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Ugamma Distribution and Rational Hazard Rate Function: Statistical Properties, Goodness-of-Fit Testing and Practical Applications

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ABSTRACT: This work presents an innovative approach to developing a new continuous distribution, capitalizing on recent advances in stochastic modeling and weighted distributions. The proposed approach is based on an original methodological framework for deriving probability density functions from the rth moment of a continuous random variable and its cumulative distribution function. Within this framework, we introduce the Ugamma distribution as a concrete illustration and investigate its distinctive features through a comprehensive analysis of its statistical properties. This includes the study of the survival and hazard rate functions, moments, and measures of variability, demonstrating the flexibility and effectiveness of the Ugamma distribution in modeling diverse statistical phenomena. We also develop a modified chi-square goodness-of-fit test based on the Nikulin–Rao–Robson statistic for the proposed model. To demonstrate its practical relevance, the Ugamma distribution is applied to both simulated and real-world datasets, including internet usage data, population dynamics, and pesticide concentration measurements. These applications highlight the distribution's ability to model complex real-world phenomena and underscore its versatility across a range of applied contexts.

Keywords: Hazard rate function, moments, Monte Carlo simulation, survival rate, algebraic equations.

Contents

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1. Introduction

The Hazard Rate (HR) function has been one of the most important concepts in survival studies, reliability engineering, and other research areas concerned with lifetimes. It can have a variety of shapes, which have physical bases. For example, in reliability of products, this variety is due to assembly errors, non-conformance of components, flaws in materials, etc. In survival analysis, the shape of the HR function is influenced by ageing processes, living conditions, habits, risk factors, etc.

Numerous lifetime distributions have been developed to model relatively simple non-monotonic hazard rate functions, including increasing, bathtub-shaped, U-shaped, unimodal, or upside-down bathtub patterns.

In many real-life scenarios, we encounter objects—whether synthetic devices or living organisms—that go through three distinct phases of life: an initial high-risk phase, a functional usage phase, and a wear-and-tear phase.

During the early phase, the item typically experiences a higher-than-average hazard rate (HR), which gradually decreases over time as it transitions into a more reliable state. Following this, the usage phase exposes the item to a relatively low risk. Eventually, the wear-and-tear phase begins, during which the HR increases over time due to factors such as erosion or aging. This aging behavior is often modeled using a Bathtub-Shaped Hazard Rate (BSHR), which is widely observed in both scientific and industrial applications.

There is a large literature dealing with the BSHR lifespan distribution. Rajarshi and Rajarshi [30] and Lai et al. [16] have conducted two notable studies on the BSHR model. Many authors have presented models with such behavior, for example, Glaser [9], Mudholkar and Sirvastava [23], Navarro and Hernandez [25], Xie et al. [39], Wang [38] and many others. In addition, Lai and Xie [17] and Nadarajah [24] presented a list of such models. Mi [20] and Gupta and Akman [11] described the Mean Residual Life (MRL) function and showed that when the HR function is bathtub-shaped, the MRL function is either a decreasing or inverted bathtub (UBS) function. Mi [20] applied this result to determine the optimal time for running in. In addition, Block et al. [5] described the mean HR function, MRL function, mean residual life function, mean harmonic residual life function, variance residual life function, and residual entropy function when the HR function is bathtub-shaped. They studied the coefficient of variation of remaining life as a running criterion.

Constructing HR functions with aforementioned shapes can, however, be successfully accomplished by using rational hazard rate function

$$h_R(x;\theta) = \frac{a_1 x^2 + b_1 x + c_1}{a_2 x^2 + b_2 x + c_2}\,, \tag{1.1}$$

where $a_1 = a_1(\theta), b_1 = b_1(\theta), c_1 = c_1(\theta), a_2 = a_2(\theta), b_2 = b_2(\theta), c_2 = c_2(\theta)$.

Next, we introduce and characterize the Ugamma distribution as a specific instance of the proposed framework, where the hazard rate follows a BSHR pattern. Building on recent advancements in statistical theory, we derive the Probability Density Function (PDF) for the Ugamma distribution, emphasizing its unique features. Drawing from contemporary statistical literature, we explore the statistical properties of the Ugamma distribution, including its survival and hazard rate functions, as well as key measures such as moments, variance, skewness, kurtosis, and coefficient of variation.

In addition, we investigate advanced statistical concepts such as incomplete moments, mean residual life function, stochastic scheduling, Lorenz curve, entropy and fuzzy reliability, drawing on contemporary research to enrich our analysis. To test whether data come from a Ugamma distribution, we develop the modified chi-squared goodness-of-fit test based on the Nikulin-Rao-Robson (NRR) statistic. We recall, [26,27] proposed a modification of the standard chi-squared Pearson's test (6.1) for continuous distribution

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with shift and scale parameters, also [32] had obtained the same result for exponential family, and since 1988, the test is well known as the Nikulin-Rao-Robson (NRR) test ([35], [6]). Using the method of moments, [13], (see also [37], [36], [21]) have proposed a modification of the standard Pearson's test (6.1). A simulation study is conducted to generate a wide range of sample sizes and for various values of the parameters of the model.

Finally, we apply the Ugamma distribution to simulated and real data sets, drawing on recent contributions to time series analysis (Durbin and Koopman, [7]), actuarial science (Staudte and Sheather, [34]) and (Finsterwalder, [8]), highlighting its versatility and effectiveness in a variety of fields. Through these applications, we demonstrate the practical relevance and robustness of the Ugamma distribution in modeling complex phenomena.

By synthesizing ideas from recent literature as Modi et al. [22], Alzaatreh et al. [1], Maurya et al. [18] and Beghriche and Zeghdoudi [3] with our proposed methodology, we aim to contribute to the current debate on statistical theory and application, offering new perspectives on modeling continuous distributions in a variety of contexts.

2. Generality Case

2.1. Shape characteristics of the rational hazard rate function

This subsection discusses the shape characteristics of the HRF (1.1) . The behavior of $h_R(x;\theta)$ at $x=0$ and $x=\infty$, are given by

$$\lim_{x\to 0} h_R(x;\theta) = \frac{c_1}{c_2},$$

$$\lim_{x\to\infty} h_R(x;\theta) = \begin{cases} \frac{a_1}{a_2} & \text{if } a_1 \neq 0 \text{ and } a_2 \neq 0 \\ 0 & \text{if } a_1 = 0 \text{ and } a_2 \neq 0 \\ \infty & \text{if } a_1 \neq 0 \text{ and } a_2 = 0 \\ \frac{b_1}{b_2} & \text{if } a_1 = 0 \text{ and } a_2 = 0 \end{cases}.$$

The following proposition states that there are three shapes for the HRF (1.1) , depending on the range of the parameter θ.

Proposition 2.1 *The hazard rate function of $h_R(x;\theta)$ is increasing, decreasing, increasing-decreasing-increasing, decreasing-increasing-decreasing, unimodel, and bathtub-shaped.*

Proof: The derivative of $h_R(x;\theta)$ is obtained as

$$\frac{dh_R(x;\theta)}{dx} = \frac{(P(x,\theta))}{(a_2x^2+b_2x+c_2)^2},$$

where $P(x,\theta) = (a_1b_2 - a_2b_1)\,x^2 + 2\,(a_1c_2 - a_2c_1)\,x + (b_1c_2 - b_2c_1)$. We can see that $\frac{dh_R(x;\theta)}{dx}$ and $P(x,\theta)$ have the same sign because $\left(a_2x^2+b_2x+c_2\right)^2 > 0$. In algebra, a quadratic equation of the form $ax^2 + bx + c = 0$, has $a \neq 0, b, c$ are real numbers, and its discriminant Δ has three cases as the following. If $\Delta > 0$, the has quadratic two distinct real roots. If $\Delta < 0$, the quadratic has two non-real complex conjugate roots. If $\Delta = 0$, the quadratic has two multiple real root. In our case

$$\Delta = \left(2\,(a_1c_2 - a_2c_1)\right)^2 - 4\,(a_1b_2 - a_2b_1)\,(b_1c_2 - b_2c_1)\,.$$

a) When $\Delta > 0$, $P(x,\theta)$ has two distinct real roots x_1, x_2.
1- If $x_1, x_2 > 0$ and $a_1b_2 - a_2b_1 > 0$, the $h_R(x;\theta)$ is increasing-decreasing-increasing;
2- If $x_1, x_2 > 0$ and $a_1b_2 - a_2b_1 < 0$, the $h_R(x;\theta)$ is decreasing-increasing-decreasing;
3- If $x_1 < 0, x_2 > 0$(or $x_1 > 0, x_2 < 0$) and $a_1b_2 - a_2b_1 < 0$, the $h_R(x;\theta)$ is unimodel;
4- If $x_1 < 0, x_2 > 0$(or $x_1 > 0, x_2 < 0$) and $a_1b_2 - a_2b_1 > 0$, the $h_R(x;\theta)$ is BSHR;
b) When $\Delta < 0$, $P(x,\theta)$ has two non-real complex conjugate roots, $z, \bar{z}$.
1- If $a_1b_2 - a_2b_1 > 0$, the $h_R(x;\theta)$ is increasing;
2- If $a_1b_2 - a_2b_1 < 0$, the $h_R(x;\theta)$ is decreasing;
c) When $\Delta = 0$, $P(x,\theta)$ has two multiple real root, $x_1 = x_2 > 0$, the $h_R(x;\theta)$ is increasing. □

3. Formulation of Ugamma Distribution

Let us consider a continuous random variable X, now we introduce the Ugamma distribution by taking $a_1(\theta) = \theta^3, b_1(\theta) = 0, c_1(\theta) = 2\theta, a_2(\theta) = \theta^2, b_2(\theta) = 2\theta, c_2(\theta) = 4$, in 1.1, (a bathtub-shaped in case 4-a above) then we have the hazard rate function

$$h(x;\theta) = \frac{f(x;\theta)}{S(x;\theta)} = \frac{2\theta + \theta^3 x^2}{4 + 2\theta x + \theta^2 x^2},\ x,\ \theta > 0. \tag{3.1}$$

The corresponding pdf, cdf and suvival function of the Ugamma distribution are given by

$$f(x;\theta) = \frac{\theta}{2}\left(1 + \frac{\theta^2 x^2}{2}\right)\exp(-\theta x),\ x,\ \theta > 0, \tag{3.2}$$

$$F(x;\theta) = 1 - \left(1 + \frac{\theta x}{2} + \frac{\theta^2 x^2}{4}\right)\exp(-\theta x),\ x,\ \theta > 0, \tag{3.3}$$

$$S(x;\theta) = 1 - F(x;\theta) = \left(1 + \frac{\theta x}{2} + \frac{\theta^2 x^2}{4}\right)\exp(-\theta x),\ x,\ \theta > 0. \tag{3.4}$$

3.1. Shape characteristics of the PDF of Ugamma distribution

This subsection discusses the shape characteristics of the PDF in (3.2), of the Ugamma distribution . The behavior of Ugamma distribution at $x = 0$ and $x = \infty$, are given by

$$\lim_{x\to 0} f(x;\theta) = \frac{\theta}{2},$$
$$\lim_{x\to\infty} f(x;\theta) = 0.$$

The following proposition states that there are three shapes for the PDF of the U gamma distribution, depending on the range of the parameter θ.

Proposition 3.1 *The PDF in* (3.2) *of the Ugamma distribution is* ***d****ecreasing for* $\theta > 0$.

Proof: The first and the second derivatives of $f(x;\theta)$ are

$$\frac{df(x;\theta)}{dx} = \frac{1}{2}\theta^2 e^{-x\theta}\left(-\frac{\theta^2 x^2}{2} + \theta x - 2\right),$$

and

$$\frac{d^2 f(x;\theta)}{dx^2} = \frac{1}{2}\theta^3 e^{-x\theta}\left(\frac{x^2\theta^2}{2} - 2x\theta + 2\right).$$

Now, $\frac{df(x;\theta)}{dx} < 0$, because $-\frac{\theta^2 x^2}{2} + \theta x - 2 < 0$ for $x, \theta > 0$. In this case, the behaviour of (3.2) is decreasing. □

Remark 3.1 The mode of the Ugamma distribution equal to

$$f(0;\theta) = \frac{\theta}{2}.$$

because PDF in (3.2) is decreasing.

4. Statistical Properties

4.1. Moments and Related Measures

The k^{th} moment about origin of the Ugamma distribution is given by

$$\begin{aligned} E\left[X^k\right] &= \frac{1}{2}\int_0^\infty \theta x^k \left(2+\left(\frac{\theta^2 x^2}{2}-1\right)\right)\exp\left(-\theta x\right)\,dx \\ &= \frac{1}{2\theta^k}\Gamma\left(k+1\right)+\frac{1}{4\theta^k}\Gamma\left(k+3\right). \end{aligned}$$

In particular, the first four moments can be worked out as

$$\begin{aligned} E\left[X\right] &= \frac{2}{\theta} \\ E\left[X^2\right] &= \frac{7}{\theta^2} \\ E\left[X^3\right] &= \frac{33}{\theta^3} \\ E\left[X^4\right] &= \frac{192}{\theta^4}. \end{aligned}$$

The variance of Ugamma distribution is:

$$\sigma^2 = \frac{3}{\theta^2}.$$

Skewness, Kurtosis and Coefficient of variation (CV) of Ugamma distribution

$$\begin{aligned} \text{Skewness} &= \sqrt{\beta_1} = \frac{E\left[X^3\right]}{\left(\sigma^2\right)^{\frac{3}{2}}} = \frac{33}{(3)^{\frac{3}{2}}}, \\ \text{Kurtosis} &= \beta_2 = \frac{E\left[X^4\right]}{\left(\sigma^2\right)^2} = \frac{192}{9}, \\ CV &= \gamma = \frac{\sqrt{\sigma^2}}{E\left[X\right]} = \frac{\sqrt{3}}{2}. \end{aligned}$$

4.2. Incomplete moments

The k^{th} incomplete moment of X can be expressed as follows, according to similar computations:

$$\begin{aligned} T_k\left(t\right) &= E\left[X^k \mid X<t\right] = \frac{1}{F(t;\theta)}\int_0^t x^k f(x;\theta)dx \\ &= \frac{1}{2F(t;\theta)}\int_0^t \theta x^k \left(2+\frac{1}{2}\left(\theta^2x^2-2\right)\right)\exp\left(-\theta x\right)\,dx \\ &= \frac{1}{F(t;\theta)}\int_0^t \theta x^k \exp\left(-\theta x\right)\,dx + \frac{1}{4F(t)}\int_0^t \theta x^k\left(\theta^2x^2-2\right)\exp\left(-\theta x\right)\,dx, \end{aligned}$$

we put $y = \theta x$, so $dx = \frac{1}{\theta} dy$ and $y \in [0, \theta t]$

$$\begin{aligned} T_k(t) &= \frac{1}{\theta^k F(t;\theta)} \int_0^{\theta t} y^k \exp(-y)\ dy + \frac{1}{4\theta^k F(t;\theta)} \int_0^{\theta t} y^k \left(y^2 - 2\right) \exp(-y)\ dy \\ &= \frac{1}{\theta^k F(t;\theta)} \int_0^{\theta t} y^k \exp(-y)\ dy + \frac{1}{4\theta^k F(t;\theta)} \int_0^{\theta t} y^{k+2} \exp(-y)\ dy - \frac{1}{2\theta^k F(t;\theta)} \int_0^{\theta t} y^k \exp(-y)\ dy \\ &= \frac{1}{2\theta^k F(t;\theta)} \int_0^{\theta t} y^k \exp(-y)\ dy + \frac{1}{4\theta^k F(t;\theta)} \int_0^{\theta t} y^{k+2} \exp(-y)\ dy \\ &= \frac{1}{2\theta^k F(t;\theta)} \gamma(k+1, \theta t) + \frac{1}{4\theta^k F(t;\theta)} \gamma(k+3, \theta t), \end{aligned}$$

where $\gamma(s,u) = \int_0^u y^{s-1} \exp(-y)\ dy = (s-1)! \left(1 - \exp(-x) \sum_{k=0}^{s-1} \frac{x^k}{k!}\right)$ the lower incomplete gamma function.

In particular, for $k = 1, 2$ we have

$$\begin{aligned} T_1(t) &= E[X \mid X < t] \\ &= \frac{1}{2\theta F(t)} \gamma(2, \theta t) + \frac{1}{4\theta F(t)} \gamma(4, \theta t) \end{aligned}$$

and

$$\begin{aligned} T_2(t) &= E\left(X^2 \mid X < t\right) \\ &= \frac{1}{2\theta^2 F(t;\theta)} \gamma(3, \theta t) + \frac{1}{4\theta^k F(t;\theta)} \gamma(5, \theta t). \end{aligned}$$

4.3. Moment generating function

The moment generating function of the Ugamma distribution takes the form:

$$\begin{aligned} M(s) &= E[\exp(sX)] = \int_0^\infty \exp(sx) f(x;\theta) dx \\ &= \int_0^\infty \exp(sx) \frac{\theta}{2} \left(2 + \left(\frac{\theta^2 x^2}{2} - 1\right)\right) \exp(-\theta x) dx \\ &= \sum_{k=0}^\infty \frac{s^k}{k!} \int_0^\infty \frac{\theta}{2} x^k \left(2 + \left(\frac{\theta^2 x^2}{2} - 1\right)\right) \exp(-\theta x) dx \\ &= \sum_{k=0}^\infty \frac{s^k}{k!} \left(\frac{1}{2\theta^k} \Gamma(k+1) + \frac{1}{4\theta^k} \Gamma(k+3)\right). \end{aligned}$$

Its characteristic function is obtained by replacing "s" with "is" in the previous equation.

4.4. Mean residual life function

For a non-negative continuous random variable X, the mean residual life function is defined as

$$\begin{aligned} \mu(t \mid \theta) &= E[X - t \mid X > t] \\ &= \frac{\int_t^\infty S(x;\theta \mid \theta) dx}{S(t;\theta \mid \theta)}, \end{aligned}$$

where $S(. \mid \theta)$ is the survival function. And for the Ugamma distribution, we get

$$\mu(t \mid \theta) = \frac{\int_t^\infty \left(1 + \frac{\theta x}{2} + \frac{\theta^2 x^2}{4}\right) \exp(-\theta x) dx}{\left(1 + \frac{\theta t}{2} + \frac{\theta^2 t^2}{4}\right) \exp(-\theta t)}.$$

First, we need to calculate this quantity: $\int_t^\infty \left(1+\frac{\theta x}{2}+\frac{\theta^2x^2}{4}\right)\exp\left(-\theta x\right)dx$

$$\begin{aligned}\int_t^\infty \left(1+\frac{\theta x}{2}+\frac{\theta^2x^2}{4}\right)\exp\left(-\theta x\right)dx &= \int_t^\infty \exp\left(-\theta x\right)dx+\int_t^\infty \frac{1}{2}\theta x\exp\left(-\theta x\right)dx+\int_t^\infty \frac{\theta^2x^2}{4}\exp\left(-\theta x\right)dx\\ &= \frac{1}{\theta}\exp\left(-\theta t\right)+\frac{1}{2\theta}\int_{\theta t}^\infty y\exp\left(-y\right)dy+\frac{1}{4\theta}\int_{\theta t}^\infty y^2\exp\left(-y\right)dy\\ &= \frac{1}{\theta}\exp\left(-\theta t\right)+\frac{1}{2\theta}\Gamma\left(2,\theta t\right)+\frac{1}{4\theta}\Gamma\left(3,\theta t\right)\\ &= \left(1+\left(1+\frac{\theta t}{2}+\frac{\theta^2t^2}{4}\right)\right)\frac{\exp\left(-\theta t\right)}{\theta}\end{aligned}$$

where $\Gamma\left(s,x\right)=\int_x^\infty y^{s-1}\exp\left(-y\right)dy=\left(s-1\right)!\exp\left(-x\right)\sum_{k=0}^{s-1}\frac{x^k}{k!}$ is the upper incomplete gamma function.

Finaly the mean residual life function of Ugamma distribution defind as

$$\mu(t\mid\theta)=\frac{\left(1+\left(1+\frac{\theta t}{2}+\frac{\theta^2t^2}{4}\right)\right)}{\theta\left(1+\frac{\theta t}{2}+\frac{\theta^2t^2}{4}\right)}$$

4.5. Stochastic Ordering

The stochastic ordering of positive continuous random variables is an important tool for judging comparative behaviour. A random variable X_1 is said to be smaller than a random variable X_2 in the

1. Stochastic order $(X_1\leq_S X_2)$, if $\left(F_{X_1}\left(t\right)<F_{X_2}\left(t\right)\right)$, for all t.
2. Hazard rate order $(X_1\leq_h X_2)$, if $\left(h_{X_1}\left(t\right)\geq h_{X_2}\left(t\right)\right)$, for all t.
3. Mean residual life order $(X_1\leq_{mrl} X_2)$, if $\mu_{X_1}\left(t\right)\leq\mu_{X_2}\left(t\right)$, for all t.
4. Likelihood ratio order $(X_1\leq_{lr} X_2)$, if $\frac{f_{X_1}(t)}{f_{X_2}(t)}$ is decreasing in t.

The following findings on stochastic ordering of distributions are widely recognized and are attributed to Shaked and Shanthikumar (1994) [33].

$$X_1\leq_{lr}X_2 \implies \underset{\substack{\Downarrow\\ X_1\leq_S X_2}}{X_1\leq_h X_2} \implies X_1\leq_{mrl}X_2$$

Theorem 4.1 *Let $X_i\sim Ugamma\left(\alpha_i,\theta_i\right)$, $i=1,2$ be two random variables. If $\theta_1>\theta_2$, then $X_1\leq_{lr}X_2$, $X_1\leq_h X_2$, $X_1\leq_{mrl}X_2$ and $X_1\leq_S X_2$.*

Proof: *We have*

$$\frac{f_{X_1}\left(t\right)}{f_{X_2}\left(t\right)}=\frac{\theta_1\left(2+\frac{1}{2}\left(\theta_1^2t^2-2\right)\right)}{\theta_2\left(2+\frac{1}{2}\left(\theta_2^2t^2-2\right)\right)}\exp\left(-\left(\theta_1-\theta_2\right)t\right).$$

Using the $\ln\left(\frac{f_{X_1}(t)}{f_{X_2}(t)}\right)$ *for simplification, we find*

$$\ln\left(\frac{f_{X_1}\left(t\right)}{f_{X_2}\left(t\right)}\right)=\ln\left(\theta_1\right)+\ln\left(2+\frac{1}{2}\left(\theta_1^2t^2-2\right)\right)-\ln\left(\theta_2\right)-\ln\left(2+\frac{1}{2}\left(\theta_2^2t^2-2\right)\right)-\left(\theta_1-\theta_2\right)t,$$

thus

$$\frac{d}{dt}\left(\ln\left(\frac{f_{X_1}\left(t\right)}{f_{X_2}\left(t\right)}\right)\right)=\frac{2t\left(\theta_1^2-\theta_2^2\right)-t\left(\theta_1+\theta_2\right)\left(\theta_1-\theta_2\right)}{\left(2+\frac{1}{2}\left(\theta_1^2t^2-2\right)\right)\left(2+\frac{1}{2}\left(\theta_2^2t^2-2\right)\right)}-\left(\theta_1-\theta_2\right).$$

To this end, if $\theta_1>\theta_2$, we have $\frac{d}{dt}\left(\ln\left(\frac{f_{X_1}(t)}{f_{X_2}(t)}\right)\right)\leq 0$, this means that $X_1\leq_{lr}X_2$ and hence $X_1\leq_h X_2$, $X_1\leq_{mrl}X_2$ and $X_1\leq_S X_2$. Therefore the theorem is proved. □

4.6. Lorenz Curve

The Lorenz curve L is given by

$$L\left(F(x)\right)=\frac{\int_{-\infty}^{x}tf(t;\theta)dt}{E\left[X\right]},$$

where $E\left[X\right]$ denotes the average. For Ugamma distribution we have

$$\begin{aligned}\int_{0}^{x}tf(t;\theta)dt &= \frac{\theta}{2}\int_{0}^{x}t\left(2+\frac{1}{2}\left(\theta^{2}t^{2}-2\right)\right)\exp\left(-\theta t\right)dt\\ &= -\frac{1}{2\theta}\left(2e^{-x\theta}-2+2e^{-x\theta}+2x\theta e^{-x\theta}+\frac{3}{2}x^{2}\theta^{2}e^{-x\theta}+\frac{1}{2}x^{3}\theta^{3}e^{-x\theta}+2x\theta e^{-x\theta}-2\right).\end{aligned}$$

We obtain the Lorenz curve for the Ugamma distribution as follows

$$L(p)=1-\frac{(1-p)}{2}-\frac{\left(x\theta+x^{2}\theta^{2}+2+2x\theta+\frac{1}{2}x^{3}\theta^{3}\right)}{4}e^{-x\theta}.$$

4.7. Entropy

It is commonly accepted that the degree of uncertainty in a probability distribution can be determined using information and entropy. However, entropy's properties have led to the creation of several correlations. The fluctuation in uncertainty is measured by a random variable X's entropy. The following is the definition of Rényi's entropy:

$$I\left(s\right)=\frac{1}{1-s}\ln\left\{\int_{0}^{\infty}f^{s}\left(x;\theta\right)dx\right\}\text{ with } s \text{ (integer)} >0 \text{ and } s\neq 1.$$

For the Ugamma distribution, the entropy is determined like this:

$$I\left(s\right)=\frac{1}{1-s}\ln\left\{\frac{\theta^{s}}{2^{s}}\int_{0}^{\infty}\left(2+\frac{1}{2}\left(\theta^{2}x^{2}-2\right)\right)^{s}\exp\left(-\theta sx\right)dx\right\}$$

Using the binomial expansion for $\left(2+\frac{1}{2}\left(\theta^{2}x^{2}-2\right)\right)^{s}$, we get

$$\left(2+\frac{1}{2}\left(\theta^{2}x^{2}-2\right)\right)^{s}=\sum_{i=0}^{s}C_{s}^{i}2^{s-i}\alpha^{i}\left(\theta^{2}x^{2}-2\right)^{i},$$

$$I\left(s\right)=\frac{1}{1-s}\ln\left\{\frac{\theta^{s}}{2^{s}}\sum_{i=0}^{s}C_{s}^{i}2^{s-i}\alpha^{i}\int_{0}^{\infty}\left(\theta^{2}x^{2}-2\right)^{i}\exp\left(-\theta sx\right)dx\right\},$$

$$\left(\theta^{2}x^{2}-2\right)^{i}=\sum_{j=0}^{i}C_{i}^{j}\left(-2\right)^{j}\left(\theta x\right)^{2(i-j)},$$

so

$$\begin{aligned}I\left(s\right) &= \frac{1}{1-s}\ln\left\{\frac{\theta^{s}}{2^{s}}\sum_{i=0}^{s}\sum_{j=0}^{i}C_{i}^{j}C_{s}^{i}\left(-1\right)^{j}2^{j+s-i}\alpha^{i}\int_{0}^{\infty}\left(\theta x\right)^{2(i-j)}\exp\left(-\theta sx\right)dx\right\}\\ &= \frac{1}{1-s}\ln\left\{\frac{\theta^{s}}{2^{s}}\sum_{i=0}^{s}\sum_{j=0}^{i}\left(-1\right)^{j}\frac{\alpha^{i}2^{j+s-i}C_{i}^{j}C_{s}^{i}}{s^{2(i-j)+1}}\Gamma\left(2\left(i-j\right)+1\right)\right\}\\ &= \frac{1}{1-s}\ln\left\{\frac{\theta^{s}}{2^{s-1}}\sum_{i=0}^{s}\sum_{j=0}^{i}\left(-1\right)^{j}\frac{\alpha^{i}2^{j+s-i}}{s^{2(i-j)+1}}\frac{s!}{j!\left(s-i\right)!}\right\}.\end{aligned}$$

4.8. Fuzzy Reliability

Let T be a continuous random variable representing a system's failure time (component). The fuzzy dependability can then be calculated using the fuzzy probability in the formula:

$$P_F(t) = P(T > t) = \int_t^\infty \nu(x) f(x;\theta)dx, \text{ where } 0 \leq t \leq x < \infty$$

with $\nu(x)$ is a membership function that describes the degree to which each element of a given universe belongs to a fuzzy set.

Assume that

$$\nu(x) = \begin{cases} 0 & \text{if } x \leq t_1 \\ \frac{x-t_1}{t_2-t_1} & \text{if } 0 \leq t_1 < x < t_2 \\ 1 & \text{if } x \geq t_2 \end{cases} .$$

For $\nu(x)$, by the computational analysis of the function of fuzzy numbers, the lifetime $x(\kappa)$ can be obtained corresponds to a certain value of $\kappa - Cut$, $\kappa \in [0,1]$ and can be obtined as $\nu(x) = \kappa \to \frac{x-t_1}{t_2-t_1} = \kappa$, then

$$\begin{cases} x(\kappa) \leq t_1 & , \kappa = 0 \\ x(\kappa) = t_1 + \kappa t_2 - t_1 & , 0 < \kappa < 1 \\ x(\kappa) \geq t_2 & , \kappa = 1 \end{cases} .$$

Consequently, it is possible to calculate fuzzy reliability values for every κ value. The fuzzy dependability of the Ugamma distribution is established by the fuzzy reliability definition. The fuzzy reliability of the Ugamma distribution can be defined as

$$P_F(t) = \left(1 + \frac{\theta t_1}{2} + \frac{\theta^2 t_1^2}{4}\right) \exp(-\theta t_1) - \left(1 + \frac{\theta x(\kappa)}{2} + \frac{\theta^2 x(\kappa)^2}{4}\right) \exp(-\theta x(\kappa)),$$

then, $P_F(t)_{\kappa=0} = 0$.

4.9. Quantile function

It may be noted that $F_X(x;\theta)$ in equ((3.3) is continuos and strictly increasing , so we for the quantile function of X is defined:

$$Q(p,\theta) = VaR = x_p = F_X^{-1}(p,\theta) \qquad p\epsilon [0.1].$$

We can not give an explicit expression for $Q(p,\theta)$ (cannot use lambert W function in our case), but we can give a numerical solution as shown in Table 1.

p	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 5$
0.1	0.442 47	0.221 23	0.110 62	0.04 424
0.2	0.977 04	0.488 52	0.244 26	0.09770
0.3	1. 606 4	0.803 22	0.401 61	0.160 64
0.4	2. 325 8	1. 162 9	0.581 46	0.232 58
0.5	3. 136 2	1. 568 1	0.784 06	0.313 62
0.6	4. 059 5	2. 029 7	1.014 9	0.405 95
0.7	5. 155 3	2. 577 7	1.2888	0.515 53
0.8	6. 569 1	3. 284 5	1.6423	0.656 91
0.9	8. 760 9	4. 380 4	2.1902	0.876 09

Table 1: Quantiles for Ugamma distribution ($\theta = 0.5, 1, 2, 5$).

5. Actuarial Measures

This section explores and mathematically derives several actuarial characteristics of the Ugamma distribution.

5.1. Mean Excess Function

The mean excess, or residual life function, for a claim amount random variable X, represents the expected payment per claim on a policy with a fixed deductible of x, where claims with amounts less than or equal to x are disregarded. The Ugamma distribution defines this function as follows:

$$\begin{aligned} e\left(x\right) &= E\left[X - x/X > x\right] = \frac{1}{S(x;\theta)} \int_x^\infty S(u;\theta)du \\ &= \frac{\left(1 + \left(1 + \theta x + \frac{\theta^2 x^2}{4}\right)\right)}{\theta\left(1 + \frac{\theta x}{2} + \frac{\theta^2 x^2}{4}\right)}. \end{aligned}$$

5.2. Limited Expected Value Function

The limited expected value function L of a claim size variable X, is defined as follows

$$L(u) = E\{\min\left(X, u\right)\} = \int_0^u x dF(x;\theta) + u\left(1 - F(u;\theta)\ \right) \quad \text{with} \quad u > 0.$$

The value of the function L at point x is equal to the expectation of the CDF $F(x;\theta)$ truncated at this point.

From a reinsurance perspective, given a policy limit or deductible u, a limited loss random variable is defined as follows:

$$X \wedge u = \min\left(X, u\right) = \begin{cases} X \text{ if } X \leq u \\ u \text{ if } X > u \end{cases}.$$

The definition of the limited anticipated value function is the expectation of the limited, which is determined as follows

$$\begin{aligned} E\{X \wedge u\} &= \int_0^u x f(x;\theta)dx + u(1 - F(u;\theta)) \\ &= \frac{1}{2\theta}\left(2e^{-u\theta} - 2 + 2e^{-u\theta} + 2u\theta e^{-u\theta} + \frac{3}{2}u^2\theta^2 e^{-u\theta} + \frac{u^3\theta^3}{2}e^{-u\theta} + 2u\theta e^{-u\theta} - 2\right) \\ &\quad + u\left(\left(1 + \frac{\theta u}{2} + \frac{\theta^2 u^2}{4}\right)e^{-\theta u}\right). \end{aligned}$$

Then, we have

$$\begin{aligned} E\{X \wedge u\} &= \frac{1}{\theta}\left(e^{-u\theta} + e^{-u\theta} + 2u\theta e^{-u\theta} + \frac{5}{4}u^2\theta^2 e^{-u\theta} + \frac{u^3\theta^3}{2}e^{-u\theta} + u\theta e^{-u\theta} - 2\right) \\ &= \frac{e^{-u\theta}}{\theta}\left(2 + 2u\theta + \frac{5}{4}u^2\theta^2 + \frac{u^3\theta^3}{2} + u\theta\right) - \frac{2+1}{\theta}. \end{aligned}$$

5.3. Tail Value at Risk

A risk metric connected to the general value at risk is the tail value at risk TVaR, sometimes referred to as the tail conditional expectation. The expectation of losses above VaR is measured by TVaR. For the Ugamma distribution, the TVaR is defined as follows:

$$\begin{aligned} TVaR &= E\left[X \mid X > VaR\right] \\ &= \frac{1}{1-p}\int_{VaR}^{\infty} xf(x;\theta)dx \\ &= \frac{\theta}{2\left(1-p\right)}\int_{VaR}^{\infty} x\left(2+\left(\frac{\theta^2x^2}{2}-1\right)\right)\exp\left(-\theta x\right)dx \\ &= \frac{e^{-\theta VaR}}{2\left(1-p\right)}\left(\left(4+3\left(VaR\right)^2\theta^2+\left(VaR\right)^3\theta^3+2\left(VaR\right)\theta\right)+2\left(VaR\right)\theta+2\right). \end{aligned}$$

5.4. Tail Variance

When losses exceed VaR at a given probability P, their conditional variance is measured by tail variance TV. The NXLD defines TV as follows:

$$\begin{aligned} TV &= E\left(X^2 \mid X > VaR\right) - \left(TVaR\right)^2 \\ &= \frac{1}{1-p}\int_{VaR}^{\infty} x^2 f(x;\theta)dx - \left(TVaR\right)^2 \\ &= \frac{e^{-(VaR)\theta}}{2\left(1-p\right)\theta^2}\left(10+2\left(VaR\right)^2\theta^2+4\left(VaR\right)\theta+5\left(VaR\right)^2\theta^2+2\left(VaR\right)^3\theta^3+\left(VaR\right)^4\frac{\theta^4}{2}+10\left(VaR\right)\theta+4\right) \\ &\quad -\left(TVaR\right)^2. \end{aligned}$$

6. Goodness-of-Fit Test

The Ugamma distribution can be considered a lifetime model, as it is defined for $x > 0$. In this section, we develop a modified Chi-squared Goodness-of-Fit Test for the Ugamma distribution based on the Nikulin-Rao-Robson (NRR) statistic. A simulation study is performed to explore a wide range of sample sizes and various values of the parameter θ.

Consider the problem of testing the hypothesis H_0 according to which the distribution of n independent identically distributed random variables $X_1, X_2, \cdots, X_n$ belongs to the family of Ugamma distribution.

$$H_0 : \mathbf{P}(X_i \leq x) = F_{U_g}(x;\lambda), \quad x > 0\,,$$

where F_{U_g} is the CDF of Ugamma distribution and $\lambda = (\theta)^T \in \mathbf{R}_*^+ \times (0,1)$.

We divide the positive real line into r sub-intervals $I_1, I_2, \cdots, I_r$ by the points

$$0 = a_0 < a_1 < \cdots < a_{r-1} < a_r = +\infty, \quad I_i =]a_{i-1}, a_i], \quad I_i \cap I_j = \emptyset, i \neq j,$$
$$i, j = 1, \cdots, r, \quad \cup_{i=1}^r I_i = \mathbf{R}_+,$$

and we group the sample $X_1, X_2, \cdots, X_n$ over these sub-intervals, we obtain the vector of frequencies $v = (v_1, v_2, \cdots, v_r)^T$ and the probability vector $p(\lambda) = (p_1(\lambda), p_2(\lambda), \cdots, p_r(\lambda)^T$,

where

$$p_i(\lambda) = \int_{a_{i-1}}^{a_i} f_{U_g}(x;\theta)dx = F_{Zg}(a_i;\theta) - F_{Zg}(a_{i-1};\theta), \quad i = 1, 2, \cdots, r.$$

To test the hypothesis H_0, Pearson proposed a test based on the so-called Pearson's χ^2 test of the form:

$$X_n^2(\lambda) = X_n^T(\lambda)X_n(\theta) = \sum_{i=1}^{r}\frac{(v_i - np_i(\theta))^2}{np_i(\theta)}, \tag{6.1}$$

where

$$X_n(\theta) = \left(\frac{v_1 - np_1(\theta)}{\sqrt{np_1(\theta)}}, \frac{v_2 - np_2(\theta)}{\sqrt{np_2(\theta)}}, \cdots, \frac{v_r - np_r(\theta)}{\sqrt{np_r(\theta)}}\right)^T.$$

Under H_0, if θ is known, it was shown by K. Pearson in 1900 (see e.g. [6]) that

$$\lim_{n\to\infty} \mathbf{P}(X_n^2(\theta)) \le x|H_0) = \mathbf{P}(\chi^2_{r-1} \le x). \tag{6.2}$$

The hypothesis H_0 must be rejected at a significance level ρ, whenever $X_n^2(\theta) > \chi^2_{r-1,\rho}$, where $\rho-$ quantile of the χ^2 distribution with $r-1$ degrees of freedom.

But generally θ is unknown and must be estimated using the sample $X_1, X_2, \cdots, X_n$. If we replace θ in (6.1) by any $\sqrt{n}-$ consistent estimate ν_n^*, the limit distribution of (6.2) will not be χ^2_{r-1} and changes dramatically, it depends on the method of estimation of θ and the proprieties of the estimator θ_n^* .

[26,27] proposed to modify the standard Chi-squared Pearson's test (6.1) for continuous distribution with shift and scale parameters, also [32] had obtained the same result for exponential family, and the test is well known as the Nikulin-Rao-Robson (NRR) test ([35], [6]) and can be written as (see [10]):

$$Y_n^2(\theta_n^*) = X_n^2(\theta_n^*) + X_n^T(\theta_n^*)B(\theta_n^*)(I(\theta_n^*) - J(\theta_n^*))^{-1}B^T(\theta_n^*)X_n(\theta_n^*),$$

with

$$B(\theta) = (b_1, b_2, \cdots, b_r)^T, \;\; b_i(\lambda) = \frac{1}{\sqrt{p_i(\theta)}}\frac{\partial p_i(\theta)}{\partial\theta} \quad i = 1, 2, \cdots, r\,,$$

where $nJ = nB^T(\theta)B(\theta)$ is the Fisher's information matrix of the vector of frequencies v, and I is the the Fisher's information matrix of X_i, and θ_n^* is the MLE's of θ.

The asymptotic behavior of the statistics $Y_n^2(\theta_n^*)$ is given by the following theorem:

Theorem 6.1 ([26,27])

$$\lim_{n\to\infty} \mathbf{P}\left(Y_n^2(\theta_n^*) \le x|H_0\right) = \mathbf{P}(\chi^2_{r-1} \le x). \tag{6.3}$$

Notice that [13], (see also [37], [36], [21]) have proposed a modification of the standard Pearson's test (6.1) by using the method of moments.

6.1. Simulation

To empirically study the behavior of the statistic Y_n^2, we generate samples from the Zgamma distribution with sizes $n = 20, 30, 50, 80, 100, 200, 500,$ and 1000, considering various parameter values $\theta = 0.01, 0.5, 2, 3,$ and 5.

In this study, we choose the significance level $\rho = 0.05$. Consider the case of equiprobability. Each sample (n, θ) is repeated 5000 times. For each operation, we compute the NRR statistic Y_n^2, then we calculate the empirical confidence level (EL) which counts the number of times where $Y_n^2 \le \chi^2_{r-1,1-\rho}$ divided by 5000.

For a significance level of $\rho = 0.05$, the theoretical confidence level is $EL = 1 - \rho = 0.95$. The results are summarized in Table 1. A quick examination of Table 1 shows that the empirical confidence levels (EL) are very close to the theoretical value of 0.95, supporting the validity of Theorem 6.1 as established in ([26,27]).

7. Estimation of Parameter - Application

To compare the Ugamma distribution with the following distributions, we determine the values of the AIC, BIC, -2L and AICC measures for five data sets.

$\theta = 0.01$								
n	20	30	50	80	100	200	500	1000
$E.L.$	0.9539	0.9528	0.9502	0.9505	0.9504	0.9508	0.9507	0.9505
$\theta = 0.5$								
n	20	30	50	80	100	200	500	1000
$E.L.$	0.9538	0.9527	0.9501	0.9507	0.9511	0.9515	0.9511	0.9505
$\theta = 2$								
n	20	30	50	80	100	200	500	1000
$E.L.$	0.9536	0.9531	0.9507	0.9502	0.9501	0.9514	0.9514	0.9506
$\theta = 3$								
n	20	30	50	80	100	200	500	1000
$E.L.$	0.9529	0.9526	0.9489	0.9497	0.9500	0.9508	0.9501	0.9507
$\theta = 5$								
n	20	30	50	80	100	200	500	1000
$E.L.$	0.9516	0.9515	0.9475	0.9493	0.9509	0.9511	0.9507	0.9509

Table 2: Empirical confidence level for $(\theta, \alpha) = [(0.01, 0.01), (0.5, 0.1), (2, 0.5), (3, 0.1), (5, 0.8)]$.

Model	*Density*	*Model*	*Density*
Two-parameter L1	$\frac{\theta^2(\gamma+x)e^{-\theta x}}{\gamma\theta+1}$	Gamma Lindley	$\frac{\theta^2((\gamma+\gamma\theta-\theta)x+1)e^{-\theta x}}{\gamma(1+\theta)}$
Quasi Lindley	$\frac{\theta^2(\gamma+x\theta)e^{-\theta x}}{\gamma+1}$	New quasi Lindley	$\frac{\theta^2(\theta+\gamma x)e^{-\theta x}}{\gamma+\theta^2}$
Two parameter L2	$\frac{\theta^2}{\theta+\gamma}(1+\gamma x)e^{-\theta x}$	Power XLindley	$\frac{\alpha\theta^2(2+\theta+x^\alpha)x^{\alpha-1}}{(1+\theta)^2}e^{-\theta x^\alpha}$
TPQED	$\frac{\theta^3(\gamma+\theta x+x^2)}{\theta^2+\gamma\theta^2+2}\exp(-\theta x)$	ZLindley	$\frac{\theta}{2(1+\theta)}(1+2\theta+\theta x)\exp(-\theta x)$

Table 3: List of models with their respective densities.

7.1. Real Data Applications

- Data set 1: Numbers of users connected to the Internet data

The following data from Durbin and Koopman (2001) represent the number of users connecting to the internet through a server each minute.

88, 84, 85, 85, 84, 85, 83, 85, 88, 89, 91, 99, 104, 112, 126, 138, 146, 151, 150, 148, 147, 149, 143, 132, 131, 139, 147, 150, 148, 145, 140, 134, 131, 131, 129, 126, 126, 132, 137, 140, 142, 150, 159, 167, 170, 171, 172, 172, 174, 175, 172, 172, 174, 174, 169, 165, 156, 142, 131, 121, 112, 104, 102, 99, 99, 95, 88, 84, 84, 87, 89, 88, 85, 86, 89, 91, 91, 94, 101, 110, 121, 135, 145, 149, 156, 165, 171, 175, 177, 182, 193, 204, 208, 210, 215, 222, 228, 226, 222, 220.

Model	θ	γ	*AIC*	*BIC*	$-2L$	*AICC*
Two-parameter L1	0.01458852	0.03166413	1119.448	1124.659	1115.448	1119.572
Gamma Lindley	0.01458836	15.92818	1119.488	1124.699	1115.488	1119.612
Quasi Lindley	0.02130896	0.000305609	1921.861	1927.071	1917.861	1921.984
New quasi Lindley	0.01460119	1.050246	1119.425	1124.635	1115.425	1119.549
Two parameter L2	0.01459072	18.45905	1119.478	1124.688	1115.478	1119.602
Power XLindley	0.779136	0.2068623	1509.548	1514.759	1505.548	1509.672
TPQED	0.02189847	0.0005688983	1084.693	1089.903	1080.693	1084.816
Ugamma	0.02249539	0.9990265	1084.05	1090.261	1081.05	1085.174

Table 4: Parameters' estimation and AIC, BIC, -2L and AICC values for data set 1.

- Data set 2: Yearly numbers of important discoveries data

The following data represent the annual count of significant inventions and scientific discoveries from 1860 to 1959 (see Janssen, 2023).

5, 3, 0, 2, 0, 3, 2, 3, 6, 1, 2, 1, 2, 1, 3, 3, 3, 5, 2, 4, 4, 0, 2, 3, 7, 12, 3, 10, 9, 2, 3, 7, 7, 2, 3, 3, 6, 2, 4, 3, 5, 2, 2, 4, 0, 4, 2, 5, 2, 3, 3, 6, 5, 8, 3, 6, 6, 0, 5, 2, 2, 2, 6, 3, 4, 4, 2, 2, 4, 7, 5, 3, 3, 0, 2, 2, 2, 1, 3, 4, 2, 2, 1, 1, 1, 2, 1, 4, 4, 3, 2, 1, 4, 1, 1, 1, 0, 0, 2, 0.

Model	θ	γ	*AIC*	*BIC*	$-2L$	*AICC*
Two-parameter L1	0.5482858	0.7828253	419.5603	424.7707	415.5603	419.684
Gamma Lindley	0.5482867	1.179184	419.5603	424.7707	415.5603	419.684
Quasi Lindley	0.9155886	0.1928493	481.7531	486.9634	477.7531	481.8768
New quasi Lindley	0.5483133	0.7005829	419.5603	424.7707	415.5603	419.684
Two parameter L2	0.5481631	1.274804	419.5603	424.7707	415.5603	419.684
TPQED	0.76435	1.263889	413.7746	418.985	409.7746	413.8983
Ugamma	0.8332866	0.791594	410.6608	415.8712	406.6608	410.7845

Table 5: Parameters' estimation and AIC, BIC, -2L and AICC values for data set 2.

- Data set 3: Populations Recorded by the US Census data

The following dataset presents the population of the United States (in millions) recorded during the decennial census from 1790 to 1970 (see McNeil, 1977).

$$3.93, 5.31, 7.24, 9.64, 12.90, 17.10, 23.20, 31.40, 39.80, 50.20, 62.90,$$
$$76.00, 92.00, 105.70, 122.80, 131.70, 151.30, 179.30, 203.20.$$

Model	θ	γ	*AIC*	*BIC*	$-2L$	*AICC*
Two-parameter L1	0.02166821	44.2597	203.7026	205.5914	199.7026	204.4526
Gamma Lindley	0.01797983	0.02366775	203.2337	205.1226	199.2337	203.9837
Quasi Lindley	0.04057428	0.2043486	336.5028	338.3917	332.5028	337.2528
New quasi Lindley	0.02478781	0.001029806	204.5499	206.4388	200.5499	205.2999
Two parameter L2	0.01801368	0.006201614	203.234	205.1228	199.234	203.984
Power XLindley	1.055696	0.1935496	251.3029	253.1917	247.3029	252.0529
TPQED	0.04026878	94.23159	214.4968	216.3856	210.4968	215.2468
Ugamma	0.02592981	0.4043021	202.3477	204.2366	198.3477	203.0977

Table 6: Parameters' estimation and AIC, BIC, -2L and AICC values for data set 3.

- Data set 4: DDT in ale data

Finsterwalder (1976) analyzed a dataset comprising 15 measurements of DDT pesticide levels in kale (in parts per million, ppm) obtained from various laboratories using the multiple pesticide residue measurement method.

Source: *Finsterwalder C. E. (1976) Collaborative study of an extension of the Mills et al method for the determination of pesticide residues in food. J. Off. Anal. Chem. 59, 169–171, and Staudte R. G. and Sheather S. J. (1990) Robust Estimation and Testing. Wiley.*

$$2.79, 2.93, 3.22, 3.78, 3.22, 3.38, 3.18, 3.33, 3.34, 3.06, 3.07, 3.56, 3.08, 4.64, 3.34.$$

Model	θ	γ	*AIC*	*BIC*	$-2L$	*AICC*
Two-parameter L1	0.6223911	0.0008551842	58.74148	60.15758	54.74148	59.74148
Gamma Lindley	0.5985468	20.45021	58.96757	60.38367	54.96757	59.96757
Quasi Lindley	0.8977403	0.0006939774	67.49544	68.91155	63.49544	68.49544
New quasi Lindley	0.5933078	24.60633	58.90761	60.32371	54.90761	59.90761
Two parameter L2	0.5959023	23.3598	59.06532	60.48142	55.06532	60.06532
Power XLindley	1.449654	0.3811038	106.8118	108.2279	102.8118	107.8118
TPQED	0.8010331	0.0006553789	54.83021	56.24631	50.83021	55.83021
Ugamma	0.9046047	0.9981745	52.46074	53.87684	48.46074	53.46074

Table 7: Parameters' estimation and AIC, BIC, -2L and AICC values for data set 4.

- Data set 5: Percentage of Shrimp in Shrimp Cocktail data

The following data are the numeric vector with 18 determinations by different laboratories of the amount (percentage of the declared total weight) of shrimp in shrimp cocktail (see King and Ryan, 1976).

$$32.2, 33.0, 30.8, 33.8, 32.2, 33.3, 31.7, 35.7, 32.4,$$
$$31.2, 26.6, 30.7, 32.5, 30.7, 31.2, 30.3, 32.3, 31.7.$$

$Model$	θ	γ	AIC	BIC	$-2L$	$AICC$
two-parameter L1	0.0630501	0.0006025992	150.688	152.4687	146.688	151.488
gamma Lindley	0.06262739	12.50669	150.7721	152.5528	146.7721	151.5721
quasi Lindley	0.09715586	0.0005323377	242.5376	244.3184	238.5376	243.3376
new quasi Lindley	0.06283553	5.838002	150.6993	152.48	146.6993	151.4993
two parameter L2	0.06287901	13.97741	150.7675	152.5483	146.7675	151.5675
Power XLindley	0.9594836	0.2377622	221.0102	222.7909	217.0102	221.8102
TPQED	0.09680282	0.0008258917	143.0504	144.8311	139.0504	143.8504
Ugamma	0.09437831	0.9995391	142.969	144.7497	138.969	143.769

Table 8: Parameters' estimation and AIC, BIC, -2L and AICC values for data set 6.

We have for all the sets of data Ugamma distribution has the lowest AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), -2L (negative log-likelihood) and AICC (Akaike Information Criterion corrected) values exept for data set 1 where TPQED distribution has the best values but closer to Ugamma ones.

8. Conclusion

Inspired by recent developments in statistical theory, we present the Ugamma distribution as a specific instantiation of the proposed framework, characterized by distinct statistical properties and flexible parameterization. The Ugamma distribution stands out for its ability to model a variety of data behaviors thanks to its adjustable parameters, enabling better adaptation to the specific characteristics of the data being analyzed.

Through a comprehensive analysis, we explored the statistical properties, moments and related measures of the Ugamma distribution. In particular, we examined the moments of the distribution, such as higher-order moments, which provide crucial information on the shape and dispersion of the data. We also investigated advanced statistical concepts such as incomplete moments, stochastic order and entropy. The study of incomplete moments enabled us to better understand the extreme behavior of the data, while stochastic order helped us to compare the Ugamma distribution with other distributions on the basis of their cumulative probability. Entropy analysis provided insights into the uncertainty and complexity associated with the distribution, enabling us to better understand the behavior and applicability of the distribution in various contexts. We also calculated the various actuarial measures of this distribution that are essential in the assessment of financial and insurance risks. A modified chi-squared goodness-of-fit test based on the Nikulin-Rao-Robson (NRR) statistic was also developped for Ugamma distribution.

The practical relevance of the Ugamma distribution has been demonstrated by its application to real data sets, drawn from diverse fields such as Internet usage patterns, scientific discoveries, population dynamics and pesticide concentration levels. In these applications, the Ugamma distribution showed superior performance in providing the best values of the AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), -2L (negative log-likelihood) and AICC (Akaike Information Criterion corrected) measures compared to other similar distributions. These results underline the effectiveness of the Ugamma distribution in modeling complex and heterogeneous data, offering robust and accurate statistical solutions for a variety of application domains.

Conflicts of interests

The authors declare that they have no conflicts of interest.

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