



## Multi-Criteria Decision-Making Approach Based on Bipolar Complex Neutrosophic N-Soft Sets

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**ABSTRACT:** This paper investigates the concept of bipolar complex neutrosophic N-soft sets, which integrates bipolar N-soft sets and bipolar complex neutrosophic sets. This model is designed for use in real-world circumstances where decision-makers deal with bipolarity of truth, indeterminacy, and falsity membership grades in two-dimensional unreliable data with grading assessment. An illustrative example is offered, which helps us understand the new concept. Next, we describe some of the core operations of this new model. An application part provides a corresponding real-life problem. We also propose two significant algorithms for decision-making problems. We present a possible comparison analysis that compares existing algorithms to demonstrate the efficiency and advantages of our algorithms.

**Keywords:** Neutrosophic set, N-soft set, bipolar N-soft set, bipolar complex neutrosophic set, bipolar complex neutrosophic N-soft set, MCDM.

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### 1. Introduction

In problem-solving cases through various sciences such as medical, economic, environmental, and engineering, uncertainty and imprecision have been challenging to deal with. A primary mathematical instrument to handle the above situations was the theory of fuzzy sets (FSs), introduced by Zadeh [40] in 1965. After that, a variety of mathematical tools were provided for this aim. In 1998, Smarandache [36] directed the attention of researchers to an important discovery called Neutrosophic sets (NSs). The neutrosophic theory includes elements of probability, set theory, logic, and statistics. Thus, it confronts actual life episodes marked by gradation of satisfaction, dissatisfaction, and indeterminacy. Thus, it is recognized for giving a generalization of all traditional sets, FSs, intuitionistic fuzzy sets (IFs), interval-valued intuitionistic fuzzy sets (IVIFs), and Pythagorean fuzzy sets (PFs) [14,15,39,40] with further applications in [20,22]. The neutrosophic-inspired sets have been categorized into many subclasses, such as interval-valued neutrosophic sets (IVNSs), single-valued neutrosophic sets (SVNSs), simplified neutrosophic sets, and so on. The SVNSs were introduced by Wang and Smarandache [37,38]. They can be represented by three real-valued functions, with their values taken from the unit closed intervals  $[0, 1]$ ; thus, it is more convenient and applicable to many areas of science and engineering. Then the

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two dimensional NSs were defined by Mumtaz and Smarandache [10], called complex neutrosophic sets (CNSs).

In 1999, dealing with uncertainty, a new mathematical tool, soft sets (SSs), is presented by Molodtsov [30]. Consequently, Then, one more updated version of SSs was the design of N-soft sets (N-SSs) introduced by Fatimah et al. [21], since, in decision-making problems, binary assessments do not necessarily define people's opinions. Recently, the N-bipolar hypersoft set (N-BHS) model captures the spirit of embracing uncertainty and at the same time multiple criteria decision making in complex systems by permitting a more structured and flexible expression of binary as well as non-binary information [31]. Often, by non-binary evaluations, making decisions is easier, and this goes incredibly quickly for qualitative information, such as how our banking industry is doing? How good a person is? Etc.

Thus, researchers apply N-SSs to decision-making techniques that are non-binary judgment-based. The idea underlying the construction of N-SSs broadens the context of SSs, whose origin is that a finite selection of attributes characterizes any alternative. Some well-known application examples in the real world can be found in [9,21]. Inspired by the new model of N-SSs, Akram et al. introduced fuzzy N-soft sets (FN-SSs) [2]. Also Akram et al. [3] utilized a hybrid fusion of FSs with N-SSs to address decisional issues to optimize the efficiency of fuzzy soft sets (FSSs) [8]. Maji and Jana et al. introduced new concepts of neutrosophic soft sets (NSSs) [29] and single-valued neutrosophic soft sets (SVNSSs) [23], respectively. Later, intuitionistic neutrosophic soft sets (INSSs) were highlighted by Broumi and Smarandache [17]. In 2017, Broumi et al. presented the complex neutrosophic soft sets (CNSSs) [18]. For further extensions and applications of N-SSs, we refer the readers to [2,3,4,32]

Moreover, another aspect is considered in many applications, which is bipolar, and therefore, bipolar fuzzy sets (BFSs) were presented by Zhang [41]. In 2000, Lee introduced the idea of bipolar-valued fuzzy sets (BVSs) on changing the range from the positive interval  $[0, 1]$  to  $[-1, 1]$  [26]. After that, many articles on bipolarity in different structures were published, [19] is one of them, which studied the bipolar neutrosophic sets (BNSs), extending FSs, IFSs, BFSs, and NSs. Consequently, the concept of bipolar motivated researchers to present new sets such as bipolar fuzzy soft sets (BFSSs) [1] and fuzzy bipolar soft sets (FBSSs) [33]. In addition, bipolar complex fuzzy soft sets (BCFSSs) (in polar form) and bipolar complex fuzzy soft sets (BCFSSs) (in Cartesian form) are defined by Alqaraleh et al. [12] and Mahmood et al. [28], respectively.

Then, the concept of bipolar neutrosophic soft sets (BNSSs) was defined by Mumtaz et al. [11] which combined the SSs and BNSs. Ali et al. [10] added a second dimension to NSs and named them complex neutrosophic sets (CNSs). Broumi et al. [16] used bipolar complex neutrosophic sets (BCNSs) to solve a medical diagnosis problem. Al-Quran et al. established the complex bipolar-valued neutrosophic soft sets (CBVNSSs), which is suggested for use in real life situations where the decision makers are interested to deal with bipolarity as well as neutrosophic memberships grades [7]. A major article raised a new notion when Shabir and Naz and Karaaslan and Karataş illustrated the bipolar of soft sets (BSSs) [25,35]. As result of need of the online rating system that often used bipolar words such as (good-bad, like-dislike, upvote-downvote, etc.) lead to establish the concept of bipolar N-soft sets (BN-SSs) by Hüeyin and Subramanian [24] where they used bipolar grade space for assessment of alternatives according the parameters. Then, a new set called neutrosophic N-soft sets (NN-SSs) was introduced by Riaz et al. [34], and its single-valued version was proposed by Ashraf et al. [13] with applications in the area of business and the field of medicine. Later, Akram et al. defined a new model called a complex neutrosophic N-soft set (CNN-SS) [5]. The parameterized nature of the universe may be complex single-valued neutrosophic (composed of functions for satisfaction degree, indeterminacy degree, and dissatisfaction degree) whose values are drawn on the complex unit circle. However, recently, Mahmood [27] has awoken us about what is being played in bipolarity because to make the distance negative contradicts the basic definition of distance in complex numbers. Thus, in their article, they preferred to use complex numbers in Cartesian form, taking a unit square instead of a unit circle in the complex plane. After that, they introduced a new model named a bipolar complex fuzzy N-soft set (BCFN-SS) by employing complex numbers in Cartesian form, where the proposed method was utilized in illustrative examples to decrease information loss risk.

### 1.1. Contribution

In this paper, we describe an extremely broad model called a bipolar complex neutrosophic N-soft set (BCNN-SS). It demonstrates the possibility of classifying the parameter nature of the universe into graded system with complex-valued functions for positive truth membership, indeterminacy membership, and membership falsity, as well as complex-valued functions for negative truth membership, indeterminacy membership, and membership falsity. The degree complex-valued and ordered grades offer the BCNN-SS with superior capabilities, allowing it to perform better than the earlier model. Further, we studied the set-theoretic operations of BCNN-SS, which include complements, unions, intersections, AND-operations, and OR-operations. We introduced the concepts of the column comparison matrix as well as the row comparison matrix. Two algorithms were illustrated. A comparison of BCNN-SS with existing methods was presented at the end.

### 1.2. Motivation

The motivation for this study is based on the following criteria:

- ▷ The BN-SSs can express situations containing bipolar-ordered grades but not fuzzy and parametric information. At the same time, CNN-SSs are designed for two dimensional but still cannot tackle information in bipolar.
- ▷ The CBVNSSs and BCNSs can deal with the degree of indeterminacy in human judgment with second-dimensional information, but they cannot assist us in decision-making problems based on non-binary evaluations or ranking systems.
- ▷ These shortcomings led us to introduce the concept of BCNN-SSs, which effectively processes bipolar ordered grades, 2-dimensional situations, and bipolar components of a NS.

Noticeably, BCNN-SS is astonishing combinations which valuable for modeling uncertainty but too complex for most practical applications in the real world. Also it is challenging to have computational efficiency since it has fourteen membership values per element.

### 1.3. Outline

The remainder of this paper is organized as follows: The second section reviews the definitions of previous models, which will be required for continuing. In the third section, the fundamental operations on BCNN-SSs, alongside a real-life example, are discussed. The fourth part illustrates a possible application of BCNN-SSs in decision-making and highlights it in different algorithms. The fifth part is used for a comparative analysis and ideas for matching the outputs of the several models. The last few sentences are given to conclude this research.

## 2. Preliminaries

In this section, some basic definitions will be presented. Throughout the paper, let  $\mathbb{U}$  be a universe of alternatives and let  $\ddot{E}$  be a set of parameters. Then we start with:

**Definition 2.1** [30] Let  $\ddot{A} \subseteq \ddot{E}$ . A soft set  $(\check{f}, \ddot{A})$  on  $\mathbb{U}$  is defined by a mapping  $\check{f} : \ddot{A} \rightarrow 2^{\mathbb{U}}$ , where  $2^{\mathbb{U}}$  is the set of all subsets of  $\mathbb{U}$ .

**Definition 2.2** [21] Let  $\ddot{A} \subseteq \ddot{E}$ . Suppose  $R^+ = \{0, 1, 2, \dots, N-1\}$  is a set of ordered grades with  $N \in \{2, 3, \dots\}$ . A triple  $(\check{f}, \ddot{A}, N)$  is an N-soft set (N-SS) on  $\mathbb{U}$  if  $\check{f} : \ddot{A} \rightarrow 2^{\mathbb{U} \times R^+}$  with the property that for each  $a \in \ddot{A}$  and there is a unique  $(u, r_a) \in \mathbb{U} \times R^+$  such that  $(u, r_a) \in \check{f}(a), u \in \mathbb{U}, r_a \in R^+$ .

**Definition 2.3** [24] Let  $\ddot{A} \subseteq \ddot{E}$ . Let  $R = \langle R^+, R^- \rangle = \{\langle r^+, r^- \rangle, r^+ \in R^+, r^- \in R^-\}$  be a set of ordered bipolar grades, where,  $R^+ = \{0, 1, 2, \dots, N-1\}$  and  $R^- = \{-(N-1), \dots, -2, -1, 0\}$  highlight the grade of positive data and the grade of negative data, respectively, with  $N \in \{2, 3, \dots\}$ .

Then, we call a triple  $(\ddot{F}, \ddot{A}, N)$  bipolar N-soft set (BN-SS) on  $\mathbb{U}$ , where  $\ddot{F}$  is a mapping  $\ddot{F} : \ddot{A} \rightarrow 2^{\mathbb{U} \times R}$  and for every  $a \in \ddot{A}$ ,

$$\ddot{F}(a) = \left\{ \left( u_i, \langle r_a^{u_i^+}, r_a^{u_i^-} \rangle \right) : u_i \in \mathbb{U} \text{ and } \langle r_a^{u_i^+}, r_a^{u_i^-} \rangle \in R \right\},$$

such that for every  $u_i \in \mathbb{U}$ , there exists a unique bipolar grade  $\langle r_a^{u_i^+}, r_a^{u_i^-} \rangle$ .

**Definition 2.4** [36] A neutrosophic set (NS)  $Y$  on  $\mathbb{U}$  is defined by

$$Y = \left\{ \left\langle u, \mathbb{T}_Y(u), \mathbb{I}_Y(u), \mathbb{F}_Y(u) \right\rangle : u \in \mathbb{U} \right\},$$

where  $\mathbb{T}_Y, \mathbb{I}_Y, \mathbb{F}_Y : \mathbb{U} \rightarrow ]^{-0}, 1^{+}[$  are truth membership function, indeterminacy membership function, and falsity membership function, respectively, and their sum is restricted by the condition  $^{-0} \leq \mathbb{T}_Y(u) + \mathbb{I}_Y(u) + \mathbb{F}_Y(u) \leq 3^{+}$ .

**Remark 2.1** *Philosophically and in reality, the utilize of neutrosophic set with values from standard and non-standard subset of  $]^{-0}, 1^{+}[$  in real-world application problems is challenging. Therefore, throughout this work, we assume the neutrosophic set that picks up the value from  $[0, 1]$ .*

**Definition 2.5** [38] A single-valued neutrosophic set (SVNS)  $S$  on  $\mathbb{U}$  is defined by

$$S = \left\{ \left\langle u, \mathbb{T}_S(u), \mathbb{I}_S(u), \mathbb{F}_S(u) \right\rangle : u \in \mathbb{U} \right\},$$

where  $\mathbb{T}_S, \mathbb{I}_S, \mathbb{F}_S : \mathbb{U} \rightarrow [0, 1]$  are truth membership function, indeterminacy membership function, and falsity membership function, respectively, and their sum is restricted by the conditions  $0 \leq \mathbb{T}_S(u) + \mathbb{I}_S(u) + \mathbb{F}_S(u) \leq 3$ . The triplet  $(\mathbb{T}_S, \mathbb{I}_S, \mathbb{F}_S)$  is named a single-valued neutrosophic number (SVNN).

**Definition 2.6** [10] A complex single-valued neutrosophic set (CSVNS)  $C$  on  $\mathbb{U}$  is defined by

$$C = \left\{ \left\langle u, \mathbb{T}_C(u), \mathbb{I}_C(u), \mathbb{F}_C(u) \right\rangle : u \in \mathbb{U} \right\},$$

where  $\mathbb{T}_C(u) = p_C(u)e^{i2\pi t_C(u)}$ ,  $\mathbb{I}_C(u) = q_C(u)e^{i2\pi w_C(u)}$  and  $\mathbb{F}_C(u) = r_C(u)e^{i2\pi f_C(u)}$ , represent the degree of truth, degree of indeterminacy, and degree of falsity, respectively, and the sum of amplitude terms  $p_C, q_C, r_C : \mathbb{U} \rightarrow [0, 1]$  or of phase terms  $t_C, w_C, f_C : \mathbb{U} \rightarrow [0, 1]$  for every  $u \in \mathbb{U}$  is not restricted by any condition. The triplet  $(\mathbb{T}_C(u), \mathbb{I}_C(u), \mathbb{F}_C(u))$  is a named complex single-valued neutrosophic number (CSVNN).

**Definition 2.7** [16] A bipolar complex neutrosophic set (BCNS)  $Y$  on  $\mathbb{U}$  is defined by

$$Y = \left\{ \left\langle u, T_1^+ e^{iT_2^+}, I_1^+ e^{iI_2^+}, F_1^+ e^{iF_2^+}, T_1^- e^{iT_2^-}, I_1^- e^{iI_2^-}, F_1^- e^{iF_2^-} \right\rangle : u \in \mathbb{U} \right\},$$

where  $T_1^+, I_1^+, F_1^+ : \mathbb{U} \rightarrow [0, 1]$  and  $T_1^-, I_1^-, F_1^- : \mathbb{U} \rightarrow [-1, 0]$ . The positive amplitude membership degrees of  $T_1^+(u), I_1^+(u), F_1^+(u)$  denote the truth, indeterminate, and false memberships of an element  $u \in \mathbb{U}$  corresponding to a BCNS  $Y$  while the negative amplitude membership degrees of  $T_1^-(u), I_1^-(u), F_1^-(u)$  denote the truth membership, indeterminate membership, and false membership of an element  $u \in \mathbb{U}$  to some implicit counter-property to a BCNS  $Y$ . The positive phase terms of BCNS  $T_2^+(u), I_2^+(u), F_2^+(u)$  are in  $(0, 2\pi)$  while the negative phase terms of BCNS  $T_2^-(u), I_2^-(u), F_2^-(u)$  are in  $(-2\pi, 0)$ . A bipolar complex neutrosophic number (BCNN) is highlighted by  $(T_1^+ e^{iT_2^+}, I_1^+ e^{iI_2^+}, F_1^+ e^{iF_2^+}, T_1^- e^{iT_2^-}, I_1^- e^{iI_2^-}, F_1^- e^{iF_2^-})$ .

**Definition 2.8** [5] Let  $\check{A} \subseteq \check{E}$ . Suppose  $R^+ = \{0, 1, 2, \dots, N-1\}$  is a set of ordered grades with  $N \in \{2, 3, \dots\}$ . A triple  $(\check{\Phi}_{\check{\Psi}}, \check{A}, N)$  is named a complex single-valued neutrosophic N-soft set (CSVNN-SS) on  $\mathbb{U}$ , if  $(\check{\Phi}, \check{A}, N)$  is an N-SS on  $\mathbb{U}$ , and  $\check{\Phi}_{\check{\Psi}} : \check{A} \rightarrow 2^{\mathbb{U} \times R^+} \times \check{C}\check{S}\check{V}\check{N}\check{N}$  is a mapping which is defined as follows:

$$\begin{aligned} \check{\Phi}_{\check{\Psi}}(a) &= \left\{ \left\langle \left( \check{\Phi}(a), \check{\Psi}(a) \right) \right\rangle : a \in \check{A} \right\}, \\ &= \left\{ \left\langle \left( (u, r_a^u), (\mathbb{T}_{au}, \mathbb{I}_{au}, \mathbb{F}_{au}) \right) \right\rangle \right\}, \\ &= \left\{ \left\langle \left( (u, r_a^u), (p_{au}e^{i2\pi t_{au}}, q_{au}e^{i2\pi w_{au}}, r_{au}e^{i2\pi f_{au}}) \right) \right\rangle \right\}, \end{aligned}$$

where  $\check{\Phi} : \check{A} \rightarrow 2^{\mathbb{U} \times R^+}$ ,  $\check{\Psi} : \check{A} \rightarrow \check{C}\check{S}\check{V}\check{N}\check{N}$  such that  $\check{C}\check{S}\check{V}\check{N}\check{N}$  is the class of all complex single-valued neutrosophic numbers of  $\mathbb{U}$ ,  $r_a^u$  highlights the order of parameters for the alternative  $u$  and  $p_{au}, q_{au}, r_{au}, t_{au}, w_{au}, f_{au} \in [0, 1]$  without conditions on their sum.

**Definition 2.9** [6] A bipolar complex fuzzy set (BCFS)  $F$  on  $\mathbb{U}$  is defined by

$$F = \left\{ \left\langle u, p_F^+ e^{i\theta_F^+}, p_F^- e^{i\theta_F^-} \right\rangle : u \in \mathbb{U} \right\},$$

where  $p_F^+ : \mathbb{U} \rightarrow [0, 1]$  and  $p_F^- : \mathbb{U} \rightarrow [-1, 0]$  are mappings.  $p_F^+ e^{i\theta_F^+}$  is the positive complex membership degree and  $p_F^- e^{i\theta_F^-}$  is the negative complex membership degree. Furthermore, the phase terms of bipolar complex positive membership function and bipolar complex negative membership function is in  $(0, 2\pi]$  and  $p_F^+ \in [0, 1]$ ,  $p_F^- \in [-1, 0]$ . A pair  $(p_F^+ e^{i\theta_F^+}, p_F^- e^{i\theta_F^-})$  is called a bipolar complex fuzzy number (BCFN).

**Definition 2.10** [27] Let  $\check{A} \subseteq \check{E}$  and  $R^+ = \{0, 1, 2, \dots, N-1\}$  is a set of ordered grades with  $N \in \{2, 3, \dots\}$ . A triple  $(\check{T}, \check{D}, N)$  is a BCNN-SS on  $\mathbb{U}$ , where  $\check{D} = (\check{f}, \check{A}, N)$  with  $\check{f} : \check{A} \rightarrow 2^{\mathbb{U} \times R^+}$  is an N-SS on  $\mathbb{U}$ , and  $\check{f}$  is a mapping so that  $\check{T} : \check{A} \rightarrow 2^{\mathbb{U} \times R^+} \times \check{B}\check{C}\check{F}\check{N}$ , which is

$$\begin{aligned} (\check{T}, \check{D}, N) &= \left\{ \left\langle a, \left( \check{M}(a), \check{N}(a) \right) \right\rangle : a \in \check{A}, \left( \check{M}(a), \check{N}(a) \right) \in 2^{\mathbb{U} \times R^+} \times \check{B}\check{C}\check{F}\check{N} \right\} \\ &= \left\{ \left\langle a, \left( (u, r_a^u), \mathbb{T}_a^+, \mathbb{T}_a^- \right) \right\rangle : a \in \check{A}, u \in \mathbb{U}, r_a^u \in R^+ \right\} \\ &= \left\{ \left\langle a, \left( (u, r_a^u), T_{xa}^+ + iT_{ya}^+, T_{xa}^- + iT_{ya}^- \right) \right\rangle : a \in \check{A}, u \in \mathbb{U}, r_a^u \in R^+ \right\}, \end{aligned}$$

where  $\check{M} : \check{A} \rightarrow 2^{\mathbb{U}}$ ,  $\check{N} : \check{A} \rightarrow \check{B}\check{C}\check{F}\check{N}$ , and  $\check{B}\check{C}\check{F}\check{N}$  represents the collection of all BCNNs.

Now after we have taken some basic definitions, we will introduce the proposed set which is named by BCNN-SS as follow:

### 3. Bipolar Complex Neutrosophic N-Soft Sets

**Definition 3.1** Let  $\check{A} \subseteq \check{E}$ . Let  $R = \langle R^+, R^- \rangle = \{ \langle r^+, r^- \rangle, r^+ \in R^+, r^- \in R^- \}$  be a set of ordered bipolar grades, where,  $R^+ = \{0, 1, 2, \dots, N-1\}$  and  $R^- = \{-(N-1), \dots, -2, -1, 0\}$  with  $N \in \{2, 3, \dots\}$ . A pair  $(\check{F}_{\check{\Theta}}, \check{A})$  is a bipolar complex neutrosophic N-soft set, where  $\check{\Theta} = (\check{F}, \check{A}, N)$  is a BN-SS on  $\mathbb{U}$  and  $\check{F}_{\check{\Theta}}$  is a mapping such that

$$\check{F}_{\check{\Theta}} : \check{A} \rightarrow 2^{\mathbb{U} \times R} \times \check{B}\check{C}\check{N}\check{N}$$

which is

$$\begin{aligned} (\check{F}_{\check{\Theta}}, \check{A}) &= \left\{ \left\langle a, \left( \check{F}(a), \check{Y}(a) \right) \right\rangle : a \in \check{A}, \left( \check{F}(a), \check{Y}(a) \right) \in 2^{\mathbb{U} \times R} \times \check{B}\check{C}\check{N}\check{N} \right\} \\ &= \left\{ \left\langle a, \left( (u, \langle r_a^{u+}, r_a^{u-} \rangle), (\mathbb{T}_a^+, \mathbb{I}_a^+, \mathbb{F}_a^+, \mathbb{T}_a^-, \mathbb{I}_a^-, \mathbb{F}_a^-) \right) \right\rangle : a \in \check{A}, u \in \mathbb{U}, \langle r_a^{u+}, r_a^{u-} \rangle \in R \right\} \\ &= \left\{ \left\langle a, \left( (u, \langle r_a^{u+}, r_a^{u-} \rangle), (T_{xa}^+ + iT_{ya}^+, I_{xa}^+ + iI_{ya}^+, F_{xa}^+ + iF_{ya}^+, \right. \right. \right. \\ &\quad \left. \left. T_{xa}^- + iT_{ya}^-, I_{xa}^- + iI_{ya}^-, F_{xa}^- + iF_{ya}^-) \right) \right\rangle : a \in \check{A}, u \in \mathbb{U}, \langle r_a^{u+}, r_a^{u-} \rangle \in R \right\}, \end{aligned}$$

where  $\ddot{F} : \ddot{A} \rightarrow 2^{\mathbb{U} \times R}$ ,  $\ddot{Y} : \ddot{A} \rightarrow \check{B}\check{C}\check{N}\check{N}$ , and  $\check{B}\check{C}\check{N}\check{N}$  is the collection of all  $BCNN$ 's of  $\mathbb{U}$ .

The name of the definition will be abbreviated by BCNN-SS. The next example lead the way to understand more the definition of a BCNN-SS.

**Example 3.1** *A private school is hiring a full-time mathematical teacher for grade seven. Let  $\mathbb{U} = \{u_1, u_2, u_3, u_4\}$  be the universe set of four applicants. Let the set of parameters be  $\ddot{A} = \{e_1 = \text{experience}, e_2 = \text{age}, e_3 = \text{management skills}, e_4 = \text{creativity}\}$ . The purpose is to sequence the applicants according to the best evaluation. Furthermore, in Table 1, the opinions are expressed by positive signs " + ", black circle " • " and negative signs " - ". For shorten in Table 2, we illustrate the relationships among the symbols, numbers, and words. So, the positive sign can be written up to four times " + + + + " and similarly for the negative sign " - - - - ". Moreover, the increase of positive symbol times raises the evaluation grade under the parameters, and the decrease of negative symbol times dedicates the disadvantages under the parameters.*

Table 1: Initial Evaluation

$\mathbb{U}/\ddot{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$\langle + + +, - \rangle$	$\langle ++, - - - \rangle$	$\langle + + + +, - \rangle$	$\langle + + +, - - - \rangle$
$u_2$	$\langle + + + +, - - \rangle$	$\langle ++, - \rangle$	$\langle + + +, - - \rangle$	$\langle + + + +, - \rangle$
$u_3$	$\langle ++, - \rangle$	$\langle + + + +, - - - \rangle$	$\langle ++, - - - \rangle$	$\langle ++, - - - - \rangle$
$u_4$	$\langle \bullet, - \rangle$	$\langle + + +, - - \rangle$	$\langle +, \bullet \rangle$	$\langle ++, - - \rangle$

Table 2: Meaning and Representation

+ signs=numbers	representations	-signs=numbers	representations
• = 0	Poor	• = 0	Poor
+ = 1	OK	- = -1	Not OK
++ = 2	Accept	-- = -2	Unaccepted
+ + + = 3	Great	- - - = -3	Not Great
+ + + + = 4	Exceptional	- - - - = -4	Unexceptional

Now, we convert the initial evaluation in Table 1 with respect to the meaning and representation in Table 2 to construct a B5-SS as bellow:

Table 3: B5-SS Representation

$\mathbb{U}/\ddot{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$\langle 3, -1 \rangle$	$\langle 2, -3 \rangle$	$\langle 4, -1 \rangle$	$\langle 3, -3 \rangle$
$u_2$	$\langle 4, -2 \rangle$	$\langle 2, -1 \rangle$	$\langle 3, -2 \rangle$	$\langle 4, -1 \rangle$
$u_3$	$\langle 2, -1 \rangle$	$\langle 4, -3 \rangle$	$\langle 2, -3 \rangle$	$\langle 2, -4 \rangle$
$u_4$	$\langle 0, -1 \rangle$	$\langle 3, -2 \rangle$	$\langle 1, 0 \rangle$	$\langle 2, -2 \rangle$

In terms of Definition 3.1, for our explanation Table 3 gives  $(u_1, \langle 3, -1 \rangle) \in \ddot{F}(e_1)$ , and  $(u_2, \langle 4, -1 \rangle) \in \ddot{F}(e_2)$ . This data is enough when it is gathered from real data. However, when the data are bipolar, uncertain, fuzzy, and two-dimensional, then the BCNN-SS is needed. It can highlight how the process of giving grades to teachers is computed. Therefore, by combining a BCNS with BN-SS, we compose

BCNN-SS. We adhere to the following criteria, which are based on the teacher's application, as shown in Table 4:

when  $r = 0$ , then  $0.0 \leq \mathbb{T}^+ < 0.2$ ,  $0.8 \leq \mathbb{I}^+ \leq 1.0$ , and  $0.8 \leq \mathbb{F}^+ \leq 1.0$

when  $r = 1$ , then  $0.2 \leq \mathbb{T}^+ < 0.4$ ,  $0.6 \leq \mathbb{I}^+ < 0.8$ , and  $0.6 \leq \mathbb{F}^+ < 0.8$

when  $r = 2$ , then  $0.4 \leq \mathbb{T}^+ < 0.6$ ,  $0.4 \leq \mathbb{I}^+ < 0.6$ , and  $0.4 \leq \mathbb{F}^+ < 0.6$

when  $r = 3$ , then  $0.6 \leq \mathbb{T}^+ < 0.8$ ,  $0.2 \leq \mathbb{I}^+ < 0.4$ , and  $0.2 \leq \mathbb{F}^+ < 0.4$

when  $r = 4$ , then  $0.8 \leq \mathbb{T}^+ \leq 1.0$ ,  $0.0 \leq \mathbb{I}^+ < 0.2$ , and  $0.0 \leq \mathbb{F}^+ < 0.2$

when  $r = 0$ , then  $-0.2 < \mathbb{T}^- \leq 0.0$ ,  $-1.0 \leq \mathbb{I}^- \leq -0.8$ , and  $-1.0 \leq \mathbb{F}^- \leq -0.8$

when  $r = -1$ , then  $-0.4 < \mathbb{T}^- \leq -0.2$ ,  $-0.8 < \mathbb{I}^- \leq -0.6$ , and  $-0.8 < \mathbb{F}^- \leq -0.6$

when  $r = -2$ , then  $-0.6 < \mathbb{T}^- \leq -0.4$ ,  $-0.6 < \mathbb{I}^- \leq -0.4$ , and  $-0.6 < \mathbb{F}^- \leq -0.4$

when  $r = -3$ , then  $-0.8 < \mathbb{T}^- \leq -0.6$ ,  $-0.4 < \mathbb{I}^- \leq -0.2$ , and  $-0.4 < \mathbb{F}^- \leq -0.2$

when  $r = -4$ , then  $-1.0 \leq \mathbb{T}^- \leq -0.8$ ,  $-0.2 < \mathbb{I}^- \leq 0.0$ , and  $-0.2 < \mathbb{F}^- \leq 0.0$

where  $\mathbb{T}^+ = \frac{T_x^+ + T_y^+}{2}$ ,  $\mathbb{I}^+ = \frac{I_x^+ + I_y^+}{2}$ ,  $\mathbb{F}^+ = \frac{F_x^+ + F_y^+}{2}$ ,  $\mathbb{T}^- = \frac{T_x^- + T_y^-}{2}$ ,  $\mathbb{I}^- = \frac{I_x^- + I_y^-}{2}$ , and  $\mathbb{F}^- = \frac{F_x^- + F_y^-}{2}$ .

Table 4: Criteria for BCN-5SS

$r^+$	$T_x^+, T_y^+$	$I_x^+, I_y^+$	$F_x^+, F_y^+$	$r^-$	$T_x^-, T_y^-$	$I_x^-, I_y^-$	$F_x^-, F_y^-$
0	[0.0, 0.2)	[0.8, 1.0]	[0.8, 1.0]	0	(-0.2, 0.0]	[-1.0, -0.8]	[-1.0, -0.8]
1	[0.2, 0.4)	[0.6, 0.8)	[0.6, 0.8)	-1	(-0.4, -0.2]	(-0.8, -0.6]	(-0.8, -0.6]
2	[0.4, 0.6)	[0.4, 0.6)	[0.4, 0.6)	-2	(-0.6, -0.4]	(-0.6, -0.4]	(-0.6, -0.4]
3	[0.6, 0.8)	[0.2, 0.4)	[0.2, 0.4)	-3	(-0.8, -0.6]	(-0.4, -0.2]	(-0.4, -0.2]
4	[0.8, 1.0]	[0.0, 0.2)	[0.0, 0.2)	-4	[-1.0, -0.8]	(-0.2, 0.0]	(-0.2, 0.0]

We have reached the situation that we can construct the BCN-5SS by Definition 3.1 as next:

$$\begin{aligned} \ddot{\Gamma}_{\ddot{\Theta}}(e_1) &= (\ddot{F}(e_1), \ddot{Y}(e_1)) \\ &= \left( \left( (u_1, \langle 3, -1 \rangle), (0.7 + i0.7, 0.21 + i0.3, 0.3 + i0.3, -0.3 - i0.2, -0.7 - i0.7, -0.79 - i0.7) \right), \right. \\ &\quad \left( (u_2, \langle 4, -2 \rangle), (0.9 + i0.95, 0.1 + i0.1, 0.1 + i0.15, -0.5 - i0.55, -0.4 - i0.4, -0.4 - i0.5) \right), \\ &\quad \left( (u_3, \langle 2, -1 \rangle), (0.5 + i0.4, 0.5 + i0.4, 0.55 + i0.55, -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.6) \right), \\ &\quad \left. \left( (u_4, \langle 0, -1 \rangle), (0.1 + i0.19, 0.9 + i1.0, 1.0 + i0.9, -0.39 - i.39, -0.7 - i0.7, -0.7 - i0.7) \right) \right), \end{aligned}$$

$$\begin{aligned} \ddot{\Gamma}_{\ddot{\Theta}}(e_2) &= (\ddot{F}(e_2), \ddot{Y}(e_2)) \\ &= \left( \left( (u_1, \langle 2, -3 \rangle), (0.55 + i0.55, 0.59 + i0.59, 0.59 + i0.59, -0.6 - i0.6, -0.3 - i0.3, -0.3 - i0.2) \right), \right. \\ &\quad \left( (u_2, \langle 2, -1 \rangle), (0.4 + i0.59, 0.5 + i0.59, 0.59 + i0.59, -0.3 - i0.3, -0.6 - i0.6, -0.7 - i0.7) \right), \\ &\quad \left( (u_3, \langle 4, -3 \rangle), (1.0 + i1.0, 0.1 + i0.1, 0.15 + i0.1, -0.7 - i0.7, -0.3 - i0.3, -0.39 - i0.39) \right), \\ &\quad \left. \left( (u_4, \langle 3, -2 \rangle), (0.7 + i0.7, 0.3 + i0.39, 0.3 + i0.3, -0.5 - i0.5, -0.5 - i0.5, -0.5 - i0.5) \right) \right), \end{aligned}$$

$$\begin{aligned}\ddot{\Gamma}_{\ddot{\Theta}}(e_3) &= \left( \ddot{F}(e_3), \ddot{Y}(e_3) \right) \\ &= \left( \left( \left( u_1, \langle 4, -1 \rangle \right), \left( 0.9 + i0.8, 0.1 + i0.1, 0.16 + i0.1, -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.7 \right) \right), \right. \\ &\quad \left( u_2, \langle 3, -2 \rangle \right), \left( 0.7 + i0.7, 0.39 + i0.39, 0.3 + i0.39, -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \right), \\ &\quad \left( u_3, \langle 2, -3 \rangle \right), \left( 0.59 + i0.59, 0.59 + i0.59, 0.5 + i0.59, -0.6 - i0.6, -0.6 - i0.6, -0.7 - i0.7 \right), \\ &\quad \left. \left( u_4, \langle 1, 0 \rangle \right), \left( 0.39 + i0.2, 0.7 + i0.79, 0.7 + i0.7, -0.1 - i0.1, -0.8 - i0.8, -0.8 - i0.8 \right) \right),\end{aligned}$$

$$\begin{aligned}\ddot{\Gamma}_{\ddot{\Theta}}(e_4) &= \left( \ddot{F}(e_4), \ddot{Y}(e_4) \right) \\ &= \left( \left( \left( u_1, \langle 3, -3 \rangle \right), \left( 0.7 + i0.75, 0.39 + i0.3, 0.3 + i0.3, -0.6 - i0.6, -0.3 - i0.2, -0.39 - i0.39 \right) \right), \right. \\ &\quad \left( u_2, \langle 4, -1 \rangle \right), \left( 1.0 + i0.8, 0.15 + i0.16, 0.19 + i0.19, -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.79 \right), \\ &\quad \left( u_3, \langle 2, -4 \rangle \right), \left( 0.4 + i0.5, 0.59 + i0.5, 0.59 + i0.59, -1.0 - i0.8, -0.1 - i0.1, -0.1 - i0.1 \right), \\ &\quad \left. \left( u_4, \langle 2, -2 \rangle \right), \left( 0.59 + i0.45, 0.4 + i0.5, 0.5 + i0.4, -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \right) \right).\end{aligned}$$

Then, the BCN-5SS information can be presented in the tabular form as in following table:

Table 5: BCN-5SS Tabular Form

$u/\bar{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$\left( (3, -1), \begin{pmatrix} 0.7 + i0.7, 0.21 + i0.3, 0.3 + i0.3, \\ -0.3 - i0.2, -0.7 - i0.7, -0.79 - i0.7 \end{pmatrix} \right)$	$\left( (2, -3), \begin{pmatrix} 0.55 + i0.55, 0.59 + i0.59, 0.59 + i0.59, \\ -0.6 - i0.6, -0.3 - i0.3, -0.3 - i0.2 \end{pmatrix} \right)$	$\left( (4, -1), \begin{pmatrix} 0.9 + i0.8, 0.1 + i0.1, 0.16 + i0.1, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.7 \end{pmatrix} \right)$	$\left( (3, -3), \begin{pmatrix} 0.7 + i0.75, 0.39 + i0.3, 0.3 + i0.3, \\ -0.6 - i0.6, -0.3 - i0.2, -0.39 - i0.39 \end{pmatrix} \right)$
$u_2$	$\left( (4, -2), \begin{pmatrix} 0.9 + i0.95, 0.1 + i0.1, 0.1 + i0.15, \\ -0.5 - i0.55, -0.4 - i0.4, -0.4 - i0.5 \end{pmatrix} \right)$	$\left( (2, -1), \begin{pmatrix} 0.4 + i0.59, 0.5 + i0.59, 0.59 + i0.59, \\ -0.3 - i0.3, -0.6 - i0.6, -0.7 - i0.7 \end{pmatrix} \right)$	$\left( (3, -2), \begin{pmatrix} 0.7 + i0.7, 0.39 + i0.39, 0.3 + i0.39, \\ -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \end{pmatrix} \right)$	$\left( (4, -1), \begin{pmatrix} 1.0 + i0.8, 0.15 + i0.16, 0.19 + i0.19, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.79 \end{pmatrix} \right)$
$u_3$	$\left( (2, -1), \begin{pmatrix} 0.5 + i0.4, 0.5 + i0.4, 0.55 + i0.55, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.6 \end{pmatrix} \right)$	$\left( (4, -3), \begin{pmatrix} 1.0 + i1.0, 0.1 + i0.1, 0.15 + i0.1, \\ -0.7 - i0.7, -0.3 - i0.3, -0.39 - i0.39 \end{pmatrix} \right)$	$\left( (2, -3), \begin{pmatrix} 0.59 + i0.59, 0.59 + i0.59, 0.5 + i0.59, \\ -0.6 - i0.6, -0.6 - i0.6, -0.7 - i0.7 \end{pmatrix} \right)$	$\left( (2, -4), \begin{pmatrix} 0.4 + i0.5, 0.59 + i0.5, 0.59 + i0.59, \\ -1.0 - i0.8, -0.1 - i0.1, -0.1 - i0.1 \end{pmatrix} \right)$
$u_4$	$\left( (0, -1), \begin{pmatrix} 0.1 + i0.19, 0.9 + i1.0, 1.0 + i0.9, \\ -0.39 - i0.39, -0.7 - i0.7, -0.7 - i0.7 \end{pmatrix} \right)$	$\left( (3, -2), \begin{pmatrix} 0.7 + i0.7, 0.3 + i0.39, 0.3 + i0.3, \\ -0.5 - i0.5, -0.5 - i0.5, -0.5 - i0.5 \end{pmatrix} \right)$	$\left( (1, 0), \begin{pmatrix} 0.39 + i0.2, 0.7 + i0.79, 0.7 + i0.7, \\ -0.1 - i0.1, -0.8 - i0.8, -0.8 - i0.8 \end{pmatrix} \right)$	$\left( (2, -2), \begin{pmatrix} 0.59 + i0.45, 0.4 + i0.5, 0.5 + i0.4, \\ -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \end{pmatrix} \right)$

**Remark 3.1** Every BCN1-SS can be considered as BCN2-SS. Furthermore, any BCNN-SS can be considered as BCN(N+1)-SS, where  $N \in \{2, 3, \dots\}$ .

**Example 3.2** The BCN-5SS in Example 3.1 can be considered the BCN6-SS since 6 is never used in grade evaluation.

**Proposition 3.1** Let  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS. If we let  $r^- = 0$  and  $\mathbb{T}^- = \mathbb{I}^- = \mathbb{F}^- = 0$  in  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  for all  $u \in \mathbb{U}$ , then  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  will be a CNN-SS.

**Proposition 3.2** Let  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS. If we let  $\mathbb{I}^- = \mathbb{F}^- = 0$  in  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  for all  $u \in \mathbb{U}$ , then  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  will be a BCFN-SS.

**Definition 3.2** Let  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS. Then

1.  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  is called an empty bipolar complex neutrosophics N-soft set (shortly, EBCNN-SS) if for all  $a \in \ddot{A}, u \in \mathbb{U}$ , and  $\langle r_a^{u+}, r_a^{u-} \rangle \in R$ , then:

$$\ddot{F}(a) = (u, \langle 0, 0 \rangle)$$

$$\ddot{Y}(a) = (0 + i0, 1 + i1, 1 + i1, 0 + i0, -1 - i1, -1 - i1),$$

where  $\ddot{\Theta} = (\ddot{F}, \ddot{A}, N)$  is a BN-SS and  $\ddot{Y} : \ddot{A} \rightarrow \check{B}\check{C}\check{N}\check{N}$ , and is denoted by  $(\ddot{\Gamma}_{\ddot{\Theta}}^\phi, \ddot{A})$ .

2.  $(\check{I}_{\check{\Theta}}, \check{A})$  is called an absolute bipolar complex neutrosophics N-soft set (shortly, ABCNN-SS) if for all  $a \in \check{A}, u \in \mathbb{U}$ , and  $\langle r_a^{u+}, r_a^{u-} \rangle \in R$ , then:

$$\begin{aligned}\check{F}(a) &= (u, \langle N - 1, -(N - 1) \rangle) \\ \check{Y}(a) &= (1 + i1, 0 + i0, 0 + i0, -1 - i1, 0 + i0, 0 + i0),\end{aligned}$$

where  $\check{\Theta} = (\check{F}, \check{A}, N)$  is a BN-SS and  $\check{Y} : \check{A} \rightarrow \check{B}\check{C}\check{N}\check{N}$ , and is denoted by  $(\check{I}_{\check{\Theta}}^{\mathbb{U}}, \check{A})$ .

For an explanation, we give the next example.

**Example 3.3** Consider the BCN-5SS given in Example 3.1. Then its EBCN-5SS and ABCN-5SS are shown in the following tables, respectively:

Table 6:  $(\check{I}_{\check{\Theta}}^{\phi}, \check{A})$  Tabular Form

$u/\check{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$
$u_2$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$
$u_3$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$
$u_4$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$	$\left( (0, 0), \left( \begin{smallmatrix} 0+i0, 1+i1, 1+i1, \\ 0+i0, -1-i1, -1-i1 \end{smallmatrix} \right) \right)$

Table 7:  $(\check{I}_{\check{\Theta}}^{\mathbb{U}}, \check{A})$  Tabular Form

$u/\check{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$
$u_2$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$
$u_3$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$
$u_4$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$	$\left( (4, -4), \left( \begin{smallmatrix} 1+i1, 0+i0, 0+i0, \\ -1-i1, 0+i0, 0+i0 \end{smallmatrix} \right) \right)$

**Definition 3.3** Let  $(\check{I}_{\check{\Theta}}, \check{A})$  and  $(\check{\Delta}_{\check{\Upsilon}}, \check{B})$  be BCNN-SSs. Then  $(\check{I}_{\check{\Theta}}, \check{A})$  is called a BCNN-S subset of  $(\check{\Delta}_{\check{\Upsilon}}, \check{B})$  if  $\check{A} \subseteq \check{B}$  and,

$$\begin{aligned}r_{\check{I}}^{u+} &\leq r_{\check{\Delta}}^{u+}, & r_{\check{I}}^{u-} &\geq r_{\check{\Delta}}^{u-}, \\ T_{\check{I}x}^+ &\leq T_{\check{\Delta}x}^+, & T_{\check{I}x}^- &\geq T_{\check{\Delta}x}^-, \\ T_{\check{I}y}^+ &\leq T_{\check{\Delta}y}^+, & T_{\check{I}y}^- &\geq T_{\check{\Delta}y}^-, \\ I_{\check{I}x}^+ &\geq I_{\check{\Delta}x}^+, & I_{\check{I}x}^- &\leq I_{\check{\Delta}x}^-, \\ I_{\check{I}y}^+ &\geq I_{\check{\Delta}y}^+, & I_{\check{I}y}^- &\leq I_{\check{\Delta}y}^-, \\ F_{\check{I}x}^+ &\geq F_{\check{\Delta}x}^+, & F_{\check{I}x}^- &\leq F_{\check{\Delta}x}^-, \\ F_{\check{I}y}^+ &\geq F_{\check{\Delta}y}^+, & F_{\check{I}y}^- &\leq F_{\check{\Delta}y}^-, \end{aligned}$$

$\forall a \in \check{A}, u \in \mathbb{U}, \langle r^+, r^- \rangle \in R$ , where  $\check{\Theta} = (\check{F}, \check{A}, N)$  and  $\check{\Upsilon} = (\check{H}, \check{B}, N)$  are two BN-SSs. It is denoted by  $(\check{I}_{\check{\Theta}}, \check{A}) \sqsubseteq (\check{\Delta}_{\check{\Upsilon}}, \check{B})$ . Also  $(\check{\Delta}_{\check{\Upsilon}}, \check{B})$  is said to be a BCNN-S superset of  $(\check{I}_{\check{\Theta}}, \check{A})$ .

**Example 3.4** Let  $\mathbb{U}_0 = \{u_1, u_2\}$  and  $\ddot{E} = \{e_1, e_2, e_3, e_4\}$  as in the Example 3.1. Suppose  $\ddot{E} \supseteq \ddot{A} = \{e_1, e_2\}$  and  $\ddot{E} \supseteq \ddot{B} = \{e_1, e_2, e_3\}$ , where a B5-SS of  $\ddot{A}$  and  $\ddot{B}$  are given below:

Table 8: B5-SS Tabular Form of  $\ddot{A}$ 

$\mathbb{U}_0/\ddot{A}$	$e_1$	$e_2$
$u_1$	$\langle 3, -1 \rangle$	$\langle 2, -3 \rangle$
$u_2$	$\langle 4, -2 \rangle$	$\langle 2, -1 \rangle$

Table 9: B5-SS Tabular Form of  $\ddot{B}$ 

$\mathbb{U}_0/\ddot{B}$	$e_1$	$e_2$	$e_3$
$u_1$	$\langle 4, -1 \rangle$	$\langle 3, -4 \rangle$	$\langle 4, -1 \rangle$
$u_2$	$\langle 4, -3 \rangle$	$\langle 3, -2 \rangle$	$\langle 3, -2 \rangle$

Then, we now define BCN-5SSs  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  and  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$  with respect to B5-SSs of  $\ddot{A}$  and  $\ddot{B}$ , respectively, as next:

Table 10: BCN-5SS Tabular Form of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ 

$\mathbb{U}_0/\ddot{A}$	$e_1$	$e_2$
$u_1$	$\left( \langle 3, -1 \rangle, \begin{pmatrix} 0.7 + i0.7, & 0.21 + i0.3, & 0.3 + i0.3, \\ -0.3 - i0.2, & -0.75 - i0.75, & -0.79 - i0.75 \end{pmatrix} \right)$	$\left( \langle 2, -3 \rangle, \begin{pmatrix} 0.55 + i0.55, & 0.59 + i0.59, & 0.59 + i0.59, \\ -0.6 - i0.6, & -0.3 - i0.3, & -0.3 - i0.2 \end{pmatrix} \right)$
$u_2$	$\left( \langle 4, -2 \rangle, \begin{pmatrix} 0.9 + i0.95, & 0.1 + i0.1, & 0.1 + i0.15, \\ -0.5 - i0.55, & -0.4 - i0.4, & -0.4 - i0.5 \end{pmatrix} \right)$	$\left( \langle 2, -1 \rangle, \begin{pmatrix} 0.4 + i0.59, & 0.5 + i0.59, & 0.59 + i0.59, \\ -0.3 - i0.3, & -0.6 - i0.6, & -0.7 - i0.7 \end{pmatrix} \right)$

Table 11: BCN-5SS Tabular Form of  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ 

$\mathbb{U}_0/\ddot{B}$	$e_1$	$e_2$	$e_3$
$u_1$	$\left( \langle 4, -1 \rangle, \begin{pmatrix} 1 + i1, & 0.1 + i0.1, & 0.1 + i0.1, \\ -0.3 - i0.3, & -0.7 - i0.7, & -0.75 - i0.7 \end{pmatrix} \right)$	$\left( \langle 3, -4 \rangle, \begin{pmatrix} 0.79 + i0.79, & 0.39 + i0.39, & 0.39 + i0.39, \\ -1 - i1, & -0.1 - i0.1, & -0.1 - i0.1 \end{pmatrix} \right)$	$\left( \langle 4, -1 \rangle, \begin{pmatrix} 0.8 + i0.8, & 0.1 + i0.1, & 0.1 + i0.1, \\ -0.2 - i0.2, & -0.6 - i0.6, & -0.6 - i0.6 \end{pmatrix} \right)$
$u_2$	$\left( \langle 4, -3 \rangle, \begin{pmatrix} 1 + i1, & 0.15 + i0.19, & 0.15 + i0.19, \\ -0.7 - i0.7, & -0.3 - i0.3, & -0.3 - i0.3 \end{pmatrix} \right)$	$\left( \langle 3, -2 \rangle, \begin{pmatrix} 0.7 + i0.7, & 0.3 + i0.3, & 0.3 + i0.3, \\ -0.5 - i0.5, & -0.4 - i0.4, & -0.5 - i0.5 \end{pmatrix} \right)$	$\left( \langle 3, -2 \rangle, \begin{pmatrix} 0.6 + i0.6, & 0.2 + i0.3, & 0.3 + i0.3, \\ -0.4 - i0.4, & -0.5 - i0.5, & -0.5 - i0.5 \end{pmatrix} \right)$

From Table 10 and Table 11, it can be noticed clearly that  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqsubseteq (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ .

**Definition 3.4** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  and  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$  be two BCNN-SSs. Then,  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  is equal to  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$  if and only if  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqsubseteq (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$  and  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B}) \sqsubseteq (\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ .

**Proposition 3.3** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ ,  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ , and  $(\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})$  be three BCNN-SSs, then:

1.  $(\ddot{I}_{\ddot{\Theta}}^{\phi}, \ddot{A}) \sqsubseteq (\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqsubseteq (\ddot{I}_{\ddot{\Theta}}^{\cup}, \ddot{A})$ .
2. If  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqsubseteq (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$  and  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B}) \sqsubseteq (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})$ , then  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqsubseteq (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})$ .

**Proof:** Clearly, we can prove them by Definitions 3.2, 3.3, and 3.4.  $\square$

**Definition 3.5** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS. The weak complement of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ , denoted by  $(\ddot{I}_{\ddot{\Theta}}^w, \ddot{A})^w = (\ddot{I}_{\ddot{\Theta}^w}, \ddot{A})$ , is defined as follow:

$$(\ddot{I}_{\ddot{\Theta}^w}, \ddot{A}) = \left\{ \left\langle a, \left( \ddot{F}^w(a), \ddot{Y}(a) \right) \right\rangle : a \in \ddot{A}, (\ddot{F}^w(a), \ddot{Y}(a)) \in 2^{\mathbb{U} \times \mathbb{R}} \times \check{B}\check{C}\check{N}\check{N} \right\},$$

such that

$$\ddot{F}^w(a) \cap \ddot{F}(a) = \phi, \quad \forall a \in \ddot{A},$$

where  $\ddot{\Theta}^w = (\ddot{F}^w, \ddot{A}, N)$  is the weak complement of the BN-SS  $\ddot{\Theta} = (\ddot{F}, \ddot{A}, N)$  [24] and  $\ddot{F}^w : \ddot{A} \rightarrow 2^{\mathbb{U} \times \mathbb{R}}$ .

**Example 3.5** Suppose  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  is the BCN-5SS as in Example 3.1 So, its weak complement is given in the following table:

Table 12: The Weak Complement  $(\ddot{I}_{\ddot{\Theta}^w}, \ddot{A})$  of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$

$u/\ddot{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$(2, -2), \left( \begin{array}{c} 0.7 + i0.7, 0.21 + i0.3, 0.3 + i0.3, \\ -0.3 - i0.2, -0.7 - i0.7, -0.79 - i0.7 \end{array} \right)$	$(1, -1), \left( \begin{array}{c} 0.55 + i0.55, 0.59 + i0.59, 0.59 + i0.59, \\ -0.6 - i0.6, -0.3 - i0.3, -0.3 - i0.2 \end{array} \right)$	$(0, -3), \left( \begin{array}{c} 0.9 + i0.8, 0.1 + i0.1, 0.16 + i0.1, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.7 \end{array} \right)$	$(2, -1), \left( \begin{array}{c} 0.7 + i0.75, 0.39 + i0.3, 0.3 + i0.3, \\ -0.6 - i0.6, -0.3 - i0.2, -0.39 - i0.39 \end{array} \right)$
$u_2$	$(1, -4), \left( \begin{array}{c} 0.9 + i0.95, 0.1 + i0.1, 0.1 + i0.15, \\ -0.5 - i0.55, -0.4 - i0.4, -0.4 - i0.5 \end{array} \right)$	$(4, -2), \left( \begin{array}{c} 0.4 + i0.59, 0.5 + i0.59, 0.59 + i0.59, \\ -0.3 - i0.3, -0.6 - i0.6, -0.7 - i0.7 \end{array} \right)$	$(4, -1), \left( \begin{array}{c} 0.7 + i0.7, 0.39 + i0.39, 0.3 + i0.39, \\ -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \end{array} \right)$	$(2, -2), \left( \begin{array}{c} 1.0 + i0.8, 0.15 + i0.16, 0.19 + i0.19, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.79 \end{array} \right)$
$u_3$	$(3, -2), \left( \begin{array}{c} 0.5 + i0.4, 0.5 + i0.4, 0.55 + i0.55, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.6 \end{array} \right)$	$(2, -1), \left( \begin{array}{c} 1.0 + i1.0, 0.1 + i0.1, 0.15 + i0.1, \\ -0.7 - i0.7, -0.3 - i0.3, -0.39 - i0.39 \end{array} \right)$	$(1, -2), \left( \begin{array}{c} 0.59 + i0.59, 0.59 + i0.59, 0.5 + i0.59, \\ -0.6 - i0.6, -0.6 - i0.6, -0.7 - i0.7 \end{array} \right)$	$(1, -1), \left( \begin{array}{c} 0.4 + i0.5, 0.59 + i0.5, 0.59 + i0.59, \\ -1.0 - i0.8, -0.1 - i0.1, -0.1 - i0.1 \end{array} \right)$
$u_4$	$(1, -2), \left( \begin{array}{c} 0.1 + i0.19, 0.9 + i1.0, 1.0 + i0.9, \\ -0.39 - i.39, -0.7 - i0.7, 0.7 - i0.7 \end{array} \right)$	$(1, -1), \left( \begin{array}{c} 0.7 + i0.7, 0.3 + i0.39, 0.3 + i0.3, \\ -0.5 - i0.5, -0.5 - i0.5, -0.5 - i0.5 \end{array} \right)$	$(2, -1), \left( \begin{array}{c} 0.39 + i0.2, 0.7 + i0.79, 0.7 + i0.7, \\ -0.1 - i0.1, -0.8 - i0.8, -0.8 - i0.8 \end{array} \right)$	$(1, -1), \left( \begin{array}{c} 0.59 + i0.45, 0.4 + i0.5, 0.5 + i0.4, \\ -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \end{array} \right)$

**Definition 3.6** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS. Then, the BCNS complement of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ , represented by  $(\ddot{I}_{\ddot{\Theta}}^c, \ddot{A})^c = (\ddot{I}_{\ddot{\Theta}}^c, \ddot{A})$ , is defined by

$$(\ddot{I}_{\ddot{\Theta}}^c, \ddot{A}) = \left\{ \left\langle a, \left( \ddot{F}(a), \ddot{Y}^c(a) \right) \right\rangle : a \in \ddot{A}, (\ddot{F}(a), \ddot{Y}^c(a)) \in 2^{\mathbb{U} \times \mathbb{R}} \times \check{B}\check{C}\check{N}\check{N} \right\},$$

where  $\ddot{\Theta} = (\ddot{F}, \ddot{A}, N)$  is a BN-SS and  $\ddot{Y}^c : \ddot{A} \rightarrow \check{B}\check{C}\check{N}\check{N}$  such that

$$\ddot{Y}^c(a) = \left( F_{xa}^+ + iF_{ya}^+, (1 - I_{xa}^+) + i(1 - I_{ya}^+), T_{xa}^+ + iT_{ya}^+, F_{xa}^- + iF_{ya}^-, (-1 - I_{xa}^-) + i(-1 - I_{ya}^-), T_{xa}^- + iT_{ya}^- \right)$$

for all  $a \in \ddot{A}$ .

**Example 3.6** The BCNS complement  $(\ddot{I}_{\ddot{\Theta}}^c, \ddot{A})$  of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  in Example 3.1 is given in the table bellow:

Table 13: The BCNS Complement  $(\ddot{I}_{\ddot{\Theta}}^c, \ddot{A})$  of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$

$u/\ddot{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$(3, -1), \left( \begin{array}{c} 0.3 + i0.3, 0.79 + i0.7, 0.7 + i0.7, \\ -0.79 - i0.7, -0.3 - i0.3, -0.3 - i0.2 \end{array} \right)$	$(2, -3), \left( \begin{array}{c} 0.59 + i0.59, 0.41 + i0.41, 0.55 + i0.55, \\ -0.3 - i0.2, -0.7 - i0.7, -0.6 - i0.6 \end{array} \right)$	$(4, -1), \left( \begin{array}{c} 0.16 + i0.1, 0.9 + i0.9, 0.9 + i0.8, \\ -0.7 - i0.7, -0.3 - i0.3, -0.2 - i0.2 \end{array} \right)$	$(3, -3), \left( \begin{array}{c} 0.3 + i0.3, 0.61 + i0.7, 0.7 + i0.75, \\ -0.39 - i0.39, 0.7 - i0.8, -0.6 - i0.6 \end{array} \right)$
$u_2$	$(4, -2), \left( \begin{array}{c} 0.1 + i0.15, 0.9 + i0.9, 0.9 + i0.95, \\ -0.4 - i0.5, -0.6 - i0.6, -0.5 - i0.55 \end{array} \right)$	$(2, -1), \left( \begin{array}{c} 0.59 + i0.59, 0.5 + i0.41, 0.4 + i0.59, \\ -0.7 - i0.7, -0.4 - i0.4, -0.3 - i0.3 \end{array} \right)$	$(3, -2), \left( \begin{array}{c} 0.3 + i0.39, 0.61 + i0.61, 0.7 + i0.7, \\ -0.5 - i0.5, -0.5 - i0.5, -0.4 - i0.4 \end{array} \right)$	$(4, -1), \left( \begin{array}{c} 0.19 + i0.19, 0.85 + i0.84, 1.0 + i0.8, \\ -0.7 - i0.79, -0.3 - i0.3, -0.2 - i0.2 \end{array} \right)$
$u_3$	$(2, -1), \left( \begin{array}{c} 0.55 + i0.55, 0.5 + i0.6, 0.5 + i0.4, \\ -0.7 - i0.6, -0.3 - i0.3, -0.2 - i0.2 \end{array} \right)$	$(4, -3), \left( \begin{array}{c} 0.15 + i0.1, 0.9 + i0.9, 1.0 + i1.0, \\ -0.39 - i0.39, -0.7 - i0.7, -0.7 - i0.7 \end{array} \right)$	$(2, -3), \left( \begin{array}{c} 0.5 + i0.59, 0.41 + i0.41, 0.59 + i0.59, \\ -0.7 - i0.7, -0.4 - i0.4, -0.6 - i0.6 \end{array} \right)$	$(2, -4), \left( \begin{array}{c} 0.59 + i0.59, 0.41 + i0.5, 0.4 + i0.5, \\ -0.1 - i0.1, -0.9 - i0.9, -1.0 - i0.8 \end{array} \right)$
$u_4$	$(0, -1), \left( \begin{array}{c} 1.0 + i0.9, 0.1 + i0, 0.1 + i0.19, \\ -0.7 - i0.7, -0.3 - i0.3, -0.39 - i.39 \end{array} \right)$	$(3, -2), \left( \begin{array}{c} 0.3 + i0.3, 0.7 + i0.61, 0.7 + i0.7, \\ -0.5 - i0.5, -0.5 - i0.5, -0.5 - i0.5 \end{array} \right)$	$(1, 0), \left( \begin{array}{c} 0.7 + i0.7, 0.3 + i0.21, 0.39 + i0.2, \\ -0.8 - i0.8, -0.2 - i0.2, -0.1 - i0.1 \end{array} \right)$	$(2, -2), \left( \begin{array}{c} 0.5 + i0.4, 0.6 + i0.5, 0.59 + i0.45, \\ -0.5 - i0.5, -0.5 - i0.5, -0.4 - i0.4 \end{array} \right)$

**Definition 3.7** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS. Then, the weak BCNS complement of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ , denoted by  $(\ddot{I}_{\ddot{\Theta}}^w, \ddot{A})^{wc} = (\ddot{I}_{\ddot{\Theta}^w}^c, \ddot{A})$ , is defined by

$$(\ddot{I}_{\ddot{\Theta}^w}^c, \ddot{A}) = \left\{ \left\langle a, \left( \ddot{F}^w(a), \ddot{Y}^c(a) \right) \right\rangle : a \in \ddot{A}, (\ddot{F}^w(a), \ddot{Y}^c(a)) \in 2^{\mathbb{U} \times \mathbb{R}} \times \check{B}\check{C}\check{N}\check{N} \right\}$$

such that

1.  $\ddot{F}^w(a) \cap \ddot{F}(a) = \phi$ .
2.  $\ddot{Y}^c(a) = \left( F_{xa}^+ + iF_{ya}^+, (1 - I_{xa}^+) + i(1 - I_{ya}^+), T_{xa}^+ + iT_{ya}^+, F_{xa}^- + iF_{ya}^-, (-1 - I_{xa}^-) + i(-1 - I_{ya}^-), T_{xa}^- + iT_{ya}^- \right)$ ,

where  $\ddot{\Theta} = (\ddot{F}, \ddot{A}, N)$  is a BN-SS,  $\ddot{F}^w : \ddot{A} \rightarrow 2^{\mathbb{U} \times R}$ , and  $\ddot{Y}^c : \ddot{A} \rightarrow \check{B}\check{C}\check{N}\check{N}$ ,  $\forall a \in \ddot{A}$ .

**Example 3.7** Suppose  $(\ddot{F}_{\ddot{\Theta}}, \ddot{A})$  is the BCN-5SS as in Example 3.1. So, its weak BCNS complement  $(\ddot{F}_{\ddot{\Theta}^w}^c, \ddot{A})$  is given in the following table:

Table 14: The Weak BCNS Complement  $(\ddot{F}_{\ddot{\Theta}^w}^c, \ddot{A})$  of  $(\ddot{F}_{\ddot{\Theta}}, \ddot{A})$ .

$u/\ddot{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$(1, -2), \left( \begin{array}{c} 0.3 + i0.3, 0.79 + i0.7, 0.7 + i0.7, \\ -0.79 - i0.7, -0.3 - i0.3, -0.3 - i0.2 \end{array} \right)$	$(3, -1), \left( \begin{array}{c} 0.59 + i0.59, 0.41 + i0.41, 0.55 + i0.55, \\ -0.3 - i0.2, -0.7 - i0.7, -0.6 - i0.6 \end{array} \right)$	$(1, -2), \left( \begin{array}{c} 0.16 + i0.1, 0.9 + i0.9, 0.9 + i0.8 \\ -0.7 - i0.7, -0.3 - i0.3, -0.2 - i0.2 \end{array} \right)$	$(1, -2), \left( \begin{array}{c} 0.3 + i0.3, 0.61 + i0.7, 0.7 + i0.75, \\ -0.39 - i0.39, 0.7 - i0.8, -0.6 - i0.6 \end{array} \right)$
$u_2$	$(2, -4), \left( \begin{array}{c} 0.1 + i0.15, 0.9 + i0.9, 0.9 + i0.95, \\ -0.4 - i0.5, -0.6 - i0.6, -0.5 - i0.55 \end{array} \right)$	$(1, -2), \left( \begin{array}{c} 0.59 + i0.59, 0.5 + i0.41, 0.4 + i0.59, \\ -0.7 - i0.7, -0.4 - i0.4, -0.3 - i0.3 \end{array} \right)$	$(2, -3), \left( \begin{array}{c} 0.3 + i0.39, 0.61 + i0.61, 0.7 + i0.7, \\ -0.5 - i0.5, -0.5 - i0.5, -0.4 - i0.4 \end{array} \right)$	$(3, -2), \left( \begin{array}{c} 0.19 + i0.19, 0.85 + i0.84, 1.0 + i0.8, \\ -0.7 - i0.79, -0.3 - i0.3, -0.2 - i0.2 \end{array} \right)$
$u_3$	$(1, -2), \left( \begin{array}{c} 0.55 + i0.55, 0.5 + i0.6, 0.5 + i0.4, \\ -0.7 - i0.6, -0.3 - i0.3, -0.2 - i0.2 \end{array} \right)$	$(1, -2), \left( \begin{array}{c} 0.15 + i0.1, 0.9 + i0.9, 1.0 + i1.0, \\ -0.39 - i0.39, -0.7 - i0.7, -0.7 - i0.7 \end{array} \right)$	$(1, -1), \left( \begin{array}{c} 0.5 + i0.59, 0.41 + i0.41, 0.59 + i0.59, \\ -0.7 - i0.7, -0.4 - i0.4, -0.6 - i0.6 \end{array} \right)$	$(1, -3), \left( \begin{array}{c} 0.59 + i0.59, 0.41 + i0.5, 0.4 + i0.5, \\ -0.1 - i0.1, -0.9 - i0.9, -1.0 - i0.8 \end{array} \right)$
$u_4$	$(1, 0), \left( \begin{array}{c} 1.0 + i0.9, 0.1 + i0.1, 0.1 + i0.19, \\ -0.7 - i0.7, -0.3 - i0.3, -0.39 - i.39 \end{array} \right)$	$(1, -1), \left( \begin{array}{c} 0.3 + i0.3, 0.7 + i0.61, 0.7 + i0.7, \\ -0.5 - i0.5, -0.5 - i0.5, -0.5 - i0.5 \end{array} \right)$	$(0, -1), \left( \begin{array}{c} 0.7 + i0.7, 0.3 + i0.21, 0.39 + i0.2, \\ -0.8 - i0.8, -0.2 - i0.2, -0.1 - i0.1 \end{array} \right)$	$(3, -1), \left( \begin{array}{c} 0.5 + i0.4, 0.6 + i0.5, 0.59 + i0.45, \\ -0.5 - i0.5, -0.5 - i0.5, -0.4 - i0.4 \end{array} \right)$

**Proposition 3.4** Let  $(\ddot{F}_{\ddot{\Theta}}, \ddot{A})$  and  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$  be two BCNN-SSs. Then

1.  $(\ddot{F}_{\ddot{\Theta}}^\phi, \ddot{A})^w \neq (\ddot{F}_{\ddot{\Theta}}^\mathbb{U}, \ddot{A})$ .
2.  $(\ddot{F}_{\ddot{\Theta}}^\mathbb{U}, \ddot{A})^w \neq (\ddot{F}_{\ddot{\Theta}}^\phi, \ddot{A})$ .
3.  $(\ddot{F}_{\ddot{\Theta}}^\phi, \ddot{A})^{ww} \neq (\ddot{F}_{\ddot{\Theta}}^\phi, \ddot{A})$ .
4.  $(\ddot{F}_{\ddot{\Theta}}^\phi, \ddot{A})^c \neq (\ddot{F}_{\ddot{\Theta}}^\mathbb{U}, \ddot{A})$ .
5.  $(\ddot{F}_{\ddot{\Theta}}^\mathbb{U}, \ddot{A})^c \neq (\ddot{F}_{\ddot{\Theta}}^\phi, \ddot{A})$ .
6.  $(\ddot{F}_{\ddot{\Theta}}, \ddot{A})^{cc} = (\ddot{F}_{\ddot{\Theta}}, \ddot{A})$ .
7. If  $(\ddot{F}_{\ddot{\Theta}}, \ddot{A}) \sqsubseteq (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ , then  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})^c \sqsubseteq (\ddot{F}_{\ddot{\Theta}}, \ddot{A})^c$ .

**Proof:** Clearly, we can prove them by Definitions 3.4, 3.5, and 3.6.  $\square$

**Definition 3.8** Let  $(\ddot{F}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS. Then, the top weak complement of  $(\ddot{F}_{\ddot{\Theta}}, \ddot{A})$ , illustrated by  $(\ddot{F}_{\ddot{\Theta}}, \ddot{A})^t = (\ddot{F}_{\ddot{\Theta}^t}, \ddot{A})$ , is defined as follow:

$$\begin{aligned} (\ddot{F}_{\ddot{\Theta}^t}, \ddot{A}) &= \left\{ \left\langle a, \left( \ddot{F}^t(a), \ddot{Y}(a) \right) \right\rangle : a \in \ddot{A}, \left( \ddot{F}^t(a), \ddot{Y}(a) \right) \in 2^{\mathbb{U} \times R} \times \check{B}\check{C}\check{N}\check{N} \right\} \\ &= \left\{ \left\langle a, \left( (u, \langle r_{ia}^{+t}(u), r_{ia}^{-t}(u) \rangle), (\mathbb{T}_a^+, \mathbb{I}_a^+, \mathbb{F}_a^+, \mathbb{T}_a^-, \mathbb{I}_a^-, \mathbb{F}_a^-) \right) \right\rangle : a \in \ddot{A}, \right. \\ &\quad \left. u \in \mathbb{U}, \langle r_{ia}^{+t}(u), r_{ia}^{-t}(u) \rangle \in R \right\} \end{aligned}$$

such that

$$r_{ia}^{+t}(u) = \begin{cases} 0 & \text{if } r_i^+ = N - 1 \\ N - 1 & \text{otherwise} \end{cases}$$

and

$$r_{ia}^{-t}(u) = \begin{cases} -(N-1) & \text{if } r_i^- = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where  $\check{\Theta}^t = (\check{F}^t, \check{A}, N)$  is the top weak complement of the BN-SS  $\check{\Theta} = (\check{F}, \check{A}, N)$  and  $\check{F}^t : \check{A} \rightarrow 2^{\mathbb{U} \times R}$ ,  $\forall u \in \mathbb{U}$ .

**Definition 3.9** Let  $(\check{I}_{\check{\Theta}}, \check{A})$  be a BCNN-SS. Then, the bottom weak complement of  $(\check{I}_{\check{\Theta}}, \check{A})$ , illustrated by  $(\check{I}_{\check{\Theta}}, \check{A})^b = (\check{I}_{\check{\Theta}^b}, \check{A})$  is defined as follow:

$$\begin{aligned} (\check{I}_{\check{\Theta}^b}, \check{A}) &= \left\{ \left\langle a, \left( \check{F}^b(a), \check{Y}(a) \right) \right\rangle : a \in \check{A}, \left( \check{F}^b(a), \check{Y}(a) \right) \in 2^{\mathbb{U} \times R} \times \check{B}\check{C}\check{N}\check{N} \right\} \\ &= \left\{ \left\langle a, \left( (u, \langle r_{ia}^{+b}(u), r_{ia}^{-b}(u) \rangle), (\mathbb{T}_a^+, \mathbb{I}_a^+, \mathbb{F}_a^+, \mathbb{T}_a^-, \mathbb{I}_a^-, \mathbb{F}_a^-) \right) \right\rangle : a \in \check{A}, \right. \\ &\quad \left. u \in \mathbb{U}, \langle r_{ia}^{+b}(u), r_{ia}^{-b}(u) \rangle \in R \right\} \end{aligned}$$

such that

$$r_i^{+b}(u) = \begin{cases} N-1 & \text{if } r_i^+ = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$r_i^{-b}(u) = \begin{cases} 0 & \text{if } r_i^- = -(N-1) \\ -(N-1) & \text{otherwise,} \end{cases}$$

where  $\check{\Theta}^b = (\check{F}^b, \check{A}, N)$  is the bottom weak complement of the BN-SS  $\check{\Theta} = (\check{F}, \check{A}, N)$  and  $\check{F}^b : \check{A} \rightarrow 2^{\mathbb{U} \times R}$ ,  $\forall u \in \mathbb{U}$ .

**Example 3.8** We find both top and bottom weak complement of  $(\check{I}_{\check{\Theta}}, \check{A})$ , given in Example 3.1, are presented in the following Tables 15 and 16, respectively:

Table 15: The Top Weak Complement  $(\check{I}_{\check{\Theta}^t}, \check{A})$  of  $(\check{I}_{\check{\Theta}}, \check{A})$ .

$u/\check{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$(4, 0), \left( \begin{smallmatrix} 0.7 + i0.7, 0.21 + i0.3, 0.3 + i0.3, \\ -0.3 - i0.2, -0.7 - i0.7, -0.79 - i0.7 \end{smallmatrix} \right)$	$(4, 0), \left( \begin{smallmatrix} 0.55 + i0.55, 0.59 + i0.59, 0.59 + i0.59, \\ -0.6 - i0.6, -0.3 - i0.3, -0.3 - i0.2 \end{smallmatrix} \right)$	$(0, 0), \left( \begin{smallmatrix} 0.9 + i0.8, 0.1 + i0.1, 0.16 + i0.1, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.7 \end{smallmatrix} \right)$	$(4, 0), \left( \begin{smallmatrix} 0.7 + i0.75, 0.39 + i0.3, 0.3 + i0.3, \\ -0.6 - i0.6, -0.3 - i0.2, -0.39 - i0.39 \end{smallmatrix} \right)$
$u_2$	$(0, 0), \left( \begin{smallmatrix} 0.9 + i0.95, 0.1 + i0.1, 0.1 + i0.15, \\ -0.5 - i0.55, -0.4 - i0.4, -0.4 - i0.5 \end{smallmatrix} \right)$	$(4, 0), \left( \begin{smallmatrix} 0.4 + i0.59, 0.5 + i0.59, 0.59 + i0.59, \\ -0.3 - i0.3, -0.6 - i0.6, -0.7 - i0.7 \end{smallmatrix} \right)$	$(4, 0), \left( \begin{smallmatrix} 0.7 + i0.7, 0.39 + i0.39, 0.3 + i0.39, \\ -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \end{smallmatrix} \right)$	$(0, 0), \left( \begin{smallmatrix} 1.0 + i0.8, 0.15 + i0.16, 0.19 + i0.19, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.79 \end{smallmatrix} \right)$
$u_3$	$(4, 0), \left( \begin{smallmatrix} 0.5 + i0.4, 0.5 + i0.4, 0.55 + i0.55, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.6 \end{smallmatrix} \right)$	$(0, 0), \left( \begin{smallmatrix} 1.0 + i1.0, 0.1 + i0.1, 0.15 + i0.1, \\ -0.7 - i0.7, -0.3 - i0.3, -0.39 - i0.39 \end{smallmatrix} \right)$	$(4, 0), \left( \begin{smallmatrix} 0.59 + i0.59, 0.59 + i0.59, 0.5 + i0.59, \\ -0.6 - i0.6, -0.6 - i0.6, -0.7 - i0.7 \end{smallmatrix} \right)$	$(4, 0), \left( \begin{smallmatrix} 0.4 + i0.5, 0.59 + i0.5, 0.59 + i0.59, \\ -1.0 - i0.8, -0.1 - i0.1, -0.1 - i0.1 \end{smallmatrix} \right)$
$u_4$	$(4, 0), \left( \begin{smallmatrix} 0.1 + i0.19, 0.9 + i1.0, 1.0 + i0.9, \\ -0.39 - i.39, -0.7 - i0.7, -0.7 - i0.7 \end{smallmatrix} \right)$	$(4, 0), \left( \begin{smallmatrix} 0.7 + i0.7, 0.3 + i0.39, 0.3 + i0.3, \\ -0.5 - i0.5, -0.5 - i0.5, -0.5 - i0.5 \end{smallmatrix} \right)$	$(4, -4), \left( \begin{smallmatrix} 0.39 + i0.2, 0.7 + i0.79, 0.7 + i0.7, \\ -0.1 - i0.1, -0.8 - i0.8, -0.8 - i0.8 \end{smallmatrix} \right)$	$(4, 0), \left( \begin{smallmatrix} 0.59 + i0.45, 0.4 + i0.5, 0.5 + i0.4, \\ -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \end{smallmatrix} \right)$

Table 16: The Bottom Weak Complement  $(\check{I}_{\check{\Theta}^b}, \check{A})$  of  $(\check{I}_{\check{\Theta}}, \check{A})$ .

$u/\check{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$(0, -4), \left( \begin{smallmatrix} 0.7 + i0.7, 0.21 + i0.3, 0.3 + i0.3, \\ -0.3 - i0.2, -0.7 - i0.7, -0.79 - i0.7 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 0.55 + i0.55, 0.59 + i0.59, 0.59 + i0.59, \\ -0.6 - i0.6, -0.3 - i0.3, -0.3 - i0.2 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 0.9 + i0.8, 0.1 + i0.1, 0.16 + i0.1, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.7 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 0.7 + i0.75, 0.39 + i0.3, 0.3 + i0.3, \\ -0.6 - i0.6, -0.3 - i0.2, -0.39 - i0.39 \end{smallmatrix} \right)$
$u_2$	$(0, -4), \left( \begin{smallmatrix} 0.9 + i0.95, 0.1 + i0.1, 0.1 + i0.15, \\ -0.5 - i0.55, -0.4 - i0.4, -0.4 - i0.5 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 0.4 + i0.59, 0.5 + i0.59, 0.59 + i0.59, \\ -0.3 - i0.3, -0.6 - i0.6, -0.7 - i0.7 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 0.7 + i0.7, 0.39 + i0.39, 0.3 + i0.39, \\ -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 1.0 + i0.8, 0.15 + i0.16, 0.19 + i0.19, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.79 \end{smallmatrix} \right)$
$u_3$	$(0, -4), \left( \begin{smallmatrix} 0.5 + i0.4, 0.5 + i0.4, 0.55 + i0.55, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.6 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 1.0 + i1.0, 0.1 + i0.1, 0.15 + i0.1, \\ -0.7 - i0.7, -0.3 - i0.3, -0.39 - i0.39 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 0.59 + i0.59, 0.59 + i0.59, 0.5 + i0.59, \\ -0.6 - i0.6, -0.6 - i0.6, -0.7 - i0.7 \end{smallmatrix} \right)$	$(0, 0), \left( \begin{smallmatrix} 0.4 + i0.5, 0.59 + i0.5, 0.59 + i0.59, \\ -1.0 - i0.8, -0.1 - i0.1, -0.1 - i0.1 \end{smallmatrix} \right)$
$u_4$	$(4, -4), \left( \begin{smallmatrix} 0.1 + i0.19, 0.9 + i1.0, 1.0 + i0.9, \\ -0.39 - i.39, -0.7 - i0.7, -0.7 - i0.7 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 0.7 + i0.7, 0.3 + i0.39, 0.3 + i0.3, \\ -0.5 - i0.5, -0.5 - i0.5, -0.5 - i0.5 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 0.39 + i0.2, 0.7 + i0.79, 0.7 + i0.7, \\ -0.1 - i0.1, -0.8 - i0.8, -0.8 - i0.8 \end{smallmatrix} \right)$	$(0, -4), \left( \begin{smallmatrix} 0.59 + i0.45, 0.4 + i0.5, 0.5 + i0.4, \\ -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \end{smallmatrix} \right)$

**Definition 3.10** Let  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS. Then, the top weak BCNS complement of  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$ , illustrated by  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})^{tc} = (\ddot{\Gamma}_{\ddot{\Theta}^t}, \ddot{A})$ , is defined as follow:

$$\begin{aligned} (\ddot{\Gamma}_{\ddot{\Theta}^t}, \ddot{A}) &= \left\{ \left\langle a, \left( \ddot{F}^t(a), \ddot{Y}^c(a) \right) \right\rangle : a \in \ddot{A}, (\ddot{F}^t(a), \ddot{Y}^c(a)) \in 2^{\mathbb{U} \times R} \times \check{B}\check{C}\check{N}\check{N} \right\} \\ &= \left\{ \left\langle a, \left( (u, \langle r_{ia}^{+t}(u), r_{ia}^{-t}(u) \rangle), (\mathbb{T}_a^+, \mathbb{I}_a^+, \mathbb{F}_a^+, \mathbb{T}_a^-, \mathbb{I}_a^-, \mathbb{F}_a^-) \right) \right\rangle : a \in \ddot{A}, u \in \mathbb{U}, \langle r_{ia}^{+t}(u), r_{ia}^{-t}(u) \rangle \in R \right\} \end{aligned}$$

such that

1.

$$r_i^{+t}(u) = \begin{cases} 0 & \text{if } r_i^+ = N - 1 \\ N - 1 & \text{otherwise} \end{cases}$$

and

$$r_i^{-t}(u) = \begin{cases} -(N - 1) & \text{if } r_i^- = 0 \\ 0 & \text{otherwise} \end{cases}$$

2.  $\ddot{F}^t(a) \cap \ddot{F}(a) = \phi$

3.  $\ddot{Y}^c(a) = \left( F_{xa}^+ + iF_{ya}^+, (1 - I_{xa}^+) + i(1 - I_{ya}^+), T_{xa}^+ + iT_{ya}^+, F_{xa}^- + iF_{ya}^-, (-1 - I_{xa}^-) + i(-1 - I_{ya}^-), T_{xa}^- + iT_{ya}^- \right),$

where  $\ddot{\Theta}^t = (\ddot{F}^t, \ddot{A}, N)$  is the top weak complement of the BN-SS  $\ddot{\Theta} = (\ddot{F}, \ddot{A}, N)$ ,  $\ddot{F}^t : \ddot{A} \rightarrow 2^{\mathbb{U} \times R}$ , and  $\ddot{Y}^c : \ddot{A} \rightarrow \check{B}\check{C}\check{N}\check{N}$ ,  $\forall a \in \ddot{A}, u \in \mathbb{U}, \langle r_{ia}^{+t}(u), r_{ia}^{-t}(u) \rangle \in R$ .

**Definition 3.11** Let  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS. Then, the bottom weak BCNS complement, illustrated by  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})^{bc} = (\ddot{\Gamma}_{\ddot{\Theta}^b}, \ddot{A})$ , is defined as follow:

$$\begin{aligned} (\ddot{\Gamma}_{\ddot{\Theta}^b}, \ddot{A}) &= \left\{ \left\langle a, \left( \ddot{F}^b(a), \ddot{Y}^c(a) \right) \right\rangle : a \in \ddot{A}, (\ddot{F}^b(a), \ddot{Y}^c(a)) \in 2^{\mathbb{U} \times R} \times \check{B}\check{C}\check{N}\check{N} \right\} \\ &= \left\{ \left\langle a, \left( (u, \langle r_{ia}^{+b}(u), r_{ia}^{-b}(u) \rangle), (\mathbb{T}_a^+, \mathbb{I}_a^+, \mathbb{F}_a^+, \mathbb{T}_a^-, \mathbb{I}_a^-, \mathbb{F}_a^-) \right) \right\rangle : a \in \ddot{A}, u \in \mathbb{U}, \langle r_{ia}^{+b}(u), r_{ia}^{-b}(u) \rangle \in R \right\} \end{aligned}$$

such that

1.

$$r_i^{+b}(u) = \begin{cases} N - 1 & \text{if } r_i^+ = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$r_i^{-b}(u) = \begin{cases} 0 & \text{if } r_i^- = -(N - 1) \\ -(N - 1) & \text{otherwise} \end{cases}$$

2.  $\ddot{F}^b(a) \cap \ddot{F}(a) = \phi$

3.  $\ddot{Y}^c(a) = \left( F_{xa}^+ + iF_{ya}^+, (1 - I_{xa}^+) + i(1 - I_{ya}^+), T_{xa}^+ + iT_{ya}^+, F_{xa}^- + iF_{ya}^-, (-1 - I_{xa}^-) + i(-1 - I_{ya}^-), T_{xa}^- + iT_{ya}^- \right),$

where  $\ddot{\Theta}^b = (\ddot{F}^b, \ddot{A}, N)$  is the bottom weak complement of the BN-SS  $\ddot{\Theta} = (\ddot{F}, \ddot{A}, N)$ ,  $\ddot{F}^b : \ddot{A} \rightarrow 2^{\mathbb{U} \times R}$ , and  $\ddot{Y}^c : \ddot{A} \rightarrow \check{B}\check{C}\check{N}\check{N}$ ,  $\forall a \in \ddot{A}, u \in \mathbb{U}, \langle r_{ia}^{+b}(u), r_{ia}^{-b}(u) \rangle \in R$ .

**Example 3.9** We find both top and bottom weak complement of  $(\ddot{\Gamma}_{\ddot{\Theta}}, \ddot{A})$  as in Example 3.1 in the following Tables 17 and 18, respectively, as bellow:

Table 17: The Top Weak BCNS Complement  $(\check{I}_{\check{\Theta}t}^c, \check{A})$  of  $(\check{I}_{\check{\Theta}}, \check{A})$ 

$u/\check{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$(4,0), \left( \begin{array}{c} 0.3+i0.3, 0.79+i0.7, 0.7+i0.7, \\ -0.79-i0.7, -0.3-i0.3, -0.3-i0.2 \end{array} \right)$	$(4,0), \left( \begin{array}{c} 0.59+i0.59, 0.41+i0.41, 0.55+i0.55, \\ -0.3-i0.2, -0.7-i0.7, -0.6-i0.6 \end{array} \right)$	$(0,0), \left( \begin{array}{c} 0.16+i0.1, 0.9+i0.9, 0.9+i0.8 \\ -0.7-i0.7, -0.3-i0.3, -0.2-i0.2 \end{array} \right)$	$(4,0), \left( \begin{array}{c} 0.3+i0.3, 0.61+i0.7, 0.7+i0.75, \\ -0.39-i0.39, 0.7-i0.8, -0.6-i0.6 \end{array} \right)$
$u_2$	$(0,0), \left( \begin{array}{c} 0.1+i0.15, 0.9+i0.9, 0.9+i0.95, \\ -0.4-i0.5, -0.6-i0.6, -0.5-i0.55 \end{array} \right)$	$(4,0), \left( \begin{array}{c} 0.59+i0.59, 0.5+i0.41, 0.4+i0.59, \\ -0.7-i0.7, -0.4-i0.4, -0.3-i0.3 \end{array} \right)$	$(4,0), \left( \begin{array}{c} 0.3+i0.39, 0.61+i0.61, 0.7+i0.7, \\ -0.5-i0.5, -0.5-i0.5, -0.4-i0.4 \end{array} \right)$	$(0,0), \left( \begin{array}{c} 0.19+i0.19, 0.85+i0.84, 1.0+i0.8, \\ -0.7-i0.79, -0.3-i0.3, -0.2-i0.2 \end{array} \right)$
$u_3$	$(4,0), \left( \begin{array}{c} 0.55+i0.55, 0.5+i0.6, 0.5+i0.4, \\ -0.7-i0.6, -0.3-i0.3, -0.2-i0.2 \end{array} \right)$	$(0,0), \left( \begin{array}{c} 0.15+i0.1, 0.9+i0.9, 1.0+i1.0, \\ -0.39-i0.39, -0.7-i0.7, -0.7-i0.7 \end{array} \right)$	$(4,0), \left( \begin{array}{c} 0.5+i0.59, 0.41+i0.41, 0.59+i0.59, \\ -0.7-i0.7, -0.4-i0.4, -0.6-i0.6 \end{array} \right)$	$(4,0), \left( \begin{array}{c} 0.59+i0.59, 0.41+i0.5, 0.4+i0.5, \\ -0.1-i0.1, -0.9-i0.9, -1.0-i0.8 \end{array} \right)$
$u_4$	$(4,0), \left( \begin{array}{c} 1.0+i0.9, 0.1+i0, 0.1+i0.19, \\ -0.7-i0.7, -0.3-i0.3, -0.39-i.39 \end{array} \right)$	$(4,0), \left( \begin{array}{c} 0.3+i0.3, 0.7+i0.61, 0.7+i0.7, \\ -0.5-i0.5, -0.5-i0.5, -0.5-i0.5 \end{array} \right)$	$(4,-4), \left( \begin{array}{c} 0.7+i0.7, 0.3+i0.21, 0.39+i0.2, \\ -0.8-i0.8, -0.2-i0.2, -0.1-i0.1 \end{array} \right)$	$(4,0), \left( \begin{array}{c} 0.5+i0.4, 0.6+i0.5, 0.59+i0.45, \\ -0.5-i0.5, -0.5-i0.5, -0.4-i0.4 \end{array} \right)$

Table 18: The Bottom Weak BCNS Complement  $(\check{I}_{\check{\Theta}b}^c, \check{A})$  of  $(\check{I}_{\check{\Theta}}, \check{A})$ 

$u/\check{A}$	$e_1 = \text{experience}$	$e_2 = \text{age}$	$e_3 = \text{management skills}$	$e_4 = \text{creativity}$
$u_1$	$(0,-4), \left( \begin{array}{c} 0.3+i0.3, 0.79+i0.7, 0.7+i0.7, \\ -0.79-i0.7, -0.3-i0.3, -0.3-i0.2 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.59+i0.59, 0.41+i0.41, 0.55+i0.55, \\ -0.3-i0.2, -0.7-i0.7, -0.6-i0.6 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.16+i0.1, 0.9+i0.9, 0.9+i0.8 \\ -0.7-i0.7, -0.3-i0.3, -0.2-i0.2 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.3+i0.3, 0.61+i0.7, 0.7+i0.75, \\ -0.39-i0.39, 0.7-i0.8, -0.6-i0.6 \end{array} \right)$
$u_2$	$(0,-4), \left( \begin{array}{c} 0.1+i0.15, 0.9+i0.9, 0.9+i0.95, \\ -0.4-i0.5, -0.6-i0.6, -0.5-i0.55 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.59+i0.59, 0.5+i0.41, 0.4+i0.59, \\ -0.7-i0.7, -0.4-i0.4, -0.3-i0.3 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.3+i0.39, 0.61+i0.61, 0.7+i0.7, \\ -0.5-i0.5, -0.5-i0.5, -0.4-i0.4 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.19+i0.19, 0.85+i0.84, 1.0+i0.8, \\ -0.7-i0.79, -0.3-i0.3, -0.2-i0.2 \end{array} \right)$
$u_3$	$(0,-4), \left( \begin{array}{c} 0.55+i0.55, 0.5+i0.6, 0.5+i0.4, \\ -0.7-i0.6, -0.3-i0.3, -0.2-i0.2 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.15+i0.1, 0.9+i0.9, 1.0+i1.0, \\ -0.39-i0.39, -0.7-i0.7, -0.7-i0.7 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.5+i0.59, 0.41+i0.41, 0.59+i0.59, \\ -0.7-i0.7, -0.4-i0.4, -0.6-i0.6 \end{array} \right)$	$(0,0), \left( \begin{array}{c} 0.59+i0.59, 0.41+i0.5, 0.4+i0.5, \\ -0.1-i0.1, -0.9-i0.9, -1.0-i0.8 \end{array} \right)$
$u_4$	$(4,-4), \left( \begin{array}{c} 1.0+i0.9, 0.1+i0, 0.1+i0.19, \\ -0.7-i0.7, -0.3-i0.3, -0.39-i.39 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.3+i0.3, 0.7+i0.61, 0.7+i0.7, \\ -0.5-i0.5, -0.5-i0.5, -0.5-i0.5 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.7+i0.7, 0.3+i0.21, 0.39+i0.2, \\ -0.8-i0.8, -0.2-i0.2, -0.1-i0.1 \end{array} \right)$	$(0,-4), \left( \begin{array}{c} 0.5+i0.4, 0.6+i0.5, 0.59+i0.45, \\ -0.5-i0.5, -0.5-i0.5, -0.4-i0.4 \end{array} \right)$

**Proposition 3.5** For the given BCNN-SSs  $(\check{I}_{\check{\Theta}}^\phi, \check{A})$  and  $(\check{I}_{\check{\Theta}}^\cup, \check{A})$ , we have:

1.  $((\check{I}_{\check{\Theta}}^\phi, \check{A})^t)^t = (\check{I}_{\check{\Theta}}^\phi, \check{A})$ .
2.  $((\check{I}_{\check{\Theta}}^\cup, \check{A})^t)^t = (\check{I}_{\check{\Theta}}^\cup, \check{A})$ .
3.  $((\check{I}_{\check{\Theta}}^\phi, \check{A})^b)^b = (\check{I}_{\check{\Theta}}^\phi, \check{A})$ .
4.  $((\check{I}_{\check{\Theta}}^\cup, \check{A})^b)^b = (\check{I}_{\check{\Theta}}^\cup, \check{A})$ .
5.  $(\check{I}_{\check{\Theta}}^\phi, \check{A})^{tc} = (\check{I}_{\check{\Theta}}^\phi, \check{A})^{bc} = (\check{I}_{\check{\Theta}}^\cup, \check{A})$ .
6.  $(\check{I}_{\check{\Theta}}^\cup, \check{A})^{tc} = (\check{I}_{\check{\Theta}}^\cup, \check{A})^{bc} = (\check{I}_{\check{\Theta}}^\phi, \check{A})$ .
7.  $((\check{I}_{\check{\Theta}}^\phi, \check{A})^{tc})^{tc} = ((\check{I}_{\check{\Theta}}^\phi, \check{A})^{bc})^{bc} = (\check{I}_{\check{\Theta}}^\phi, \check{A})$ .
8.  $((\check{I}_{\check{\Theta}}^\cup, \check{A})^{tc})^{tc} = ((\check{I}_{\check{\Theta}}^\cup, \check{A})^{bc})^{bc} = (\check{I}_{\check{\Theta}}^\cup, \check{A})$ .

**Proof:** Clearly, we can prove them by Definitions 3.4, 3.5, and 3.6.  $\square$

**Definition 3.12** Let  $(\check{I}_{\check{\Theta}}, \check{A})$  and  $(\check{\Delta}_{\check{Y}}, \check{B})$  be two BCNN-SSs. Then, we define their restricted union as  $(\check{I}_{\check{\Theta}}, \check{A}) \sqcup_R (\check{\Delta}_{\check{Y}}, \check{B}) = (\check{\Sigma}_{\check{M}}, \check{A} \cap \check{B})$ , where  $\check{\Theta} = (\check{F}, \check{A}, N_1)$ ,  $\check{Y} = (\check{H}, \check{B}, N_2)$  are BN-SSs and  $\check{M} = \check{\Theta} \sqcup_R \check{Y} = (\check{F} \cup \check{H}, \check{A} \cap \check{B}, \max\{N_1, N_2\})$  such that  $\forall a \in \check{A} \cap \check{B}$  (with  $\check{A} \cap \check{B} \neq \phi$ ),  $u \in \mathbb{U}$ ,  $\langle r_a^{u+}, r_a^{u-} \rangle \in R$ , we have that  $\left( (u, \langle r_a^{u+}, r_a^{u-} \rangle), (T_{\check{x}a}^+ + iT_{\check{y}a}^+, I_{\check{x}a}^+ + iI_{\check{y}a}^+, F_{\check{x}a}^+ + iF_{\check{y}a}^+, T_{\check{x}a}^- + iT_{\check{y}a}^-, I_{\check{x}a}^- + iI_{\check{y}a}^-, F_{\check{x}a}^- + iF_{\check{y}a}^-) \right) \in \check{\Sigma}_{\check{M}}(a)$  if and only if  $\left( (u, \langle r_{\check{F}}^{u+}, r_{\check{F}}^{u-} \rangle), (T_{\check{F}a}^+ + iT_{\check{F}a}^+, I_{\check{F}a}^+ + iI_{\check{F}a}^+, F_{\check{F}a}^+ + iF_{\check{F}a}^+, T_{\check{F}a}^- + iT_{\check{F}a}^-, I_{\check{F}a}^- + iI_{\check{F}a}^-, F_{\check{F}a}^- + iF_{\check{F}a}^-) \right) \in \check{I}_{\check{\Theta}}(a)$  and  $\left( (u, \langle r_a^{u+}, r_a^{u-} \rangle), (T_{\check{\Delta}a}^+ + iT_{\check{\Delta}a}^+, I_{\check{\Delta}a}^+ + iI_{\check{\Delta}a}^+, F_{\check{\Delta}a}^+ + iF_{\check{\Delta}a}^+, T_{\check{\Delta}a}^- + iT_{\check{\Delta}a}^-, I_{\check{\Delta}a}^- + iI_{\check{\Delta}a}^-, F_{\check{\Delta}a}^- + iF_{\check{\Delta}a}^-) \right) \in \check{\Delta}_{\check{Y}}(a)$ , where

$$\begin{aligned}
r_a^{u+} &= \max\{r_{\tilde{F}}^{u+}, r_{\tilde{\Delta}}^{u+}\}, & r_a^{u-} &= \min\{r_{\tilde{F}}^{u-}, r_{\tilde{\Delta}}^{u-}\} \\
T_{xa}^+ &= \max\{T_{\tilde{F}xa}^+, T_{\tilde{\Delta}xa}^+\}, & T_{xa}^- &= \min\{T_{\tilde{F}xa}^-, T_{\tilde{\Delta}xa}^-\}, \\
T_{ya}^+ &= \max\{T_{\tilde{F}ya}^+, T_{\tilde{\Delta}ya}^+\}, & T_{ya}^- &= \min\{T_{\tilde{F}ya}^-, T_{\tilde{\Delta}ya}^-\}, \\
I_{xa}^+ &= \min\{I_{\tilde{F}xa}^+, I_{\tilde{\Delta}xa}^+\}, & I_{xa}^- &= \max\{I_{\tilde{F}xa}^-, I_{\tilde{\Delta}xa}^-\}, \\
I_{ya}^+ &= \min\{I_{\tilde{F}ya}^+, I_{\tilde{\Delta}ya}^+\}, & I_{ya}^- &= \max\{I_{\tilde{F}ya}^-, I_{\tilde{\Delta}ya}^-\}, \\
F_{xa}^+ &= \min\{F_{\tilde{F}xa}^+, F_{\tilde{\Delta}xa}^+\}, & F_{xa}^- &= \max\{F_{\tilde{F}xa}^-, F_{\tilde{\Delta}xa}^-\}, \\
F_{ya}^+ &= \min\{F_{\tilde{F}ya}^+, F_{\tilde{\Delta}ya}^+\} \quad \text{and} \quad & F_{ya}^- &= \max\{F_{\tilde{F}ya}^-, F_{\tilde{\Delta}ya}^-\}.
\end{aligned}$$

**Definition 3.13** Let  $(\tilde{F}_{\tilde{\Theta}}, \tilde{A})$  and  $(\tilde{\Delta}_{\tilde{\Upsilon}}, \tilde{B})$  be two BCNN-SSs. Then, we define their extended union as  $(\tilde{F}_{\tilde{\Theta}}, \tilde{A}) \sqcup_E (\tilde{\Delta}_{\tilde{\Upsilon}}, \tilde{B}) = (\tilde{\Sigma}_{\tilde{M}}, \tilde{A} \cup \tilde{B})$ , where  $\tilde{\Theta} = (\tilde{F}, \tilde{A}, N_1)$ ,  $\tilde{\Upsilon} = (\tilde{H}, \tilde{B}, N_2)$  are BN-SSs and  $\tilde{M} = \tilde{\Theta} \sqcup_E \tilde{\Upsilon} = (\tilde{F} \cup \tilde{H}, \tilde{A} \cup \tilde{B}, \max\{N_1, N_2\})$  such that  $\forall a \in \tilde{A} \cup \tilde{B}$  (with  $\tilde{A} \cap \tilde{B} \neq \phi$ ),  $u \in \mathbb{U}$ ,  $\langle r_a^{u+}, r_a^{u-} \rangle \in R$ , we have that  $\left( (u, \langle r_a^{u+}, r_a^{u-} \rangle), (T_{xa}^+ + iT_{ya}^+, I_{xa}^+ + iI_{ya}^+, F_{xa}^+ + iF_{ya}^+, T_{xa}^- + iT_{ya}^-, I_{xa}^- + iI_{ya}^-, F_{xa}^- + iF_{ya}^-) \right) \in \tilde{\Sigma}_{\tilde{M}}(a)$  if and only if

$$\tilde{\Sigma}_{\tilde{M}}(a) = \begin{cases} \tilde{F}_{\tilde{\Theta}}(a) & \text{if } a \in \tilde{A} - \tilde{B} \\ \tilde{\Delta}_{\tilde{\Upsilon}}(a) & \text{if } a \in \tilde{B} - \tilde{A} \\ \left( (u, \langle r_a^{u+}, r_a^{u-} \rangle), (T_{xa}^+ + iT_{ya}^+, I_{xa}^+ + iI_{ya}^+, F_{xa}^+ + iF_{ya}^+, T_{xa}^- + iT_{ya}^-, I_{xa}^- + iI_{ya}^-, F_{xa}^- + iF_{ya}^-) \right) & \text{if } a \in \tilde{A} \cap \tilde{B}, \end{cases}$$

where

$$\begin{aligned}
r_a^{u+} &= \max\{r_{\tilde{F}}^{u+}, r_{\tilde{\Delta}}^{u+}\}, & r_a^{u-} &= \min\{r_{\tilde{F}}^{u-}, r_{\tilde{\Delta}}^{u-}\} \\
T_{xa}^+ &= \max\{T_{\tilde{F}xa}^+, T_{\tilde{\Delta}xa}^+\}, & T_{xa}^- &= \min\{T_{\tilde{F}xa}^-, T_{\tilde{\Delta}xa}^-\}, \\
T_{ya}^+ &= \max\{T_{\tilde{F}ya}^+, T_{\tilde{\Delta}ya}^+\}, & T_{ya}^- &= \min\{T_{\tilde{F}ya}^-, T_{\tilde{\Delta}ya}^-\}, \\
I_{xa}^+ &= \min\{I_{\tilde{F}xa}^+, I_{\tilde{\Delta}xa}^+\}, & I_{xa}^- &= \max\{I_{\tilde{F}xa}^-, I_{\tilde{\Delta}xa}^-\}, \\
I_{ya}^+ &= \min\{I_{\tilde{F}ya}^+, I_{\tilde{\Delta}ya}^+\}, & I_{ya}^- &= \max\{I_{\tilde{F}ya}^-, I_{\tilde{\Delta}ya}^-\}, \\
F_{xa}^+ &= \min\{F_{\tilde{F}xa}^+, F_{\tilde{\Delta}xa}^+\}, & F_{xa}^- &= \max\{F_{\tilde{F}xa}^-, F_{\tilde{\Delta}xa}^-\}, \\
F_{ya}^+ &= \min\{F_{\tilde{F}ya}^+, F_{\tilde{\Delta}ya}^+\} \quad \text{and} \quad & F_{ya}^- &= \max\{F_{\tilde{F}ya}^-, F_{\tilde{\Delta}ya}^-\}.
\end{aligned}$$

such that

$$\left( (u, \langle r_{\tilde{F}}^{u+}, r_{\tilde{F}}^{u-} \rangle), (T_{\tilde{F}xa}^+ + iT_{\tilde{F}ya}^+, I_{\tilde{F}xa}^+ + iI_{\tilde{F}ya}^+, F_{\tilde{F}xa}^+ + iF_{\tilde{F}ya}^+, T_{\tilde{F}xa}^- + iT_{\tilde{F}ya}^-, I_{\tilde{F}xa}^- + iI_{\tilde{F}ya}^-, F_{\tilde{F}xa}^- + iF_{\tilde{F}ya}^-) \right) \in \tilde{F}_{\tilde{\Theta}}(a) \text{ and } \left( (u, \langle r_{\tilde{\Delta}}^{u+}, r_{\tilde{\Delta}}^{u-} \rangle), (T_{\tilde{\Delta}xa}^+ + iT_{\tilde{\Delta}ya}^+, I_{\tilde{\Delta}xa}^+ + iI_{\tilde{\Delta}ya}^+, F_{\tilde{\Delta}xa}^+ + iF_{\tilde{\Delta}ya}^+, T_{\tilde{\Delta}xa}^- + iT_{\tilde{\Delta}ya}^-, I_{\tilde{\Delta}xa}^- + iI_{\tilde{\Delta}ya}^-, F_{\tilde{\Delta}xa}^- + iF_{\tilde{\Delta}ya}^-) \right) \in \tilde{\Delta}_{\tilde{\Upsilon}}(a).$$

**Definition 3.14** Let  $(\tilde{F}_{\tilde{\Theta}}, \tilde{A})$  and  $(\tilde{\Delta}_{\tilde{\Upsilon}}, \tilde{B})$  be two BCNN-SSs. Then, we define their restricted intersection as  $(\tilde{F}_{\tilde{\Theta}}, \tilde{A}) \cap_R (\tilde{\Delta}_{\tilde{\Upsilon}}, \tilde{B}) = (\tilde{\Pi}_{\tilde{N}}, \tilde{A} \cap \tilde{B})$ , where  $\tilde{\Theta} = (\tilde{F}, \tilde{A}, N_1)$ ,  $\tilde{\Upsilon} = (\tilde{H}, \tilde{B}, N_2)$  are BN-SSs and  $\tilde{N} = \tilde{\Theta} \cap_R \tilde{\Upsilon} = (\tilde{F} \cap \tilde{H}, \tilde{A} \cap \tilde{B}, \min\{N_1, N_2\})$  such that  $\forall a \in \tilde{A} \cap \tilde{B}$  (with  $\tilde{A} \cap \tilde{B} \neq \phi$ ),  $u \in \mathbb{U}$ ,  $\langle r_a^{u+}, r_a^{u-} \rangle \in R$ , we have that  $\left( (u, \langle r_a^{u+}, r_a^{u-} \rangle), (T_{xa}^+ + iT_{ya}^+, I_{xa}^+ + iI_{ya}^+, F_{xa}^+ + iF_{ya}^+, T_{xa}^- + iT_{ya}^-, I_{xa}^- + iI_{ya}^-, F_{xa}^- + iF_{ya}^-) \right) \in \tilde{\Pi}_{\tilde{N}}(a)$  if and only if  $\left( (u, \langle r_{\tilde{F}}^{u+}, r_{\tilde{F}}^{u-} \rangle), (T_{\tilde{F}xa}^+ + iT_{\tilde{F}ya}^+, I_{\tilde{F}xa}^+ + iI_{\tilde{F}ya}^+, F_{\tilde{F}xa}^+ + iF_{\tilde{F}ya}^+, T_{\tilde{F}xa}^- + iT_{\tilde{F}ya}^-, I_{\tilde{F}xa}^- + iI_{\tilde{F}ya}^-, F_{\tilde{F}xa}^- + iF_{\tilde{F}ya}^-) \right) \in \tilde{F}_{\tilde{\Theta}}(a)$  and  $\left( (u, \langle r_{\tilde{\Delta}}^{u+}, r_{\tilde{\Delta}}^{u-} \rangle), (T_{\tilde{\Delta}xa}^+ + iT_{\tilde{\Delta}ya}^+, I_{\tilde{\Delta}xa}^+ + iI_{\tilde{\Delta}ya}^+, F_{\tilde{\Delta}xa}^+ + iF_{\tilde{\Delta}ya}^+, T_{\tilde{\Delta}xa}^- + iT_{\tilde{\Delta}ya}^-, I_{\tilde{\Delta}xa}^- + iI_{\tilde{\Delta}ya}^-, F_{\tilde{\Delta}xa}^- + iF_{\tilde{\Delta}ya}^-) \right) \in \tilde{\Delta}_{\tilde{\Upsilon}}(a)$

$iI_{\Delta ya}^+, F_{\Delta xa}^+ + iF_{\Delta ya}^+, T_{\Delta xa}^- + iT_{\Delta ya}^-, I_{\Delta xa}^- + iI_{\Delta ya}^-, F_{\Delta xa}^- + iF_{\Delta ya}^-) \in \check{\Delta}_{\check{Y}}(a)$ , where

$$\begin{aligned} r_a^{u+} &= \min\{r_{\check{F}}^{u+}, r_{\check{\Delta}}^{u+}\}, & r_a^{u-} &= \max\{r_{\check{F}}^{u-}, r_{\check{\Delta}}^{u-}\} \\ T_{xa}^+ &= \min\{T_{\check{F}xa}^+, T_{\check{\Delta}xa}^+\}, & T_{xa}^- &= \max\{T_{\check{F}xa}^-, T_{\check{\Delta}xa}^-\}, \\ T_{ya}^+ &= \min\{T_{\check{F}ya}^+, T_{\check{\Delta}ya}^+\}, & T_{ya}^- &= \max\{T_{\check{F}ya}^-, T_{\check{\Delta}ya}^-\}, \\ I_{xa}^+ &= \max\{I_{\check{F}xa}^+, I_{\check{\Delta}xa}^+\}, & I_{xa}^- &= \min\{I_{\check{F}xa}^-, I_{\check{\Delta}xa}^-\}, \\ I_{ya}^+ &= \max\{I_{\check{F}ya}^+, I_{\check{\Delta}ya}^+\}, & I_{ya}^- &= \min\{I_{\check{F}ya}^-, I_{\check{\Delta}ya}^-\}, \\ F_{xa}^+ &= \max\{F_{\check{F}xa}^+, F_{\check{\Delta}xa}^+\}, & F_{xa}^- &= \min\{F_{\check{F}xa}^-, F_{\check{\Delta}xa}^-\}, \\ F_{ya}^+ &= \max\{F_{\check{F}ya}^+, F_{\check{\Delta}ya}^+\} \quad \text{and} \quad & F_{ya}^- &= \min\{F_{\check{F}ya}^-, F_{\check{\Delta}ya}^-\}. \end{aligned}$$

**Definition 3.15** Let  $(\check{I}_{\check{\Theta}}, \check{A})$  and  $(\check{\Delta}_{\check{Y}}, \check{B})$  be two BCNN-SSs. Then, we define their extended intersection as  $(\check{I}_{\check{\Theta}}, \check{A}) \cap_E (\check{\Delta}_{\check{Y}}, \check{B}) = (\check{\Pi}_{\check{N}}, \check{A} \cup \check{B})$ , where  $\check{\Theta} = (\check{F}, \check{A}, N_1)$ ,  $\check{Y} = (\check{H}, \check{B}, N_2)$  are BN-SSs, and  $\check{N} = \check{\Theta} \cap_E \check{Y} = (\check{F} \cap \check{H}, \check{A} \cup \check{B}, \min\{N_1, N_2\})$  such that  $\forall a \in \check{A} \cup \check{B}$  (with  $\check{A} \cap \check{B} \neq \phi$ ),  $u \in \mathbb{U}$ ,  $\langle r_a^{u+}, r_a^{u-} \rangle \in R$ , we have that  $\left( (u, \langle r_a^{u+}, r_a^{u-} \rangle), (T_{xa}^+ + iT_{ya}^+, I_{xa}^+ + iI_{ya}^+, F_{xa}^+ + iF_{ya}^+, T_{xa}^- + iT_{ya}^-, I_{xa}^- + iI_{ya}^-, F_{xa}^- + iF_{ya}^-) \right) \in \check{\Pi}_{\check{N}}(a)$  if and only if

$$\check{\Pi}_{\check{N}}(a) = \begin{cases} \check{I}_{\check{\Theta}}(a) & \text{if } a \in \check{A} - \check{B} \\ \check{\Delta}_{\check{Y}}(a) & \text{if } a \in \check{B} - \check{A} \\ \left( (u, \langle r_a^{u+}, r_a^{u-} \rangle), (T_{xa}^+ + iT_{ya}^+, I_{xa}^+ + iI_{ya}^+, \right. \\ \left. F_{xa}^+ + iF_{ya}^+, T_{xa}^- + iT_{ya}^-, I_{xa}^- + iI_{ya}^-, F_{xa}^- + iF_{ya}^-) \right) & \text{if } a \in \check{A} \cap \check{B}, \end{cases}$$

where

$$\begin{aligned} r_a^{u+} &= \min\{r_{\check{F}}^{u+}, r_{\check{\Delta}}^{u+}\}, & r_a^{u-} &= \max\{r_{\check{F}}^{u-}, r_{\check{\Delta}}^{u-}\}, \\ T_{xa}^+ &= \min\{T_{\check{F}xa}^+, T_{\check{\Delta}xa}^+\}, & T_{xa}^- &= \max\{T_{\check{F}xa}^-, T_{\check{\Delta}xa}^-\}, \\ T_{ya}^+ &= \min\{T_{\check{F}ya}^+, T_{\check{\Delta}ya}^+\}, & T_{ya}^- &= \max\{T_{\check{F}ya}^-, T_{\check{\Delta}ya}^-\}, \\ I_{xa}^+ &= \max\{I_{\check{F}xa}^+, I_{\check{\Delta}xa}^+\}, & I_{xa}^- &= \min\{I_{\check{F}xa}^-, I_{\check{\Delta}xa}^-\}, \\ I_{ya}^+ &= \max\{I_{\check{F}ya}^+, I_{\check{\Delta}ya}^+\}, & I_{ya}^- &= \min\{I_{\check{F}ya}^-, I_{\check{\Delta}ya}^-\}, \\ F_{xa}^+ &= \max\{F_{\check{F}xa}^+, F_{\check{\Delta}xa}^+\}, & F_{xa}^- &= \min\{F_{\check{F}xa}^-, F_{\check{\Delta}xa}^-\}, \\ F_{ya}^+ &= \max\{F_{\check{F}ya}^+, F_{\check{\Delta}ya}^+\} \quad \text{and} \quad & F_{ya}^- &= \min\{F_{\check{F}ya}^-, F_{\check{\Delta}ya}^-\}. \end{aligned}$$

such that

$\left( (u, \langle r_{\check{F}}^{u+}, r_{\check{F}}^{u-} \rangle), (T_{\check{F}xa}^+ + iT_{\check{F}ya}^+, I_{\check{F}xa}^+ + iI_{\check{F}ya}^+, F_{\check{F}xa}^+ + iF_{\check{F}ya}^+, T_{\check{F}xa}^- + iT_{\check{F}ya}^-, I_{\check{F}xa}^- + iI_{\check{F}ya}^-, F_{\check{F}xa}^- + iF_{\check{F}ya}^-) \right) \in \check{I}_{\check{\Theta}}(a)$   
and  $\left( (u, \langle r_{\check{\Delta}}^{u+}, r_{\check{\Delta}}^{u-} \rangle), (T_{\check{\Delta}xa}^+ + iT_{\check{\Delta}ya}^+, I_{\check{\Delta}xa}^+ + iI_{\check{\Delta}ya}^+, F_{\check{\Delta}xa}^+ + iF_{\check{\Delta}ya}^+, T_{\check{\Delta}xa}^- + iT_{\check{\Delta}ya}^-, I_{\check{\Delta}xa}^- + iI_{\check{\Delta}ya}^-, F_{\check{\Delta}xa}^- + iF_{\check{\Delta}ya}^-) \right) \in \check{\Delta}_{\check{Y}}(a)$ .

**Example 3.10** Let  $(\check{I}_{\check{\Theta}}, \check{A})$  and  $(\check{\Delta}_{\check{Y}}, \check{B})$  be the BCNN-SSs given in Example 3.4 on  $\mathbb{U}$ . Then their restricted union, extended union, restricted intersection, and extended intersection are given in the Tables 19, 20, 21, and 22, respectively, below:

Table 19: Restricted Union of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_R (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ 

$\sqcup_R$	$e_1$	$e_2$
$u_1$	$\left( \langle 4, -1 \rangle, \left( \begin{array}{ccc} 1 + i1, & 0.1 + i0.1, & 0.1 + i0.1, \\ -0.3 - i0.3, & -0.7 - i0.7, & -0.75 - i0.7 \end{array} \right) \right)$	$\left( \langle 3, -4 \rangle, \left( \begin{array}{ccc} 0.79 + i0.79, & 0.39 + i0.39, & 0.39 + i0.39, \\ -1 - i1, & -0.1 - i0.1, & -0.1 - i0.1 \end{array} \right) \right)$
$u_2$	$\left( \langle 4, -3 \rangle, \left( \begin{array}{ccc} 1 + i0.95, & 0.1 + i0.1, & 0.1 + i0.15, \\ -0.7 - i0.7, & -0.3 - i0.3, & -0.3 - i0.3 \end{array} \right) \right)$	$\left( \langle 3, -2 \rangle, \left( \begin{array}{ccc} 0.7 + i0.7, & 0.3 + i0.3, & 0.3 + i0.3, \\ -0.5 - i0.5, & -0.4 - i0.4, & -0.5 - i0.5 \end{array} \right) \right)$

Table 20: Extended Union of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_E (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ 

$\sqcup_E$	$e_1$	$e_2$	$e_3$
$u_1$	$\left( \langle 4, -1 \rangle, \left( \begin{array}{ccc} 1 + i1, & 0.1 + i0.1, & 0.1 + i0.1, \\ -0.3 - i0.3, & -0.7 - i0.7, & -0.75 - i0.7 \end{array} \right) \right)$	$\left( \langle 3, -4 \rangle, \left( \begin{array}{ccc} 0.79 + i0.79, & 0.39 + i0.39, & 0.39 + i0.39, \\ -1 - i1, & -0.1 - i0.1, & -0.1 - i0.1 \end{array} \right) \right)$	$\left( \langle 4, -1 \rangle, \left( \begin{array}{ccc} 0.8 + i0.8, & 0.1 + i0.1, & 0.1 + i0.1, \\ -0.2 - i0.2, & -0.6 - i0.6, & -0.6 - i0.6 \end{array} \right) \right)$
$u_2$	$\left( \langle 4, -3 \rangle, \left( \begin{array}{ccc} 1 + i1, & 0.1 + i0.1, & 0.1 + i0.15, \\ -0.7 - i0.7, & -0.3 - i0.3, & -0.3 - i0.3 \end{array} \right) \right)$	$\left( \langle 3, -2 \rangle, \left( \begin{array}{ccc} 0.7 + i0.7, & 0.3 + i0.3, & 0.3 + i0.3, \\ -0.5 - i0.5, & -0.4 - i0.4, & -0.5 - i0.5 \end{array} \right) \right)$	$\left( \langle 3, -2 \rangle, \left( \begin{array}{ccc} 0.6 + i0.6, & 0.2 + i0.3, & 0.3 + i0.3, \\ -0.4 - i0.4, & -0.5 - i0.5, & -0.5 - i0.5 \end{array} \right) \right)$

Table 21: Restricted Intersection of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_R (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ 

$\sqcap_R$	$e_1$	$e_2$
$u_1$	$\left( \langle 3, -1 \rangle, \left( \begin{array}{ccc} 0.7 + i0.7, & 0.21 + i0.3, & 0.3 + i0.3, \\ -0.3 - i0.2, & -0.75 - i0.75, & -0.79 - i0.75 \end{array} \right) \right)$	$\left( \langle 2, -3 \rangle, \left( \begin{array}{ccc} 0.55 + i0.55, & 0.59 + i0.59, & 0.59 + i0.59, \\ -0.6 - i0.6, & -0.3 - i0.3, & -0.3 - i0.2 \end{array} \right) \right)$
$u_2$	$\left( \langle 4, -2 \rangle, \left( \begin{array}{ccc} 0.9 + i0.95, & 0.15 + i0.19, & 0.15 + i0.19, \\ -0.5 - i0.5, & -0.4 - i0.4, & -0.4 - i0.5 \end{array} \right) \right)$	$\left( \langle 2, -1 \rangle, \left( \begin{array}{ccc} 0.4 + i0.59, & 0.5 + i0.59, & 0.59 + i0.59, \\ -0.3 - i0.3, & -0.6 - i0.6, & -0.7 - i0.7 \end{array} \right) \right)$

Table 22: Extended Intersection of  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_E (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ 

$\sqcap_E$	$e_1$	$e_2$	$e_3$
$u_1$	$\left( \langle 3, -1 \rangle, \left( \begin{array}{ccc} 0.7 + i0.7, & 0.21 + i0.3, & 0.3 + i0.3, \\ -0.3 - i0.2, & -0.75 - i0.75, & -0.79 - i0.75 \end{array} \right) \right)$	$\left( \langle 2, -3 \rangle, \left( \begin{array}{ccc} 0.55 + i0.55, & 0.59 + i0.59, & 0.59 + i0.59, \\ -0.6 - i0.6, & -0.3 - i0.3, & -0.3 - i0.2 \end{array} \right) \right)$	$\left( \langle 4, -1 \rangle, \left( \begin{array}{ccc} 0.8 + i0.8, & 0.1 + i0.1, & 0.1 + i0.1, \\ -0.2 - i0.2, & -0.6 - i0.6, & -0.6 - i0.6 \end{array} \right) \right)$
$u_2$	$\left( \langle 4, -2 \rangle, \left( \begin{array}{ccc} 0.9 + i0.95, & 0.15 + i0.19, & 0.15 + i0.19, \\ -0.5 - i0.5, & -0.4 - i0.4, & -0.4 - i0.5 \end{array} \right) \right)$	$\left( \langle 2, -1 \rangle, \left( \begin{array}{ccc} 0.4 + i0.59, & 0.5 + i0.59, & 0.59 + i0.59, \\ -0.3 - i0.3, & -0.6 - i0.6, & -0.7 - i0.7 \end{array} \right) \right)$	$\left( \langle 3, -2 \rangle, \left( \begin{array}{ccc} 0.6 + i0.6, & 0.2 + i0.3, & 0.3 + i0.3, \\ -0.4 - i0.4, & -0.5 - i0.5, & -0.5 - i0.5 \end{array} \right) \right)$

**Proposition 3.6** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ ,  $(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ , and  $(\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})$  be three BCNN-SSs, then:

1.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_E (\ddot{I}_{\ddot{\Theta}}, \ddot{A}) = (\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ .
2.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_E (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B}) = (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B}) \sqcup_E (\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ .
3.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_E (\ddot{I}_{\ddot{\Theta}}, \ddot{A}) = (\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ .
4.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_E (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B}) = (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B}) \sqcap_E (\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ .
5.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_E (\ddot{I}_{\ddot{\Theta}}^\phi, \ddot{A}) = (\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ .
6.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_E (\ddot{I}_{\ddot{\Theta}}^\phi, \ddot{A}) = (\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ .
7.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_E [(\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B}) \sqcup_E (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})] = [(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_E (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})] \sqcup_E (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})$ .

8.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_R [(\ddot{\Delta}_{\ddot{Y}}, \ddot{B}) \sqcap_R (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})] = [(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_R (\ddot{\Delta}_{\ddot{Y}}, \ddot{B})] \sqcap_R (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})$ .
9.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_E [(\ddot{\Delta}_{\ddot{Y}}, \ddot{B}) \sqcap_R (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})] = [(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_E (\ddot{\Delta}_{\ddot{Y}}, \ddot{B})] \sqcap_R [(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcup_E (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})]$ .
10.  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_R [(\ddot{\Delta}_{\ddot{Y}}, \ddot{B}) \sqcup_E (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})] = [(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_R (\ddot{\Delta}_{\ddot{Y}}, \ddot{B})] \sqcup_E [(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \sqcap_R (\ddot{\Omega}_{\ddot{\Lambda}}, \ddot{C})]$ .

**Proof:** Clearly, we can prove them by Definitions 3.12, 3.13, 3.14, and 3.15.  $\square$

**Definition 3.16** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  and  $(\ddot{\Delta}_{\ddot{Y}}, \ddot{B})$  be two BCNN-SSs. Then, we define their OR-operation as  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \vee (\ddot{\Delta}_{\ddot{Y}}, \ddot{B}) = ((\ddot{I} \vee \ddot{\Delta})_{\ddot{\Theta} \vee \ddot{Y}}, \ddot{A} \times \ddot{B})$ , where  $\ddot{\Theta} \vee \ddot{Y} = (\ddot{F} \vee \ddot{H}, \ddot{A} \times \ddot{B}, \max\{N_1, N_2\})$ , such that

$$\begin{aligned}
& ((\ddot{I} \vee \ddot{\Delta})_{\ddot{\Theta} \vee \ddot{Y}}, \ddot{A} \times \ddot{B}) = \\
& \left\{ \left\langle (e_i, e_j), \left( \ddot{F}(e_i, e_j), \ddot{Y}(e_i, e_j) \right) : (e_i, e_j) \in \ddot{A} \times \ddot{B}, (\ddot{F}(e_i, e_j), \ddot{Y}(e_i, e_j)) \in 2^{\mathbb{U} \times R} \times \check{B}\check{C}\check{N}\check{N} \right\rangle \right. \\
& = \left\{ \left\langle (e_i, e_j), \left( (u_i, \langle \max(r_{(e_i)}^{u_i+}, r_{(e_j)}^{u_i+}), \min(r_{(e_i)}^{u_i-}, r_{(e_j)}^{u_i-}) \rangle), (T_{(e_i, e_j)}^+, I_{(e_i, e_j)}^+, F_{(e_i, e_j)}^+, T_{(e_i, e_j)}^-, I_{(e_i, e_j)}^-, F_{(e_i, e_j)}^-) \right) \right\rangle \right. \\
& \quad : (e_i, e_j) \in \ddot{A} \times \ddot{B}, u_i \in \mathbb{U}, \langle r_{e_i}^{u_i+}, r_{e_i}^{u_i-} \rangle \in R_{\ddot{F}}, \langle r_{e_j}^{u_i+}, r_{e_j}^{u_i-} \rangle \in R_{\ddot{\Delta}} \left. \right\} \\
& = \left\{ \left\langle (e_i, e_j), \left( (u_i, \langle \max(r_{\ddot{F}(e_i)}^{u_i+}, r_{\ddot{\Delta}(e_j)}^{u_i+}), \min(r_{\ddot{F}(e_i)}^{u_i-}, r_{\ddot{\Delta}(e_j)}^{u_i-}) \rangle), (T_{x(e_i, e_j)}^+ + iT_{y(e_i, e_j)}^+, I_{x(e_i, e_j)}^+ + iI_{y(e_i, e_j)}^+, \right. \right. \\
& \quad \left. \left. F_{x(e_i, e_j)}^+ + iF_{y(e_i, e_j)}^+, T_{x(e_i, e_j)}^- + iT_{y(e_i, e_j)}^-, I_{x(e_i, e_j)}^- + iI_{y(e_i, e_j)}^-, F_{x(e_i, e_j)}^- + iF_{y(e_i, e_j)}^-) \right) \right\rangle : (e_i, e_j) \in \ddot{A} \times \ddot{B}, \\
& \quad u \in \mathbb{U}, \langle r_{e_i}^{u_i+}, r_{e_i}^{u_i-} \rangle \in R_{\ddot{F}}, \langle r_{e_j}^{u_i+}, r_{e_j}^{u_i-} \rangle \in R_{\ddot{\Delta}} \left. \right\},
\end{aligned}$$

where  $\ddot{\Theta} \vee \ddot{Y} : \ddot{A} \times \ddot{B} \rightarrow 2^{\mathbb{U} \times R}$ ,  $\ddot{Y} : \ddot{A} \times \ddot{B} \rightarrow \check{B}\check{C}\check{N}\check{N}$  such that

$$\begin{aligned}
T_{x(e_i, e_j)}^+(u_i) &= \max(T_{x(e_i)}^+(u_i), T_{x(e_j)}^+(u_j)), \\
T_{y(e_i, e_j)}^+(u_i) &= \max(T_{y(e_i)}^+(u_i), T_{y(e_j)}^+(u_j)), \\
I_{x(e_i, e_j)}^+(u_i) &= \left( \frac{I_{x(e_i)}^+(u_i) + I_{x(e_j)}^+(u_j)}{2} \right), \\
I_{y(e_i, e_j)}^+(u_i) &= \left( \frac{I_{y(e_i)}^+(u_i) + I_{y(e_j)}^+(u_j)}{2} \right), \\
F_{x(e_i, e_j)}^+(u_i) &= \min(F_{x(e_i)}^+(u_i), F_{x(e_j)}^+(u_j)), \\
F_{y(e_i, e_j)}^+(u_i) &= \min(F_{y(e_i)}^+(u_i), F_{y(e_j)}^+(u_j)), \\
T_{x(e_i, e_j)}^-(u_i) &= \min(T_{x(e_i)}^-(u_i), T_{x(e_j)}^-(u_j)), \\
T_{y(e_i, e_j)}^-(u_i) &= \min(T_{y(e_i)}^-(u_i), T_{y(e_j)}^-(u_j)), \\
I_{x(e_i, e_j)}^-(u_i) &= \left( \frac{I_{x(e_i)}^-(u_i) + I_{x(e_j)}^-(u_j)}{2} \right), \\
I_{y(e_i, e_j)}^-(u_i) &= \left( \frac{I_{y(e_i)}^-(u_i) + I_{y(e_j)}^-(u_j)}{2} \right), \\
F_{x(e_i, e_j)}^-(u_i) &= \max(F_{x(e_i)}^-(u_i), F_{x(e_j)}^-(u_j)), \text{ and} \\
F_{y(e_i, e_j)}^-(u_i) &= \max(F_{y(e_i)}^-(u_i), F_{y(e_j)}^-(u_j)).
\end{aligned}$$

**Definition 3.17** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  and  $(\ddot{\Delta}_{\ddot{Y}}, \ddot{B})$  be two BCNN-SSs. Then, we define their AND-operation as  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \wedge (\ddot{\Delta}_{\ddot{Y}}, \ddot{B}) = ((\ddot{I} \wedge \ddot{\Delta})_{\ddot{\Theta} \wedge \ddot{Y}}, \ddot{A} \times \ddot{B})$ ,  $\ddot{\Theta} \wedge \ddot{Y} = (\ddot{F} \wedge \ddot{H}, \ddot{A} \times \ddot{B}, \min(N_1, N_2))$ , such that

$$\begin{aligned}
& ((\ddot{F} \wedge \ddot{\Delta})_{\ddot{\Theta} \wedge \ddot{Y}}, \ddot{A} \times \ddot{B}) = \\
& \left\{ \left\langle (e_i, e_j), \left( \ddot{F}(e_i, e_j), \ddot{Y}(e_i, e_j) \right) \right\rangle : (e_i, e_j) \in \ddot{A} \times \ddot{B}, (\ddot{F}(e_i, e_j), \ddot{Y}(e_i, e_j)) \in 2^{\mathbb{U} \times R} \times \check{B}\check{C}\check{N}\check{N} \right\} \\
& = \left\{ \left\langle (e_i, e_j), \left( (u_i, \langle \min(r_{\ddot{F}(e_i)}^{u_i+}, r_{\ddot{F}(e_j)}^{u_i+}), \max(r_{\ddot{F}(e_i)}^{u_i-}, r_{\ddot{F}(e_j)}^{u_i-}) \rangle), (\mathbb{T}_{(e_i, e_j)}^+, \mathbb{I}_{(e_i, e_j)}^+, \mathbb{F}_{(e_i, e_j)}^+, \mathbb{T}_{(e_i, e_j)}^-, \mathbb{I}_{(e_i, e_j)}^-, \mathbb{F}_{(e_i, e_j)}^-) \right) \right\rangle \right. \\
& \quad \left. : (e_i, e_j) \in \ddot{A} \times \ddot{B}, u_i \in \mathbb{U}, \langle r_{e_i}^{u_i+}, r_{e_i}^{u_i-} \rangle \in R_{\ddot{F}}, \langle r_{e_j}^{u_i+}, r_{e_j}^{u_i-} \rangle \in R_{\ddot{\Delta}} \right\} \\
& = \left\{ \left\langle (e_i, e_j), \left( (u_i, \langle \min(r_{\ddot{F}(e_i)}^{u_i+}, r_{\ddot{\Delta}(e_j)}^{u_i+}), \max(r_{\ddot{F}(e_i)}^{u_i-}, r_{\ddot{\Delta}(e_j)}^{u_i-}) \rangle), (T_{x(e_i, e_j)}^+ + iT_{y(e_i, e_j)}^+, I_{x(e_i, e_j)}^+ + iI_{y(e_i, e_j)}^+, \right. \right. \\
& \quad \left. \left. F_{x(e_i, e_j)}^+ + iF_{y(e_i, e_j)}^+, T_{x(e_i, e_j)}^- + iT_{y(e_i, e_j)}^-, I_{x(e_i, e_j)}^- + iI_{y(e_i, e_j)}^-, F_{x(e_i, e_j)}^- + iF_{y(e_i, e_j)}^-) \right) \right\rangle : (e_i, e_j) \in \ddot{A} \times \ddot{B}, \\
& \quad u \in \mathbb{U}, \langle r_{e_i}^{u_i+}, r_{e_i}^{u_i-} \rangle \in R_{\ddot{F}}, \langle r_{e_j}^{u_i+}, r_{e_j}^{u_i-} \rangle \in R_{\ddot{\Delta}} \right\},
\end{aligned}$$

where  $\ddot{\Theta} \vee \ddot{Y} : \ddot{A} \times \ddot{B} \rightarrow 2^{\mathbb{U} \times R}$ ,  $\ddot{Y} : \ddot{A} \times \ddot{B} \rightarrow \check{B}\check{C}\check{N}\check{N}$  such that:

$$\begin{aligned}
T_{x(e_i, e_j)}^+(u_i) &= \min(T_{x(e_i)}^+(u_i), T_{x(e_j)}^+(u_j)) \\
T_{y(e_i, e_j)}^+(u_i) &= \min(T_{y(e_i)}^+(u_i), T_{y(e_j)}^+(u_j)) \\
I_{x(e_i, e_j)}^+(u_i) &= \left( \frac{I_{x(e_i)}^+(u_i) + I_{x(e_j)}^+(u_j)}{2} \right) \\
I_{y(e_i, e_j)}^+(u_i) &= \left( \frac{I_{y(e_i)}^+(u_i) + I_{y(e_j)}^+(u_j)}{2} \right) \\
F_{x(e_i, e_j)}^+(u_i) &= \max(F_{x(e_i)}^+(u_i), F_{x(e_j)}^+(u_j)) \\
F_{y(e_i, e_j)}^+(u_i) &= \max(F_{y(e_i)}^+(u_i), F_{y(e_j)}^+(u_j)) \\
T_{x(e_i, e_j)}^-(u_i) &= \max(T_{x(e_i)}^-(u_i), T_{x(e_j)}^-(u_j)) \\
T_{y(e_i, e_j)}^-(u_i) &= \max(T_{y(e_i)}^-(u_i), T_{y(e_j)}^-(u_j)) \\
I_{x(e_i, e_j)}^-(u_i) &= \left( \frac{I_{x(e_i)}^-(u_i) + I_{x(e_j)}^-(u_j)}{2} \right) \\
I_{y(e_i, e_j)}^-(u_i) &= \left( \frac{I_{y(e_i)}^-(u_i) + I_{y(e_j)}^-(u_j)}{2} \right) \\
F_{x(e_i, e_j)}^-(u_i) &= \min(F_{x(e_i)}^-(u_i), F_{x(e_j)}^-(u_j)) \text{ and} \\
F_{y(e_i, e_j)}^-(u_i) &= \min(F_{y(e_i)}^-(u_i), F_{y(e_j)}^-(u_j)).
\end{aligned}$$

**Example 3.11** Let  $(\ddot{F}_{\ddot{\Theta}}, \ddot{A})$  and  $(\ddot{\Delta}_{\ddot{Y}}, \ddot{B})$  be the BCNN-SSs as in Example 3.4 on  $\mathbb{U}$ . Then their OR-operation and AND-operation are given below:

Table 23: OR-Operation  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \vee (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ 

$\mathbb{U}/\ddot{A}$	$(e_1, e_1)$	$(e_1, e_2)$	$(e_1, e_3)$
$u_1$	$(4, -1), \left( \begin{array}{c} 1+i, 0.15+i0.2, 0.1+i0.1, \\ -0.3-i0.3, -0.72-i0.72, -0.75-i0.7 \end{array} \right)$	$(3, -4), \left( \begin{array}{c} 0.79+i0.79, 0.3+i0.34, 0.3+i0.3, \\ -1-i, -0.42-i0.42, -0.1-i0.1 \end{array} \right)$	$(4, -1), \left( \begin{array}{c} 0.8+i0.8, 0.15+i0.2, 0.1+i0.1, \\ -0.3-i0.2, -0.67-i0.67, -0.6-i0.6 \end{array} \right)$
$u_2$	$(4, -3), \left( \begin{array}{c} 1+i, 0.12+i0.17, 0.1+i0.1, \\ -0.7-i0.7, -0.35-i0.35, -0.3-i0.3 \end{array} \right)$	$(4, -2), \left( \begin{array}{c} 0.9+i0.95, 0.22+i0.22, 0.15+i0.15, \\ -0.5-i0.55, -0.4-i0.4, -0.4-i0.5 \end{array} \right)$	$(4, -2), \left( \begin{array}{c} 0.9+i0.95, 0.17+i0.24, 0.15+i0.15, \\ -0.5-i0.55, -0.45-i0.45, -0.4-i0.5 \end{array} \right)$
$\mathbb{U}/\ddot{A}$	$(e_2, e_1)$	$(e_2, e_2)$	$(e_2, e_3)$
$u_1$	$(4, -3), \left( \begin{array}{c} 1+i, 0.34+i0.34, 0.1+i0.1, \\ -0.6-i0.6, -0.5-i0.5, -0.3-i0.2 \end{array} \right)$	$(3, -4), \left( \begin{array}{c} 0.79+i0.79, 0.49+i0.49, 0.39+i0.39, \\ -1-i, -0.2-i0.2, -0.1-i0.1 \end{array} \right)$	$(4, -3), \left( \begin{array}{c} 0.8+i0.8, 0.34+i0.34, 0.1+i0.1, \\ -0.6-i0.6, -0.45-i0.45, -0.3-i0.2 \end{array} \right)$
$u_2$	$(4, -3), \left( \begin{array}{c} 1+i, 0.3+i0.36, 0.1+i0.1, \\ -0.7-i0.7, -0.45-i0.45, -0.3-i0.3 \end{array} \right)$	$(3, -2), \left( \begin{array}{c} 0.7+i0.7, 0.4+i0.44, 0.3+i0.3, \\ -0.5-i0.5, -0.5-i0.5, -0.5-i0.5 \end{array} \right)$	$(3, -2), \left( \begin{array}{c} 0.6+i0.6, 0.35+i0.44, 0.3+i0.3, \\ -0.4-i0.4, -0.55-i0.55, -0.5-i0.5 \end{array} \right)$

Table 24: AND-Operation  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A}) \wedge (\ddot{\Delta}_{\ddot{\Upsilon}}, \ddot{B})$ 

$\mathbb{U}/\ddot{A}$	$(e_1, e_1)$	$(e_1, e_2)$	$(e_1, e_3)$
$u_1$	$(3, -1), \left( \begin{array}{c} 0.7+i0.7, 0.15+i0.2, 0.3+i0.3, \\ -0.3-i0.2, -0.72-i0.72, -0.79-i0.75 \end{array} \right)$	$(3, -1), \left( \begin{array}{c} 0.7+i0.7, 0.3+i0.34, 0.39+i0.39, \\ -0.3-i0.2, -0.42-i0.42, -0.79-i0.75 \end{array} \right)$	$(3, -1), \left( \begin{array}{c} 0.7+i0.7, 0.15+i0.2, 0.3+i0.3, \\ -0.2-i0.2, -0.65-i0.65, -0.79-i0.75 \end{array} \right)$
$u_2$	$(4, -2), \left( \begin{array}{c} 0.9+i0.95, 0.12+i0.17, 0.15+i0.15, \\ -0.5-i0.55, -0.35-i0.35, -0.4-i0.5 \end{array} \right)$	$(3, -2), \left( \begin{array}{c} 0.7+i0.7, 0.22+i0.24, 0.3+i0.3, \\ -0.5-i0.5, -0.4-i0.4, -0.5-i0.5 \end{array} \right)$	$(3, -2), \left( \begin{array}{c} 0.6+i0.6, 0.17+i0.24, 0.3+i0.3, \\ -0.4-i0.4, -0.45-i0.45, -0.5-i0.5 \end{array} \right)$
$\mathbb{U}/\ddot{A}$	$(e_2, e_1)$	$(e_2, e_2)$	$(e_2, e_3)$
$u_1$	$(2, -1), \left( \begin{array}{c} 0.55+i0.55, 0.34+i0.34, 0.59+i0.59, \\ -0.3-i0.3, -0.5-i0.5, -0.75-i0.7 \end{array} \right)$	$(2, -3), \left( \begin{array}{c} 0.55+i0.55, 0.49+i0.49, 0.59+i0.59, \\ -0.6-i0.6, -0.2-i0.2, -0.3-i0.2 \end{array} \right)$	$(2, -1), \left( \begin{array}{c} 0.55+i0.55, 0.34+i0.34, 0.59+i0.59, \\ -0.2-i0.2, -0.45-i0.45, -0.6-i0.6 \end{array} \right)$
$u_2$	$(2, -1), \left( \begin{array}{c} 0.4+i0.59, 0.3+i0.37, 0.59+i0.59, \\ -0.3-i0.3, -0.45-i0.45, -0.7-i0.7 \end{array} \right)$	$(2, -1), \left( \begin{array}{c} 0.4+i0.59, 0.4+i0.44, 0.59+i0.59, \\ -0.3-i0.3, -0.5-i0.5, -0.7-i0.7 \end{array} \right)$	$(2, -1), \left( \begin{array}{c} 0.4+i0.59, 0.35+i0.44, 0.59+i0.59, \\ -0.3-i0.3, -0.55-i0.55, -0.7-i0.7 \end{array} \right)$

#### 4. Applications

We want to solve a problem related to choosing the best alternatives under consideration. To reach our aim, we present the column comparison matrix, row comparison matrix, and the score function definitions. Therefore, modified algorithms are introduced based on BCNN-SS. These algorithms are an adaptation of the related concepts presented in CBNSS and CFN-SS, respectively. A numerical example is presented for the decision-making process. Thus, we introduce the following definitions:

**Definition 4.1** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS on  $\mathbb{U}$ . Then the column comparison matrix have the rows which are composed of alternatives  $\mathbb{U} = \{u_1, u_2, \dots, u_m\}$  and it has columns highlight the parameters  $\ddot{A} = \{e_1, e_2, \dots, e_n\}$ . The entries are defined,  $m_{ij}$ , for all  $u_i, u_k \in \mathbb{U}$  so that  $i \neq k$  as:

$$m_{ij} = r_{ij}^+ + (r_{T_x^+} + r_{I_x^+} - r_{F_x^+}) + (r_{T_y^+} + r_{I_y^+} - r_{F_y^+}) - r_{ij}^- - (r_{T_x^-} + r_{I_x^-} - r_{F_x^-}) - (r_{T_y^-} + r_{I_y^-} - r_{F_y^-}),$$

where

$$\begin{aligned} r_{ij}^+ &= \text{the number of times that } r_{e_j}^+(u_i) \geq r_{e_j}^+(u_k), \\ r_{T_x^+} &= \text{the number of times that } T_{xe_j}^+(u_i) \geq T_{xe_j}^+(u_k), \\ r_{I_x^+} &= \text{the number of times that } I_{xe_j}^+(u_i) \geq I_{xe_j}^+(u_k), \\ r_{F_x^+} &= \text{the number of times that } F_{xe_j}^+(u_i) \geq F_{xe_j}^+(u_k), \\ r_{T_y^+} &= \text{the number of times that } T_{ye_j}^+(u_i) \geq T_{ye_j}^+(u_k), \\ r_{I_y^+} &= \text{the number of times that } I_{ye_j}^+(u_i) \geq I_{ye_j}^+(u_k), \\ r_{F_y^+} &= \text{the number of times that } F_{ye_j}^+(u_i) \geq F_{ye_j}^+(u_k), \end{aligned}$$

$$\begin{aligned}
r_{ij}^- &= \text{the number of times that } : r_{e_j}^-(u_i) \geq r_{e_j}^-(u_k), \\
r_{T_x^-} &= \text{the number of times that } T_{x_{e_j}}^-(u_i) \geq T_{x_{e_j}}^-(u_k), \\
r_{I_x^-} &= \text{the number of times that } I_{x_{e_j}}^-(u_i) \geq I_{x_{e_j}}^-(u_k), \\
r_{F_x^-} &= \text{the number of times that } F_{x_{e_j}}^-(u_i) \geq F_{x_{e_j}}^-(u_k), \\
r_{T_y^-} &= \text{the number of times that } T_{y_{e_j}}^-(u_i) \geq T_{y_{e_j}}^-(u_k), \\
r_{I_y^-} &= \text{the number of times that } I_{y_{e_j}}^-(u_i) \geq I_{y_{e_j}}^-(u_k), \\
r_{F_y^-} &= \text{the number of times that } F_{y_{e_j}}^-(u_i) \geq F_{y_{e_j}}^-(u_k).
\end{aligned}$$

**Definition 4.2** A squared matrix is called the row comparison matrix such that both its rows and columns are written by the name of alternatives of the set  $\mathbb{U} = \{u_1, u_2, \dots, u_m\}$  and the entries are defined by

$$d_{ij} = (a_{ij}^+ + b_{ij}^+ + c_{ij}^+) - (a_{ij}^- + b_{ij}^- + c_{ij}^-),$$

where

$$\begin{aligned}
a_{ij}^+ & \text{ is the number of parameters that } T_{x_{e_i}}^+(u_i) \geq T_{x_{e_j}}^+(u_j), \\
b_{ij}^+ & \text{ is the number of parameters that } I_{x_{e_i}}^+(u_i) \leq I_{x_{e_j}}^+(u_j), \\
c_{ij}^+ & \text{ is the number of parameters that } F_{x_{e_i}}^+(u_i) \leq F_{x_{e_j}}^+(u_j), \\
a_{ij}^- & \text{ is the number of parameters that } T_{y_{e_i}}^-(u_i) \geq T_{y_{e_j}}^-(u_j), \\
b_{ij}^- & \text{ is the number of parameters that } I_{y_{e_i}}^-(u_i) \leq I_{y_{e_j}}^-(u_j) \text{ and} \\
c_{ij}^- & \text{ is the number of parameters that } F_{y_{e_i}}^-(u_i) \leq F_{y_{e_j}}^-(u_j).
\end{aligned}$$

**Definition 4.3** Let  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  be a BCNN-SS on  $\mathbb{U}$ . Then the score function  $\alpha_i$  of elements  $u_i$  is defined as:

$$\alpha_i = \sum_{j=1} m_{ij}.$$

Now the algorithm for decision-making in BCNN-SS theory is illustrated as follows:

### Algorithm 1: For Column Comparison Matrix

**Step:1:** Input the BCNN-SS  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$ .

**Step:2:** Construct the matrix for  $(\ddot{I}_{\ddot{\Theta}}, \ddot{A})$  by Definition 3.1.

**Step:3:** Compute the column comparison matrix Definition 4.1.

**Step:4:** Calculate the score functions for all alternatives in  $\mathbb{U}$  by Definition 4.3.

**Step:5:** Choose the maximum score of the alternative as optimal solution.  
If there are many maximum scores, then any one can be chosen.

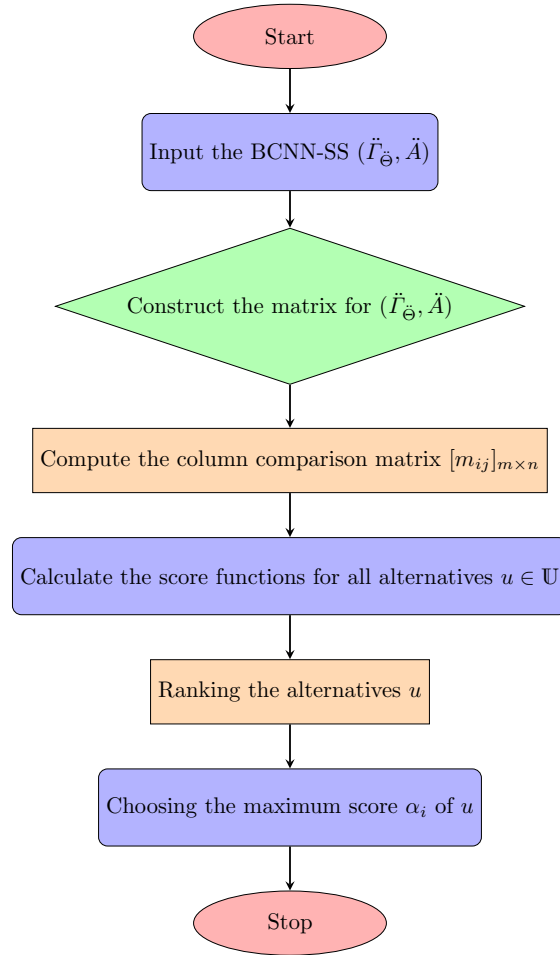


Figure 1: Flowchart for Algorithm 1

**Algorithm 2: For Row Comparison Matrix**

**Step:1:** Input the BCNN-SS  $(\check{I}_{\check{\Theta}}, \check{A})$ .

**Step:2:** Construct the matrix for  $(\check{I}_{\check{\Theta}}, \check{A})$  by Definition 3.1.

**Step:3:** Compute the row comparison matrix Definition 4.2.

**Step:4:** Calculate the row sum and column sum for all alternatives in  $\mathbb{U}$ , then subtract column sum from row sum.

**Step:5:** Choose the maximum score of the alternative as optimal solution in step 4. If there are many maximum scores, then any one can be chosen.

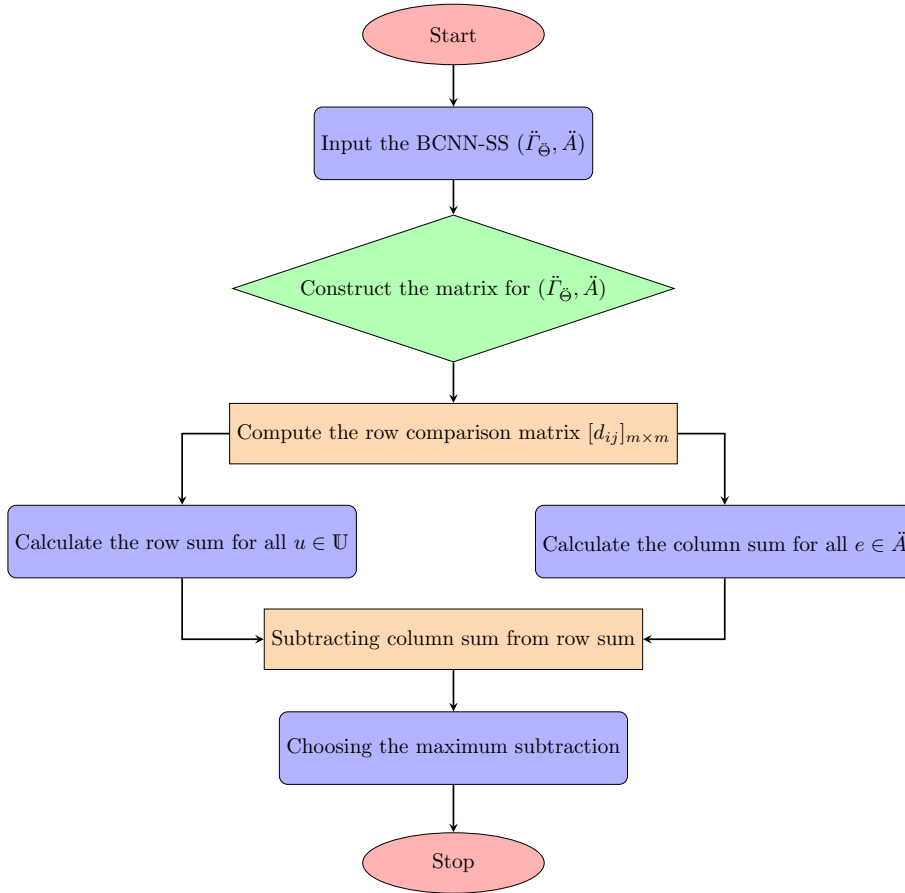


Figure 2: Flowchart for Algorithm 2

To explain application of Algorithms 1 and 2, we apply both of them on a new presented example as next:

**Example 4.1** A specialist company making tablets for reading wants to produce and test three new tablets at two different times, before and after the test. Let  $\mathbb{U} = \{u_1, u_2, u_3\}$  illustrate the set of alternative tablets. Then the best of them will be chosen for their advertisement campaign. Let  $A = \{e_1 = \text{price}, e_2 = \text{size}, e_3 = \text{charge}, e_4 = \text{consistency}\}$ .

The decision-making process needs information to be taken after the examination of the tablets. Based on the parameters, we use check( $\checkmark$ ) and cross( $\times$ ) marks, as shown in Table 5. Similar to Example 3.1, we construct a N5-SS as shown in Table 5:

Table 25: B5-SS Form

$\mathbb{U}/\check{A}$	$e_1 = \text{price}$	$e_2 = \text{size}$	$e_3 = \text{charge}$	$e_4 = \text{consistency}$
$u_1$	$\langle \checkmark, \times \rangle$	$\langle \checkmark\checkmark, \times\times \rangle$	$\langle \checkmark, \times \rangle$	$\langle \checkmark\checkmark\checkmark, \times \rangle$
$u_2$	$\langle \checkmark\checkmark, \times \rangle$	$\langle \checkmark\checkmark\checkmark, \times \rangle$	$\langle \checkmark, \times\times \rangle$	$\langle \checkmark\checkmark\checkmark, \times\times \rangle$
$u_3$	$\langle \checkmark, \times\times \rangle$	$\langle \checkmark\checkmark, \times\times\times \rangle$	$\langle \checkmark\checkmark, \times \rangle$	$\langle \checkmark\checkmark, \times\times \rangle$

Table 26: Meaning and Representation

+ signs=numbers	representations	-signs=numbers	representations
● = 0	Poor	● = 0	Poor
✓ = 1	OK	× = -1	Not OK
✓✓ = 2	Accept	×× = -2	Unaccepted
✓✓✓ = 3	Great	××× = -3	Not Great
✓✓✓✓ = 4	Exceptional	×××× = -4	unexceptional

We construct the B-5SS from Tables 25 and 26, as bellow:

Table 27: B5-SS Form

$U/\ddot{A}$	$e_1$	$e_2$	$e_3$	$e_4$
$u_1$	$\langle 1, -1 \rangle$	$\langle 2, -2 \rangle$	$\langle 1, -1 \rangle$	$\langle 3, -1 \rangle$
$u_2$	$\langle 2, -1 \rangle$	$\langle 3, -1 \rangle$	$\langle 1, -2 \rangle$	$\langle 3, -2 \rangle$
$u_3$	$\langle 1, -2 \rangle$	$\langle 2, -3 \rangle$	$\langle 2, -1 \rangle$	$\langle 2, -2 \rangle$

We construct the BCN-5SS by Definition 3.1 and Table 4 as shown in the following tabular representation:

Table 28: BCN-5SS Form

$U/\ddot{A}$	$e_1 = price$	$e_2 = size$	$e_3 = charge$	$e_4 = consistency$
$u_1$	$\left( (1, -1), \begin{pmatrix} 0.39 + i0.39, 0.7 + i0.7, 0.6 + i0.6, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.7 \end{pmatrix} \right)$	$\left( (2, -2), \begin{pmatrix} 0.5 + i0.5, 0.5 + i0.59, 0.4 + i0.4, \\ -0.4 - i0.4, -0.5 - i0.5, -0.5 - i0.5 \end{pmatrix} \right)$	$\left( (1, -1), \begin{pmatrix} 0.39 + i0.39, 0.75 + i0.75, 0.6 + i0.6, \\ -0.3 - i0.3, -0.7 - i0.7, -0.7 - i0.7 \end{pmatrix} \right)$	$\left( (3, -1), \begin{pmatrix} 0.79 + i0.79, 0.3 + i0.3, 0.2 + i0.2, \\ -0.2 - i0.2, -0.7 - i0.7, -0.7 - i0.7 \end{pmatrix} \right)$
$u_2$	$\left( (2, -1), \begin{pmatrix} 0.59 + i0.59, 0.59 + i0.59, 0.4 + i0.4, \\ -0.2 - i0.2, -0.7 - i0.6, -0.7 - i0.7 \end{pmatrix} \right)$	$\left( (3, -1), \begin{pmatrix} 0.79 + i0.79, 0.3 + i0.3, 0.3 + i0.3, \\ -0.2 - i0.2, -0.6 - i0.6, -0.7 - i0.7 \end{pmatrix} \right)$	$\left( (1, -2), \begin{pmatrix} 0.3 + i0.3, 0.7 + i0.7, 0.7 + i0.7, \\ -0.4 - i0.4, -0.55 - i0.55, -0.55 - i0.55 \end{pmatrix} \right)$	$\left( (3, -2), \begin{pmatrix} 0.79 + i0.79, 0.3 + i0.3, 0.35 + i0.3, \\ -0.5 - i0.5, -0.4 - i0.4, -0.5 - i0.5 \end{pmatrix} \right)$
$u_3$	$\left( (1, -2), \begin{pmatrix} 0.39 + i0.3, 0.7 + i0.7, 0.6 + i0.7, \\ -0.4 - i0.4, -0.4 - i0.4, -0.5 - i0.5 \end{pmatrix} \right)$	$\left( (2, -3), \begin{pmatrix} 0.4 + i0.5, 0.5 + i0.5, 0.5 + i0.5, \\ -0.6 - i0.6, -0.2 - i0.2, -0.3 - i0.3 \end{pmatrix} \right)$	$\left( (2, -1), \begin{pmatrix} 0.45 + i0.5, 0.55 + i0.55, 0.4 + i0.4, \\ -0.3 - i0.3, -0.7 - i0.7, -0.7 - i0.7 \end{pmatrix} \right)$	$\left( (2, -2), \begin{pmatrix} 0.5 + i0.5, 0.5 + i0.59, 0.4 + i0.4, \\ -0.4 - i0.4, -0.4 - i0.4, -0.5 - i0.5 \end{pmatrix} \right)$

Then its column comparison matrix is calculated by the formula stated in Definition 4.1 as below:

Table 29: Column Comparison Matrix of BCN-5SS

$U/\ddot{A}$	$e_1 = price$	$e_2 = size$	$e_3 = charge$	$e_4 = consistency$
$u_1$	-1	2	-1	2
$u_2$	0	0	-1	5
$u_3$	2	1	0	-3

Now, we calculate the score of all of the alternatives in  $U$ , which are presented as below:

Table 30: Scores of BCN-5SS by Algorithm 1

$U$	Score
$u_1$	2
$u_2$	4
$u_3$	0

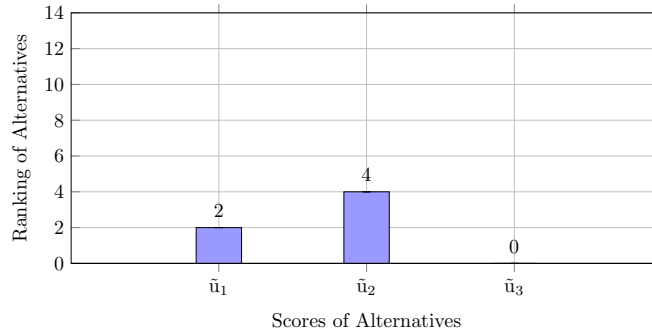


Figure 3: BCN-5SS Ranking by Algorithm 1

Clearly, the maximum score is recorded by the alternative  $u_2$ , which has a score of 4. The optimal solution is tablet number two. So, this is the decision-making that solves our problem. In Algorithm 2, the steps will be satisfied, and then we get the row comparison matrix as stated in 4.2 in the following table:

Table 31: Row Comparison Matrix of BCN-5SS

	$u_1$	$u_2$	$u_3$
$u_1$	0	-4	-3.5
$u_2$	3.5	0	1
$u_3$	2.5	3	0

Now, we go through the step 4 in Algorithm 2 as in the next table:

Table 32: Row Comparison Matrix of BCN-5SS

$\mathbb{U}$	Grade Sum	Row Sum	Column Sum	Their Subtraction
$u_1$	$\langle 7, -5 \rangle$	-7.5	6	-13.5
$u_2$	$\langle 9, -6 \rangle$	4.5	-7	11.5
$u_3$	$\langle 7, -8 \rangle$	-0.5	-2.5	2

Now we calculate all score functions of the alternatives in  $\mathbb{U}$ , which are presented below:

Table 33: Scores of BCN-5SS by Algorithm 2

$\mathbb{U}$	Score
$u_1$	-13.5
$u_2$	11.5
$u_3$	2

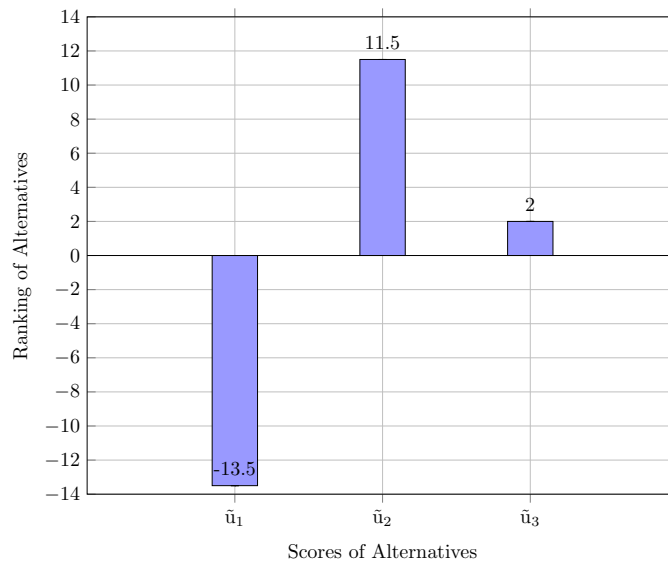


Figure 4: BCN-5SS Ranking by Algorithm 2

Finally and easily, the maximum score is recorded by the alternative  $u_2$ , tablet two, which has the score of 11.5.

## 5. Comparison Analysis

In this comparative section, we discuss the comparison of the two algorithms (Algorithm 1 and Algorithm 2) proposed with other previous related models in the literature. The first algorithm called the column comparison matrix, is compared with the models (NSS, CNSS, INSS, and CBVNSS). The proposed Example 4.1 in the previous section is used for the stated previous models (NSS, CNSS, INSS, and CBVNSS). Three of them, NSS, CNSS, and INSS, have a similar ranking of the alternatives of the set  $\mathbb{U}$  shown as  $u_1 > u_2 > u_3$ . So, the decision-making for the best option is in favor of  $u_1$ , whereas the model CBVNSS has a different option for the best choice among the alternatives, which is  $u_2$ . The BCNN-SS model is similar in ranking with FBSS and CBVNSS in that they have the ranking,  $u_2 > u_1 > u_3$  (see Table 34). On the other hand of comparison in Algorithm 2, it can be noticed that the ranking of the models (BFSS, BCFSS [Cartesian form], BCFSS [polar form], and BCNN-SS [proposed]) have been expressed in the same order,  $u_2 > u_3 > u_1$ . It keeps the alternative  $u_2$  that has the highest value among others. Then, we have the same ranking  $u_2 > u_1 > u_3$  for FN-SS, CFN-SS, and CNN-SS. As a result of the comparison, the BCNN-SS in both algorithms shows that  $u_2$  is the optimal solution for decision makers to be picked but different in other alternatives (see Table 35).

Table 34: Comparison in Algorithm 1

Models	Methods	Ranking
NSS	Algorithm 1	$u_1 > u_2 > u_3$
CNSS	Algorithm 1	$u_1 > u_2 > u_3$
INSS	Algorithm 1	$u_1 > u_2 > u_3$
CBVNSS	Algorithm 1	$u_2 > u_1 > u_3$
BCNN-SS (proposed)	Algorithm 1	$u_2 > u_1 > u_3$

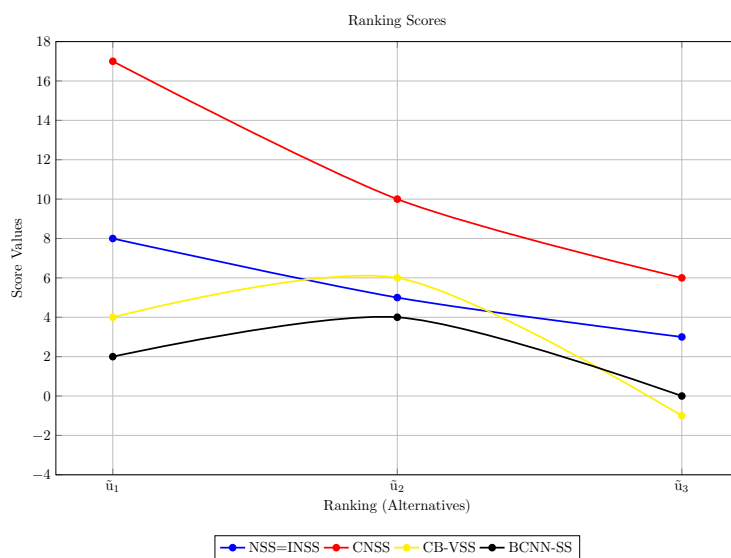


Figure 5: Comparative Analysis of BVPHFS with the Mentioned Models

Table 35: Comparison in Algorithm 2

Models	Methods	Ranking
BFSS	Algorithm 2	$u_2 > u_3 > u_1$
FBSS	Algorithm 2	$u_2 > u_1 > u_3$
BCFSS (Cartesian form)	Algorithm 2	$u_2 > u_3 > u_1$
BCFSS (polar form)	Algorithm 2	$u_2 > u_3 > u_1$
BCNN-SS (proposed)	Algorithm 2	$u_2 > u_3 > u_1$
FN-SS	Algorithm 2	$u_2 > u_1 > u_3$
CFN-SS	Algorithm 2	$u_2 > u_1 > u_3$

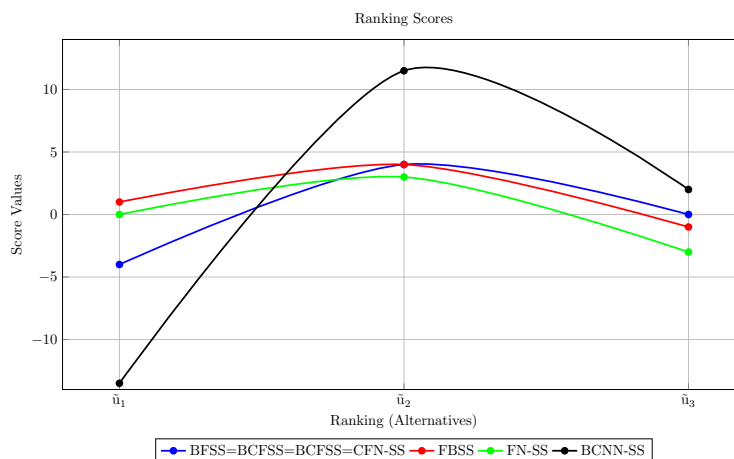


Figure 6: Comparative Analysis of BVPHFS with the Mentioned Models

## 6. Conclusion and Future Direction

In this study, we have discussed the concept of BCNN-SSs, which is the combination of BN-SSs and BCNSs. Necessarily, the Cartesian form of complex numbers is used for the proposed set since the amplitude term can not be negative in the polar form of complex numbers. The BCNN-SS is eventually the extension of CNN-SS and CBVNSS. In addition, a real-world scenario is given to help us visualize this new idea. We then have described some of the basic operations around this new model. An application section tells a matching real-world example with the given two proposed fundamental algorithms for decision-making problems. Then, we have implemented a comparative analysis that compares existing algorithms to show the efficiency and advantages of our algorithms. Furthermore, Algorithm 1 has similar results to CBVNSS of ranking alternatives while it has similar results without bipolar as NSS, CNSS, and INSS. Since the bipolar is taken into consideration in the set, the result of ranking is getting more accurate for decision-making problems. In Algorithm 2, the best option of alternatives is the same for all stated models. Ultimately, the BCNN-SS model enables decision-makers to analyze complex problems more effectively and base their decisions on better-informed choices as well as achieve optimal solutions according to its structured approach in handling uncertainty and multiple criteria.

The proposed approach is likely to be of higher utility when combined with other MCDM methods, such as TOPSIS, VIKOR, PROMETHEE and ELECTRE, particularly in the optimization phase for big data scenarios and real-time applications alongside AI-based decision support systems. Moreover, furthering the computational tools along with the model's extension to dynamic and time-dependent decision-making will greatly enhance its practical usability.

### List of Abbreviations

Table 36: List of Concepts and Their Abbreviations

Concepts	Abbreviations
Absolute bipolar complex neutrosophic N-soft set	ABCNN-SS
Bipolar complex fuzzy N-soft set	BCFN-SS
Bipolar complex fuzzy number	BCF $\acute{N}$
Bipolar complex fuzzy soft set (In both polar and Cartesian forms)	BCFSS
Bipolar complex neutrosophic N-soft set	BCNN-SS
Bipolar complex neutrosophic number	BCN $\acute{N}$
Bipolar complex neutrosophic set	BCNS
Bipolar fuzzy set	BFS
Bipolar fuzzy soft set	BFSS
Bipolar neutrosophic set	BNS
Bipolar neutrosophic soft set	BNSS
Bipolar N-soft set	BN-SS
Bipolar soft set	BSS
Bipolar-valued fuzzy set	BVFS
Class of all bipolar complex neutrosophic numbers	$\check{B}\check{C}\check{N}\acute{N}$
Class of all bipolar complex fuzzy numbers	$\check{B}\check{C}\check{F}\acute{N}$
Class of all complex single-valued neutrosophic numbers	$\check{C}\check{S}\check{V}\acute{N}\acute{N}$
Complex bipolar-valued neutrosophic soft set	CBVNSS
Complex neutrosophic N-soft set	CNN-SS
Complex neutrosophic set	CNS
Complex single-valued neutrosophic N-soft set	CSVNN-SS
Complex single-valued neutrosophic number	CSVN $\acute{N}$

Concepts	Abbreviations
Complex single-valued neutrosophic set	CSVNS
Fuzzy bipolar soft set	FBSS
Fuzzy N-soft set	FN-SS
Fuzzy set	FS
Fuzzy soft set	FSS
Interval-valued intuitionistic fuzzy set	IVIFS
Interval-valued neutrosophic set	IVNS
Intuitionistic fuzzy set	IFS
Intuitionistic neutrosophic soft set	INSS
Neutrosophic N-soft set	NN-SS
Neutrosophic set	NS
Neutrosophic soft set	NSS
N-bipolar hypersoft set	N-BHS
N-soft set	N-SS
Pythagorean fuzzy set	PFS
Single-valued neutrosophic set	SVNS
Single-valued neutrosophic number	SVN $\mathcal{N}$
Single-valued neutrosophic soft set	SVNSS
Soft set	SS

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