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Some New Coupled Fixed Point Results and Applications to a System of Integral Equations

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ABSTRACT: This paper concerns with the solvability and asymptotic stability of solutions for a class of systems of integral equations in $BC(\mathbb{R}_+)$ which is the space of real valued, continuous and bounded functions defined on the set of nonnegative real numbers. We firstly present a theorem about existence of coupled fixed point of an operator and then we give a corollary depending on this theorem. By using this corollary we show that a class of systems of integral equations has at least one solution which is asymptotically stable. Also we give an example showing applicability of our main result. Finally we mention an open problem worth focusing on at the future.

Key Words: Coupled fixed point theorem, integral equations, measure of noncompactness, existence of solutions, asymptotic stability.

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1. Introduction

Integral and differential equations, especially nonlinear ones are frequently used in the modelling of some problems in mechanics, physics, medicine, etc. [27,32,26,11,12,14,15,16,17,18]. In addition to which equation can represent the problem, another important question is whether the equation defining the problem has a solution or not in the relevant space. It is at this point that fixed point theory is generally used as a vital tool. By using fixed point theorems, some results can be obtained as to whether a solution exists or not, with the help of a transformation defined in accordance with the relevant equation, [30,23,28,1,21]. Thus, the problem of investigating the existence of a solution of the equation turns into the problem of investigating whether the mentioned transformation has a fixed point or not. We should immediately point out that with this technique, some sufficient conditions can be given for the existence of a solution but, if these sufficient conditions are not satisfied then no conclusion can be obtained about whether a solution exists. Although this may seem like a disadvantage, sometimes knowing what conditions will be sufficient for the existence of a solution can be as valuable as knowing what the solution is.

On the other hand, by combining the concept of measure of noncompactness with fixed point theorems, some existence theorems in which extremely useful results can be obtained has been presented, [8,2,31, 3,9,4,5,6,7].

Some of fixed point theorems that help determine the sufficient conditions for an operator equation to have at least one solution can provide some results about the behaviour of the solutions as well as the existence of the solution of the equation. For example, Darbo type fixed point theorems associated with measures of noncompactness can give results about the increase or decrease of the solutions of the equation according to the independent variable, as well as about the asymptotic stability, [20,22,25,13,24,29].

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2. Definitions and Auxiliary Facts

Let $(E, \|.\|)$ be an infinite Banach space with zero element θ. We write $B(x,r)$ to denote the closed ball centered at x with radius r and especially, we write B_r in case of $x=\theta$. We write $\overline{X}$ and $\mathrm{Conv}\, X$ to denote the closure and convex closure of X, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty, bounded subsets of E and $\mathfrak{N}_E$ indicate its subfamily of all relatively compact sets. Finally, the standard algebraic operations on sets are denoted by λX and $X+Y$, respectively.

We use the following definition of the measure of noncompactness on a Banach space E, given in [8].

Definition 2.1 *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0,\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

1. *The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$,*
2. *$X \subset Y \Rightarrow \mu(X) \le \mu(Y)$,*
3. *$\mu(X) = \mu(\overline{X}) = \mu(Conv\, X)$,*
4. *$\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$,*
5. *If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1,2,...)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the intersection set $\cap_{n=1}^{\infty} X_n$ is nonempty.*

Let X be a fixed nonempty and bounded subset of the space $BC(\mathbb{R}_+)$ and T be a positive number. For $\varepsilon > 0$ and $x \in X$, we denote by $\omega^T(x,\varepsilon)$ the modulus of continuity of the function x on the interval $[0,T]$ i.e.

$$\omega^T(x,\varepsilon) = \sup\{|x(t_1) - x(t_2)| : t_1, t_2 \in [0,T] \text{ and } |t_1 - t_2| \le \varepsilon\}.$$

Furthermore, let $\omega^T(X,\varepsilon)$, $\omega_0^T(X)$ and $\omega_0(X)$ are defined by

$$\begin{aligned} \omega^T(X,\varepsilon) &= \sup\left\{\omega^T(x,\varepsilon) : x \in X\right\}, \\ \omega_0^T(X) &= \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon) \end{aligned}$$

and

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X). \tag{2.1}$$

Moreover for a fixed $t \in \mathbb{R}_+$ let us define

$$X(t) = \{x(t) : x \in X\}$$

and

$$\mathrm{diam}\; X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

Then, the authors have shown in [8] the function μ given by

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \mathrm{diam}\; X(t) \tag{2.2}$$

is a measure of noncompactness in the space $BC(\mathbb{R}_+)$.

Definition 2.2 *An element $(x,y) \in X \times X$ is called coupled fixed point of the mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$, [10].*

Theorem 2.1 *Suppose $\mu_1, \mu_2, ..., \mu_n$ are measures of noncompactness in $E_1, E_2, ..., E_n$, respectively. Moreover assume that the function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ is convex and $F(x_1, x_2, ..., x_n) = 0$ if and only if $x_1 = x_2 = ... = x_n = 0$. Then*

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), ..., \mu_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times ... \times E_n$ where X_i denote the natural projection of X into E_i for $i = 1,2,...,n$, [8].

Corollary 2.1 *Let μ be a measure of noncompactness in the Banach space E and $F:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ be define as $F(x,y)=x+y$. Then*

$$\bar{\mu}(X)=\mu(X_1)+\mu(X_2)$$

is a measure of noncompactness in the space $E\times E$, where X_i denote the natural projection of X into E for $i=1,2$, [3].

Theorem 2.2 *Let C be a nonempty, bounded, closed and convex subset of a Banach space E. Assume $T:C\to C$ is a continuous operator such that*

$$\theta(\psi(\mu(T(X))))+f(\psi(\mu(T(X))))\le f(\phi(\psi(\mu(X)),\psi(\mu(X))))$$

hold for all nonempty subset X of C. Where μ is an arbitrary measure of noncompactness on E, $\psi:[0,\infty)\to[0,\infty)$ is nondecreasing such that $\psi(t)=0$ if and only $t=0$, $\phi\in\Theta$ and $(\theta,f)\in\Delta$. Then T has a fixed point in C, [31].

We should note that the reader can find the definitions of the notions Θ and Δ by looking "Definition 3 and Definition 4" respectively in the paper [31].

3. Main Results

3.1. Some New Results on Coupled Fixed Points

Now we present a coupled fixed point theorem which will be used in showing of the existence of solution of a system of integral equations as main tool.

Theorem 3.1 *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $\zeta>0$. Also, assume $F:C\times C\to C$ is a continuous operator such that*

$$\zeta+f(\psi(\mu(F(X_1\times X_2)))+\psi(\mu(F(X_2\times X_1))))\le f(\phi(\psi(\mu(X_1)+\mu(X_2)),\psi(\mu(X_1)+\mu(X_2)))) \tag{3.1}$$

hold for all nonempty subsets X_1,X_2 of C. Where μ is an arbitrary measure of noncompactness on E, $\phi\in\Theta$ and $f:(0,\infty)\to\mathbb{R}$ is a function satisfying the properties of the second element of Δ. Also $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing function such that $\psi(t+s)\le\psi(t)+\psi(s)$ for all $t,s\ge 0$ and $\psi(t)=0$ if and only $t=0$. Then operator F has at least one coupled fixed point.

Proof: Let we define operator $T:C\times C\to C\times C$ as $T(x,y)=(F(x,y),F(y,x))$. Then it is obvious that F has a coupled fixed point (x,y) if and only if T has a fixed point (x,y). So we should focus on the existence of fixed point of T.

Taking into account the continuity of F we get that T is continuous. On the other hand $T(X)\subset F(X_1\times X_2)\times F(X_2\times X_1)$ hold such that X_i denote the natural projection of X into E. Also, by considering the facts that f,ψ are nondecreasing and $\psi(t+s)\le\psi(t)+\psi(s)$, we can write

$$\begin{aligned} f(\psi(\bar{\mu}(T(X)))) &\le f(\psi(\bar{\mu}(F(X_1\times X_2)\times F(X_2\times X_1)))) \\ &= f(\psi(\mu(F(X_1\times X_2))+\mu(F(X_2\times X_1)))) \\ &\le f(\psi(\mu(F(X_1\times X_2)))+\psi(\mu(F(X_2\times X_1)))) \end{aligned} \tag{3.2}$$

for any nonempty subset X of $C\times C$. Then by using (3.1) and (3.2) we get

$$\begin{aligned} \zeta+f(\psi(\bar{\mu}(T(X)))) &\le \zeta+f(\psi(\mu(F(X_1\times X_2)))+\psi(\mu(F(X_2\times X_1)))) \\ &\le f(\phi(\psi(\mu(X_1)+\mu(X_2)),\psi(\mu(X_1)+\mu(X_2)))) \\ &= f(\phi(\psi(\bar{\mu}(X)),\psi(\bar{\mu}(X)))). \end{aligned}$$

Where $\bar{\mu}$ is the measure of noncompactness on $E\times E$ defined by $\bar{\mu}(X)=\mu(X_1)+\mu(X_2)$. So all the conditions of Theorem (2.2) with $\theta(t)=\zeta$ and T has a fixed point (x,y). Thus F has at least one coupled fixed point. □

Corollary 3.1 *Let us take $\psi(t)=t$, $\phi(u,v)=u$ and $f(t)=\ln t$. Then, by using Theorem (3.1), we say that if $F: C\times C\to C\subset E$ is a continuous operator and satisfies the inequality*

$$\zeta+\ln\left(\mu\left(F\left(X_1\times X_2\right)\right)+\mu\left(F\left(X_2\times X_1\right)\right)\right)\leq\ln\left(\mu\left(X_1\right)+\mu\left(X_2\right)\right) \tag{3.3}$$

for all nonempty subsets X_1, X_2 of C then F has at least one coupled fixed point.

3.2. Existence and Asymptotic Stability of the Solutions of a System of Integral Equations

In this section we consider the following system of integral equations in $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$

$$\begin{aligned} x(t) &= T_1(x,y)(t)+T_2(x,y)(t)\int_0^{\varphi(t)} u\left(t,s,x\left(\alpha(s)\right),y\left(\alpha(s)\right)\right)ds \\ y(t) &= T_1(y,x)(t)+T_2(y,x)(t)\int_0^{\varphi(t)} u\left(t,s,y\left(\alpha(s)\right),x\left(\alpha(s)\right)\right)ds. \end{aligned} \tag{3.4}$$

Now we give some assumptions to handle the equation (3.4)

(a_1) The functions $\alpha,\varphi:\mathbb{R}_+\to\mathbb{R}_+$ are continuous and $\alpha(s)\to\infty$ as $s\to\infty$.

(a_2) The operators $T_1,T_2: BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\to BC(\mathbb{R}_+)$ are continuous and there exist the constants k_1,k_2,l_1,l_2 such that the following inequalities are satisfied

$$\begin{aligned} \left|T_1(x_1,y_1)(t)-T_1(x_2,y_2)(t)\right| &\leq k_1\left|x_1(t)-x_2(t)\right|+k_2\left|y_1(t)-y_2(t)\right|, \\ \left|T_2(x_1,y_1)(t)-T_2(x_2,y_2)(t)\right| &\leq l_1\left|x_1(t)-x_2(t)\right|+l_2\left|y_1(t)-y_2(t)\right|. \end{aligned}$$

(a_3) $u:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exist continuous functions $p,q:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\lim_{t\to\infty} p(t)\int_0^{\varphi(t)} q(s)\,ds=0$$

and

$$|u(t,s,x,y)|\leq p(t)\,q(s)$$

hold for all $t,s\in\mathbb{R}_+$ $(s\leq t)$ and $x,y\in\mathbb{R}$.

(a_4) The inequality

$$0<\max\left\{k_1+Ul_1,k_2+Ul_2\right\}<\frac{1}{2}$$

hold, where $U=\sup\left\{p(t)\int_0^{\varphi(t)} q(s)\,ds : t\geq 0\right\}$.

Theorem 3.2 *Equation (3.4) has at least one solution in the space $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$. Also the solutions of the equation (3.4) are asymptotically stable.*

Proof: Let us define the operator $F: BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\to BC(\mathbb{R}_+)$ by

$$F(x,y)(t)=T_1(x,y)(t)+T_2(x,y)(t)\int_0^{\varphi(t)} u\left(t,s,x\left(\alpha(s)\right),y\left(\alpha(s)\right)\right)ds.$$

Taking into account above assumptions it is obvious that F is continuous.

Also, for all $x,y\in BC(\mathbb{R}_+)$ we can write

$$\begin{aligned}
|F(x,y)(t)| &= \left| T_1(x,y)(t) + T_2(x,y)(t)\int_0^{\varphi(t)} u(t,s,x(\alpha(s)),y(\alpha(s)))\,ds \right| \\
&\leq |T_1(x,y)(t) - T_1(0,0)(t)| + |T_1(0,0)(t)| \\
&\quad + |T_2(x,y)(t) - T_2(0,0)(t)| \left| \int_0^{\varphi(t)} u(t,s,x(\alpha(s)),y(\alpha(s)))\,ds \right| \\
&\quad + |T_2(0,0)(t)| \left| \int_0^{\varphi(t)} u(t,s,x(\alpha(s)),y(\alpha(s))) \right| ds \\
&\leq k_1|x(t)| + k_2|y(t)| + K \\
&\quad + (l_1|x(t)| + l_2|y(t)| + L)\,p(t)\int_0^{\varphi(t)} q(s)\,ds \\
&\leq k_1\|x\|_\infty + k_2\|y\|_\infty + K + [l_1\|x\|_\infty + l_2\|y\|_\infty + L]\,U, \qquad (3.5)
\end{aligned}$$

where $K=\sup_{t\geq 0}|T_1(0,0)(t)|$ and $L=\sup_{t\geq 0}|T_2(0,0)(t)|$.

If we take

$$r_0 \geq \frac{K+LU}{1-[(k_1+k_2)+U(l_1+l_2)]}$$

then we conclude from (3.5) that

$$\begin{aligned}
|F(x,y)(t)| &\leq k_1\|x\|_\infty + k_2\|y\|_\infty + K + [l_1\|x\|_\infty + l_2\|y\|_\infty + L]\,U \\
&\leq k_1r_0 + k_2r_0 + K + [l_1r_0 + l_2r_0 + L]\,U \\
&= [(k_1+k_2) + U(l_1+l_2)]\,r_0 + K + LU \\
&\leq r_0
\end{aligned}$$

for $x,y\in B_{r_0}$. So we infer that $F(B_{r_0}\times B_{r_0})\subset B_{r_0}$.

Now we will focus on the continuity of F on $B_{r_0}\times B_{r_0}$. Let us take an arbitrary but fixed $\varepsilon>0$ and $(x_1,y_1),(x_2,y_2)\in B_{r_0}\times B_{r_0}$ such that $\|x_1-x_2\|_\infty<\varepsilon$, $\|y_1-y_2\|_\infty<\varepsilon$. Then we can write

$$\begin{aligned}
&|F(x_1,y_1)(t) - F(x_2,y_2)(t)| \\
&= \left| T_1(x_1,y_1)(t) + T_2(x_1,y_1)(t)\int_0^{\varphi(t)} u(t,s,x_1(\alpha(s)),y_1(\alpha(s)))\,ds \right. \\
&\qquad \left. -T_1(x_2,y_2)(t) - T_2(x_2,y_2)(t)\int_0^{\varphi(t)} u(t,s,x_2(\alpha(s)),y_2(\alpha(s)))\,ds \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \left|T_1(x_1,y_1)(t)-T_1(x_2,y_2)(t)\right| \\
&\quad +\left|T_2(x_1,y_1)(t)-T_2(x_2,y_2)(t)\right|\int\limits_0^{\varphi(t)}\left|u(t,s,x_1(\alpha(s)),y_1(\alpha(s)))\right|ds \\
&\quad +\left|T_2(x_2,y_2)(t)\right|\int\limits_0^{\varphi(t)}\left|u(t,s,x_1(\alpha(s)),y_1(\alpha(s)))-u(t,s,x_2(\alpha(s)),y_2(\alpha(s)))\right|ds \\
&\leq k_1\left|x_1(t)-x_2(t)\right|+k_2\left|y_1(t)-y_2(t)\right| \\
&\quad +\left(l_1\left|x_1(t)-x_2(t)\right|+l_2\left|y_1(t)-y_2(t)\right|\right)p(t)\int\limits_0^{\varphi(t)}q(s)\,ds \\
&\quad +\left(l_1\left|x_2(t)\right|+l_2\left|y_2(t)\right|+L\right)\int\limits_0^{\varphi(t)}\left|u(t,s,x_1(\alpha(s)),y_1(\alpha(s)))-u(t,s,x_2(\alpha(s)),y_2(\alpha(s)))\right|ds \\
&\leq k_1\left\|x_1-x_2\right\|_\infty+k_2\left\|y_1-y_2\right\|_\infty \\
&\quad \left(l_1\left\|x_1-x_2\right\|_\infty+l_2\left\|y_1-y_2\right\|_\infty\right)p(t)\int\limits_0^{\varphi(t)}q(s)\,ds \\
&\quad +\left(l_1\left\|x_2\right\|+l_2\left\|y_2\right\|+L\right)\int\limits_0^{\varphi(t)}\left|u(t,s,x_1(\alpha(s)),y_1(\alpha(s)))-u(t,s,x_2(\alpha(s)),y_2(\alpha(s)))\right|ds \qquad (3.6)
\end{aligned}$$

On the other hand, considering $\lim_{t\to\infty} p(t)\int\limits_0^{\varphi(t)} q(s)\,ds = 0$, we can find $T>0$ such that

$$p(t)\int\limits_0^{\varphi(t)} q(s)\,ds < \varepsilon \tag{3.7}$$

hold for $t>T$.

So, taking into

$$\int\limits_0^{\varphi(t)}\left|u(t,s,x_1(\alpha(s)),y_1(\alpha(s)))-u(t,s,x_2(\alpha(s)),y_2(\alpha(s)))\right|ds \leq 2p(t)\int\limits_0^{\varphi(t)} q(s)\,ds$$

account, we obtain the followings from (3.6) and (3.7) when $t>T$.

$$\left|F(x_1,y_1)(t)-F(x_2,y_2)(t)\right| \leq (k_1+k_2)\varepsilon+(l_1+l_2)\varepsilon^2+((l_1+l_2)r_0+L)2\varepsilon \tag{3.8}$$

Also we can write

$$\left|F(x_1,y_1)(t)-F(x_2,y_2)(t)\right| \leq (k_1+k_2)\varepsilon+U(l_1+l_2)\varepsilon+((l_1+l_2)r_0+L)\,\omega^T(u,\varepsilon)$$

for $t\in[0,T]$ from (3.6). Where

$$\begin{aligned}
&\omega^T(u,\varepsilon) \\
=\;&\sup\left\{\left|u(t,s,x_1,y_1)-u(t,s,x_2,y_2)\right| : t\in[0,T],\ 0\leq s\leq \sup\{\varphi(t):0\leq t\leq T\},\right. \\
&\qquad \left. x_1,y_1,x_2,y_2\in[-r_0,r_0],\ \ |x_1-x_2|+|y_1-y_2|<2\varepsilon\right\}.
\end{aligned}$$

By considering the continuity of u on $[0,T]\times\left[0,\sup_{0\leq t\leq T}\varphi(t)\right]\times[-r_0,r_0]\times[-r_0,r_0]$, we obtain that $\omega^T(u,\varepsilon)\to 0$ as $\varepsilon\to 0$. So, due to this fact and the result (3.8) we conclude that F is continuous on $B_{r_0}\times B_{r_0}$.

Let us take $\varepsilon, T > 0$, $X \times Y \subset B_{r_0} \times B_{r_0}$, $t_1, t_2 \in [0,T]$ such that $|t_1 - t_2| \leq \varepsilon$ and, without loss of generality, assume that $\varphi(t_2) \leq \varphi(t_1)$. Then we obtain the following inequalities for $(x,y) \in X \times Y$

$$\begin{aligned}
& |F(x,y)(t_1) - F(x,y)(t_2)| \\
= & \Bigg| T_1(x,y)(t_1) + T_2(x,y)(t_1) \int_0^{\varphi(t_1)} u(t_1, s, x(\alpha(s)), y(\alpha(s)))\, ds \\
& + T_2(x,y)(t_1) \int_0^{\varphi(t_2)} u(t_2, s, x(\alpha(s)), y(\alpha(s)))\, ds - T_2(x,y)(t_1) \int_0^{\varphi(t_2)} u(t_2, s, x(\alpha(s)), y(\alpha(s)))\, ds \\
& - T_1(x,y)(t_2) - T_2(x,y)(t_2) \int_0^{\varphi(t_2)} u(t_2, s, x(\alpha(s)), y(\alpha(s)))\, ds \Bigg| \\
\leq & |T_1(x,y)(t_1) - T_1(x,y)(t_2)| \\
& + |T_2(x,y)(t_1) - T_2(x,y)(t_2)| \int_0^{\varphi(t_2)} |u(t_2, s, x(\alpha(s)), y(\alpha(s)))|\, ds \\
& + |T_2(x,y)(t_1)| \left| \int_0^{\varphi(t_1)} u(t_1, s, x(\alpha(s)), y(\alpha(s)))\, ds - \int_0^{\varphi(t_2)} u(t_2, s, x(\alpha(s)), y(\alpha(s)))\, ds \right| \\
\leq & k_1 |x(t_1) - x(t_2)| + k_2 |y(t_1) - y(t_2)| \\
& + (l_1 |x(t_1) - x(t_2)| + l_2 |y(t_1) - y(t_2)|)\, p(t_2) \int_0^{\varphi(t_2)} q(s)\, ds \\
& + |T_2(x,y)(t_1)| \int_0^{\varphi(t_2)} |u(t_1, s, x(\alpha(s)), y(\alpha(s)))\, ds - u(t_2, s, x(\alpha(s)), y(\alpha(s)))|\, ds \\
& + |T_2(x,y)(t_1)| \int_{\varphi(t_2)}^{\varphi(t_1)} |u(t_1, s, x(\alpha(s)), y(\alpha(s)))|\, ds \\
\leq & k_1 |x(t_1) - x(t_2)| + k_2 |y(t_1) - y(t_2)| + (l_1 |x(t_1) - x(t_2)| + l_2 |y(t_1) - y(t_2)|)\, U \\
& + (l_1 |x(t_1)| + l_2 |y(t_1)| + L)\, \tilde{\varphi}_T \omega_{r_0}^T(u_1, \varepsilon) \\
& + (l_1 |x(t_1)| + l_2 |y(t_1)| + L)\, \omega^T(\varphi, \varepsilon)\, U \\
\leq & k_1 \omega^T(x, \varepsilon) + k_2 \omega^T(y, \varepsilon) + \left(l_1 \omega^T(x, \varepsilon) + l_2 \omega^T(y, \varepsilon)\right) U \\
& + ((l_1 + l_2) r_0 + L) \left[\tilde{\varphi}_T \omega_{r_0}^T(u_1, \varepsilon) + \omega^T(\varphi, \varepsilon)\, U\right]. \qquad (3.9)
\end{aligned}$$

Here

$$\begin{aligned}
\omega_{r_0}^T(u_1, \varepsilon) &= \sup \Big\{ |u(t_1, s, x, y) - u(t_2, s, x, y)| : t_1, t_2 \in [0,T],\ 0 \leq s \leq \sup_{0 \leq t \leq T} \varphi(t), \\
& \qquad x, y \in [-r_0, r_0],\ |t_1 - t_2| \leq \varepsilon \Big\}, \\
\tilde{\varphi}_T &= \sup\{|\varphi(t)| : t \in [0,T]\}
\end{aligned}$$

and

$$\omega^T(z_i, \varepsilon) = \sup\{|z_i(t_1) - z_i(t_2)| : t_1, t_2 \in [0,T], |t_1 - t_2| \leq \varepsilon\}$$

for $i = 1, 2, 3$ and $z_1 = x$, $z_2 = y$, $z_3 = \varphi$.

By taking the supremum over all $x \in X$ and $y \in Y$ we can write

$$\begin{aligned}\omega^T\left(F\left(X\times Y\right),\varepsilon\right) &\leq (k_1+Ul_1)\,\omega^T\left(X,\varepsilon\right)+(k_2+Ul_2)\,\omega^T\left(Y,\varepsilon\right)\\ &\quad +\left((l_1+l_2)\,r_0+L\right)\left[\tilde{\varphi}_T\omega^T_{r_0}\left(u_1,\varepsilon\right)+\omega^T\left(\varphi,\varepsilon\right)U\right]\end{aligned} \tag{3.10}$$

from (3.9). Taking into account the fact that $\omega^T_{r_0}\left(u_1,\varepsilon\right)\to 0$, $\omega^T\left(\varphi,\varepsilon\right)$ as $\varepsilon\to 0$, we obtain

$$\omega^T_0\left(F\left(X\times Y\right)\right)\leq (k_1+Ul_1)\,\omega^T_0\left(X\right)+(k_2+Ul_2)\,\omega^T_0\left(Y\right)$$

from (3.10). Then, by taking limit of both sides of the last inequality for $T\to\infty$, we get

$$\omega_0\left(F\left(X\times Y\right)\right)\leq (k_1+Ul_1)\,\omega_0\left(X\right)+(k_2+Ul_2)\,\omega_0\left(Y\right). \tag{3.11}$$

On the other hand, let us choose arbitrary elements $(x_1,y_1),(x_2,y_2)\in X\times Y$ and $t\geq 0$. Then we can write

$$\begin{aligned}&\left|F\left(x_1,y_1\right)(t)-F\left(x_2,y_2\right)(t)\right|\\ =&\left|T_1\left(x_1,y_1\right)(t)+T_2\left(x_1,y_1\right)(t)\int_0^{\varphi(t)}u\left(t,s,x_1\left(\alpha\left(s\right)\right),y_1\left(\alpha\left(s\right)\right)\right)ds\right.\\ &\left.-T_1\left(x_2,y_2\right)(t)-T_2\left(x_2,y_2\right)(t)\int_0^{\varphi(t)}u\left(t,s,x_2\left(\alpha\left(s\right)\right),y_2\left(\alpha\left(s\right)\right)\right)ds\right|\\ \leq&\left|T_1\left(x_1,y_1\right)(t)-T_1\left(x_2,y_2\right)(t)\right|\\ &+\left|T_2\left(x_1,y_1\right)(t)-T_2\left(x_2,y_2\right)(t)\right|\int_0^{\varphi(t)}\left|u\left(t,s,x_1\left(\alpha\left(s\right)\right),y_1\left(\alpha\left(s\right)\right)\right)\right|ds\\ &+\left|T_2\left(x_2,y_2\right)(t)\right|\int_0^{\varphi(t)}\left|u\left(t,s,x_1\left(\alpha\left(s\right)\right),y_1\left(\alpha\left(s\right)\right)\right)-u\left(t,s,x_2\left(\alpha\left(s\right)\right),y_2\left(\alpha\left(s\right)\right)\right)\right|ds\\ \leq&\; k_1\left|x_1(t)-x_2(t)\right|+k_2\left|y_1(t)-y_2(t)\right|\\ &+\left(l_1\left|x_1(t)-x_2(t)\right|+l_2\left|y_1(t)-y_2(t)\right|\right)p(t)\int_0^{\varphi(t)}q(s)\,ds\\ &+2\left(l_1\left|x_2(t)\right|+l_2\left|y_2(t)\right|+L\right)p(t)\int_0^{\varphi(t)}q(s)\,ds\\ \leq&\;(k_1+Ul_1)\,\mathrm{diam}X(t)+(k_2+Ul_2)\,\mathrm{diam}Y(t)+2\left((l_1+l_2)\,r_0+L\right)p(t)\int_0^{\varphi(t)}q(s)\,ds.\end{aligned} \tag{3.12}$$

So, taking into account the fact that $p(t)\int_0^{\varphi(t)}q(s)\,ds\to 0$ as $t\to\infty$, we get

$$\limsup_{t\to\infty}\mathrm{diam}F\left(X\times Y\right)(t)\leq (k_1+Ul_1)\limsup_{t\to\infty}\;\mathrm{diam}X(t)+(k_2+Ul_2)\limsup_{t\to\infty}\mathrm{diam}Y(t) \tag{3.13}$$

from (3.12).

Combining (3.11) and (3.13) we obtain that

$$\mu\left(F\left(X\times Y\right)\right)\leq (k_1+Ul_1)\,\mu\left(X\right)+(k_2+Ul_2)\,\mu\left(Y\right). \tag{3.14}$$

On the other hand we can write

$$\mu\left(F\left(X\times Y\right)\right)\leq\max\left\{k_1+Ul_1,k_2+Ul_2\right\}\left(\mu\left(X\right)+\mu\left(Y\right)\right) \tag{3.15}$$

and similarly

$$\mu\left(F\left(Y\times X\right)\right)\leq\max\left\{k_1+Ul_1,k_2+Ul_2\right\}\left(\mu\left(Y\right)+\mu\left(X\right)\right) \tag{3.16}$$

from (3.14). Since $0<\max\left\{k_1+Ul_1,k_2+Ul_2\right\}<1/2$, we can find a positive constant ζ such that $e^{-\zeta}=2\max\left\{k_1+Ul_1,k_2+Ul_2\right\}$. So we obtain

$$\mu\left(F\left(X\times Y\right)\right)+\mu\left(F\left(Y\times X\right)\right)\leq e^{-\zeta}\left(\mu\left(X\right)+\mu\left(Y\right)\right)$$

from (3.15), (3.16) and then by passing to logarithms we have

$$\zeta+\ln\left(\mu\left(F\left(X\times Y\right)\right)+\mu\left(F\left(Y\times X\right)\right)\right)\leq\ln\left(\mu\left(X\right)+\mu\left(Y\right)\right).$$

Hence by using Corollary (3.1) which is a result of Theorem (3.1) we conclude that the operator F has at least one coupled fixed point in $B_{r_0}\times B_{r_0}\subset BC\left(\mathbb{R}_+\right)\times BC\left(\mathbb{R}_+\right)$.

Now we consider the asymptotic stability of the solutions of (3.4). Let us define $\Omega\subset BC\left(\mathbb{R}_+\right)\times BC\left(\mathbb{R}_+\right)$ as

$$\Omega=\left\{(x,y)\in B_{r_0}\times B_{r_0}\subset BC\left(\mathbb{R}_+\right)\times BC\left(\mathbb{R}_+\right):F\left(x,y\right)=x,\ F\left(y,x\right)=y\right\}.$$

That is the set Ω consists of coupled fixed points of F. Also, if we define

$$\tilde{\Omega}=\left\{(y,x)\in B_{r_0}\times B_{r_0}\subset BC\left(\mathbb{R}_+\right)\times BC\left(\mathbb{R}_+\right):(x,y)\in\Omega\right\}$$

then $\Omega=\tilde{\Omega}$. Indeed, it is obvious that $(x,y)\in\Omega$ implies $(y,x)\in\Omega$. On the other hand let us define the sets

$$\Omega_1=\left\{F\left(x,y\right):(x,y)\in\Omega\right\}$$

and

$$\Omega_2=\left\{F\left(y,x\right):(x,y)\in\Omega\right\}.$$

Then we can say $\Omega_1=\Omega_2$ and $\Omega\subset\Omega_1\times\Omega_2\subset B_{r_0}\times B_{r_0}$. Also we get the facts that

$$F\left(\Omega_1\times\Omega_2\right)\supset\Omega_1 \tag{3.17}$$

and

$$F\left(\Omega_2\times\Omega_1\right)\supset\Omega_2 \tag{3.18}$$

from the definitions of Ω,Ω_1 and Ω_2. Since the inequality (3.14) hold for all nonempty subsets of $B_{r_0}\times B_{r_0}$, we can write

$$\begin{aligned}\mu\left(F\left(\Omega_1\times\Omega_2\right)\right) &\leq \left(k_1+Ul_1\right)\mu\left(\Omega_1\right)+\left(k_2+Ul_2\right)\mu\left(\Omega_2\right)\\ &\leq \left(k_1+Ul_1\right)\mu\left(F\left(\Omega_1\times\Omega_2\right)\right)+\left(k_2+Ul_2\right)\mu\left(F\left(\Omega_2\times\Omega_1\right)\right)\\ &= \left(k_1+k_2+U\left(l_1+l_2\right)\right)\mu\left(F\left(\Omega_1\times\Omega_2\right)\right)\end{aligned} \tag{3.19}$$

from the results (3.17) and (3.18). Taking into account $k_1+k_2+U\left(l_1+l_2\right)<1$ and (3.17), (3.18), (3.19) we conclude that

$$\mu\left(F\left(\Omega_1\times\Omega_2\right)\right)=0$$

and so

$$\mu\left(\Omega_1\right)=\mu\left(\Omega_2\right)=0.$$

This result means that

$$\omega_0\left(\Omega_1\right)+\limsup_{t\to\infty}\ \mathrm{diam}\Omega_1\left(t\right)=0$$

and then

$$\limsup_{t\to\infty}\mathrm{diam}\Omega_1\left(t\right)=0.$$

On the other hand, considering the fact

$$\liminf_{t\to\infty} \operatorname{diam}\Omega_1(t) \leq \limsup_{t\to\infty} \operatorname{diam}\Omega_1(t) = 0,$$

we can conlude

$$\begin{aligned} \lim_{t\to\infty} \operatorname{diam}\Omega_1(t) &= 0, \\ \lim_{t\to\infty} \operatorname{diam}\Omega_2(t) &= 0. \end{aligned} \tag{3.20}$$

Let us take arbitrary elements $x_1, x_2 \in \Omega_1$ and $y_1, y_2 \in \Omega_2$ such that (x_1, y_1) and (x_2, y_2) are coupled fixed points of F. Then, by using (3.20) and the definition of μ, we can say that the limits

$$\lim_{t\to\infty} (x_1(t) - x_2(t)) = 0$$

and

$$\lim_{t\to\infty} (y_1(t) - y_2(t)) = 0$$

are satisfied uniformly according to Ω_1 and Ω_2, respectively. Thus we get

$$\lim_{t\to\infty} \|(x_1(t), y_1(t)) - (x_2(t), y_2(t))\|_{\mathbb{R}^2} = \lim_{t\to\infty} \sqrt{(x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2} = 0.$$

This result means that the coupled fixed points of F in $B_{r_0} \times B_{r_0}$ has the same asymptote. That is the solutions of (3.4) are asymptotical stable. □

Example 3.1 *Let us consider the following system in* $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$

$$\begin{aligned} x(t) &= \frac{1 + x(t) - y(t)}{5} + \frac{\sin(x(t))}{3+t} \int_0^t \frac{e^{-(t+s)}}{1 + x^2(s) + y^2(s)} ds \\ y(t) &= \frac{1 + y(t) - x(t)}{5} + \frac{\sin(y(t))}{3+t} \int_0^t \frac{e^{-(t+s)}}{1 + y^2(s) + x^2(s)} ds. \end{aligned} \tag{3.21}$$

In this equation

$$T_1(x, y)(t) = \frac{1 + x(t) - y(t)}{5},$$

$$T_2(x, y)(t) = \frac{\sin(x(t))}{3+t},$$

$$u(t, s, x, y) = \frac{e^{-(t+s)}}{1 + x^2 + y^2}, \ \varphi(t) = t.$$

On the other hand, since

$$\begin{aligned} |T_1(x_1, y_1)(t) - T_1(x_2, y_2)(t)| &= \left| \frac{1 + x_1(t) - y_1(t)}{5} - \frac{(1 + x_2(t) - y_2(t))}{5} \right| \\ &\leq \frac{1}{5} (|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|), \end{aligned}$$

$$\begin{aligned} |T_2(x_1, y_1)(t) - T_2(x_2, y_2)(t)| &= \left| \frac{\sin(x_1(t))}{3+t} - \frac{\sin(x_2(t))}{3+t} \right| \\ &\leq \frac{1}{3} |x_1(t) - x_2(t)|, \end{aligned}$$

and

$$\begin{aligned} |u(t,s,x,y)| &= \left| \frac{e^{-(t+s)}}{1+x^2+y^2} \right| \\ &\le e^{-(t+s)} \end{aligned}$$

hold, we can take

$$k_1 = k_2 = K = \frac{1}{5},\ l_1 = \frac{1}{3},\ l_2 = L = 0,\ p(t) = e^{-t},\ q(s) = e^{-s}.$$

Also

$$\begin{aligned} \lim_{t\to\infty} p(t) \int_0^t q(s)\,ds &= \lim_{t\to\infty} e^{-t} \int_0^t e^{-s} ds \\ &= \lim_{t\to\infty} \left(e^{-t} - e^{-2t}\right) = 0, \end{aligned}$$

$$U = \sup\left\{ p(t) \int_0^t q(s)\,ds : t \ge 0 \right\} = \frac{1}{4}$$

and

$$\max\{k_1 + Ul_1, k_2 + Ul_2\} = \frac{17}{60} < \frac{1}{2}.$$

So if we take

$$r_0 \ge \frac{K + LU}{1 - [(k_1+k_2) + U(l_1+l_2)]} = \frac{12}{31}$$

then by using Theorem (3.2), we concude that equation (3.21) has at least one solution in the unit ball B_1. Also the solutions of equation (3.21) are asymptotically stable.

3.3. A Problem on a System of Integral Equations Arising from Mathematical Modelling in Epidemiology

We will consider a systems of integral equations arising from modelling of spread of a disease in a population. We assume that, the population consists of non-intersecting three compartments; Susceptible (S), Infected (I) and Recovered (R). We assume that all new members of the population get involved in the susceptible class S at a constant rate b. In this model when a susceptible individual gets in effective contact with an infective one, the pathogen can be transmitted from the infectious to the susceptible. In this case the susceptible individual becomes candidate for class I with a certain probability (β) changing according to some rates. But the susceptible individual may not be infectious immediately. The period of after effective contact before becoming infective is defined as incubation period. In this model the incubation period is denoted by τ. λ is natural death rate in each compartment, δ is death rate derived from the pathogen and γ represents the rate of recovery. Also $N(t)$ shows the total number of the population at time t and $S(t) + I(t) + R(t) = N(t)$ for all $t \ge 0$. Naturally, the functions S, I, R and N are nonnegative and bounded.

Under the above assumptions the mathematical modelling of spread of a disease in a population can be given with the following system of differential equation:

$$\begin{aligned} \frac{dS}{dt} &= b - \beta e^{-\lambda\tau} S(t-\tau) I(t-\tau) - \lambda S(t), \\ \frac{dI}{dt} &= \beta e^{-\lambda\tau} S(t-\tau) I(t-\tau) - (\lambda+\delta+\gamma) I(t), \\ \frac{dR}{dt} &= \gamma I(t) - \lambda R(t). \end{aligned}$$

It can be seen that $R(t)$ do not appear in other remainder equations, we can only analysis the following system and it will be enough as mathematically.

$$\begin{aligned}\frac{dS}{dt} &= b - \beta e^{-\lambda\tau} S(t-\tau) I(t-\tau) - \lambda S(t),\\ \frac{dI}{dt} &= \beta e^{-\lambda\tau} S(t-\tau) I(t-\tau) - (\lambda+\delta+\gamma) I(t). \end{aligned} \tag{3.22}$$

On the other hand, in general, the studies on stability analysis of this model depends on basic reproduction number $\mathcal{R}_0$ which is defined as the number of secondary cases generated by an infected individual in a completely susceptible population. If $\mathcal{R}_0 < 1$ then the disease will not give rise to large outbreaks, and gradually dies out. On the other hand, if $\mathcal{R}_0 > 1$ then the disease continues to spread in the population.

Example 3.2 *Let us take the following system of integral equations in* $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$

$$\begin{aligned} S(t) &= bt - \beta \int_0^{t-\tau} e^{-\lambda\tau} S(z) I(z)\,dz - \lambda \int_0^t S(z)\,dz,\\ I(t) &= \beta \int_0^{t-\tau} e^{-\lambda\tau} S(z) I(z)\,dz - (\lambda+\delta+\gamma) \int_0^t I(z)\,dz. \end{aligned} \tag{3.23}$$

It can be seen that the equation (3.22) can be obtained from the equation (3.23) by taking derivative according to t. So, considering of the equation (3.23) will be enough to analyze of the equation (3.22).

If we take

$$T_1(x,y)(t) = \frac{b}{2}\left[1 + sgn(x(0) - y(0))\right] t - \left(\lambda + \frac{(\delta+\gamma)}{2}\left[1 - sgn(x(0) - y(0))\right]\right) \int_0^t x(z)\,dz,$$

$$T_2(x,y)(t) = \beta sgn(x(0) - y(0)),$$

$$u(t,s,x,y) = e^{-\lambda\tau} xy,\ \ \varphi(t) = t - \tau,\ \ \alpha(z) = z$$

and

$$x = S,\ \ y = I$$

then the equation (3.4) that is the system

$$\begin{aligned} S(t) &= T_1(S,I)(t) + T_2(S,I)(t) \int_0^{\varphi(t)} u(t,s,S(\alpha(z)),I(\alpha(z)))\,dz,\\ I(t) &= T_1(I,S)(t) + T_2(I,S)(t) \int_0^{\varphi(t)} u(t,s,I(\alpha(z)),S(\alpha(z)))\,dz \end{aligned}$$

turns to the system (3.23). Here, it is assumed that $S(0) > I(0)$*. Indeed, it is an obvious fact that at the beginning of an disease, the number of susceptible individuals* $(S(0))$ *is more than the number of infected individuals* $(I(0))$*.*

Problem 1 *The fact that equation (3.23) can be obtained by appropriate selection of the operators in equation (3.4) is an extremely important result to see the effectiveness of equation (3.4). However, obtaining detailed results about the solutions of equation (3.23) is a different problem in itself. For interested researchers, this is a problem worth focusing on.*

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