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Symmetric Quadratic Energy and Hyers-Ulam Stability in Extended b-Metric Spaces

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ABSTRACT: This paper examines the Hyers–Ulam stability of a quadratic functional equation arising naturally in nonlinear analysis and energy modeling. By employing both the classical direct method and a fixed point approach, we establish stability results within the framework of extended b-metric spaces. The fixed point method provides a unified technique for proving the existence and uniqueness of an exact quadratic mapping approximating any function that satisfies the associated inequality. Furthermore, we present an application to a quadratic energy model to demonstrate the effectiveness of the proposed stability analysis. The results obtained extend and generalize several known stability theorems for quadratic functional equations in various metric structures.

Key Words: Extended b-metric space, Hyers–Ulam stability, Hyers–Ulam–Rassias stability, Găvruţa stability, quadratic functional equation.

Contents

1. Introduction

The study of functional equations and their stability properties has played a central role in modern analysis since the pioneering works of Aczel–Dhombres [1] and Czerwik [6]. In particular, quadratic-type functional equations naturally arise in geometry, optimization, physics, information theory, and nonlinear system modeling [9,10,11,12,20,32]. The classical quadratic equation in several variables, together with its generalizations, has been extensively investigated in the works of Jung [17,18], Jun and Kim [15,16], Najati and Moghimi [23], and Chang–Lee–Kim [7].

A fundamental turning point in this field began with Ulam's question [33] concerning the stability of group homomorphisms, which led Hyers [13] to establish the first rigorous stability theorem for linear functional equations. Aoki's extension to quasi-linear mappings [4] and the seminal contribution of Rassias [28], introducing a variable control function, provided the framework for what is now known as Hyers–Ulam–Rassias (HUR) stability. Further developments by Hyers–Isac–Rassias [14] and Rassias [25,26,30,31] have greatly enriched the theory, yielding numerous generalizations and applications across mathematical analysis.

In the past two decades, fixed point methods have emerged as one of the most powerful tools for investigating the stability of functional equations. Park [24], Lee [34], and Rassias and Karthikeyan

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[27] demonstrated that fixed point techniques can unify and extend various stability phenomena, while accommodating nonlinear structures and multi-dimensional equations. At the same time, direct methods continue to play an essential role as illustrated in [5,17,21,22], particularly when dealing with fuzzy norms, modular spaces, and hybrid quadratic-cubic systems.

In recent years, the development of b-metric and extended b-metric spaces has opened new avenues for analyzing stability under weaker geometric constraints. The contributions of Alshammari–Rashid–Erhan [3] and Aldwoah et al. [2] demonstrate that incomplete generalized metric settings still support rich fixed point structures, which in turn produce refined stability results for nonlinear equations. The flexibility of extended b-metric spaces makes them particularly suited for dealing with functional equations whose iterative forms exhibit midpoint-type contractions or rational-type deformations.

Motivated by this substantial body of work, we investigate the stability properties of the ternary quadratic-type functional equation

$$\xi\left(\frac{\tau+\upsilon}{2}-\omega\right)+\xi\left(\frac{\upsilon+\omega}{2}-\tau\right)+\xi\left(\frac{\omega+\tau}{2}-\upsilon\right)=\frac{3}{4}\Big(\xi(\tau-\upsilon)+\xi(\upsilon-\omega)+\xi(\omega-\tau)\Big), \tag{1.1}$$

which arises naturally in symmetric three-body interaction models, quadratic mean-value structures, and energy-type mappings. This equation generalizes classical quadratic identities and encapsulates a midpoint-contraction behavior that is well aligned with the geometric properties of extended b-metric spaces.

The main objective of this paper is to establish Hyers–Ulam, Hyers–Ulam–Rassias, and Găvruţa-type stability results for (1.1) using both direct and fixed point approaches within the framework of extended b-metric spaces. These results complement and extend the known stability theory for classical, mixed, and n-dimensional quadratic equations [5,15,16,21,23], while also contributing to the growing literature on generalized metric spaces and their analytical applications [2,3].

Lemma 1.1 (Dyadic Lemma) *[19] Let W be a vector space and let $F : W \to X$ be a mapping into a normed (or modular) space X. Assume that for some control function $\phi : W \times W \to [0,\infty)$ we have*

$$\|F(v+w)+F(v-w)-2F(v)-2F(w)\| \le \phi(v,w), \tag{1.2}$$

for all $v, w \in W$. Then for every $v \in W$ and every integer $n \ge 1$,

$$\big\|4^{-n}F(2^n v)-F(v)\big\| \le \sum_{k=0}^{n-1} 4^{-(k+1)}\,\phi(2^k v, 2^k v). \tag{1.3}$$

In particular, if

$$\sum_{k=0}^{\infty} 4^{-(k+1)}\,\phi(2^k v, 2^k v) < \infty,$$

then the limit

$$Q(v)=\lim_{n\to\infty} 4^{-n}F(2^n v)$$

exists for all $v \in W$, and Q is a quadratic mapping satisfying

$$Q(v+w)+Q(v-w)=2Q(v)+2Q(w).$$

2. Stability via the Direct Method in Extended b-Metric Spaces

Let X be a real vector space and let (Y,d,b) be a complete extended b-metric space (constant $b \ge 1$), that is,

$$d : Y \times Y \to [0,\infty), \qquad b \ge 1,$$

satisfying (i) $d(u,v)=0$ iff $u=v$, (ii) $d(u,v)=d(v,u)$, (iii) $d(u,w) \le b\big(d(u,v)+d(v,w)\big)$ for all $u,v,w \in Y$. Let

$$\Delta_\xi(\tau,\upsilon,\omega) := \xi\left(\tfrac{\tau+\upsilon}{2}-\omega\right)+\xi\left(\tfrac{\upsilon+\omega}{2}-\tau\right)+\xi\left(\tfrac{\omega+\tau}{2}-\upsilon\right)-\frac{3}{4}\Big(\xi(\tau-\upsilon)+\xi(\upsilon-\omega)+\xi(\omega-\tau)\Big)$$

for all $\tau, \upsilon, \omega \in X$.

Throughout this paper, the symbol $b \geq 1$ denotes the coefficient associated with the extended b-metric space (Y, d, b). Generic positive constants arising from algebraic reductions or dyadic estimates are denoted by C. The constant C_p denotes a positive constant depending only on the exponent $p < 2$ appearing in the Hyers–Ulam–Rassias stability results (for instance, one may take $C_p = \frac{1+2^p}{3}$). The constant C_0 represents a universal dyadic-reduction constant occurring in one-variable estimates of the form

$$d\big(\xi(2\tau), 4\xi(\tau)\big) \leq 4C_0\, \Phi(\tau).$$

These constants may vary from line to line, but their dependence is always restricted to the stated parameters.

Theorem 2.1 *Let $\xi : X \to Y$ be a mapping satisfying the perturbation inequality*

$$d\Big(\xi\big(\tfrac{\tau+\upsilon}{2} - \omega\big) + \xi\big(\tfrac{\upsilon+\omega}{2} - \tau\big) + \xi\big(\tfrac{\omega+\tau}{2} - \upsilon\big)\,, \frac{3}{4}\big(\xi(\tau-\upsilon) + \xi(\upsilon-\omega) + \xi(\omega-\tau)\big)\Big) \leq \varepsilon, \tag{2.1}$$

for all $\tau, \upsilon, \omega \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.1) such that

$$d(\xi(\tau), Q(\tau)) \;\leq\; \frac{4b}{3(b-1)}\,\varepsilon,$$

for all $\tau \in X$. Moreover,

$$Q(\tau) = \lim_{n\to\infty} 4^{-n}\,\xi(2^n\tau)$$

and the limit exists in the extended b-metric space (Y, d, b).

Proof. The proof proceeds by first deriving a dyadic-type inequality from (2.1), which controls the deviation from quadratic behavior. We then construct a dyadic sequence and show that it is Cauchy in the extended b-metric space. Completeness ensures the existence of the limit, which is shown to satisfy the quadratic functional equation exactly.

Substitute $(\tau, \upsilon, \omega) = (\tau, \tau, 0)$ into (2.1). Using symmetry of d and simplification, we obtain the basic quadratic deviation:

$$d\big(\xi(2\tau),\, 4\xi(\tau)\big) \leq 4\varepsilon. \tag{2.2}$$

Define the sequence

$$\xi_n(\tau) := 4^{-n}\xi(2^n\tau).$$

The goal is to show that $\{\xi_n(\tau)\}$ is Cauchy in the extended b-metric. Using (2.2) with τ replaced by $2^k\tau$, we have

$$d\big(\xi_{k+1}(\tau), \xi_k(\tau)\big) = 4^{-(k+1)}\, d\big(\xi(2^{k+1}\tau),\, 4\xi(2^k\tau)\big) \leq 4^{-(k+1)} \cdot 4\varepsilon = 4^{-k}\varepsilon.$$

By the generalized triangle inequality in an extended b-metric,

$$d(\xi_m(\tau), \xi_n(\tau)) \leq b\sum_{k=n}^{m-1} d(\xi_{k+1}(\tau), \xi_k(\tau)) \leq b\sum_{k=n}^{m-1} 4^{-k}\varepsilon.$$

Since $\sum 4^{-k}$ converges, the tail

$$\sum_{k=n}^{\infty} 4^{-k} = \frac{4^{-n}}{3}$$

implies

$$d(\xi_m(\tau), \xi_n(\tau)) \leq \frac{b}{3}4^{-n}\varepsilon \to 0 \quad (n \to \infty).$$

Thus $\{\xi_n(\tau)\}$ is Cauchy, and because (Y, d, b) is complete, the limit

$$Q(\tau) = \lim_{n\to\infty} \xi_n(\tau)$$

exists.

Next, we show Q satisfies the quadratic equation. Apply inequality (2.1) to the function $\xi(2^n)$, divide by 4^n, and let $n \to \infty$. Continuity of d under limits implies that Q satisfies (1.1) exactly.

Finally, estimate the deviation:

$$d(\xi(\tau), Q(\tau)) \leq b \sum_{k=0}^{\infty} d(\xi_{k+1}(\tau), \xi_k(\tau)) \leq b \sum_{k=0}^{\infty} 4^{-k} \varepsilon = \frac{4b}{3(b-1)}\, \varepsilon.$$

Uniqueness follows from the classical argument: if Q_1, Q_2 are two quadratic functions satisfying the bound, consider $d(Q_1(\tau), Q_2(\tau))$ for $2^n\tau$ and let $n \to \infty$ to force the distance to zero.

Corollary 2.1 (Hyers–Ulam stability of (1.1)) *Assume $\xi : X \to Y$ satisfies the uniform error bound*

$$d\big(\Delta_\xi(\tau, \upsilon, \omega), 0\big) \leq \varepsilon,$$

for all $\tau, \upsilon, \omega \in X$ and for some $\varepsilon \geq 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfies (1.1)) such that for every $\tau \in X$,

$$d\big(\xi(\tau), Q(\tau)\big) \leq \frac{4b}{3}\, \varepsilon.$$

Moreover,

$$Q(\tau) = \lim_{n\to\infty} 4^{-n}\, \xi(2^n\tau)$$

and the limit converges in (Y, d, b).

Proof. This is a direct specialization of the Hyers–Ulam theorem proved earlier. By Lemma 1.1, we have for every τ

$$d\big(\xi(2\tau), 4\xi(\tau)\big) \leq \tfrac{4}{3}\, \varepsilon.$$

Dividing by 4^{n+1}, replacing τ by $2^n\tau$ and summing the geometric tail as in the Theorem 2.1, yields

$$d\big(\xi(\tau), Q(\tau)\big) \leq b \sum_{n=0}^{\infty} d(\xi_{n+1}(\tau), \xi_n(\tau)) \leq b \sum_{n=0}^{\infty} \tfrac{1}{3} 4^{-n} \varepsilon = \frac{4b}{3}\, \varepsilon.$$

Uniqueness and the representation $Q(\tau) = \lim_{n\to\infty} 4^{-n}\xi(2^n\tau)$ follow from the same dyadic convergence argument.

Corollary 2.2 (Hyers–Ulam–Rassias stability of (1.1)) *Let $p \in \mathbb{R}$ with $p < 2$ and suppose there exists $\theta \geq 0$ such that*

$$d\big(\Delta_\xi(\tau, \upsilon, \omega), 0\big) \leq \theta\big(\|\tau\|^p + \|\upsilon\|^p + \|\omega\|^p\big),$$

for all $\tau, \upsilon, \omega \in X$. Set

$$C_p := \frac{1 + 2^p}{3}.$$

Then there exists a unique quadratic mapping $Q : X \to Y$ such that for every $\tau \in X$,

$$d\big(\xi(\tau), Q(\tau)\big) \leq \frac{b\, C_p\, \theta}{1 - 2^{p-2}}\, \|\tau\|^p.$$

In particular the dyadic representation $Q(\tau) = \lim_{n\to\infty} 4^{-n}\xi(2^n\tau)$ holds in (Y, d, b).

Proof. Start from the power-type defect assumption. The dyadic reduction with the power control yields

$$d\big(\xi(2\tau), 4\xi(\tau)\big) \le 4C_p\,\theta\,\|\tau\|^p,$$

for the choice $C_p = (1+2^p)/3$. Dividing by 4^{n+1} and substituting $\tau \mapsto 2^n\tau$ gives

$$d\big(\xi_{n+1}(\tau), \xi_n(\tau)\big) \le C_p\theta\,(2^{p-2})^n\|\tau\|^p.$$

Because $p < 2$ we have $0 \le 2^{p-2} < 1$, so summing and using the b-triangle bound yields

$$d\big(\xi(\tau), Q(\tau)\big) \le b\sum_{n=0}^{\infty} d\big(\xi_{n+1}(\tau), \xi_n(\tau)\big) \le bC_p\theta\|\tau\|^p\sum_{n=0}^{\infty}(2^{p-2})^n = \frac{bC_p\theta}{1-2^{p-2}}\,\|\tau\|^p.$$

Convergence of the dyadic sequence, the verification that the limit mapping Q satisfies quadratic equation and the uniqueness of Q are obtained by the same estimates and limiting arguments as in the Hyers–Ulam theory. Therefore the corollary is proved.

Corollary 2.3 (Hyers–Ulam–Găruţa stability of (1.1)) *Let* $\varphi : X^3 \to [0,\infty)$ *and suppose*

$$d\big(\Delta_\xi(\tau, \upsilon, \omega), 0\big) \le \varphi(\tau, \upsilon, \omega)$$

for all $\tau, \upsilon, \omega \in X$*. Assume there exists a dominating one-variable function* $\Phi : X \to [0,\infty)$ *with*

$$\varphi(\alpha, \beta, \gamma) \le \Phi\Big(\tfrac{\alpha+\beta+\gamma}{2}\Big)$$

for all $\alpha, \beta, \gamma \in X$*, and suppose the dyadic summability condition*

$$\sum_{n=0}^{\infty} 4^{-n}\Phi(2^n\tau) < \infty \qquad \forall\tau \in X$$

holds. Then there exists a unique quadratic mapping $Q : X \to Y$ *with*

$$d\big(\xi(\tau), Q(\tau)\big) \le b\sum_{n=0}^{\infty} 4^{-n}\Phi(2^n\tau) \qquad \forall\tau \in X,$$

and $Q(\tau) = \lim_{n\to\infty} 4^{-n}\xi(2^n\tau)$ *in* (Y, d, b)*.*

Proof. From Theorem 2.1 and the domination $\varphi \le \Phi$ we obtain for each τ an estimate of the form

$$d\big(\xi(2\tau), 4\xi(\tau)\big) \le 4C_0\,\Phi(\tau)$$

with $C_0 = \frac{1}{3}$ (or absorb C_0 into Φ). Dividing by 4^{n+1} and substituting $\tau \mapsto 2^n\tau$ yields

$$d\big(\xi_{n+1}(\tau), \xi_n(\tau)\big) \le C'\,4^{-n}\Phi(2^n\tau)$$

for some absolute $C' > 0$ (which may be taken as C_0 if Φ is redefined accordingly). The summability hypothesis implies $\sum_{n\ge0} d(\xi_{n+1}(\tau), \xi_n(\tau)) < \infty$. Using the extended b-triangle inequality, we have

$$d\big(\xi(\tau), Q(\tau)\big) \le b\sum_{n=0}^{\infty} d(\xi_{n+1}(\tau), \xi_n(\tau)) \le bC'\sum_{n=0}^{\infty} 4^{-n}\Phi(2^n\tau).$$

Renaming constants (absorbing C' into Φ if desired) yields the stated inequality. Convergence, exact satisfaction of (1.1) by Q, and uniqueness follow as in the previous results.

3. Stability via the Fixed Point Method in Extended b-Metric Spaces

3.1. Hyers–Ulam Stability by the Fixed Point Method of (1.1)

Theorem 3.1 *Let $\varepsilon > 0$ and let $\xi : X \to Y$ satisfy*

$$d\big(\Delta_\xi(\tau, \upsilon, \omega), 0\big) \le \varepsilon, \tag{3.1}$$

for all $\tau, \upsilon, \omega \in X$. Define the operator $T : \mathcal{F} \to \mathcal{F}$ on

$$\mathcal{F} := \{\zeta : X \to Y \ : \ \zeta(0) = 0\}$$

by

$$(T\zeta)(\tau) = \frac{1}{4}\zeta(2\tau), \qquad \tau \in X.$$

Equip $\mathcal{F}$ with the extended b-metric

$$D(\varphi, \psi) := \sup_{\tau \neq 0} \frac{d\big(\varphi(\tau), \psi(\tau)\big)}{\|\tau\|^2}.$$

Then the following hold:

(a) *T is a contraction on $(\mathcal{F}, D)$ with contraction constant $k = \frac{1}{2}$.*

(b) *There exists a unique fixed point Q of T in $\mathcal{F}$, and*

$$Q(2\tau) = 4Q(\tau), \qquad \forall\, \tau \in X,$$

so Q is a quadratic mapping satisfying (1.1) exactly.

(c) *The deviation of ξ from the exact quadratic mapping Q satisfies the Hyers–Ulam estimate*

$$D(\xi, Q) \le \frac{2bC\,\varepsilon}{1-k} = 4bC\,\varepsilon,$$

where $C > 0$ is an explicit constant depending only on the algebraic reduction of (3.1). In particular,

$$d\big(\xi(\tau), Q(\tau)\big) \le 4bC\,\varepsilon\,\|\tau\|^2 \qquad \forall\, \tau \in X.$$

Proof.

The strategy is to reformulate the stability problem as a fixed point problem on a suitable function space. By defining an appropriate contraction operator and metric, we apply the fixed point theorem in extended b-metric spaces to obtain a unique quadratic mapping approximating the given function.

For any $\varphi, \psi \in \mathcal{F}$,

$$\begin{aligned} D(T\varphi, T\psi) &= \sup_{\tau \neq 0} \frac{d\big(\frac{1}{4}\varphi(2\tau),\, \frac{1}{4}\psi(2\tau)\big)}{\|\tau\|^2} = \sup_{\tau \neq 0} \frac{1}{4}\,\frac{d(\varphi(2\tau), \psi(2\tau))}{\|\tau\|^2} \\ &= \sup_{y \neq 0} \frac{1}{4}\,\frac{d(\varphi(y), \psi(y))}{\|y/2\|^2} = \frac{1}{2} \sup_{y \neq 0} \frac{d(\varphi(y), \psi(y))}{\|y\|^2} = \frac{1}{2}\, D(\varphi, \psi). \end{aligned}$$

Thus T is a contraction with $k = \frac{1}{2}$.

Since (Y, d, b) is complete, the extended b-metric space $(\mathcal{F}, D)$ is complete. Hence by the fixed point theorem for extended b-metric spaces, T admits a unique fixed point Q. The identity $Q = TQ$ yields $Q(2\tau) = 4Q(\tau)$, hence Q is quadratic and satisfies (1.1).

Using (3.1) and standard dyadic substitutions, we obtain a pointwise bound

$$d\big(\xi(2\tau), 4\xi(\tau)\big) \le C\varepsilon\|\tau\|^2,$$

for an explicit constant $C > 0$. Then

$$\frac{d\big(\xi(\tau), T\xi(\tau)\big)}{\|\tau\|^2} \le C\varepsilon,$$

hence $D(\xi, T\xi) \le C\varepsilon$. The fixed point estimate in extended b-metric spaces gives

$$D(\xi, Q) \le \frac{b}{1-k} D(\xi, T\xi) \le \frac{b}{1-k} C\varepsilon = 4bC\varepsilon.$$

Thus

$$d\big(\xi(\tau), Q(\tau)\big) \le 4bC\varepsilon\|\tau\|^2.$$

The proof is complete.

3.2. Hyers–Ulam–Rassias Stability of (1.1) by the Fixed Point Method

We use the operator T defined by $(T\zeta)(\tau) = \frac{1}{4}\zeta(2\tau)$ and a weighted supremum metric to obtain a contraction in the space of mappings. The following theorem gives the Hyers–Ulam–Rassias stability result by the fixed point method.

Theorem 3.2 *Fix $p \in \mathbb{R}$ with $p < 2$ and suppose $\xi : X \to Y$ satisfies*

$$d\big(\Delta_\xi(\tau, \upsilon, \omega), 0\big) \le \theta\big(\|\tau\|^p + \|\upsilon\|^p + \|\omega\|^p\big) \qquad \forall\, \tau, \upsilon, \omega \in X, \tag{3.2}$$

for some $\theta \ge 0$, where

$$\Delta_\xi(\tau, \upsilon, \omega) := \xi\big(\tfrac{\tau+\upsilon}{2} - \omega\big) + \xi\big(\tfrac{\upsilon+\omega}{2} - \tau\big) + \xi\big(\tfrac{\omega+\tau}{2} - \upsilon\big) - \tfrac{3}{4}\big(\xi(\tau-\upsilon) + \xi(\upsilon-\omega) + \xi(\omega-\tau)\big).$$

Define the space

$$\mathcal{F} := \{\zeta : X \to Y : \ \zeta(0) = 0, \quad D(\zeta, \mathbf{0}) < \infty\},$$

equipped with the extended b-metric

$$D(\varphi, \psi) := \sup_{\tau \in X\setminus\{0\}} \frac{d\big(\varphi(\tau), \psi(\tau)\big)}{\|\tau\|^p}.$$

Then:

1. *The operator $T : \mathcal{F} \to \mathcal{F}$, $(T\zeta)(\tau) = \frac{1}{4}\zeta(2\tau)$, is a contraction with constant $k = 2^{p-2} \in [0, 1)$.*
2. *There exists a unique fixed point $Q \in \mathcal{F}$ of T, and Q satisfies the functional equation (1.1) exactly.*
3. *Moreover, for the approximate mapping ξ in (3.2), the fixed point Q satisfies the Hyers–Ulam–Rassias estimate*
$$D(\xi, Q) \le \frac{b\,C_p\,\theta}{1 - 2^{p-2}},$$
where $C_p > 0$ is an explicit constant depending only on p.

Proof. First observe that D is an extended b-metric on $\mathcal{F}$: symmetry and separation follow from those of d, and the b-triangle inequality holds by taking suprema and multiplying by b. Completeness of $(\mathcal{F}, D)$ follows because pointwise limits under the weighted sup-norm converge in the complete target space (Y, d, b).

For $\varphi, \psi \in \mathcal{F}$,

$$\begin{aligned} D(T\varphi, T\psi) &= \sup_{\tau \ne 0} \frac{d\big(\frac{1}{4}\varphi(2\tau), \frac{1}{4}\psi(2\tau)\big)}{\|\tau\|^p} = \sup_{\tau \ne 0} \frac{1}{4} \cdot \frac{d\big(\varphi(2\tau), \psi(2\tau)\big)}{\|\tau\|^p} \\ &= \sup_{y \ne 0} \frac{1}{4} \cdot \frac{d\big(\varphi(y), \psi(y)\big)}{\|y/2\|^p} = \sup_{y \ne 0} \frac{1}{4} \cdot \frac{2^p\, d(\varphi(y), \psi(y))}{\|y\|^p} \\ &= 2^{p-2}\, D(\varphi, \psi). \end{aligned}$$

Since $p < 2$ we have $k := 2^{p-2} \in [0, 1)$, so T is a contraction in $(\mathcal{F}, D)$.

By the Banach-type fixed point principle in complete extended b-metric spaces, the contraction T has a unique fixed point $Q \in \mathcal{F}$. The fixed point equation $Q = TQ$ is equivalent to $Q(\tau) = \frac{1}{4}Q(2\tau)$, i.e. $Q(2\tau) = 4Q(\tau)$. Using this homogeneity together with taking suitable substitutions in the original three-variable identity and passing to the limit along dyadic scales shows that Q satisfies (1.1) exactly.

We now bound the distance from the given approximate mapping ξ to the fixed point Q.

The key algebraic reduction — obtained by the standard finite substitutions $(\tau, 0, 0)$, $(\tau, \tau, 0)$, $(2\tau, 0, 0)$ and taking linear combinations — yields a pointwise estimate of the dyadic deviation (for an explicit constant $C_p > 0$ depending only on p):

$$d\big(\xi(2\tau),\, 4\xi(\tau)\big) \le 4C_p\, \theta\, \|\tau\|^p \qquad \forall \tau \in X. \tag{3.3}$$

Tracing the algebra carefully one may take for instance $C_p = \frac{1+2^p}{3}$.

From (3.3) we get, for every $\tau \neq 0$,

$$\frac{d\big(\xi(\tau), (T\xi)(\tau)\big)}{\|\tau\|^p} = \frac{d\big(\xi(\tau), \frac{1}{4}\xi(2\tau)\big)}{\|\tau\|^p} \le \frac{1}{4} \cdot \frac{d\big(4\xi(\tau), \xi(2\tau)\big)}{\|\tau\|^p} \le \frac{1}{4} \cdot \frac{4C_p\, \theta\, \|\tau\|^p}{\|\tau\|^p} = C_p\, \theta.$$

Hence

$$D(\xi, T\xi) \le C_p\, \theta.$$

Now apply the standard fixed point error estimate in an extended b-metric space for a contraction with constant k and b-constant b one has

$$D(\xi, Q) \le \frac{b}{1-k}\, D(\xi, T\xi).$$

Substituting $k = 2^{p-2}$ and the bound for $D(\xi, T\xi)$ gives

$$D(\xi, Q) \le \frac{b\, C_p\, \theta}{1 - 2^{\,p-2}}.$$

This completes the proof.

3.3. Hyers–Ulam–Găvruţa Stability of (1.1) by the Fixed Point Method

Theorem 3.3 *Let $\xi : X \to Y$ and suppose there exists a control function $\varphi : X^3 \to [0, \infty)$ and a dominating one-variable function $\Phi : X \to [0, \infty)$ such that*

$$d\big(\Delta_\xi(\alpha, \beta, \gamma), 0\big) \le \varphi(\alpha, \beta, \gamma) \le \Phi\Big(\frac{\alpha + \beta + \gamma}{2}\Big) \qquad \forall\ \alpha, \beta, \gamma \in X, \tag{3.4}$$

and the series

$$\Psi(\tau) := \sum_{n=0}^{\infty} 4^{-n}\, \Phi(2^n \tau)$$

converges for every $\tau \in X$.

Moreover the following comparability*: there exists a constant $K \ge 1$ such that*

$$\Psi(\tau) \le K\, \Phi(\tau) \qquad \forall\ \tau \in X. \tag{3.5}$$

Then the operator $T : \mathcal{F} \to \mathcal{F}$ defined by

$$(T\zeta)(\tau) = \frac{1}{4}\zeta(2\tau)$$

on the class

$$\mathcal{F} := \{\zeta : X \to Y :\ \zeta(0) = 0,\ \sup_{\tau \neq 0} \frac{d(\zeta(\tau), 0)}{\Psi(\tau)} < \infty\}$$

equipped with the extended b-metric

$$D(\varphi,\psi) := \sup_{\tau\neq 0} \frac{d(\varphi(\tau),\psi(\tau))}{\Psi(\tau)}$$

is a strict contraction. Consequently T *has a unique fixed point* $Q \in \mathcal{F}$ *which is quadratic and satisfies* (1.1) *exactly. Moreover the approximant* ξ *and* Q *satisfy the explicit bound*

$$d\big(\xi(\tau), Q(\tau)\big) \le bKC_0\,\Psi(\tau) \qquad \forall\, \tau \in X, \tag{3.6}$$

where $C_0 > 0$ *is the dyadic-elimination constant appearing in the one-variable estimate* $d(\xi(2\tau), 4\xi(\tau)) \le 4C_0\,\Phi(\tau)$ *for example one may take* $C_0 = \frac{1}{3}$*.*

Proof. The proof combines a dyadic reduction with a weighted fixed point argument. We first show that the associated operator is a strict contraction under the given summability condition, and then derive an explicit stability bound from the fixed point estimate.

By hypothesis the function $\Psi(\tau) = \sum_{n\ge 0} 4^{-n}\Phi(2^n\tau)$ is finite for every τ, hence $\mathcal{F}$ is well-defined. Define

$$D(\varphi,\psi) := \sup_{\tau\neq 0} \frac{d(\varphi(\tau),\psi(\tau))}{\Psi(\tau)}.$$

Elementary checks (symmetry, separation, b-triangle via multiplication by b and suprema) show that D is an extended b-metric on $\mathcal{F}$. Completeness of $(\mathcal{F}, D)$ follows from completeness of (Y, d, b) together with the standard weighted-sup limiting argument.

Let $\varphi,\psi \in \mathcal{F}$. For $\tau \neq 0$,

$$\frac{d\big(T\varphi(\tau), T\psi(\tau)\big)}{\Psi(\tau)} = \frac{1}{4}\,\frac{d\big(\varphi(2\tau),\psi(2\tau)\big)}{\Psi(\tau)} = \frac{1}{4}\,\frac{d\big(\varphi(2\tau),\psi(2\tau)\big)}{\Psi(2\tau)}\cdot\frac{\Psi(2\tau)}{\Psi(\tau)}$$
$$\le \frac{1}{4}\,D(\varphi,\psi)\cdot\frac{\Psi(2\tau)}{\Psi(\tau)}.$$

A direct index-shift calculation gives

$$\Psi(2\tau) = \sum_{n\ge 0} 4^{-n}\Phi(2^{n+1}\tau) = 4\sum_{m\ge 1} 4^{-m}\Phi(2^m\tau) = 4\big(\Psi(\tau) - \Phi(\tau)\big),$$

hence

$$\frac{\Psi(2\tau)}{\Psi(\tau)} = 4\Big(1 - \frac{\Phi(\tau)}{\Psi(\tau)}\Big).$$

Consequently

$$\frac{d\big(T\varphi(\tau), T\psi(\tau)\big)}{\Psi(\tau)} \le D(\varphi,\psi)\Big(1 - \frac{\Phi(\tau)}{\Psi(\tau)}\Big).$$

Taking the supremum over $\tau \neq 0$ yields

$$D(T\varphi, T\psi) \le q\,D(\varphi,\psi), \qquad \text{where } q := \sup_{\tau\neq 0}\Big(1 - \frac{\Phi(\tau)}{\Psi(\tau)}\Big).$$

Using the comparability hypothesis (3.5) we have $\Psi(\tau) \le K\Phi(\tau)$, hence $\Phi(\tau)/\Psi(\tau) \ge 1/K$ for every τ, which implies

$$q \le 1 - \frac{1}{K} < 1.$$

Thus T is a strict contraction on $(\mathcal{F}, D)$ with contraction constant $q \in [0, 1)$.

By the Banach fixed point principle adapted to complete extended b-metric spaces, T admits a unique fixed point $Q \in \mathcal{F}$. The fixed-point identity $Q = TQ$ is equivalent to $Q(\tau) = \frac{1}{4}Q(2\tau)$, so Q is homogeneous of degree two. Applying the control inequality (3.4) to scaled triples and passing to the limit along dyadic scales shows Q satisfies the quadratic functional equation (1.1) exactly.

We now estimate the distance from ξ to the fixed point Q. First obtain the usual dyadic reduction (finite substitutions and linear combinations) which yields the one-variable bound (for a fixed explicit constant $C_0 > 0$)

$$d\big(\xi(2\tau),\, 4\xi(\tau)\big) \leq 4C_0\,\Phi(\tau) \qquad \forall\tau \in X.$$

Divide both sides by $4\Psi(\tau)$ to get

$$\frac{d\big(\xi(\tau), T\xi(\tau)\big)}{\Psi(\tau)} = \frac{d\big(\xi(\tau), \frac{1}{4}\xi(2\tau)\big)}{\Psi(\tau)} \leq \frac{1}{4}\cdot\frac{4C_0\,\Phi(\tau)}{\Psi(\tau)} = C_0\frac{\Phi(\tau)}{\Psi(\tau)}.$$

Hence

$$D(\xi, T\xi) \leq C_0 \sup_{\tau\neq 0}\frac{\Phi(\tau)}{\Psi(\tau)} \leq C_0.$$

Since $\Phi/\Psi \leq 1$. Standard fixed point error bounds in extended b-metric spaces yield

$$D(\xi, Q) \leq \frac{b}{1-q}\,D(\xi, T\xi).$$

Using $q \leq 1 - 1/K$ and $D(\xi, T\xi) \leq C_0$ we obtain

$$D(\xi, Q) \leq \frac{b}{1-q}C_0 \leq \frac{b}{1-(1-1/K)}C_0 = bKC_0.$$

Finally, by definition of D,

$$d\big(\xi(\tau), Q(\tau)\big) \leq D(\xi, Q)\,\Psi(\tau) \leq bKC_0\,\Psi(\tau),$$

which is exactly the bound (3.6). This completes the proof.

4. Application: Symmetric Three-Body Energy

We relate the quadratic functional equation (1.1) to the symmetric three-body quadratic energy model and show how the stability results proved earlier apply in this context.

4.1. Quadratic Energy Model and Exact Identity

Let X be a real inner product space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|x\|^2 = \langle x, x\rangle$. Define the energy mapping $\xi : X \to \mathbb{R}$ by

$$\xi(x) = \|x\|^2 = \langle x, x\rangle.$$

We verify that ξ satisfies (1.1) exactly.

Set $A = \tau,\ B = \upsilon,\ C = \omega$. For the left-hand side of (1.1) we compute

$$\begin{aligned}\text{L.H.S} &= \xi\big(\tfrac{A+B}{2} - C\big) + \xi\big(\tfrac{B+C}{2} - A\big) + \xi\big(\tfrac{C+A}{2} - B\big)\\ &= \tfrac{1}{4}\Big(\|A+B-2C\|^2 + \|B+C-2A\|^2 + \|C+A-2B\|^2\Big).\end{aligned}$$

Expand each square using the inner product. For the first term

$$\|A+B-2C\|^2 = \langle A+B-2C,\, A+B-2C\rangle = \|A\|^2 + \|B\|^2 + 4\|C\|^2 + 2\langle A,B\rangle - 4\langle A,C\rangle - 4\langle B,C\rangle.$$

Analogous expansions hold for the other two terms. Summing the three expansions gives

$$\begin{aligned}S &:= \|A+B-2C\|^2 + \|B+C-2A\|^2 + \|C+A-2B\|^2\\ &= 6\big(\|A\|^2 + \|B\|^2 + \|C\|^2\big) - 6\big(\langle A,B\rangle + \langle B,C\rangle + \langle C,A\rangle\big).\end{aligned}$$

On the other hand, the sum of pairwise squared differences equals

$$\begin{aligned} \mathrm{R} &:= \|A-B\|^2+\|B-C\|^2+\|C-A\|^2 \\ &= 2\big(\|A\|^2+\|B\|^2+\|C\|^2\big)-2\big(\langle A,B\rangle+\langle B,C\rangle+\langle C,A\rangle\big). \end{aligned}$$

Comparing the two expressions we obtain $S=3\mathrm{R}$. Therefore,

$$\text{L.H.S} = \tfrac{1}{4}S = \tfrac{3}{4}\mathrm{R} = \tfrac{3}{4}\Big(\|A-B\|^2+\|B-C\|^2+\|C-A\|^2\Big).$$

Since $\xi(x)=\|x\|^2$, the right-hand side of (1.1) is exactly $\frac{3}{4}\mathrm{R}$. Hence ξ satisfies (1.1) identically (see Figure 1).

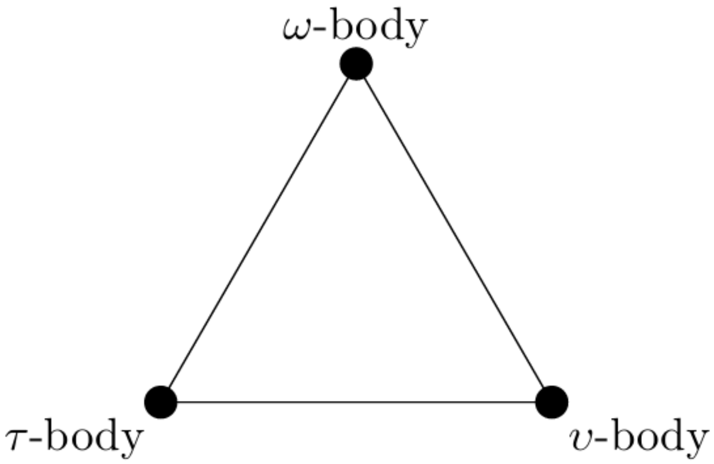

Figure 1: Symmetric three-body configuration.

Physical interpretation of the energy reduction factor. The factor $\frac{3}{4}$ appearing in (1.1) has a clear physical meaning in the context of symmetric three-body quadratic energy models. When each particle is replaced by the midpoint of the other two, the pairwise distances between the particles are uniformly contracted. Since the energy function $\xi(x)=\|x\|^2$ is quadratic, this geometric contraction results in a proportional reduction of the total interaction energy. Specifically, the midpoint transformation reduces the total pairwise quadratic energy by exactly 25%, yielding the factor $\frac{3}{4}$. This reflects a dissipative mechanism whereby symmetric averaging drives the system toward a lower-energy equilibrium configuration.

4.2. Interpretation: Midpoint Contraction and Energy Reduction

The three arguments on the left of (1.1),

$$\tfrac{A+B}{2}-C, \quad \tfrac{B+C}{2}-A, \quad \tfrac{C+A}{2}-B,$$

are the relative midpoints obtained by replacing each vertex by the midpoint of the other two. For the quadratic energy $\xi(x)=\|x\|^2$ this midpoint contraction reduces the total pairwise energy by the factor 3/4:

$$\xi\big(\tfrac{A+B}{2}-C\big)+\xi\big(\tfrac{B+C}{2}-A\big)+\xi\big(\tfrac{C+A}{2}-B\big)=\tfrac{3}{4}\big(\xi(A-B)+\xi(B-C)+\xi(C-A)\big).$$

Thus the functional identity encodes a precise *energy reduction law* under symmetric barycentric contraction for quadratic interactions (see figure 2).

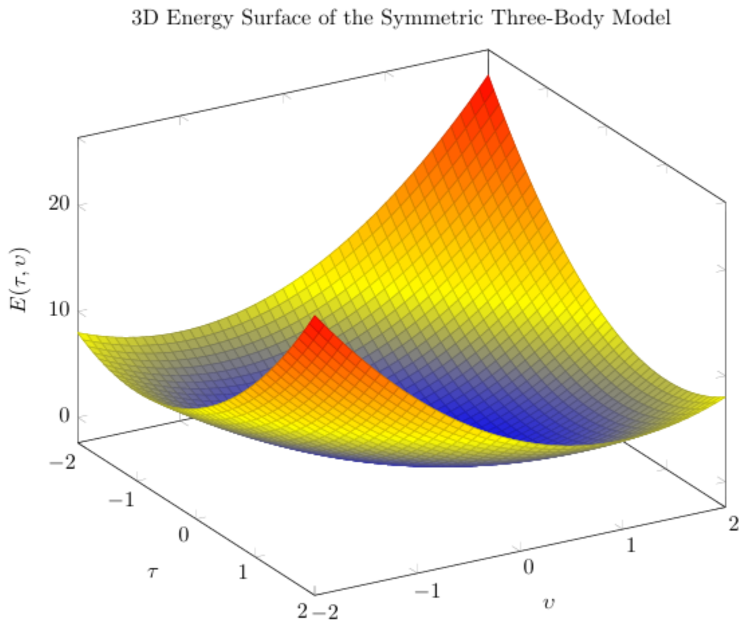
3D Energy Surface of the Symmetric Three-Body Model

Figure 2: Energy surface corresponding to $E(\tau, \upsilon) = (\tau - \upsilon)^2 + \tau^2 + \upsilon^2$.

4.3. Characterization of Quadratic Mappings

The identity (1.1) is a quadratic functional identity: any mapping $Q : X \to Y$ which is a quadratic form (i.e. homogeneous of degree two $Q(2x) = 4Q(x)$ and induced by a symmetric bilinear form) satisfies (1.1). Conversely, under mild regularity assumptions (e.g. continuity at a point or boundedness on a set of positive measure) the family of solutions of (1.1) coincides with the quadratic mappings. Hence (1.1) both models quadratic energies and characterizes them up to the usual quadratic class.

4.4. Stability Interpretation and Physical Consequences

In practice the measured or computed energy $\widetilde{\xi}$ will not satisfy (1.1) exactly due to measurement noise, higher-order interactions, or modeling errors. The stability results established earlier (Hyers-Ulam, Hyers-Ulam-Rassias, Hyers-Ulam-Gărŭta) give quantitative guarantees:

- If $\widetilde{\xi}$ satisfies the uniform defect bound $\|\Delta_{\widetilde{\xi}}(\tau, \upsilon, \omega)\| \leq \varepsilon$ for all τ, υ, ω, then by the Hyers-Ulam corollary there exists a unique quadratic mapping Q with

  $$d(\widetilde{\xi}(\tau), Q(\tau)) \leq C\,\varepsilon \quad \text{for all } \tau,$$

  where C is an explicit constant depending only on the extended b-metric constants. Physically, this implies the observed energy is uniformly close to an exact quadratic energy.

- If the defect grows polynomially, $\|\Delta_{\widetilde{\xi}}(\tau, \upsilon, \omega)\| \leq \theta(\|\tau\|^p + \|\upsilon\|^p + \|\omega\|^p)$ with $p < 2$, then the Hyers-Ulam-Rassias corollary provides a bound of the form

  $$d(\widetilde{\xi}(\tau), Q(\tau)) \leq C'\,\theta\,\|\tau\|^p,$$

  so large-scale deviations are controlled in a polynomial manner.

- For general control functions $\varphi(\tau, \upsilon, \omega)$ satisfying the required dyadic summability, the Hyers-Ulam-Gărŭta corollary yields

  $$d(\widetilde{\xi}(\tau), Q(\tau)) \leq C'' \sum_{n=0}^{\infty} 4^{-n} \Phi(2^n \tau),$$

where Φ dominates φ. This covers highly nonuniform or data-driven perturbations.

Thus the stability theory ensures that if the measured energy approximately satisfies the midpoint contraction law (1.1), then there is a true quadratic energy law nearby. In applied terms, one may:

1. *Infer* the underlying quadratic model Q from noisy measurements $\widetilde{\xi}$ with explicit error bounds.
2. *Justify* the use of quadratic approximations (Hessian, harmonic oscillator models) when deviations satisfy the HUR or HUG hypotheses.
3. *Estimate* model parameters (e.g. spring constants, stiffness tensors) by projecting $\widetilde{\xi}$ onto the quadratic class, with quantitative residual control coming from the above corollaries.

The identity (1.1) is therefore both a *structural* equation describing exact quadratic energy behavior under symmetric midpoint contraction and a *diagnostic* device: its approximate validity (measured by the defect) yields, via the stability theorems, a provably close quadratic model. This link provides a rigorous justification for quadratic modeling in symmetric three-body systems and a precise method to quantify and control modeling errors.

5. Conclusion

In this work, we investigated the Hyers–Ulam–Găvruţa stability of a quadratic functional equation within the framework of extended b-metric spaces. By employing both the direct method and a fixed point approach, we established explicit stability bounds that guarantee the existence of a unique quadratic mapping approximating any function satisfying the equation with controlled deviation. The extended b-metric setting enabled a broader and more flexible geometric analysis compared to classical normed spaces, thereby refining and generalizing several earlier stability results.

To highlight the relevance of these findings, we illustrated that the quadratic identity naturally corresponds to the equilibrium configuration in a symmetric three-body quadratic energy model. In this context, the derived stability inequalities quantify how perturbations in the functional equation translate into energy distortions, while the fixed point interpretation reveals that midpoint contractions act as the underlying mechanism driving the system toward its minimal-energy quadratic configuration.

Overall, the combination of analytic and geometric viewpoints demonstrates that the quadratic functional equation remains robust under perturbations even in generalized metric environments. The results enrich the theory of functional equation stability in non-standard metric settings and provide a foundation for further applications in nonlinear analysis, dynamical geometry, and energy-based modeling.

Future research directions: Several directions for future research naturally arise from the present work. One possible extension is to investigate the stability of equation (1.1) in other generalized metric frameworks, such as partial metric spaces, controlled metric-type spaces, or probabilistic and fuzzy metric spaces. Another direction is to study analogous stability problems for higher-order or mixed-type functional equations arising from multi-body interaction models and non-quadratic energy functionals. In addition, it would be of interest to examine stability under weaker regularity assumptions, such as local boundedness or measurability, and to explore numerical and data-driven approaches for reconstructing quadratic models from approximate measurements in applied settings.

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