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Well-Posedness and Stability of a Lamé System with Internal Fractional Damping *

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ABSTRACT: In this paper, we consider polynomial stabilization for a Lamé system in a bounded domain under an internal fractional damping. We reformulate the system into an augmented model and prove the well-posedness of it by using semigroup method. Based on a general criteria of Arendt-Batty, we show that the system is strongly stable. By combining frequency domain method and multiplier techniques, we establish an optimal polynomial energy decay rate.

Keywords: Lamé system, internal fractional damping, optimal polynomial decay rate.

Contents

1. Introduction

In this paper, we consider a Lamé system under fractional dampings, which is described by:

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + a_1 \partial_t^{\sigma,\kappa} u(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{in } \Gamma \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{P}$$

where μ, λ are Lamé constants, $u = (u_1, u_2, ..., u_n)^T$. Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. Moreover, $a_1 > 0$. The notation $\partial_t^{\sigma,\kappa}$ stands for the exponential fractional derivative operator of order σ. It is defined by

$$\partial_t^{\sigma,\kappa} w(t) = \frac{1}{\Gamma(1-\sigma)} \int_0^t (t-s)^{-\sigma} e^{-\kappa(t-s)} \frac{dw}{ds}(s)\, ds \qquad 0 < \sigma < 1, \qquad \kappa \geq 0.$$

Recently, in [1], Ammari et al., studied the wave equation with internal fractional damping. The system considered is as follows:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \gamma \partial_t^{\sigma,\kappa} u(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega. \end{cases}$$

The authors proved that the energy decays polynomially as $t^{-2/(1-\sigma)}$.

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Very recently in [6], Benaissa and Boudaoud extended the result of Ammari to higher-space dimension and internal control of diffusive type defined by:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \zeta \int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi,t)d\xi = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ \partial_t \psi(x,\xi,t) + (\xi^2+\kappa)\psi(x,\xi,t) - u_t(x,t)\omega(\xi) = 0 & \text{in } \Omega \times (-\infty,\infty) \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega, \\ \psi(x,\xi,0) = 0 & \text{on } \Omega \times (-\infty,\infty). \end{cases}$$

The authors established a less decay estimate by adopting the multiplier method.

In [12], Oliveira et al., studied the porous-elastic system with two internal fractional dampings. The system considered is as follows:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x + \partial_t^{\sigma,\kappa} u = 0 & \text{in } (0,L) \times (0,+\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \partial_t^{\sigma,\kappa}\phi = 0 & \text{in } (0,L) \times (0,+\infty). \\ u(0,t) = \phi(0,t) = 0 & \text{on } (0,+\infty), \\ u(1,t) = \phi(1,t) = 0 & \text{on } (0,+\infty) \\ u(x,0) = u_0(x), \quad \phi(x,0) = \phi_0(x) & \text{on } (0,L) \\ u_t(x,0) = u_1(x), \quad \phi_t(x,0) = \phi_1(x) & \text{on } (0,L), \end{cases}$$

They proved a less precise polynomial decay of the energy.

The stability of the elastic Lamé system with different types of dissipative has been intensively studied. We start by recall some results. In [3], Guesmia et al. considered Lamé system with infinite memories acting in the all equations of the system. They established asymptotic stability results under some conditions on the relaxation functions. Benaissa and Gaouar in [4] considered an elastic Lamé system subject to internal frictionnel dissipation. They proved that the system is exponentially stable.

We should mention here that, to the best of our knowledge, there is no result concerning the Lamé system with the presence of a fractional damping. In addition to being nonlocal, fractional derivatives involve singular and nonintegrable kernels. This makes the problem very delicate.

Our aim in this work is to prove that the stability of our system holds with fractional damping and to obtain an optimal polynomial decay.

The remainder of the paper falls into five sections. In Section **2**, we show that the above system can be replaced by an augmented one obtained by coupling a system with a suitable diffusion, and we study of energy functional associated to system. In section **3**, we state a well-posedness result for problem (P). In section **4**, we prove the strong asymptotic stability of solutions. In section **5**, we show the lack of exponential stability by spectral analysis. Finally, in section **6** we show the polynomial stability using the Borichev-Tomilov theorem.

2. Preliminary

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.1 (see [11]) *Let ω be the function:*

$$\omega(\xi) = |\xi|^{(2\sigma-1)/2}, \quad -\infty < \xi < +\infty, \ 0 < \sigma < 1. \tag{2.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \psi(\xi,t) + (\xi^2+\kappa)\psi(\xi,t) - U(t)\omega(\xi) = 0, \quad -\infty < \xi < +\infty, \kappa > 0, t > 0, \tag{2.2}$$

$$\psi(\xi,0) = 0, \tag{2.3}$$

$$O(t) = (\pi)^{-1}\sin(\sigma\pi)\int_{-\infty}^{+\infty}\omega(\xi)\psi(\xi,t)\,d\xi \tag{2.4}$$

is given by

$$O = I^{1-\sigma,\kappa}U = D^{\sigma,\kappa}U, \tag{2.5}$$

where

$$[I^{\sigma,\kappa} f](t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} e^{-\kappa(t-s)} f(s)\, ds.$$

Proof: From (2.2) and (2.3), we have

$$\psi(\xi,t) = \int_0^t \omega(\xi) e^{-(\xi^2+\kappa)(t-s)} U(s) ds. \tag{2.6}$$

Hence, by using (2.4), we get

$$O(t) = (\pi)^{-1} sin(\sigma\pi) e^{-\kappa t} \int_0^t \left[2 \int_0^{+\infty} |\xi|^{2\sigma-1} e^{-\xi^2(t-s)} d\xi \right] e^{\kappa s} U(s) ds. \tag{2.7}$$

Thus,

$$\begin{aligned} O(t) &= (\pi)^{-1} sin(\sigma\pi) e^{-\kappa t} \int_0^t \left[(t-s)^{-\sigma} \Gamma(\sigma) \right] e^{\kappa s} U(s) ds \\ &= (\pi)^{-1} sin(\sigma\pi) \int_0^t \left[(t-s)^{-\sigma} \Gamma(\sigma) \right] e^{-\kappa(t-s)} U(s) ds \end{aligned} \tag{2.8}$$

which completes the proof. Indeed, we know that $(\pi)^{-1} sin(\sigma\pi) = \dfrac{1}{\Gamma(\sigma)\Gamma(1-\sigma)}$. □

Lemma 2.1 (see [9]) *If* $\lambda \in D_\kappa = \mathbb{C} \backslash]-\infty, -\kappa]$ *then*

$$\int_{-\infty}^{+\infty} \frac{\omega^2(\xi)}{\lambda + \kappa + \xi^2}\, d\xi = \frac{\pi}{\sin \sigma\pi} (\lambda+\kappa)^{\sigma-1}.$$

Consequently, by using Theorem 2.1, the system (P) is equivalent to

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div} \;\; u) & \\ \qquad\qquad + \zeta \displaystyle\int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi,t)\, d\xi = 0 & \text{in } \Omega \times (0,+\infty), \\ \psi_t(x,\xi,t) + (\xi^2+\kappa)\psi(x,\xi,t) - u_t(x,t)\omega(\xi) = 0 & \text{in } \Omega \times (-\infty,\infty) \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega, \\ \psi(x,\xi,0) = 0 & \text{on } \Omega \times (-\infty,\infty), \end{cases} \tag{P'}$$

where $\zeta = (\pi)^{-1} \sin(\sigma\pi) a_1$.

We define the energy of the solution by:

$$\begin{aligned} E(t) = \frac{1}{2} \sum_{j=1}^{n} \left(\|u_{jt}\|^2_{L^2(\Omega)} + \mu \|\nabla u_j\|^2_{L^2(\Omega)} + \zeta \int_\Omega \int_{-\infty}^{+\infty} |\psi_j(x,\xi,t)|^2\, d\xi dx \right) \\ + \frac{(\mu+\lambda)}{2} \|\text{div } u\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.9}$$

Lemma 2.2 *Let* (u, ψ, z) *be a regular solution of the problem* (P')*. Then there exists a positive constant* C *such that the energy functional defined by (2.9) satisfies*

$$E'(t) - \zeta \sum_{j=1}^{n} \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\, d\xi\, dx. \tag{2.10}$$

Proof: Multiplying the first equation in (P) by $\overline{u}_{jt}$, integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|u_{jt}\|_2^2 - \mu\Re\int_\Omega \Delta u_j\overline{u}_{jt}\,dx - (\mu+\lambda)\Re\int_\Omega \frac{\partial}{\partial x_j}(\text{div } u)\overline{u}_{jt}\,dx$$
$$+\zeta\int_\Omega \overline{u}_{jt}\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx.$$

Then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{j=1}^n\left(\|u_{jt}\|^2_{L^2(\Omega)} + \mu\|\nabla u_j\|^2_{L^2(\Omega)}\right) + \frac{(\mu+\lambda)}{2}\|\text{div } u\|^2_{L^2(\Omega)}\\ &\quad+\zeta\Re\sum_{j=1}^n\int_\Omega \overline{u}_{jt}\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx = 0.\end{aligned} \tag{2.11}$$

Multiplying the second equation in (P') by $\zeta\overline{\psi}_j$ and integrating over $\Omega\times(-\infty,+\infty)$, we obtain:

$$\begin{aligned}&\frac{\zeta}{2}\frac{d}{dt}\sum_{j=1}^n\|\psi_j\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \zeta\sum_{j=1}^n\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\,dx\\ &\qquad -\zeta\Re\sum_{j=1}^n\int_\Omega u_{jt}(x,t)\int_{-\infty}^{+\infty}\omega(\xi)\overline{\psi}_j(x,\xi,t)\,d\xi\,dx = 0.\end{aligned} \tag{2.12}$$

From (2.9), (2.11) and (2.12) we get

$$E'(t) = -\zeta\sum_{j=1}^n\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\,dx. \tag{2.13}$$

This completes the proof of the lemma. □

3. Well-Posedness of the Problem

We discuss (P') in the state space

$$\mathcal{H} = \left(H_0^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega\times(-\infty,+\infty))\right)^n$$

with the inner product:

$$\begin{aligned}< U,\tilde{U} >_{\mathcal{H}} = &\sum_{j=1}^n\int_\Omega\left(v_j\overline{\tilde{v}}_j + \mu\nabla u_j\nabla\overline{\tilde{u}}_j\right)dx + (\mu+\lambda)\int_\Omega(\text{div } u)(\text{div }\overline{\tilde{u}})\,dx\\ &+\zeta\sum_{j=1}^n\int_\Omega\int_{-\infty}^{+\infty}\psi_j(x,\xi)\overline{\tilde{\psi}}_j(x,\xi)d\xi dx\end{aligned}$$

for any $U=(u,v,\psi)^T\in\mathcal{H}, \tilde{U}=(\tilde{u},\tilde{v},\tilde{\psi})^T\in\mathcal{H}$.
Set $U=(u,v,\psi)^T$, where $v=u_t$. Then, one can rewrite system (P') into an evolutionary equation in $\mathcal{H}$:

$$\begin{cases}U' = \mathcal{A}U, & t>0,\\ U(0) = (u_0,u_1,\psi_0),\end{cases} \tag{3.1}$$

where the operator $\mathcal{A}: D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is defined as

$$\mathcal{A}\begin{pmatrix}u\\ v\\ \psi\end{pmatrix} = \begin{pmatrix}v\\ \mu\Delta u + (\mu+\lambda)\nabla(\text{div } u) - \zeta\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi)\,d\xi\\ -(\xi^2+\kappa)\psi + v(x)\omega(\xi)\end{pmatrix}, \tag{3.2}$$

$$D(\mathcal{A}) = \left\{\begin{array}{l}(u,v,\psi)^T \text{ in } \mathcal{H}: u\in\left(H^2(\Omega)\cap H_0^1(\Omega)\right)^n, v\in\left(H^1(\Omega)\right)^n,\\ -(\xi^2+\kappa)\psi + v(x)\omega(\xi)\in\left(L^2(\Omega\times(-\infty,+\infty))\right)^n,\\ |\xi|\psi\in\left(L^2(\Omega\times(-\infty,+\infty))\right)^n \text{ in } \Omega\end{array}\right\}. \tag{3.3}$$

Proposition 3.1 *The operator $\mathcal{A}$ is m-dissipative in the state space $\mathcal{H}$.*

Proof: For all $U \in D(\mathcal{A})$, we have

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \sum_{j=1}^{n} \int_{\Omega} \int_{-\infty}^{+\infty} (\xi^2 + \kappa)|\psi_j(x,\xi)|^2 \, d\xi \, dx, \tag{3.4}$$

where we have used (3.1), (2.10) and the fact that

$$E(t) = \frac{1}{2}\|U\|^2_{\mathcal{H}}. \tag{3.5}$$

Now let

$$G = (G_1, G_2, G_3)^T \in \mathcal{H},$$

where $G_i = (g_i^1, g_i^2, ..., g_i^n)^T$. we prove the existence of

$$U = (u, v, \psi)^T \in D(\mathcal{A})$$

unique solution of the equation

$$(\tilde{\lambda} I - \mathcal{A})U = G \quad \forall \tilde{\lambda} > 0. \tag{3.6}$$

Equivalently, we have the following system

$$\begin{cases} \tilde{\lambda} u - v = G_1(x), \\ \tilde{\lambda} v - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \zeta \displaystyle\int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\, d\xi = G_2(x), \\ \tilde{\lambda}\psi + (\xi^2+\kappa)\psi - v(x)\omega(\xi) = G_3(x,\xi). \end{cases} \tag{3.7}$$

Suppose u is found with the appropriate regularity. Then, $(3.7)_1$ and $(3.7)_3$ yield

$$v = \tilde{\lambda} u - G_1(x) \in \left(H^1(\Omega)\right)^n \tag{3.8}$$

and

$$\psi = \frac{G_3(x,\xi) + \omega(\xi)v(x)}{\xi^2 + \kappa + \tilde{\lambda}}. \tag{3.9}$$

Inserting (3.8) in $(3.7)_2$, we get

$$\begin{aligned} &(\tilde{\lambda}^2 + a_1\tilde{\lambda}(\tilde{\lambda}+\kappa)^{\sigma-1})u - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \\ &= G_2(x) + (\tilde{\lambda} + a_1(\tilde{\lambda}+\kappa)^{\sigma-1})G_1(x) - \zeta \int_{-\infty}^{+\infty} \frac{\omega(\xi)G_3(x,\xi)}{\tilde{\lambda}+\xi^2+\kappa}\, d\xi. \end{aligned} \tag{3.10}$$

Let $w \in \left(H_0^1(\Omega)\right)^n$ be a test function. Multiplying (3.10) by w. Consequently, (3.10) can be written after integrating by parts in the following form

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \int_{\Omega} \left((\tilde{\lambda}^2 + a_1\tilde{\lambda}(\tilde{\lambda}+\kappa)^{\sigma-1})u_j\overline{w}_j + \mu\nabla u_j \nabla\overline{w}_j \, dx\right) + (\mu+\lambda)\int_{\Omega} (\text{div } u)(\text{div } \overline{w})\, dx \\ = \displaystyle\sum_{j=1}^{n} \int_{\Omega} \left(g_2^j(x) + (\tilde{\lambda} + a_1(\tilde{\lambda}+\kappa)^{\sigma-1})g_1^j(x)\right)\overline{w}_j\, dx - \zeta \sum_{j=1}^{n} \int_{\Omega} \overline{w}_j \left(\int_{-\infty}^{\infty} \frac{\omega(\xi) g_3^j(x,\xi)}{\xi^2+\kappa+\tilde{\lambda}}\, d\xi\right) dx. \end{cases} \tag{3.11}$$

Problem (3.11) is of the form

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{3.12}$$

where $\mathcal{B} : \left(H_0^1(\Omega)\right)^n \times \left(H_0^1(\Omega)\right)^n \to \mathbb{C}$ is the sesquilinear form defined by

$$\mathcal{B}(u,w) = \sum_{j=1}^{n} \int_{\Omega} \left((\tilde{\lambda}^2 + a_1\tilde{\lambda}(\tilde{\lambda}+\kappa)^{\sigma-1})u_j\overline{w}_j + \mu\nabla u_j \nabla\overline{w}_j \, dx\right) + (\mu+\lambda)\int_{\Omega} (\text{div } u)(\text{div } \overline{w})\, dx$$

and $\mathcal{L} : \left(H_0^1(\Omega)\right)^n \to \mathbb{C}$ is the antilinear functional given by

$$\mathcal{L}(w) = \sum_{j=1}^{n} \int_\Omega \left(g_2^j(x) + (\tilde{\lambda} + a_1(\tilde{\lambda}+\kappa)^{\sigma-1}) g_1^j(x)\right) \overline{w}_j \, dx - \zeta \sum_{j=1}^{n} \int_\Omega \overline{w}_j \left(\int_{-\infty}^{\infty} \frac{\omega(\xi) g_3^j(x,\xi)}{\xi^2 + \kappa + \tilde{\lambda}} \, d\xi \right) dx.$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram theorem, we conclude that for all $w \in \left(H_0^1(\Omega)\right)^n$, the system (3.12) has a unique solution $u \in \left(H_0^1(\Omega)\right)^n$. By the regularity theory for the linear elliptic equations, it follows that $u \in \left(H^2(\Omega)\right)^n$. Therefore, the operator $(\tilde{\lambda} I - \mathcal{A})$ is surjective for any $\tilde{\lambda} > 0$. Consequently, using Hille-Yosida Theorem (see [7]), we have the following existence result:

Theorem 3.1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (3.1) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (3.1) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

□

4. Strong Stability of the System

In this section, we use a general criteria of Arendt-Batty and Lyubich-Vũ (see [2] and [10]) to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associate to the Lamé system in the absence of the compactness of the resolvent of $\mathcal{A}$.

Theorem 4.1 ([2]- [10]) *Let X be a reflexive Banach space and $(T(t))_{t\geq 0}$ be a C_0-semigroup generated by A on X. Assume that $(T(t))_{t\geq 0}$ is bounded and that no eigenvalues of A lie on the imaginary axis. If $r(A) \cap iR$ is countable, then $(T(t))_{t\geq 0}$ is stable.*

Our main result is the following theorem

Theorem 4.2 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (3.1) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 4.2, we need the following two lemmas.

Lemma 4.1 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof: First we show the result for $\tilde{\lambda} \neq 0$. Suppose that there exist a real number $\tilde{\lambda} \neq 0$ and $0 \neq U = (u, v, \psi)^T \in D(\mathcal{A})$, such that

$$\mathcal{A}U = i\tilde{\lambda}U. \tag{4.1}$$

Our goal is to find a contradiction by proving that $U = 0$. Detailing (4.1) we get

$$\begin{cases} i\tilde{\lambda}u - v = 0, \\ i\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \zeta \displaystyle\int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\, d\xi = 0, \\ i\tilde{\lambda}\psi + (\xi^2+\kappa)\psi - v(x)\omega(\xi) = 0. \end{cases} \tag{4.2}$$

Then, from (3.4) we have

$$\psi = 0, \quad v = 0. \tag{4.3}$$

Next, inserting (4.3) in $(4.2)_1$ and using the fact that $\tilde{\lambda} \neq 0$, we get

$$u \equiv 0. \tag{4.4}$$

Hence $U \equiv 0$.

Now if $\tilde{\lambda} = 0$, we have $v = 0$ and we deduce that

$$\begin{cases} -\mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) = 0, \\ u = 0 \text{ in } \Gamma. \end{cases} \tag{4.5}$$

Multiplying by $\overline{u}$, integrating over Ω we have

$$\|\nabla u\|^2_{L^2(\Omega)} + \|\text{div } u\|^2_{L^2(\Omega)} = 0. \tag{4.6}$$

Hence $u = 0$. Then $U \equiv 0$. □

Lemma 4.2 *We have If $\tilde{\lambda} \neq 0$, the operator $i\tilde{\lambda}I - \mathcal{A}$ is surjective.*
If $\tilde{\lambda} = 0$ and $\kappa \neq 0$, the operator $i\tilde{\lambda}I - \mathcal{A}$ is surjective.

Proof:
Case 1: $\tilde{\lambda} \neq 0$. Let $G = (G_1, G_2, G_3)^T \in \mathcal{H}$ be given, and let $X = (u, v, \psi)^T \in D(\mathcal{A})$ be such that

$$(i\tilde{\lambda}I - \mathcal{A})X = G. \tag{4.7}$$

Equivalently, we have

$$\begin{cases} i\tilde{\lambda}u - v = G_1, \\ i\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \zeta \displaystyle\int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\, d\xi = G_2, \\ i\tilde{\lambda}\psi + (\xi^2+\kappa)\psi - v(x)\omega(\xi) = G_3, \end{cases} \tag{4.8}$$

From $(4.8)_1$ and $(4.8)_2$, we have

$$-\tilde{\lambda}^2 u - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \zeta \int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\, d\xi = (G_2 + i\tilde{\lambda}G_1) \tag{4.9}$$

with $u_{|\Gamma} = 0$. Solving system (4.9) is equivalent to finding $u \in (H^2 \cap H_0^1(\Omega))^n$ such that

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \int_\Omega \left((-\tilde{\lambda}^2 + ia_1\tilde{\lambda}(i\tilde{\lambda}+\kappa)^{\sigma-1}) u_j \overline{w}_j + \mu\nabla u_j \nabla\overline{w}_j \, dx \right) + (\mu+\lambda) \int_\Omega (\text{div } u)(\text{div } \overline{w})\, dx \\ = \displaystyle\sum_{j=1}^{n} \int_\Omega \left(g_2^j(x) + (i\tilde{\lambda} + a_1(i\tilde{\lambda}+\kappa)^{\sigma-1}) g_1^j(x) \right) \overline{w}_j \, dx - \zeta \sum_{j=1}^{n} \int_\Omega \overline{w}_j \left(\int_{-\infty}^{\infty} \frac{\omega(\xi) g_3^j(x,\xi)}{\xi^2+\kappa+i\tilde{\lambda}}\, d\xi \right) dx \end{cases} \tag{4.10}$$

for all $w \in (H_0^1(\Omega))^n$. We can rewrite (4.10) as

$$-(L_{\tilde{\lambda}} u, w)_{((H_0^1(\Omega))^n, ((H_0^1(\Omega))')^n)} + a_{(H_0^1(\Omega))^n}(u,w) = l(w) \tag{4.11}$$

with the sesquilinear form defined by

$$a_{(H_0^1(\Omega))^n}(u,w) = \mu \sum_{j=1}^{n} \int_\Omega \nabla u_j \nabla\overline{w}_j \, dx + (\mu+\lambda) \int_\Omega (\text{div } u)(\text{div } \overline{w})\, dx + ia_1\tilde{\lambda}(i\tilde{\lambda}+\kappa)^{\sigma-1} \sum_{j=1}^{n} \int_\Omega u_j \overline{w}_j \, dx$$

and

$$(L_{\tilde{\lambda}}u,w)_{((H_0^1(\Omega))^n,((H_0^1(\Omega))')^n)}=\sum_{j=1}^{n}\int_{\Omega}\tilde{\lambda}^2u_j\overline{w}_j\,dx.$$

One can easily see that $L_\lambda, a_{(H_0^1(\Omega))^n}$ and l are bounded. Furthermore

$$\begin{array}{rcl}\Re a_{(H_0^1(\Omega))^n}(u,u) &=& \mu\|\nabla u\|_2^2+(\mu+\lambda)\|\text{div } u\|_2^2+a_1\lambda\Re\left(i(i\lambda+\omega)^{\tau-1}\right)\|u\|_2^2\\ &\geq& \mu\|\nabla u\|_2^2+(\mu+\lambda)\|\text{div } u\|_2^2,\end{array}$$

where we have used the fact that

$$a_1\lambda\Re\left(i(i\lambda+\omega)^{\tau-1}\right)=\zeta\lambda^2\int_{-\infty}^{+\infty}\frac{\omega(\xi)^2}{\lambda^2+(\omega+\xi^2)^2}\,d\xi>0.$$

Thus $a_{(H_0^1(\Omega))^n}$ is coercive. Consequently, following Fredholm alternative, proving the existence of U solution of (4.11) reduces to proving that (4.11) with $l\equiv 0$ has a notrivial solution. Indeed if there exists $U\neq 0$, such that

$$-(L_{\tilde{\lambda}}u,w)_{((H_0^1(\Omega))^n,((H_0^1(\Omega))')^n)}+a_{(H_0^1(\Omega))^n}(u,w)=0\quad\forall w\in(H_0^1(\Omega))^n. \tag{4.12}$$

In particular for $w=u$, it follows that

$$\lambda^2\|u\|^2_{L^2(0,L)}-i\gamma\lambda(i\lambda+\omega)^{\tau-1}\|u\|_2^2=\mu\|\nabla u\|_2^2+(\mu+\lambda)\|\text{div } u\|_2^2.$$

Hence, we have

$$u=0. \tag{4.13}$$

Now, if $\tilde{\lambda}=0$, the system (4.8) is reduced to the following system

$$\begin{cases}v=-G_1,\\ -\mu\Delta u-(\mu+\lambda)\nabla(\text{div } u)+\zeta\displaystyle\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi)\,d\xi=G_2,\\ (\xi^2+\kappa)\psi-v(x)\omega(\xi)=G_3,\end{cases} \tag{4.14}$$

Solving system (4.14) is equivalent to finding $u\in(H^2\cap H_0^1(\Omega))^n$ such that

$$\begin{array}{l}\mu\displaystyle\sum_{j=1}^{n}\int_\Omega\nabla u_j\nabla\overline{w}_j\,dx+(\mu+\lambda)\int_\Omega(\text{div } u)(\text{div } \overline{w})\,dx=\sum_{j=1}^{n}\int_\Omega g_2^j\overline{w}_j\,dx\\ +\zeta\displaystyle\int_{-\infty}^{\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\sum_{j=1}^{n}\int_\Omega g_1^j\overline{w}_j\,dx-\zeta\sum_{j=1}^{n}\int_\Omega\overline{w}_j\int_{-\infty}^{\infty}\frac{\omega(\xi)g_3^j(x,\xi)}{\xi^2+\kappa}\,d\xi\,dx.\end{array} \tag{4.15}$$

for all $w\in(H_0^1(\Omega))^n$. Consequently, problem (4.15) is equivalent to the problem

$$\mathcal{B}(u,w)=\mathcal{L}(w), \tag{4.16}$$

where the sesquilinear form $\mathcal{B}:(H_0^1(\Omega))^n\times(H_0^1(\Omega))^n\to\mathbb{C}$ and the antilinear form $\mathcal{L}:(H_0^1(\Omega))^n\to\mathbb{C}$ are defined by

$$\mathcal{B}(u,w)=\mu\sum_{j=1}^{n}\int_\Omega\nabla u_j\nabla\overline{w}_j\,dx+(\mu+\lambda)\int_\Omega(\text{div } u)(\text{div } \overline{w})\,dx \tag{4.17}$$

and

$$\begin{array}{l}\mathcal{L}(w)=\displaystyle\sum_{j=1}^{n}\int_\Omega g_2^j\overline{w}_j\,dx+\zeta\int_{-\infty}^{\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\sum_{j=1}^{n}\int_\Omega g_1^j\overline{w}_j\,dx\\ \qquad -\zeta\displaystyle\sum_{j=1}^{n}\int_\Omega\overline{w}_j\int_{-\infty}^{\infty}\frac{\omega(\xi)g_3^j(x,\xi)}{\xi^2+\kappa}\,d\xi\,dx.\end{array} \tag{4.18}$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in (H_0^1(\Omega))^n$ problem (4.16) admits a unique solution $u \in (H_0^1(\Omega))^n$. Applying the classical elliptic regularity, it follows from (4.15) that $u \in (H^2(\Omega))^n$. Therefore, the operator $\mathcal{A}$ is surjective. □

We deduce that $r(\mathcal{A}) \cap i\mathbb{R} \subset \{0\}$.Thus, we get the conclusion by applying Theorem 4.1. The proof is thus complete.

5. Lack of Uniform Stabilization

In this section we shall prove that system is not uniformly exponentially stable. This result is due to the fact that a sub-sequence of eigenvalues of $\mathcal{A}$ which is close to the imaginary axis. According to a Theorem due to Huang [8] and Pruss [13], it is sufficient to prove that the resolvent of the operator $\mathcal{A}$ is not uniformly bounded on the imaginary axis.

Our main result is the following.

Theorem 5.1 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let $\tilde{\lambda}$ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, \psi)^T$. To solve $\mathcal{A}U = \tilde{\lambda}U$ is enough to solve

$$\begin{cases} \tilde{\lambda}u - v = 0, \\ \tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \zeta \displaystyle\int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\, d\xi = 0, \\ \tilde{\lambda}\psi + (\xi^2+\kappa)\psi - v(x)\omega(\xi) = 0. \end{cases} \tag{5.1}$$

Next, by eliminating v from the above system we get the following system:

$$(\tilde{\lambda}^2 + a_1\tilde{\lambda}(\tilde{\lambda}+\kappa)^{\sigma-1})u - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) = 0. \tag{5.2}$$

As a consequence of (5.2), div u verifies the scalar equation

$$(\tilde{\lambda}^2 + a_1\tilde{\lambda}(\tilde{\lambda}+\kappa)^{\sigma-1})\text{div } u - (2\mu+\lambda)\Delta(\text{div } u) = 0 \tag{5.3}$$

From (5.3) we arrive at

$$\begin{cases} (\tilde{\lambda}^2 + a_1\tilde{\lambda}(\tilde{\lambda}+\kappa)^{\sigma-1})E - (2\mu+\lambda)\Delta E = 0 & \text{in } \Omega \\ E = 0 & \text{in } \partial\Omega, \end{cases} \tag{5.4}$$

where $E = \text{div } u$. Our goal is to find large eigenvalues which are closed to the imaginary axis and to give their expansion.

Lemma 5.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\tilde{\lambda}_m\}_{m\in\mathbf{Z}^*, |m|\geq N} \subset \sigma(\mathcal{A}), \tag{5.5}$$

where

$$\tilde{\lambda}_m = i\sqrt{2\mu+\lambda}\beta_m - \frac{a_1}{2\sqrt{2\mu+\lambda}^{1-\sigma}\beta_m^{(1-\sigma)}}\left(\cos(1-\sigma)\frac{\pi}{2} - i\sin(1-\sigma)\frac{\pi}{2}\right) + o\left(\frac{1}{\beta_m^{(1-\sigma)}}\right), m \geq N,$$

$$\tilde{\lambda}_m = \overline{\tilde{\lambda}_{-m}} \text{ if } m \leq -N.$$

Moreover for all $|m| \geq N$, the eigenvalues $\tilde{\lambda}_m$ are simple.

Proof: Let $-\beta_m^2 = (i\beta_m)^2$ be a sequence of eigenvalues corresponding to the sequence of normalized eigenfunctions y_m of the operator Δ such that

$$|\beta_m| \longrightarrow \infty \text{ as } m \longrightarrow \infty$$

and

$$\begin{cases} \Delta y_m = -\beta_m^2 y_m & \text{in } \Omega, \\ y_m = 0 & \text{on } \partial\Omega. \end{cases}$$

then

$$\Delta y_m = -\beta_m^2 y_m = (\tilde{\lambda}^2 + a_1\tilde{\lambda}(\tilde{\lambda}+\kappa)^{\sigma-1})y_m \tag{5.6}$$

$$\beta_m^2 = -\frac{1}{2\mu+\lambda}(\tilde{\lambda}^2 + a_1\tilde{\lambda}(\tilde{\lambda}+\kappa)^{\sigma-1}) \tag{5.7}$$

Where

$$\tilde{\lambda}^2 + (2\mu+\lambda)\beta_m^2 + a_1\tilde{\lambda}(\tilde{\lambda}+\kappa)^{\sigma-1} = 0. \tag{5.8}$$

then

$$\tilde{\lambda}_m = i\sqrt{2\mu+\lambda}\beta_m + \varepsilon_m. \tag{5.9}$$

Form equation and (5.8) and (5.9) we get

$$\varepsilon_m = -\frac{a_1}{2(i\sqrt{2\mu+\lambda}\beta_m)^{1-\sigma}} + o(\frac{1}{\beta_m^{(1-\sigma)}}) \tag{5.10}$$

Using (5.9) and (5.10), we obtain

$$\tilde{\lambda}_m = i\sqrt{2\mu+\lambda}\beta_m - \frac{a_1}{2\sqrt{2\mu+\lambda}^{1-\sigma}\beta_m^{(1-\sigma)}}\left(\cos(1-\sigma)\frac{\pi}{2} - i\sin(1-\sigma)\frac{\pi}{2}\right) + o(\frac{1}{\beta_m^{(1-\sigma)}}). \tag{5.11}$$

From (5.11) we have in that case $\beta_m^{(1-\sigma)}\Re\tilde{\lambda}_m \sim \beta$, with

$$\beta = -\frac{a_1}{2\sqrt{2\mu+\lambda}^{1-\sigma}}\cos(1-\sigma)\frac{\pi}{2} \tag{5.12}$$

The operator $\mathcal{A}$ has a non exponential decaying branche of eigenvalues. Thus the proof is complete. □

6. Optimal Polynomial Stability

The necessary and suficient conditions for the plynomial stability of the C_0- semigroup of contractions on a Hilbert space were obtained by Borichev and Tomilov [5].

Theorem 6.1 ([5]) *Let $S(t) = e^{\mathcal{A}t}$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{|\beta|\geq 1}\frac{1}{\beta^\delta}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$

for some $\delta > 0$, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{\delta}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Our main result is as follows.

Theorem 6.2 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{(1-\sigma)}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Moreover, the rate of energy decay $t^{2/(1-\sigma)}$ is optimal for general initial data in $D(\mathcal{A})$.

Proof: We will need to study the resolvent equation $(i\tilde{\lambda}-\mathcal{A})U=G$, for $\lambda\in\mathbb{R}$, namely

$$\begin{cases} i\tilde{\lambda}u-v=G_1, \\ i\tilde{\lambda}v-\mu\Delta u-(\mu+\lambda)\nabla(\text{div }u)+\zeta\displaystyle\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi)\,d\xi=G_2, \\ i\tilde{\lambda}\psi+(\xi^2+\kappa)\psi-v(x)\omega(\xi)=G_3, \end{cases} \tag{6.1}$$

where $G=(G_1,G_2,G_3)^T$. Taking inner product in $\mathcal{H}$ with U and using (3.4) we get

$$|Re\langle\mathcal{A}U,U\rangle|\leq\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}. \tag{6.2}$$

This implies that

$$\zeta\sum_{j=1}^{n}\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\,dx\leq C\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}. \tag{6.3}$$

From $(6.1)_3$ we obtain

$$v(x)\,\omega(\xi)=(\xi^2+\kappa+i\,\tilde{\lambda})\,\psi-G_3(\xi), \tag{6.4}$$

By multiplying (6.4) by $(i\gamma+\xi^2+\kappa)^{-2}|\xi|$, we get

$$(i\tilde{\lambda}+\xi^2+\kappa)^{-2}v(x)\omega(\xi)|\xi|=(i\tilde{\lambda}+\xi^2+\kappa)^{-1}|\xi|\psi-(i\tilde{\lambda}+\xi^2+\kappa)^{-2}|\xi|G_3(x,\xi). \tag{6.5}$$

Hence, by taking absolute values of both sides of (6.5), integrating over the interval $]-\infty,+\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$\mathcal{S}|v_j(x)|\leq\sqrt{2}\mathcal{U}\left(\int_{-\infty}^{+\infty}\xi^2|\psi_j|^2\,d\xi\right)^{\frac{1}{2}}+2\mathcal{V}\left(\int_{-\infty}^{+\infty}|g_3^j(x,\xi)|^2\,d\xi\right)^{\frac{1}{2}}, \tag{6.6}$$

where

$$\mathcal{S}=\left|\int_{-\infty}^{+\infty}(i\tilde{\lambda}+\xi^2+\kappa)^{-2}|\xi|\omega(\xi)\,d\xi\right|=\frac{|1-2\sigma|}{4}\frac{\pi}{|\sin\frac{(2\sigma+3)}{4}\pi|}|i\tilde{\lambda}+\kappa|^{\frac{(2\sigma-5)}{4}},$$

$$\mathcal{U}=\left(\int_{-\infty}^{+\infty}(|\tilde{\lambda}|+\xi^2+\kappa)^{-2}\,d\xi\right)^{\frac{1}{2}}=(\frac{\pi}{2})^{1/2}||\tilde{\lambda}|+\kappa|^{-\frac{3}{4}},$$

$$\mathcal{V}=\left(\int_{-\infty}^{+\infty}(|\tilde{\lambda}|+\xi^2+\kappa)^{-4}|\xi|^2\,d\xi\right)^{\frac{1}{2}}=\left(\frac{\pi}{16}||\tilde{\lambda}|+\kappa|^{-\frac{5}{2}}\right)^{1/2}.$$

Thus, by using the inequality $2PQ\leq P^2+Q^2, P\geq0, Q\geq0$, again, we get

$$\mathcal{S}^2\int_0^L|v_j(x)|^2\,dx\leq2\mathcal{U}^2\left(\int_0^L\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j|^2\,d\xi\,dx\right)+4\mathcal{V}^2\left(\int_0^L\int_{-\infty}^{+\infty}|g_3^j(x,\xi)|^2\,d\xi\,dx\right). \tag{6.7}$$

We deduce that

$$\int_{\Omega}|v_j(x)|^2\,dx\leq c|\tilde{\lambda}|^{1-\sigma}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+c\|F\|_{\mathcal{H}}^2. \tag{6.8}$$

Let us introduce the following notation

$$\mathcal{I}_u(x)=\sum_{j=1}^{n}\left(|v_j(x)|^2+\mu|\nabla u_j(x)|^2\right)+(\mu+\lambda)|\text{div }u(x)|^2$$

and

$$\mathcal{E}_u=\int_{\Omega}\mathcal{I}_u(x)\,dx.$$

Lemma 6.1 *We have that*

$$\mathcal{E}_u \leq c\|v\|^2 + c\|G\|_{\mathcal{H}}^2 + c'\|G\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{6.9}$$

for positive constants c and c'.

Proof: Multiplying the equation $(6.1)_2$ by $\bar{u}$, integrating on Ω we obtain

$$\begin{aligned} &-\int_\Omega v_j(\overline{i\tilde{\lambda}u_j})\,dx + \mu\int_\Omega |\nabla u_j|^2\,dx + (\mu+\lambda)\int_\Omega (\text{div } u)\frac{\partial \bar{u_j}}{\partial x_j}\,dx \\ &+\zeta\int_\Omega \bar{u_j}(\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi)\,dx = \int_\Omega \bar{u} g_2^j\,dx. \end{aligned} \tag{6.10}$$

From $(6.1)_1$, we have $i\tilde{\lambda}u_j = v_j + g_1^j$. Then

$$\begin{aligned} &-\int_\Omega |v_j|^2\,dx + \mu\int_\Omega |\nabla u_j|^2\,dx + (\mu+\lambda)\int_\Omega (\text{div } u)\frac{\partial \bar{u_j}}{\partial x_j}\,dx \\ &+\zeta\int_\Omega \bar{u_j}(\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi)\,dx = \int_\Omega \bar{u_j} g_2^j\,dx + \int_\Omega v_j\bar{g_1^j}\,dx. \end{aligned} \tag{6.11}$$

Hence

$$\begin{aligned} &-\sum_{j=1}^n\int_\Omega |v_j|^2\,dx + \mu\sum_{j=1}^n\int_\Omega |\nabla u_j|^2\,dx + (\mu+\lambda)\int_\Omega |\text{div } u|^2\,dx \\ &+\zeta\sum_{j=1}^n\int_\Omega \bar{u_j}(\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi)\,dx = \sum_{j=1}^n\int_\Omega \bar{u_j} g_2^j\,dx + \sum_{j=1}^n\int_\Omega v_j\bar{g_1^j}\,dx. \end{aligned} \tag{6.12}$$

We can estimate

$$\begin{aligned} &\left|\int_\Omega \bar{u_j}\left(\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi\right)dx\right| \\ &\leq \|u_j\|_{L^2(\Omega)}\left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right)^{\frac{1}{2}}\left(\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi)|^2\,d\xi\,dx\right)^{\frac{1}{2}} \\ &\leq \frac{\varepsilon}{2}\left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right)\|u_j\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi)|^2\,d\xi\,dx \\ &\leq \frac{\varepsilon}{2}C(\Omega)\left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right)\|\nabla u_j\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi)|^2\,d\xi\,dx, \end{aligned}$$

$$\left|\int_\Omega \bar{u_j}g_2^j\,dx\right| \leq \frac{\varepsilon}{2}C(\Omega)\|\nabla u_j\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon}\|g_2^j\|_{L^2(\Omega)}^2,$$

$$\left|\int_\Omega v_j\bar{g_1^j}\,dx\right| \leq \frac{\varepsilon}{2}\|v_j\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon}\|g_1^j\|_{L^2(\Omega)}^2.$$

□

Choosing ε small enough, we conclude (6.9). Since $\kappa > 0$, from (6.3), we have

$$\|\psi_j\|_{L^2(\Omega\times(-\infty,+\infty))}^2 = \int_\Omega\int_{-\infty}^{+\infty}|\psi_j|^2 d\xi\,dx \leq \frac{1}{\kappa}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j|^2 d\xi\,dx \leq c\|U\|\,\|F\|.$$

We conclude that

$$\|U\|^2 \leq c\|v\|^2 + c'\|F\|^2 + c''\|U\|\,\|F\|. \tag{6.13}$$

Finally, (6.8) and (6.9) imply that

$$\|U\|_{\mathcal{H}} \leq C|\tilde{\lambda}|^{1-\sigma}\|F\|_{\mathcal{H}}.$$

for a positive constant C. The conclusion then follows by applying Theorem 6.1.

□

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