



## Novel Iterative Approaches: Examining Stability, and Convergence

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**ABSTRACT:** In this paper, we propose a new iterative method for approximating fixed points of contraction mappings. We prove a convergence theorem and evaluate its rate of convergence regarding the KF, AA, Piri, and  $S^*$  iterative algorithms. In addition, we present findings on stability. These findings contribute to the ongoing development of iterative algorithms for nonlinear problems in various mathematical and applied contexts.

**Key Words:** Contraction mappings, fixed-point, iterative process.

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### 1. Introduction

Fixed point theory is a fundamental area of nonlinear analysis, with wide-ranging applications in mathematics, optimization, economics, and engineering. Specifically, it focuses on identifying points that remain unchanged under specific mappings, referred to as fixed points. Iterative methods have long been the preferred approach for approximating fixed points due to their simplicity and effectiveness. These methods play a pivotal role in solving complex problems such as nonlinear equations, variational inequalities, and optimization tasks; see [1,2,3,4,5,6,7,8,9,10]

The development of efficient iterative algorithms, especially for contraction mappings, has garnered significant attention in recent years. The key considerations in designing such algorithms include ensuring strong convergence to a fixed point and analyzing the stability of the iteration process. Convergence guarantees that the sequence generated by the iterative method approaches the true fixed point, while stability ensures that small perturbations in the initial conditions or parameters do not lead to large deviations in the solution. Together, convergence and stability are crucial properties that determine the practical applicability of an iterative method. Numerous iterative procedures have been extensively studied in the literature by various authors, such as Picard [14], Mann [12], Ishikawa [13], Noor [15], SP [16], Agarwal [17], Piri [18], and Ullah et al. [19], among others.

In 2013, Karahan and Ozdemir [20] introduced a three-step iterative method, namely  $S^*$  iteration

$$\begin{cases} y_1 = y \in \Upsilon, \\ y_{n+1} = (1 - \varepsilon_n)\Pi y_n + \varepsilon_n \Pi v_n, \\ v_n = (1 - \sigma_n)\Pi y_n + \sigma_n \Pi u_n, \\ u_n = (1 - \omega_n)y_n + \omega_n \Pi y_n, \end{cases} \quad (1.1)$$

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where  $\{\varepsilon_n\}, \{\sigma_n\}$  and  $\{\omega_n\}$  are sequences in  $[0, 1]$  with  $\sum_{n=0}^{\infty} \varepsilon_n = \infty$ .

In 2013, Karahan and Ozdemir [20] showed that their iteration converges faster than Picard, Maan, and Agarwal-O'Regan-Sahu iterations for nonexpansive mapping in Banach space.

In 2019, Piri et al. [18] introduced the following iteration process

$$\begin{cases} y_1 = y \in \Upsilon, \\ y_{n+1} = (1 - \varepsilon_n)\Pi u_n + \varepsilon_n \Pi v_n, \\ v_n = \Pi u_n, \\ u_n = \Pi((1 - \sigma_n)y_n + \sigma_n \Pi y_n), \end{cases} \quad (1.2)$$

where  $\{\varepsilon_n\}$  and  $\{\sigma_n\}$  are sequences in  $(0, 1)$  with  $1 - \varepsilon_n < \varepsilon_n$  and  $1 - \sigma_n < \sigma_n$ .

Piri et al. [21] showed that their iteration converges quicker than Picard [14], SP [16], Agarwal [17], Noor [15], Ishikawa [13], Mann [12], Thakur [22] and Abbas and Nazir [23] for contractive mappings.

In 2022, Ullah et al. [19] introduced a three-step iteration namely KF iteration

$$\begin{cases} y_1 = y \in \Upsilon, \\ y_{n+1} = \Pi((1 - \varepsilon_n)\Pi y_n + \varepsilon_n \Pi v_n), \\ v_n = \Pi u_n, \\ u_n = \Pi((1 - \sigma_n)y_n + \sigma_n \Pi y_n) \end{cases} \quad (1.3)$$

where  $\{\varepsilon_n\}, \{\sigma_n\}$  are sequences in  $[0, 1]$  with  $\sum_{n=0}^{\infty} \varepsilon_n = \infty$ .

In 2022, Beg et al. [24] introduced the following four-step iterative scheme named as AA-iteration scheme.

$$\begin{cases} y_1 = y \in \Upsilon, \\ y_{n+1} = \Pi v_n, \\ v_n = \Pi[(1 - \varepsilon_n)u_n + \varepsilon_n \Pi u_n], \\ u_n = \Pi[(1 - \omega_n)w_n + \omega_n \Pi w_n], \\ w_n = \Pi[(1 - \vartheta_n)y_n + \vartheta_n \Pi y_n], \end{cases} \quad (1.4)$$

where  $\{\varepsilon_n\}, \{\omega_n\}$  and  $\{\vartheta_n\}$  are in  $[0, 1]$  and  $\sum_{n=0}^{\infty} \varepsilon_n = \infty$ .

Motivated and inspired by the above-mentioned results, we introduce a new iterative algorithm (New Algo.)

$$\begin{cases} a_1 = a \in \Upsilon, \\ a_{n+1} = \Pi^3[(1 - \varepsilon_n)\Pi c_n + \varepsilon_n \Pi b_n], \\ b_n = \Pi^3[(1 - \sigma_n)\Pi d_n + \sigma_n \Pi c_n], \\ c_n = \Pi^3[d_n], \\ d_n = \Pi[(1 - \vartheta_n)a_n + \vartheta_n \Pi a_n], \end{cases} \quad (1.5)$$

In this paper, we propose a new iterative method for approximating fixed points of contraction mappings in real normed linear spaces. We establish a strong convergence theorem, proving that the proposed method converges to a fixed point under standard conditions. Additionally, we investigate the stability of the algorithm, providing insights into its robustness and reliability in practical scenarios. To evaluate the effectiveness of the proposed method, we compare its performance with several well-known iterative algorithms, including the  $S^*$ , Piri, KF, and AA iterations. Theoretical analysis demonstrates that our method achieves strong convergence and exhibits superior stability.

The results of this study contribute to the ongoing development of iterative techniques for fixed-point problems, offering an efficient and stable alternative to existing methods.

## 2. Preliminaries

In this section, we present some fundamental definitions and results that will be instrumental in establishing our main findings. Let  $\Lambda$  be a real normed linear space, and let  $\Upsilon$  be a nonempty closed convex subset of  $\Lambda$ . Consider  $\Pi$  as a self-contraction mapping from  $\Lambda$  to itself, and let  $F(\Pi) = \{x \in \Lambda : \Pi x = x\}$  denote the set of fixed points of the mapping  $\Pi$ .

**Definition 2.1** Let  $\Pi : \Lambda \rightarrow \Lambda$  be an operator. We say that  $\Pi$  is a contraction operator if it satisfies the condition

$$\|\Pi u - \Pi v\| \leq \zeta \|u - v\| \quad \text{for all } u, v \in \Lambda,$$

where  $\zeta \in [0, 1)$ .

**Definition 2.2** Let  $\{u_n\}_{n=0}^{\infty}$  be a sequence in  $\Lambda$ . An iteration of the form  $x_{n+1} = f(\Pi, x_n)$ , converging to a fixed point  $v$ , is said to be  $\Pi$ -stable, or stable with respect to  $\Pi$ , if for  $\sigma_n = \|x_{n+1} - f(\Pi, u_n)\|$ ,  $n = 0, 1, 2, 3, \dots$ , the following holds:

$$\lim_{n \rightarrow \infty} u_n = v \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} \sigma_n = 0.$$

**Definition 2.3** Let  $\{u_n\}$  and  $\{v_n\}$  be two sequences of positive real numbers that approach  $u$  and  $v$ , respectively.

$$l = \lim_{n \rightarrow \infty} \frac{|u_n - u|}{|v_n - v|}$$

1. If  $l = \infty$ , this implies that  $\{v_n\}$  converges faster than  $\{u_n\}$ .
2. If  $0 < l < \infty$ , it indicates that  $\{u_n\}$  and  $\{v_n\}$  converge at the same rate, meaning both sequences approach their respective limits at a similar pace.
3. If  $l = 0$ , it means that  $\{u_n\}$  converges to  $u$  more quickly than  $\{v_n\}$  converges to  $v$ .

**Lemma 2.1** [25] Let  $\{u_n\}_{n=0}^{\infty}$  and  $\{v_n\}_{n=0}^{\infty}$  be sequences of non-negative real numbers that satisfy the inequality

$$u_{n+1} \leq (1 - w_n)u_n + v_n,$$

where  $w_n \in (0, 1)$  for all  $n \in \mathbb{N}$ ,  $\sum_{n=0}^{\infty} w_n = \infty$ , and  $\frac{v_n}{w_n} \rightarrow 0$  as  $n \rightarrow \infty$ . Under these conditions, we have  $\lim_{n \rightarrow \infty} u_n = 0$ .

## 3. Main Results

In this section, we begin with the convergence and stability results of a new four-step iterative scheme using contraction mapping under some parametrical conditions. Suppose  $(\Lambda, \|\cdot\|)$  is a real normed linear space and  $\Upsilon$  is a nonempty closed convex subset of  $\Lambda$ . Let  $\Pi : \Lambda \rightarrow \Lambda$  be a contraction mapping with a contraction factor  $\zeta \in [0, 1)$ ,  $F(\Pi) \neq \emptyset$ . Furthermore, we compare the rate of convergence of our results to that of Piri, KF, AA,  $S^*$ .

### 3.1. Convergence Result

**Theorem 3.1** Let  $\{a_n\}_{n=0}^{\infty}$  be a sequence defined by (1.5) where  $\{\varepsilon_n\}$ ,  $\{\omega_n\}$  and  $\{\vartheta_n\}$  are in  $[0, 1]$  and  $\sum_{n=0}^{\infty} \varepsilon_n = \infty$ . Then  $\{a_n\}$  converges strongly to a unique fixed point of  $\Pi$ .

**Proof:** Assume that  $F(\Pi) \neq \phi$  and  $v \in F(\Pi)$ , using Algorithm (1.5), we have

$$\begin{aligned}
\|d_n - v\| &= \|\Pi[(1 - \vartheta_n)a_n + \vartheta_n\Pi a_n] - v\| \\
&\leq \zeta\|(1 - \vartheta_n)a_n + \vartheta_n\Pi a_n - v\| \\
&\leq \zeta\|(1 - \vartheta_n)(a_n - v) + \vartheta_n(\Pi a_n - v)\| \\
&\leq \zeta[(1 - \vartheta_n)\|a_n - v\| + \vartheta_n\|\Pi a_n - v\|] \\
&\leq \zeta[(1 - \vartheta_n)\|a_n - v\| + \zeta\vartheta_n\|a_n - v\|] \\
&\leq \zeta[(1 - \vartheta_n) + \zeta\vartheta_n]\|a_n - v\| \\
&\leq \zeta[1 - \vartheta_n + \zeta\vartheta_n]\|a_n - v\| \\
&\leq \zeta\|a_n - v\|.
\end{aligned} \tag{3.1}$$

Using Algorithm (1.5) and equation (3.1), we get

$$\begin{aligned}
\|c_n - v\| &= \|\Pi^3(d_n) - v\| \\
&\leq \zeta^3\|d_n - v\| \\
&\leq \zeta^4\|a_n - v\|.
\end{aligned} \tag{3.2}$$

Using Algorithm (1.5), equations (3.1) and (3.2), we get

$$\begin{aligned}
\|b_n - v\| &= \|\Pi^3[(1 - \sigma_n)\Pi d_n + \sigma_n\Pi c_n] - v\| \\
&\leq \zeta^3\|(1 - \sigma_n)\Pi d_n + \sigma_n\Pi c_n - v\| \\
&\leq \zeta^3\|(1 - \sigma_n)(\Pi d_n - v) + \sigma_n(\Pi c_n - v)\| \\
&\leq \zeta^3[(1 - \sigma_n)\zeta\|d_n - v\| + \sigma_n\zeta\|c_n - v\|] \\
&\leq \zeta^4[(1 - \sigma_n)\zeta\|a_n - v\| + \sigma_n\zeta^3\|a_n - v\|] \\
&\leq \zeta^5[(1 - \sigma_n)\|a_n - v\| + \sigma_n\|a_n - v\|] \\
&\leq \zeta^5\|a_n - v\|.
\end{aligned} \tag{3.3}$$

Using Algorithm (1.5), equations (3.2) and (3.3), we have

$$\begin{aligned}
\|a_{n+1} - v\| &= \|\Pi^3[(1 - \varepsilon_n)\Pi c_n + \varepsilon_n\Pi b_n] - v\| \\
&\leq \zeta^3\|(1 - \varepsilon_n)\Pi c_n + \varepsilon_n\Pi b_n - v\| \\
&\leq \zeta^3\|(1 - \varepsilon_n)(\Pi c_n - v) + \varepsilon_n(\Pi b_n - v)\| \\
&\leq \zeta^3[(1 - \varepsilon_n)\zeta\|c_n - v\| + \varepsilon_n\zeta\|b_n - v\|] \\
&\leq \zeta^4[(1 - \varepsilon_n)\zeta^4\|a_n - v\| + \varepsilon_n\|b_n - v\|] \\
&\leq \zeta^4[(1 - \varepsilon_n)\zeta^4\|a_n - v\| + \varepsilon_n\zeta^5\|a_n - v\|] \\
&\leq \zeta^8[(1 - \varepsilon_n)\|a_n - v\| + \varepsilon_n\zeta\|a_n - v\|] \\
&\leq \zeta^8[1 - (1 - \zeta)\varepsilon_n]\|a_n - v\|.
\end{aligned} \tag{3.4}$$

Inductively, we get

$$\|a_{n+1} - v\| \leq \zeta^{8(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|. \tag{3.5}$$

Since  $0 \leq \zeta < 1$ ,  $1 - \varepsilon_k(1 - \zeta) \in (0, 1]$  and  $\varepsilon_k \in [0, 1]$ , for all  $k \in \mathbb{N}$ , from equation (3.5) we have

$$\|a_{n+1} - v\| \leq \frac{\zeta^{8(n+1)} \|a_0 - v\|}{e^{(1-\zeta)\sum_{k=0}^n \varepsilon_k}}. \tag{3.6}$$

Since  $\sum_{n=0}^{\infty} \varepsilon_n = \infty$ . Hence,  $\{a_n\}$  converges strongly to  $v \in F(\Pi)$ . This completes the proof.  $\square$

### 3.2. Stability Result

**Theorem 3.2** Let  $\{a_n\}_{n=0}^{\infty}$  be a sequence defined by Algorithm (1.5),  $\sum_{n=0}^{\infty} \varepsilon_n = \infty$ . Then, iterative algorithm  $\{a_n\}_{n=0}^{\infty}$  is  $\Pi$ -stable.

**Proof:** Since  $\Pi$  is a contraction mapping with contraction factor  $\zeta \in [0, 1)$ . Let  $v$  be a fixed point of  $\Pi$ . Let us assume that  $\{l_n\}_{n=0}^{\infty}$  and  $\{\sigma_n\}_{n=0}^{\infty}$  are sequences in  $E$  and  $[0, \infty)$  respectively. Define  $\sigma_n = \|l_{n+1} - f(\Pi, l_n)\|$ . We have to show that  $\lim_{n \rightarrow \infty} \sigma_n = 0 \iff \lim_{n \rightarrow \infty} l_n = v$ . Assume that  $\lim_{n \rightarrow \infty} \sigma_n = 0$  and using Algorithm (1.5) we get

$$\|l_{n+1} - v\| \leq \|l_{n+1} - f(\Pi, l_n)\| + \|f(\Pi, l_n) - v\|. \quad (3.7)$$

Subsequently, we have

$$\|l_{n+1} - v\| \leq \sigma_n + \zeta^8 [1 - (1 - \zeta)\varepsilon_n] \|l_n - v\|. \quad (3.8)$$

So, we get

$$u_{n+1} \leq (1 - w_n)u_n + v_n, \quad (3.9)$$

where  $u_n = \|l_n - v\|$ ,  $w_n = (1 - \zeta)\varepsilon_n \in (0, 1]$  and  $v_n = \sigma_n$ . As  $\lim_{n \rightarrow \infty} \sigma_n = 0$ , this implies  $\frac{v_n}{w_n} = 0$  as  $n \rightarrow \infty$ .

By using Lemma (2.1), we have

$$\lim_{n \rightarrow \infty} \|l_{n+1} - v\| = 0. \quad (3.10)$$

$$\implies \lim_{n \rightarrow \infty} l_{n+1} = v. \quad (3.11)$$

Conversely, let  $\lim_{n \rightarrow \infty} l_n = v$ .

Then

$$\begin{aligned} \sigma_n &= \|l_{n+1} - f(\Pi, l_n)\| \\ &\leq \|l_{n+1} - v\| + \|v - f(\Pi, l_n)\| \\ &\leq \|l_{n+1} - v\| + \zeta^8 [1 - (1 - \zeta)\varepsilon_n] \|l_n - v\|. \end{aligned} \quad (3.12)$$

Since  $\lim_{n \rightarrow \infty} l_{n+1} = v$  and  $\zeta \in [0, 1)$ , we get  $\lim_{n \rightarrow \infty} \sigma_n = 0$ .  $\square$

### 3.3. Rate of Convergence

**Theorem 3.3** Let  $\{y_n\}_{n=0}^{\infty}$  and  $\{a_n\}_{n=0}^{\infty}$  be two sequences defined by Algorithms (1.1) and (1.5) respectively, where  $\{\varepsilon_n\}$ ,  $\{\sigma_n\}$ ,  $\{\omega_n\}$  and  $\{\vartheta_n\}$  are in  $[0, 1]$  and  $\sum_{n=0}^{\infty} \varepsilon_n = \infty$ . Then the iteration defined by (1.5) converges faster than (1.1) to a fixed point  $v$  of  $\Pi$ .

**Proof:** From equation (3.5) as in Theorem (3.1), we have

$$\|a_{n+1} - v\| \leq \zeta^{8(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|. \quad (3.13)$$

So, we have

$$\|a_{n+1} - v\| \leq \lambda_n, \quad (3.14)$$

where  $\lambda_n = \zeta^{8(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|$ .  
Using Algorithm (1.1), we get

$$\begin{aligned} \|u_n - v\| &= \|(1 - \omega_n)y_n + \omega_n \Pi y_n - v\| \\ &\leq (1 - \omega_n)\|y_n - v\| + \omega_n \|\Pi y_n - v\| \\ &\leq (1 - \omega_n)\|y_n - v\| + \zeta \omega_n \|y_n - v\| \\ &< (1 - \omega_n)\|y_n - v\| + \omega_n \|y_n - v\| \\ &\leq \zeta \|y_n - v\|. \end{aligned} \tag{3.15}$$

Using Algorithm (1.1) and equation (3.15), we get

$$\begin{aligned} \|v_n - v\| &= \|(1 - \sigma_n)\Pi y_n + \sigma_n \Pi u_n - v\| \\ &\leq (1 - \sigma_n)\|\Pi y_n - v\| + \sigma_n \|\Pi u_n - v\| \\ &\leq \zeta [(1 - \sigma_n)\|y_n - v\| + \sigma_n \|u_n - v\|] \\ &\leq \zeta [(1 - \sigma_n)\|y_n - v\| + \sigma_n \zeta \|y_n - v\|] \\ &\leq \zeta \|y_n - v\|. \end{aligned} \tag{3.16}$$

From Algorithm (1.1) and equation (3.16), we get

$$\begin{aligned} \|y_{n+1} - v\| &= \|(1 - \varepsilon_n)\pi y_n + \varepsilon_n \Pi v_n - v\| \\ &\leq (1 - \varepsilon_n)\|\pi y_n - v\| + \varepsilon_n \|\Pi v_n - v\| \\ &\leq (1 - \varepsilon_n)\zeta \|y_n - v\| + \zeta^2 \varepsilon_n \|y_n - v\| \\ &\leq \zeta [1 - (1 - \zeta)\varepsilon_n] \|y_n - v\|. \end{aligned} \tag{3.17}$$

Inductively, we get:

$$\|y_{n+1} - v\| \leq \zeta^{1(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|.$$

So, we have

$$\|y_{n+1} - v\| \leq \nu_n, \tag{3.18}$$

where  $\nu_n = \zeta^{1(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|$ .  
Using equations (3.14) and (3.18), we get

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\nu_n} = \lim_{n \rightarrow \infty} \zeta^{7(n+1)} \frac{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|}{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|}. \tag{3.19}$$

As  $\zeta \in [0, 1)$  and  $\lim_{n \rightarrow \infty} \frac{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|}{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|}$  are finite, which implies

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\nu_n} = \lim_{n \rightarrow \infty} \zeta^{7(n+1)} = 0. \tag{3.20}$$

This implies that Algorithm 1.5 converges faster than Algorithm 1.1 to  $v$ .  $\square$

**Theorem 3.4** *Let  $\{y_n\}_{n=0}^{\infty}$  and  $\{a_n\}_{n=0}^{\infty}$  be two sequences defined by Algorithms (1.2) and (1.5) respectively, where  $\{\varepsilon_n\}$ ,  $\{\sigma_n\}$ ,  $\{\omega_n\}$  and  $\{\vartheta_n\}$  are in  $[0, 1]$  and  $\sum_{n=0}^{\infty} \varepsilon_n = \infty$ . Then the iteration defined by (1.5) converges faster than (1.2) to a fixed point  $v$  of  $\Pi$ .*

**Proof:** From equation (3.5) as in Theorem (3.1), we have

$$\|a_{n+1} - v\| \leq \zeta^{8(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|. \quad (3.21)$$

So, we have

$$\|a_{n+1} - v\| \leq \lambda_n, \quad (3.22)$$

where  $\lambda_n = \zeta^{8(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|$ .

Using Algorithm (1.2), we get

$$\begin{aligned} \|v_n - v\| &= \|\Pi u_n - v\| \\ &\leq \zeta \|u_n - v\|. \end{aligned} \quad (3.23)$$

Using Algorithm (1.2), we get

$$\begin{aligned} \|u_n - v\| &= \|\Pi[(1 - \sigma_n)y_n + \sigma_n \Pi y_n] - v\| \\ &\leq \zeta \|(1 - \sigma_n)y_n + \sigma_n \Pi y_n - v\| \\ &\leq \zeta [(1 - \sigma_n)\|y_n - v\| + \sigma_n \|\Pi y_n - v\|] \\ &\leq \zeta [(1 - \sigma_n)\|y_n - v\| + \zeta \sigma_n \|y_n - v\|] \\ &\leq \zeta [(1 - \sigma_n)\|y_n - v\| + \sigma_n \|y_n - v\|] \\ &\leq \zeta \|y_n - v\|. \end{aligned} \quad (3.24)$$

From Algorithm (1.2) and equations (3.23) and (3.24), we get

$$\begin{aligned} \|y_{n+1} - v\| &= \|(1 - \varepsilon_n)\Pi u_n + \varepsilon_n \Pi v_n - v\| \\ &\leq (1 - \varepsilon_n)\|\Pi u_n - v\| + \varepsilon_n \|\Pi v_n - v\| \\ &\leq (1 - \varepsilon_n)\zeta \|u_n - v\| + \zeta \varepsilon_n \|v_n - v\| \\ &\leq \zeta^2 [(1 - \varepsilon_n)\|y_n - v\| + \varepsilon_n \zeta \|y_n - v\|] \\ &\leq \zeta^2 [1 - (1 - \zeta)\varepsilon_n] \|y_n - v\|. \end{aligned} \quad (3.25)$$

Inductively, we get:

$$\|y_{n+1} - v\| \leq \zeta^{2(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|.$$

So, we have

$$\|y_{n+1} - v\| \leq \nu_n, \quad (3.26)$$

where  $\nu_n = \zeta^{2(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|$ .

Using equations (3.22) and (3.26), we get

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\nu_n} = \lim_{n \rightarrow \infty} \zeta^{6(n+1)} \frac{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|}{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|}. \quad (3.27)$$

As  $\zeta \in [0, 1)$  and  $\lim_{n \rightarrow \infty} \frac{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|}{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|}$  is finite, which implies

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\nu_n} = \lim_{n \rightarrow \infty} \zeta^{6(n+1)} = 0. \quad (3.28)$$

This implies Algorithm 1.5 converges faster than Algorithm 1.2 to  $v$ .  $\square$

**Theorem 3.5** Let  $\{y_n\}_{n=0}^{\infty}$  and  $\{a_n\}_{n=0}^{\infty}$  be two sequences defined by Algorithms (1.4) and (1.5) respectively, where  $\{\varepsilon_n\}$ ,  $\{\sigma_n\}$ ,  $\{\omega_n\}$  and  $\{\vartheta_n\}$  are in  $[0, 1]$  and  $\sum_{n=0}^{\infty} \varepsilon_n = \infty$ . Then the iteration defined by (1.5) converges faster than (1.4) to a fixed point  $v$  of  $\Pi$ .

**Proof:** From equation (3.5) as in Theorem (3.1), we have

$$\|a_{n+1} - v\| \leq \zeta^{8(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|. \quad (3.29)$$

So, we have

$$\|a_{n+1} - v\| \leq \lambda_n, \quad (3.30)$$

where  $\lambda_n = \zeta^{8(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|$ .

Using Algorithm 1.3, we have

$$\begin{aligned} \|v_n - v\| &= \|\Pi u_n - v\| \\ &\leq \zeta \|u_n - v\|. \end{aligned} \quad (3.31)$$

Using Algorithm 1.3 and equation (3.31), we have

$$\begin{aligned} \|u_n - v\| &= \|\Pi[(1 - \sigma_n)y_n + \sigma_n \pi y_n] - v\| \\ &\leq \zeta \|(1 - \sigma_n)y_n + \sigma_n \pi y_n - v\| \\ &\leq \zeta [(1 - \sigma_n)\|y_n - v\| + \sigma_n \zeta \|y_n - v\|] \\ &\leq \zeta \|y_n - v\|. \end{aligned} \quad (3.32)$$

Using Algorithm 1.3 and equation (3.32), we have

$$\begin{aligned} \|y_{n+1} - v\| &= \|\Pi((1 - \varepsilon_n)\Pi y_n + \varepsilon_n \Pi v_n) - v\| \\ &\leq \zeta(1 - \varepsilon_n)\|\Pi y_n - v\| + \varepsilon_n \|\Pi v_n - v\| \\ &\leq \zeta(1 - \varepsilon_n)\|y_n - v\| + \zeta \varepsilon_n \|v_n - v\| \\ &\leq \zeta(1 - \varepsilon_n)\|y_n - v\| + \zeta^3 \varepsilon_n \|y_n - v\| \\ &\leq \zeta(1 - \varepsilon_n)\|y_n - v\| + \zeta^2 \varepsilon_n \|y_n - v\|. \end{aligned} \quad (3.33)$$

Using equations (3.31), (3.32) and (3.33), we have

$$\|y_{n+1} - v\| \leq \zeta^1 [1 - (1 - \zeta)\varepsilon_n] \|y_n - v\|.$$

Inductively, we get:

$$\|y_{n+1} - v\| \leq \zeta^{1(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|.$$

So, we have

$$\|y_{n+1} - v\| \leq \nu_n, \quad (3.34)$$

where  $\nu_n = \zeta^{1(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|$ .

Using equations (3.30) and (3.34), we get

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\nu_n} = \lim_{n \rightarrow \infty} \zeta^{7(n+1)} \frac{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|}{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|}. \quad (3.35)$$

As  $\zeta \in [0, 1)$  and  $\lim_{n \rightarrow \infty} \frac{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|}{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|}$  is finite, which implies

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\nu_n} = \lim_{n \rightarrow \infty} \zeta^{7(n+1)} = 0. \quad (3.36)$$

This implies Algorithm 1.5 converges faster than Algorithm 1.3 to  $v$ .  $\square$

**Theorem 3.6** *Let  $\{y_n\}_{n=0}^\infty$  and  $\{a_n\}_{n=0}^\infty$  be two sequences defined by Algorithms (1.4) and (1.5) respectively, where  $\{\varepsilon_n\}$ ,  $\{\sigma_n\}$ ,  $\{\omega_n\}$  and  $\{\vartheta_n\}$  are in  $[0, 1]$  and  $\sum_{n=0}^\infty \varepsilon_n = \infty$ . Then the iteration defined by (1.5) converges faster than (1.4) to a fixed point  $v$  of  $\Pi$ .*

**Proof:** From equation (3.5) as in Theorem (3.1), we have

$$\|a_{n+1} - v\| \leq \zeta^{8(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|. \quad (3.37)$$

So, we have

$$\|a_{n+1} - v\| \leq \lambda_n, \quad (3.38)$$

where  $\lambda_n = \zeta^{8(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|$ .

Using Algorithm 1.4, we have

$$\begin{aligned} \|w_n - v\| &= \|(1 - \vartheta_n)y_n + \vartheta_n \Pi y_n - v\| \\ &\leq \|(1 - \vartheta_n)y_n + \vartheta_n \Pi y_n - v\| \\ &\leq (1 - \vartheta_n) \|y_n - v\| + \vartheta_n \|\Pi y_n - v\| \\ &\leq (1 - \vartheta_n) \|y_n - v\| + \zeta \vartheta_n \|y_n - v\| \\ &\leq (1 - \vartheta_n) \|y_n - v\| + \vartheta_n \|y_n - v\| \\ &\leq \|y_n - v\|. \end{aligned} \quad (3.39)$$

Using Algorithm 1.4 and equation (3.39), we have

$$\begin{aligned} \|u_n - v\| &= \|\Pi[(1 - \omega_n)w_n + \omega_n \Pi w_n] - v\| \\ &\leq \zeta \|(1 - \omega_n)w_n + \sigma_n \Pi w_n - v\| \\ &\leq \zeta [(1 - \omega_n) \|w_n - v\| + \omega_n \|\Pi w_n - v\|] \\ &\leq \zeta [(1 - \omega_n) \|w_n - v\| + \zeta \omega_n \|w_n - v\|] \\ &\leq \zeta [(1 - \omega_n) \|w_n - v\| + \omega_n \|w_n - v\|] \\ &\leq \zeta \|w_n - v\| \\ &\leq \zeta \|y_n - v\|. \end{aligned} \quad (3.40)$$

Using Algorithm 1.4 and equation (3.40), we have

$$\begin{aligned} \|v_n - v\| &= \|\Pi[(1 - \varepsilon_n)u_n + \varepsilon_n \Pi u_n] - v\| \\ &\leq \zeta \|(1 - \varepsilon_n)u_n + \varepsilon_n \Pi u_n - v\| \\ &\leq \zeta [(1 - \varepsilon_n) \|u_n - v\| + \varepsilon_n \|\Pi u_n - v\|] \\ &\leq \zeta [(1 - \varepsilon_n) \|u_n - v\| + \zeta \varepsilon_n \|u_n - v\|] \\ &\leq \zeta [(1 - \varepsilon_n) \|u_n - v\| + \varepsilon_n \|u_n - v\|] \\ &\leq \zeta [1 - (1 - \zeta)\varepsilon_n] \|u_n - v\| \\ &\leq \zeta^2 [1 - (1 - \zeta)\varepsilon_n] \|y_n - v\|. \end{aligned} \quad (3.41)$$

Using Algorithm 1.4 and equation (3.41), we have

$$\begin{aligned}
\|y_{n+1} - v\| &= \|\pi v_n - v\| \\
&\leq \zeta \|v_n - v\| \\
&\leq \zeta^3 [1 - (1 - \zeta)\varepsilon_n] \|v_n - v\| \\
&\leq \zeta^3 [1 - (1 - \zeta)\varepsilon_n] \|v_n - v\|.
\end{aligned} \tag{3.42}$$

Inductively, we get:

$$\|y_{n+1} - v\| \leq \zeta^{3(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|.$$

So, we have

$$\|y_{n+1} - v\| \leq \nu_n, \tag{3.43}$$

where  $\nu_n = \zeta^{3(n+1)} \prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|$ .

Using equations (3.38) and (3.43), we get

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\nu_n} = \lim_{n \rightarrow \infty} \zeta^{5(n+1)} \frac{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|}{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|}. \tag{3.44}$$

As  $\zeta \in [0, 1)$  and  $\lim_{n \rightarrow \infty} \frac{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|a_0 - v\|}{\prod_{k=0}^n [1 - (1 - \zeta)\varepsilon_k] \|y_0 - v\|}$  is finite, which implies

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{\nu_n} = \lim_{n \rightarrow \infty} \zeta^{5(n+1)} = 0. \tag{3.45}$$

This implies Algorithm 1.5 converges faster than Algorithm 1.4 to  $v$ .

□

#### 4. Numerical Example

**Example 4.1** Assume that  $\Lambda = \mathbb{R}$  is equipped with the usual norm, and let  $\Upsilon = [1, 20]$ . Define the operator  $\psi : \Upsilon \rightarrow \Upsilon$  by:

$$\psi(x) = \sqrt{x^2 - 8x + 16}, \quad \text{for all } x \in \Upsilon.$$

Clearly,  $x^* = 2$  is the unique fixed point. The sequences  $\{\xi_n\} = \{\epsilon_n\} = \{\rho_n\} = \{\chi_n\} = \frac{1}{(n+1)^{1/4}}$  are used, with an initial guess of  $x_0 = 10$ . Table 1 and Figure 1 illustrate that the new iteration method converges to  $x^* = 2$  more rapidly than the  $S^*$ , KF, AA, and Piri iterative methods.

Table 1: Comparison of different Algorithms.

Algorithm	Iter.	Sec.
New Algo.	10	0.1967
AA	11	0.5006
KF	13	0.6853
Piri	18	1.2742
$S^*$	20	1.4196

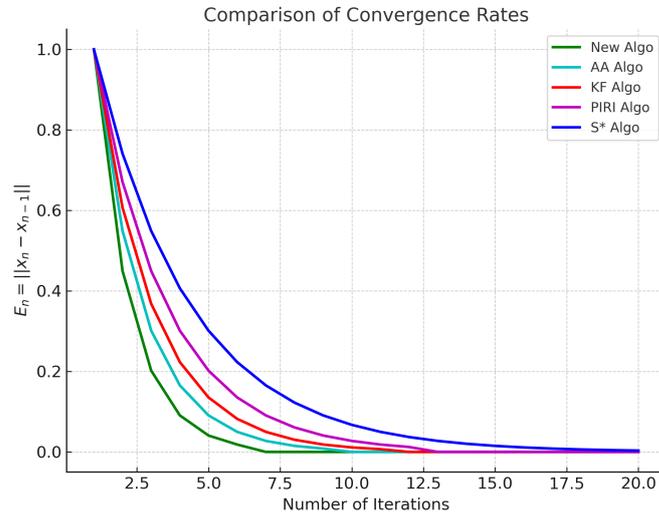


Figure 1: Comparison of different Algorithms

**Example 4.2** Assume that  $\Lambda = \mathbb{R}$  is equipped with the usual norm, and let  $\Upsilon = [0, 10]$ . Define the operator  $\phi : \Upsilon \rightarrow \Upsilon$  by:

$$\phi(x) = \frac{x + 3}{2}, \quad \text{for all } x \in \Upsilon.$$

Clearly,  $x^* = 3$  is the unique fixed point since  $\phi(3) = 3$ . Table 2 and Figure 2 illustrate that the new iteration method achieves convergence to one of the fixed points more efficiently than the  $S^*$ , KF, AA, and Piri iterative methods.

Table 2: Comparison of different Algorithms.

<i>Algorithm</i>	<i>Iter.</i>	<i>Sec.</i>
New Algo.	16	0.5196
$S^*$	18	0.6006
piri	20	0.9853
AA	24	1.3742
KF	27	1.5196

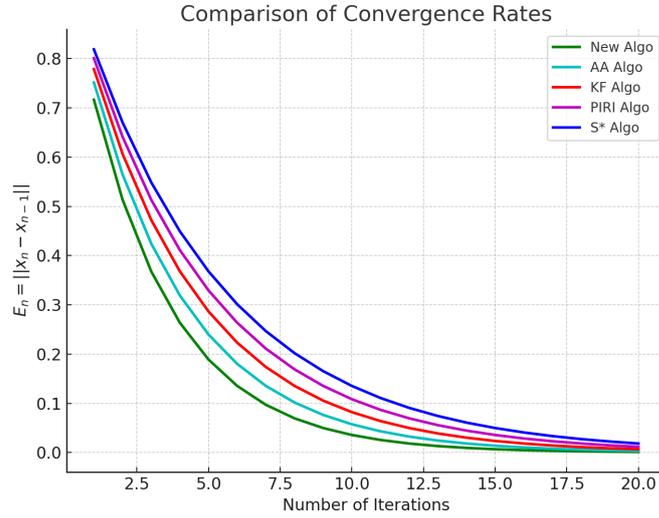


Figure 2: Comparison of different Algorithms

**Example 4.3** Assume that  $\Lambda = \mathbb{R}$  is equipped with the usual norm, and let  $\Upsilon = [1, 100]$ . Define the operator  $\psi : \Upsilon \rightarrow \Upsilon$  by:

$$\psi(x) = \sqrt{8x - 9}, \quad \text{for all } x \in \Upsilon.$$

Clearly,  $x^* = 4 \pm \sqrt{7}$  are the unique fixed points. The sequences  $\{\xi_n\} = \{\epsilon_n\} = \{\rho_n\} = \{\chi_n\} = \frac{\sqrt{n}}{1+1}$  are used, with an initial guess of  $x_0 = 10$ . Table 3 and Figure 3 illustrate that the new iteration method achieves convergence to one of the fixed points more efficiently compared to the  $S^*$ , KF, AA, and Piri iterative methods.

Table 3: Comparison of different Algorithms.

<i>Algorithm</i>	<i>Iter.</i>	<i>Sec.</i>
New Algo.	30	0.6196
AA	34	0.8006
KF	40	0.9853
Piri	45	1.4742
$S^*$	69	1.7196

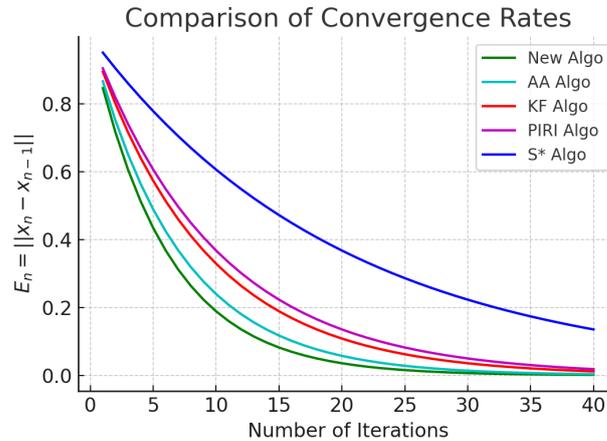


Figure 3: Comparison of different Algorithms

## 5. Conclusion

In this paper, we introduced a new iterative method for approximating fixed points of contraction mappings in real normed linear spaces. We established a strong convergence theorem for the proposed method, demonstrating its efficiency in reaching a fixed point under standard conditions. Additionally, we analyzed the stability of the iteration process, showing that the method is robust and reliable when small perturbations are present. Our comparisons with several existing iterative algorithms, such as the  $S^*$ , Piri, KF, and AA iterations, reveal that the new method exhibits superior convergence behavior and stability.

These results contribute to the growing body of research on fixed-point iterative methods, offering a practical and efficient alternative for solving nonlinear problems in various fields. Future work may focus on extending this method to more general classes of mappings and investigating its performance in higher-dimensional and more complex spaces.

## References

1. Chugh, R. Gupta, N., *Strong convergence of new split general system of monotone variational inclusion problem*. Appl. Anal., 1-28, (2023).
2. Gupta, N., Jolaoso, L. O., Nandal, A., Chugh, R., *Convergence analysis of multiple-sets split equality common fixed point problem with applications*. Numer. Algebra Control Optim., 1-27, (2023).
3. Batra, C., Gupta, N., Chugh, R., Kumar, R., *Generalized viscosity extragradient algorithm for pseudomonotone equilibrium and fixed point problems for finite family of demicontractive operators*. J. Appl. Math. Comput., 68(6), 4195-4222, (2022).
4. Batra, C., Chugh, R., Sajid, M., Gupta, N., Kumar, R., *Generalized viscosity approximation method for solving split generalized mixed equilibrium problem with application to compressed sensing*. AIMS Mathematics, 9(1), 1718-1754, (2024).
5. Chugh, R., Batra, C., *Fixed point theorems of enriched Ciric's type and enriched Hardy-Rogers contractions*. Numerical Algebra, Control and Optimization, 1-23, (2023).
6. Chugh, R., Gupta, N., Batra, C., *Modified subgradient extragradient method for solution of split variational inequality problem with application to generalized Nash equilibrium problem*. Optimization, 1-35, (2024).
7. Chugh, R., Kumar, R., Batra, C., *A novel inertial Tseng's method for solving generalized variational inequality problem*. Journal of Applied Mathematics and Computing, 69(6), 4525-4551, (2023).
8. Batra, C., Chugh, R., Kumar, R., Suwais, K., Almanasra, S., Mlaiki, N., *Strong convergence of split equality variational inequality, variational inclusion, and multiple sets fixed point problems in Hilbert spaces with application*. Journal of Inequalities and Applications, 2024(1), 40, (2024).
9. Mehra, S., Chugh, R., Batra, C., *A novel iterative algorithm for solving variational inequality, finite family of monotone inclusion and fixed point problems*. Tamkang Journal of Mathematics, (2024).
10. Gupta, N., Postolache, M., Nandal, A., Chugh, R., *A Cyclic Iterative Algorithm for Multiple-Sets Split Common Fixed Point Problem of Demicontractive Mappings without Prior Knowledge of Operator Norm*. Mathematics, 9, 372, (2021).

11. Banach, S., *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*. Fund. Math., 3(1), 133-181, (1922).
12. Mann, W. R., *Mean Value methods in iteration*. Proc. Amer. Math. Soc., 4, 506–510.
13. Ishikawa, S., (1974). *Fixed points by a new iteration method*. Proc. Amer. Math. Soc., 44, 147–150, (1953).
14. Berinde, V., *Stability of Picard iteration for contractive mappings satisfying an implicit relation*. Carpathian J. Math., 27, 13-23, (2011).
15. Noor, M. A., *New approximation schemes for general variational inequalities*. J. Math. Anal. Appl., 251, 217–229, (2000).
16. Chugh, R., Kumar, V., *Strong convergence of SP iterative scheme for quasi-contractive operators*. Int. J. Comput. Appl., 31(5), 21-27, (2011).
17. Agarwal, R. P., O'Regan, Donal, Sahu, D. R., *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*. J. Nonlinear Convex Anal., 8, 61–79, (2007).
18. Piri, H., Daraby, B., Rahrovi, S., Ghasemi, M., *Approximating fixed points of generalized-nonexpansive mappings in Banach spaces by new faster iteration process*. Numer. Algorithms, 81, 1129–1148, (2019).
19. Ullah, K., Ahmad, J., Khan, F. M., *Numerical reckoning fixed points via new faster iteration process*. Applied General Topology, 23(1), 213-223, (2022).
20. Karahan, I., Ozdemir, M. *A general iterative method for approximation of fixed points and their applications*. Adv. Fixed Point Theory, 3(3), 510-526, (2013).
21. Karakaya, V., Gürsoy, F., Ertürk, M., *Some convergence and data dependence results for various fixed point iterative methods*. Kuwait Journal of Science, 43(1), 112-128, (2016).
22. Thakur, B. S., Thakur, D., Postolache, M., *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*. Appl. Math. Comp. 275, 147-155, (2016).
23. Abbas, M., Nazir, T., *A new faster iteration process applied to constrained minimization and feasibility problems*. Mat. Vesn. 66, 223-234, (2014).
24. Beg, I., Abbas, M., Asghar, M. W., *Convergence of AA-Iterative Algorithm for Generalized o-Nonexpansive Mappings with an Application*. (2022)..
25. Harder, A. M., *Fixed point theory and stability results for fixed points iteration procedures [Ph. D. thesis]*. University of Missouri-Rolla (1987).
26. Soltuz, S.M., Grosan T., *Data dependence for Ishikawa iteration when dealing with contractive like operators*. Fixed Point Theory Appl., 2008, 1-7, (2008).

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