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A Novel Ranking Approach for Solving Fully Fuzzy Linear Fractional Programming Problem via Pentagonal Fuzzy Number

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ABSTRACT: Fuzzy programming comes in handy when dealing with unclear coefficients. Various methods have emerged recently to tackle ambiguity effectively. This article introduces a novel ranking function approach that leverages numbers to address fully fuzzy fractional linear programming (FFLFP) challenges. The process involves a membership function, for fuzzy numbers and a new solving technique for FFLFP. Subsequently changing the FFLFP problem into a fully fuzzy linear programming (FLFP) issue using a proposed method we apply arithmetic operations on pentagonal integers to update the simplex table iteratively until reaching the optimal fuzzy solution. An illustrative arithmetical example is provided to demonstrate the phases involved the result of solution, for the given problem.

Keywords: Fuzzy set, fuzzy number, pentagonal fuzzy number, ranking function, fuzzy, linear fractional programming.

Contents

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1. Introduction

There are often difficulties in making decisions in the actual world include uncertainty or ambiguity. Fuzzy numbers are used in many domains, such as fuzzy process modeling, control theory, decision making, and expert system reasoning. Prior researchers have focused on examining the arithmetic and algebraic properties of triangular and trapezoidal fuzzy numbers. Fuzzy systems, such as fuzzy set model [16] Zadeh, and fuzzy logic, have been widely used with great success. Fuzzy set theory has been used in several fields, ranging from fuzzy topological spaces to medicine and beyond. Nevertheless, manipulating the matrix formulation is a straightforward task when examining different mathematical models. In order to account for the inherent uncertainty in mathematical formulations across Multiple disciplines within the realms of technological advancement and science, we have created the notion of pentagonal fuzzy numbers (PFN) besides the accompanying pentagonal fuzzy matrices (PFMs). Multiple writers have reported findings on the characteristics of determinants, the adjoint of fuzzy matrices, and the process of power sequences converging using fuzzy matrices. Below is a concise overview of fuzzy matrices.

Panda and Pal [13] Introduce a generalized introduction is given to the concept of pentagonal fuzzy number (PFN). Several studies have been written on this issue; however, they have uncertainties in defining this particular form of imprecise number. In this study, we provide a precise concept for constructing a pentagonal fuzzy number, as well as the corresponding arithmetic operations. The study focuses on the construction and fundamental features of pentagonal fuzzy matrices (PFMs) using the Pentagonal Fuzzy Number (PFN) framework. Nawkhass and Sulaiman [12] Presents a method to change and solve a specific issue by making modifications to the symmetric fuzzy method. It proposes an algorithm and shows how the fuzzy linear fractional programming problem (FLFPP) may be solved deprived of increasing the computational complexity. Additionally, it presents a method that employs an optimum average to transform MOLFPP into a singular LFPP by the alteration of the symmetric fuzzy method. Hasan and Al Kanani [6] Introduces a unique approach using decagonal fuzzy integers as variables to solve fully fuzzy fractional linear programming (FFFLP) problems. This approach relies on the incorporation of a novel membership function for a ten-sided fuzzy number and the use of a comprehensive fuzzy simplex technique. By using a supplementary approach, the FFFLP issue is changed into the completely fuzzy linear programming (FFLP) problem. Afterward, the problem is solved using fully fuzzy simplex tables, where all the values are represented as fuzzy decagonal integers. Nawkhass and Sulaiman [12] For the purpose of dealing with non-linear fractional programming, the notion of geometric arithmetic mean should be used. To do this, they get the inequality by using the traditional optimizing statement, which entails defining the essential and sufficient circumstances for locating the stationary points of the over-all inequality constraint optimization challenges. This allows them to accomplish their goal. The geometric arithmetic mean inequality is used in this process, one may determine the best solution to non-linear fractional issues.

Mitlif, Sabri, and Ouda [8] Pentagonal fuzzy variables (PFV) are used in the formulation of "linear programming problems" (LPP). In this context, we will focus on a method for tackling these problems known as the simplex methodology. Linear programming problems with pentagonal fuzzy numbers are the two fundamental classifications of these problems. This work aims to determine the optimum solution for Linear Programming Problem using Parameterized Family of Norms on the goal function and right-hand side. Nawkhass, and Sulaiman [12] Present a method to transform and resolve the issue using a symmetric fuzzy method. Propose an approach and demonstrate how fuzzy LFPP may be solved deprived of increasing computational power. The purpose of our approach is to transform a complementarity multi-objective linear fractional programming problem (CMOLFPP) into a linear programming problem by using a symmetric fuzzy linear fractional problem (SFLFP). In order to showcase the effectiveness of the proposed methodology. Sahoo, Tripathy and Pati [15] The issue known as IFMOLFPP has been turned into the intuitionistic fuzzy multi objective linear programming problem (IFMOLPP). Subsequently, the resulting IFMOLPP has been translated into the crisp multi goal linear programming problem (CMOLPP) using a well-defined accurateness function. They have formulated propositions that demonstrate the equivalence among the optimal solution of (CMOLPP) and the optimal solutions of (IFMOLPP) and (IFMOLFPP). By using Zimmermann's methodology with appropriate nonlinear membership functions, they have transformed the (CMOLPP) into a solitary objective LPP that can be readily solved using any LPP procedure or software. Mustafa and Sulaiman [9] Present two novel ranking function methods for addressing fully fuzzy linear fractional programming (FFLFP) issues, where the goal function and constraint coefficients are triangular fuzzy numbers (TrFNs). Utilizing a ranking function is a very effective strategy for working with TrFNs. The imprecise values are transformed into precise ones by the use of the recommended ranking function process. The approach developed by Charnes and Cooper converts issues in linear fractional programming (LFP) into problems in linear programming (LP). Chakraborty and Gupta [4] Introduce a novel notion of using the pentagonal neutrosophic (PN) technique for solving linear programming (LP) problems. Based on our understanding, there is currently no known method for tackling the PNLP challenge. This study focuses on the PNLP issue, which involves considering the goals and constraints as pentagonal neutrosophic numbers (PNN) for the first time. In order to develop our method, we outlined the principles of PN arithmetic operations and mathematical calculations inside the PNN environment. This suggested solution utilizes a ranking function and transforms it into an analogous crisp LP (CrLP) issue.

The structure of this article is as follows; In Sector 2 we present the idea of fuzzy number in a well-defined manner. In Sector 3, numerous preliminaries concerning pentagonal fuzzy number are presented. In Sector 4, Ranking function and proposed pentagonal ranking function are established. Methodology and proposed method to convert FLFPP into fuzzy linear programming problem FLPP is presented in Sector 5. In Sector 6, proposed method to find the fuzzy optimal solution of FLFP problems is presented. In Sector 7, Proposed Algorithm for Solving Fuzzy Linear Fractional Programming Problem is addressed. In Sector 8, introduce flowchart for solving fuzzy linear fractional programming problem. In Sector 9, present two real live numerical examples, in section 10 results and discussion is presented. Lastly, conclusions are addressed in Sect. 11.

2. Preliminaries

Before we begin, we will briefly review some fundamental concepts of fuzzy numbers as well as some fundamental conclusions.

Definition 1. Fuzzy set. [12] The membership function of a fuzzy set is considered to be its defining characteristic. This function takes values from the domain, space, or universe of discourse and maps them into the unit interval $[0, 1]$. To define a fuzzy set A inside the universal set X, the equation $A = (x, \mu(x); x \in X)$ would be used. In this context, the grade of the membership function is denoted by $\mu_A : A \to [0, 1]$, and the grade value of $x \in X$ in the fuzzy set A is denoted by μ_A (x respectively).

Definition 2. Normal fuzzy set. [12] A fuzzy set A is said to be normal if there is an element x belonging to X that has a membership value of one, which is denoted by the term $\mu_A(x) = 1$.

Definition 3. Fuzzy number. [12] A fuzzy number A, which is a subset of the real line R, is characterized by the membership function μ_A that satisfies the conditions listed below:

(i) The function μ_A inside its domain is piecewise continuous.

(ii) The set A is considered to be normal, meaning that there exists an x_0 that belongs to A and $\mu_A(x_0)$ equals 1.

(iii) The set A is convex if $\mu_A(\lambda x_1 + (1 + \lambda)x_2)$ is greater than or equal to the minimum of $\mu_A(x_1)$ and $\mu_{A_2}(x_2)$. Given x_1 and x_2 in the set X.

In the realm of fuzzy algebra, two varieties of uncertain numbers, namely the triangular fuzzy number and the trapezoidal fuzzy number, have been established as a result of the many applications of fuzzy numbers.

3. Pentagonal Fuzzy Number

A Pentagonal Fuzzy Number (PFN) is a kind of number that is depicted by a five-sided shape known as a pentagon. It serves as an extension of the triangular number providing increased versatility, in depicting vagueness and inaccuracy.

A Pentagonal Fuzzy Number of a fuzzy set $\mu_A(x)$ is demarcated as $\mu_A(x) = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$, and its membership function can be defined by follows, [13,14]

$$\mu_A(x) = \begin{cases} M_1(x) = \frac{x-\upsilon_1}{\upsilon_2-\upsilon_1} & for \quad \upsilon_1 \leqslant \upsilon_2 \\ M_2(x) = \frac{x-\upsilon_2}{\upsilon_3-\upsilon_2} & for \quad \upsilon_2 \leqslant \upsilon_3 \\ 1 & for \quad x = \upsilon_3 \\ N_1(x) = \frac{\upsilon_4-x}{\upsilon_4-\upsilon_3} & for \quad \upsilon_3 \leqslant \upsilon_4 \\ N_2(x) = \frac{\upsilon_5-x}{\upsilon_5-\upsilon_4} & for \quad \upsilon_4 \leqslant \upsilon_5 \\ 0 & otherwise \end{cases}$$

3.1. Graphical Representation

The PFN can be visualized as a five shape, with corners located at the coordinates $(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$.

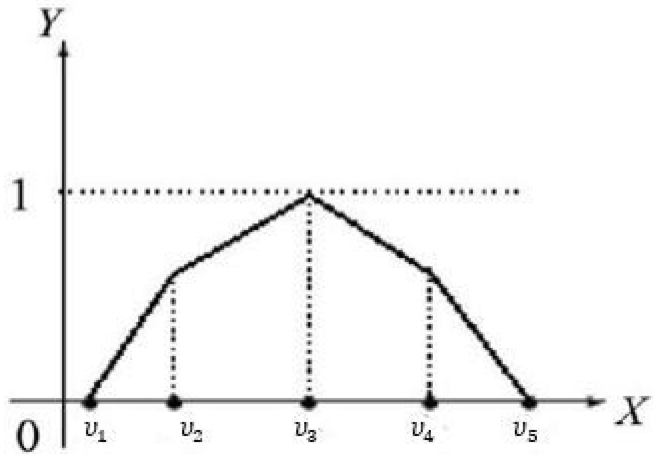

Figure 1: Pentagonal fuzzy number.

3.2. Arithmetic operations of PFN

In the study of fuzzy numbers, the formation of an arithmetic process is of the utmost importance; the author makes an effort to develop some fundamental arithmetic operations of PFN. There are two weights that are connected with each and every PFN, and they are w_1 and w_2. We use the representation w_iA for $i = 1, 2$ to accurately reflect w_1 besides w_2 as the weights of the PFN A. This is done to prevent any mistake that may arise.

(1) Addition: Let $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ and $B = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ be two PFNs; then, $A + B = (\upsilon_1 + \nu_1, \upsilon_2 + \nu_2, \upsilon_3 + \nu_3, \upsilon_4 + \nu_4, \upsilon_5 + \nu_5)$, beside $w_{i(A+B)}) \geq max(w_{iA}, w_{iB})$ for $i = 1, 2$.

(2) Subtraction: Describe the subtraction of two PFNs $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ and $B = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ as $A - B = (\upsilon_1 - \nu_1, \upsilon_2 - \nu_2, \upsilon_3 - \nu_3, \upsilon_4 - \nu_4, \upsilon_5 - \nu_5)$, beside $w_{i(A-B)}) \geq max(w_{iA}, w_{iB})$ for $i = 1, 2$.

(3) Scalar Multiplication: Assume $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ be a PFN besides $k \in R$ be any scalar. If $k \geq 0$,$kA = (k\upsilon_1, k\upsilon_2, k\upsilon_3, k\upsilon_4, k\upsilon_5)$ $k \leqslant 0$, $kA = (k\upsilon_5, k\upsilon_4, k\upsilon_3, k\upsilon_2, k\upsilon_1)$

(4) Multiplication: Let $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ and $B = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ be two PFNs; then, $AB = (\upsilon_1\nu_1, \upsilon_2\nu_2, \upsilon_3\nu_3, \upsilon_4\nu, \upsilon_5\nu_5)$ with $w_{(i(AB))} \geq max(w_{iA}, w_{iB})$, $i = 1, 2$.

(5) Inverse: Describe the opposite of a PFN when all its components are non-zero. Assume $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ is a PFN; then, $A^{-1} \approx \frac{1}{A} \approx \left(\frac{1}{\upsilon_1}, \frac{1}{\upsilon_2}, \frac{1}{\upsilon_3}, \frac{1}{\upsilon_4}. \frac{1}{\upsilon_5}\right)$.If one of the apparatuses of a PFN converts zero, so we cannot find its opposite.

(6) Division: The division of two PFNs $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ and $B = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ is approached as the multiplication through opposite. $\frac{A}{B} \approx AB^{-1} \approx \left(\frac{\upsilon_1}{\nu_1}, \frac{\upsilon_2}{\nu_2}, \frac{\upsilon_3}{\nu_3}, \frac{\upsilon_4}{\nu_4}, \frac{\upsilon_5}{\nu_5}\right)$. Note that a PFN A is divisible by B only when B is a non-null PFN having non-zero components.

(7) Exponent: The proponent of a PFN $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ is demarcated as the power of its components. $A^n \approx (\upsilon_1^n, \upsilon_2^n, \upsilon_3^n, \upsilon_4^n, \upsilon_5^n)$, with n being a real number.

Definition 4. Positive PFN. A PFN $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ is supposed to be positive if all its entries are positive. Similarly, $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ is negative if all of its entries are negative.
Definition 5. Null PFN. A PFN A is entitled a Null PFN if all of its entrances are zero, i.e., $A = (0, 0, 0, 0, 0)$.
Definition 6. Null equivalent PFN. A PFN $A = (\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$ is supposed to be null equal if its middle entry is at the point 0 , i.e., of the form $(\delta_1, \varepsilon_1, 0, \varepsilon_2, \delta_2)$, wherever $\delta_1 \cdot \varepsilon_1 0, \delta_2 \cdot \varepsilon_2 \neq 0$. It is signified by $\tilde{0}$.
Definition 7. Unit equivalent PFN. A PFN A is supposed to be a unit equal PFN when its middle entry is at 1, i.e., of the form $(\delta_1, \varepsilon_1, 1, \varepsilon_2, \delta_2), where \delta_1 \cdot \varepsilon_1 \neq 0, \delta_2 \cdot \varepsilon_2 \neq 0$. Following the completion of our earlier arithmetic operations on PFN, we made the observation that the subtraction of two PFNs that share a middle entry result in a null equivalent PFN, however the division of these PFNs results in another unit equivalent PFN product.

4. Ranking Function

The use of "ranking functions" is a method that is ideal for the purpose of comparing number sets that include fuzzy numbers. In order to do this, it is necessary to establish a "ranking function $\Re : F(R) \to R$, which maps each fuzzy number into the real line ". Taking into consideration the fact that $\tilde{\upsilon}$ and $\tilde{\nu}$ are two "pentagonal fuzzy numbers," we can proceed to characterize the ordering on $F(\Re)$ in the following manner:
$\tilde{\upsilon} \geq \tilde{\nu}$ iff $\Re(\tilde{\upsilon}) \geq \Re(\tilde{\nu})$
$\tilde{\upsilon} > \tilde{\nu}$ iff $\Re(\tilde{\upsilon}) > \Re(\tilde{\nu})$
$\tilde{\upsilon} = \tilde{\nu}$ iff $\Re(\tilde{\upsilon}) = \Re(\tilde{\nu})$
where $\tilde{\upsilon}$ and $\tilde{\nu}$ are in $F(\Re)$, also we write $\tilde{\upsilon} \leq \tilde{\nu}$ $\Re \leq$ iff $\tilde{\nu} \geq \tilde{\upsilon}$. [3]

Lemma 4.1 [15,16] Adopt $\Re$ be any "linear ranking function". So:
i) $\tilde{\upsilon} \geq \tilde{\nu}$ iff $\tilde{\upsilon} - \tilde{\nu} \geq 0$ iff $-\tilde{\nu} \geq -\tilde{\upsilon}$.
ii) if $\tilde{\upsilon} \geq \tilde{\nu}$ iff $\tilde{k} \geq \tilde{i}$, then $\tilde{\upsilon} + \tilde{k} \geq \tilde{\nu} + \tilde{i}$.
Several numbers are included here. For the purpose of comparing fuzzy numbers, the linear ranking function $\Re$ is used.
$\Re(k\tilde{\upsilon} + \tilde{\nu}) = k\Re(\tilde{\upsilon}) + \Re(\tilde{\nu})$. Where $k \in \Re$.

4.1. Proposed Pentagonal Ranking Function Approach

We introduce a novel ranking method in this study with a weighted pentagonal membership function. Now, we propose the following ranking function:

$$\Re(\tilde{\sigma}) = \frac{\int_0^\gamma \varphi^2 \left[\frac{M_1^{-1}(\varphi)}{2} + \frac{M_2^{-1}(\varphi)}{2} + \frac{N_1^{-1}(\varphi)}{2} + \frac{N_2^{-1}(\varphi)}{2}\right]}{\int_0^\gamma \varphi^2 d\varphi}$$

$$\Re(\tilde{\sigma}) = \frac{\frac{1}{2}\int_0^\gamma \varphi^2 [[\upsilon_1 + \upsilon_2 + \upsilon_4 + \upsilon_5] + [-\upsilon_1 + 2\upsilon_3 - \upsilon_5]\,\varphi] d\varphi}{\int_0^\gamma \varphi^2 d\varphi}$$

By using the $\alpha - cut$ where $0 \leq \gamma \leq 1$

$$\Re(\tilde{\sigma}) = \frac{\frac{1}{6}[\upsilon_1 + \upsilon_2 + \upsilon_4 + \upsilon_5] + \frac{1}{8}[-\upsilon_1 + 2\upsilon_5 - \upsilon_5]}{\frac{1}{3}}$$

$$\Re(\tilde{\sigma}) = \frac{1}{8}\upsilon_1 + \frac{1}{2}\upsilon_2 + \frac{3}{4}\upsilon_3 + \frac{1}{2}\upsilon_4 + \frac{1}{8}\upsilon_5 \tag{4.1}$$

5. Methodology

5.1. Fuzzy Linear Fractional Programming Problem FLFPP with Fuzzy Numbers

It is possible to illustrate the programming problems associated with a fuzzy number linear fractional as follows:

$$Maximize \frac{p'x + \alpha}{q'x + \beta}$$

Subject to

$$Ax \leq b$$
$$x \geq 0.$$

Where $b \in \Re^m$, $x \in \Re^n, x \in X, A \in \Re^{(m+n)}, p, d, \alpha, \beta \in (F(\Re))$ and $\Re$ is the linear ranking function. [7,14]

5.2. Proposed Method to Convert FLFPP into Fuzzy Linear Programming Problem FLPP

This part will cover the process by which fuzzy linear fractional programming (FLFP) issues are transformed into linear fractional programming (LFP) using the suggested pentagonal ranking function that was discussed in section (4.1) As a result of the fact that the fuzzy linear fractional programming (FLFP) issues in this scenario have a generic form that is represented by formula (4.1), we obtain the equation that follows:

$$Maximize \frac{(\upsilon_{1,1}, \upsilon_{1,2}, \upsilon_{1,3}, \upsilon_{1,4}, \upsilon_{1,5})x_1 + (\upsilon_{1,6}, \upsilon_{1,7}, \upsilon_{1,8}, \upsilon_{1,9}, \upsilon_{1,10})x_2 + (\upsilon_{1,11}, \upsilon_{1,12}, \upsilon_{1,13}, \upsilon_{1,14}, \upsilon_{1,15})x_3}{(\upsilon_{2,1}, \upsilon_{2,2}, \upsilon_{2,3}, \upsilon_{2,4}, \upsilon_{2,5})x_1 + (\upsilon_{2,6}, \upsilon_{2,7}, \upsilon_{2,8}, \upsilon_{2,9}, \upsilon_{2,10})x_2 + (\upsilon_{2,11}, \upsilon_{2,12}, \upsilon_{2,13}, \upsilon_{2,14}, \upsilon_{2,15})x_3}$$

Subject to

$$Ax \leq b$$
$$x \geq 0.$$

Following this, the goal function (numerator and denominator) that was presented earlier is transformed by using the suggested pentagonal ranking function that was discussed in section (4.1), we get:

$$(\upsilon_{1,1}, \upsilon_{1,2}, \upsilon_{1,3}, \upsilon_{1,4}, \upsilon_{1,5}) = \frac{1}{8}\upsilon_{1,1} + \frac{1}{2}\upsilon_{1,2} + \frac{3}{4}\upsilon_{1,3} + \frac{1}{2}\upsilon_{1,4} + \frac{1}{8}\upsilon_{1,5}$$
$$(\upsilon_{1,6}, \upsilon_{1,7}, \upsilon_{1,8}, \upsilon_{1,9}, \upsilon_{1,10}) = \frac{1}{8}\upsilon_{1,6} + \frac{1}{2}\upsilon_{1,7} + \frac{3}{4}\upsilon_{1,8} + \frac{1}{2}\upsilon_{1,9} + \frac{1}{8}\upsilon_{1,10}$$
$$(\upsilon_{1,11}, \upsilon_{1,12}, \upsilon_{1,13}, \upsilon_{1,14}, \upsilon_{1,15}) = \frac{1}{8}\upsilon_{1,11} + \frac{1}{2}\upsilon_{1,12} + \frac{3}{4}\upsilon_{1,13} + \frac{1}{2}\upsilon_{1,14} + \frac{1}{8}\upsilon_{1,15}$$
$$(\upsilon_{2,1}, \upsilon_{2,2}, \upsilon_{2,3}, \upsilon_{2,4}, \upsilon_{2,5}) = \frac{1}{8}\upsilon_{2,1} + \frac{1}{2}\upsilon_{2,2} + \frac{3}{4}\upsilon_{2,3} + \frac{1}{2}\upsilon_{2,4} + \frac{1}{8}\upsilon_{2,5}$$
$$(\upsilon_{2,6}, \upsilon_{2,7}, \upsilon_{2,8}, \upsilon_{2,9}, \upsilon_{2,10}) = \frac{1}{8}\upsilon_{2,6} + \frac{1}{2}\upsilon_{2,7} + \frac{3}{4}\upsilon_{2,8} + \frac{1}{2}\upsilon_{2,9} + \frac{1}{8}\upsilon_{2,10}$$
$$(\upsilon_{2,11}, \upsilon_{2,12}, \upsilon_{2,13}, \upsilon_{2,14}, \upsilon_{2,15}) = \frac{1}{8}\upsilon_{2,11} + \frac{1}{2}\upsilon_{2,12} + \frac{3}{4}\upsilon_{2,13} + \frac{1}{2}\upsilon_{2,14} + \frac{1}{8}\upsilon_{2,15}$$

In light of the change described above, the FLFPP may now be expressed as follows:

$$Maximize[\left[[\frac{1}{8}\upsilon_{1,1} + \frac{1}{2}\upsilon_{1,2} + \frac{3}{4}\upsilon_{1,3} + \frac{1}{2}\upsilon_{1,4} + \frac{1}{8}\upsilon_{1,5}]x_1 + [\frac{1}{8}\upsilon_{1,6} + \frac{1}{2}\upsilon_{1,7} + \frac{3}{4}\upsilon_{1,8} + \frac{1}{2}\upsilon_{1,9} + \frac{1}{8}\upsilon_{1,10}]x_2\right]$$
$$+ \left[[\frac{1}{8}\upsilon_{1,11} + \frac{1}{2}\upsilon_{1,12} + \frac{3}{4}\upsilon_{1,13} + \frac{1}{2}\upsilon_{1,14} + \frac{1}{8}\upsilon_{1,15}]\right]] \div [\left[[\frac{1}{8}\upsilon_{2,1} + \frac{1}{2}\upsilon_{2,2} + \frac{3}{4}\upsilon_{2,3} + \frac{1}{2}\upsilon_{2,4} + \frac{1}{8}\upsilon_{2,5}]x_1\right] +$$
$$\left[[\frac{1}{8}\upsilon_{2,6} + \frac{1}{2}\upsilon_{2,7} + \frac{3}{4}\upsilon_{2,8} + \frac{1}{2}\upsilon_{2,9} + \frac{1}{8}\upsilon_{2,10}]x_2 + [\frac{1}{8}\upsilon_{2,11} + \frac{1}{2}\upsilon_{2,12} + \frac{3}{4}\upsilon_{2,13} + \frac{1}{2}\upsilon_{2,14} + \frac{1}{8}\upsilon_{2,15}]\right]]$$

Subject to

$$Ax \le b$$
$$x \ge 0.$$

6. Proposed Method to Find the Fuzzy Optimal Solution of FLFP Problems

The technique applied the simplex algorithm for solving a LFPP. Assuming the next problem:

$$Maximize \quad \frac{p'x+\alpha}{q'x+\beta}$$

Subject to

$$Ax \le b$$
$$x \ge 0.$$

Let us assume that the set S, which is defined as $x : Ax \le b$ and $x \ge 0$, is compact. Furthermore, let us assume that the value of $q^t x + \beta$ is greater than zero for every x that belongs to S. The following linear programming (LP) is obtained by letting z equal to $1/(q^t x + \beta)$ and y equal to zx, and then multiplying the restraints $Ax \le b$ by z.

$$Minimize \quad \alpha z + p^t y$$

Subject to

$$\begin{aligned} Ay - bz &\le 0 \\ \beta z + q^t y &= 1 \\ y &\ge 0 \\ z &\ge 0 \end{aligned}$$

To begin, it is important to keep in mind that if (y, z) is a workable solution to the problem described above, then z is greater than zero. Given that if z is equal to zero, then y must be such that Ay is less than zero and y is more than zero. This implies that y is a direction of S, which is contrary to the compactness assumption. Given that $(\bar{y}, \bar{z})$ is an ideal solution to the LP mentioned before, we are now able to establish that $\bar{x} = \bar{y}/\bar{z}$ is also an ideal solution to the fractional program presented earlier.

It is important to observe that the inequality $A\bar{x} \le b$ holds, and also that $\bar{x} \le 0$. This implies that $\bar{x}$ is a valid solution to the fractional program. In order to demonstrate the optimality of $\bar{x}$, consider x to be a solution that satisfies the conditions $Ax \le b$ and $x \le 0$. It should be noted that the inequality $q^t x + \beta > 0$ is assumed, and that the vector (y, z) is a valid solution to the LP, where $y = x/(q^t x + \beta)$ and $z = 1/(q^t x + \beta)$. Given that $(\bar{y}, \bar{z})$ is an optimum solution to the linear program, it follows that $p^t \bar{y} + \alpha\bar{z}$ is less than or equal to $p^t y + \alpha z$. By replacing the variables $\bar{y}, y$, and z, the inequality may be expressed as $\bar{z}(p^t \bar{x}\alpha) \le (p^t x + \alpha)/(q^t x + \beta)$. The result may be obtained by partitioning the left-hand side by $1 = q^t \bar{y} + \beta\bar{z}$

If $q^t x + \beta$ is negative for every $x \in S$, then by defining $-z$ as the reciprocal of $(q^t x + \beta)$ and y as the product of z and x, we get the LP as follow:

$$Minimize \quad -\alpha z - p^t y$$

Subject to

$$\begin{aligned} Ay - bz &\le 0 \\ -\beta z - q^t y &= 1 \\ y &\ge 0 \\ z &\ge 0 \end{aligned}$$

In a manner that is comparable to the previous statement, if the linear program $(\bar{y}, \bar{z})$ is solved, then the fractional programming issue may be solved by multiplying $\bar{x}$ by $\bar{y}$ divided by $\bar{z}$.

7. Proposed Algorithm for Solving Fuzzy Linear Fractional Programming Problem

This algorithm utilizes the pentagonal fuzzy number ranking function and the proposed method mentioned in sections (4.1) , (5.2) and (6) to solve Fuzzy Linear Fractional Programming Problems (FLFPP).
Step 1: Formulate the problem to the mathematical model and Express the model to FLFPP form.
Step 2: Conversion to Fuzzy Linear Programming Problem (FLPP).
I: Utilize the proposed pentagonal ranking function (Equation 1) to convert the fuzzy factors in the goal function and restraints into equivalent real numbers.
II: The FFLPP now becomes a standard FLPP.
Step 3: Initialization:
I: Create the initial table.
II: The objective function will be the converted objective function from Step 2.
III: The constraints will be the converted constraints from Step 2.
Step 4: Optimality Check:
I: Calculate the objective function value for the current basic solution.
II: Check if the objective function value is optimal.
Step 5: Iteration:
I: If the solution is not optimal, select the entering variable.
II: Select the leaving variable using the minimum ratio test.
III: Update the simplex table by performing the pivot operation.
IV: Go back to Step 4 and repeat until an optimal solution is reached.
Step 6: Solution Interpretation:
The optimal solution obtained from the simplex process represents the optimum fuzzy solution for the original FFLPP.

8. Flowchart for Solving Fuzzy Linear Fractional Programming Problem

Here is a visual diagram that outlines the steps of the suggested approach, for addressing FLFPP. This diagram illustrates the processes and strategies utilized in the proposed technique aiding in comprehension and implementation of the algorithm. It serves as a tool to improve clarity and facilitate navigation through the algorithm.
A [Problem Formulation] → B Convert to FLPP
B → C Initialize Simplex Table
C → D Check Optimality
D – No → E Select Entering Variable
D – Yes → F Solution Interpretation
E → G Select Leaving Variable
G → H Pivot Operation
H → C
F → I Convert Solution to Fuzzy Numbers
I → J End
Here is a visual diagram that illustrates the suggested method, for addressing FLFPP concerns. It helps illustrate the step-by-step process involved in the conversion and solution of the problem.

9. Numerical Examples

Using an issue that really exists in the world, we will demonstrate the suggested method in this section. It is clear that the LFPP is an unpredictable optimization problematic because of the fluctuations in the maximum daily needs that it presents. It is thus impossible to predict how much of any produce or constituent will be used. For this reason, we shall treat the issue as a FLFP problem. In order to account for any unknown value, we use pentagonal fuzzy numbers. The solution that has been given will also be successful in solving the mathematical programming challenge.

Problem 1: (Production Planning)

Phoenix Woodworks, located in Phoenix, Arizona, is a successful furniture manufacturer that focuses on producing two specific types of products: handcrafted tables (Product A) and sophisticated chairs (Product B). Product A generates a profit of approximately 3$, 4$, 5$, 7$ or 8$ per unit, while Product B generates a profit of approximately 2$, 3$, 4$, 6$ or 9$ per unit. Due to expensive machine and maintenance equipment to the time during the manufacture procedure, it is projected that a fixed profit of about (8, 12, 41, 16, 19) dollars is added to the profit function. The cost for each unit of the above items is around 1$, 3$, 4$, 6$ and 8$ respectively, and approximately 1$, 2$, 5$, 7$ and 9$, respectively. Due to the anticipated duration of the manufacture procedure, it is presumable that a immovable cost of about 4$, 6$, 9$, 13$, 15$ are added to the cost function. Presumptuous that the amount of raw material required to production product A and B is about (2,5,6,8,9) units per pound and (1,3,7,9,10) units per pound correspondingly, the available supply of this raw material is limited to approximately (5,6,15,20,25) pounds. The labor required to produce one unit of product A is about 1, 2, 7, 9 or 10 hours, whereas for product B it is approximately 2, 3, 5, 8, 10 or 12 hours per unit. However, the total amount of labor available every day is approximately 3, 6, 11, 15 or 21 hours. The company's objective is to optimize profitability via smart resource allocation.

Solution: Let x_1 and x_2 to be the amount of units of A besides B respectively to be produced. Then the above problem can be expressed as:

$$Maximize Z = \frac{(3,4,5,7,8)x_1 + (2,3,4,6,9)x_2 + (8,12,14,16,19)}{(1,3,4,6,8)x_1 + (1,2,5,7,9)x_2 + (4,6,9,13,15)}$$

Subject to

$$(2,5,6,8,9)x_1 + (1,3,7,9,10)x_2 \leq (5,6,15,20,25)$$
$$(1,2,7,9,10)x_1 + (2,3,5,8,12)x_2 \leq (3,6,11,15,21)$$
$$x_1 \geq 0, x_2 \geq 0$$

By using a Proposed Pentagonal ranking function formula (4.1):

$$\Re(\tilde{\sigma}) = \frac{1}{8}\upsilon_1 + \frac{1}{2}\upsilon_2 + \frac{3}{4}\upsilon_3 + \frac{1}{2}\upsilon_4 + \frac{1}{8}\upsilon_5$$

The FLFPP is converted into crisp linear fractional programming, therefore we have the following:

$$Maximize Z = \frac{(10.625)x_1 + (8.875)x_2 + (27.875)}{(8.625)x_1 + (9.5)x_2 + (18.625)}$$

Subject to

$$(12.375)x_1 + (11)x_2 \leq (18.625)$$
$$(12.125)x_1 + (11)x_2 \leq (21.75)$$
$$x_1 \geq 0, x_2 \geq 0$$

By using a proposed method mentioned in section 6 we find optimal solution of FLFP problems, as follows:

let $k \leq \frac{1}{(8.625)x_1+(9.5)x_2+(18.625)}$ and multiply first and second constraint by k and putting $m_1 = kx_1, m_2 = kx_2$, we get the following linear programming problem:

$$Maximize Z = (10.625)m_1 + (8.875)m_2 + (27.875)k$$

Subject to

$$(12.375)m_1 + (11)m_2 \leq (18.625)k$$
$$(12.125)m_1 + (11)m_2 \leq (21.75)k$$
$$(8.625)m_1 + (9.5)m_2 + (18.625)k \leq 1$$

$$x_1, x_2, k \geq 0$$

Using simplex algorithm to solve the problem, we get $m_1 = 0, m_2 = 0.028, k = 0.015$ Therefore, the optimal solution is $x_1 = 0, x_2 = 1.86$ $Max.Z = 44.382$.

Problem 2: (Production Planning)

The establishment produces two types of goods, A besides B, with multi-objective profit. Product A has a profit of around 2\$, 3\$, 5\$, 7\$ and 10\$ per unit, whereas product B has a profit of approximately 1\$, 3\$, 5\$, 8\$ and 11\$ per unit. Due to expensive machine and maintenance equipment to the time during the manufacture procedure, it is projected that a fixed profit of about (9, 12, 14, 18, 20) dollars is added to the profit function. The cost for each unit of the above items is about 2\$, 4\$, 6\$, 8\$ and 11\$ respectively, and approximately 2\$, 3\$, 6\$ 9\$ and 12\$ correspondingly. Due to the expected period of the manufacture procedure, it is supposable that a fixed cost of about 5\$, 8\$, 11\$, 12\$, 15\$ are added to the cost function. The raw material required for producing product A and B is about (3, 4, 8, 10, 12) units per pound and (2, 6, 7, 9, 14) units per pound, correspondingly. The available source of this raw material is limited to approximately (5 ,9 ,12 ,15, 22) pounds. The labor required to produce one unit of product A is about 1, 2, 4, 7, or 8 hours, whereas for product B it is approximately 1, 3, 7, 9, or 12 hours per unit. However, the total amount of labor available every day is approximately 4, 6, 8, 12, or 17 hours. Compute the ideal quantity of items A and B to produce in order to maximize the overall income.

Solution: Let us assume that the amount of components of A and B that are to be created is denoted by x_1 and x_2, respectively. The issue described above may therefore be represented as follows:

$$MaximizeZ = \frac{(2,3,5,7,10)x_1 + (1,3,5,8,11)x_2 + (9,12,14,18,20)}{(2,4,6,8,11)x_1 + (2,3,6,9,12)x_2 + (5,8,11,12,15)}$$

Subject to

$$(3,4,8,10,12)x_1 + (2,6,7,9,14)x_2 \leq (5,9,12,15,22)$$
$$(1,2,4,7,8)x_1 + (1,3,7,9,12)x_2 \leq (4,6,8,12,17)$$

$$x_1 \geq 0, x_2 \geq 0$$

By using a Proposed Pentagonal ranking function formula (4.1):

$$\Re(\tilde{\sigma}) = \frac{1}{8}\upsilon_1 + \frac{1}{2}\upsilon_2 + \frac{3}{4}\upsilon_3 + \frac{1}{2}\upsilon_4 + \frac{1}{8}\upsilon_5$$

The FLFPP is converted into crisp linear fractional programming, therefore we have the following:

$$MaximizeZ = \frac{(10.25)x_1 + 10.75)x_2 + (29.125)}{(12)x_1 + (12.25)x_2 + (20.75)}$$

Subject to

$$(14.875)x_1 + (14.75)x_2 \leq (24.375)$$
$$(8.625)x_1 + (12.875)x_2 \leq (17.625)$$

$$x_1 \geq 0, x_2 \geq 0$$

By using a proposed method mentioned in section 6 we find optimal solution of FLFP problems, as follows:
let $k \leq \frac{1}{(12)x_1+(12.25)x_2+(20.75)}$ and multiply first and second constraint by k and putting $m_1 = kx_1, m_2 = kx_2$, we get the following linear programming problem:

$$MaximizeZ = (10.25)m_1 + (10.75)m_2 + (29.125)k$$

Subject to

$$\begin{aligned}(14.875)m_1 + (14.75)m_2 &\leq (24.375)k\\(8.625)m_1 + (12.875)m_2 &\leq (17.625)k\\(12)m_1 + (12.25)m_2 + (20.75)k &\leq 1\end{aligned}$$

$$x_1, x_2, k \geq 0$$

Using simplex algorithm to solve the problem, we get $m_1 = 0.01, m_2 = 0.02, k = 0.02$ Therefore, the optimal solution is $x_1 = 0.5, x_2 = 1$ $Max.Z = 1.153$.

10. Results and Discussion

10.1. Main Discoveries

The study presents a method to address challenges, in linear fractional programming (FFLFP) by utilizing a ranking function based on pentagonal fuzzy numbers. This approach offers advantages;

1. Simplification; The proposed method simplifies FFLFP problems by converting them into linear programming (FFLP) problems allowing the use of the known simplex algorithm for finding solutions.

2. Enhanced Precision; Pentagonal fuzzy numbers provide an way to represent uncertainty compared to traditional triangular or trapezoidal fuzzy numbers. This enhanced representation aids in decision making under uncertainty by facilitating solution discovery.

3. Flexibility; The method can be customized for scenarios involving uncertainty and vagueness. The proposed ranking system adeptly handles objectives and constraints making it suitable for applications.

10.2. Discussion

The article effectively uses two scenarios to show how the proposed method can solve fuzzy linear fractional programming (FFLFP) problems. In the scenario focusing on making handcrafted tables (Product A) maximizes profits, for Phoenix Woodworks demonstrating how fuzzy parameters impact the solution. The second scenario illustrates that producing Product B leads to the income for a business even with multiple profit goals emphasizing the importance of fuzzy resource constraints in decision making. These examples highlight how the method offers easy to understand solutions while addressing uncertainties in real life situations showcasing its potential for optimizing resources, in fuzzy environments.

11. Conclusion

This research introduces an approach, to tackling FFLFP challenges using a ranking system based on pentagonal fuzzy numbers.By transforming these challenges into linear programming (FFLP) puzzles this approach effectively deals with the complexities of FFLFP issues. It allows for the use of the known simplex algorithm to find the solutions. This method comes with benefits such, as improved problem handling, better accuracy in representing uncertainty using numbers and versatility in various practical situations. These advantages aid in reaching solutions and gaining insight into decision making processes in settings.To demonstrate how this method is applied the study presents instances related to production planning challenges, with goals and limitations. These cases highlight how the suggested approach optimizes resource distribution and decision making in real world scenarios where uncertainties play a role.

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