Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 10** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.81622

Domain of the Matrix A_λ in the Space of p-Bounded Variation Sequences

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ABSTRACT: In this paper, we defined the sequence space $\mathcal{A}_\lambda(bv_p)$ of p-bounded variation using the triangle matrix $\mathcal{A}_\lambda$ of non-absolute type. We analyzed the topological properties and defined the Schauder basis of the sequence space $\mathcal{A}_\lambda(bv_p)$. Also, the Köthe duals of $\mathcal{A}_\lambda(bv_p)$ have been computed. Finally, we characterize certain classes of matrix transformations concerning this sequence space.

Keywords: Sequence space, $\mathcal{A}_\lambda$-matrix, Köthe duals, matrix transformations.

Contents

1. Introduction

For s as the set of all infinite sequences in the set of complex numbers, $\mathbb{C}$, any subspace of s is known as a sequence space. The symbols bv, c, c_0, and ℓ_∞ denote the set of all sequence spaces of bounded variation, convergent, null and bounded sequences, respectively. Furthermore, we shall denote the spaces of all absolutely convergent series and p-absolutely convergent series by ℓ_1 and ℓ_p, respectively. Throughout the text, $\mathbb{N}$ is the set of natural numbers including zero.

If $\mathcal{A}$ is an infinite matrix with complex entries a_{rv}, for every $r, v \in \mathbb{N}$ (see [16]). Also, a sequence $x = (x_v)$ is called as $\mathcal{A}$-summable to ξ in $\mathbb{C}$ if $\mathcal{A}x \to \xi$, where ξ is known as $\mathcal{A}$-limit of x. Further, the matrix domain $X_\mathcal{A}$ of $\mathcal{A}$ is defined as

$$X_\mathcal{A} = \{x \in s : \mathcal{A}x \in X\} \tag{1.1}$$

Several researchers have constructed the approach to study the sequences spaces by using the matrix domain of a triangle matrix. They studied the spaces c_{N_q} and $(\ell_\infty)_{N_q}$ [23], $\ell(p)_{C_1} = X_p$ and $(\ell_\infty)_{C_1} = X_\infty$ [20], $(\ell_\infty)_{R^t} = r^t_\infty$, $c_{R^t} = r^t_c$ and $(c_0)_{R^t} = r^t_0$ [1,17], and $(\ell_p)_{E^t} = e^t_p$ and $(\ell_\infty)_{E^t} = e^t_\infty$ [2,3,19] for $1 \le p \le \infty$, where C_1, N_q, E^t and R^t indicate the Cesàro means of order one, Nörlund means, Euler means of order t and Riesz means, respectively. Başar and Altay [8] introduced the sequence space bv_p as follows:

$$bv_p = \left\{ x = (x_v) \in s : \sum_v |x_v - x_{v-1}|^p < \infty \right\},\ 1 \le p < \infty$$

and

$$bv_\infty = \left\{ x = (x_v) \in s : \sup_{v \in \mathbb{N}} |x_v - x_{v-1}| \right\}.$$

2020 *Mathematics Subject Classification:* 40A05, 46A45.
Submitted January 21, 2026. Published April 17, 2026

Başar et al. [9] introduced and studied $bv(u,p)$ and $bv_\infty(u,p)$ as the spaces of non-absolute type generated from Maddox's spaces $\ell(p)$ and $\ell_\infty(p)$ as follows:

$$bv_\infty(u,p) = \left\{ x = (x_v) \in s : \sum_v |u_v \Delta x_v|^{p_v} < \infty \right\},\ 0 < p_v \leq H < \infty,$$
$$\ell_\infty(u,p) = \left\{ x = (x_v) \in s : \sup_{v\in\mathbb{N}} |u_v \Delta x_v|^{p_v} < \infty \right\}.$$

For a strictly increasing unbounded sequence (λ_v) of positive reals, the matrix $\mathcal{A}_\lambda = (a_{rv}(\lambda))$ and the backward difference matrix $\Delta = (\delta_{rv})$ are defined as

$$a_{rv}(\lambda) = \begin{cases} \dfrac{\lambda_v - 2\lambda_{v-1} + \lambda_{v-2}}{\lambda_r - \lambda_{r-1}}, & 0 \leq v \leq r \\ 0, & v > r \end{cases}$$

and

$$\delta_{rv}(\lambda) = \begin{cases} (-1)^{r-v}, & r-1 \leq v \leq r \\ 0, & 0 \leq v < r-1 \ or\ v > r \end{cases},$$

for all $v,\ r \in \mathbb{N}$.

In this paper, we introduce the sequence space $\mathcal{A}_\lambda(bv_p)$ of p-bounded variation and investigate the behaviors of new sequence space according to its topological properties. Further, we construct the basis and examine the Köthe duals of the new space. We have also characterized some matrix classes related to the space $\mathcal{A}_\lambda(bv_p)$.

2. The Sequence Space $\mathcal{A}_\lambda(bv_p)$

In this section, we defined the sequence space $\mathcal{A}_\lambda(bv_p)$ and proved it as a BK-space of non-absolute type. It is also verified that $\mathcal{A}_\lambda(bv_p)$ is isomorphic to ℓ_p.

Braha and Başar [13] have defined the convergent, null and bounded sequences by means of the matrix domain of a triangle matrix over a normed space as

$$\mathcal{A}_\lambda(c_0) = \left\{ x \in s : \lim_{r\to\infty} (\mathcal{A}_\lambda x)_r = 0 \right\},$$
$$\mathcal{A}_\lambda(c) = \left\{ x \in s : \exists\ \xi \in \mathbb{R} \ni \lim_{r\to\infty} (\mathcal{A}_\lambda x)_r = \xi \right\},$$
$$\mathcal{A}_\lambda(\ell_\infty) = \left\{ x \in s : \sup_{r\in\mathbb{N}} |(\mathcal{A}_\lambda x)_r| < \infty \right\}.$$

For $\mu \in \{c_0,\ c,\ \ell_\infty\}$, the norm $\|\cdot\|_{\mathcal{A}_\lambda(\mu)}$ on $\mathcal{A}_\lambda(\mu)$ is defined as

$$\|x\|_{\mathcal{A}_\lambda(\mu)} = \sup_{r\in\mathbb{N}} \frac{1}{\lambda_r - \lambda_{r-1}} \sum_{v=0}^{r} |(\lambda_v - 2\lambda_{v-1} + \lambda_{v-2})x_v|. \tag{2.1}$$

Trivially, $\|x\|_{\mathcal{A}_\lambda(\mu)} < \infty$, for $x \in \mathcal{A}_\lambda(\mu)$. Also, the bv-norm $\|\cdot\|_{bv}$ is defined as

$$\|x\|_{bv} = \sum_v |x_v - x_{v-1}|. \tag{2.2}$$

Yeşilkayagil and Başar [25] have examined the $\mathcal{A}_\lambda$-almost null and $\mathcal{A}_\lambda$-almost convergent sequence spaces respectively, i.e.,

$$\mathcal{A}_\lambda(f_0) = \left\{ x = (x_v) \in s : \lim_{n\to\infty} \frac{1}{n+1} \sum_{v=0}^{n} (\mathcal{A}_\lambda x)_{r+v} = 0, \text{uniformly in } r \right\},$$
$$\mathcal{A}_\lambda(f) = \left\{ x = (x_v) \in s : \exists\, \xi \in \mathbb{R},\ \lim_{n\to\infty} \frac{1}{n+1} \sum_{v=0}^{n} (\mathcal{A}_\lambda x)_{r+v} = \xi, \text{ uniformly in } r \right\}.$$

Note that $\mathcal{A}_\lambda(f_0)$ and $\mathcal{A}_\lambda(f)$ are the domains of $\mathcal{A}_\lambda$ with respect to f_0 and f, respectively.

Following [6,8], we constructed a new sequence space by using matrix domain over a triangle limitation method. We define $\mathcal{A}_\lambda(bv_p)$ as the set of all sequences such that $\mathcal{A}_\lambda$-transforms belong to $\ell_p (1 \le p < \infty)$ as

$$\mathcal{A}_\lambda(bv_p) = \left\{ x = (x_v) \in s : \sum_v |F_r(x)|^p < \infty \right\},$$

where

$$F_r(x) = \frac{1}{\lambda_r - \lambda_{r-1}} \sum_{v=0}^{r} (\lambda_v - 2\lambda_{v-1} + \lambda_{v-2})(x_v - x_{v-1}),\ v \in \mathbb{N}. \tag{2.3}$$

With the notation of (1.1), we can write

$$bv_p = (\ell_p)_{\mathcal{A}_\lambda}, \quad 1 \le p < \infty. \tag{2.4}$$

It is evident that the space $\mathcal{A}_\lambda(bv_p)$ can be reduced to some new sequence space $\mathcal{A}_\lambda(bv)$ for $p = 1$, (see, [21]).

Throughout the paper, we shall assume the terms of the sequences $x = (x_v)$ and $y = (y_v)$ are related as: y is F-transform of x, i.e,

$$y_v = F_r(x) = \sum_{v=0}^{r} \left(\frac{\lambda_v - 2\lambda_{v-1} + \lambda_{v-2}}{\lambda_r - \lambda_{r-1}} \right) (x_v - x_{v-1}),\ r \in \mathbb{N}. \tag{2.5}$$

Theorem 2.1 *The space $\mathcal{A}_\lambda(bv_p)$ is a linear space with respect to the usual coordinatewise addition and scalar multiplication and also BK-space with norm $\|x\|_{\mathcal{A}_\lambda(bv_p)} = \|\mathcal{A}_\lambda x\|_{\ell_p}$, $1 \le p \le \infty$.*

Proof: Since (2.4) holds, ℓ_p is BK-space with respect to their natural norms (see, [12]) and the matrix $\mathcal{A}_\lambda$ is triangle. Therefore, Theorem 4.3.2 of Wilansky [24] gives the fact that the space $\mathcal{A}_\lambda(bv_p)$ is a BK-space, where $1 \le p \le \infty$. □

Remark 2.1 *It is an easy part to prove that the absolute property is not true for the space $\mathcal{A}_\lambda(bv_p)$ that is, $\|x\|_{\mathcal{A}_\lambda(bv_p)} = \||x|\|_{\mathcal{A}_\lambda(bv_p)}$ for at least one sequence in the space $\mathcal{A}_\lambda(bv_p)$. Thus, $\mathcal{A}_\lambda(bv_p)$ is of non-absolute type.*

Theorem 2.2 *The space $\mathcal{A}_\lambda(bv_p)$ of p-bounded variation of non-absolute type is norm isomorphic to ℓ_p.*

Proof: Define the linear map $\mathcal{A}_\lambda(bv_p) \to \ell_p$ by $x \to \mathcal{A}_\lambda y = Tx = F(x)$. It is obvious that $x = \theta$ whenever $Tx = \theta$, and hence T is injective. Further, for $y \in \ell_p$, $1 \le p < \infty$, define the sequence $x = (x_v)$ by

$$x_v = \sum_{j=0}^{v} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i - \lambda_{i-1}}{(\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})} y_i, \tag{2.6}$$

so that

$$x_v - x_{v-1} = \sum_{i=v-1}^{v} (-1)^{v-i} \frac{\lambda_i - \lambda_{i-1}}{(\lambda_v - 2\lambda_{v-1} + \lambda_{v-2})} y_i.$$

Thus, from (2.3), we get

$$F_r(x) = \frac{1}{\lambda_r - \lambda_{r-1}} \sum_{v=0}^{n} \sum_{i=v-1}^{v} (-1)^{v-i} \lambda_i - \lambda_{i-1} y_i$$
$$= \sum_{v=0}^{r} \frac{(\lambda_v - \lambda_{v-1}) y_v - (\lambda_{v-1} - \lambda_{v-2}) y_{v-1}}{\lambda_r - \lambda_{r-1}} = y_r,$$

for all $v \in \mathbb{N}$. This implies that $F(x) \in \ell_p$. Consequently T is bijective and norm preserving. Hence, T is linear bijection which implies the spaces $\mathcal{A}_\lambda(bv_p)$ and ℓ_p are linearly isomorphic. □

Remark 2.2 *The space $\mathcal{A}_\lambda(bv_p)$ is not a Hilbert space for $1 \le p \le \infty$, except for $p = 2$.*

3. The Basis and Köthe Duals of $\mathcal{A}_\lambda(bv_p)$

In this section, we discuss about the sequence of the elements of the space $\mathcal{A}_\lambda(bv_p)$ which form the basis for this space. We also analyze the results which determine the Köthe duals of the space $\mathcal{A}_\lambda(bv_p)$. Here we consider the case for $1 < p \le \infty$ and leave the case $p = 1$ to the reader as it can be proved in the similar way.

A sequence (y_r) in a normed space U is called a Schauder basis for U if for each $x \in U$, there exists a unique sequence of scalars (γ_r) such that $x = \sum_v \gamma_v y_v$, that is

$$\lim_r \|x - \sum_{v=0}^{r} \gamma_v y_v\| = 0.$$

Thus, by Theorem 2.2, T is bijective, therefore the inverse image of the basis $e^{(v)}$ of the space ℓ_p is the basis for the space $\mathcal{A}_\lambda(bv_p)$.

Theorem 3.1 *Suppose that $e^{(r)}(\lambda) \in \mathcal{A}_\lambda(bv_p)$ for every fixed $r \in \mathbb{N}$ is a sequence defined by*

$$e_v^{(r)} = \begin{cases} (-1)^{v-1} \dfrac{\lambda_r - \lambda_{r-1}}{(\lambda_v - 2\lambda_{v-1} + \lambda_{v-2})}, & r \le v \le r-1 \\ 0, & v > r \end{cases} . \tag{3.1}$$

Then, $\{e_v^{(r)}\}$ is a basis for space $\mathcal{A}_\lambda(bv_p)$ and every $x \in \mathcal{A}_\lambda(bv_p)$ is uniquely expressed as $x = \sum_r \gamma_r(\lambda) {e_v}^{(r)}(\lambda)$, where $\gamma_r(\lambda) = F_r(x)$, $\forall\ r \in \mathbb{N}$.

Remark 3.1 Let μ be any sequence space. Then the α-dual, the β-dual and the γ-dual of μ denoted as μ^α, μ^β and μ^γ are as follows:

$$\mu^\alpha = \{a = (a_v) \in s : ax = (a_v x_v) \in \ell_1\ \forall\ x = (x_v) \in \mu\},$$
$$\mu^\beta = \{a = (a_v) \in s : ax = (a_v x_v) \in cs\ \forall\ x = (x_v) \in \mu\},$$
$$\mu^\gamma = \{a = (a_v) \in s : ax = (a_v x_v) \in bs\ \forall\ x = (x_v) \in \mu\}.$$

We now begin with the following Lemmas which are needed to examine the duals.

Lemma 3.1 *[22] $\mathcal{A} \in (\ell_p : \ell_1)$ if and only if $\sup_{K \in \mathcal{G}} \sum_v \left|\sum_{r \in K} a_{rv}\right|^q < \infty$, where $\mathcal{G}$ is the family of all finite subset of $\mathbb{N}$.*

Lemma 3.2 *[22] $\mathcal{A} \in (\ell_p : c)$ if and only if*

$$\lim_{r \to \infty} a_{rv} \text{ exists for all } v \in \mathbb{N}, \tag{3.2}$$

$$\sup_{r \in \mathbb{N}} \sum_v |a_{rv}|^q < \infty, \quad (1 < p < \infty). \tag{3.3}$$

Lemma 3.3 *[22] $\mathcal{A} \in (\ell_\infty : c)$ if and only if (3.2) holds and*

$$\lim_{r\to\infty} \sum_v |a_{rv}| = \sum_v \left| \lim_{r\to\infty} a_{rv} \right|. \tag{3.4}$$

Lemma 3.4 *[22] $\mathcal{A} \in (\ell_p : \ell_\infty)$ if and only if (3.3) holds with $1 < p \leq \infty$.*

Theorem 3.2 *The α-dual of the space $\mathcal{A}_\lambda(bv_p)$ is the set*

$$d_1^q(\lambda) = \left\{ a = (a_r) \in s : \sup_{K\in\mathcal{G}} \sum_r \left| \sum_{v\in K} b_{rv}(\lambda) \right|^q < \infty \right\},$$

where $B = \{b_{rv}(\lambda)\}$ is defined by

$$b_{rv}(\lambda) = \begin{cases} (-1)^{r-v} \dfrac{\lambda_v - \lambda_{v-1}}{(\lambda_r - 2\lambda_{r-1} + \lambda_{r-2})} a_r, & r-1 \leq v \leq r \\ 0, & v > r \end{cases}.$$

Proof: Suppose that $a = (a_r) \in s$. Then, using the equations (2.5) and (2.6), we obtain

$$a_r x_r = a_r \sum_{v=r-1}^{r} (-1)^{r-v} \frac{\lambda_v - \lambda_{v-1}}{(\lambda_r - 2\lambda_{r-1} + \lambda_{r-2})} y_v = (By)_r. \tag{3.5}$$

Thus, $ax = (a_r x_r) \in \ell_1$, whenever $x \in \mathcal{A}_\lambda(bv_p)$ if and only if $By \in \ell_1$ whenever $y \in \ell_p$. Thus $a \in (A_\lambda(bv_p))^\alpha$ whenever $B \in (\ell_p : \ell_1)$. By using Lemma 3.1, we derive with B instead of $\mathcal{A}$ that the sequence $a = (a_r)$ is in the α-dual of $A_\lambda(bv_p)$ if and only if

$$\sup_{K\in\mathcal{G}} \sum_r \left| \sum_{v\in K} b_{rv}(\lambda) \right|^q < \infty, \tag{3.6}$$

which gives us that $(A_\lambda(bv_p))^\alpha = d_1^q(\lambda)$. □

Theorem 3.3 *Suppose that*

$$d_2^q(\lambda) = \left\{ a = (a_r) \in s : \sup_r \sum_{v=0}^{r-1} |\phi_v(r)|^q < \infty \right\}$$

and

$$d_3^q(\lambda) = \left\{ a = (a_r) \in s : \sup_v \left| \frac{\lambda_v - \lambda_{v-1}}{\lambda_v - 2\lambda_{v-1} + \lambda_{v-2}} a_v \right|^q < \infty \right\},$$

where

$$\phi_v(r) = (\lambda_v - \lambda_{v-1}) \left[\frac{a_v}{\lambda_v - 2\lambda_{v-1} + \lambda_{v-2}} + \left(\frac{1}{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}} - \frac{1}{\lambda_{r+1} - 2\lambda_r + \lambda_{r-1}} \right) \sum_{k=v+1}^{r} a_k \right],$$

for $v < r$. Then, $(\mathcal{A}_\lambda(bv_p))^\beta = d_2^q(\lambda) \cap d_3^q(\lambda)$.

Proof: Consider $a = (a_v) \in s$. Using the relation (2.6) between (x_v) and (y_v), we get

$$\sum_{v=0}^{r} a_v x_v = \sum_{v=0}^{r} a_v \sum_{k=0}^{v} \left(\sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j - \lambda_{j-1}}{(\lambda_k - 2\lambda_{k-1} + \lambda_{k-2})} y_j \right)$$

$$= \sum_{v=0}^{r-1} (\lambda_v - \lambda_{v-1}) \left[\frac{a_v}{\lambda_v - 2\lambda_{v-1} + \lambda_{v-2}} + \left(\frac{1}{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}} - \frac{1}{\lambda_{r+1} - 2\lambda_r + \lambda_{r-1}} \right) \sum_{k=v+1}^{r} a_k \right] y_v \tag{3.7}$$

$$+ \frac{\lambda_r - \lambda_{r-1}}{(\lambda_r - 2\lambda_{r-1} + \lambda_{r-2})} a_r y_r$$

$$= \sum_{v=0}^{r-1} \phi_v(r) y_v + \frac{\lambda_r - \lambda_{r-1}}{(\lambda_r - 2\lambda_{r-1} + \lambda_{r-2})} a_r y_r$$

$$= (Cy)_r, \quad \text{for all } r \in \mathbb{N},$$

where the matrix $C = \{c_{rv}(\lambda)\}$ is given as

$$c_{rv}(\lambda) = \begin{cases} \phi_v(r), & 0 \le v \le r-1, \\ \dfrac{(\lambda_r - \lambda_{r-1})\, a_r}{(\lambda_r - 2\lambda_{r-1} + \lambda_{r-2})}, & v = r, \\ 0, & v > r \end{cases}.$$

Thus, from equation (3.7), we deduce that $ax = (a_v x_v) \in cs$, whenever $x \in A_\lambda(bv_p)$ if and only if $Cy \in c$, whenever $y \in \ell_p$. This implies that $a \in A_\lambda(bv_p)^\beta$ if and only if $C \in (\ell_p : c)$. Therefore, from Lemma 3.1 and Lemma 3.2, we obtain

$$\sup_r \sum_{v=0}^{r-1} |\phi_v(r)|^q < \infty$$

and

$$\sup_v \left| \frac{\lambda_v - \lambda_{v-1}}{\lambda_v - 2\lambda_{v-1} + \lambda_{v-2}} a_v \right|^q < \infty.$$

Hence, $(A_\lambda(bv_p))^\beta = d_2^q(\lambda) \cap d_3^q(\lambda)$.
It is trivial to prove the case $p = \infty$ in similar fashion by using Lemma 3.3, so we omit the details. □

Theorem 3.4 *$(\mathcal{A}_\lambda(bv_p))^\gamma = d_2^q(\lambda)$, where $1 < p \le \infty$.*

Proof: This can be obtained by a similar way using Lemma 3.4 and Theorem 3.3. □

4. Matrix Transformations of the Space $\mathcal{A}_\lambda(bv_p)$

In this section we characterize certain matrix transformation from the space $\mathcal{A}_\lambda(bv_p)$ into the spaces ℓ_∞, ℓ_p and ℓ_1, respectively. First, we recall some notations. Let U and V be any subsets of s. Let $\mathcal{A} = (a_{rv})$ be an infinite matrix with complex entries (a_{rv}). By $\mathcal{A}(x) = (\mathcal{A}_r(x))$, we write the $\mathcal{A}$-transform of a sequence (x_v) if the series

$$\mathcal{A}_r(x) = \sum_v a_{rv} x_v$$

is convergent for $r \ge 0$. If $\mathcal{A}x \in V$ with $x \in U$, then we say that $\mathcal{A}$ defines a matrix mapping from U into V. Further, (U, V) indicates the family of all infinite matrices that maps U into V.
For brevity in notation we shall write $\Delta x_v = x_v - x_{v-1}$, with

$$\tilde{a}_{rv} = (\lambda_v - \lambda_{v-1}) \left(\frac{a_{rv}}{\lambda_v - 2\lambda_{v-1} + \lambda_{v-2}} \right),$$

and

$$a(r, v) = \sum_{i=0}^{r} a_{iv}, \text{ for all } r, v \in \mathbb{N}.$$

Lemma 4.1 *[24] The matrix mappings between the BK-spaces are continuous.*

Theorem 4.1 *Let $1 < p < \infty$. Then $\mathcal{A} = (a_{rv}) \in (\mathcal{A}_\lambda(bv_p) : \ell_\infty)$ if and only if*

$$\sup_{m\in\mathbb{N}} \sum_v \left| \sum_{j=v}^{m} \left(\frac{\lambda_j - \lambda_{j-1}}{\lambda_v - 2\lambda_{v-1} + \lambda_{v-2}} a_{rj} \right) \right|^p < \infty, \quad r \in \mathbb{N}. \tag{4.1}$$

$$\sup_{r\in\mathbb{N}} \sum_v |\tilde{a}_{rv}|^p < \infty. \tag{4.2}$$

Proof: Take $\mathcal{A} \in (\mathcal{A}_\lambda(bv_p) : \ell_\infty)$ and $1 < p < \infty$. Then, $\mathcal{A}x$ exist and is in ℓ_∞ for every $x \in (\mathcal{A}_\lambda(bv_p)$. Thus, $(a_{rv}) \in (\mathcal{A}_\lambda(bv_p)^\beta \ \forall\ r \in \mathbb{N}$ by hypothesis, which implies the necessity of equation (4.1).
Consider the following equality and the sequences x_v and y_v are derived by using the relation (2.6) from the mth partial sum of the series $\sum_v a_{rv}x_v$, we obtain

$$\sum_{v=0}^{m} a_{rv}x_v = \sum_{v=0}^{m-1} \tilde{a}_{rv}y_v + \frac{\lambda_m - \lambda_{m-1}}{(\lambda_m - 2\lambda_{m-1} + \lambda_{m-2})} a_{rm}y_m, \ \forall\ m, r \in \mathbb{N}. \tag{4.3}$$

Therefore, passing $m \to \infty$ in equation (4.3), we obtain

$$\sum_v a_{rv}x_v = \sum_v \tilde{a}_{rv}y_v, \ \forall\ r \in \mathbb{N}. \tag{4.4}$$

Since, $\mathcal{A}_\lambda(bv_p)$ and ℓ_∞ are BK-spaces and by using Lemma 4.1, there exist a constant M such that

$$\|\mathcal{A}x\|_{\ell_\infty} \le M \|x\|_{\mathcal{A}_\lambda(bv_p)},$$

for every $x \in \mathcal{A}_\lambda(bv_p)$. Therefore, by using the Hölder's inequality in equation (4.4), it is immediate to see that

$$\frac{\|(\mathcal{A}x)\|_{\ell_\infty}}{\|y\|_{\ell_p}} \le \sup_{r\in\mathbb{N}} \left(\sum_v |\tilde{a}_{rv}|^q \right)^{1/q} < \infty,$$

for every $x \in \mathcal{A}_\lambda(bv_p)$, which yields the necessity of (4.2).
Conversely, suppose that conditions (4.1) and (4.2) hold and let any $x = (x_r) \in (\mathcal{A}_\lambda(bv_p)$. Then the sequence $a_{rv} \in \mathcal{A}_\lambda(bv_p)^\beta$ for every $r \in \mathbb{N}$ and this implies the existence of $\mathcal{A}$-transforms of x. Furthermore, by bearing in mind that $y = (y_v) \in \ell_p$ and using Theorem 2.2, and by virtue of Hölder's inequality to equation (4.2), we obtain

$$\|\mathcal{A}x\|_{l_\infty} = \sup \left| \sum \tilde{a}_{rv}y_v \right| \le \sup_{r\in\mathbb{N}} \left(\sum |\tilde{a}_{rv}|^q \right)^{1/q} \left(\sum |y_v|^p \right)^{1/p} < \infty,$$

which shows $\mathcal{A} \in (\mathcal{A}_\lambda(bv_p) : \ell_\infty)$. □

Now, we make use of the following Lemma given in [22].

Lemma 4.2 *$\mathcal{A} \in (\ell_\infty : \ell_p)$ if and only if*

$$\sup_{K\in\mathcal{G}} \sum_r \left| \sum_{v\in K} a_{rv} \right|^p < \infty, \ 1 \le p < \infty. \tag{4.5}$$

Theorem 4.2 *$\mathcal{A} = (a_{rv}) \in (\mathcal{A}_\lambda(bv_\infty) : \ell_p)$ if and only if*

$$\lim_{m\to\infty} \sum_v \left| \sum_{j=v}^{m} \frac{\lambda_j - \lambda_{j-1}}{\lambda_r - 2\lambda_{r-1} + \lambda_{r-2}} a_{rj} \right| = \sum_v |\tilde{a}_{rv}|. \tag{4.6}$$

$$\sup_{K\in\mathcal{G}}\sum_r\left|\sum_{v\in K}\tilde{a}_{rv}\right|^p<\infty,\ 1\le p<\infty. \tag{4.7}$$

$$\sup_{r\in\mathbb{N}}\sum_r|\tilde{a}_{rv}|<\infty,\ p=\infty. \tag{4.8}$$

Proof: Let $\mathcal{A}\in(\mathcal{A}_\lambda(bv_\infty):\ell_p)$. Then since inclusion $\ell_p\subset\ell_\infty$ holds, the necessity of (4.7) and (4.8) are immediately obtained from Theorem 4.1. To prove the necessity of condition (4.6) for both the cases $1\le p<\infty$ and $p=\infty$, consider the sequence $e^{(r)}(\lambda)=\{e_v^{(r)}(\lambda)\}_{r\in\mathbb{N}}$ defined by equation (3.1), which is in the space $\mathcal{A}_\lambda(bv_\infty)$ for each fixed $v\in\mathbb{N}$. Because $\mathcal{A}x$ exist and is in ℓ_p for each $x\in\mathcal{A}_\lambda(bv_\infty)$, it is immediate to see that $\mathcal{A}e^{(r)}(\lambda)=\tilde{a}_{rv}$ is in ℓ_p for every $v\in\mathbb{N}$, which shows that the condition (4.6) is necessary.
Conversely, suppose the conditions (4.6) and (4.7) are satisfied and $x\in\mathcal{A}_\lambda(bv_\infty)$. Then $\mathcal{A}x$ exists. Since $x\in\mathcal{A}_\lambda(bv_\infty)$ if and only if $y\in\ell_\infty$ by Theorem 2.2. Reconsider the equality $\mathcal{A}x=By$ obtained from (4.4) with b_{rv} instead of $\tilde{a}_{rv}$ and hence by Lemma 4.2 that $\mathcal{A}x\in\ell_p$ whenever $By\in\ell_p$, which shows the sufficiency of (4.6) and (4.7).
Furthermore, suppose (4.6) and (4.8) are satisfied and take $x\in\mathcal{A}_\lambda(bv_\infty)$. Then $\mathcal{A}x$ exist and again by using (4.4), we see that

$$\|\mathcal{A}x\|_{\ell_\infty}=\sup_{r\in\mathbb{N}}\left|\sum_v\tilde{a}_{rv}y_v\right|\le\|y\|_{\ell_\infty}\sup_{r\in\mathbb{N}}\sum_v|\tilde{a}_{rv}|<\infty$$

which prove the sufficiency of (4.6) and (4.8). □

Proposition 4.1 *$\mathcal{A}=(a_{rv})\in(\mathcal{A}_\lambda(bv_p):\ell_1)$ if and only if the condition (4.1) holds and the condition*

$$\sup_{K\in\mathcal{G}}\sum_v\left|\sum_{r\in K}\tilde{a}_{rv}\right|^q<\infty,\ (1<p<\infty). \tag{4.9}$$

Lemma 4.3 *Let U and V be any two sequence spaces, $\mathcal{A}$ is an infinite matrix and B is a triangle matrix. Then $\mathcal{A}\in(U:V_B)$ if, and only if $B\mathcal{A}\in(U:V)$.*

Continuing in the similar manner, we have the following results.

Corollary 4.1 *Let $\mathcal{A}=(a_{rv})$ is an infinite matrix over $\mathbb{C}$. Then the following are hold:*

(i) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):bv_\infty)$ *if and only if (4.1) and (4.2) hold with e_{rv}, where* $e_{rv}=a_{rv}-a_{(r-1)v}\ \ \forall\ r,v\in\mathbb{N}$ *instead of a_{rv}.*

(ii) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):X_\infty)$ *if and only if (4.1) and (4.2) hold with e_{rv}, where* $e_{rv}=\sum_{j=0}^{r}\frac{a_{jv}}{1+r},\forall\ r,v\in\mathbb{N}$ *instead of a_{rv}.*

(iii) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):r_\infty^q)$ *if and only if (4.1) and (4.2) hold with e_{rv}, where* $e_{rv}=\sum_{j=0}^{r}\frac{q_j}{Q_r}a_{jv},\forall\ r,v\in\mathbb{N}$ *instead of a_{rv}.*

(iv) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):a_\infty^t)$ *if and only if (4.1) and (4.2) hold with e_{rv}, where* $e_{rv}=\sum_{j=0}^{r}\frac{1+t^j}{1+r}a_{jv},\forall r,v\in\mathbb{N}$ *instead of a_{rv}.*

(v) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):e_\infty^t)$ *if and only if (4.1) and (4.2) hold with e_{rv}, where* $e_{rv}=\sum_{j=0}^{r}\binom{r}{j}(1-t)^{r-j}t^ja_{jv},\forall\ r,v\in\mathbb{N}$ *instead of a_{rv}.*

Corollary 4.2 *Let $\mathcal{A}=(a_{rv})$ is an infinite matrix over $\mathbb{C}$. Then the following are hold:*

(i) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):bv_p)$ *if and only if (4.6), (4.7) and (4.8) hold with* e_{rv}, *for* $e_{rv}=a_{rv}-a_{r-1,v}$ *instead of* a_{rv}.

(ii) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):X_p)$ *if and only if (4.6), (4.7) and (4.8) hold with* e_{rv}, *for* $e_{rv}=\dfrac{c(r,v)}{r+1}$ *instead of* a_{rv}.

(iii) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):r_p^q)$ *if and only if (4.6), (4.7) and (4.8) hold with* e_{rv}, *for* $e_{rv}=\sum_{j=0}^{r}\dfrac{q_j}{Q_r}a_{jv}$ *instead of* a_{rv}.

(iv) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):a_p^t)$ *if and only if (4.6), (4.7) and (4.8) hold with* e_{rv}, *for* $e_{rv}=\sum_{j=0}^{r}\dfrac{1+t^j}{1+r}a_{jv}$ *instead of* a_{rv}.

(v) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):e_p^t)$ *if and only if (4.6), (4.7) and (4.8) hold with* e_{rv}, *for* $e_{rv}=\sum_{j=0}^{r}\binom{r}{j}(1-t)^{r-j}t^ja_{jv}$ *instead of* a_{rv}.

Corollary 4.3 *Let $\mathcal{A}=(a_{rv})$ is an infinite matrix over $\mathbb{C}$. Then the following are hold:*

(i) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):bv_1)$ *if and only if (4.1) and (4.9) hold with* e_{rv}, *for* $e_{rv}=e_{rv}-e_{r-1,v}$ *instead of* a_{rv}.

(ii) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):X_1)$ *if and only if (4.1) and (4.9) hold with* e_{rv}, *for* $e_{rv}=\dfrac{c(r,v)}{1+r}$ *instead of* a_{rv}.

(iii) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):r_1^q)$ *if and only if (4.1) and (4.9) hold with* e_{rv}, *for* $e_{rv}=\sum_{j=0}^{r}\dfrac{q_j}{Q_r}a_{jv}$ *instead of* a_{rv}.

(iv) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):a_1^t)$ *if and only if (4.1) and (4.9) hold with* e_{rv}, *for* $e_{rv}=\sum_{j=0}^{r}\dfrac{1+t^j}{1+r}a_{jv}$ *instead of* a_{rv}.

(v) $\mathcal{E}=(e_{rv})\in(\mathcal{A}_\lambda(bv_p):e_1^t)$ *if and only if (4.1) and (4.9) hold with* e_{rv}, *for* $e_{rv}=\sum_{j=0}^{r}\binom{r}{j}(1-t)^{r-j}t^ja_{jv}$ *instead of* a_{rv}.

Acknowledgments

The authors are very grateful to all the referees for their careful reading and insightful suggestions.

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