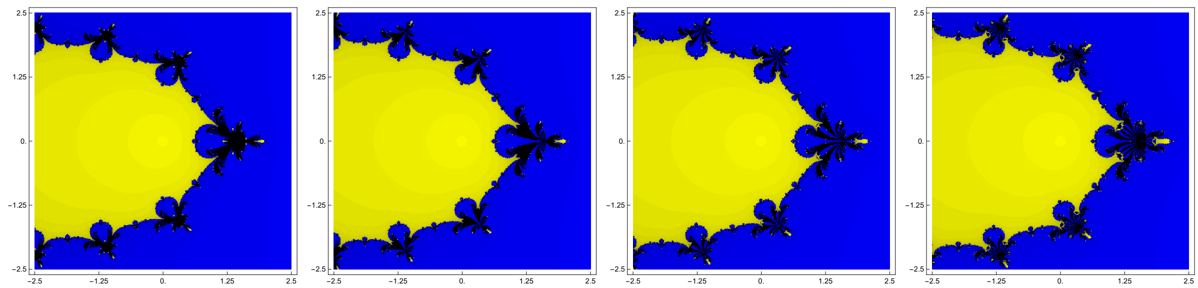
(a) $\alpha = 1$, (95.86% convergence)

(b) $\alpha = 0.75$ (95.22% convergence)

(c) $\alpha = 0.5$ (95.38% convergence)

(d) $\alpha = 0.25$ (94.92% convergence)

Figure 12: Convergence planes for $f_2(z)$ by using $TeCO$



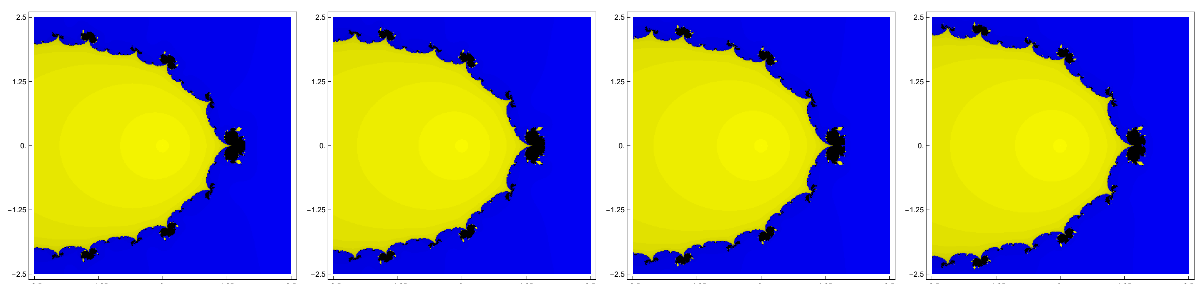
(a) $\alpha = 1$, (95.11% convergence)

(b) $\alpha = 0.75$ (95.25% convergence)

(c) $\alpha = 0.5$ (95.52% convergence)

(d) $\alpha = 0.25$ (95.21% convergence)

Figure 13: Convergence planes for $f_2(z)$ by using $CKeCO$



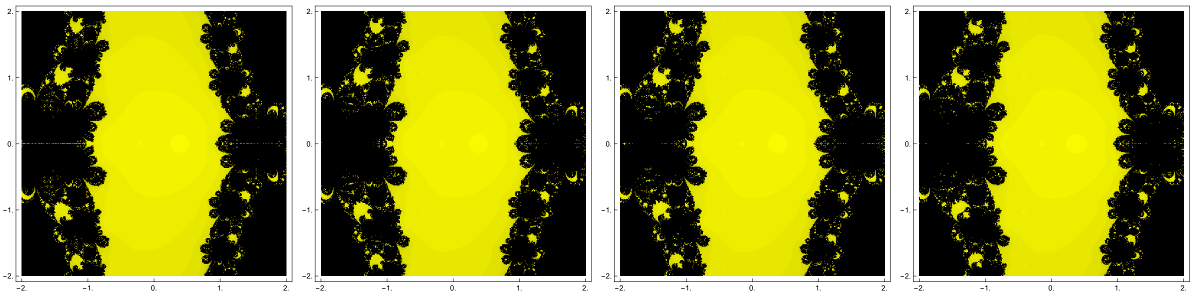
(a) $\alpha = 1$ (97.91% convergence)

(b) $\alpha = 0.75$ (97.89% convergence)

(c) $\alpha = 0.5$ (97.89% convergence)

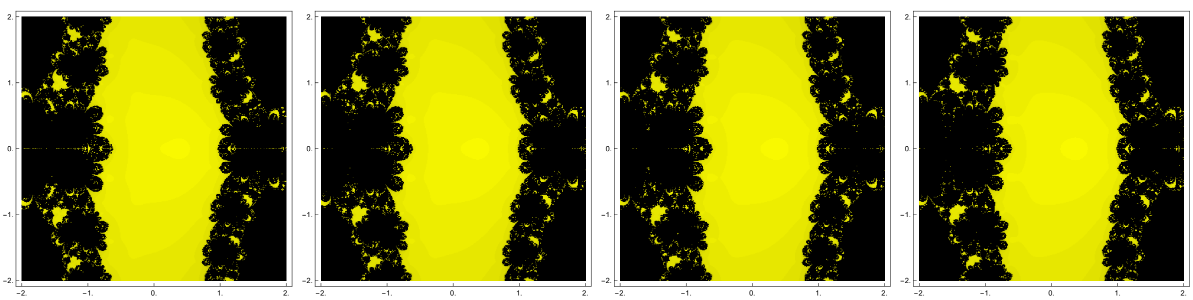
(d) $\alpha = 0.25$ (97.88% convergence)

Figure 14: Convergence planes for $f_2(z)$ by using $CWFM$



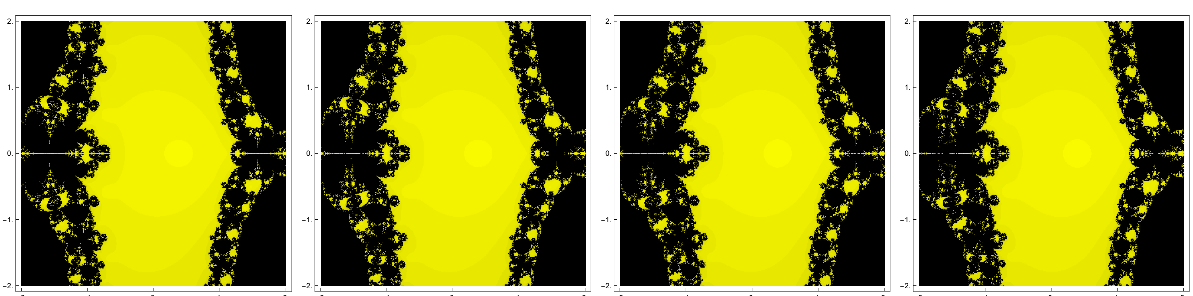
(a) $\alpha = 1$, (47.02% convergence)
(b) $\alpha = 0.75$ (46.85% convergence)
(c) $\alpha = 0.5$ (46.79% convergence)
(d) $\alpha = 0.25$ (46.55% convergence)

Figure 15: Convergence planes for $f_3(z)$ by using $TeCO$



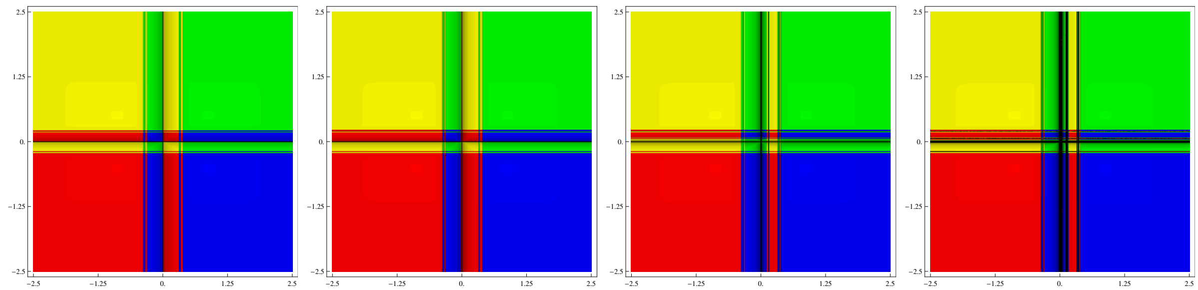
(a) $\alpha = 1$, (45.87% convergence)
(b) $\alpha = 0.75$ (45.56% convergence)
(c) $\alpha = 0.5$ (45.41% convergence)
(d) $\alpha = 0.25$ (45.21% convergence)

Figure 16: Convergence planes for $f_3(z)$ by using $CKeCO$



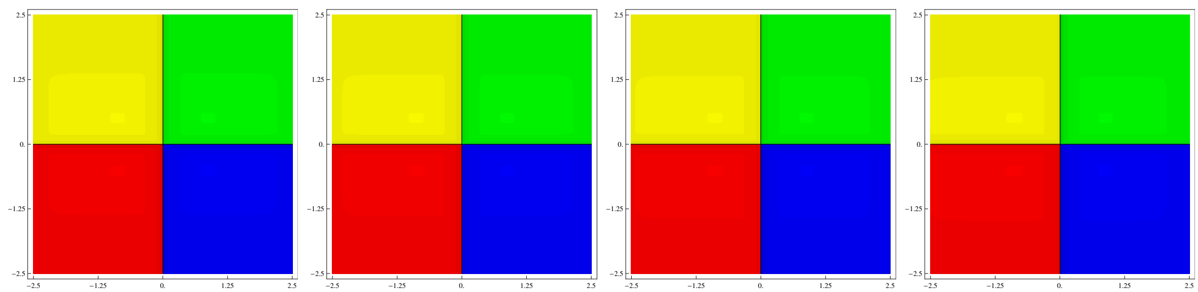
(a) $\alpha = 1$, (52.44% convergence)
(b) $\alpha = 0.75$ (52.32% convergence)
(c) $\alpha = 0.5$ (52.20% convergence)
(d) $\alpha = 0.25$ (51.94% convergence)

Figure 17: Convergence planes for $f_3(z)$ by using $CWFM$



(a) $\alpha = 1$ (99.50% convergence)
(b) $\alpha = 0.75$ (99.50% convergence)
(c) $\alpha = 0.5$ (99.00% convergence)
(d) $\alpha = 0.25$ (96.18% convergence)

Figure 18: Convergence planes for $F_1(y_1, y_2)$ by using $CVTM$



(a) $\alpha = 1$ (99.50% convergence)
(b) $\alpha = 0.75$ (99.50% convergence)
(c) $\alpha = 0.5$ (99.50% convergence)
(d) $\alpha = 0.25$ (99.50% convergence)

Figure 19: Convergence planes for $F_1(y_1, y_2)$ by using $CVWFM$

5. Conclusion

In this study, we present a conformable Weerakoon-Fernando method to solve nonlinear equations, noting that the traditional Weerakoon-Fernando method is encompassed as a special case of proposed method. We also generalized the proposed scheme for solving system of nonlinear equations. An analysis of the convergence order of the proposed schemes is examined, confirming that it achieves third-order convergence. To examine the performance of the proposed iterative techniques and validate the theoretical results, a set of numerical experiments is carried out, including several problems inspired by real-world applications. The numerical results of the proposed methods are presented in terms of number of iterations, errors in consecutive iterations, function value at the last iteration, approximated computational order of convergence (ACOC), and graphical error plots illustrating error behaviour across iterations. The numerical results provide strong evidence that the proposed methods are not only more efficient than existing approaches but also outperform their corresponding classical versions in many instances, while also achieving the predicted theoretical convergence order in practical computations. Furthermore, it was observed that the proposed method can converge to both real and complex roots from real initial estimates and can obtain distinct solutions starting from the same initial guess, a feature that classical methods lack. This characteristic underscores broader capability and flexibility of the proposed method in handling nonlinear problems. To thoroughly assess the stability of the proposed methods, the impact of various initial guesses was examined using convergence planes and it was observed that the proposed methods exhibit a more stable and well-structured basin of attraction compared to existing iterative techniques. Overall, the proposed methods demonstrate high reliability and efficiency, making them a strong alternative for solving nonlinear equations.

Acknowledgments: The authors thank anonymous referees for valuable suggestions and comments. Both the authors are also thankful to the Department of Science and Technology, New Delhi, India for approving the proposal under the scheme FIST program(Ref.No.SR/FST/MS/2022dated19.12.2022).

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Toshan Kumar Shriwas,
Department of Mathematics,
Guru Ghasidas Vishwavidyalaya (A Central University) Bilaspur (C.G.),
India, 495009.
E-mail address: toshan.shriwas786@gmail.com

and

Jai Prakash Jaiswal,
Department of Mathematics,
Guru Ghasidas Vishwavidyalaya (A Central University) Bilaspur (C.G.),
India, 495009.
E-mail address: asstprofjpmanit@gmail.com