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An Elliptic Partial Differential Equations System and its Application Involving the Biharmonic Operator

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ABSTRACT: This paper investigates a novel system of coupled fourth-order nonlinear partial differential equations defined on the whole space $\mathbb{R}^N$. The system is characterized by a unique combination of the biharmonic operator, the Laplacian, and nonlinear terms involving the square of the gradient. The coupling is linear, involving positive parameters λ_i, k_i, and a_i, alongside continuous source functions f_i. We establish the existence of solutions using the sub- and super-solution method. Additionally, we discuss an engineering application to illustrate the physical relevance of the problem. To the best of our knowledge, this is the first study addressing a system that integrates this specific configuration of biharmonic, Laplacian, and gradient terms.

Key Words: Nonlinear, fourth-order, biharmonic, laplacian, grandient, flliptic system, partial differential equations, sub-and super-solution, thin plates.

Contents

1. Introduction

In 2020 Dragos-Patru Covei and Traian A. Pirvu in [5] showed the existence of a positive solution for the problem

$$\begin{cases} -\frac{k_1}{2}\Delta z_1 + \frac{|\nabla z_1(x)|^2}{2} = f_1(x) - (\lambda_1 + a_1) z_1(x) + a_1 z_2(x), \\ -\frac{k_2}{2}\Delta z_2 + \frac{|\nabla z_2(x)|^2}{2} = f_2(x) - (\lambda_2 + a_2) z_2(x) + a_2 z_1(x), \end{cases} \quad x \in \mathbb{R}^N,$$

for $N \geq 1$, $\lambda_i > 0 (i = 1, 2)$ and $f_i : \mathbb{R}^N \to [0, \infty)$ $(i = 1, 2)$ are continuous convex functions satisfying

$$\text{there exists } M_i > 0 \text{ such that } f_i(x) \leq M_i \left(|x|^2 + 1 \right).$$

Using the same approach, we investigate the existence of positive solution for the following system of partial differential equations,

$$\begin{cases} -\frac{k_1^3}{2}\Delta^2 z_1 + \frac{\left(-k_1\Delta z_1(x) + |\nabla z_1(x)|^2\right)^2}{2} + \frac{k_1^2}{2}\Delta|\nabla z_1|^2 = f_1(x) - (\lambda_1 + a_1) z_1(x) + a_1 z_2(x), \\ -\frac{k_2^3}{2}\Delta^2 z_2 + \frac{\left(-k_2\Delta z_2(x) + |\nabla z_2(x)|^2\right)^2}{2} + \frac{k_2^2}{2}\Delta|\nabla z_2|^2 = f_2(x) - (\lambda_2 + a_2) z_2(x) + a_2 z_1(x), \\ \qquad x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

here $N \geq 1$ represents the space dimension, $|\cdot|$ is the Euclidean norm of $\mathbb{R}^N$, let $f_i : \mathbb{R}^N \to [0, \infty)$ $(i = 1, 2)$ are continuous convex functions satisfying

$$\text{there exists } M_i > 0 \text{ such that } f_i(x) \leq M_i \left(|x|^6 + 1 \right), \tag{1.2}$$

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$\lambda_i > 0$ $(i = 1, 2)$ are some real constants suitable chosen and $k_i > 0$, $a_i > 0$ $(i = 1, 2)$ are some real arbitrary constants.

As is well known, the biharmonic operator is one of the best tools for modeling the deformation of elastic plates under external forces and for describing the flow of incompressible fluids, particularly in two-dimensional flow problems. The biharmonic equation captures the behavior of systems where the second derivative of the Laplacian is crucial, making it an essential mathematical model in elasticity theory and fluid dynamics. In elasticity, it describes plate bending, while in fluid dynamics, it is used to model flow over surfaces and in problems related to fluid stability.

The biharmonic operator can also be an effective tool for modeling systems where two objects interact with each other under the influence of external forces. This is particularly relevant in scenarios where the objects are coupled, such as in the case of two elastic bodies in contact or interacting via a shared boundary. The biharmonic equation can describe the deformation of each object, as well as the interaction forces between them, by capturing the complex relationship between their respective displacements. This makes it a valuable model for studying coupled systems, such as the contact mechanics of plates or the interaction between different materials subjected to external loads or forces.

Mathematically, the existence of a positive solution ensures that the system's behavior is well-defined and physically plausible, where both objects experience some degree of deformation that satisfies the system's governing equations.

In real-life applications, the existence of a positive solution for the system of biharmonic equations would correspond to a physically realistic scenario where two interacting objects (such as two elastic bodies, plates, or structures) deform in response to external forces, while maintaining positive deformations throughout the interaction.

For example, imagine two elastic plates placed in contact, subjected to external loads or forces. These forces could come from various sources such as weight, pressure, or mechanical impacts. The system of biharmonic equations would model how the plates bend and deform due to these forces, taking into account both their individual properties (like stiffness and elasticity) and the interaction between the plates (through the coupling terms, $a_1 z_2(x)$ and $a_2 z_1(x)$).

In this case, a "positive solution" would mean that both plates experience non-negative deformations at every point—there is no point at which the deformation goes below zero (which would be physically unrealistic for most structures, as negative deformation could imply compression or collapse). This suggests that the plates remain in a stable, deformed state where they bend but do not buckle or collapse under the applied forces.

In practical terms, this could be applied to:

- **Engineering structures:** Modeling the behavior of coupled beams, plates, or shells subjected to external loads where the interaction between components is crucial for stability and design.

- **Contact mechanics:** Describing how two objects, such as mechanical parts, interact under load, with one object exerting force on the other while maintaining a positive deformation state.

- **Material science:** Understanding the behavior of materials under stress or deformation, ensuring that the system remains within acceptable limits for design and safety.

In essence, a positive solution guarantees that the system behaves in a physically plausible way, where deformations are positive (non-collapsing) and consistent with the applied forces, ensuring stability in real-world scenarios like the bending of plates, vibration modes, or stress distribution in coupled systems.

This paper will be structured as follows

- Main result: We will prove the existence of a positive the solution of the system (1.1).

- Example of Coupled Thin Elastic Plates: we will present an example of this system and clarify the meaning of the existence of a solution.

2. Main Result

In this part, we focus on the problem

$$\begin{cases} -\frac{k_1^3}{2}\Delta^2 z_1 + \dfrac{\left(-k_1\Delta z_1(x)+|\nabla z_1(x)|^2\right)^2}{2} + \dfrac{k_1^2}{2}\Delta|\nabla z_1|^2 = f_1(x)-(\lambda_1+a_1)z_1(x)+a_1z_2(x), \\ -\frac{k_2^3}{2}\Delta^2 z_2 + \dfrac{\left(-k_2\Delta z_2(x)+|\nabla z_2(x)|^2\right)^2}{2} + \dfrac{k_2^2}{2}\Delta|\nabla z_2|^2 = f_2(x)-(\lambda_2+a_2)z_2(x)+a_2z_1(x), \\ x\in\mathbb{R}^N, \end{cases}$$

where $\lambda_i > 0$ $(i = 1, 2)$ are some real constants suitable chosen and $k_i > 0$, $a_i > 0$ $(i = 1, 2)$ are some real arbitrary constants, and f_i are some continuous functions satisfying

$$\text{there exists } M_i > 0 \text{ such that } f_i(x) \le M_i\left(|x|^6+1\right).$$

We have the following result,

Theorem 2.1 *For all $\lambda_1, \lambda_2 \in (0,\infty)$ the system of equations (1.1) has at least one pisitive solution with quartic growth, i.e.,*

$$z_i(x) \le K_i(1+|x|^4),\ \text{for some } K_i > 0, \quad i = 1, 2, \tag{2.1}$$

and, such that

$$|\nabla z_i(x)| \le \overline{C}_i(1+|x|^3),\ \text{for } x\in\mathbb{R}^N \text{ and for some positive constant } \overline{C}_i. \tag{2.2}$$

3. Proof of Main Results

To prove the result of Theorem 2.1, we require the following lemmas.

Lemma 3.1 *The system (1.1) is equivalent to the semilinear elliptic system*

$$\begin{cases} \Delta^2 u = u(x)\left[\frac{2}{k_1^4}\left(f_1(x)+(\lambda_1+a_1)k_1\ln u - a_1k_2\ln v\right)\right], \\ \Delta^2 v = v(x)\left[\frac{2}{k_2^4}\left(f_2(x)+(\lambda_2+a_2)k_2\ln v - a_2k_1\ln u\right)\right], \end{cases} \quad x\in\mathbb{R}^N. \tag{3.1}$$

Proof: With the variable change

$$z_1(x) = k_1w_1(x) \text{ and } z_2(x) = k_2w_2(x),$$

we transform the system (1.1) into

$$\begin{cases} -\frac{k_1^4}{2}\Delta^2\omega_1 + \dfrac{k_1^4\left(-\Delta\omega_1(x)+|\nabla\omega_1(x)|^2\right)^2}{2} + \dfrac{k_1^4}{2}\Delta|\nabla\omega_1|^2 = f_1(x)-k_1(\lambda_1+a_1)\omega_1(x)+k_2a_1\omega_2(x), \\ -\frac{k_2^4}{2}\Delta^2\omega_2 + \dfrac{k_2^4\left(-\Delta\omega_2(x)+|\nabla\omega_2(x)|^2\right)^2}{2} + \dfrac{k_2^4}{2}\Delta|\nabla\omega_2|^2 = f_2(x)-k_2(\lambda_2+a_2)\omega_2(x)+k_1a_2\omega_1(x), \\ x\in\mathbb{R}^N, \end{cases}$$

or, equivalently

$$\begin{cases} -\Delta^2\omega_1 + \left(-\Delta\omega_1(x)+|\nabla\omega_1(x)|^2\right)^2 + \Delta|\nabla\omega_1|^2 = \frac{2}{k_1^4}\left[f_1(x)-k_1(\lambda_1+a_1)\omega_1(x)+k_2a_1\omega_2(x)\right], \\ -\Delta^2\omega_2 + \left(-\Delta\omega_2(x)+|\nabla\omega_2(x)|^2\right)^2 + \Delta|\nabla\omega_2|^2 = \frac{2}{k_2^4}\left[f_2(x)-k_2(\lambda_2+a_2)\omega_2(x)+k_1a_2\omega_1(x)\right], \\ x\in\mathbb{R}^N, \end{cases} \tag{3.2}$$

The variable change

$$u(x) = e^{-\omega_1(x)} \text{ and } v(x) = e^{-\omega_2(x)},$$

transform the system (3.2) into

$$\begin{cases} \Delta^2 u = u\left[\frac{2}{k_1^4}\left(f_1(x)+(\lambda_1+a_1)k_1\ln u - a_1k_2\ln v\right)\right], \\ \Delta^2 v = v\left[\frac{2}{k_2^4}\left(f_2(x)+(\lambda_2+a_2)k_2\ln v - a_2k_1\ln u\right)\right], \end{cases}$$

since

$$\Delta u\left(x\right)=e^{-\omega_1\left(x\right)}(-\Delta w_1\left(x\right)+\left|\nabla\omega_1\left(x\right)\right|^2),$$
$$\Delta v\left(x\right)=e^{-\omega_2\left(x\right)}(-\Delta w_2\left(x\right)+\left|\nabla\omega_2\left(x\right)\right|^2).$$

and

$$\Delta^2u\left(x\right)=e^{-\omega_1}\left(-\Delta^2\omega_1+\left(-\Delta\omega_1(x)+|\nabla\omega_1(x)|^2\right)^2+\Delta|\nabla\omega_1|^2\right),$$
$$\Delta^2v\left(x\right)=e^{-\omega_1}\left(-\Delta^2\omega_2+\left(-\Delta\omega_2(x)+|\nabla\omega_2(x)|^2\right)^2+\Delta|\nabla\omega_2|^2\right).$$

□

The existence of a solution $(u\left(x\right),v\left(x\right))\in C^4\left(\mathbb{R}^N\right)\times C^4\left(\mathbb{R}^N\right)$ for the problem (3.1), such that $0<u\left(x\right)\leq 1$ and $0<v\left(x\right)\leq 1$, for all $x\in\mathbb{R}^N$, is proved in the following:

Theorem 3.1 *If there exist functions $\underline{u}$, $\underline{v}$, $\overline{u}$, $\overline{v}:\mathbb{R}^N\to(0,1]$ of class $C^4\left(\mathbb{R}^N\right)$ such that*

$$\begin{cases}-\Delta^2\underline{u}\left(x\right)+\underline{u}\left(x\right)[\frac{2}{k_1^4}\left(f_1\left(x\right)+\left(\lambda_1+a_1\right)k_1\ln\underline{u}\left(x\right)\right)]\leq 2a_1\frac{k_2}{k_1^4}\underline{u}\left(x\right)\ln\underline{v}\left(x\right),\\ -\Delta^2\underline{v}\left(x\right)+\underline{v}\left(x\right)[\frac{2}{k_2^4}\left(f_2\left(x\right)+\left(\lambda_2+a_2\right)k_2\ln\underline{v}\left(x\right)\right)]\leq 2a_2\frac{k_1}{k_2^4}\underline{v}\left(x\right)\ln\underline{u}\left(x\right),\\ -\Delta^2\overline{u}\left(x\right)+\overline{u}\left(x\right)[\frac{2}{k_1^4}\left(f_1\left(x\right)+\left(\lambda_1+a_1\right)k_1\ln\overline{u}\left(x\right)\right)]\geq 2a_1\frac{k_2}{k_1^4}\overline{u}\left(x\right)\ln\overline{v}\left(x\right),\\ -\Delta^2\overline{v}\left(x\right)+\overline{v}\left(x\right)[\frac{2}{k_2^4}\left(f_2\left(x\right)+\left(\lambda_2+a_2\right)k_2\ln\overline{v}\left(x\right)\right)]\geq 2a_2\frac{k_1}{k_2^4}\overline{v}\left(x\right)\ln\overline{u}\left(x\right),\\ \underline{u}\left(x\right)\leq\overline{u}\left(x\right),\ \ \underline{v}\left(x\right)\leq\overline{v}\left(x\right),\end{cases}\tag{3.3}$$

in the entire Euclidean space $\mathbb{R}^N$, then system (3.1) possesses an entire solution $(u,v)\in C^4\left(\mathbb{R}^N\right)\times C^4\left(\mathbb{R}^N\right)$ with $\underline{u}\left(x\right)\leq u\left(x\right)\leq\overline{u}\left(x\right)$ in $\mathbb{R}^N$ and $\underline{v}\left(x\right)\leq v\left(x\right)\leq\overline{v}\left(x\right)$ in $\mathbb{R}^N$.

Let us point out that the functions $(\underline{u},\underline{v})$ (resp. $(\overline{u},\overline{v})$) are called sub-solution (resp. super-solution) for the system (3.1).

Proof: In the following, we construct the functions $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ that satisfy the inequalities (3.1) in $\mathbb{R}^N$. We follow the approach outlined in Bensoussan, Sethi, Vickson, and Derzko [4] and Dragos and Traian in [5] for the scalar case. Specifically, we note that there exist

$$\left(\underline{u}\left(x\right),\underline{v}\left(x\right)\right)=\left(e^{B_1|x|^4+D_1},e^{B_2|x|^4+D_2}\right),\ \text{with}\ B_1,B_2,D_1,D_2\in\left(-\infty,0\right),\tag{3.4}$$

such that for all $\lambda_1>0$ and $\lambda_2>0$ the following hold

$$\begin{cases}-\Delta^2\underline{u}\left(x\right)+\underline{u}\left(x\right)[\frac{2}{k_1^4}\left(f_1\left(x\right)+\left(\lambda_1+a_1\right)k_1\ln\underline{u}\left(x\right)\right)]\leq 2a_1\frac{k_2}{k_1^4}\underline{u}\left(x\right)\ln\underline{v}\left(x\right),\\ -\Delta^2\underline{v}\left(x\right)+\underline{v}\left(x\right)[\frac{2}{k_2^4}\left(f_2\left(x\right)+\left(\lambda_2+a_2\right)k_2\ln\underline{v}\left(x\right)\right)]\leq 2a_2\frac{k_1}{k_2^4}\underline{v}\left(x\right)\ln\underline{u}\left(x\right),\end{cases}\tag{3.5}$$

i.e. $\left(\underline{u}\left(x\right),\underline{v}\left(x\right)\right)$ is a sub-solution for the problem (3.1). Indeed, using (1.2), we find B_1, B_2, D_1, $D_2\in(-\infty,0)$ such that

$$\begin{cases}24NB_1+12B_1|x|^2-16B_1^2|x|^6-192B_1^2(N+2)|x|^2+\frac{2}{k_1^4}\left[M_1(|x|^6+1)\right.\\ \left.+(\lambda_1+a_1)k_1(B_1|x|^4+D_1)\right]=2a_1\frac{k_2}{k_1^4}(B_2|x|^4+D_2)\\ 24NB_2+12B_2|x|^2-16B_2^2|x|^6-192B_2^2(N+2)|x|^2+\frac{2}{k_2^4}\left[M_2(|x|^6+1)\right.\\ \left.+(\lambda_2+a_2)k_2(B_2|x|^4+D_2)\right]=2a_2\frac{k_1}{k_2^4}(B_1|x|^4+D_1)\end{cases}$$

or, after rearranging the terms

$$\begin{cases}|x|^6[\frac{2}{k_1^4}M_1-16B_1^2]+|x|^4[\frac{2}{k_1^3}\lambda_1B_1+\frac{2}{k_1^3}a_1B_1-2a_1\frac{k_2}{k_1^4}B_2]+|x|^2[12B_1-192B_1^2(N+2)]\\ +2NB_1+\frac{2}{k_1^4}(M_1+\lambda_1D_1k_1+a_1k_1D_1)-2a_1\frac{k_2}{k_1^4}D_2=0\\ |x|^6[\frac{2}{k_2^4}M_2-16B_2^2]+|x|^4[\frac{2}{k_2^3}\lambda_2B_2+\frac{2}{k_2^3}a_2B_2-2a_2\frac{k_1}{k_2^4}B_1]+|x|^2[12B_2-192B_2^2(N+2)]\\ +2NB_2+\frac{2}{k_2^4}(M_2+\lambda_2D_2k_2+a_2k_2D_2)-2a_2\frac{k_1}{k_2^4}D_1=0\end{cases}$$

Now, we consider the system of equations

$$\begin{cases} \frac{2}{k_1^4}M_1 - 16B_1^2 = 0 \\ \frac{2}{k_1^3}\lambda_1 B_1 + \frac{2}{k_1^3}a_1 B_1 - 2a_1\frac{k_2}{k_1^4}B_2 = 0 \\ B_1 - 16B_1^2(N+2) = 0 \\ 2NB_1 + \frac{2}{k_1^4}(M_1 + \lambda_1 D_1 k_1 + a_1 k_1 D_1) - 2a_1\frac{k_2}{k_1^4}D_2 = 0 \\ \frac{2}{k_2^4}M_2 - 16B_2^2 = 0 \\ \frac{2}{k_2^3}\lambda_2 B_2 + \frac{2}{k_2^3}a_2 B_2 - 2a_2\frac{k_1}{k_2^4}B_1 = 0 \\ B_2 - 16B_2^2(N+2) = 0 \\ 2NB_2 + \frac{2}{k_2^4}(M_2 + \lambda_2 D_2 k_2 + a_2 k_2 D_2) - 2a_2\frac{k_1}{k_2^4}D_1 = 0. \end{cases}$$

then adding line (1) to line (2), (3) to (4), (5) to (6), and (7) to (8) we have

$$\begin{cases} 16B_1^2 - \frac{2}{k_1^4}M_1 - (\frac{2}{k_1^3}\lambda_1 + \frac{2}{k_1^3}a_1)B_1 + 2a_1\frac{k_2}{k_1^4}B_2 = 0 \\ 16B_1^2(N+2) - (2N+1)B_1 - \frac{2}{k_1^4}M_1 - \frac{2}{k_1^4}(\lambda_1 k_1 + a_1 k_1)D_1 + 2a_1\frac{k_2}{k_1^4}D_2 = 0 \\ 16B_2^2 - \frac{2}{k_2^4}M_2 - (\frac{2}{k_2^3}\lambda_2 + \frac{2}{k_2^3}a_2)B_2 + 2a_2\frac{k_1}{k_2^4}B_1 = 0 \\ 16B_2^2(N+2) - (2N+1)B_2 - \frac{2}{k_2^4}M_2 - \frac{2}{k_2^4}(\lambda_2 k_2 + a_2 k_2)D_2 + 2a_2\frac{k_1}{k_2^4}D_1 = 0. \end{cases} \tag{3.6}$$

Since we wish to analyze the existence of $B_1,\ B_2,\ D_1,\ D_2 \in (-\infty, 0)$ that solve (3.6) we couple the Equations (1) and (3) together

$$\begin{pmatrix} 16B_1^2 - \frac{2M_1}{k_1^4} \\ 16B_2^2 - \frac{2M_2}{k_2^4} \end{pmatrix} = \begin{pmatrix} \frac{2}{k_1^4}(\lambda_1 + a_1)k_1 & -2a_1\frac{k_2}{k_1^4} \\ -2a_2\frac{k_1}{k_2^4} & \frac{2}{k_2^4}(\lambda_2 + a_2)k_2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{3.7}$$

and, similarly for the Equations (2) and (4)

$$\begin{pmatrix} 16B_1^2(N+2) - (2N+1)B_1 - \frac{2}{k_1^4}M_1 \\ 16B_2^2(N+2) - (2N+1)B_2 - \frac{2}{k_2^4}M_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{k_1^4}(\lambda_1 + a_1)k_1 & -2a_1\frac{k_2}{k_1^4} \\ -2a_2\frac{k_1}{k_2^4} & \frac{2}{k_2^4}\left(\lambda_2 + a_2\right)k_2 \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}. \tag{3.8}$$

Since

$$\begin{vmatrix} \frac{2}{k_1^4}(\lambda_1 + a_1)k_1 & -2a_1\frac{k_2}{k_1^4} \\ -2a_2\frac{k_1}{k_2^4} & \frac{2}{k_2^4}\left(\lambda_2 + a_2\right)k_2 \end{vmatrix} = \frac{1}{k_1^3 k_2^3}\left(4\lambda_1\lambda_2 + 4\lambda_1 a_2 + 4\lambda_2 a_1\right) > 0,$$

the system (3.7) can be written equivalently as

$$\begin{pmatrix} -B_1 \\ -B_2 \end{pmatrix} = \frac{1}{2\lambda_1\lambda_2 + 2\lambda_1 a_2 + 2\lambda_2 a_1} \begin{pmatrix} k_1^3(\lambda_2 + a_2) & \frac{a_1 k_2^3}{k_1} \\ \frac{a_2 k_1^4}{k_2} & k_2^3(a_1 + \lambda_1) \end{pmatrix} \begin{pmatrix} \frac{2M_1}{k_1^4} - 16B_1^2 \\ \frac{2M_2}{k_2^4} - 16B_2^2 \end{pmatrix}. \tag{3.9}$$

From the results of [10,13] there exist and are unique $B_1,\ B_2 \in (-\infty, 0)$ that solve the system of equations (3.9). Next, we observe that the system (3.8) can be written equivalently as

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \frac{1}{2\lambda_1\lambda_2 + 2\lambda_1 a_2 + 2\lambda_2 a_1} \begin{pmatrix} k_1^3(\lambda_2 + a_2) & \frac{a_1 k_2^3}{k_1} \\ \frac{a_2 k_1^4}{k_2} & k_2^3(a_1 + \lambda_1) \end{pmatrix}$$
$$\times \begin{pmatrix} 16B_1^2(N+2) - (2N+1)B_1 - \frac{2}{k_1^4}M_1 \\ 16B_2^2(N+2) - (2N+1)B_2 - \frac{2}{k_2^4}M_2 \end{pmatrix},$$

from where we can see that there exist $B_1,\ B_2,\ D_1,\ D_2 \in (-\infty, 0)$ that solve (3.6) and then $(\underline{u}\,(x)\,, \underline{v}\,(x))$ are such that the inequalities in (3.5) hold.

To construct a super-solution it is useful to remember that $\ln 1 = 0$ and then a simple calculation shows that

$$\left(\overline{u}\,(x)\,,\ \overline{v}\,(x)\right) = (1, 1)\,,$$

is a super-solution of the problem (3.1).

Until now, we constructed the corresponding sub- and super-solutions employed in the scalar case by [4]. Clearly, (3.3) holds and then in Theorem 3.1 it remains to prove that there exists $(u(x), v(x)) \in C^4(\mathbb{R}^N) \times C^4(\mathbb{R}^N)$ with $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ in $\mathbb{R}^N$ and $\underline{v}(x) \leq v(x) \leq \overline{v}(x)$ in $\mathbb{R}^N$ satisfying (3.1).

To do this, let B_k be the ball whose center is the origin of $\mathbb{R}^N$ and which has radius $k = 1, 2, ...$ We consider the boundary value problem

$$\begin{cases} \Delta^2 u = u[\frac{2}{k_1^4}(f_1(x) + (\lambda_1 + a_1) k_1 \ln u - a_1 k_2 \ln v)], \ x \in B_k, \\ \Delta^2 v = v[\frac{2}{k_2^4}(f_2(x) + (\lambda_2 + a_2) k_2 \ln v - a_2 k_1 \ln u)], \ x \in B_k, \\ u(x) = \underline{u}_k(x), \ v(x) = \underline{v}_k(x), \ x \in \partial B_k, \end{cases} \tag{3.10}$$

where $\underline{u}_k = \underline{u}_{|B_k}$ and $\underline{v}_k = \underline{v}_{|B_k}$. In a similar way, we define $\overline{u}_k = \overline{u}_{|B_k}$ and $\overline{v}_k = \overline{v}_{|B_k}$ then $\underline{u}_k$, $\overline{u}_k$, $\underline{v}_k$, $\overline{v}_k \in C^4(\overline{B}_k)$.

Observing that

$$\begin{aligned} \inf_{x\in\mathbb{R}^N} \underline{u}(x) &\leq \min_{x\in\overline{B}_k} \underline{u}_k(x) \text{ and } \sup_{x\in\mathbb{R}^N} \overline{u}(x) \geq \max_{x\in\overline{B}_k} \overline{u}_k(x), \\ \inf_{x\in\mathbb{R}^N} \underline{v}(x) &\leq \min_{x\in\overline{B}_k} \underline{v}_k(x) \text{ and } \sup_{x\in\mathbb{R}^N} \overline{v}(x) \geq \max_{x\in\overline{B}_k} \overline{v}_k(x), \end{aligned}$$

by the result of Yasuhiro Furusho and Takasi Kusano in [9] and using the same methode as in [12], we can easily prove that existence of a solution $(u_k, v_k) \in [C^4(B_k) \cap C(\overline{B}_k)]^2$ satisfying the system (3.10). The functions (u_k, v_k) also satisfy

$$\begin{aligned} \underline{u}_k(x) &\leq u_k(x) \leq \overline{u}_k(x), \ x \in \overline{B}_k, \\ \underline{v}_k(x) &\leq v_k(x) \leq \overline{v}_k(x), \ x \in \overline{B}_k. \end{aligned}$$

By a standard regularity argument based on Schauder estimates, see Tolksdorf [17](proposition 3.7, p. 806), we can see that for all integers $k \geq n + 1$ there are $\alpha_1, \alpha_2 \in (0, 1)$ and positive constants C_1, C_2, independent of k such that

$$\begin{cases} u_k \in C^{4,\alpha_1}(\overline{B}_n) \text{ and } |u_k|_{C^{4,\alpha_1}(\overline{B}_n)} < C_1, \\ v_k \in C^{4,\alpha_2}(\overline{B}_n) \text{ and } |v_k|_{C^{4,\alpha_2}(\overline{B}_n)} < C_2, \end{cases} \tag{3.11}$$

where $|\cdot|_{C^{4,\circ}}$ is the usual norm of the space $C^{4,\circ}(\overline{B}_n)$. Moreover, there exist constants: C_3 independent of u_k, C_4 independent of v_k and such that

$$\begin{cases} \max_{x\in\overline{B}_n} |\Delta u_k(x)| \leq C_3 \max_{x\in\overline{B}_k} |u_k(x)|, \\ \max_{x\in\overline{B}_n} |\Delta v_k(x)| \leq C_4 \max_{x\in\overline{B}_k} |v_k(x)|. \end{cases} \tag{3.12}$$

The information from (3.11) and (3.12) implies that $\{(\Delta^2 u_k, \Delta^2 v_k)\}_k$ as well as $\{(u_k, v_k)\}_k$ are uniformly bounded on $\overline{B}_n$. We wish to show that this sequence $\{(u_k, v_k)\}_k$ contains a subsequence converging to a desired entire solution of (3.1).By the same arrguments as in [5] we can say that $u(x) = \lim_{n\to\infty} u_n(x)$ and $v(x) = \lim_{n\to\infty} v_n(x)$, such that the solution (u, v) belongs to $C^4(\mathbb{R}^N) \times C^4(\mathbb{R}^N)$ and satisfies (3.1).

This completes the proof of Theorem 3.1.

□

Proof: [Theorem 2.1] As easily verified, the existence of solutions is proved by Lemma 3.1 and Theorem 3.1. Then it remains to prove (2.1).

A recapitulation of the changes of variables say that

$$z_1(x) = -k_1 \ln u(x) \ \text{ and } z_2(x) = -k_2 \ln v(x),$$

is a solution for (1.1). Observing that

$$\underline{u}\left(x\right)=e^{B_1|x|^4+D_1}\leq u\left(x\right)\leq\ \overline{u}\left(x\right)=1,\,x\in\mathbb{R}^N,$$

it follows that

$$B_1\left|x\right|^4+D_1\leq\ln u\left(x\right)\leq\ln 1,$$

and, then

$$0\leq -k_1\ln u\left(x\right)\leq -k_1B_1\left|x\right|^4-k_1D_1,$$

or equivalently

$$0\leq z_1\left(x\right)\leq K_1\left(\left|x\right|^4+1\right),\text{ for } x\in\mathbb{R}^N \text{ and } K_1=\max\{-k_1B_1,-k_1D_1\}.$$

In the same way

$$0\leq z_2\left(x\right)\leq K_2\left(\left|x\right|^4+1\right),\text{ for } x\in\mathbb{R}^N \text{ and } K_2=\max\{-k_2B_2,-k_2D_2\},$$

and the proof is completed. □

Remark 3.1

- We can probably get the convexity of the solution by the same arguments as in [3, Theorem 3, p. 278].
- Since $(z_1(x), z_2(x))$ verifies (2.1) the inequality (2.2) follows from [7, Lemma 1, p. 24] (see also the arguments in [8, Theorem 1, p. 236]).
- We can also try to prove the uniqueness of such a solution using the result of [2,6,11] (see also the former papers of [14,15]).

4. Exemple of Coupled Thin Elastic Plates

We study the example of two coupled thin elastic plates, where the displacements of the plates are denoted by $z_1(x)$ and $z_2(x)$, respectively. The governing system of partial differential equations for the displacements is as follows:

$$\begin{cases} -\frac{k_1^3}{2}\Delta^2 z_1+\frac{\left(-k_1\Delta z_1(x)+|\nabla z_1(x)|^2\right)^2}{2}+\frac{k_1^2}{2}\Delta|\nabla z_1|^2=f_1(x)-(\lambda_1+a_1)z_1(x)+a_1z_2(x),\\ -\frac{k_2^3}{2}\Delta^2 z_2+\frac{\left(-k_2\Delta z_2(x)+|\nabla z_2(x)|^2\right)^2}{2}+\frac{k_2^2}{2}\Delta|\nabla z_2|^2=f_2(x)-(\lambda_2+a_2)z_2(x)+a_2z_1(x),\end{cases}$$

where $x\in\mathbb{R}^N$.

To understand the utility of this system, we can provide the following physical interpretation:

- **Biharmonic Operator:** The term $-\frac{k_i^3}{2}\Delta^2 z_i$ represents the elastic restoring force due to the bending stiffness of the plate, where k_i is a material and geometric constant specific to the i-th plate.

- **Nonlinear Terms:** The nonlinear terms, such as $\frac{\left(-k_i\Delta z_i(x)+|\nabla z_i(x)|^2\right)^2}{2}$ and $\frac{k_i^2}{2}\Delta|\nabla z_i|^2$, capture the effects of large deformations, including in-plane forces and stretching.

- **Coupling Between Plates:** The interaction terms $-(\lambda_i+a_i)z_i(x)+a_iz_j(x)$ model the mutual influence between the two plates, where λ_i and a_i represent the individual and coupling effects, respectively.

To ensure the problem is well-posed, suitable boundary and initial conditions need to be defined:

- **Boundary Conditions:** These could include clamped, simply supported, or free boundary conditions, depending on the specific setup of the plates.
- **Initial Conditions:** For time-dependent problems, initial conditions for $z_i(x, 0)$ and $\frac{\partial z_i}{\partial t}(x, 0)$ are required.

The solutions $z_1(x)$ and $z_2(x)$ represent the equilibrium or dynamic configurations of the plates under the applied loads and boundary conditions. These solutions provide valuable insights into:

- The stress distribution and deformation patterns of the plates.
- Potential failure points or regions of instability in the system.
- The coupled behavior and interaction effects between the plates.

For more detailed examples, refer to [1,16,18].

The existence of solutions to the system of equations governing the coupled thin elastic plates is significant both physically and mathematically. Below, we summarize the key interpretations:

- **Equilibrium Configuration:** The existence of solutions implies that the coupled system of plates can reach a stable equilibrium under the applied forces $f_1(x)$ and $f_2(x)$, boundary conditions, and material properties. The solutions $z_1(x)$ and $z_2(x)$ describe the deformed shapes of the plates at equilibrium.
- **Plate Interaction:** The coupling terms $a_1 z_2(x)$ and $a_2 z_1(x)$ represent the mutual influence between the plates. The existence of solutions ensures that this interaction results in a physically realistic deformation pattern.
- **Nonlinear Effects:** The nonlinear terms, such as $|\nabla z_i|^2$ and $\Delta|\nabla z_i|^2$, capture large deformations and in-plane forces. The existence of solutions ensures that these nonlinearities do not produce unphysical behaviors, such as infinite displacements or instabilities.

Finding the solution is very useful, as it can help with

- **Design and Safety:** The solutions help engineers predict the behavior of the coupled plates under various loadings, ensuring that the design is safe and reliable.
- **Stability Analysis:** The existence of solutions guarantees that the plates will not undergo unbounded or catastrophic deformations, such as buckling or collapse, under the given conditions.
- **Material and Geometric Properties:** The solutions reveal how the interplay between material properties (e.g., bending stiffness, denoted by k_1, k_2) and geometric constraints influences the system's overall behavior.

In the end, the existence of a solution to the governing equations confirms that the coupled thin elastic plate system can reach a stable, physically realistic, and mathematically consistent state under the specified conditions. This validates the mathematical model and ensures its applicability in real-world scenarios for analyzing and designing systems involving coupled plates.

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