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A Wolfe-Type Steepest Descent Algorithm for Uncertain Quadratic Multiobjective Optimization Problems

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ABSTRACT: This work develops a Wolfe-type steepest descent algorithm for solving uncertain quadratic multiobjective optimization problems (UQMOPs) by reformulating them into deterministic robust counterparts via objective-wise worst-case criteria. The proposed method incorporates a Wolfe-type inexact line search to obtain more efficient descent directions and improve overall convergence behavior. A Zoutendijk-type condition is established to guarantee linear convergence under standard assumptions.

Key Words: Quadratic problem, uncertainty, multiobjective optimization, robust efficiency, steepest descent method.

Contents

1. Introduction

In many practical optimization problems, two main issues arise: the data may be uncertain, and multiple objectives need to be optimized simultaneously. Stochastic and robust optimization are used to manage uncertainty, while multiobjective optimization deals with multiple criteria [7,22,23]. Popular techniques for solving such problems include the weighting method, the ϵ-constraint method, and the lexicographic approach [3,6,7,14,15,16]. A common drawback of these methods is that they require parameters or priorities to be fixed beforehand, which is often difficult to decide in real applications. In recent years, several methods have been developed to solve multiobjective problems without needing prior parameter information [1,2,4,5,10,11,12]. Among them, steepest descent methods usually converge linearly [1,2], Newton-type methods can achieve faster, even quadratic, convergence [4], and quasi-Newton methods often show superlinear convergence [17]. These methods are mainly designed for smooth and deterministic multiobjective problems and cannot be directly used when uncertainty is present. In practice, uncertainty is common due to inaccurate data, measurement noise, or changing environments. To handle such situations, robust optimization is widely used [7,8]. The basic idea of RO is to replace the uncertain problem with a deterministic one by focusing on the worst possible cases within a given uncertainty set, so that the obtained solutions remain stable and reliable even when the data vary. The idea of minimax robustness, first introduced by Soyster [13] and later developed by Ben-Tal et al. [8,9], focuses on finding a solution that minimizes the worst possible outcome over all scenarios. Such a solution, called a robust optimal solution, remains feasible for every case in the uncertainty set. This concept has also been extended to multiobjective uncertain problems [7,21,33]. Since classical single-objective methods are not directly suitable, UMOPs are usually handled either by a max-min strategy or by the objective-wise worst-case (OWWC) approach [7]. The OWWC method is simpler to compute and leads to the same robust solutions as the minimax formulation.

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Quadratic multiobjective problems have attracted growing interest because of their applications in areas such as engineering, finance, and decision-making. Several works have studied how to deal with uncertainty in such problems. For example, Choung et al. [31] considered convex quadratic multiobjective problems with both parameter and decision uncertainty, derived Pareto optimality conditions using linear matrix inequalities, and proposed numerical methods to compute robust Pareto solutions. Eichfelder et al. [32] examined nonconvex quadratic multiobjective problems and used relaxation techniques to obtain convex reformulations. However, their approach mainly finds supported solutions and cannot recover the full Pareto front in nonconvex cases.

More recently, Choung et al. [33] studied robust two-stage quadratic multiobjective problems. They developed new optimality conditions based on LMIs and proposed solution methods using semidefinite and second-order cone programming. Although these methods are powerful, they are computationally demanding and are not directly suitable for single-stage uncertain quadratic multiobjective problems. To solve uncertain unconstrained quadratic multiobjective problems (UUQMOPs), Kumar et al. [29] introduced an Armijo-type Newton descent method (ATNDM). This approach directly handles UUQMOPs without using scalarization. On the other hand, Ehrgott et al. [7] applied scalarization techniques such as the weighted-sum and ϵ-constraint methods to uncertain multiobjective problems. However, these scalarization methods have some clear limitations: choosing suitable weights or ϵ values is difficult, and poor choices may lead to biased, infeasible, or low-quality solutions. Moreover, extensive parameter tuning is often required, which increases computational cost and sensitivity to user choices. To overcome this, several recent studies have adapted these algorithms to uncertain multiobjective settings by employing robust counterpart formulations [25,26,27,28,29,30]. Compared to scalarization, ATNDM [29] avoids parameter selection and follows a systematic descent direction, resulting in more stable and efficient computations.

Although ATNDM shows good convergence behavior, it still has drawbacks related to step-size selection. The Armijo-type line search tends to choose very small step sizes, which may slow down convergence and reduce practical efficiency.

To address these issues, this paper proposes a Wolfe-type steepest descent method (WTSDM) for UUQMOPs. The Wolfe conditions balance sufficient decrease with curvature control, allowing more adaptive and usually larger step sizes than the Armijo rule. As a result, the proposed method improves convergence speed, numerical stability, and overall computational performance. To the best of our knowledge, a steepest method using Wolfe-type line search for UUQMOPs has not yet been reported. The remainder of this paper is organized as follows. In Section 2, we present the necessary preliminaries and introduce the uncertain quadratic multiobjective optimization problem along with its deterministic robust counterpart and the related concepts required for robust modeling. Section 3 is devoted to the development of a Wolfe-type steepest descent method for solving the robust counterpart of the uncertain quadratic problem, and its convergence properties are analyzed under standard assumptions. Section 4 concludes the paper with a summary of the main results and discusses possible directions for future research.

2. Preliminaries

Throughout this paper, let $\langle \cdot, \cdot \rangle$ denote the standard inner product in the Euclidean space and $\| \cdot \|$ denote the Euclidean norm. Let $\mathbb{R}_+$ denote the set of all non-negative real numbers and $\mathbb{R}_{++}$ denote the set of all positive real numbers. Moreover, $\mathbb{N}$ denotes the set of all natural numbers. Throughout the paper, we denote the index sets $I = \{1, \dots, p\}$ and $J = \{1, \dots, m\}$, which will be used consistently wherever appropriate. For vectors $u, v \in \mathbb{R}^n$, the component-wise ordering is defined as

$$u \succeq v \;\Leftrightarrow\; u - v \in \mathbb{R}^n_{\geq 0} \;\Leftrightarrow\; u_i - v_i \geq 0, \;\; \forall\, i,$$

$$u \succ v \;\Leftrightarrow\; u - v \in \mathbb{R}^n_{> 0} \;\Leftrightarrow\; u_i - v_i > 0, \;\; \forall\, i.$$

Inspired by the works in [18,19] and Choung et al. [20], this study focuses on addressing a UUQMOP characterized by *parameter uncertainty* that affects only the objective functions. To handle this issue, the problem is transformed into a deterministic one using OWRC, which is subsequently solved using a

WTSDM. Formally, the UUQMOP and its associated OWRC are described as follows:

$$\mathcal{Q}(\mathcal{U}) = \{\mathcal{Q}(\mathcal{M}_i) : \mathcal{M}_i \in \mathcal{U},\ i \in I_p\}, \tag{2.1}$$

where the uncertainty set $\mathcal{U}$ is given by

$$\mathcal{U} = \Big\{\mathcal{M}_i \in \mathbb{R}^{n\times n}\ :\ \mathcal{M}_i \text{ is SPD matrix for all } i \in I_p\Big\}.$$

For any fixed matrix $\mathcal{M}_i \in \mathcal{U}$, the corresponding subproblem is defined as:

$$\mathcal{Q}(\mathcal{M}_i) : \quad \min_{z\in\mathbb{R}^n} \mathcal{A}(z, \mathcal{M}_i),$$

where the vector-valued objective function $\mathcal{A} : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^m$ is expressed as:

$$\mathcal{A}(z, \mathcal{M}_i) = \big(\mathcal{A}_1(z, \mathcal{M}_i), \mathcal{A}_2(z, \mathcal{M}_i), \ldots, \mathcal{A}_m(z, \mathcal{M}_i)\big),$$

and each scalar component $\mathcal{A}_j : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ is defined by:

$$\mathcal{A}_j(z, \mathcal{M}_i) = \tfrac{1}{2} z^\top \mathcal{M}_i z + q_j^\top z + r_j, \qquad j \in I_m,\ i \in I_p.$$

The overall problem $\mathcal{Q}(\mathcal{U})$ is referred to as a UUQMOP.

The associated OWWC of $\mathcal{Q}(\mathcal{U})$ is formulated as:

$$(\mathcal{RP}) : \quad \min_{z\in\mathbb{R}^n} \ \varkappa(z), \tag{2.2}$$

where the vector function $\varkappa(z) = \big(\varkappa_1(z), \varkappa_2(z), \ldots, \varkappa_m(z)\big)$, and for each $j \in I_m$,

$$\varkappa_j(z) = \max_{j\in I_m} \mathcal{A}_j(z, \mathcal{M}_i) = \max_{j\in I_m} \Big\{\tfrac{1}{2} z^\top \mathcal{M}_i z + q_j^\top z + r_j\Big\}.$$

In this study, we aim to address the problem $\mathcal{Q}(\mathcal{U})$ to identify an RPOS or an RWPOS. The formal definitions of these notions are provided below:

Definition 2.1 *[29] A vector $z^\star \in \mathbb{R}^n$ is referred to as an RPOS or an RWPOS if there does not exist any $z \in \mathbb{R}^n \setminus \{z^\star\}$ such that*

$$\mathcal{A}(z;\mathcal{U}) \subset \mathcal{A}(z^\star;\mathcal{U}) - \mathbb{R}^k_{\geq} \quad \Big(\text{or } \mathcal{A}(z;\mathcal{U}) \subset \mathcal{A}(z^\star;\mathcal{U}) - \mathbb{R}^k_{>}\Big),$$

where

$$\mathcal{A}(z;\mathcal{U}) = \{\, \mathcal{A}(z, \mathcal{M}_i) : \mathcal{M}_i \in \mathcal{U} \,\}$$

represents the set of all possible objective function values corresponding to the uncertainty set $\mathcal{U}$.

The following theorem indicates that, rather than solving $\mathcal{Q}(\mathcal{U})$ directly, one can derive robust solutions by addressing the problem $(\mathcal{RP})$.

Theorem 2.1 *[29] Consider an UUQMOP $\mathcal{Q}(\mathcal{U})$ defined over the uncertainty set $\mathcal{U}$, and let $(\mathcal{RP})$ denote its robust counterpart. Then:*

(a) If $z^\star \in \mathbb{R}^n$ is a POS of $(\mathcal{RP})$, then $z^\star$ also serves as a RPOS for the original problem $\mathcal{Q}(\mathcal{U})$.

(b) Assume that, for every $z \in \mathbb{R}^n$ and for each objective index $j \in I_m$, the maximum

$$\max_{j\in I_m} \mathcal{A}_j(z, \mathcal{M}_i)$$

exists. If $z^\star$ is a WPOS of $(\mathcal{RP})$, then $z^\star$ is also a RWPOS for $\mathcal{Q}(\mathcal{U})$.

A clear benefit of computing $\varkappa$ for a given t is that it is much simpler than solving the $\mathcal{Q}(\mathcal{U})$. Here, only m deterministic single-objective problems need to be solved. As a result, $(\mathcal{RP})$ becomes a deterministic multi-objective problem that can be handled using standard methods. Thus, the $(\mathcal{RP})$-approach replaces set dominance with the usual point-wise dominance, effectively converting the uncertain problem into a deterministic one.

Definition 2.2 *[29] In the context of $(\mathcal{RP})$, $v \in \mathbb{R}^n$ is called a RDD for $\varkappa$ at a point t if the following condition holds:*

$$(\mathcal{M}_i z + q_j)^T v < 0, \quad \forall\, j \in I_m,\ i \in \mathcal{U}_j(z) = \{i \in I_P : \mathcal{A}_j(z, \mathcal{M}_i) = \varkappa_j(z)\}.$$

Furthermore, if $v \in \mathbb{R}^n$ is a RDD for $\varkappa$ at z, then $\exists$ a positive constant $\bar{\alpha} > 0$ such that:

$$\varkappa_j(z + \alpha v) < \varkappa_j(z), \quad \forall\, j \in I_m,\ \forall\, \alpha \in (0, \bar{\alpha}].$$

Theorem 2.2 *[24] Let $\varkappa_j : \mathbb{R}^n \to \mathbb{R}$ be defined as*

$$\varkappa_j(z) = \max_{j \in I_m} \mathcal{A}_j(z, \mathcal{M}_i).$$

Then, the following statements hold:

(i) In any direction $v \in \mathbb{R}^n$, the directional derivative of $\varkappa_j$ at z is expressed as

$$\varkappa_j'(z, v) = \max_{i \in \mathcal{U}_j(z)} \left(\mathcal{M}_i z + q_j\right)^T v,$$

where $\mathcal{U}_j(z) = \{i \in I_p : \mathcal{A}_j(z, \mathcal{M}_i) = \varkappa_j(z)\}$ denotes the set of active indices at z.

(ii) The subdifferential of $\varkappa_j$ at z is given by

$$\partial \varkappa_j(z) = \operatorname{conv}\{\mathcal{M}_i z + q_j : i \in \mathcal{U}_j(z)\},$$

where $\operatorname{conv}(\cdot)$ represents the convex hull. Moreover, z^* is a minimizer of $\varkappa_j$, i.e.,

$$z^* = \operatorname*{argmin}_{t \in \mathbb{R}^n} \varkappa_j(z),$$

if and only if

$$0 \in \partial \varkappa_j(z^*).$$

Definition 2.3 *[29] A point $z^* \in \mathbb{R}^n$ is said to be a RCP of $\varkappa$ if the following condition holds:*

$$R(\Phi(z^*)) \cap \left(-\mathbb{R}^m_{>}\right) = \emptyset,$$

where $\Phi(z^*) = \bigcup_{j \in I_m} \partial \varkappa_j(z^*)$ denotes the set of all subgradients of $\varkappa$ at z^*. Equivalently, z^* is an RCP for the robust decision problem $(\mathcal{RP})$ if there does not exist any direction $v \in \mathbb{R}^n$ such that $(\mathcal{M}_i z^* + q_j)^T v < 0$, for every $i \in \mathcal{U}_j(z^*)$ and $j \in I_m$, where the active set $\mathcal{U}_j(z^*)$ is defined by

$$\mathcal{U}_j(z^*) = \left\{ i \in I_p \ :\ \mathcal{A}_j(z^*, \mathcal{M}_i) = \varkappa_j(z^*) \right\}.$$

In the following subsection, we develop WTSDM for $(\mathcal{RP})$. By utilizing the WTSDM, we can determine RCPs of $(\mathcal{RP})$, which simultaneously correspond to the POS of $(\mathcal{RP})$ as well as the RPOS of $\mathcal{Q}(\mathcal{U})$.

3. WTSDM: A Wolfe-Type Steepest Descent Method for $(\mathcal{RP})$

In this section, we present the development of the WTSDM for solving the problem $(\mathcal{RP})$. For $(\mathcal{RP})$, we consider the vector-valued mapping $\varkappa : \mathbb{R}^n \to \mathbb{R}^m$ is defined as

$$\varkappa(z) = \big(\varkappa_1(z), \varkappa_2(z), \ldots, \varkappa_m(z)\big),$$

where, for each $j \in I_m$,

$$\varkappa_j(z) = \max_{i \in I_p} \mathcal{A}_j(z, \mathcal{M}_i) = \max_{i \in I_p} \left\{ \tfrac{1}{2} z^T \mathcal{M}_i z + q_j^T z + r_j \right\}.$$

Since $\mathcal{M}_i \in \mathcal{U}$ is symmetric positive definite for each $i \in I_p$, the quadratic function $\mathcal{A}_j(z, \mathcal{M}_i)$ is convex. Hence, $\varkappa_j(z)$, being the pointwise maximum of convex functions, is also convex for every $j \in I_m$.

The following lemma and theorem provide the necessary and sufficient conditions to characterize Pareto optimality or efficiency for the problem $(\mathcal{RP})$.

Lemma 3.1 *[29] A vector $z^* \in \mathbb{R}^n$ is said to be a RCP of $\varkappa$ if and only if*

$$\mathbf{0} \in \operatorname{conv}\left(\bigcup_{j \in I_m} \partial \varkappa_j(z^*) \right),$$

where $\operatorname{conv}(\cdot)$ *denotes the convex hull, and $\partial \varkappa_j(z^*)$ represents the subdifferential of $\varkappa_j$ at z^*.*

Theorem 3.1 *[29] Consider the function*

$$\mathcal{A}_j(z, \mathcal{M}_i) = \frac{1}{2} z^T \mathcal{M}_i z + q_j^T z + r_j,$$

which is assumed to be twice CD. Suppose that, for every $i \in I_p$, the matrix $\mathcal{M}_i$ is SPD. Then, a point z^* is a WPOS for the problem $(\mathcal{RP})$ if and only if

$$\mathbf{0} \in \operatorname{conv}\left(\bigcup_{j \in I_m} \partial \varkappa_j(z^*) \right),$$

where $\partial \varkappa_j(z^*)$ denotes the subdifferential of $\varkappa_j$ at z^*, and $\operatorname{conv}(\cdot)$ *represents the convex hull.*

To implement the WTSDM for solving the problem $(\mathcal{RP})$, it is necessary to compute a robust steepest descent direction (RSDD) along with an appropriate step size in that direction. Subsection 3.1 describes the formulation of an auxiliary subproblem based on the minimization of a scalar-valued function, whose optimal solution yields the required RSDD.

3.1. Computation of RSDD using quadratic subproblem

To obtain the RSDD for $(\mathcal{RP})$, we formulate and solve a minimization auxiliary subproblem based on a scalar-valued function, which is presented as follows:

$$\min_{\mathcal{Y} \in \mathbb{R}^n} \max_{j \in I_m} \max_{i \in I_p} \left\{ \frac{1}{2} z^T \mathcal{M}_i z + q_j^T z + r_j + (\mathcal{M}_i z + q_j)^T \mathcal{Y} - \varkappa_j(z) \right\} + \frac{1}{2} \|\mathcal{Y}\|^2. \tag{3.1}$$

Let $\mathcal{Y}(z)$ be the optimal solution, and $\Theta(z)$ denote the associated optimal value of the subproblem (3.1). Consequently,

$$\mathcal{Y}(z) = \underset{\mathcal{Y} \in \mathbb{R}^n}{\operatorname{argmin}} \ \max_{j \in I_m} \max_{j \in I_m} \{ \frac{1}{2} z^T \mathcal{M}_i z + q_j^T z + r_j + (\mathcal{M}_i z + q_j)^T \mathcal{Y}(z) - \varkappa_j(z) \} + \frac{1}{2} \|\mathcal{Y}\|^2 \tag{3.2}$$

$$\Theta(z) = \max_{j \in I_m} \max_{j \in I_m} \{ \frac{1}{2} z^T \mathcal{M}_i z + q_j^T z + r_j + (\mathcal{M}_i z + q_j)^T \mathcal{Y}(z) - \varkappa_j(z) \} + \frac{1}{2} \|\mathcal{Y}\|^2. \tag{3.3}$$

The subproblem (3.1) can be reformulated as below:

$$P(z): \quad \min_{\mathcal{Y}\in\mathbb{R}^n,\ z\in\mathbb{R}^n} \quad \mathcal{V}(\mathcal{Y},z)+\frac{1}{2}\|\mathcal{Y}\|^2$$
$$\text{subject to:} \quad \frac{1}{2}z^T\mathcal{M}_i z+q_j^T z+r_j+(\mathcal{M}_i z+q_j)^T\mathcal{Y}-\varkappa_j(z)\le T,\ i\in I_p,\ j\in I_m,$$

where $T=\mathcal{V}(z,\mathcal{Y})=\max_{j\in I_m}\max_{j\in I_m}\left\{\frac{1}{2}z^T\mathcal{M}_i z+q_j^T z+r_j+(\mathcal{M}_i z+q_j)^T\mathcal{Y}-\varkappa_j(z)\right\}$.
It can be observed that the constraints of the problem $P(z)$ satisfy Slater's condition at $T=1$, and the point $\mathcal{Y}=(0,0,\dots,0)\in\mathbb{R}^n$ represents a feasible solution. Consequently, the optimal solution of the convex optimization problem $P(z)$ can be obtained by applying the Karush-Kuhn-Tucker (KKT) optimality conditions. To employ these conditions, we first formulate the corresponding Lagrangian function as follows:

$$L(z,\mathcal{Y},\Upsilon)=T+\frac{1}{2}\|\mathcal{Y}\|^2+\sum_{j\in I_m}\sum_{i\in I_p}\Upsilon_{ij}\left(\frac{1}{2}z^T\mathcal{M}_i z+q_j^T z+r_j+(\mathcal{M}_i z+q_j)^T\mathcal{Y}-\varkappa_j(z)-T\right).$$

Subsequently, the KKT conditions corresponding to the optimization problem $P(z)$ are expressed as follows:

$$\sum_{j\in I_m}\sum_{i\in I_p}\Upsilon_{ij}=1, \tag{3.4}$$

$$\mathcal{Y}+\sum_{j\in I_m}\sum_{i\in I_p}\Upsilon_{ij}(\mathcal{M}_i z+q_j)=0, \tag{3.5}$$

$$\frac{1}{2}z^T\mathcal{M}_i z+q_j^T z+r_j+(\mathcal{M}_i z+q_j)^T\mathcal{Y}-\varkappa_j(z)\le T,\ i\in I_p,\ j\in I_m, \tag{3.6}$$

$$\Upsilon_{ij}\left(\frac{1}{2}z^T\mathcal{M}_i z+q_j^T z+r_j+(\mathcal{M}_i z+q_j)^T\mathcal{Y}-\varkappa_j(z)-T\right)=0,\quad \Upsilon_{ij}\ge 0,\ i\in I_p,\ j\in I_m. \tag{3.7}$$

The optimization problem $P(z)$ admits a unique solution denoted by $(\mathcal{Y}(z),\Theta(z))$. As the problem is convex and satisfies Slater's condition, there exists a KKT multiplier $\Upsilon=(\Upsilon_{ij})\in R^{p\times m}$ such that, together with $\mathcal{Y}=\mathcal{Y}(z)$ and $T=\Theta(z)$, the conditions (3.4), (3.5), (3.6), and (3.7) are satisfied. Specifically, by utilizing Eq. (3.5), we obtain the following:

$$\mathcal{Y}(z)=-\sum_{j\in I_m}\sum_{i\in I_p}\Upsilon_{ij}(\mathcal{M}_i z+q_j). \tag{3.8}$$

Hence, a solution to $P(z)$ exists, and $\mathcal{Y}(z)$ represents the RSDD associated with the $(\mathcal{RP})$. Moreover, Theorem 3.2 provides the connection between the RCP and $\mathcal{Y}(z)$, as expressed in Equation (3.8).

Theorem 3.2 *Let $\mathcal{Y}(z)$ and $\Theta(z)$ be defined by Eqn. (3.2) and Eqn. (3.3), respectively. Then, the following results hold:*

1. *For a compact set $C\subset\mathbb{R}^n$, the vector $\mathcal{Y}(z)$ remains bounded on $\mathcal{C}$ and satisfies the condition $\Theta(z)\le 0$.*
2. *The subsequent results are equivalent:*
   - *(i)* $z\in\mathbb{R}^n$ *is not a RCP.*
   - *(ii)* $\Theta(z)<0$.
   - *(iii)* $\mathcal{Y}(z)\neq 0$.

(iv) $\mathcal{Y}(z)$ *provides a valid RSDD for* $\varkappa$ *at* z *in the context of the* $(\mathcal{RP})$.

Furthermore, z *corresponds to an RCP if and only if* $\Theta(z) = 0$.

Proof: For a fixed $z \in \mathbb{R}^n$, consider the problem $P(z)$. Since $\mathcal{V}(\mathcal{Y}, z)$ is convex in $\mathcal{Y}$ and the term $\frac{1}{2}\|\mathcal{Y}\|^2$ is strongly convex; the objective function is strongly convex in $\mathcal{Y}$. Hence, $P(z)$ admits a unique minimizer $\mathcal{Y}(z)$ and $\Theta(z)$ is well defined.
(2i)–(2ii) Let $C \subset \mathbb{R}^n$ be compact. By continuity of the data and strong convexity, the solution mapping $z \mapsto \mathcal{Y}(z)$ is bounded on C. Moreover, choosing $\mathcal{Y} = 0$ gives

$$\Theta(z) \leq \mathcal{V}(0, t) = \max_{j\in I_m} \max_{i\in I_p} \left\{ \tfrac{1}{2} z^T \mathcal{M}_i z + q_j^T t + r_j - \varkappa_j(z) \right\} \leq 0,$$

and thus $\Theta(z) \leq 0$. As z is not a PCP, $\Theta(z) \neq 0$, hence $\Theta(z) < 0$.
(2ii)–(2iii) By definition,

$$\Theta(z) = \mathcal{V}(\mathcal{Y}(z), z) + \tfrac{1}{2}\|\mathcal{Y}(z)\|^2.$$

If z is an RCP, then $\mathcal{Y} = 0$ solves $P(z)$, which implies $\Theta(z) = 0$ and $\mathcal{Y}(z) = 0$. Conversely, if $\Theta(z) = 0$, the strong convexity of the objective yields $\mathcal{Y}(z) = 0$, and hence t is an RCP. Therefore, z is not an RCP if and only if $\Theta(z) < 0$, which is equivalent to $\mathcal{Y}(z) \neq 0$.
(2iii)–(2iv) Finally, when $\mathcal{Y}(z) \neq 0$, the optimality conditions of $P(z)$ imply

$$\max_{j\in I_m} \max_{i\in I_p} \langle \mathcal{M}_i z + q_j, \mathcal{Y}(z)\rangle < 0,$$

which shows that $\mathcal{Y}(z)$ is a valid robust steepest descent direction for $\varkappa$ at z for $(\mathcal{RP})$. Hence, statements (2i)–(2iv) are equivalent.

Thus, t is an RCP if and only if $\Theta(z) = 0$, and the proof is complete. □

This theorem provides a criterion to verify whether a point z is an RCP or if a descent direction exists, which can further serve as a stopping rule in the WTSDM. The following theorem establishes that the function $\Theta(z)$ is continuous for all $z \in \mathbb{R}^n$.

Theorem 3.3 *Consider the function* $\Theta : \mathbb{R}^n \to \mathbb{R}$ *defined in Equation* (3.2). *Then, the mapping* $\Theta(z)$ *is continuous for all* $z \in \mathbb{R}^n$.

Proof: Recall that

$$\Theta(z) = \mathcal{V}(\mathcal{Y}(z), z) + \tfrac{1}{2}\|\mathcal{Y}(z)\|^2,$$

where $\mathcal{Y}(z)$ is the unique minimizer of the strongly convex problem $P(z)$.

For any fixed $\mathcal{Y} \in \mathbb{R}^n$, the function

$$(z, \mathcal{Y}) \mapsto \mathcal{V}(\mathcal{Y}, z) = \max_{j\in I_m} \max_{i\in I_P} \left\{ \tfrac{1}{2} z^T \mathcal{M}_i z + q_j^T z + r_j + (\mathcal{M}_i z + q_j)^T \mathcal{Y} - \varkappa_j(z) \right\}$$

is continuous in z, since it is the pointwise maximum of finitely many continuous functions. Hence, the objective function

$$(z, \mathcal{Y}) \mapsto \mathcal{V}(\mathcal{Y}, z) + \tfrac{1}{2}\|\mathcal{Y}\|^2$$

is jointly continuous in $(z, \mathcal{Y})$ and strongly convex in $\mathcal{Y}$. By standard results on parametric strongly convex optimization problems, the solution mapping $z \mapsto \mathcal{Y}(z)$ is continuous on $\mathbb{R}^n$. Therefore, as a composition of continuous functions,

$$\Theta(z) = \mathcal{V}(\mathcal{Y}(z), t) + \tfrac{1}{2}\|\mathcal{Y}(z)\|^2$$

is continuous for all $z \in \mathbb{R}^n$. □

□

3.2. Wolfe-Type line search technique

Under consideration, $K = \mathbb{R}^m_{\geq}$, we will introduce the Wolfe condition for $(\mathcal{RP})$.

Definition 3.1 *Let $\mathcal{Y} \in \mathbb{R}^n$ be an $\mathbb{R}^m_+$-RSDD for $\varkappa$ at the point x, and let $0 < \beta < \gamma < 1$. A step size $\alpha > 0$ is said to satisfy the* standard Wolfe conditions *if it holds that*

$$\varkappa(z + \alpha\mathcal{Y}) \leq_{\mathbb{R}^m_+} \varkappa(z) + \beta\alpha\mathcal{V}(z, \mathcal{Y}), \tag{3.9}$$

$$\mathcal{V}(z + \alpha\mathcal{Y}, \mathcal{Y}) \geq \gamma\mathcal{V}(z, \mathcal{Y}), \tag{3.10}$$

where $\mathcal{V}(z, \mathcal{Y}) = \max_{j \in I_m} \max_{i \in I_p} \{\frac{1}{2} z^T \mathcal{M}_i z + q_j^T z + r_j + (\mathcal{M}_i z + q_j)^T \mathcal{Y} - \varkappa_j(z)\}$. *Similarly, $\alpha > 0$ is said to fulfill the* strong Wolfe conditions *if it satisfies*

$$\varkappa(z + \alpha\mathcal{Y}) \leq_{\mathbb{R}^m_+} \varkappa(z) + \beta\alpha\mathcal{V}(z, \mathcal{Y}), \tag{3.11}$$

$$|\mathcal{V}(z + \alpha\mathcal{Y}, \mathcal{Y})| \leq \gamma|\mathcal{V}(z, \mathcal{Y})|. \tag{3.12}$$

Theorem 3.4 *Suppose that for all $i \in I_p$ and $j \in [m]$, the function $\mathcal{A}_j(z, \mathcal{M}_i)$ is CD with respect to t. Let $\mathcal{Y}$ be a $\mathbb{R}^m_+$-RSDD for $\varkappa$ at z, and assume there exists a vector $v \in \mathbb{R}^m$ such that*

$$\varkappa(z + \alpha\mathcal{Y}) \geq v, \quad \textit{for all } \alpha > 0. \tag{3.13}$$

Under these conditions, there exists a positive interval of step sizes for which both the standard Wolfe conditions and the strong Wolfe conditions for $(\mathcal{RP})$ are satisfied.

Proof: Let $\mathcal{Y}$ be an $\mathbb{R}^m_+$-RSDD for $\varkappa$ at z. Define the scalar function

$$\phi(\alpha) := \mathcal{V}(z + \alpha\mathcal{Y}, \mathcal{Y}), \qquad \alpha \geq 0.$$

Since $\mathcal{A}_j(z, \mathcal{M}_i)$ is continuously differentiable in t for all $i \in I_p$, $j \in I_m$, it follows that ϕ is continuous in α.

Because $\mathcal{Y}$ is a $\mathbb{R}^m_+$-RSDD at z, we have

$$\phi(0) = \mathcal{V}(z, \mathcal{Y}) < 0.$$

Hence, for sufficiently small $\alpha > 0$, by continuity,

$$\varkappa(z + \alpha\mathcal{Y}) \leq_{\mathbb{R}^m_+} \varkappa(z) + \beta\alpha\mathcal{V}(z, \mathcal{Y}),$$

so the sufficient decrease condition (3.9) holds.

From assumption (3.13), the vector function $\varkappa(z + \alpha\mathcal{Y})$ is bounded below for all $\alpha > 0$. This implies that $\phi(\alpha)$ cannot tend to $-\infty$ as $\alpha \to \infty$. Hence, there exists $\bar{\alpha} > 0$ such that

$$\phi(\bar{\alpha}) \geq \gamma\phi(0),$$

which yields the curvature condition (3.10). Combining the above arguments, we conclude that there exists an interval $(0, \bar{\alpha}]$ on which both (3.9) and (3.10) are satisfied. Therefore, the standard Wolfe conditions hold for some positive step sizes.

For the strong Wolfe conditions, note that by continuity of ϕ and the facts $\phi(0) < 0$ and $\phi(\bar{\alpha}) \geq \gamma\phi(0)$, the intermediate value theorem ensures the existence of $\alpha^* \in (0, \bar{\alpha}]$ such that

$$|\phi(\alpha^*)| \leq \gamma|\phi(0)|,$$

which gives (3.12). Since (3.11) coincides with (3.9), The strong Wolfe conditions are also satisfied. Hence, there exists a positive interval of step sizes for which both the standard and strong Wolfe conditions for ($\mathcal{RP}$) hold. □

This theorem represents the standard Wolfe-line search condition.

A RSDD has been identified, and the WTILS is applied to compute the step size. Based on these components, the steepest descent algorithm for ($\mathcal{RP}$) is formulated as follows.

Algorithm 1 ***(Wolfe-Type Steepest Descent Algorithm)***

1. ***Initialization:*** *Choose a tolerance $\epsilon > 0$, a line search parameter $\beta \in (0,1)$, and an initial point $z^0 \in \mathbb{R}^n$. Set the iteration counter $k := 0$.*
2. ***Computation of the search direction:*** *Solve the subproblem $P(z^k)$ to obtain the descent direction $\mathcal{Y}^k$ and compute the corresponding value $\Theta(z^k)$.*
3. ***Stopping test:*** *If $|\Theta(z^k)| < \epsilon$, then stop; otherwise, go to Step 4.*
4. ***Step size determination (Wolfe line search):*** *Find the largest step size*
$$\alpha_k \in \{2^{-r} : r = 0, 1, 2, \dots\}$$
such that both Wolfe conditions (3.9) and (3.10) are satisfied.
5. ***Update:*** *Update the iterate by*
$$z^{k+1} := z^k + \alpha_k \mathcal{Y}^k,$$
set $k := k + 1$, and return to Step 2.

Explanation of the Wolfe-Type Steepest Descent Algorithm. The Wolfe-type steepest descent algorithm is an iterative procedure for computing an approximate solution of the robust problem. In Step 1, the algorithm is initialized by choosing a tolerance $\epsilon > 0$, a line search parameter $\beta \in (0,1)$, and an initial point $z^0 \in \mathbb{R}^n$, and by setting the iteration counter $k = 0$.

In Step 2, the quadratic subproblem $P(z^k)$ is solved to obtain the search direction $\mathcal{Y}^k$, and the corresponding optimality measure $\Theta(z^k)$ is evaluated. Step 3 checks the stopping criterion. If $|\Theta(z^k)| < \epsilon$, the current iterate z^k is accepted as an approximate solution and the algorithm terminates. Otherwise, the procedure continues.

In Step 4, a step size α_k is determined by a backtracking line search. Specifically, the largest $\alpha_k \in \{2^{-r} : r = 0, 1, 2, \dots\}$ is selected such that both Wolfe-type conditions (3.9) and (3.10) are satisfied.

Finally, in Step 5, the new iterate is computed as $z^{k+1} = z^k + \alpha_k \mathcal{Y}^k$, the iteration counter is updated as $k := k + 1$, and the algorithm returns to Step 2. This process is repeated until the stopping criterion is met, ensuring sufficient descent and stability through the Wolfe-type inexact line search.

Having presented the structure and detailed description of the proposed algorithm, we now turn our attention to its convergence behavior. In the upcoming section, we analyze the sequence $\{z^k\}$ generated by the algorithm and establish the conditions under which it converges to a stationary point of the ($\mathcal{RP}$).

3.3. Convergence Analysis of the Proposed Algorithm

The purpose of this convergence analysis is to show that the sequence z^k generated by the proposed steepest descent method with Wolfe-type line search converges to a stationary point of the problem ($\mathcal{RP}$) under suitable assumptions. To this end, we first introduce the set of assumptions required for the analysis. Based on these assumptions, a Zoutendijk-type condition adapted to the proposed iterative scheme is established. This condition serves as a key ingredient in the convergence proof and ensures the stability of the method over different problem settings. The section then culminates in a main convergence theorem, which confirms the theoretical validity and reliability of the proposed algorithm.

Consider the iterative process defined by the steepest descent method with a Wolfe-type line search, expressed as

$$z^{k+1} = z^k + \alpha_k \mathcal{Y}^k, \quad k \geq 0, \tag{3.14}$$

where $\mathcal{Y}^k$ denotes a $\mathbb{C}$-descent direction for the function T at the point z^k, and α_k represents a suitable step size. We make use of the following assumptions to support the convergence analysis:

Assumption 1 *Assume that the cone $\mathbb{C}$ is finitely generated, and there exists an open set O such that*

$$L = \{x \in \mathbb{R}^n \mid \varkappa(x) \preceq_{\mathbb{C}} \varkappa(z^0)\} \subset O,$$

and that the Jacobian matrix $J\varkappa$ of $\varkappa$ is Lipschitz continuous on O with some constant $L > 0$, i.e., for any $x, y \in O$,

$$\|J\varkappa(z) - J\varkappa(y)\| \leq L\|x - y\|.$$

Assumption 2 *Assume further that every $\mathbb{C}$-monotone nonincreasing sequence in $\varkappa(L)$ is bounded from below. Specifically, if $\{G_k\}_{k\in\mathbb{N}} \subset \varkappa(L)$ satisfies $G_{k+1} \preceq_{\mathbb{C}} G_k$ for all k, then there exists a vector $G \in \mathbb{R}^m$ such that $G \preceq_{\mathbb{C}} G_k$ for every k.*

With these assumptions in place, we proceed to derive a Zoutendijk-type condition corresponding to the iteration scheme in (3.14).

Lemma 3.2 *Suppose that Assumptions 1 and 2 are satisfied. Let the sequence $\{z^k\}$ be generated by the iterative scheme in (3.14), where each direction $\mathcal{Y}^k$ is a $\mathbb{C}$-descent direction for $\varkappa$ at z^k, and the step size α_k satisfies the standard Wolfe conditions. Then, the following series is convergent:*

$$\sum_{k\geq 0} \frac{\mathcal{V}^2(z^k, \mathcal{Y}^k)}{\|\mathcal{Y}^k\|^2} < \infty. \tag{3.15}$$

Proof: From the Wolfe condition (3.10), in conjunction with Assumption 1 and the result in Lemma 2.3 of [34], it can be deduced that

$$(\sigma - 1)\mathcal{V}(z^k, \mathcal{Y}^k) \leq \mathcal{V}(z^{k+1}, \mathcal{Y}^k) - \mathcal{V}(z^k, \mathcal{Y}^k) \leq L\alpha_k\|\mathcal{Y}^k\|^2.$$

Since $\mathcal{V}(z^k, \mathcal{Y}^k) < 0$ and $\mathcal{Y}^k \neq 0$, it follows that

$$\frac{\mathcal{V}(z^k, \mathcal{Y}^k)}{\|\mathcal{Y}^k\|^2} \leq L\alpha_k \cdot \frac{\mathcal{V}(z^k, \mathcal{Y}^k)}{\sigma - 1}. \tag{3.16}$$

From Wolfe condition (3.9), the sequence $\{\varkappa(z^k)\}$ is $\mathbb{C}$-monotonically decreasing. That is,

$$\varkappa(z^{k+1}) - \varkappa(z^0) \preceq_{\mathbb{C}} \rho \sum_{j=0}^{k} \alpha_j \mathcal{V}(z^j, \mathcal{Y}^k)e,$$

for all $k \geq 0$. By Assumption 2, since $\{\varkappa(z^k)\} \subset L$, there exists $\bar{\varkappa} \in \mathbb{R}^m$ such that

$$\bar{\varkappa} - \varkappa(z^0) \preceq_{\mathbb{C}} \rho \sum_{j=0}^{k} \alpha_j \mathcal{V}(z^j, \mathcal{Y}^j)e.$$

Hence, for all $w \in K$,

$$\langle \bar{\varkappa} - \varkappa(z^0), w\rangle \leq \rho\langle e, w\rangle \sum_{j=0}^{k} \alpha_j \mathcal{V}(z^j, \mathcal{Y}^j).$$

Taking the minimum over all $w \in K$ with $\|w\| = 1$, we obtain

$$\frac{1}{\sigma - 1} \min_{\bar{w}\in K} \langle \bar{\varkappa} - \varkappa(z^0), \bar{w}\rangle \geq \frac{\rho\langle e, w\rangle}{\sigma - 1} \sum_{j=0}^{k} \alpha_j \mathcal{V}(z^j, \mathcal{Y}^j) > 0.$$

Thus,

$$\sum_{k\geq 0} \alpha_k \cdot \frac{\mathcal{V}(z^k, \mathcal{Y}^k)}{\sigma - 1} < \infty.$$

Combining this with inequality (3.16) yields the desired result.

$$\sum_{k\geq 0} \frac{\mathcal{V}(z^k, \mathcal{Y}^k)}{\|\mathcal{Y}^k\|^2} < \infty.$$

□

Remark 3.1 The Zoutendijk-type condition given in (3.15) plays a central role in the convergence analysis of line search methods. In particular, when the step sizes are chosen according to the standard Wolfe conditions, the steepest descent algorithm guarantees that

$$\lim_{k\to\infty} \|\mathcal{Y}(z^k)\| = 0,$$

which shows that the norm of the descent direction converges to zero along the generated sequence.

Under the stated assumptions, the established Zoutendijk-type condition ensures a sufficient decrease of the objective function values along the sequence generated by the algorithm. This result provides a fundamental tool for analyzing the convergence behavior of the proposed method. Based on this condition, we can now state the following theorem, which guarantees that the sequence z^k produced by the algorithm converges to a Pareto critical point of the problem $\mathcal{RP}$.

Theorem 3.5 *Let the sequence $\{z^k\}$ be generated by the steepest descent algorithm. Suppose that Assumptions 1 and 2 hold, and the following condition is satisfied:*

$$\sum_{k\geq 0} \frac{1}{\|\mathcal{Y}^k\|^2} = \infty. \tag{3.17}$$

If each direction $\mathcal{Y}^k$ fulfills a sufficient descent condition and the corresponding step size α_k satisfies the standard Wolfe conditions (3.9) and (3.10), then the sequence satisfies

$$\liminf_{k\to\infty} \|\mathcal{Y}(z^k)\| = 0.$$

Proof: Suppose, for the sake of contradiction, that there exists a constant $\gamma > 0$ such that $\|\mathcal{Y}(z^k)\| \geq \gamma$ holds for all $k \geq 0$. Let us show the result. Observe that, by the inequality

$$\mathcal{V}(z^k, \mathcal{Y}(z^k)) + \frac{1}{2}\|\mathcal{Y}(z^k)\|^2 < 0. \tag{3.18}$$

As $\mathcal{V}(z^k, \mathcal{Y}(z^k)) < 0$, d^k satisfies a sufficient descent condition, i.e. there exists, $c > 0$ such that

$$\mathcal{V}(z^k, \mathcal{Y}(z^k)) < c\mathcal{V}(z^k, \mathcal{Y}(z^k)). \tag{3.19}$$

By using (3.18) and (3.19), we obtain:

$$\frac{c_2\gamma^4}{4\|\mathcal{Y}^k\|^2} \leq \frac{c_2\|\mathcal{Y}(z^k)\|^4}{4\|\mathcal{Y}^k\|^2} \leq \frac{c_2\mathcal{V}(z^k, \mathcal{Y}(z^k))}{\|\mathcal{Y}^k\|^2} \leq \frac{\mathcal{V}(z^k, \mathcal{Y}^k)}{\|\mathcal{Y}^k\|^2}.$$

Since, under the hypothesis of the theorem, the Zoutendijk condition (3.3) holds, we have a contradiction to (3.17). Hence, $\liminf_{k\to\infty} \|\mathcal{Y}(z^k)\| = 0$. □

4. Conclusions

In this paper, we studied an uncertain multiobjective optimization problem (UMOP) and transformed it into a deterministic formulation using objective-wise worst-case robust counterparts. To solve the resulting robust problem, we developed a steepest descent algorithm equipped with a Wolfe-type inexact line search. The descent direction is obtained by solving an associated subproblem, while the step size is selected using the Wolfe conditions to ensure both sufficient decrease and curvature control. A Zoutendijk-type condition was established for the robust counterpart, which forms the basis for proving the convergence of the proposed method under standard assumptions.

The present study focuses on UMOPs with finite uncertainty sets. An interesting direction for future research is to extend the proposed framework to problems with infinite uncertainty sets, which would further improve the applicability of the method to more general and realistic uncertain environments.

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Conflict of interest

The authors declare that there are no competing interests associated with this work.

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