Mixed Hemivariational Inequality Arising in a Thermo-Elastic Frictional Contact Problem

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ABSTRACT: This paper presents a new mathematical model for the analysis of frictional contact between a thermoelastic body and a foundation. The contact interaction is described through a combination of unilateral frictional conditions, nonmonotone multivalued contact laws, and friction laws formulated via the Clarke subdifferential. A variational formulation of the problem is developed, and the existence and uniqueness of a weak solution are rigorously proved.

Key Words: Mixed hemivariational inequality, Lagrange multiplier, elastic contact problem, thermal effects.

Contents

1. Introduction

In recent years, the theory of variational inequalities has garnered significant scholarly interest owing to its diverse applications. As one of the most dynamically evolving branches of applied mathematics, it has undergone substantial extensions across multiple research directions. Whereas variational inequalities address models characterized by monotone processes, hemivariational inequalities encompass non-monotone relations governed by nonconvex subdifferentials, with considerable evolution occurring through the framework of multivalued operators. This latter domain originated with Panagiotopoulos' foundational contribution [1], employing the Clarke generalized subgradient for locally Lipschitz functions [2]. Hemivariational inequalities have subsequently proven instrumental in modeling phenomena across diverse disciplines including contact mechanics, physics, and engineering; see [3,4,5,6,7,8,9,11,20,?,?] and associated references.

Conversely, substantial contributions have emerged in this domain. Significant results concerning generalized mixed variational-hemivariational inequalities were recently established by Bai, Migórski, and Zeng [10], who demonstrated abstract properties of solution sets including existence, uniqueness, boundedness, convexity, closedness, and stability. The investigation in [12] examines an elliptic system governed by mixed boundary conditions, featuring a nonhomogeneous partial differential operator coupled with both a hemivariational inequality and a Lagrange multiplier inequality constraint. This system provides a mathematical framework for mechanical frictional contact phenomena. Subsequent research [13] formulates the elastic frictional contact problem within the theoretical structure of mixed variational-hemivariational inequalities. Thermo-elastic frictional contact problems incorporating temperature-dependent material behavior were analyzed in [14]. Further developments include [15], which presents numerical approximations for frictional contact between thermoelectric locking piezoelectric materials and rigid foundations. Quasi-static thermoelectric contact problems combining Signorini conditions with Coulomb friction laws were investigated in [16]. Building upon prior work in [17], recent

advances propose novel mixed hemivariational inequality approaches to thermo-elastic frictional contact problems, extending established methodologies in this field.

This work is organized as follows. Section 2 presents essential definitions and foundational results required for subsequent developments. Section 3 investigates a complex thermomechanical frictional contact problem within elasticity theory, deriving its variational formulation. Finally, Section 4 establishes the existence and uniqueness of solutions to the proposed model through the framework of mixed hemivariational inequalities.

2. Background

This section establishes essential notations, definitions, and foundational results employed throughout this work. We first recall the generalized (Clarke) directional derivative and the Clarke generalized subgradient.

Definition 2.1 *Let X be a Banach space and $J \colon X \to \mathbb{R}$ be a locally Lipschitz functional. The Clarke generalized directional derivative of J at a point $u \in X$ in the direction $v \in X$ is defined by:*

$$J^0(u; v) = \limsup_{\substack{y \to u \\ t \downarrow 0}} \frac{J(y + tv) - J(u)}{t}.$$

The generalized subgradient (or Clarke subdifferential) of J at u is the subset of the dual space X^* given by:

$$\partial J(u) = \left\{ \xi \in X^* \mid J^0(u; v) \geq \langle \xi, v \rangle \quad \forall v \in X \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X^* and X.

Next, we present a fundamental result on a mixed hemivariational inequality that will be employed to establish the unique solvability of an elastic frictional contact problem incorporating thermal effects. Let $(V, \| \cdot \|_V)$ and $(Z, \| \cdot \|_Z)$ be reflexive Banach spaces, and let $\Lambda \subseteq Z$ be a nonempty subset. Denote by $\langle \cdot, \cdot \rangle_{V^* \times V}$ the duality pairing between V^* and V. Consider further a Banach space $(X, \| \cdot \|_X)$. Given an operator $A \colon V \to V^*$, a locally Lipschitz functional $J \colon X \to \mathbb{R}$, a bilinear form $b \colon V \times Z \to \mathbb{R}$, a trace operator $\gamma \colon V \to X$, and an element $f \in V^*$, we analyze the following generalized mixed hemivariational inequality:

Problem 1 Find $u \in V$ and $\lambda \in \Lambda$ such that the following system holds:

$$\langle Au, v - u \rangle + b(v - u, \lambda) + J^0(\gamma u; \gamma v - \gamma u) \geq \langle f, v - u \rangle \qquad \text{for all } v \in V, \qquad (2.1)$$
$$b(u, \rho - \lambda) \leq 0 \qquad \text{for all } \rho \in \Lambda. \qquad (2.2)$$

Here, the problem's data A, h, b, J and γ are assumed to verify the following assumptions.

$H(A)$: The operator $A \colon V \longrightarrow V^*$ is bounded, hemi-continuous and there exists a positive constant $m_A > 0$ such that

$$\langle Au - Av, u - v \rangle \geq m_A \|u - v\|_V^2 \qquad \text{for all } u, v \in V.$$

$H(J)$: The mapping $J \colon X \longrightarrow \mathbb{R}$ satisfies the following conditions

   (a) $J(.)$ is locally Lipschitz continuous,
   (b) there exists $m_J > 0$ such that, for all $v_i \in X$, $i = 1, 2$,

   $$J^0(v_1; v_2 - v_1) + J^0(v_2; v_1 - v_2) \leq m_J \|v_1 - v_2\|_V^2,$$

   (c) $\|\partial J(v)\|_V^* \leq c_{0J} + c_{1J}\|v\|_Y$, for all $v \in X$, with $c_{0J}, c_{1J} \leq 0$.

$H(\Lambda)$: Λ is nonempty, closed, convex, and contains 0_Z.

$H(b)$: Bilinear form $b \colon V \times Z \to \mathbb{R}$ satisfies:

(a) there exists $M_b > 0$ such that $|b(u,v)| \leq M_b \|u\|_V \|v\|_E$,

(b) there exists $\alpha_b > 0$ such that, for all $v_i \in X$, $i = 1, 2$,

$$\inf_{\rho \in Z, \rho \neq 0_Z} \sup_{v \in V, v \neq 0_V} \frac{b(v,\rho)}{\|v\|_V \|\rho\|_Z} \geq \alpha_b.$$

$H(\gamma)$: $\gamma \colon V \longrightarrow Z$ is a linear compact operator, and $\|\gamma\|$ is the norm of γ.

$H(f)$: $f \in V^*$.

H_0 : $m_A > m_J \|\gamma\|^2$.

Theorem 2.1 *Under hypotheses $H(J)$, $H(A)$, $H(b)$, $H(\gamma)$, $H(f)$, $H(\Lambda)$, and H_0, Problem 1 admits at least one solution $(u, \lambda) \in V \times \Lambda$. The solution component u is unique.*

Theorem 2.1 is a direct consequence of [18, Theorem 15] since it is easy to verify that all the hypotheses of [18, Theorem 15] hold under the assumptions of Theorem 2.1.

3. Physical setting and weak formulation

This section implements the abstract theory of mixed hemivariational inequalities to model a static frictional contact problem for a thermomechanically coupled nonlinear elastic body. The contact interface is characterized by Signorini's unilateral conditions (frictionless normal compliance) and a nonmonotone multivalued friction law described via a locally Lipschitz superpotential.

Consider the physical configuration of the contact model. The thermo-elastic body occupies an open, bounded, and connected domain $\Omega \subset \mathbb{R}^d$ ($d \in \{2, 3\}$). The boundary $\Gamma := \partial\Omega$ is Lipschitz continuous and admits a decomposition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ where $\{\Gamma_i\}_{i=1}^4$ are pairwise disjoint measurable subsets satisfying $\mathrm{meas}(\Gamma_1) > 0$. The following standard notation is then adopted: ν denotes the unit outward normal on the boundary Γ, $(\mathbb{S}^d, \|.\|_{\mathbb{S}^d})$ denotes the space of the symmetric tensor of the second order on $\mathbb{R}^d$ and the inner products and norms in $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by:

$$u.v = u_i v_i, \quad \|v\|_{\mathbb{R}^d} = (v.v)^{\frac{1}{2}} \quad \text{for all } u, v \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\|_{\mathbb{S}^d} = (\tau.\tau)^{\frac{1}{2}} \quad \text{for all } \sigma, \tau \in \mathbb{S}^d.$$

Then, the classical formulation of the elastic frictional contact problem with thermal effect described in a following way.

Problem 2 Find a displacement field $u \colon \Omega \longrightarrow \mathbb{R}^d$, a stress field $\sigma \colon \Omega \longrightarrow \mathbb{S}^d$ and a temperature

$\theta : \Omega \longrightarrow \mathbb{R}$ such that:

$$\sigma = \mathbb{A}\varepsilon(u) - \mathbb{M}\theta \qquad\qquad\qquad in \ \Omega, \qquad (3.1)$$

$$q = -\mathbb{K}\nabla\theta \qquad\qquad\qquad in \ \Omega, \qquad (3.2)$$

$$Div\,\sigma + f_0 = 0 \qquad\qquad\qquad in \ \Omega, \qquad (3.3)$$

$$-\nabla q = h_0 + \mathbb{M}\varepsilon(u) \qquad\qquad\qquad in \ \Omega, \qquad (3.4)$$

$$u = 0 \qquad\qquad\qquad on \ \Gamma_1, \qquad (3.5)$$

$$\theta = 0 \qquad\qquad\qquad on \ \Gamma_1, \qquad (3.6)$$

$$\sigma.\nu = f_2 \qquad\qquad\qquad on \ \Gamma_2, \qquad (3.7)$$

$$q.\nu = q_2 \qquad\qquad\qquad on \ \Gamma_2, \qquad (3.8)$$

$$\begin{cases} u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu.u_\nu = 0, \\ \sigma_\tau = 0. \end{cases} \qquad on \ \Gamma_3, \qquad (3.9)$$

$$\begin{cases} \theta_\nu \leq 0, \quad q_\nu \leq 0, \quad q_\nu.\theta_\nu = 0, \\ q_\tau = 0. \end{cases} \qquad on \ \Gamma_3, \qquad (3.10)$$

$$\begin{cases} -\sigma_\nu \in \partial j_\nu(u_\nu), \\ -\sigma_\tau \in \mu(\|u_\tau\|)\partial j_\tau(u_\tau). \end{cases} \qquad on \ \Gamma_4, \qquad (3.11)$$

$$\begin{cases} -q_\nu \in \partial j_{\theta_\nu}(\theta_\nu), \\ -q_\tau \in \partial j_{\theta_\tau}(\theta_\tau) - h_\tau(\|u_\tau\|). \end{cases} \qquad on \ \Gamma_4, \qquad (3.12)$$

We now provide a concise interpretation of the governing equations and boundary conditions (3.1)-(3.12). Equations (3.1) and (3.2) define the thermoelastic constitutive law, where $\mathbb{A}$ denotes the nonlinear elasticity operator, $\mathbb{M}$ represents the thermal expansion operator, σ is the Cauchy stress tensor, and $\varepsilon(u)$ signifies the linearized strain tensor for displacement field u:

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^\top\right),$$

with ∇u denoting the displacement gradient and $(\nabla u)^\top$ its transpose. The thermal conductivity tensor $\mathbb{K}$ relates to heat flux vector q through Fourier's law. Equation (3.3) expresses mechanical equilibrium with f_0 representing volumetric force density, while (3.4) governs heat conduction with h_0 as the volumetric heat source.

Boundary conditions are specified as follows: On Γ_1, the body is rigidly clamped (zero displacement, (3.5)) and thermally insulated (zero heat flux, (3.6)). Surface traction f_2 acts on Γ_2 ((3.7)), with prescribed heat flux q_2 on Γ_2 ((3.8)). The contact boundary partitions into two distinct regimes: Γ_3 features Signorini-type frictionless unilateral contact conditions (3.9)-(3.10), while Γ_4 exhibits multivalued subdifferential contact conditions (3.11)-(3.12). Here h_τ denotes frictional heating, with ∂j_ν and ∂j_τ representing Clarke subdifferentials of locally Lipschitz functions $j_\nu(\cdot;\cdot)$ and $j_\tau(\cdot;\cdot)$ with respect to their final arguments.

To establish solution existence and uniqueness for Problem 2, we impose the following hypotheses: The elasticity operator $\mathbb{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies the following conditions

$$\begin{cases} (a) \quad \text{for all } \varepsilon \in \mathbb{S}^d, \ x \longmapsto \mathbb{A}(x,\varepsilon) \text{ is measurable on } \Omega \text{ with } \mathbb{A}(x,0_{\mathbb{S}^d}) = 0_{\mathbb{S}^d}; \\ (b) \quad \text{there is } L_A > 0 \text{ such that, for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ \qquad \|\mathbb{A}(x,\varepsilon_1) - \mathbb{A}(x,\varepsilon_2)\|_{\mathbb{S}^d} \leq L_\mathbb{A}\|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d} \qquad \text{for a.e., } x \in \Omega; \\ (c) \quad \text{there exists a positive constant } m_\mathbb{A} > 0 \text{ such that, for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \\ \qquad (\mathbb{A}(x,\varepsilon_1) - \mathbb{A}(x,\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq m_\mathbb{A}\|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d} \quad \text{for a.e., } x \in \Omega. \end{cases} \qquad (3.13)$$

The thermal conductivity operator $\mathbb{K} : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ verifies the following conditions:

$$\begin{cases} (a) \text{ for all } \xi \in \mathbb{R}^d, \ x \longmapsto \mathbb{K}(x,\xi) \text{ is measurable on } \Omega \text{ with } \mathbb{K}(x,0_{\mathbb{R}^d}) = 0_{\mathbb{R}^d}; \\ (b) \text{ there exists a positive constant } m_\mathbb{K} > 0 \text{ such that, for all } \xi_1, \xi_2 \in \mathbb{R}^d, \\ \qquad (\mathbb{K}(x,\xi_1) - \mathbb{K}(x,\xi_2)) : (\xi_1 - \xi_2) \geq m_\mathbb{K}\|\xi_1 - \xi_2\|_{\mathbb{R}^d} \quad \text{for a.e., } x \in \Omega. \end{cases} \qquad (3.14)$$

The thermal expansion operator $\mathbb{M} : \Omega \times \mathbb{R} \longrightarrow \mathbb{S}^d$ satsfies

$$
\begin{cases}
(a) \text{ for all } r \in \mathbb{R}, \ x \longmapsto \mathbb{M}(x, \kappa) \text{ is measurable on } \Omega \text{ with } \mathbb{M}(x, 0) = 0; \\
(b) \text{ there is } L_M > 0 \text{ such that, for all } r_1, r_2 \in \mathbb{R}, \\
\quad \|\mathbb{M}(x, \kappa_1) - \mathbb{M}(x, \kappa_2)\|_{\mathbb{S}^d} \leq L_{\mathbb{M}} \|\kappa_1 - \kappa_2\|_{\mathbb{R}} \qquad \text{for a.e., } x \in \Omega;
\end{cases}
\tag{3.15}
$$

The functions $j_a : \Gamma_4 \times \mathbb{R} \longrightarrow \mathbb{R}$, with $a \in (\nu, \theta_\nu)$ satisfy the following conditions:

$$
\begin{cases}
(a) \ \ j_a(., \kappa) \text{ is measurable on } \Gamma_4 \text{ for all } \kappa \in \mathbb{R}, \text{ with } j_a(., 0) \in L^1(\Gamma_4); \\
(a) \ \ j_a(., \kappa) \text{ is locally Lipschitz on } \mathbb{R} \text{ for all } x \in \Gamma_4; \\
(b) \ \ \text{there exists } c_{0a}, c_{1a} \geq 0 \text{ such that} \\
\quad |\partial j_a(x, \kappa)| \leq c_{0a} + c_{1a}|\kappa| \qquad \text{for all } \kappa \in \mathbb{R}, \text{ a.e., } x \in \Omega; \\
(c) \ \ \text{there exists a constant } \alpha_{j_a} > 0 \text{ such that, for all } \kappa_1, \kappa_2 \in \mathbb{R}, \\
\quad j_a^0(x, \kappa_1; \kappa_2 - \kappa_1) + j_a^0(x, \kappa_2; \kappa_1 - \kappa_2) \leq \alpha_{j_a} |\kappa_1 - \kappa_2|^2 \quad \text{for a.e., } x \in \Omega.
\end{cases}
\tag{3.16}
$$

The functions $j_s : \Gamma_4 \times \mathbb{R}^d \longrightarrow \mathbb{R}$, with $s \in (\tau, \theta_\tau)$ satisfy the following conditions:

$$
\begin{cases}
(a) \ \ j_s(., \xi) \text{ is measurable on } \Gamma_4 \text{ for all } \xi \in \mathbb{R}^d, \text{ with } j_s(., 0) \in L^1(\Gamma_4); \\
(a) \ \ j_s(., \xi) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for all } x \in \Gamma_4; \\
(b) \ \ \text{there exists } c_{0s}, c_{1s} \geq 0 \text{ such that} \\
\quad \|\partial j_s(x, \xi)\| \leq c_{0s} + c_{1s}\|\xi\| \qquad \text{for all } \xi \in \mathbb{R}^d, \text{ a.e., } x \in \Omega; \\
(c) \ \ \text{there exists a constant } \alpha_{j_s} > 0 \text{ such that, for all } \xi_1, \xi_2 \in \mathbb{R}, \\
\quad j_s^0(x, \xi_1; \xi_2 - \xi_1) + j_s^0(x, \xi_2; \xi_1 - \xi_2) \leq \alpha_{j_s} \|\xi_1 - \xi_2\|^2 \quad \text{for a.e., } x \in \Omega.
\end{cases}
\tag{3.17}
$$

The function $\mu : \Gamma_4 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satsfies

$$
\begin{cases}
(a) \ \mu(., \kappa) \text{ is measurable on } \Gamma_4 \text{ for all } \kappa \in \mathbb{R}^+; \\
(b) \ \text{there is } L_\mu > 0 \text{ such that, for all } \kappa_1, \kappa_2 \in \mathbb{R}, \\
\quad |\mu(x, \kappa_1) - \mu(x, \kappa_2)| \leq L_\mu \|\kappa_1 - \kappa_2\| \qquad \text{for a.e., } x \in \Gamma_4; \\
(c) \ \text{there exists } \mu_0 \geq 0 \text{ such that } \mu(x, r) \leq \mu_0 \text{ for all } r \in \mathbb{R}^+, \text{ a.e., } x \in \Gamma_4.
\end{cases}
\tag{3.18}
$$

The heat function $h_\tau : \Gamma_4 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is such that

$$
\begin{cases}
(a) \ \ |h_\tau(x, \kappa_1) - h_\tau(x, \kappa_2)| \leq L_{h_\tau} \|\kappa_1 - \kappa_2\|, \ \forall \kappa_1, \kappa_2 \in \mathbb{R}, \text{ with } L_{h_\tau} > 0, \\
(b) \ \ h_\tau(., \kappa) \in L^2(\Gamma_4) \text{ for all } \kappa \in \mathbb{R}^+, \\
(c) \ \ (h_\tau(x, \kappa_1) - h_\tau(x, \kappa_1)) . (\kappa_1 - \kappa_2) \geq 0, \text{ for all } \kappa_1, \kappa_2 \in \mathbb{R}, \text{ a.e., } x \in \Gamma_4.
\end{cases}
\tag{3.19}
$$

The densities of volume forces, surface tractions, and heat sources satisfy the following:

$$
f_0 \in L^2(\Omega; \mathbb{R}^d), \quad f_2 \in L^2(\Gamma_2; \mathbb{R}^d), \quad q_0 \in L^2(\Omega), \quad q_2 \in L^2(\Gamma_2).
\tag{3.20}
$$

To derive the weak formulation of Problem 2, we introduce the following function spaces:

$$
H = L^2(\Omega; \mathbb{R}^d), \qquad \mathcal{H} = L^2(\Omega; \mathbb{S}^d), \qquad \mathcal{H}_1 = \{\sigma \in \mathcal{H} \ : \ \text{Div}\,\sigma \in \mathcal{H}\}
$$

It is known that H, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed respectively with the following inner products and associated norms:

$$
\langle u, v \rangle_H = \int_\Omega u(x)v(x)dx, \qquad \|v\|_H = \langle v, v \rangle_H^{\frac{1}{2}} \qquad \text{for all } u, \ v \in H,
$$

$$
\langle \sigma, \tau \rangle_{\mathcal{H}} = \int_\Omega \sigma(x)\tau(x)dx, \qquad \|\tau\|_{\mathcal{H}} = \langle \tau, \tau \rangle_{\mathcal{H}}^{\frac{1}{2}} \qquad \text{for all } \sigma, \ \tau \in \mathcal{H}.
$$

The displacement field is defined on the space

$$
E = \left\{ v \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_1 \right\},
$$

which forms a closed subspace of $H^1(\Omega; \mathbb{R}^d)$. Given that $\text{meas}(\Gamma_1) > 0$, Korn's inequality ensures that E is a Hilbert space when equipped with the inner product

$$\langle u, v \rangle_E = \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}$$

and corresponding norm

$$\|v\|_E = \|\varepsilon(v)\|_{\mathcal{H}},$$

for all $u, v \in E$. The trace operator on E is denoted by $\gamma_1 : E \to L^2(\Gamma; \mathbb{R}^d)$ and a map $\alpha : E \to L^2(\Gamma)$ is defined by:

$$\alpha(v)(x) = v_\nu(x) = \gamma_1 v(x).\nu \qquad \text{on } \Gamma, \text{ for all } v \in E,$$

and $X = H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$, the image of E under the function α is defined as follows:

$$X = \{w \in L^2(\Gamma)\} : \exists v \in E \text{ such that } w = \alpha(v) \text{ on } \Gamma\}.$$

By the trace theorem, it is easy to find that the space X endowed with the norm:

$$\|w\|_X^2 = \|w\|_{L^2(\Gamma)}^2 + \int \int \frac{|w(x) - w(y)|^2}{\|x - y\|_{\mathbb{R}^d}^2} \, dx \, dy,$$

is a reflexive Banach space. Then, a subset U of E are defined as a closed and convex subset $\mathcal{U}$ of X^*. For this reason;

$$U = \{v \in E \quad \text{such that } v_\nu \leq 0 \text{ on } \Gamma_3\}, \tag{3.21}$$

$$\mathcal{U} = \{\rho \in X^* \quad \text{such that } \langle \rho, v_n \rangle_{X^* \times X} \leq 0 \quad \text{for all } v \in E\}. \tag{3.22}$$

The temperature field is defined on the space

$$Q = \{\eta \in H^1(\Omega) \mid \eta = 0 \text{ on } \Gamma_1\},$$

which constitutes a closed subspace of $H^1(\Omega)$. Given $\text{meas}(\Gamma_1) > 0$, the Poincaré inequality guarantees that Q is a Hilbert space when equipped with the inner product

$$\langle \theta, \eta \rangle_Q = \langle \nabla\theta, \nabla\eta \rangle_{\mathcal{H}}$$

and corresponding norm

$$\|\eta\|_Q = \|\nabla\eta\|_{\mathcal{H}},$$

for all $\theta, \eta \in Q$. Let $\gamma_2 : Q \to L^2(\Gamma)$ be the trace operator on Q. Define the normal component operator $\beta : Q \to L^2(\Gamma)$ by

$$\beta(\eta)(x) = \eta_\nu(x) = \gamma_2\eta(x) \cdot \nu \quad \text{on } \Gamma, \quad \forall \eta \in Q.$$

Analogous to the displacement field treatment, define $Y = \beta(Q)$ as the range space:

$$Y = \{\xi \in L^2(\Gamma) \mid \exists\eta \in Q \text{ such that } \xi(x) = \beta(\eta)(x) \text{ a.e. } x \in \Gamma\},$$

which forms a reflexive Banach space when endowed with the Sobolev-Slobodeckij norm:

$$\|\xi\|_Y^2 = \|\xi\|_{L^2(\Gamma)}^2 + \iint_\Gamma \frac{|\xi(x) - \xi(y)|^2}{\|x - y\|_{\mathbb{R}^d}^2} \, dx \, dy.$$

Furthermore, define the closed convex set $\Theta \subset Q$ and the admissible set $S \subset Y^*$ as:

$$\Theta = \{\eta \in Q \mid \eta_\nu \leq 0 \text{ on } \Gamma_3\}, \tag{3.23}$$

$$S = \{\psi \in Y^* \mid \langle \psi, \eta_\nu \rangle_{Y^* \times Y} \leq 0 \quad \forall \eta \in Q\}. \tag{3.24}$$

To derive the weak formulation of Problem 2, we assume sufficient regularity of the solution components (u, σ, θ, q) on Ω. Consider test functions $v \in E$ for the mechanical system. Multiplying (3.1)

by $v - u$ and applying Green's formula yields: The weak formulation derivation begins with sufficient regularity assumptions on the solution components (u, σ, θ, q). For the displacement field, consider test functions $v \in E$. Multiplying (3.1) by $v - u$ and applying Green's formula yields:

$$\langle \sigma, \varepsilon(v) - \varepsilon(u) \rangle_{E^* \times E} = \langle f_0, v - u \rangle_{E^* \times E} + \int_\Gamma \sigma\nu \cdot (v - u) da.$$

Decomposing the boundary integral over $\Gamma = \bigcup_{i=1}^4 \Gamma_i$ and applying boundary conditions (3.5) (displacement constraint) and (3.7) (traction condition) gives:

$$\langle \sigma, \varepsilon(v) - \varepsilon(u) \rangle_{E^* \times E} = \langle f_0, v - u \rangle_{E^* \times E} + \int_{\Gamma_2} f_2 \cdot (v - u) da$$
$$+ \int_{\Gamma_3} \sigma\nu \cdot (v - u) da + \int_{\Gamma_4} \sigma\nu \cdot (v - u) da.$$

The Riesz representation theorem defines $f \in E^*$ satisfying:

$$\langle f, v - u \rangle_{E^* \times E} = \langle f_0, v - u \rangle_{E^* \times E} + \int_{\Gamma_2} f_2 \cdot (v - u) da,$$

resulting in:

$$\langle \sigma, \varepsilon(v) - \varepsilon(u) \rangle_{E^* \times E} = \langle f, v - u \rangle_{E^* \times E} + \sum_{i=3}^4 \int_{\Gamma_i} \sigma\nu \cdot (v - u) da. \tag{3.25}$$

Decomposing into normal/tangential components and applying contact conditions (3.9), (3.11) yields:

$$\int_{\Gamma_3} \sigma\nu \cdot (v - u) da = \int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu) da, \tag{3.26}$$

$$\int_{\Gamma_4} \sigma\nu \cdot (v - u) da \geq - \int_{\Gamma_4} j_\nu^0(u_\nu; v_\nu - u_\nu) da - \int_{\Gamma_4} \mu(\|u_\tau\|) j_\tau^0(u_\tau; v_\tau - u_\tau) da. \tag{3.27}$$

Combining (3.25)-(3.27) produces the inequality:

$$\langle \sigma, \varepsilon(v) - \varepsilon(u) \rangle_{E^* \times E} + \int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu) da$$
$$+ \int_{\Gamma_4} j_\nu^0(u_\nu; v_\nu - u_\nu) da + \int_{\Gamma_4} \mu(\|u_\tau\|) j_\tau^0(u_\tau; v_\tau - u_\tau) da \geq \langle f, v - u \rangle_{E^* \times E}. \tag{3.28}$$

Introducing a multiplier $\bar{\lambda} \in X^*$ via:

$$\langle \bar{\lambda}, w \rangle_{X^* \times X} \leq - \int_{\Gamma_3} \sigma_\nu w_\nu da \quad \forall w \in X, \tag{3.29}$$

and bilinear form $\bar{b} : E \times X^* \to \mathbb{R}$:

$$\bar{b}(v, \rho) = \langle \rho, v_\nu \rangle_{X^* \times X} \quad \forall v \in E, \rho \in X^*, \tag{3.30}$$

we express the Γ_3 integral as:

$$\int_{\Gamma_3} \sigma\nu \cdot (v - u) da = \bar{b}(u - v, \bar{\lambda}). \tag{3.31}$$

Substitution into (3.28) yields:

$$\langle \sigma, \varepsilon(v) - \varepsilon(u) \rangle_{E^* \times E} + \bar{b}(u - v, \bar{\lambda})$$
$$+ \int_{\Gamma_4} j_\nu^0(u_\nu; v_\nu - u_\nu) da + \int_{\Gamma_4} \mu(\|u_\tau\|) j_\tau^0(u_\tau; v_\tau - u_\tau) da \geq \langle f, v - u \rangle_{E^* \times E}. \tag{3.32}$$

Contact conditions imply $u \in U$ and $\bar{\lambda} \in \mathcal{U}$, with the variational inequality:

$$\bar{b}(u, \rho - \bar{\lambda}) = \int_{\Gamma_3} \sigma_\nu u_\nu da + \langle \rho, u_\nu \rangle_{X^* \times X} \leq 0 \quad \forall \rho \in \mathcal{U}. \tag{3.33}$$

Incorporating the constitutive law (3.1), we obtain the displacement formulation:

Problem 3 *Find* $(u, \bar{\lambda}) \in U \times \mathcal{U}$ *satisfying:*

$$\langle \mathbb{A}\varepsilon(u), \varepsilon(v) - \varepsilon(u) \rangle_{E^* \times E} - \langle \mathbb{M}\theta, \varepsilon(v) - \varepsilon(u) \rangle_{E^* \times E} + \bar{b}(v - u, \bar{\lambda})$$
$$+ \int_{\Gamma_4} j_\nu^0(u_\nu; v_\nu - u_\nu) da + \int_{\Gamma_4} \mu(\|u_\tau\|) j_\tau^0(u_\tau; v_\tau - u_\tau) da \geq \langle f, v - u \rangle_{E^* \times E}, \tag{3.34}$$

$$\bar{b}(u, \rho - \bar{\lambda}) \leq 0, \tag{3.35}$$

for all $v \in U$, $\rho \in \mathcal{U}$.

For the temperature field, multiply (3.4) by $\eta - \theta$ ($\eta \in \Theta$) and apply Green's formula:

$$\langle -q, \nabla\eta - \nabla\theta \rangle_{Q^* \times Q} - \langle \mathbb{M}\varepsilon(u), \nabla\eta - \nabla\theta \rangle_{Q^* \times Q} + \int_\Gamma q \cdot \nu(\eta - \theta) da = \langle h_0, \eta - \theta \rangle_{Q^* \times Q}.$$

Incorporating boundary conditions (3.6), (3.8) and defining $h \in Q^*$ to obtain

$$\langle h, \eta \rangle_{Q^* \times Q} = \langle h_0, \eta \rangle_{Q^* \times Q} + \int_{\Gamma_2} q_2 \eta da,$$

we obtain:

$$\langle -q, \nabla\eta - \nabla\theta \rangle_{Q^* \times Q} - \langle \mathbb{M}\varepsilon(u), \nabla\eta - \nabla\theta \rangle_{Q^* \times Q}$$
$$+ \int_{\Gamma_3} q \cdot \nu(\eta - \theta) da + \int_{\Gamma_4} q \cdot \nu(\eta - \theta) da = \langle h, \eta - \theta \rangle_{Q^* \times Q}. \tag{3.36}$$

Using conditions (3.10), (3.12) and Clarke's calculus:

$$\int_{\Gamma_4} q \cdot \nu(\eta - \theta) da \geq - \int_{\Gamma_4} j_{\theta_\nu}^0(\theta_\nu; \eta_\nu - \theta_\nu) da - \int_{\Gamma_4} j_{\theta_\tau}^0(\theta_\tau; \eta_\tau - \theta_\tau) da$$
$$- \int_{\Gamma_4} h_\tau(\|u_\tau\|)(\eta - \theta) da. \tag{3.37}$$

Introducing multiplier $\tilde{\lambda} \in Y^*$:

$$\langle \tilde{\lambda}, \eta \rangle_{Y^* \times Y} = \int_{\Gamma_3} q_\nu \eta_\nu da \quad \forall \eta \in Y, \tag{3.38}$$

and bilinear form $\tilde{b} : Q \times Y^* \to \mathbb{R}$:

$$\tilde{b}(\eta, \zeta) = \langle \zeta, \eta_\nu \rangle_{Y^* \times Y} \quad \forall \eta \in Q, \zeta \in Y^*, \tag{3.39}$$

the Γ_3 integral becomes:

$$\int_{\Gamma_3} q \cdot \nu(\eta - \theta) da = \tilde{b}(\eta - \theta, \tilde{\lambda}). \tag{3.40}$$

Combining (3.2), (3.36), (3.37), and (3.40) yields:

$$\langle \mathbb{K}\nabla\theta, \nabla\eta - \nabla\theta \rangle_{Q^* \times Q} + \langle \mathbb{M}\varepsilon(u), \eta - \theta \rangle_{Q^* \times Q} + \tilde{b}(\eta - \theta, \tilde{\lambda})$$
$$+ \int_{\Gamma_4} j_{\theta_\tau}^0(\theta_\tau; \eta_\tau - \theta_\tau) da + \int_{\Gamma_4} j_{\theta_\nu}^0(\theta_\nu; \eta_\nu - \theta_\nu) da$$
$$- \int_{\Gamma_4} h_\tau(\|u_\tau\|)(\eta - \theta) da = \langle h, \eta - \theta \rangle_{Q^* \times Q}. \tag{3.41}$$

The variational inequality for the multiplier is given by

$$\tilde{b}(\eta, \zeta - \tilde{\lambda}) = \langle \zeta, \eta_\nu \rangle_{Y^* \times Y} - \int_{\Gamma_3} q_\nu \eta_\nu \, da \leq 0 \quad \forall \eta \in Q, \zeta \in Y^*. \tag{3.42}$$

This concludes the mixed hemivariational formulation for the temperature field.

Problem 4 Find $\theta \in \Theta$ and $\tilde{\lambda} \in S$ satisfying:

$$\langle \mathbb{K}\nabla\theta, \ \nabla\eta - \nabla\theta \rangle_{Q^* \times Q} + \langle \mathbb{M}\varepsilon(\mathbf{u}), \ \eta - \theta \rangle_{Q^* \times Q} + \tilde{b}(\eta - \theta, \ \tilde{\lambda})$$

$$+ \int_{\Gamma_4} j^0_{\theta_\tau}(\theta_\tau; \ \eta_\tau - \theta_\tau) da + \int_{\Gamma_4} j^0_{\theta_\nu}(\theta_\nu; \ \eta_\nu - \theta_\nu) da \tag{3.43}$$

$$- \int_{\Gamma_4} h_\tau(\|\mathbf{u}_\tau\|) \cdot (\eta - \theta) da \geq \langle h, \ \eta - \theta \rangle_{Q^* \times Q},$$

$$\tilde{b}(\theta, \ \zeta - \tilde{\lambda}) \leq 0, \tag{3.44}$$

for all $\eta \in \Theta, \zeta \in S$.

Finally, combining Problems 3 and 4, we obtain the full variational formulation of Problem 2:

Problem 5 Find $(\mathbf{u}, \theta) \in U \times \Theta$ and $(\bar{\lambda}, \tilde{\lambda}) \in \mathcal{U} \times S$ that simultaneously solve systems (3.34), (3.35), (3.43), and (3.44).

4. Existence and uniqueness of weak solution

The main existence and uniqueness result in the study of Problem 5 are represented in the following Theorem 4.1.

Theorem 4.1 *Assume that the conditions* (3.13)-(3.20), *and the following smallness condition*

$$\min\{m_\mathbb{A}, m_\mathbb{K}\} > \max\{(\alpha_{j_\nu} + \alpha_{j_\tau})\bar{\mu}\|\gamma_1\|^2, \ (\alpha_{j_{\theta_\nu}} + \alpha_{\theta_\tau})\|\gamma_2\|^2\}, \tag{4.1}$$

hold. Then, Problem 5 has at least one solution $((u, \theta), (\bar{\lambda}, \tilde{\lambda})) \in (U \times \Theta) \times (\mathcal{U} \times S)$, *where its first component* (u, θ) *is unique.*

The proof is based on application of Theorem 4.1, and it is carried out into several steps. First, it is evident to Problem 5 is equivalent to the following problem. Find $(u, \theta) \in U \times \Theta$ and $(\bar{\lambda}, \tilde{\lambda}) \in \mathcal{U} \times S$ such that

$$\langle \mathbb{A}\varepsilon(u), \varepsilon(v) - \varepsilon(u) \rangle_\mathcal{H} + \langle \mathbb{K}\nabla\theta, \nabla\eta - \nabla\theta \rangle_{Q^* \times Q} - \langle \mathbb{M}\theta, \varepsilon(v) - \varepsilon(u) \rangle_{E^* \times E}$$

$$+ \langle \mathbb{M}\varepsilon(u), \eta - \theta \rangle_{E^* \times E} + \bar{b}(v - u, \bar{\lambda}) + \tilde{b}(\eta - \theta, \tilde{\lambda}) - \int_{\Gamma_4} h_\tau(\|u_\tau\|).(\eta - \theta) \, da$$

$$+ \int_{\Gamma_4} j^0_\nu(u_\nu; v_\nu - u_\nu) \, da + \int_{\Gamma_4} \mu(\|u_\tau\|)j^0_\tau(u_\tau; v_\tau - u_\tau) \, da \tag{4.2}$$

$$+ \int_{\Gamma_4} j^0_{\theta_\nu}(\theta_\nu; \eta_\nu - \theta_\nu) \, da + \int_{\Gamma_4} j^0_{\theta_\tau}(\theta_\tau; \eta_\tau - \theta_\tau) \, da$$

$$\geq \langle f, v - u \rangle_{E^* \times E} + \langle h, \eta - \theta \rangle_{Q^* \times Q},$$

$$\bar{b}(u, \rho - \bar{\lambda}) + \tilde{b}(\theta, \xi - \tilde{\lambda}) \leq 0, \tag{4.3}$$

for all $v \in V$, $\eta \in Q$, $\rho \in \mathcal{U}$, $\xi \in S$.
It is easily recognized that Problem 5 is equivalent to (4.2)-(4.3). Then, to provide the existence and

uniqueness of solution of Problem 5 , it suffices to demonstrate the unique solvability of (4.2)-(4.3). To this end, let $V = U \times \Theta$ and $\Lambda = \mathcal{U} \times S$ be the Hilbert spaces endowed with the following inner products

$$\langle y, z \rangle_V = \langle u, v \rangle_E + \langle \theta, \eta \rangle_{Q^* \times Q} \qquad\qquad \text{for all } y = (u, \theta), z = (v, \eta),$$

$$\langle \lambda, \zeta \rangle_\Lambda = \langle \bar{\lambda}, \rho \rangle_\mathcal{U} + \langle \tilde{\lambda}, \varrho \rangle_S \qquad\qquad \text{for all } \lambda = (\bar{\lambda}, \tilde{\lambda}), \zeta = (\rho, \varrho),$$

and define the operator $A : V \longrightarrow V^*$ as follows

$$\langle Ay, z \rangle_{V^* \times V} = \langle \mathbb{A}\varepsilon(u), \varepsilon(v) \rangle_\mathcal{H} + \langle \mathbb{K}\nabla\theta, \nabla\eta \rangle_{Q^* \times Q} - \langle \mathbb{M}\theta, \varepsilon(v) \rangle_\mathcal{H}$$
$$+ \langle \mathbb{M}\varepsilon(u), \eta \rangle_\mathcal{H} - \int_{\Gamma_4} h_\tau(\|u_\tau\|).\eta \, da, \tag{4.4}$$

for all $y = (u, \theta), z = (v, \eta) \in V$. Moreover, we define a mapping $J : V \longrightarrow \mathbb{R}$ as follows:

$$J(z) = \int_{\Gamma_4} j_\nu(v_\nu) \, da + \int_{\Gamma_4} \mu(\|v_\tau\|) j_\tau(v_\tau) \, da + \int_{\Gamma_4} j_{\theta_\nu}(\theta_\nu) \, da$$
$$+ \int_{\Gamma_4} j_{\theta_\tau}(\theta_\tau) \, da, \tag{4.5}$$

for all $z = (v, \eta) \in V$. Additionally, a bilinear form $b : V \times Z^* \longrightarrow \mathbb{R}$ defined by

$$b(z, \zeta) = \bar{b}(v, \rho) + \tilde{b}(\eta, \varrho) \qquad \text{for all } z = (v, \eta) \in V, \zeta = (\rho, \varrho) \in Z, \tag{4.6}$$

and an element $f \in V^*$ as follows

$$\langle \tilde{f}, z \rangle_{V^* \times V} = \langle f, v \rangle_{V^* \times V} + \langle h, \eta \rangle_{Q^{*`} \times Q} \qquad \text{for all } z = (v, \eta) \in V. \tag{4.7}$$

From V, Λ and assumptions (3.13)-(3.20), the Problem (4.2)-(4.3) can be formulated by:

Problem 6 Find $y \in V$ and $\lambda \in \Lambda$ such that

$$\langle Ay, z - y \rangle_{V^* \times V} + b(z - y, \lambda) + J^0(y; z - y) \geq \langle f, z - y \rangle_{V^* \times V}, \tag{4.8}$$
$$b(y, \rho - \lambda) \leq 0, \tag{4.9}$$

for every $z \in V$ and $\rho \in \Lambda$.

To derive the unique solvability of Problem 6, it is sufficient to show that under the hypotheses (3.13)-(3.20) the assumptions $H(A), H(J), H(b), H(\gamma), H(\Lambda), H(f)$ and H_0 of Theorem 1 hold.

1. From the definitions of $\mathcal{U}$ and S, it can easily be seen that Λ is a nonempty, closed and convex subset of Z^*. Then hypothesis $H(\Lambda)$ is valid.

2. By using the conditions (3.16), (3.17) with $a, s \in \{\tau, \theta_\tau\}$ and (3.18), the functional $J : V \times V \longrightarrow \mathbb{R}$ is given, as defined by (4.5). Moreover, we define the maps $j_i : \Gamma_4 \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, 2$ and $j_l : \Gamma_4 \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $l = 3, 4$ as follows:

$$\begin{aligned} j_1(x, r) &= j_\nu(x, r) && \text{for all } x \in \Gamma_4, \, r \in \mathbb{R}, \\ j_2(x, r) &= j_{\theta_\nu}(x, r) && \text{for all } x \in \Gamma_4, \, r \in \mathbb{R}, \\ j_3(x, r, \xi) &= j_{\theta_\tau}(x, \xi) && \text{for all } x \in \Gamma_4, \, r \in \mathbb{R}, \, \xi \in \mathbb{R}^d, \\ j_3(x, r, \xi) &= \mu(r) j_\tau(x, \xi) && \text{for all } x \in \Gamma_4, \, r \in \mathbb{R}, \, \xi \in \mathbb{R}^d. \end{aligned}$$

Then, J can be reformulated as $J(z) = \sum_{i=1}^{4} J_i(z)$, where J_i are given by

$$J_1(z) = \int_{\Gamma_4} j_1(x, v_\nu) da,$$

$$J_2(z) = \int_{\Gamma_4} j_2(x, \eta_\nu) da,$$

$$J_3(z) = \int_{\Gamma_4} j_3(x, \eta_\tau) da,$$

$$J_4(z) = \int_{\Gamma_4} j_4(x, \|v_\nu\|, \eta_\tau) da,$$

for all $z = (v, \eta) \in V$. First, by using the aforementioned hypotheses and [6, Theorem 3.47], it is clear that J is well defined and J is locally Lipschitz on V. Thus $H(J)(a)$ is valid. Moreover, it follows from conditions (3.16)(c), (3.17)(c), (3.18)(c), [7, Corollary 4.15], the continuity of trace operators γ_1, γ_2, and the inequality $|\xi_\nu| \leq \|\xi\|_{\mathbb{R}^d}$ for all $\xi \in \mathbb{R}^d$ that

$$\|\partial J_1(z)\|_{V^*} \leq c_{0\nu} \mathrm{meas}(\Gamma_4) + c_{1\nu} \|\gamma_1\| \sqrt{\mathrm{meas}(\Gamma_4)} \|z\|_V,$$

$$\|\partial J_2(z)\|_{V^*} \leq c_{0\theta_\nu} \mathrm{meas}(\Gamma_4) + c_{1\theta_\nu} \|\gamma_2\| \sqrt{\mathrm{meas}(\Gamma_4)} \|z\|_V,$$

$$\|\partial J_3(z)\|_{V^*} \leq c_{0\theta_\tau} \mathrm{meas}(\Gamma_4) + c_{1\theta_\tau} \|\gamma_2\| \sqrt{\mathrm{meas}(\Gamma_4)} \|z\|_V,$$

$$\|\partial J_4(z)\|_{V^*} \leq \bar{\mu} c_{0\tau} \mathrm{meas}(\Gamma_4) + \bar{\mu} c_{1\tau} \|\gamma_1\| \sqrt{\mathrm{meas}(\Gamma_4)} \|z\|_V,$$

for all for all $z = (v, \eta) \in V$. Hence, it is deduced that

$$\|\partial J(z)\|_{V^*} \leq c_{0J} + c_{1J} \|z\|_V \qquad \text{for every } z = (v, \eta) \in V,$$

where c_{0J} and c_{1J} are chosen as follows

$$c_{0J} = (c_{0\nu} + c_{0\theta_\nu} + c_{0\theta_\tau} + \bar{\mu} c_{0\tau}) \mathrm{meas}(\Gamma_1),$$

$$c_{1J} = ((c_{1\nu} + \bar{\mu} c_{1\tau}) \|\gamma_1\| + (c_{1\theta_\nu} + \bar{\mu} c_{1\theta_\tau}) \|\gamma_2\|) \sqrt{\mathrm{meas}(\Gamma_1)}.$$

Then, hypothesis $H(J)(b)$ is fulfilled. Additionally, Due to [7, Corollary 4.15] and the assumptions (3.16)(c), (3.17)(c) and (3.18)(c), for all $y_1 = (u_1, \theta_1)$, $y_2 = (u_2, \theta_2) \in V$, the following results

obtained

$$
\begin{aligned}
& J_1^0(y_1; y_2 - y_1) + J_1^0(y_2; y_1 - y_2) \\
& \leq \int_{\Gamma_4} j_1^0(x, u_{1\nu}; u_{2\nu} - u_{1\nu}) da + \int_{\Gamma_4} j_1^0(x, u_{2\nu}; u_{1\nu} - u_{2\nu}) \\
& \leq \alpha_{j_\nu} \int_{\Gamma_4} |u_{1\nu} - u_{2\nu}|^2 da \\
& \leq \alpha_{j_\nu} \|\gamma_1\|^2 \|u_1 - u_2\|_E^2,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
& J_2^0(y_1; y_2 - y_1) + J_2^0(y_2; y_1 - y_2) \\
& \leq \int_{\Gamma_4} j_2^0(x, \theta_{1\nu}; \theta_{2\nu} - \theta_{1\nu}) da + \int_{\Gamma_4} j_2^0(x, \theta_{2\nu}; \theta_{1\nu} - \theta_{2\nu}) \\
& \leq \alpha_{j_{\theta_\nu}} \int_{\Gamma_4} |\theta_{1\nu} - \theta_{2\nu}|^2 da \\
& \leq \alpha_{j_{\theta_\nu}} \|\gamma_2\|^2 \|\theta_1 - \theta_2\|_Q^2,
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
& J_3^0(y_1; y_2 - y_1) + J_3^0(y_2; y_1 - y_2) \\
& \leq \int_{\Gamma_4} j_3^0(x, \theta_{1\tau}; \theta_{2\tau} - \theta_{1\tau}) da + \int_{\Gamma_4} j_3^0(x, \theta_{2\tau}; \theta_{1\tau} - \theta_{2\tau}) \\
& \leq \alpha_{j_{\theta_\tau}} \int_{\Gamma_4} |\theta_{1\tau} - \theta_{2\tau}|^2 da \\
& \leq \alpha_{j_{\theta_\tau}} \|\gamma_2\|^2 \|\theta_1 - \theta_2\|_Q^2,
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
& J_4^0(y_1; y_2 - y_1) + J_4^0(y_2; y_1 - y_2) \\
& \leq \int_{\Gamma_4} j_4^0(x, u_{1\tau}; u_{2\tau} - u_{1\tau}) da + \int_{\Gamma_4} j_4^0(x, u_{2\tau}; u_{1\tau} - u_{2\tau}) da \\
& \leq \bar{\mu} \alpha_{j_\tau} \int_{\Gamma_4} |u_{1\tau} - u_{2\tau}|^2 da \\
& \leq \bar{\mu} \alpha_{j_\tau} \|\gamma_1\|^2 \|u_1 - u_2\|_E^2,
\end{aligned}
\tag{4.13}
$$

Combining the inequalities (4.10)-(4.13), one gets

$$
J^0(y_1; y_2 - y_1) + J^0(y_2; y_1 - y_2) \leq \alpha_J \|y_1 - y_2\|_V^2 \qquad \text{for all } y_1\, y_2 \in V,
\tag{4.14}
$$

where $\alpha_J = \max\{(\bar{\mu}\alpha_{j_\tau} + \alpha_{j_\nu})\|\gamma_1\|^2, (\alpha_{j_{\theta_\nu}} + \alpha_{j_{\theta_\tau}})\|\gamma_2^2\|\}$. Then, $H(J)(c)$ holds.

3. We claim that the operator A defined by (4.4) satisfies the hypothesis $H(A)$, for each $y = (u, \theta)$, $z = (v, \eta) \in V$, we obtain

$$
\begin{aligned}
\langle Ay - Az, y - z \rangle_{V^* \times V} \leq & \langle \mathbb{A}\varepsilon(u) - \mathbb{A}\varepsilon(v), u - v \rangle_{\mathcal{H}} + \langle \mathbb{K}\nabla\theta - \mathbb{K}\nabla\eta, \nabla\theta - \nabla\eta \rangle_H \\
& - \langle \mathbb{M}\theta - \mathbb{M}\eta, \theta - \eta \rangle_{\mathcal{H}} + \langle \mathbb{M}\varepsilon(u) - \mathbb{M}\varepsilon(v), \theta - \eta \rangle_H \\
& - \int_{\Gamma_4} (h_\tau(\|u_\tau\| - \|v_\tau\|)).(u - v)\, da.
\end{aligned}
$$

By using the relation

$$
\langle \mathbb{M}\theta - \mathbb{M}\eta, \theta - \eta \rangle_{\mathcal{H}} = \langle \mathbb{M}\varepsilon(u) - \mathbb{M}\varepsilon(v), \theta - \eta \rangle_H,
$$

we obtain

$$
\begin{aligned}
\langle Ay - Az, y - z \rangle_{V^* \times V} \leq & \langle \mathbb{A}\varepsilon(u) - \mathbb{A}\varepsilon(v), u - v \rangle_{\mathcal{H}} + \langle \mathbb{K}\nabla\theta - \mathbb{K}\nabla\eta, \nabla\theta - \nabla\eta \rangle_H \\
& - \int_{\Gamma_4} (h_\tau(\|u_\tau\| - \|v_\tau\|)).(u - v)\, da.
\end{aligned}
$$

The assumptions (3.13)(b), (3.14)(b) and (3.19)(a), one gets

$$\langle Ay - Az, y - z\rangle_{V^* \times V}$$
$$\leq (L_{\mathbb{A}} + \frac{L_{h_\tau}}{2}\|\gamma_1\|^2)\|u - v\|_E^2 + (L_{\mathbb{K}} + \frac{L_{h_\tau}}{2}\|\gamma_2\|^2)\|\theta - \eta\|_Q^2,$$

then, $H(A)(a)$ is valid with $L_A = \max\{L_{\mathbb{A}} + \frac{L_{h_\tau}}{2}\|\gamma_1\|^2,\ L_{\mathbb{A}} + \frac{L_{h_\tau}}{2}\|\gamma_1\|^2\}$. Next, we demonstrate the strongly monotonicity of the operator A. To do so, we take $y_1 = (u_1, \theta_1)$, $y_2 = (u_2, \theta_2) \in V$, using assumptions (3.13)(c), (3.14)(b) and (3.19)(c), we find

$$\langle Ay_1 - Ay_2, y_1 - y_2\rangle_{V^* \times V}$$
$$\geq \langle \mathbb{A}\varepsilon(u_1) - \mathbb{A}\varepsilon(u_1), u_1 - u_2\rangle_{\mathcal{H}} + \langle \mathbb{K}\nabla\theta_1 - \mathbb{K}\nabla\theta_2, \nabla\theta_1 - \nabla\theta_2\rangle_H$$
$$+ \int_{\Gamma_4} (h_\tau(\|u_{1\tau}\| - \|u_{2\tau}\|).(u_1 - u_2))\, da$$
$$\geq m_{\mathbb{A}}\|u_1 - u_2\|_E^2 + m_{\mathbb{K}}\|\theta_1 - \theta_2\|_Q^2.$$

Then, $H(A)(c)$ is verified with $L_A = \min\{m_{\mathbb{A}}, m_{\mathbb{K}}\}$.

4. From the definition (4.6) of b, it is clear that b is a bounded bilinear form. Now, we discuss the inf-sup condition. Therefore, $Z = X \times Y$ a Hilbert space endowed with the following inner product

$$\langle \zeta, \rho\rangle_Z = \langle \psi, \varphi\rangle_X + \langle \varrho, \varsigma\rangle_Y \qquad \text{for all } \zeta = (\psi, \varphi),\ \rho = (\varrho, \varsigma) \in Z,$$

and an operator $P : V \longrightarrow Z$ is given as follows

$$P(z) = (\alpha(v), \beta(\eta)) = (v_\nu, \eta_\nu) \qquad \text{for all } z = (v, \eta) \in V.$$

It is evident that P is a linear, bounded and surjective operator. Then by using the converse theorem [19, Theorem 5.7], the operator P has a right inverse bounded and linear operator $B : Z \longrightarrow V$, then we get

$$P \circ B(\zeta) = \zeta \qquad \text{for all } \zeta \in Z.$$

Thus,

$$P \circ B(P(y)) = P(y), \quad B \circ B \in V, \qquad B(y) \in V,$$

for all $y = (u, \theta) \in V$ and $\zeta = (\bar{\zeta}, \tilde{\zeta})$. Additionally, taking $\rho = (\bar{\rho}, \tilde{\rho}) \in Z^*$ to obtain

$$\|\rho\|_{Z^*} = \sup_{\zeta \in Z\backslash\{0_Z\}} \frac{\langle \rho, \zeta\rangle_{Z^* \times Z}}{\|\zeta\|_Z}$$

$$\leq \sup_{\substack{y=(u,\theta)\in V\backslash\{0_V\} \\ u\in E,\, \theta\in Q}} \frac{\langle \rho, P(y)\rangle_{Z^* \times Z}}{\|P(y)\|_Z}$$

$$= \sup_{\substack{y=(u,\theta)\in V\backslash\{0_V\} \\ u\in E,\, \theta\in Q}} \frac{\langle \bar{\rho}, \alpha(u)\rangle_{X^* \times X} + \langle \tilde{\rho}, \beta(\theta)\rangle_{Y^* \times Y}}{\|\alpha(u)\|_X + \|\beta(\theta)\|_Y}$$

$$= \sup_{\substack{y=(u,\theta)\in V\backslash\{0_V\} \\ u\in E,\, \theta\in Q}} \frac{\bar{b}(u, \bar{\rho}) + \tilde{b}(\theta, \tilde{\rho})}{\|\alpha(u)\|_X + \|\beta(\theta)\|_Y}$$

$$\leq c(\Omega) \sup_{\substack{y=(u,\theta)\in V\backslash\{0_V\} \\ u\in E,\, \theta\in Q}} \frac{\bar{b}(B_{|_V} \circ \alpha(u), \bar{\rho}) + \tilde{b}(B_{|_Q} \circ \beta(\theta), \tilde{\rho})}{\|B_{|_V} \circ \alpha(u)\|_X + \|B_{|_Q} \circ \beta(\theta)\|_Y}$$

$$= c(\Omega) \sup_{\substack{y=(u,\theta)\in V\backslash\{0_V\} \\ u\in E,\, \theta\in Q}} \frac{b(y, \rho)}{\|y\|_V}.$$

Then, the Poincare inequality and the Sobolev embedding theorem ensure that

$$\inf_{\rho \in Z \setminus \{0_Z\}} \sup_{\substack{y=(u,\theta) \in V \setminus \{0_V\} \\ u \in E, \, \theta \in Q}} \frac{b(y,\rho)}{\|y\|_V} \geq \frac{1}{c(\Omega)} := \alpha_b.$$

Thus, the hypothesis $H(b)$ is fulfilled. Thus, the assumption of Theorem 2.1 holds, and Problem 6 has at least one solution $(y, \lambda) \in V \times \Lambda$ with $y = (u, \theta)$ and $\lambda = (\bar{\lambda}, \tilde{\lambda})$.

5. Finally, it remains to prove that $y = (u, \theta)$ is unique. Indeed, assume that (y_1, λ_1) and (y_2, λ_2) in $V \times \Lambda$, where $y_l = (u_l, \theta_l)$, $\lambda_l = (\bar{\lambda}_l, \tilde{\lambda}_l)$, $l = 1, 2$ are two solutions of Problem 6. Then, we find

$$\langle \mathbb{A}\varepsilon(u_l), \varepsilon(v) - \varepsilon(u_l) \rangle_{\mathcal{H}} + \langle \mathbb{K}\nabla\theta_l, \nabla\eta - \nabla\theta_l \rangle_H - \langle \mathbb{M}\theta_l, \varepsilon(v) - \varepsilon(u_l) \rangle_{\mathcal{H}}$$

$$+ \langle \mathbb{M}\varepsilon(u_l), \eta - \theta_l \rangle_H - \int_{\Gamma_4} h_\tau(\|u_{l\tau}\|).(\eta - \theta_l) da + \bar{b}(v - u_l, \bar{\lambda})$$

$$+ \tilde{b}(\eta - \theta, \tilde{\lambda}) + \int_{\Gamma_4} j_\nu(u_{l\nu}; v_\nu - u_{l\nu}) \, da + \int_{\Gamma_4} \mu(\|v_{l\tau}\|)j_\tau(u_{l\tau}; v_\tau - u_{l\tau}) \, da \qquad (4.15)$$

$$+ \int_{\Gamma_4} j_{\theta_\nu}(\theta_{l\nu}; \eta_\nu - \theta_{l\nu}) \, da + \int_{\Gamma_4} j_{\theta_\tau}(\theta_{l\tau}; \eta_\tau - \theta_{l\tau}) \, da$$

$$\geq \langle f, v - u_l \rangle_{V^* \times V} + \langle h, \eta - \theta_l \rangle_{Q^* \times Q},$$

$$\bar{b}(u_l, \rho - \bar{\lambda}_l) + \tilde{b}(\theta_l, \varrho - \tilde{\lambda}_l) \leq 0, \qquad (4.16)$$

for all $v \in E, \eta \in Q, \rho \in \mathcal{U}$, and $\varrho \in S$.

By taking $(\rho, v) = (\bar{\lambda}_l, u_l)$, $(\varrho, \eta) = (\tilde{\lambda}_l, \theta_l)$ dans (4.15) and (4.16), and adding the obtained results, we obtain

$$\langle \mathbb{A}\varepsilon(u_1) - \mathbb{A}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2) \rangle_{\mathcal{H}} + \langle \mathbb{K}\nabla\theta_1 - \mathbb{K}\nabla\theta_2, \nabla\theta_1 - \nabla\theta_2 \rangle_H$$

$$+ \int_{\Gamma_4} (h_\tau(\|u_{1\tau}\| - h_\tau(\|u_{2\tau}\|).(\theta_1 - \theta_2) da$$

$$\leq \bar{b}(u_1 - u_2, \bar{\lambda}_1 - \bar{\lambda}_2) + \bar{b}(u_1 - u_2, \bar{\lambda}_1 - \bar{\lambda}_2)$$

$$+ \int_{\Gamma_4} [j_\nu(u_{1\nu}; u_{2\nu} - u_{1\nu}) + j_\nu(u_{2\nu}; u_{1\nu} - u_{2\nu})] da$$

$$+ \int_{\Gamma_4} [\mu(\|u_{1\tau}\|)j_\tau(u_{1\tau}; u_{2\tau} - u_{1\tau}) + \mu(\|u_{2\tau}\|)j_\tau(u_{2\tau}; u_{1\tau} - u_{2\tau})] da$$

$$+ \int_{\Gamma_4} [j_{\theta_\nu}(\theta_{1\nu}; \theta_{2\nu} - \theta_{1\nu}) + j_{\theta_\nu}(\theta_{2\nu}; \theta_{1\nu} - \theta_{2\nu})] da$$

$$+ \int_{\Gamma_4} [j_{\theta_\tau}(\theta_{1\tau}; \theta_{2\tau} - \theta_{1\tau}) + j_{\theta_\tau}(\theta_{1\tau}; \theta_{2\tau} - \theta_{1\tau})] da.$$

By using conditions (3.13)-(3.18) and inequality (4.16), we obtain

$$\min\{m_{\mathbb{A}}, m_{\mathbb{K}}\} \|y_1 - y_2\|_V^2$$
$$\leq \max\{(\alpha_{j_\nu} + \bar{\mu}\alpha_{j_\tau})\|\gamma_1\|^2 + (\alpha_{j_{\theta_\nu}} + \alpha_{j_{\theta_\tau}})\|\gamma_2\|)\} \|y_1 - y_2\|_V^2. \qquad (4.17)$$

Then

$$\left(\min\{m_{\mathbb{A}}, m_{\mathbb{K}}\} - \max\{(\alpha_{j_\nu} + \bar{\mu}\alpha_{j_\tau})\|\gamma_1\|^2 + (\alpha_{j_{\theta_\nu}} + \alpha_{j_{\theta_\tau}})\|\gamma_2\|^2)\}\right) \|y_1 - y_2\|_V^2$$
$$\leq 0. \qquad (4.18)$$

Therefore, by employing the smallness condition (4.1), we obtain $y_1 = y_2$. Then, $u_1 = u_2$ and $\theta_1 = \theta_2$. This finishes the proof.

5. Conclusion

This paper presents a mathematical model for frictional contact between a thermoeleastic body and a foundation, incorporating Signorini conditions and non-monotone friction laws via Clarke subdifferentials. Existence and uniqueness are established using mixed hemivariational inequalities. Numerical analysis of the model represents an interesting future direction.

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Declarations

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Conflict of interest

No potential conflicts of interest to declare.

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Authors' contributions

All authors are contributed equally to the work.

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