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Entropy Analysis of Multivariate Fractional Controlled ARMA Systems

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ABSTRACT: In this paper, entropy measure is used as effective tool to analysis the behavior of controlled autoregressive moving average system of fractional order (FCARMA). The characteristic function technique is employed of output state, through it, entropy is computed in its three types Shannon, Rényi and Tsallis. The important aspect of this work is that we consider and discuss the controlled ARMA System when it is affected by fractional order for the entire system, not just for a specific part as in ARIMA. Additionally, we introduce an algorithm to calculate the values of entropy when it is influenced by Gaussian and Cauchy processes. Finally, examples are given to illustrative the information behavior and proposed algorithm of system under Gaussian process and Cauchy processes.

Keywords: FCARMA system, Gaussian process, entropy and characteristic function.

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1. Introduction

The ARMA model and their extensions are widely used for analyzing of time series and accurate forecasting of their future behavior. One of these extensions is MCARFIMA system, which is characterized by the fractional order and control process. This system has been used throughout various fields, such as energy [1], economics [2], industry [3], trade and sales [4], medicine and health [5], insurance and social security [6].

In the literature, [7] determined the characteristic function of univariate AR(2) state from its residual functions. [8] generalized this work to $\mathrm{ARMA(p,q)}$ state. [9] presented recursive algorithm for impulse response ARMA modelling, it was applied for modelling fractional order ARMA system described by fractional powers and bilinear transformation. [10] used the entropy measure to characterize the uncertainty of the tracking error for stochastic ARMA system over a communication network. They proposed an algorithm based on the probability density function of the tracking error to compute entropy. (2019) [11] presented ARTFIMA model as generalization of ARIMA model , which includes a flexibility operation to handle non-summability of its covariance function, making it more mathematically tractable. (2021) [12] considered the parameters estimation of ARFIMA model by using the minimizing of mean squared error between the simulated and empirical characteristic function. [13] (2022) investigated the equivalence between classical ARMA and fractional ARMA (FARMA) systems in both continuous time and discrete time frameworks, establishing conditions under which fractional dynamics can be represented by equivalent integer order models. [14] presented the method based on rapidly maximum entropy for estimating the ARMA model parameters from observed data while [15] used maximum entropy optimization

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to estimate and analysis the ARMA parameters and studied the statistical properties of model. [16] extended the method based on maximum entropy to represent ARMA models, and discuss the effectiveness in parameter estimation and probabilistic modeling. [17] introduced multivariate controlled ARMA system and proposed based on the characteristic function of output state of system to determine the Rényi entropy formulation, through which an effective algorithm for computing the systems entropy was developed. [18] extended the previous study [17] on Rényi entropy calculation to incorporate VCARFIMA system. While the study in [17] focus on the classical integer order and fractional integrated formulation of controlled ARMA system, this work generalizes their results by considering a multivariate fractional order controlled ARMA (MFCARMA) system. The characteristic function of output process is computed by using its residual characteristic functions through which an explicit form of the entropy can be obtained under stationery and independence conditions of process. Also, the information of this system is discussed according to Shannon, Rényi and Tsallis measures. To further support the theoretical analysis, two examples are provided and discussed, the first corresponds to a system perturbed by a Gaussian process and the second by a Cauchy process.

2. Preliminaries

In this section, we introduce the definitions, notations, fundamental concepts and theorems that will be used throughout this paper. The Shannon entropy of a random variable y is defined as [19]

$$^{S}H\left(y\right)=-E\left(log\left(f\left(y;\theta\right)\right)\right) \tag{2.1}$$

where, f is a probability function and θ is a petameters vector. As a generalizations of Shannon entropy, the Rényi and Tsallis entropies can be defined as follows respectively [20],

$$^{\mathcal{R}}H_{\alpha}\left(y\right)=\begin{cases}\frac{1}{1-\alpha}log\left(E\left(f\left(y;\theta\right)^{\alpha-1}\right)\right), & 0<\alpha\neq1,\\ ^{S}H\left(y\right) & ,\quad \alpha=1\end{cases} \tag{2.2}$$

$$^{T}H_{\beta}\left(y\right)=\begin{cases}\frac{1}{\beta-1}\left(1-E\left(f\left(y;\theta\right)^{\beta-1}\right)\right), & 0<\beta\neq1,\\ ^{S}H\left(y\right) & ,\quad \beta=1\end{cases} \tag{2.3}$$

where, α *and* β are positive real numbers and referred to as entropic index. The relationship between these entropies and Shannon entropy can be described as [21] $\lim_{\beta\to1}{}^{T}H_{\beta}\left(y\right)=\lim_{\alpha\to1}{}^{\mathcal{R}}H_{\alpha}\left(y\right)={}^{S}H\left(y\right)$.

Lemma 2.1 *[22] let* $\mathrm{y}\in\mathbb{R}^{\mathrm{d}}$ *be a random with covariance matrix* $\mathbf{\Sigma}_{\mathrm{y}}$ *then, , then the following inequality holds*

$$^{S}H\left(y\right)\leq\frac{1}{2}\ln\left(\left(2\pi e\right)^{d}\det\left(\mathbf{\Sigma}_{\mathrm{y}}\right)\right) \tag{2.4}$$

If y *is distributed multivariate Gaussian* $\mathrm{MG_d}(0,\mathbf{\Sigma}_{\mathrm{y}})$, *then equality is hold.*

Lemma 2.2 *[17] let* $\mathrm{y}\in\mathbb{R}^{\mathrm{d}}$ *be a random with covariance matrix* $\mathbf{\Sigma}_{\mathrm{y}}$ *then, , then the following inequalities are hold*

$$^{\mathcal{R}}H_{\alpha}\left(y\right)\leq\mathcal{K}_{d}\left(\alpha\right)+\frac{1}{2}\log\left(\det\left(\mathbf{\Sigma}_{\mathrm{y}}\right)\right) \tag{2.5}$$

$$^{T}H_{\beta}\left(y\right)\leq\frac{1}{\left(\beta-1\right)}\left(1-e^{\left(1-\beta\right)\left(\mathcal{K}_{d}\left(\beta\right)+\frac{1}{2}\ln\left(\det\left(\mathbf{\Sigma}_{\mathrm{y}}\right)\right)\right)}\right) \tag{2.6}$$

where,

$$\mathcal{K}_{\mathrm{d}}\left(\gamma\right)=\begin{cases}\frac{\mathrm{d}}{2}\log\left(\frac{\pi(\gamma(\mathrm{d}+2)-\mathrm{d})}{\gamma-1}\right)+\frac{1}{\gamma-1}\log\left(\frac{(\gamma(\mathrm{d}+2)-\mathrm{d})}{2\gamma}\right)+\log\left(\frac{\Gamma\left(\frac{\gamma}{\gamma-1}\right)}{\Gamma\left(\frac{(\gamma(\mathrm{d}+2)-\mathrm{d})}{2(\gamma-1)}\right)}\right), & \gamma>1\\ \frac{\mathrm{d}}{2}\log\left(\frac{\pi(\gamma(\mathrm{d}+2)-\mathrm{d})}{1-\gamma}\right)-\frac{\gamma}{1-\gamma}\log\left(\frac{(\gamma(\mathrm{d}+2)-\mathrm{d})}{2\gamma}\right)-\log\left(\frac{\Gamma\left(\frac{\gamma}{1-\gamma}\right)}{\Gamma\left(\frac{(\gamma(\mathrm{d}+2)-\mathrm{d})}{2(1-\gamma)}\right)}\right), & \frac{\mathrm{d}}{\mathrm{d}+2}<\gamma<1\\ \frac{\mathrm{d}}{2}\log\left(2\pi\mathrm{e}\right) & ,\quad \gamma=1\end{cases} \tag{2.7}$$

The symbol Γ *represents gamma function.*

Lemma 2.3 *[17] Consider* y $\sim$MG$_\mathrm{d}$(0,$\boldsymbol{\Sigma}_\mathrm{y}$). *Then,*

$$^{S}H\left(y\right)=\frac{1}{2}log\left(\det\left(2\pi\ \boldsymbol{\Sigma}_{\mathrm{y}}\right)\right) \tag{2.8}$$

$$^{\mathcal{R}}H_{\alpha}\left(y\right)=\frac{1}{2}log\left(\det\left(2\pi\ \boldsymbol{\Sigma}_{\mathrm{y}}\right)\right)+\frac{\mathrm{d}}{2\left(\alpha-1\right)}log\left(\alpha\right) \tag{2.9}$$

$$^{T}H_{\beta}\left(y\right)=\frac{\beta^{\frac{d}{2}}}{(1-\beta)}\left(\det\left(2\pi\boldsymbol{\Sigma}_{\mathrm{y}}\right)\right)^{\frac{1-\beta}{2}}-\frac{1}{(1-\beta)} \tag{2.10}$$

Lemma 2.4 *[23] Consider* y *to be Cauchy distributed* y $\sim$MC$_\mathrm{d}$(0,$\boldsymbol{\Sigma}_\mathrm{y}$). *Then,*

$$^{S}H\left(y\right)=log\left(\frac{\Gamma\left(\frac{\mathrm{d}+1}{2}\right)\det\left(4\pi e^{2}\boldsymbol{\Sigma}_{\mathrm{y}}\right)^{\frac{1}{2}}}{\sqrt{\pi}}\right) \tag{2.11}$$

$$^{\mathcal{R}}H_{\alpha}\left(y\right)=log\left(\frac{\alpha^{\ \frac{\mathrm{d}}{\alpha-1}}\ \Gamma\left(\frac{\mathrm{d}+1}{2}\right)\det\left(4\pi\boldsymbol{\Sigma}_{\mathrm{y}}\right)^{\frac{1}{2}}}{\sqrt{\pi}}\right) \tag{2.12}$$

$$^{T}H_{\beta}\left(y\right)=\frac{1}{(1-\beta)}\left(\frac{\beta^{\ \frac{\mathrm{d}}{\beta-1}}\ \Gamma\left(\frac{\mathrm{d}+1}{2}\right)\det\left(4\pi\boldsymbol{\Sigma}_{\mathrm{y}}\right)^{\frac{1}{2}}}{\sqrt{\pi}}\right)^{(1-\beta)}-\frac{1}{(1-\beta)} \tag{2.13}$$

Now, consider N$_\tau$ *is the scale of isolated time,* N$_\tau$ = {τ,τ+1,τ+2, ...}, (τ is fixed real number). $\mathbb{Z}$ *is an integer numbers set.*

Definition 2.1 *[24] let* ω:N$_\tau\to$R *be a function. Then the fractional operator* Δ_τ^γ *is defined as follows*

$$\Delta_{\tau}^{\gamma}\omega\left(\mathrm{s}\right)=\frac{1}{\Gamma\left(\gamma\right)}\sum_{\mathrm{t}=\tau}^{\mathrm{s}-\gamma}\left(\mathrm{s}-\delta\left(\mathrm{t}\right)\right)^{\left(\gamma-1\right)}\omega\left(\mathrm{t}\right)\qquad,\quad \mathrm{s}\in\mathrm{N}_{\tau+\delta} \tag{2.14}$$

Where, $\delta\left(\mathrm{t}\right)=\mathrm{t}+1$ *and* $\mathrm{s}^{(\gamma)}=\frac{\Gamma(\mathrm{s}+1)}{\Gamma(\mathrm{s}+1-\gamma)}$.
The Caputo fractional operator is [24]

$$^{C}\Delta_{\tau}^{\gamma}\omega\left(\mathrm{s}\right)=\frac{1}{\Gamma\left(\mathrm{m}-\gamma\right)}\sum_{\mathrm{t}=\tau}^{\mathrm{s}-(\mathrm{m}-\gamma)}\left(\mathrm{s}-\delta\left(\mathrm{t}\right)\right)^{\left(\mathrm{m}-\gamma-1\right)}\Delta^{m}\omega\left(\mathrm{t}\right),\mathrm{s}\in\mathrm{N}_{\tau+\mathrm{m}+\gamma} \tag{2.15}$$

Here, $m=\lceil\gamma\rceil+1$*, ⌈ ⌉ is a ceiling function. As special case, by taking m=1, we have*

$$^{C}\Delta_{\tau}^{\gamma}\omega\left(\mathrm{s}\right)=\frac{1}{\Gamma\left(1-\gamma\right)}\sum_{\mathrm{t}=\tau}^{\mathrm{s}-(1-\gamma)}\left(\mathrm{s}-\delta\left(\mathrm{t}\right)\right)^{\left(-\gamma\right)}\Delta\omega\left(\mathrm{t}\right)\qquad,\quad \mathrm{s}\in\mathrm{N}_{\tau+1+\gamma} \tag{2.16}$$

Theorem 2.1 *[25] The following equations are equivalent*

$$\left.\begin{array}{r}^{C}\Delta_{0}^{\gamma}\omega\left(\mathrm{t}\right)=\mathcal{F}\left(\mathrm{t}+\gamma-1,\omega\left(\mathrm{t}+\gamma-1\right)\right)\\ \Delta^{(k)}\omega\left(0\right)=\omega_{0}\qquad,\mathrm{k}\ =\ 0,\ 1,\ 2,...,\ \mathrm{m}\ \ --1\end{array}\right\} \tag{2.17}$$

And

$$\omega\left(\mathrm{t}\right)=\omega_{0}\left(t\right)+\frac{1}{\Gamma\left(\gamma\right)}\sum_{\mathrm{s}=\mathrm{m}-\gamma}^{\mathrm{t}-\gamma}\left(\mathrm{t}-\delta\left(\mathrm{s}\right)\right)^{\gamma-1}\mathcal{F}\left(\mathrm{s}+\gamma-1,\omega\left(\mathrm{s}+\gamma-1\right)\right),\quad \mathrm{t}\in\mathrm{N}_{\mathrm{m}} \tag{2.18}$$

where, $m=\lceil\gamma\rceil+1$*, and*

$$\omega_{0}\left(t\right)=\sum_{\mathrm{k}=0}^{\mathrm{m}-1}\frac{\left(t\right)^{(k)}}{\Gamma\left(\mathrm{k}+1\right)}\Delta^{(k)}\ \omega\left(0\right)$$

By using the formula $\left(\mathrm{t}-\delta\left(\mathrm{s}\right)\right)^{\gamma-1}=\frac{\Gamma(\mathrm{t}-\mathrm{s})}{\Gamma(\mathrm{t}-\mathrm{s}+1-\gamma)}$ *and* $j=\mathrm{s}+\gamma$*, we get*

$$\omega\left(t\right)=\omega_{0}\left(t\right)+\frac{1}{\Gamma\left(\gamma\right)}\sum_{j=1}^{t}\frac{\Gamma\left(t-j+\gamma\right)}{\Gamma\left(t-j+1\right)}\mathcal{F}\left(j-1,\omega\left(j-1\right)\right) \tag{2.19}$$

3. Multivariate Controlled Fractional Autoregressive Moving Average Systems

Bao et al. [26] presented multivariate controlled ARMA system (MCARMA) as extension of ARMA system and proposed an algorithm for estimating its parameters. We propose a further extension of the system by incorporating a fractional order, which we believe will play a significant role in elucidating the random behavior of time series . the resulting system is given as follows: . This development will enhance the system's ability to model complex dynamics and improve its performance. The proposed system would take the following form:

$$\left.\begin{array}{c} {}^{C}\Delta_0^{\gamma} \mathrm{y}(\mathrm{t}) = \sum_{\mathrm{i}=1}^{\mathrm{p}} \mathrm{C_i}\ \mathrm{y}(\mathrm{t}-\mathrm{i}+\gamma) + \sum_{\mathrm{j}=0}^{\mathrm{r}} \mathrm{D_j}\ u(\mathrm{t}-\mathrm{j}+\gamma) + \sum_{\mathrm{k}=0}^{\mathrm{q}} \mathrm{E_k}\varepsilon(\mathrm{t}-\mathrm{k}+\gamma) \\ \mathrm{y}(0)=0 \end{array}\right\} \tag{3.1}$$

where, $\mathrm{y}(\mathrm{t})$ is the system output state, $u(\mathrm{t})$ is the system input, $\varepsilon(\mathrm{t})$ is the random vector.$\mathrm{C_i} \in \mathrm{R}^{\mathrm{d}\times\mathrm{d}}$, $\mathrm{D_j}\in\mathrm{R}^{\mathrm{d}\times\mathrm{d}}$, $\mathrm{E_k}\in\mathrm{R}^{\mathrm{d}\times\mathrm{d}}$, $\mathrm{i}=1,2,\ldots,\mathrm{p}$, $\mathrm{j}=1,2,\ldots,\mathrm{r}$, $\mathrm{k}=1,,\ldots,\mathrm{q}$, $\mathrm{C_0}=\mathrm{I_d}, \mathrm{D_0}=\mathrm{I_d}$, $\mathrm{E_0}=\mathrm{I_d}$. Assume that $\boldsymbol{\Sigma}_u$ and $\boldsymbol{\Sigma}_\varepsilon$ are covariance matrices of u and ε respectively. From 2.1 and 2.1 The proposed system 3.1 can be written as

$$\mathrm{y}(\mathrm{t}) = \frac{1}{\Gamma(\gamma)} \sum_{m=1}^{\mathrm{t}} \frac{\Gamma(\mathrm{t}-m+\gamma)}{\Gamma(\mathrm{t}-m+1)} \left(\sum_{\mathrm{i}=1}^{\mathrm{p}} \mathrm{C_i}\ \mathrm{y}(m-1-\mathrm{i}) + \sum_{\mathrm{j}=0}^{\mathrm{r}} \mathrm{D_j}\ u(m-1-\mathrm{j}) + \sum_{\mathrm{k}=0}^{\mathrm{q}} \mathrm{E_k}\varepsilon(m-1-\mathrm{k}) \right) \tag{3.2}$$

Rewriting the equation 3.2 by using the lag operator $\mathcal{L}^{\mathrm{j}}\mathrm{x}(\mathrm{t}) = \mathrm{x}(\mathrm{t}-\mathrm{j})$, we give

$$\mathrm{y}(\mathrm{t}) = \frac{1}{\Gamma(\gamma)} \sum_{m=1}^{\mathrm{t}} \frac{\Gamma(\mathrm{t}-m+\gamma)}{\Gamma(\mathrm{t}-m+1)} \left(\sum_{\mathrm{i}=1}^{\mathrm{p}} \mathrm{C_i}\ \mathrm{y}(m-1-\mathrm{i}) + \sum_{\mathrm{j}=0}^{\mathrm{r}} \mathrm{D_j}\ u(m-1-\mathrm{j}) + \sum_{\mathrm{k}=0}^{\mathrm{q}} \mathrm{E_k}\varepsilon(m-1-\mathrm{k}) \right) \tag{3.3}$$

Consequently, equation 3.3 becomes

$$\mathrm{y}(\mathrm{t}) = \frac{1}{\Gamma(\gamma)} \sum_{m=1}^{\mathrm{t}} \frac{\Gamma(\mathrm{t}-m+\gamma)}{\Gamma(\mathrm{t}-m+1)} \left(\mathrm{C}(\mathcal{L})\,\mathrm{y}(m-1) + \mathrm{D}(\mathcal{L})\,u(m-1) + \mathrm{E}(\mathcal{L})\,\varepsilon(m-1)\right) \tag{3.4}$$

where,

$$\mathrm{C}(\mathcal{L}) = \mathrm{C_1}\mathcal{L} + \mathrm{C_2}\mathcal{L}^2 + \cdots + \mathrm{C_p}\mathcal{L}^{\mathrm{p}}$$
$$\mathrm{D}(\mathcal{L}) = \mathrm{I_d} + \mathrm{D_1}\mathcal{L} + \mathrm{D_2}\mathcal{L}^2 + \cdots + \mathrm{D_r}\mathcal{L}^{\mathrm{r}}$$
$$\mathrm{E}(\mathcal{L}) = \mathrm{I_d} + \mathrm{E_1}\mathcal{L} + \mathrm{E_2}\mathcal{L}^2 + \cdots + \mathrm{E_q}\mathcal{L}^{\mathrm{q}}$$

Assume that $\theta_m = \frac{\Gamma(\mathrm{t}-m+\gamma)}{\Gamma(\mathrm{t}-m+1)}$, then

$$\mathrm{y}(\mathrm{t}) = \frac{1}{\Gamma(\gamma)} \sum_{m=1}^{\mathrm{t}} \theta_m \left(\mathrm{C}(\mathcal{L})\,\mathrm{y}(m-1) + \mathrm{D}(\mathcal{L})\,u(m-1) + \mathrm{E}(\mathcal{L})\,\varepsilon(m-1)\right) \tag{3.5}$$

Rewriting equation 3.5in the following form

$$\mathrm{y}(\mathrm{t}) = \frac{1}{\Gamma(\gamma)} \left(\mathcal{N}_t \mathrm{C}(\mathcal{L})\,\mathcal{L}^t \mathrm{y}(\mathrm{t}) + \mathcal{N}_{t-1}(t)\,\mathrm{C}(\mathcal{L})\,\mathcal{L}^{t-1}\mathrm{y}(\mathrm{t}) +,,, \ + \mathcal{N}_1 \mathrm{C}(\mathcal{L})\,\mathcal{L}\mathrm{y}(\mathrm{t})\right) \tag{3.6}$$

where, $\mathcal{N}_1 = \theta_t(t)$, $\mathcal{N}_2(t) = \theta_{t-1}(t), \ldots, \mathcal{N}_t(t) = \theta_1(t)$
Therefore,

$$\mathrm{y}(\mathrm{t}) = \sum_{m=2}^{\mathrm{t+p}} \Omega_m^C \mathcal{L}^m \mathrm{y}(\mathrm{t}) + \sum_{m=1}^{\mathrm{t+r}} \Omega_m^D \mathcal{L}^m u(\mathrm{t}) + \sum_{m=1}^{\mathrm{t+q}} \Omega_m^E \mathcal{L}^m \varepsilon(\mathrm{t}) \tag{3.7}$$

where,

$$\Omega_m^C = \frac{1}{\Gamma(\gamma)} \sum_{\substack{i=1 \\ i\le p, \\ m-i\le t}}^{m-1} \mathrm{C_i}\mathcal{N}_{m-i}$$

$$\Omega_m^D = \frac{1}{\Gamma(\gamma)} \sum_{\substack{i=1 \\ i \le r, \\ m-i \le t}}^{m} \mathrm{D_i} \mathcal{N}_{m-i}$$

$$\Omega_m^E = \frac{1}{\Gamma(\gamma)} \sum_{\substack{i=1 \\ i \le q, \\ m-i \le t}}^{m} \mathrm{E_i} \mathcal{N}_{m-i}$$

assume that $\Omega_1^C = 0$ and $\Omega_0^D = \Omega_0^E = 1$, then

$$\mathrm{y}(\mathrm{t}) = \sum_{m=1}^{\mathrm{t+p}} \Omega_m^C \mathcal{L}^m \mathrm{y}(\mathrm{t}) + \sum_{m=0}^{\mathrm{t+r}} \Omega_m^D \mathcal{L}^m u(\mathrm{t}) + \sum_{m=0}^{\mathrm{t+q}} \Omega_m^E \mathcal{L}^m \varepsilon(\mathrm{t}) \tag{3.8}$$

This gives

$$\mathrm{y}(\mathrm{t}) = \Omega^C(\mathcal{L})\,\mathrm{y}(\mathrm{t}) + \Omega^D(\mathcal{L})\,u(\mathrm{t}) + \Omega^E(\mathcal{L})\,\varepsilon(\mathrm{t}) \tag{3.9}$$

where,

$$\Omega^C(\mathcal{L}) = \sum_{m=1}^{\mathrm{t+p}} \Omega_m^C \mathcal{L}^m$$

$$\Omega^D(\mathcal{L}) = \sum_{m=0}^{\mathrm{t+r}} \Omega_m^D \mathcal{L}^m$$

$$\Omega^E(\mathcal{L}) = \sum_{m=0}^{\mathrm{t+q}} \Omega_m^E$$

We assume that the state $\mathrm{y}(\mathrm{t})$ is stable to ensure that the operator $\left(\mathrm{I}-\Omega^C(\mathcal{L})\right)$ is invertible. Hence,

$$\mathrm{y}(\mathrm{t}) = \left(\mathrm{I}-\Omega^C(\mathcal{L})\right)^{-1}\Omega^D(\mathcal{L})\,u(\mathrm{t}) + \left(\mathrm{I}-\Omega^C(\mathcal{L})\right)^{-1}\Omega^E(\mathcal{L})\,\varepsilon(\mathrm{t}) \tag{3.10}$$

for simplicity, let us use the notations

$$\widetilde{\mathcal{M}} = \left(\mathrm{I}-\Omega^C(\mathcal{L})\right)^{-1}\Omega^D(\mathcal{L}),\ \mathcal{M}^*(\mathcal{L}) = \left(\mathrm{I}-\Omega^C(\mathcal{L})\right)^{-1}\Omega^E(\mathcal{L})$$

such that $\widetilde{\mathcal{M}}(\mathcal{L}) = \sum_{\mathrm{j}=0}^{\infty} \widetilde{\mathcal{M}}_j\ \mathcal{L}^{\mathrm{j}}$, $\mathcal{M}^*(\mathcal{L}) = \sum_{\mathrm{k}=0}^{\infty} \mathcal{M}_k^*\ \mathcal{L}^{\mathrm{k}}$ then the equation 3.10 appears as

$$\mathrm{y}(\mathrm{t}) = \sum_{\mathrm{j}=0}^{\infty} \widetilde{\mathcal{M}}_j\ \mathcal{L}^{\mathrm{j}} u(\mathrm{t}) + \sum_{\mathrm{k}=0}^{\infty} \mathcal{M}_k^*\ \mathcal{L}^{\mathrm{k}} \varepsilon(\mathrm{t}) \tag{3.11}$$

By Multiplying both sides of the equation 3.11 by the operator $\left(\mathrm{I}-\Omega^C(\mathcal{L})\right)$, we get

$$\left(\mathrm{I}-\Omega^C(\mathcal{L})\right)\mathrm{y}(\mathrm{t}) = \left(\mathrm{I}-\Omega^C(\mathcal{L})\right)\sum_{\mathrm{j}=0}^{\infty} \widetilde{\mathcal{M}}_j\ \mathcal{L}^{\mathrm{j}} u(\mathrm{t}) + \left(\mathrm{I}-\Omega^C(\mathcal{L})\right)\sum_{\mathrm{k}=0}^{\infty} \mathcal{M}_k^*\ \mathcal{L}^{\mathrm{k}} \varepsilon(\mathrm{t}) \tag{3.12}$$

Consequently, if $\mathcal{M}_0^* = \mathrm{I_d}$ and $\widetilde{\mathcal{M}}_0 = \mathrm{I_d}$ then the last equation can be written as

$$\left(\mathrm{I}-\Omega^C(\mathcal{L})\right)\mathrm{y}(\mathrm{t})$$

$$= \left(\mathrm{I_d} + \sum_{\mathrm{j=1}}^{\infty} \left(\widetilde{\mathcal{M}}_j - \sum_{\mathrm{s=1}}^{\mathrm{j}} \Omega_s^C \ \widetilde{\mathcal{M}}_{\mathrm{j-s}} \right) \mathcal{L}^{\mathrm{j}} \right) u\,(\mathrm{t})$$

$$+ \left(\mathrm{I_d} + \sum_{\mathrm{k=1}}^{\infty} \left(\mathcal{M}_{\mathrm{k}}^{*} - \sum_{\mathrm{s=1}}^{\mathrm{k}} \Omega_s^C \ \mathcal{M}_{\mathrm{k-s}}^{*} \right) \mathcal{L}^{\mathrm{k}} \right) \ \varepsilon\,(\mathrm{t}) \tag{3.13}$$

Again, if $\Omega_s^C = 0$ for s> p + t , $\Omega_s^D = 0$ for s > r, $\Omega_s^E = 0$ for s > q , then equation 3.7 can be rewritten in the following form

$$\left(\mathrm{I} - \Omega^C\,(\mathcal{L}) \right) \mathrm{y}\,(\mathrm{t}) = \sum_{m=0}^{\infty} \Omega_m^D \mathcal{L}^m u\,(\mathrm{t}) + \sum_{m=0}^{\infty} \Omega_m^E \mathcal{L}^m \varepsilon\,(\mathrm{t}) \tag{3.14}$$

By comparing the coefficients in equations 3.13 and 3.14 , we obtain

$$\widetilde{\mathcal{M}}_i = \Omega_i^D + \sum_{\mathrm{s=1}}^{\mathrm{i}} \Omega_s^C \ \widetilde{\mathcal{M}}_{\mathrm{i-s}} \tag{3.15}$$

$$\mathcal{M}_{\mathrm{i}}^{*} = \Omega_i^E + \sum_{\mathrm{s=1}}^{\mathrm{i}} \Omega_s^E \ \mathcal{M}_{\mathrm{i-s}}^{*} \tag{3.16}$$

4. Characteristic Function

Based on equivalent 3.11, the characteristic function from its residual functions of proposed system 3.1 is given as follows

$$\varphi_{y(\mathrm{t})}\,(\mathrm{r}) = \mathrm{E}\left(\mathrm{e}^{\mathrm{i r}' \mathrm{y(t)}} \right)$$

$$= \mathrm{E}\left(\mathrm{e}^{\mathrm{i} r' \left(\sum_{\mathrm{j=0}}^{\infty} \widetilde{\mathcal{M}}_j \ u(\mathrm{t-j}) + \sum_{\mathrm{k=0}}^{\infty} \mathcal{M}_k^{*} \ \varepsilon(\mathrm{t-k}) \right)} \right) \tag{4.1}$$

Hence,

$$\varphi_{\mathrm{y(t)}}\,(\mathrm{r}) = \varphi_{\widetilde{\mathcal{M}}_k u(\mathrm{t-k})}\,(\mathrm{r}) \ \ \varphi_{\mathcal{M}_k^{*} \varepsilon(\mathrm{t-k})}\,(\mathrm{r}) \tag{4.2}$$

where, $\varphi_{\widetilde{\mathcal{M}}_k u(\mathrm{t-k})}$, $\varphi_{\mathcal{M}_k^{*} \varepsilon(\mathrm{t-k})}$ represent the characteristic functions of$\widetilde{\mathcal{M}}_k u\,(\mathrm{t-k})$ and $\mathcal{M}_k^{*} \varepsilon\,(\mathrm{t-k})$ respectively.

5. Entropy Analysis

In this section, we derive the entropy of the proposed system 3.1, focusing exclusively on the Gaussian and Cauchy processes.

Multivariate Gaussian Distribution

Under the assumptions of independence, the covariance matrices $\boldsymbol{\Sigma}_u$ and $\boldsymbol{\Sigma}_\varepsilon$ of the Gaussian distributed processes $u\,(\mathrm{t})$ and $\varepsilon\,(\mathrm{t})$ and the form 4.2 yield the characteristic function of the proposed system 3.1 in the following form

$$\varphi_{\mathrm{y(t)}}\,(\mathrm{r}) = \prod_{\mathrm{k=0}}^{\infty} e^{-\frac{\mathrm{i}}{2} \left(\mathrm{r}' \widetilde{\mathcal{M}}_k \mathrm{S}_u \widetilde{\mathcal{M}}_k' r \right)} \ . \ e^{-\frac{\mathrm{i}}{2} \left(\mathrm{r}' \mathcal{M}_{\mathrm{k}}^{*} \mathrm{S}_\varepsilon \mathcal{M}_{\mathrm{k}}^{*\prime} r \right)} \tag{5.1}$$

The structure of the characteristic function 4.2, together with the independence assumptions, ensures that the output state of system 3.1 is also Gaussian distributed. Therefore,

$$\boldsymbol{\Sigma}_{\mathrm{y}} = \sum_{\mathrm{k=0}}^{\infty} \left(\widetilde{\mathcal{M}}_k \boldsymbol{\Sigma}_u \ \widetilde{\mathcal{M}}_k' + \mathcal{M}_{\mathrm{k}}^{*} \boldsymbol{\Sigma}_\varepsilon \mathcal{M}_{\mathrm{k}}^{*\prime} \right) \tag{5.2}$$

Therefore, from 2.3 and equation5.2 , the Shannon, Rényi **and Tsallis** entropies of process $\mathrm{y}\,(\mathrm{t})$ are obtained as follows

$${}^{S}H\,(y) = \frac{1}{2} log \left(\det \left(2\pi \sum_{\mathrm{k=0}}^{\infty} \left(\widetilde{\mathcal{M}}_k \boldsymbol{\Sigma}_u \ \widetilde{\mathcal{M}}_k' + \mathcal{M}_{\mathrm{k}}^{*} \boldsymbol{\Sigma}_\varepsilon \mathcal{M}_{\mathrm{k}}^{*\prime} \right) \right) \right) \tag{5.3}$$

$$^{\mathcal{R}}H_{\alpha}(y)=\frac{1}{2}log\left(\det\left(2\pi\sum_{\mathrm{k}=0}^{\infty}\left(\widetilde{\mathcal{M}}_{k}\boldsymbol{\Sigma}_{u}\ \widetilde{\mathcal{M}}_{k}^{\prime}+\mathcal{M}_{\mathrm{k}}^{*}\boldsymbol{\Sigma}_{\varepsilon}\mathcal{M}_{\mathrm{k}}^{*\prime}\right)\right)\right)+\frac{\mathrm{d}}{2\left(\alpha-1\right)}log\left(\alpha\right)\tag{5.4}$$

$$^{T}H_{\beta}(y)=\frac{\beta^{\frac{d}{2}}}{(1-\beta)}\left(\det\left(2\pi\sum_{\mathrm{k}=0}^{\infty}\left(\widetilde{\mathcal{M}}_{k}\boldsymbol{\Sigma}_{u}\ \widetilde{\mathcal{M}}_{k}^{\prime}+\mathcal{M}_{\mathrm{k}}^{*}\boldsymbol{\Sigma}_{\varepsilon}\mathcal{M}_{\mathrm{k}}^{*\prime}\right)\right)\right)^{\frac{1-\beta}{2}}-\frac{1}{(1-\beta)}\tag{5.5}$$

Multivariate Cauchy Distribution
The characteristic function of $\mathrm{y}\sim\mathrm{MC_d}(0,\boldsymbol{\Sigma}_{\mathrm{y}})$ is given as follows [27]

$$\varphi_{\mathrm{y(t)}}(\mathrm{r})=\mathrm{e}^{-\mathrm{i}\ \sqrt{\mathrm{r}^{\prime}\boldsymbol{\Sigma}_{y}\mathrm{r}}}\tag{5.6}$$

Hence,

$$\varphi_{\mathrm{y(t)}}(\mathrm{r})=\mathrm{e}^{-\mathrm{i}\sum_{\mathrm{k}=0}^{\infty}\left(\sqrt{\mathrm{r}^{\prime}\widetilde{\mathcal{M}}_{k}\boldsymbol{\Sigma}_{u}\widetilde{\mathcal{M}}_{k}^{\prime}r}+\sqrt{\mathrm{r}^{\prime}\mathcal{M}_{\mathrm{k}}^{*}\boldsymbol{\Sigma}_{\varepsilon}\mathcal{M}_{\mathrm{k}}^{*\prime}r}\right)}\tag{5.7}$$

Under the assumption ($\boldsymbol{\Sigma}_{u}$ and $\boldsymbol{\Sigma}_{\varepsilon}$ are positive definite), then there exists matrices L_{u} and L_{ε} in $\mathrm{R}^{\mathrm{d\times d}}$ satisfy the following form
$\boldsymbol{\Sigma}_{u}=\mathrm{L}_{u}\mathrm{L}_{u}^{\prime}$ and $\boldsymbol{\Sigma}_{\varepsilon}=\mathrm{L}_{\varepsilon}\mathrm{L}_{\varepsilon}^{\prime}$
Consequently, the characteristic function is obtained as

$$\varphi_{\mathrm{y(t)}}(\mathrm{r})=\mathrm{e}^{-\mathrm{i}\sum_{\mathrm{k}=0}^{\infty}\left(\sqrt{\mathrm{r}^{\prime}\widetilde{\mathcal{K}}_{k}\widetilde{\mathcal{K}}_{k}^{\prime}r}+\sqrt{\mathrm{r}^{\prime}\mathcal{K}_{\mathrm{k}}^{*}\mathcal{K}_{\mathrm{k}}^{*\prime}r}\right)}\tag{5.8}$$

where, $\widetilde{\mathcal{K}}_{k}=\widetilde{\mathcal{M}}_{k}\mathrm{L}_{u}$, $\mathcal{K}_{\mathrm{k}}^{*}=\mathcal{M}_{\mathrm{k}}^{*}\ \mathrm{L}_{\varepsilon}$, $\mathrm{j}=0,1,2,\ldots$
suppose the matrices $\mathcal{K}_{\mathrm{j}}\mathcal{K}_{\mathrm{j}}^{\prime}$, $\mathcal{K}_{\mathrm{j}}^{*}\mathcal{K}_{\mathrm{j}}^{*\prime}$, $\mathrm{j}=0,1,2,\ldots$ are proportional one to another then there exists a matrix $\mathcal{DD}$ such that [29]

$$\sqrt{\mathrm{r}^{\prime}\mathcal{D}\mathrm{r}}=\sum_{\mathrm{k}=0}^{\infty}\left(\sqrt{\mathrm{r}^{\prime}\widetilde{\mathcal{K}}_{k}\widetilde{\mathcal{K}}_{k}^{\prime}r}+\sqrt{\mathrm{r}^{\prime}\mathcal{K}_{\mathrm{k}}^{*}\mathcal{K}_{\mathrm{k}}^{*\prime}r}\right)\tag{5.9}$$

Hence,

$$\varphi_{\mathrm{y(t)}}(\mathrm{r})=\exp\left(-\sqrt{\mathrm{r}^{\prime}\mathcal{D}\mathrm{r}}\right)\tag{5.10}$$

Therefore, from 2.4 and equation 5.2, the Shannon, Rényi **and Tsallis** entropies of process $\mathrm{y}(\mathrm{t})$ are given as follows

$$^{S}H(y)=log\left(\frac{\Gamma\left(\frac{\mathrm{d}+1}{2}\right)\det\left(4\pi e^{2}\mathcal{D}\right)^{\frac{1}{2}}}{\sqrt{\pi}}\right)\tag{5.11}$$

$$^{\mathcal{R}}H_{\alpha}(y)=log\left(\frac{\alpha^{\frac{\mathrm{d}}{\alpha-1}}\ \Gamma\left(\frac{\mathrm{d}+1}{2}\right)\det\left(4\pi\mathcal{D}\right)^{\frac{1}{2}}}{\sqrt{\pi}}\right)\tag{5.12}$$

$$^{T}H_{\beta}(y)=\frac{1}{(1-\beta)}\left(\frac{\beta^{\frac{\mathrm{d}}{\beta-1}}\ \Gamma\left(\frac{\mathrm{d}+1}{2}\right)\det\left(4\pi\mathcal{D}\right)^{\frac{1}{2}}}{\sqrt{\pi}}\right)^{(1-\beta)}-\frac{1}{(1-\beta)}\tag{5.13}$$

Entropy Computation Algorithm For FCARMA System.

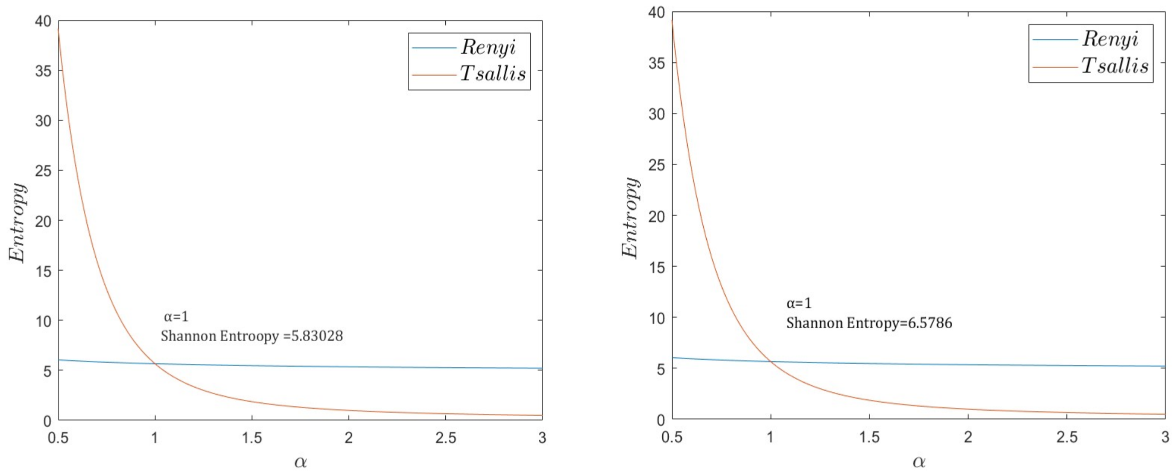

Figure 1: Entropies of MFCARMA state by Gaussian process moving average are presented with fractional orders (a) $\gamma = 0.15$. (b):$\gamma = 0.99$

Algorithm 1: Shannon, Renyi and Tsallis entropies for MFCARMA state.

Input: $\gamma, \alpha, \beta > 0$, $\boldsymbol{d} \times \boldsymbol{d}$**−matrices** $\mathrm{D}_0 = I$, $E_0 = I$ $\mathrm{C_i}$,$\mathrm{D_j}$, $\mathrm{E_k}$, $\boldsymbol{\Sigma}_u$, $\boldsymbol{\Sigma}_\varepsilon$, , i =1, 2, ..., p , j = 1, 2, ..., r, k = 1, 2, ..., q.

Set $\mathcal{M}_0^* = \widetilde{\mathcal{M}}_0 = \mathrm{I_d}$, $\Omega_1^C = 0$ and $\Omega_0^D = \Omega_0^E = 1$
Process: Calculate $\theta_m = \frac{\Gamma(\mathrm{t}-\ m+\gamma)}{\Gamma(\mathrm{t}-\ m+1)}$
Calculate $\Omega_m^C = 0$ and $\Omega_m^D = \Omega_m^E$ using equation 3.7
Compute $\mathcal{M}_i^* = \widetilde{\mathcal{M}}_i$, $i = 1, 2, \ldots,$ using equations 3.15-3.16
If $u\,(\mathrm{t}) \sim \mathrm{MG_d}(0,\boldsymbol{\Sigma}_u)$ *and* $\varepsilon\,(\mathrm{t}) \sim \mathrm{MG_d}(0,\boldsymbol{\Sigma}_\varepsilon)$, or
$u\,(\mathrm{t}) \sim \mathrm{M}C(0,\boldsymbol{\Sigma}_u)$ *and* $\varepsilon\,(\mathrm{t}) \sim \mathrm{MC_d}(0,\boldsymbol{\Sigma}_\varepsilon)$
Output: Calculate $\boldsymbol{\Sigma}_\mathrm{y}$ and entropies using equations 3.11 and 2.8-2.13
.

6. Illustrative Examples

This section includes two examples that Illustrate and support the theoretical aspects of the present work. Thuse, we consider $p = 1$,$r = 1$, $q = 1$ and the parameters $\mathbf{C}_1$, $\mathbf{D}_1$ and $\mathbf{E}_1$ of system 3.1 in the following form [22],

$$\mathbf{C}_1 = \begin{bmatrix} 0.6 & 0.5 \\ -0.8 & 1 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} 3 & -0.8 \\ -1 & 2.2 \end{bmatrix},$$

$$\mathbf{E}_1 = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 1.2 \end{bmatrix}, \quad \gamma = 0.9.$$

Example 1.
In this example, we assume that the stochastic control function $u(t)$ and the random vector $\varepsilon(t)$ follow a multivariate Gaussian distribution with its covariance matrices $\Sigma_\mathrm{u} = \mathrm{I}_d$ and $\Sigma_\varepsilon = \mathrm{I}_d$, respectively. Based on the values of the matrix C_1, it is guaranteed that the operator $\left(\mathrm{I}_d - \Omega^C\left(\mathcal{L}\right)\right)$ is invertible. Accordantly, the process $\mathrm{y}\,(\mathrm{t})$ is stable.
Example 2.
In this example, we assume that the stochastic control function $u(t)$ and the random vector $\varepsilon(t)$ follow a multivariate Cauchy distribution with its covariance matrices $\Sigma_\mathrm{u} = \mathrm{I}_d$ and $\Sigma_\varepsilon = \mathrm{I}_d$, respectively.

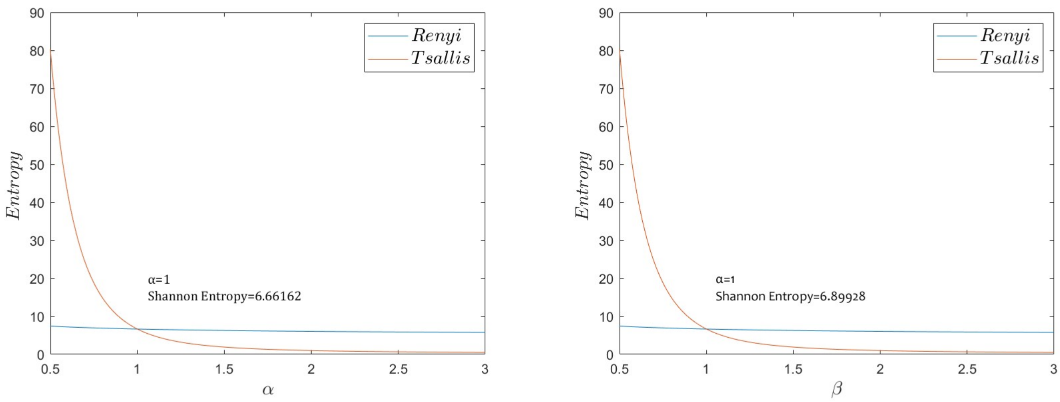

Figure 2: Entropies of MFCARMA state by Cauchy process moving average are presented with fractional orders (a) $\gamma = 0.15$. (b):$\gamma = 0.99$

Figure 1 and Figure 2, illustrate the behavior of the proposed system through entropy measures . It is observed that Tsallis entropy exhibits greater flexibility than Rényi and Shannon entropies in capturing the dynamics of time series behavior, which corresponds with the complicated of the system. Additionally, It is worth noting that the Tsallis measure is greater than those based on Shannon and Rényi entropies when $\alpha < 1$, but this behavior reverses when $\alpha > 1$ becoming smaller. This behavior provides greater flexibility for the Tsallis measure in capturing more details about the random dynamics of MFCARMA. system, making contribution of this study significant and worthy of further exploration.

7. Conclusion

In this paper, a fractional order controlled autoregressive moving average system is proposed. This system is characterized by flexibility and effectiveness in predicting time-series behavior through fractional parameters, which reveal more detailed and accurate dynamics than those captured by conventional models. The explicit expressions of Shannon, Rényi and Tsallis entropies are derived using the characteristic function of presented system state. The results in this work, refer that the fractional order is more effectiveness in the system's stochastic, highlighting its crucial role in the behavior. Finally, it can be concluded that the Tsallis entropy is more suitable than the Rényi entropy for this type of model, due to its higher flexibility, which allows it to reveal finer details of the information embedded in the system's state.

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