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On Right ISSF-Rings

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ABSTRACT: As a proper generalization of the right IF-ring, the concept of a right ISSF-ring is introduced and studied in this article. A ring R is considered a right ISSF-ring if all injective right R-modules are ss-flat. We offer several characterizations and examine different properties of right ISSF-rings.

Keywords: Injective module, IF-ring, ss-flat module, ISSF-rings.

Contents

1. Introduction

Throughout this paper, every module is a unitary R-module, where R is a ring that is associative and has identity. Mod-R represents the class of right R-modules, while R-Mod represents the class of left R-modules. The symbol f.g. will be used to indicate the finitely generated. If the tensor induced map $N \otimes_R A \longrightarrow M \otimes_R A$ is injective, for any A in Mod-R, then a submodule N of M_R is known by pure [6]. Soc(M) stands for the socle of M. The symbols $U \leq W$, (resp., $U \leq^p W, U \leq^{ss} W, U \leq^{fgss} W$) denote that U is a submodule (resp., pure, semisimple small, finitely generated semisimple small) of W. $N^* =$Hom$_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$, known as the character module of N. The symbol c.u.d.p. means closed under direct products. If $M \in$ Mod-R is pure in all modules that include it as a submodule, then M is referred to as FP-injective [9]. If for each $f \in \text{Hom}_R(C, H)$ (with $C, H \in R$-Mod) and every $g \in \text{Hom}_R(C, M)$, there is an $h \in \text{Hom}_R(H, M)$ with $g = hf$, then $M \in R$-Mod is defined to be injective [1]. Several extensions and generalizations of injective and flat modules and some related concepts have been extensively studied in the literature (we refer the reader to [2], [8,9,10,11,12,13,14,15], [17], [18], [22]) for example). In [16], the concepts of ss-injective (resp., ss-flat) modules were presented as a generalization of injective (resp., flat) modules. For $M \in R$-Mod, if every $f \in \text{Hom}_R(A, M)$ extends to ${}_RR$ for any $A \leq^{ss} {}_RR$, then M is named ss-injective. A module $T \in$ Mod-R is named ss-flat if for every $D \leq^{ss} {}_RR$ exactness holds for $0 \to T \otimes_R D \to T \otimes_R R$. The symbol ${}_R(SS\text{-}\mathbb{I})$ $(resp., (SS\text{-}\mathbb{F})_R)$ refers to the class of all modules in R-Mod that are ss-injective (resp., the class of all modules in Mod-R that are ss-flat). If $M \in$ Mod-R, then the symbol E(M) means the injective envelope of M. The concept of right IF-rings was first presented by Colby in [3]. A ring R is named a right IF-ring, if every injective module in Mod-R is flat.

As a proper generalization of a right IF-ring, we present and examine in this paper the idea of a right ISSF-ring. If it holds that all injective right R-modules are ss-flat, then the ring R is considered a right ISSF-ring. Many examples of ISSF-rings are given. Many characterizations of ISSF-ring are given, for example, Proposition 2.1 shows the equivalent of the next assertions is hold for a ring R: (1) R is a right ISSF-ring; (2) M is embedded in a module that is ss-flat, for every $M \in$ Mod-R; (3) An ss-flat module contains M embedded in it; for each injective right R-module M; (4) E(M) is embedded in an ss-flat module, for any $M \in$ Mod-R; (5) For any $M \in$ Mod-R, E(M) is an ss-flat module. Also, we prove in Proposition 2.2 that a ring R is a right ISSF-ring $\Leftrightarrow$ All FP-injective right R-modules are ss-flat $\Leftrightarrow$ If an FP-injective right module M contains an FP-injective submodule N, then M/N is an ss-flat module for that module $\Leftrightarrow$ E(M) is ss-flat for each $M \in$ Mod-R that is finitely presented $\Leftrightarrow$ F^* is ss-flat, for every free module F in R-Mod. In Corollary 2.1, we prove that if $(SS\text{-}\mathbb{F})_R$ is closed under direct products, then R is a right ISSF-ring if and only if $i_L^* : ({}_RR)^* \to L^*$ is an $(SS\text{-}\mathbb{F})_R$-precover of L^*, for every f.g.

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semisimple small $L \leq {}_RR$, where $i_L : L \to R$ is the inclusion mapping iff $({}_RR)^*$ is ss-flat. Finally, we prove in Proposition 2.5 that ${}_RR$ is ss-injective, if ${}_R(SS\text{-}\mathbb{I})$ is closed under pure submodules and R is a right ISSF-ring.

2. Right ISSF-Rings

In this section, as a generalization of right IF-ring, we introduce the concept of right ISSF-rings.

Definition 2.1 *If every injective right R-module is ss-flat, then the ring R is called a right ISSF-ring.*

Example 2.1 ***(1)*** *If $soc({}_RR) = 0$, then R is a right ISSF-ring. Since if $soc({}_RR) = 0$, then Mod-$R = (SS\text{-}\mathbb{F})_R$, and consequently, any injective module in Mod-R is ss-flat. Therefore, R is a right ISSF-ring.*
(2) *Clearly, all right IF-rings are right ISSF-rings.*
(3) *Generally, it is not true that the converse of (2), for Example: $\mathbb{Z}$ is an ISSF-ring (by (1) above). But $\mathbb{Z}$ is not an IF-ring, since $\mathbb{Q}_\mathbb{Z}$ is injective but not flat, by [5, Example (3), p. 401]. So, right ISSF-ring is a proper generalization of right IF-ring.*
(4) *All regular rings are right ISSF-rings, where a ring R is called regular if for each $a \in R$, we have $a = aba$ for some $b \in R$ [6, p.38]. If a ring R is a regular, then [6, Theorem 10.4.9, p.262] implies that all right R-module is flat. Hence all injective right R-module is ss-flat. So R is an ISSF-ring.*

In the following proposition, we will introduce some characterizations of a right ISSF-ring.

Proposition 2.1 *For a ring R, the equivalent of the next conditions is hold:*
(1) R is a right ISSF-ring.
(2) M is embedded in an ss-flat module, for every $M \in$ Mod-R.
(3) M is embedded in an ss-flat module, for any injective right R-module M.
(4) $E(M)$ is embedded in an ss-flat module, for any right R-module M.
(5) $E(M)$ is an ss-flat module, for any right R-module M.

Proof: (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.
(1) $\Rightarrow$ (2). Clearly, there is a right R-monomorphism $\alpha : M \to E(M)$, where E(M) is the injective envelope of $M \in$ Mod-R. By hypothesis, E(M) is an ss-flat module. Thus M is embedded in an ss-flat module.
(4) $\Rightarrow$ (5). If $M \in$ Mod-R, then there exist a monomorphism $\alpha : E(M) \to L$, where L is ss-flat and hence $E(M) \cong \alpha(E(M))$. From injectivity of E(M) and [1, Proposition 5.1.2, p.135], we get that $\alpha(E(M))$ is a summand of L. Since L is ss-flat, we have that $\alpha(E(M))$ is ss-flat. Hence E(M) is an ss-flat module.
(5) $\Rightarrow$ (1). If $M \in$ Mod-R is injective, then $M = E(M)$. By (5), E(M) is ss-flat and hence M is ss-flat. Therefore,R is a right ISSF-ring. □

We provide more characterizations of ISSF-ring in the following result.

Proposition 2.2 *The equivalent of the next conditions is hold for a ring R:*
(1) R is a right ISSF-ring.
(2) $E(M)$ is ss-flat, for any finitely presented right R-module M.
(3) If $F \in$ Mod-R is f.g. free and $K \leq^{ss} F$ and cyclic, then F/K is a submodule of a f.g. free module.
(4) M is ss-flat, for any $M \in$ Mod-R with M is FP-injective.
(5) If $M \in$ Mod-R is FP-injective and $L \leq^{p} M$, then M/L is ss-flat.
(6) M/L is ss-flat, for any FP-injective right module M and any FP-injective submodule L of M.
(7) F^* is ss-flat, for any free left R-module F.

Proof: (1) $\Rightarrow$ (2). Let $M \in$ Mod-R with M is finitely presented. Thus, E(M) is an ss-flat module, by Proposition 2.1.
(2) $\Rightarrow$ (3). Let $M = F/K$, where F is a f.g. free right R-module and $K \leq^{ss} F$ with K is cyclic. Thus, there is an α from M into E(M) a monomorphism. By (2), E(M) is an ss-flat module. Thus, α factors through a free module, say F_1, is f.g. and hence $\alpha = gf$ for some $f \in \text{Hom}_R(M, F_1)$ and $g \in \text{Hom}_R(F_1, E(M))$. Thus, f is a monomorphism and hence $M \leq F_1$.

(3) ⇒ (4). Given a right R-homomorphism $\alpha : F/K \to M$, with $F \in$ Mod-R is f.g. free, K is cyclic with $K \leq^{ss} F$ and M is FP-injective. By hypothesis, $F/K \leq F_1$, with F_1 is f.g. free. Since F/K is a f.g. module and F_1 is f.g. free, we have $F_1/((F/K))$ is a finitely presented module. From FP-injectivity of M and [21, 35.1(c), p.297] we get that M is injective with respect to the sequence $0 \to F/K \xrightarrow{i} F_1 \xrightarrow{\pi} F_1/(F/K) \longrightarrow 0$ where i (resp., π) is the inclusion (resp., natural epimorphism) and hence there is a homomorphism $\lambda : F_1 \to M$ such that $\lambda i = \alpha$. Thus, α factors through a f.g. free module F_1 and hence M is an ss-flat module.
(4) ⇒ (5). Let $M \in$ Mod-R be FP-injective and $L \leq^p M$. By hypothesis, M is ss-flat and hence we can get from [16, Corollary 2.4] that M/L is an ss-flat module.
(5) ⇒ (6). Let L be an FP-injective submodule of an FP-injective right R-module M. By [9, p.561], $L \leq^p E(L)$ (resp., $M \leq^p E(M)$). Since $\mathrm{E}(L) \leq \mathrm{E}(M)$ and it is injective, we have from [1, Proposition 5.1.2, p.135] that $\mathrm{E}(L)$ is a summand of $\mathrm{E}(M)$ and hence $E(L) \leq^p E(M)$. Since $L \leq^p E(L)$, we have from [21, 33.3(1), p.276] that $L \leq^p E(M)$. By [21, 33.3(2), p.276], $L \leq^p M$ (because $L \leq M \leq E(M)$). By (5), M/L is ss-flat.
(6) ⇒ (7). If $F \in R$-Mod is free, then we have from [7, Theorem, p. 239] that F^* is an injective right R-module. Since $< 0 >$ is an injective module, we have $< 0 >$ and F^* are FP-injective modules. By (6), $F^*/ < 0 >$ is an ss-flat module and hence F^* is an ss-flat module.
(7) ⇒ (1). Let M be any injective right R-module. Thus, M^* is a left R-module. By [20, Proposition 2.5, p. 10], $M^* \cong F/K$, where $F \in R$-Mod is free. Thus, there is an epimorphism $\alpha : F \to M^*$. By hypothesis, F^* is ss-flat. By [4, Lemma 17-1.7(i), p. 361], $\alpha^* : M^{**} \to F^*$ is a monomorphism. By [4, Corollary 17-1.5, p. 360], there a monomorphism β from M into M^{**}. Hence $\alpha^*\beta$ is a monomorphism. Thus $\alpha^*\beta(M)$ is a direct summand of F^* (by injectivity of M) and so $\alpha^*\beta(M)$ is ss-flat. Hence from $M \cong \alpha^*\beta(M)$, we have that M is an ss-flat module. Thus R is a right ISSF-ring. □

A homomorphism $\alpha : A \to M$ is called $\mathcal{F}$-precover of $M \in$ Mod-R with $\mathcal{F} \subseteq$ Mod-R and $A \in \mathcal{F}$ if, for any $g \in \mathrm{Hom}_R(K, M)$ with $K \in \mathcal{F}$, there is an $h \in \mathrm{Hom}_R(L, A)$ with $\alpha h = g$ [1, p.244].

Corollary 2.1 *If $(SS\text{-}\mathcal{F})_R$ is c.u.d.p., then the next statements are equivalent:*
(1) R is a right ISSF-ring.
(2) $({}_RR)^*$ is ss-flat.
(3) If $i_K : K \to R$ is the inclusion mapping, then $i_K^* : ({}_RR)^* \to K^*$ is a $(SS\text{-}\mathcal{F})_R$-precover of K^*, for every f.g. semisimple small left ideal K of R.

Proof: (1) ⇒ (2). Since ${}_RR$ is a free left R-module, $({}_RR)^*$ is ss-flat (by Proposition 2.2).
(2) ⇒ (3). Let K be a f.g. semisimple small left ideal of R. Thus $i_K^* : ({}_RR)^* \to K^*$ is an epimorphism (by [4, Lemma 17-1.7(ii), p.361],). By (2), $({}_RR)^*$ is ss-flat. Let B be an ss-flat right R-module, then the sequence $0 \to B \otimes_R K \stackrel{I_B \otimes_R i_K}{\longrightarrow} B \otimes_R R$ is exact. By [4, Lemma 17-1.7(ii), p. 361], the sequence $(B \otimes_R R)^* \stackrel{(I_B \otimes_R i_K)^*}{\longrightarrow} (B \otimes_R K)^* \to 0$ is exact. By [19, Theorem 2.75, p. 92], the sequence $\mathrm{Hom}_R(B, ({}_RR)^*) \to \mathrm{Hom}_R(B, K^*) \to 0$ is exact. Thus $i_K^* : ({}_RR)^* \to K^*$ is a $(SS\text{-}\mathcal{F})_R$-precover of K^*.
(3) ⇒ (1). By hypothesis, $({}_RR)^*$ is ss-flat. Let $F \in R$-Mod be free. Thus $F \cong ({}_RR)^{(I)}$, for some index set I and hence $F^* \cong ({}_RR^{(I)})^* \cong (({}_RR)^*)^I$ by [7, Lemma 4.3.3, p. 86]. By hypothesis, $(({}_RR)^*)^I$ is ss-flat and so F^* is ss-flat. Therefore, R is a right ISSF-ring (by Proposition 2.2). □

It is essay to prove the next lemma:

Lemma 2.1 *The class $(SS\text{-}\mathbb{F})_R$ is c.u.d.p. if and only if $({}_R(SS\text{-}\mathbb{I}))^* \subseteq (SS\text{-}\mathbb{F})_R$.*

Corollary 2.2 *If ${}_RR$ is an ss-injective left R-module and $(SS\text{-}\mathbb{F})_R$ is c.u.d.p., then R is a right ISSF-ring.*

Proof: Let ${}_RR$ be an ss-injective left R-module. Since $(SS\text{-}\mathbb{F})_R$ is c.u.d.p. (by hypothesis), $({}_RR)^*$ is ss-flat (by Lemma 2.1). By Corollary 2.1, R is a right ISSF-ring. □

Let $K \in R$-Mod and $N \in$ Mod-R. For any index set I, define $\varphi_K : N^I \otimes_R K \to (N \otimes_R K)^I$ by $\varphi((m_\alpha)_{\alpha \in I} \otimes_R n) = (m_\alpha \otimes_R n)_{\alpha \in I}$, for any $n \in K, (m_\alpha)_{\alpha \in I} \in N^I$. Thus φ_K is a natural homomorphism, by [3, p.241].

we can easily prove the next lemma:

Lemma 2.2 *Let U_R be an ss-flat module. Then the following two statements are equivalent:*
(1) *For any index set S, U^S is ss-flat.*
(2) *For any index set S and $K \leq^{fgss} {}_R R$, the natural homomorphism $\varphi_K : U^S \otimes_R K \to (U \otimes_R K)^S$ is an isomorphism.*

Proposition 2.3 *For any $K \leq^{fgss} {}_R R$ and for any index set I, if R is a right ISSF-ring, then the natural homomorphism $\varphi_K : (({}_R R)^*)^I \otimes_R K \to (({}_R R)^* \otimes_R K)^I$ is an isomorphism.*

Proof: Suppose that R is a right ISSF-ring. By Proposition 2.2, $({}_R R^{(I)})^*$ is an ss-flat right R-module, for any index set I. Since $(({}_R R)^*)^I \cong ({}_R R^{(I)})^*$ (by [6, Lemma 4.3.3, p.86]), we have $(({}_R R)^*)^I$ is ss-flat. By Lemma 2.2, $\varphi_K : (({}_R R)^*)^I \otimes_R K \to (({}_R R)^* \otimes_R K)^I$ is an isomorphism, for any index set I, □

The converse of Proposition 2.3 is discussed in the next proposition.

Proposition 2.4 *If $({}_R R)^*$ is ss-flat and the natural homomorphism $\varphi_K : (({}_R R)^*)^I \otimes_R K \to (({}_R R)^* \otimes_R K)^I$ is an isomorphism, for any $K \leq^{fgss} {}_R R$ and for an index set I, then R is a right ISSF-ring.*

Proof: Follows from [6, Lemma 4.3.3, p. 86]) and Proposition 2.2. □

Proposition 2.5 *If a ring R is a right ISSF-ring and ${}_R(SS\text{-}\mathbb{I})$ is closed under pure submodules, then ${}_R R$ is an ss-injective left R-module.*

Proof: By Proposition 2.2 and [8, Lemma 2.3]. □

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